

ON THE INVERSE OF PARABOLIC BOUNDARY VALUE PROBLEMS FOR LARGE TIMES

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ABSTRACT. We construct algebras of Volterra pseudodifferential operators that contain, in particular, the inverses of the most natural classical systems of parabolic boundary value problems of general form.

Parabolicity is determined by the invertibility of the principal symbols, and as a result is equivalent to the invertibility of the operators within the calculus. Existence, uniqueness, regularity, and asymptotics of solutions as $t \rightarrow \infty$ are consequences of the mapping properties of the operators in exponentially weighted Sobolev spaces and subspaces with asymptotics. An important aspect of this work is that the microlocal and global kernel structure of the inverse operator (solution operator) of a parabolic boundary value problem for large times is clarified. Moreover, our approach naturally yields qualitative perturbation results for the solvability theory of parabolic boundary value problems.

To achieve these results, we assign $t = \infty$ the meaning of a conical point and treat the operators as totally characteristic pseudodifferential boundary value problems.

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INTRODUCTION

The present article studies invertibility, regularity, and asymptotics of general systems of parabolic boundary value problems in an infinite space–time cylinder within a new frame of pseudodifferential analysis on singular and non-compact manifolds. More precisely, our approach is a combination of Volterra pseudodifferential calculus of boundary value problems with the transmission property at the boundary and analysis of degenerate pseudodifferential operators with meromorphic symbolic structures via interpreting $t = \infty$ as an anisotropic conical point.

The general concept of pseudodifferential analysis to derive qualitative properties such as existence, uniqueness, regularity, and asymptotics of solutions to partial differential equations via embedding the problems into an operator algebra with symbolic structure is far developed, in particular, in elliptic theory. Ellipticity is determined by the invertibility of the principal symbol — which is a tuple of scalar and operator-valued components in general — and the existence of parametrices within the pseudodifferential calculus is proved. In particular, Fredholm solvability of elliptic equations is achieved in natural scales of Sobolev spaces associated with the problems under consideration with the parametrix being a Fredholm inverse.

The literature on this topic is vast; we just refer to Hörmander [31] (especially vol. III), Cordes [10], Egorov and Schulze [13], Kumano-go [40], or Shubin [70] for introductions to the general theory; Boutet de Monvel [6], Grubb [26], Rempel and Schulze [57] are concerned with elliptic pseudodifferential boundary value problems (Boutet de Monvel’s algebra) that complete the classical elliptic differential boundary value problems satisfying Shapiro–Lopatinskij conditions as considered by Agmon, Douglis, and Nirenberg [1]; degenerate elliptic equations (point singularities and more complicated structures) are subject to the works of Schulze [63, 65], Melrose and Mendoza [48], Melrose [47], and Plamenevskij [55]. Schrohe and Schulze [59, 60] constructed an enveloping pseudodifferential operator algebra of the differential boundary value problems of Fuchs type as considered by

Kondrat'ev [36]; especially these works constitute one of the main sources of the present article.

In contrast to the elliptic theory, much less attention has been paid to the pseudodifferential analysis of parabolic equations. Much work is concerned with the resolvent analysis of operators and consequences for the associated heat equation, see, e.g., Gilkey [21], Seeley [67, 68], Grubb and Seeley [27], Shubin [70]; also the works of Grubb and Solonnikov [28], Grubb [24, 25], and Purmonen [56] are to be seen in this context. However, apart from the study of parabolic operators within the substantially larger class of anisotropic elliptic ones, there are comparatively few works that deal with parabolic operators directly.

Piriou [53, 54] introduced the calculus of Volterra pseudodifferential operators that is specifically designed for the analysis of parabolic equations. The symbols extend holomorphically with respect to the time covariable to the lower complex half-plane, and, analogously to the elliptic theory, parabolicity is determined by the invertibility of the principal symbol, but this is now to hold within the half-plane, too. The main aspect is that under this hypothesis the operator itself is invertible within the calculus, which is a much stronger structural result than to just obtain a parametrix. While Piriou's efforts were mainly concentrated to study parabolicity on a closed spatial manifold in finite time, Rempel and Schulze [57] initiated the analysis of parabolic boundary value problems in Boutet de Monvel's calculus in finite time, and Krainer and Schulze [39] studied parabolicity within a calculus of Volterra pseudodifferential operators on a closed spatial manifold with exponential weights as $t \rightarrow \infty$.

The present article is concerned with the completion of the classical theory of parabolic boundary value problems satisfying Shapiro–Lopatinskij conditions on a compact spatial manifold with smooth boundary as considered by Agranovich and Vishik [3], Ejdel'man and Zhitarashu [15] to an algebra of Volterra pseudodifferential operators on the infinite time interval with exponential weights at infinity. We thereby also control the long-time behaviour of solutions in terms of exponential long-time asymptotics (see Agmon and Nirenberg [2], Maz'ya and Plamenevskij [46], Pazy [51]) as a regularity feature of the pseudodifferential calculus (see the notes at the end of Section 4).

The main new aspect of this work is that the analysis of the relevant effects as $t \rightarrow \infty$ is included; this additional non-compactness in fact gives rise to degenerate operators and requires to establish new structures of pseudodifferential Volterra boundary value problems in order to achieve the desired results. More precisely, with the new time coordinate $r = e^{-t}$ it is natural to treat the problems as anisotropic totally characteristic ones with respect to $r = 0$, and we end up with an analytic setup similar to boundary value problems on conic manifolds close to a conical point, see Schrohe and Schulze [59, 60]; however, we have to cope with additional difficulties that are due to parabolicity and Volterra calculus. As these global aspects are the primary focus of this work, we exclude any considerations of the initial value problem and restrict ourselves to inhomogeneous right hand sides of the equations in space–time and homogeneous initial data, i.e., the operators

are supposed to act within exponentially weighted anisotropic Sobolev spaces of distributions that vanish identically near $t = -\infty$ — in fact, we often meet this simplification in the literature on parabolic operators, but, nevertheless, we shall also pursue generalizations.

In [39] the case of empty boundary was considered; the present results on boundary value problems now directly apply to problems from applications as also the classical theory of differential boundary value problems is embedded. Notice, e.g., that we may consider non-local perturbations in the calculus with vanishing principal symbols (generalized drift and potential terms) of a parabolic differential boundary value problem, and the qualitative properties of the new equation (existence, uniqueness, smoothness, and control of exponential long-time asymptotics) remain unchanged as also this equation is invertible within the calculus, and the qualitative properties are consequences from the general regularity concept. Note that — as we carry out the construction of the Volterra calculus in Boutet de Monvel's algebra with unified order to keep notation reasonably small — we actually have to use reductions of orders on the boundary of the space-time cylinder to make the differential boundary value problems fit directly into the concept; an explicit construction of suitable parabolic reductions of orders can be found in [39].

Organization of the text. We briefly recall in Section 1 the basic definitions and properties of the Volterra symbolic calculus before we enter the discussion of a parameter-dependent Boutet de Monvel's calculus in Sections 2 and 3. The parameter thereby represents the time covariable, and it is therefore necessary to consider Volterra boundary value problems, i.e., the parameter runs over a complex half-plane with holomorphic dependence. The parameter-dependent theory of Volterra boundary value problems in Boutet de Monvel's calculus is the elementary building block for what follows in the remaining parts of this work.

In Section 4 we give a first structure result about the inverse of parabolic boundary value problems. This result already admits to prove and to control existence, uniqueness and smoothness of solutions in exponentially weighted anisotropic Sobolev spaces, but not to observe exponential long-time asymptotics. The remaining sections are devoted to establish the (smaller) calculus of totally characteristic pseudodifferential Volterra boundary value problems that admits to include also this missing feature. In Section 5 the meromorphic conormal symbolic structure is studied — a central additional ingredient near the origin $r = 0$ (which corresponds to $t \rightarrow \infty$ in the original coordinates). In Section 6 we give the definitions of the weighted anisotropic Sobolev spaces and their subspaces with asymptotics, while Sections 7 and 8 are devoted to establish the calculus itself and its algebraic properties. Finally, in Section 9, we study parabolicity and give a proof of the equivalence of parabolicity and invertibility within the calculus.

Concluding remarks. The theory of boundary value problems may conceptually be regarded as a specific case of edge-degenerate problems, the boundary being the edge (see Schulze [63, 65]). Though motivated by classical questions and being embedded in a classical context, the present article indicates, in particular,

how the idea to systematically combine Volterra pseudodifferential calculus with pseudodifferential analysis of singular problems may yield insights also for degenerate parabolic equations. It seems that this concept is the most promising one to approach singular parabolic problems, where in general a hierarchy of symbols and extra conditions of trace and potential type on the singular strata of the space–time configuration as a whole are to be expected (see [66]).

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1. BASIC CALCULUS OF VOLTERRA SYMBOLS

We give a brief review over the basic definitions and properties of general anisotropic operator-valued symbols and Volterra symbols. A detailed discussion can be found, e.g., in Buchholz and Schulze [7], and Krainer [38]; we refer to Piriou [53, 54] for classical material on scalar Volterra symbols. Throughout this work, the anisotropy $\ell \in 2\mathbb{N}$ is fixed in all considerations of the symbolic and operational calculi.

For $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ we denote $\langle \xi, \lambda \rangle_\ell := (1 + |\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}}$, where $|\cdot|$ denotes the Euclidean norm. Note that there exists a constant $c > 0$ such that for all $s \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^n$, $\lambda_1, \lambda_2 \in \mathbb{R}^q$ the following inequality is fulfilled (*Peetre's inequality*):

$$\langle \xi_1 + \xi_2, \lambda_1 + \lambda_2 \rangle_\ell^s \leq c^{|s|} \langle \xi_1, \lambda_1 \rangle_\ell^{|s|} \langle \xi_2, \lambda_2 \rangle_\ell^s. \quad (1.1)$$

Moreover, for a multi-index $\beta = (\alpha, \alpha') \in \mathbb{N}_0^{n+q}$ let $|\beta|_\ell := |\alpha| + \ell \cdot |\alpha'|$, where $|\cdot|$ denotes the usual length of a multi-index as the sum of its components.

1.1. General anisotropic operator-valued symbols. Let $\Lambda \subseteq \mathbb{R}^q$ be conical, i.e., for $\lambda \in \Lambda$ and $\varrho \in \mathbb{R}_+$ we have $\varrho \cdot \lambda \in \Lambda$; moreover, assume that Λ is closed, and equals the closure of its interior points.

Let E and \tilde{E} be Hilbert spaces endowed with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. Recall that we consider strongly continuous group-actions $\kappa : (\mathbb{R}_+, \cdot) \rightarrow \mathcal{L}(E)$, i.e., for each $e \in E$ the function $\mathbb{R}_+ \ni \varrho \mapsto \kappa_\varrho e \in E$ is continuous, and $\kappa_\varrho \kappa_{\varrho'} = \kappa_{\varrho\varrho'}$ for $\varrho, \varrho' \in \mathbb{R}_+$, as well as $\kappa_1 = \text{Id}_E$. The space of $\mathcal{L}(E, \tilde{E})$ -valued anisotropic symbols of order $\mu \in \mathbb{R}$ is defined as

$$S^{\mu; \ell}(\mathbb{R}^n \times \Lambda; E, \tilde{E}) := \{a \in C^\infty(\mathbb{R}^n \times \Lambda, \mathcal{L}(E, \tilde{E})); \text{ for all } k \in \mathbb{N}_0 : \\ \sup_{\substack{(\xi, \lambda) \in \mathbb{R}^n \times \Lambda \\ |\beta|_\ell \leq k}} \|\tilde{\kappa}_{\langle \xi, \lambda \rangle_\ell}^{-1} \partial_{(\xi, \lambda)}^\beta a(\xi, \lambda) \kappa_{\langle \xi, \lambda \rangle_\ell} \| \langle \xi, \lambda \rangle_\ell^{-\mu + |\beta|_\ell} < \infty\}.$$

The subspace of *classical symbols* is given as

$$S_{cl}^{\mu; \ell}(\mathbb{R}^n \times \Lambda; E, \tilde{E}) := \left\{ a \in S^{\mu; \ell}(\mathbb{R}^n \times \Lambda; E, \tilde{E}); a \sim \sum_{k=0}^{\infty} \chi_{a(\mu-k)} \right\},$$

where $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ is a 0-excision function, i.e., $\chi \equiv 0$ near the origin and $\chi \equiv 1$ near infinity, and $a_{(\mu-k)} \in C^\infty((\mathbb{R}^n \times \Lambda) \setminus \{0\}, \mathcal{L}(E, \tilde{E}))$ are (anisotropic) homogeneous functions of degree $\mu - k$, the so-called *homogeneous components* of a .

Recall that a function $f : (\mathbb{R}^n \times \Lambda) \setminus \{0\} \rightarrow \mathcal{L}(E, \tilde{E})$ is called (anisotropic) homogeneous of degree $\mu \in \mathbb{R}$, if for $(\xi, \lambda) \in (\mathbb{R}^n \times \Lambda) \setminus \{0\}$ and $\varrho > 0$

$$f(\varrho\xi, \varrho^\ell\lambda) = \varrho^\mu \tilde{\kappa}_\varrho f(\xi, \lambda) \tilde{\kappa}_\varrho^{-1}. \quad (1.2)$$

1.2. Operator-valued Volterra symbols. The space of (classical) *Volterra symbols* of order $\mu \in \mathbb{R}$ is defined as

$$S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) := S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \cap \mathcal{A}(\mathbb{H}, C^\infty(\mathbb{R}^n, \mathcal{L}(E, \tilde{E}))),$$

where $\mathbb{H} := \{z \in \mathbb{C}; \operatorname{Im}(z) \geq 0\} \subseteq \mathbb{C}$ is the upper half-plane. We will also consider Volterra symbols with respect to the lower half-plane \mathbb{H}_- , and a right half-plane $\mathbb{H}_\beta := \{z \in \mathbb{C}; \operatorname{Re}(z) \geq \beta\}$ with the origin “shifted” to $\beta \in \mathbb{R}$.

Notice that restriction from the upper half-plane to the real line induces an embedding $S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \hookrightarrow S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$. The homogeneous components of a classical Volterra symbol are holomorphic in the interior of the half-plane \mathbb{H} .

As usual, the symbol spaces of order $-\infty$ are defined as the intersections over all symbol spaces of order $\mu \in \mathbb{R}$. Notice that these spaces are independent of the anisotropy $\ell \in 2\mathbb{N}$ and the group-actions, and they are denoted as $S_{(V)}^{-\infty}(\mathbb{R}^n \times \Lambda; E, \tilde{E})$.

All symbol spaces carry Fréchet topologies in a canonical way.

1.3. Asymptotic expansion. Recall the definition of asymptotic expansion for the spaces of (Volterra) symbols:

Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \xrightarrow[k \rightarrow \infty]{} -\infty$ and $\bar{\mu} := \max_{k \in \mathbb{N}} \mu_k$.

Moreover, let $a_k \in S_{(V)}^{\mu_k;\ell}(\mathbb{R}^n \times \Lambda; E, \tilde{E})$. A symbol $a \in S_{(V)}^{\bar{\mu};\ell}(\mathbb{R}^n \times \Lambda; E, \tilde{E})$ is called the asymptotic expansion of the a_k , if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $a - \sum_{j=1}^k a_j \in S_{(V)}^{R;\ell}(\mathbb{R}^n \times \Lambda; E, \tilde{E})$. The symbol a is uniquely

determined modulo $S_{(V)}^{-\infty}(\mathbb{R}^n \times \Lambda; E, \tilde{E})$, and for short we write $a \underset{(V)}{\sim} \sum_{k=1}^{\infty} a_k$.

Given a sequence of anisotropic operator-valued (Volterra) symbols a_k as above, there exists a symbol a such that $a \underset{(V)}{\sim} \sum_{k=1}^{\infty} a_k$. In any case, the proof relies on a Borel argument: More precisely, within the framework of general symbols a can be constructed as a convergent series $a(\xi, \lambda) := \sum_{k=1}^{\infty} \chi\left(\frac{\xi}{c_k}, \frac{\lambda}{c_k^\ell}\right) a_k(\xi, \lambda)$, where $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ is a 0-excision function, and $(c_k) \subseteq \mathbb{R}_+$ such that $c_k \rightarrow \infty$ as $k \rightarrow \infty$ sufficiently fast. This argument breaks down for Volterra symbols, for the

analyticity in the covariable is not preserved. Nevertheless, the result about asymptotic completeness of the Volterra symbol classes holds true, and we can construct a as a convergent sum $a(\xi, \zeta) := \sum_{k=1}^{\infty} (H(\varphi(c_k t)) a_k)(\xi, \zeta)$ using the (Fourier) kernel cut-off operator

$$(H(\varphi)b)(\xi, \zeta) := \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\tau} \varphi(t) b(\xi, \zeta - \tau) dt d\tau \sim \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j!} D_t^j \varphi(0) \right) \cdot \partial_{\zeta}^j b(\xi, \zeta), \quad (1.3)$$

where in this case $\varphi \in C_0^{\infty}(\mathbb{R})$ with $\varphi \equiv 1$ near $t = 0$, and $(c_k) \subseteq \mathbb{R}_+$ with $c_k \rightarrow \infty$ as $k \rightarrow \infty$ sufficiently fast. The argument is worked out in detail in Krainer [38], see also Buchholz and Schulze [7]. There are alternative arguments to prove the asymptotic completeness, see, e.g., Mikayelyan [49].

1.4. The translation operator in Volterra symbols. For $\tau \geq 0$ the translation operator $(T_{i\tau} a)(\xi, \zeta) := a(\xi, \zeta + i\tau)$ is continuous in the spaces

$$T_{i\tau} : S_{V(cl)}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow S_{V(cl)}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}),$$

and the asymptotic expansion $T_{i\tau} a \sim \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \partial_{\zeta}^k a$ holds. In particular, the operator $I - T_{i\tau}$ is continuous in the spaces

$$I - T_{i\tau} : S_{V(cl)}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow S_{V(cl)}^{\mu-1; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}).$$

An important application of the translation operator is that it provides a splitting of the principal symbol sequence in the classes of Volterra symbols. More precisely, this means the following:

Let $S_V^{(\mu; \ell)}((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E})$ denote the closed subspace of $C^{\infty}((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}, \mathcal{L}(E, \tilde{E}))$ consisting of all anisotropic homogeneous functions of degree $\mu \in \mathbb{R}$ that are holomorphic in the interior of \mathbb{H} . For every $\tau > 0$ the translation operator is continuous in the spaces

$$T_{i\tau} : S_V^{(\mu; \ell)}((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}) \longrightarrow S_{V(cl)}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}),$$

and for every 0-excision function $\chi \in C^{\infty}(\mathbb{R}^n \times \mathbb{H})$ the asymptotic expansion $T_{i\tau} a \sim \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \chi(\partial_{\zeta}^k a)$ holds. This shows, in particular, that for the principal homogeneous component of order μ we have the identity $(T_{i\tau} a)_{(\mu)} = a$.

In other words, the principal symbol sequence

$$0 \longrightarrow S_{V(cl)}^{\mu-1; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow S_{V(cl)}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow S_V^{(\mu; \ell)}((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}) \longrightarrow 0$$

for Volterra symbols is topologically exact and splits, and the operator $T_{i\tau}$ provides a splitting of this sequence.

1.5. The construction of Volterra parametrices. To illustrate the construction of parametrices within the Volterra symbol classes assume that we are given a symbol

$$a(x, \xi, \zeta) \in S_{V_{cl}}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) = C_b^\infty(\mathbb{R}^n, S_{V_{cl}}^{\mu; \ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}))$$

that is parameter-dependent elliptic with parameter-space \mathbb{H} , i.e., the homogeneous principal component $a_{(\mu)}(x, \xi, \zeta)$ is invertible for $x \in \mathbb{R}^n$ and $(\xi, \zeta) \neq 0$ with $\|a_{(\mu)}(x, \xi, \zeta)^{-1}\| = O(1)$ as $|x| \rightarrow \infty$, uniformly for $(|\xi|^{2\ell} + |\zeta|^2)^{\frac{1}{2\ell}} = 1$. Notice that within the Volterra symbol classes (parameter-dependent) elliptic symbols are called *parabolic* in general.

Then there exists a Volterra symbol $p(x, \xi, \zeta) \in S_{V_{cl}}^{-\mu; \ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ such that $a \# p - 1$ and $p \# a - 1$ are Volterra symbols of order $-\infty$.

To see this note that according to Section 1.4 we have $b := a_{(\mu)}(x, \xi, \zeta + i)^{-1} \in S_{V_{cl}}^{-\mu; \ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$, and $ab - 1$ and $ba - 1$ are Volterra symbols of order -1 , and so are $a \# b - 1$ and $b \# a - 1$. Now we may apply the formal Neumann series argument to obtain p ; notice that asymptotic expansions can be carried out within the classes of Volterra symbols (see Section 1.3).

Observe that for this example we considered ordinary parameter-dependent global pseudodifferential operators $\text{op}_x(a)(\zeta)$ on \mathbb{R}^n , i.e., the Leibniz-product of (Volterra) symbols $a(x, \xi, \zeta)$ and $b(x, \xi, \zeta)$ is given as

$$a \# b(x, \xi, \zeta) = \iint e^{-iy\eta} a(x, \xi + \eta, \zeta) b(x + y, \xi, \zeta) dy d\eta \underset{(V)}{\sim} \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} (\partial_\xi^\alpha a)(D_x^\alpha b).$$

2. VOLTERRA BOUNDARY VALUE PROBLEMS ON THE HALF-SPACE

The present section is devoted to establish the structure of boundary value problems with parameters localized on the half-space. The parameter represents the additional time-covariable which will later enter the considerations when having the structure of the operators (boundary value problems) with respect to the spatial variables at hand. For this reason it is crucial to consider Volterra boundary value problems, i.e., the operators depend holomorphically on the parameter in a half-plane.

We employ the classical calculus of Boutet de Monvel [6] for boundary value problems, where we make use of a presentation of the boundary symbolic and operational calculus due to Schulze (see, e.g., [65]) which emphasizes its pseudodifferential nature. See also Grubb [26], Rempel and Schulze [57], and Schrohe [58] for expositions on Boutet de Monvel's algebra.

2.1. Spaces of distributions on the half-space. Let $H_0^{s, \delta}(\overline{\mathbb{R}_+^n})$ be the closed subspace of all distributions $u \in H^{s, \delta}(\mathbb{R}^n) = \langle x \rangle^{-\delta} H^s(\mathbb{R}^n)$ such that $\text{supp}(u) \subseteq \overline{\mathbb{R}_+^n}$, which coincides with the closure of $C_0^\infty(\mathbb{R}_+^n)$ in $H^{s, \delta}(\mathbb{R}^n)$.

Moreover, with the restriction operator $r^+ : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}_+^n)$ we denote $H^{s, \delta}(\mathbb{R}_+^n) := r^+ H^{s, \delta}(\mathbb{R}^n)$, endowed with the quotient topology. The kernel

of r^+ in $H^{s,\delta}(\mathbb{R}^n)$ equals $H_0^{s,\delta}(\overline{\mathbb{R}}_-^n)$, and there are continuous extension operators $e_{s,\delta} : H^{s,\delta}(\mathbb{R}_+^n) \rightarrow H^{s,\delta}(\mathbb{R}^n)$. Hence the sequence $0 \rightarrow H_0^{s,\delta}(\overline{\mathbb{R}}_-^n) \rightarrow H^{s,\delta}(\mathbb{R}^n) \rightarrow H^{s,\delta}(\mathbb{R}_+^n) \rightarrow 0$ is topologically exact and splits. Embedding and complex interpolation properties carry over to the scales $\{H^{s,\delta}(\mathbb{R}_+^n)\}_{s,\delta \in \mathbb{R}}$.

The $L^2(\mathbb{R}_+^n)$ -inner product extends to a non-degenerate sesquilinear pairing $H^{s,\delta}(\mathbb{R}_+^n) \times H_0^{-s,-\delta}(\overline{\mathbb{R}}_+^n) \rightarrow \mathbb{C}$ for $s, \delta \in \mathbb{R}$, which induces an identification of the dual spaces $H^{s,\delta}(\mathbb{R}_+^n)' \cong H_0^{-s,-\delta}(\overline{\mathbb{R}}_+^n)$ and $H_0^{s,\delta}(\overline{\mathbb{R}}_+^n)' \cong H^{-s,-\delta}(\mathbb{R}_+^n)$.

Let $\mathcal{S}(\mathbb{R}_+^n) := r^+ \mathcal{S}(\mathbb{R}^n)$, endowed with the quotient topology. Then $\mathcal{S}(\mathbb{R}_+^n) \cong \text{proj-lim}_{s,\delta \in \mathbb{R}} H^{s,\delta}(\mathbb{R}_+^n)$, and $\mathcal{S}(\mathbb{R}_+^n)$ is dense in $H^{s,\delta}(\mathbb{R}_+^n)$ for all $s, \delta \in \mathbb{R}$. Moreover, we have $\mathcal{S}'(\mathbb{R}_+^n) := \mathcal{S}(\mathbb{R}_+^n)' \cong \text{ind-lim}_{s,\delta \in \mathbb{R}} H_0^{s,\delta}(\overline{\mathbb{R}}_+^n)$.

We denote e^+ to be the operator of extension by zero for functions defined on the half-space \mathbb{R}_+^n to functions defined on the full space \mathbb{R}^n . Then e^+ makes sense as an operator $e^+ : H_0^{s,\delta}(\overline{\mathbb{R}}_+^n) \rightarrow H^{s,\delta}(\mathbb{R}^n)$ for all $s, \delta \in \mathbb{R}$ which coincides with inclusion. For $-\frac{1}{2} < s < \frac{1}{2}$ the operator r^+ acts as a topological isomorphism $H_0^{s,\delta}(\overline{\mathbb{R}}_+^n) \rightarrow H^{s,\delta}(\mathbb{R}_+^n)$, and we may identify these spaces with each other. In particular, the operator e^+ is well-defined in $H^{s,\delta}(\mathbb{R}_+^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ for $s > -\frac{1}{2}$.

By passing to direct sums (or tensor products) we have for $N \in \mathbb{N}_0$ the corresponding \mathbb{C}^N -valued analogues of the spaces above. If $\delta = 0$ we drop it from the notation.

For functions u defined in a conical subset of \mathbb{R}_{x_n} we set

$$(\kappa_\varrho u)(x_n) = \varrho^{\frac{1}{2}} u(\varrho x_n) \quad (2.1)$$

for $\varrho \in \mathbb{R}_+$. Then $\{\kappa_\varrho\}$ gives rise to a strongly continuous group-action in the spaces $H^{s,\delta}(\mathbb{R}, \mathbb{C}^N)$, $H_0^{s,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^N)$, and $H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^N)$ for all $s, \delta \in \mathbb{R}$.

2.2. Volterra transmission symbols.

Definition 2.1. Let $\Lambda \subseteq \mathbb{R}^q$ be conical, and assume that Λ is the closure of its interior. A parameter-dependent classical symbol

$$a((x', x_n), (\xi', \xi_n), \lambda) \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda) = C_b^\infty(\mathbb{R}_x^n, S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \Lambda))$$

with parameter-space Λ has the transmission property if

$$a((x', x_n), (\xi', \langle \xi', \lambda \rangle_\ell \xi_n), \lambda) \in S^{\mu;\ell}(\mathbb{R}_{x'}^{n-1} \times \mathbb{R}_{\xi'}^{n-1} \times \Lambda_\lambda, S_{tr}^\mu(\mathbb{R}_{x_n} \times \mathbb{R}_{\xi_n})),$$

where $S_{tr}^\mu(\mathbb{R}_{x_n} \times \mathbb{R}_{\xi_n})$ is the space of symbols with the transmission property with respect to $x_n = 0$, i.e., $a(x, \xi, \lambda)$ has the transmission property if and only if the homogeneous components $a_{(\mu-j)}$ satisfy

$$D_{x_n}^k \partial_{(\xi', \lambda)}^\alpha a_{(\mu-j)}(x', 0, 0, -1, 0) = e^{i\pi(\mu-j-|\alpha|_\ell)} D_{x_n}^k \partial_{(\xi', \lambda)}^\alpha a_{(\mu-j)}(x', 0, 0, +1, 0) \quad (2.2)$$

for $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^{(n-1)+q}$. We denote the space of classical parameter-dependent symbols with the transmission property as $S_{tr,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda)$.

The spaces of classical $\mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ -valued parameter-dependent (Volterra) symbols with the transmission property are defined as follows:

$$\begin{aligned} S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) &:= S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \\ &\subseteq S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}), \\ S_{tr\,V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) &:= S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \cap \\ &\quad S_{V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}). \end{aligned}$$

Remark 2.2. According to the general theory of transmission symbols these spaces are closed subspaces of the general (Volterra) symbol spaces. Moreover, the (Volterra) symbols of order $-\infty$ have the transmission property, i.e., the condition of having the transmission property is empty for smoothing symbols.

The spaces of (Volterra) transmission symbols are invariant with respect to the following operations:

- pointwise products,
- taking partial derivatives,
- asymptotic expansions, i.e., if $a \underset{(V)}{\sim} \sum_{j=0}^{\infty} a_j$ with (Volterra) transmission symbols a_j then a is a (Volterra) transmission symbol,
- Leibniz-products,
- formal adjoints in case of general symbols; note that the Volterra symbol class is not preserved under this operation.

Notice that also the kernel cut-off operator and the translation operator act within the spaces of (Volterra) transmission symbols, and we may use the translation operator to associate with a principal symbol $a_{(\mu)}(x, \xi, \zeta)$ that is holomorphic with respect to $\zeta \in \mathbb{H}$ and fulfills the symmetry relation (2.2) a parameter-dependent Volterra transmission symbol $a(x, \xi, \zeta)$.

Remark 2.3. Let $a \in S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ be parameter-dependent elliptic, and let p of order $-\mu$ be such that $a \# p - 1$ and $p \# a - 1$ are of order $-\infty$. Then p has the transmission property.

In other words, the calculus of (Volterra) transmission operators is closed with respect to the construction of (Volterra) parametrices to parameter-dependent elliptic (parabolic) elements.

Definition 2.4. With a symbol $a(x, \xi, \lambda) \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ we associate the operator convention $\text{op}_x^+(a) = r^+ \text{op}_x(a)e^+$, which is apriori well-defined as a λ -dependent operator family $\mathcal{S}(\mathbb{R}_+^n, \mathbb{C}^{N_-}) \longrightarrow r^+ \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{N_+})$. We mainly need the partial action

$$\text{op}_{x_n}^+(a) = r^+ \text{op}_{x_n}(a((x', x_n), (\xi', \xi_n), \lambda))e^+ : \mathcal{S}(\mathbb{R}_+, \mathbb{C}^{N_-}) \longrightarrow r^+ \mathcal{S}'(\mathbb{R}, \mathbb{C}^{N_+}),$$

which depends as a family of operators on $(x', \xi', \lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda$.

Theorem 2.5. Let $a \in S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$. Then $\text{op}_{x_n}^+(a)$ extends by continuity from $\mathcal{S}(\mathbb{R}_+, \mathbb{C}^{N_-})$, respectively $C_0^\infty(\mathbb{R}_+, \mathbb{C}^{N_-})$, to a family of continuous

operators

$$\begin{aligned} op_{x_n}^+(a) &: H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}) \longrightarrow H^{s-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+}), \quad s > -\frac{1}{2}, \delta \in \mathbb{R}, \\ op_{x_n}^+(a) &: H_0^{t,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^{N_-}) \longrightarrow H^{t-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+}), \quad t, \delta \in \mathbb{R}. \end{aligned}$$

More precisely, $op_{x_n}^+(\cdot)$ gives rise to a continuous operator

$$\begin{aligned} op_{x_n}^+ &: S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \longrightarrow \\ &\begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), H^{s-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+})), & s > -\frac{1}{2}, \delta \in \mathbb{R}, \\ S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H_0^{t,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^{N_-}), H^{t-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+})), & t, \delta \in \mathbb{R}. \end{cases} \end{aligned}$$

For Volterra transmission symbols we obtain

$$\begin{aligned} op_{x_n}^+ &: S_{tr\,V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \longrightarrow \\ &\begin{cases} S_V^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), H^{s-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+})), & s > -\frac{1}{2}, \delta \in \mathbb{R}, \\ S_V^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; H_0^{t,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^{N_-}), H^{t-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+})), & t, \delta \in \mathbb{R}. \end{cases} \end{aligned}$$

Notice that we employ the group-action $\{\kappa_\varrho\}$ from (2.1) on the Sobolev spaces on the half-axis. For $a \in S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ we have

$$\begin{aligned} \kappa_\varrho^{-1} op_{x_n}^+(a) \kappa_\varrho &= op_{x_n}^+(a^\varrho), \\ a^\varrho((x', x_n), (\xi', \xi_n), \lambda) &= a((x', \varrho^{-1}x_n), (\xi', \varrho\xi_n), \lambda). \end{aligned}$$

In particular, if a does not depend on the variable x_n , then $op_{x_n}^+(a)$ is a classical operator-valued symbol; if $a_{(\mu-j)}$ is the homogeneous component of order $\mu-j$ of a , then $op_{x_n}^+(a_{(\mu-j)})$ is the homogeneous component of order $\mu-j$ of $op_{x_n}^+(a)$.

Remark 2.6. Since we are interested in the calculus on the half-space \mathbb{R}_+^n , we may also consider the symbol spaces

$$\begin{aligned} S_{tr\,cl}^{\mu;\ell}(\overline{\mathbb{R}}_+^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) &:= r^+ S_{tr\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}), \\ S_{tr\,V\,cl}^{\mu;\ell}(\overline{\mathbb{R}}_+^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) &:= r^+ S_{tr\,V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}), \end{aligned}$$

i.e., the dependence on the variable x is restricted to $\overline{\mathbb{R}}_+^n$.

2.3. The calculus of Volterra boundary symbols.

Notation 2.7. We denote the operator $\partial_+ := r^+ \partial_{x_n} e^+$ which gives rise to an operator-valued symbol

$$\partial_+ \in \begin{cases} S_{cl}^{1;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^N), H^{s-1,\delta}(\mathbb{R}_+, \mathbb{C}^N)), & s > -\frac{1}{2}, \delta \in \mathbb{R}, \\ S_{V\,cl}^{1;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^N), H^{s-1,\delta}(\mathbb{R}_+, \mathbb{C}^N)), & s > -\frac{1}{2}, \delta \in \mathbb{R}, \\ S_{cl}^{1;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H_0^{t,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^N), H_0^{t-1,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^N)), & t, \delta \in \mathbb{R}, \\ S_{V\,cl}^{1;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; H_0^{t,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^N), H_0^{t-1,\delta}(\overline{\mathbb{R}}_+, \mathbb{C}^N)), & t, \delta \in \mathbb{R}. \end{cases}$$

Definition 2.8. a) A *classical singular (Volterra) Green symbol* of order $\mu \in \mathbb{R}$ and type zero is an operator-valued symbol

$$g \in \begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+}), \\ S_{Vcl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+}). \end{cases}$$

Via $\mathcal{S}(\mathbb{R}_+) \cong \text{proj-lim}_{s,\delta \in \mathbb{R}} H^{s,\delta}(\mathbb{R}_+)$ and $\mathcal{S}'(\mathbb{R}_+) \cong \text{ind-lim}_{s,\delta \in \mathbb{R}} H_0^{s,\delta}(\overline{\mathbb{R}}_+)$ the space $S_{(V)cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$ is defined as

$$\bigcap_{s,s',\delta,\delta' \in \mathbb{R}} S_{(V)cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H_0^{s,\delta}(\overline{\mathbb{R}}_+), H^{s',\delta'}(\mathbb{R}_+)),$$

where the group-action (2.1) is involved on the Sobolev spaces. We shall make use of analogous conventions also below.

A classical singular (Volterra) Green symbol of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ is an operator-valued symbol g of the form $g = \sum_{j=0}^d g_j \partial_+^j$ with classical singular (Volterra) Green symbols g_j of order $\mu - j \in \mathbb{R}$ and type zero. Notice in particular that

$$g \in \begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), \mathcal{S}(\mathbb{R}_+, \mathbb{C}^{N_+})) \\ S_{Vcl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), \mathcal{S}(\mathbb{R}_+, \mathbb{C}^{N_+})) \end{cases}$$

for $s > d - \frac{1}{2}$. We endow the space of classical singular (Volterra) Green symbols of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ with the topology of the non-direct sum of Fréchet spaces.

The *regularizing singular (Volterra) Green symbols* of type $d \in \mathbb{N}_0$ consist of all $g = \sum_{j=0}^d g_j \partial_+^j$ with singular (Volterra) Green symbols g_j of order $-\infty$ and type zero.

b) A *classical (Volterra) trace symbol* of order $\mu \in \mathbb{R}$ and type zero is an operator-valued symbol

$$t \in \begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; \mathcal{S}'(\mathbb{R}_+), \mathbb{C}) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{M_+}), \\ S_{Vcl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C}) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{M_+}). \end{cases}$$

A classical (Volterra) trace symbol of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ is an operator-valued symbol t of the form $t = \sum_{j=0}^d t_j \partial_+^j$ with classical (Volterra) trace symbols of order $\mu - j \in \mathbb{R}$ and type zero. In particular, t gives rise to an element

$$t \in \begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), \mathbb{C}^{M_+}), \\ S_{Vcl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), \mathbb{C}^{M_+}) \end{cases}$$

for $s > d - \frac{1}{2}$, and we endow the space of classical (Volterra) trace symbols of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ with the topology of the non-direct sum of Fréchet spaces.

The *regularizing (Volterra) trace symbols* of type $d \in \mathbb{N}_0$ consist of all $t = \sum_{j=0}^d t_j \partial_+^j$ with (Volterra) trace symbols t_j of order $-\infty$ and type zero.

- c) A *classical (Volterra) potential symbol* of order $\mu \in \mathbb{R}$ is an operator-valued symbol

$$k \in \begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; \mathbb{C}, \mathcal{S}(\mathbb{R}_+)) \otimes \mathcal{L}(\mathbb{C}^{M_-}, \mathbb{C}^{N_+}), \\ S_{Vcl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+)) \otimes \mathcal{L}(\mathbb{C}^{M_-}, \mathbb{C}^{N_+}). \end{cases}$$

The space of classical (Volterra) potential symbols carries the Fréchet topology of the projective limit. We call the (Volterra) potential symbols of order $-\infty$ *regularizing*.

- d) The space of *classical (Volterra) boundary symbols* of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ consists of all operator-valued symbols

$$a_0 \in \begin{cases} S^{\mu;\ell} \left(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; \begin{matrix} H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}) & H^{s-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+}) \\ \oplus & \oplus \\ \mathbb{C}^{M_-} & \mathbb{C}^{M_+} \end{matrix} \right), \\ S_V^{\mu;\ell} \left(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; \begin{matrix} H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}) & H^{s-\mu,\delta}(\mathbb{R}_+, \mathbb{C}^{N_+}) \\ \oplus & \oplus \\ \mathbb{C}^{M_-} & \mathbb{C}^{M_+} \end{matrix} \right), \end{cases} \quad (2.3)$$

where $s > d - \frac{1}{2}$ and $\delta \in \mathbb{R}$, of the form

$$a_0 = \begin{pmatrix} \text{op}_{x_n}^+(a) + g & k \\ t & s \end{pmatrix} \quad (2.4)$$

with:

- a classical (Volterra) symbol with the transmission property

$$a \in \begin{cases} S_{trcl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \\ S_{trVcl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}). \end{cases}$$

a is called the *pseudodifferential part* of the boundary symbol a_0 . Note that this notion is non-canonical since the representation in the upper left corner is not unique.

- a classical singular (Volterra) Green symbol g of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$.
- a classical (Volterra) trace symbol t of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$.
- a classical (Volterra) potential symbol k of order $\mu \in \mathbb{R}$.
- a classical (Volterra) pseudodifferential symbol

$$s \in \begin{cases} S_{cl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda; \mathbb{C}^{M_-}, \mathbb{C}^{M_+}) \\ S_{Vcl}^{\mu;\ell}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{H}; \mathbb{C}^{M_-}, \mathbb{C}^{M_+}). \end{cases}$$

We endow the space of classical (Volterra) boundary symbols of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ with the topology of the non-direct sum of Fréchet spaces.

If the pseudodifferential part of a_0 equals zero we call a_0 a *generalized singular Green symbol*. If all components of a_0 are regularizing, then a_0 is called a *regularizing boundary symbol*.

Note that if the pseudodifferential part of a_0 does not depend on the variable x_n , then a_0 is a classical operator-valued symbol in the symbol spaces (2.3).

Example 2.9. The restriction operator

$$r : \mathcal{S}(\mathbb{R}_+) \longrightarrow \mathbb{C}, \quad u(x_n) \longmapsto u(0)$$

gives rise to a classical Volterra trace symbol of order $\frac{1}{2}$ and type 1. Moreover, we have $r = r(\frac{1}{2})$. To see this we first define for $(\xi', \zeta) \in (\mathbb{R}^{n-1} \times \mathbb{H}) \setminus \{0\}$ and $u \in \mathcal{S}(\mathbb{R}_+)$

$$t_0(\xi', \zeta)(u) := \int_0^\infty (|\xi'|^\ell - i\zeta)^{\frac{1}{\ell}} e^{-x_n (|\xi'|^\ell - i\zeta)^{\frac{1}{\ell}}} u(x_n) dx_n,$$

$$t_1(\xi', \zeta)(u) := - \int_0^\infty e^{-x_n (|\xi'|^\ell - i\zeta)^{\frac{1}{\ell}}} u(x_n) dx_n.$$

Hence $t_0, t_1 \in C^\infty((\mathbb{R}^{n-1} \times \mathbb{H}) \setminus \{0\}, \mathcal{L}(\mathcal{S}'(\mathbb{R}_+), \mathbb{C}))$, and we have $t_0(\varrho\xi', \varrho^\ell\zeta) = \varrho^{\frac{1}{2}} t_0(\xi', \zeta) \kappa_\varrho^{-1}$ and $t_1(\varrho\xi', \varrho^\ell\zeta) = \varrho^{-\frac{1}{2}} t_1(\xi', \zeta) \kappa_\varrho^{-1}$ for $\varrho > 0$ with the group-action $\{\kappa_\varrho\}$ from (2.1). Thus t_0, t_1 are anisotropic homogeneous and analytic in the interior of \mathbb{H} , and consequently $t_0(\xi', \zeta + i) \in S_{V_{cl}}^{\frac{1}{2}, \ell}(\mathbb{R}^{n-1} \times \mathbb{H}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$ and $t_1(\xi', \zeta + i) \in S_{V_{cl}}^{-\frac{1}{2}, \ell}(\mathbb{R}^{n-1} \times \mathbb{H}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$. By construction we have $r = t_0 + t_1 \partial_+$, and the assertion follows.

Remark 2.10. Let a_0 be a classical (Volterra) boundary symbol of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$. From Definition 2.8 we see that we may write

$$a_0 = \sum_{j=0}^d a_{0,j} \begin{pmatrix} \partial_+^j & 0 \\ 0 & 1 \end{pmatrix}$$

with classical (Volterra) boundary symbols $a_{0,j}$ of order $\mu - j$ and type zero.

Using this we see that the spaces of boundary symbols are invariant with respect to the natural manipulations of the symbolic calculus. In particular, asymptotic expansions can be carried out within (Volterra) boundary symbols of fixed type $d \in \mathbb{N}_0$.

Notation 2.11. a) For $\nu \in \mathbb{N}_0$ let γ_ν denote the operator

$$\gamma_\nu : \mathcal{S}(\mathbb{R}_+, \mathbb{C}^N) \longrightarrow \mathbb{C}^N, \quad f(x_n) \longmapsto (\partial_+^\nu f)(0).$$

In view of Example 2.9 we conclude that γ_ν gives rise to a classical Volterra trace symbol of order $\nu + \frac{1}{2}$ and type $\nu + 1$.

b) For reals $\mu \in \mathbb{R}$ let $\mu_+ := \max\{0, \mu\}$ be the “positive part” of μ .

Proposition 2.12. *a) Let g be a classical singular (Volterra) Green symbol of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$. Then g has a representation*

$$g = \sum_{j=0}^{d-1} k_j \gamma_j + g_0 \quad (2.5)$$

with classical (Volterra) potential symbols k_j of order $\mu - j - \frac{1}{2}$, and a classical singular (Volterra) Green symbol g_0 of order $\mu \in \mathbb{R}$ and type zero. The symbols k_j and g_0 in this representation are unique, and the mapping $(k_0, \dots, k_{d-1}, g_0) \mapsto g$ induced by (2.5) provides a topological isomorphism.

b) Let t be a classical (Volterra) trace symbol of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$. Then we may write

$$t = \sum_{j=0}^{d-1} s_j \gamma_j + t_0 \quad (2.6)$$

with classical (Volterra) pseudodifferential symbols s_j of order $\mu - j - \frac{1}{2}$, and a classical (Volterra) trace symbol t_0 of order $\mu \in \mathbb{R}$ and type zero. The symbols s_j and t_0 are unique, and the mapping $(s_0, \dots, s_{d-1}, t_0) \mapsto t$ induced by (2.6) is a topological isomorphism.

Proof. (2.5) and (2.6) follow from integration by parts in the representations of g and t via their associated symbol-kernels. \square

Theorem 2.13. *Let*

$$a_0 = \begin{pmatrix} op_{x_n}^+(a) + g_1 & k_1 \\ t_1 & s_1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} op_{x_n}^+(b) + g_2 & k_2 \\ t_2 & s_2 \end{pmatrix}$$

be classical (Volterra) boundary symbols of order $\mu_1, \mu_2 \in \mathbb{Z}$ and type $d_1, d_2 \in \mathbb{N}_0$. Then the pointwise product within the spaces of operator-valued symbols (2.3) gives rise to a classical (Volterra) boundary symbol $a_0 b_0$ of order $\mu_1 + \mu_2$ and type $d = \max\{\mu_2 + d_1, d_2\}$, where more precisely

$$a_0 b_0 = \begin{pmatrix} op_{x_n}^+(a \#_{x_n} b) + \tilde{g} & \tilde{k} \\ \tilde{t} & \tilde{s} \end{pmatrix},$$

i.e., the pseudodifferential part of $a_0 b_0$ equals the Leibniz-product of the pseudodifferential parts of a_0 and b_0 with respect to the variable x_n . Note that we assume that the “dimensions” fit together in order to be able to calculate the product.

Proof. In the framework of general parameter-dependent boundary symbols this is a standard result in Boutet de Monvel’s calculus. Just the case of Volterra boundary symbols requires a closer inspection.

First notice that the pseudodifferential part $a \#_{x_n} b$ of $a_0 b_0$ is a classical Volterra symbol with the transmission property. Hence we just have to show that the remaining components are classical Volterra symbols. For $a_0 b_0$ is a parameter-dependent operator-valued Volterra symbol in the symbol spaces (2.3), we at once

obtain the analyticity of \tilde{k} and \tilde{s} in the interior of \mathbb{H} , i.e., \tilde{k} is a classical Volterra potential symbol, and \tilde{s} is a classical Volterra pseudodifferential symbol. According to Proposition 2.12 we may write $\tilde{g} = \sum_{j=0}^{d-1} \hat{k}_j \gamma_j + \tilde{g}_0$ and $\tilde{t} = \sum_{j=0}^{d-1} \hat{s}_j \gamma_j + \tilde{t}_0$ in the sense of (2.5) and (2.6), respectively. Now \tilde{g} and \tilde{t} are analytic in the interior of \mathbb{H} as operator-valued symbols, and from Proposition 2.12 we conclude that \hat{k}_j , \tilde{g}_0 , \hat{s}_j and \tilde{t}_0 are necessarily also analytic in the interior of \mathbb{H} , and thus they are classical Volterra symbols. Consequently, \tilde{g} and \tilde{t} are Volterra symbols, and the proof of the theorem is finished. \square

Theorem 2.14. *Let*

$$a_0 = \begin{pmatrix} op_{x_n}^+(a) + g_1 & k_1 \\ t_1 & s_1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} op_{x_n}^+(b) + g_2 & k_2 \\ t_2 & s_2 \end{pmatrix}$$

be classical (Volterra) boundary symbols of order $\mu_1, \mu_2 \in \mathbb{Z}$ and type $d_1, d_2 \in \mathbb{N}_0$. Then the Leibniz-product $a_0 \# b_0$ is a classical (Volterra) boundary symbol of order $\mu_1 + \mu_2$ and type $d = \max\{\mu_2 + d_1, d_2\}$. More precisely, we have

$$a_0 \# b_0 = \begin{pmatrix} op_{x_n}^+(a \# b) + \tilde{g} & \tilde{k} \\ \tilde{t} & \tilde{s} \end{pmatrix},$$

i.e., the pseudodifferential part of $a_0 \# b_0$ equals the Leibniz-product of the pseudodifferential parts of a_0 and b_0 with respect to the variable $x \in \mathbb{R}^n$. The asymptotic expansion

$$a_0 \# b_0 \underset{(V)}{\sim} \sum_{\alpha \in \mathbb{N}_0^n - 1} \frac{1}{\alpha!} (\partial_{\xi'}^\alpha a_0) (D_{x'}^\alpha b_0)$$

holds within the spaces of (Volterra) boundary symbols.

Proof. The assertion follows from the general theory of (Volterra) pseudodifferential calculus with operator-valued symbols (see also [38]): We have the explicit formula

$$a_0 \# b_0(x', \xi', \zeta) = \iint e^{-iy' \eta'} a_0(x', \xi' + \eta', \zeta) b_0(x' + y', \xi', \zeta) dy' d\eta'$$

for the Leibniz-product (symbol of composition) at hand, and Theorem 2.13 guarantees that the integrand belongs to the appropriate spaces of (Volterra) boundary symbols. Componentwise analysis now reveals the assertion of the theorem. \square

Theorem 2.15. *Let a_0 be a classical boundary symbol of order $\mu \leq 0$ and type zero. Then the formal adjoint symbol*

$$a_0^{(*)}(x', \xi', \lambda) = \iint e^{-iy' \eta'} a(x' + y', \xi' + \eta', \lambda)^* dy' d\eta'$$

is a classical boundary symbol of order μ and type zero. Moreover, the pseudodifferential part of $a_0^{()}$ equals the formal adjoint of the pseudodifferential part of a_0 .*

The asymptotic expansion

$$a_0^{(*)}(x', \xi', \lambda) \sim \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{1}{\alpha!} \partial_{\xi'}^\alpha D_{x'}^\alpha a^*$$

holds within the spaces of boundary symbols.

2.4. Operator calculus on the half-space.

Definition 2.16. Let $\mathcal{B}_{(V)cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \Lambda)$ denote the space of all operator families $\mathcal{A}(\lambda) = \text{op}_{x'}(a_0)(\lambda)$, $\lambda \in \Lambda$, that are built upon parameter-dependent classical (Volterra) boundary symbols $a_0(x', \xi', \lambda)$ of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}_0$. Let $\mathcal{B}^{-\infty, d}(\mathbb{R}_+^n; \Lambda)$ and $\mathcal{B}_V^{-\infty, d}(\mathbb{R}_+^n; \mathbb{H})$ denote the spaces of operators having regularizing (Volterra) boundary symbols. Note that we suppress the “dimensions” N_- , N_+ , M_- , and M_+ from the notation for better readability.

Every $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \Lambda)$ acts as a family of continuous operators

$$\mathcal{A}(\lambda) = \begin{pmatrix} A^+(\lambda) + G(\lambda) & K(\lambda) \\ T(\lambda) & S(\lambda) \end{pmatrix} : \begin{matrix} \mathcal{S}(\mathbb{R}_+^n, \mathbb{C}^{N_-}) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, \mathbb{C}^{M_-}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\mathbb{R}_+^n, \mathbb{C}^{N_+}) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, \mathbb{C}^{M_+}) \end{matrix}. \quad (2.7)$$

The components $G(\lambda)$, $K(\lambda)$ and $T(\lambda)$ are called (parameter-dependent) (Volterra) singular Green operator, potential operator and trace operator, respectively.

The spaces of (parameter-dependent) boundary operators carry the Fréchet topology induced by the boundary symbols.

Definition 2.17. With an operator $\mathcal{A}(\lambda) \in \mathcal{B}_{cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \Lambda)$ with boundary symbol

$$a_0 = \begin{pmatrix} \text{op}_{x_n}^+(a) + g & k \\ t & s \end{pmatrix}$$

we associate the following tuple of *principal symbols*:

- First we may pass to the homogeneous principal symbol

$$\sigma_\psi^{\mu; \ell}(\mathcal{A})(x, \xi, \lambda) = a_{(\mu)}(x, \xi, \lambda) \quad (2.8)$$

of the pseudodifferential part of $\mathcal{A}(\lambda)$, where $(x, \xi, \lambda) \in \overline{\mathbb{R}}_+^n \times \mathbb{R}^n \times \Lambda$, $(\xi, \lambda) \neq 0$. We call $\sigma_\psi^{\mu; \ell}(\mathcal{A})$ the *principal pseudodifferential symbol* of $\mathcal{A}(\lambda)$.

- With the notation $a^0(x', \xi, \lambda) := a((x', 0), \xi, \lambda)$ we pass to the boundary symbol

$$\tilde{a}_0 = \begin{pmatrix} \text{op}_{x_n}^+(a^0) + g & k \\ t & s \end{pmatrix},$$

which is a classical symbol in the spaces of operator-valued symbols (2.3). Hence its homogeneous principal part is well-defined, and we set

$$\sigma_\partial^{\mu; \ell}(\mathcal{A})(x', \xi', \lambda) := \tilde{a}_{0(\mu)}(x', \xi', \lambda). \quad (2.9)$$

$\sigma_\partial^{\mu; \ell}(\mathcal{A})$ is called the *principal boundary symbol* of $\mathcal{A}(\lambda)$.

Note that these symbols are uniquely determined by $\mathcal{A}(\lambda)$, and the mapping $\mathcal{A}(\lambda) \mapsto (\sigma_\psi^{\mu;\ell}(\mathcal{A}), \sigma_\partial^{\mu;\ell}(\mathcal{A}))$ is continuous. In case of Volterra operators $\mathcal{A}(\zeta)$ the principal symbols are analytic in the interior of the half-plane \mathbb{H} .

The tuple $(\sigma_\psi^{\mu;\ell}(\mathcal{A}), \sigma_\partial^{\mu;\ell}(\mathcal{A}))$ determines the operator $\mathcal{A}(\lambda)$ up to classical parameter-dependent (Volterra) boundary operators of order $\mu - 1$ and type d .

Theorem 2.18. *a) Every $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu,d;\ell}(\mathbb{R}_+^n; \Lambda)$ extends from the spaces (2.7) to a family of continuous operators*

$$\mathcal{A}(\lambda) : \begin{array}{c} H^s(\mathbb{R}_+^n, \mathbb{C}^{N_-}) \\ \oplus \\ H^s(\mathbb{R}^{n-1}, \mathbb{C}^{M_-}) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+^n, \mathbb{C}^{N_+}) \\ \oplus \\ H^{s-\mu}(\mathbb{R}^{n-1}, \mathbb{C}^{M_+}) \end{array} \quad (2.10)$$

for $s > d - \frac{1}{2}$. Moreover, $\mathcal{B}_{(V)cl}^{\mu,d;\ell}(\mathbb{R}_+^n; \Lambda)$ embeds into spaces of operator-valued (Volterra) symbols taking values in bounded operators between (2.10).

b) The composition as operators on rapidly decreasing functions (2.7) is well-defined as a continuous bilinear mapping

$$\mathcal{B}_{(V)cl}^{\mu_1,d_1;\ell}(\mathbb{R}_+^n; \Lambda) \times \mathcal{B}_{(V)cl}^{\mu_2,d_2;\ell}(\mathbb{R}_+^n; \Lambda) \longrightarrow \mathcal{B}_{(V)cl}^{\mu_1+\mu_2,d;\ell}(\mathbb{R}_+^n; \Lambda),$$

where $d = \max\{\mu_2 + d_1, d_2\}$. The principal pseudodifferential symbol of the composition equals the product of the involved principal pseudodifferential symbols, and the principal boundary symbol equals the product of the involved principal boundary symbols.

c) Taking the formal adjoint operator with respect to the L^2 -inner product(s) induces a continuous antilinear mapping

$$* : \mathcal{B}_{cl}^{\mu,0;\ell}(\mathbb{R}_+^n; \Lambda) \longrightarrow \mathcal{B}_{cl}^{\mu,0;\ell}(\mathbb{R}_+^n; \Lambda)$$

for $\mu \leq 0$. In the classical case, the principal pseudodifferential symbol of the formal adjoint operator equals the adjoint principal pseudodifferential symbol, and the principal boundary symbol of the formal adjoint operator equals the adjoint principal boundary symbol.

Notice that the spaces of Volterra boundary operators are not invariant with respect to this operation.

Proof. a) follows from the general theorem on the boundedness of pseudodifferential operators with operator-valued symbols in abstract edge Sobolev spaces (see Seiler [69]) together with the identification of the “ordinary” Sobolev spaces as such via the group-action (2.1) (see, e.g., Schulze [65]).

b) is a consequence of Theorem 2.14, while c) follows from Theorem 2.15. \square

Notation 2.19. Let $\varphi : \overline{\mathbb{R}_+^n} \longrightarrow \mathbb{C}$ be a function, and let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a vector of functions u_1 on \mathbb{R}_+^n and u_2 on \mathbb{R}^{n-1} . We denote the “multiplication” of u with φ as

$$\varphi u = \begin{pmatrix} \varphi u_1 \\ \varphi|_{\mathbb{R}^{n-1}} u_2 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi|_{\mathbb{R}^{n-1}} \end{pmatrix} u.$$

In this sense we have in a well-defined manner the multiplications $\varphi\mathcal{A}(\lambda)$ and $\mathcal{A}(\lambda)\varphi$ of parameter-dependent boundary operators $\mathcal{A}(\lambda)$ with functions $\varphi \in C_b^\infty(\overline{\mathbb{R}_+^n})$.

Definition 2.20. Let $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu,d;\ell}(\mathbb{R}_+^n; \Lambda)$. Moreover, let $U \subseteq \overline{\mathbb{R}_+^n}$ be an open subset. Then $\mathcal{A}(\lambda)$ is called *compactly supported* in U if there exists a function $\varphi \in C_0^\infty(U)$ such that $\mathcal{A}(\lambda) = \varphi\mathcal{A}(\lambda)\varphi$.

The spaces of compactly supported operators form subspaces which are topologized as strict countable inductive limits of Fréchet spaces.

Remark 2.21. Let $U, V \subseteq \mathbb{R}^n$ be open subsets, and let $\chi : U \rightarrow V$ be a diffeomorphism such that χ restricts to a diffeomorphism $U \cap \{x_n > 0\} \rightarrow V \cap \{x_n > 0\}$. Hence χ also preserves the “boundary” $\{x_n = 0\}$, and for vectors $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of functions (distributions), where v_1 is defined in $V \cap \{x_n > 0\}$ and v_2 in $V \cap \{x_n = 0\}$, we may pass to the *pull-back* $\chi^*v = \begin{pmatrix} \chi^*v_1 \\ \chi|_{U \cap \{x_n=0\}}^* v_2 \end{pmatrix}$. Similarly, we also consider the *push-forward* χ_*u .

With an operator

$$A : \begin{array}{c} C_0^\infty(V \cap \{x_n \geq 0\})^{N-} \\ \oplus \\ C_0^\infty(V \cap \{x_n = 0\})^{M-} \end{array} \longrightarrow \begin{array}{c} C^\infty(V \cap \{x_n \geq 0\})^{N+} \\ \oplus \\ C^\infty(V \cap \{x_n = 0\})^{M+} \end{array}$$

we may now associate the operator pull-back χ^*A , where $(\chi^*A)u = \chi^*(A(\chi_*u))$, and consequently χ^*A acts in the spaces

$$\chi^*A : \begin{array}{c} C_0^\infty(U \cap \{x_n \geq 0\})^{N-} \\ \oplus \\ C_0^\infty(U \cap \{x_n = 0\})^{M-} \end{array} \longrightarrow \begin{array}{c} C^\infty(U \cap \{x_n \geq 0\})^{N+} \\ \oplus \\ C^\infty(U \cap \{x_n = 0\})^{M+} \end{array}.$$

Of course we may also start from open subsets U, V of the half-space $\overline{\mathbb{R}_+^n}$ and χ a diffeomorphism in the corresponding category.

Theorem 2.22. *With the notations from Remark 2.21 we have the following:*

The operator pull-back χ^ gives rise to a topological isomorphism from the space of compactly supported (Volterra) boundary operators of order μ and type d in V onto the space of compactly supported (Volterra) boundary operators of order μ and type d in U .*

*Let χ' denote the restriction of χ to $U \cap \{x_n = 0\}$. Hence $\chi' : U \cap \{x_n = 0\} \rightarrow V \cap \{x_n = 0\}$ is a diffeomorphism of open subsets in $\mathbb{R}^{n-1} \cong \{x_n = 0\}$. The tuple of principal symbols associated with the operator pull-back $(\chi^*A)(\lambda)$ is given as*

$$\left. \begin{array}{l} \sigma_\psi^{\mu;\ell}(\chi^*A)(x, \xi, \lambda) = \sigma_\psi^{\mu;\ell}(A)(\chi(x), [D\chi(x)^{-1}]^t \xi, \lambda), \\ \sigma_\partial^{\mu;\ell}(\chi^*A)(x', \xi', \lambda) = \sigma_\partial^{\mu;\ell}(A)(\chi'(x'), [D\chi'(x')^{-1}]^t \xi', \lambda). \end{array} \right\} \quad (2.11)$$

Consequently, the calculus of (Volterra) boundary operators can be defined on a compact manifold with boundary, and the tuple of principal symbols has an invariant meaning as sections defined on the cotangent bundles.

2.5. Ellipticity and parabolicity on the half-space.

Definition 2.23. Let $\mathcal{A}(\lambda) \in \mathcal{B}_{cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \Lambda)$ be a classical parameter-dependent boundary operator of order $\mu \in \mathbb{Z}$ and type $d \leq \mu_+$. We call $\mathcal{A}(\lambda)$ *parameter-dependent elliptic* provided that the following conditions are fulfilled:

- The homogeneous principal pseudodifferential symbol $\sigma_{\psi}^{\mu; \ell}(\mathcal{A})(x, \xi, \lambda)$ is invertible in $\mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ for all $(x, \xi, \lambda) \in \overline{\mathbb{R}_+^n} \times \mathbb{R}^n \times \Lambda$, $(\xi, \lambda) \neq 0$, and for the inverse we have $\|\sigma_{\psi}^{\mu; \ell}(\mathcal{A})(x, \xi, \lambda)^{-1}\| = O(1)$ as $|x| \rightarrow \infty$, uniformly for $(|\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}} = 1$. (In particular, we have $N_- = N_+$.)
- There exists some $s_0 > d - \frac{1}{2}$ such that the homogeneous principal boundary symbol $\sigma_{\partial}^{\mu; \ell}(\mathcal{A})(x', \xi', \lambda)$ acts as an isomorphism in the spaces

$$\sigma_{\partial}^{\mu; \ell}(\mathcal{A})(x', \xi', \lambda) : \begin{array}{c} H^{s_0}(\mathbb{R}_+, \mathbb{C}^{N_-}) \\ \oplus \\ \mathbb{C}^{M_-} \end{array} \longrightarrow \begin{array}{c} H^{s_0 - \mu}(\mathbb{R}_+, \mathbb{C}^{N_+}) \\ \oplus \\ \mathbb{C}^{M_+} \end{array}$$

for $(x', \xi', \lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda$, $(\xi', \lambda) \neq 0$, and for the inverse we have $\|\sigma_{\partial}^{\mu; \ell}(\mathcal{A})(x', \xi', \lambda)^{-1}\| = O(1)$ as $|x'| \rightarrow \infty$, uniformly for $(|\xi'|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}} = 1$.

An operator $\mathcal{A}(\zeta) \in \mathcal{B}_{V, cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \mathbb{H})$ of order $\mu \in \mathbb{Z}$ and type $d \leq \mu_+$ is called *parabolic* provided that $\mathcal{A}(\zeta)$ is parameter-dependent elliptic as an element of $\mathcal{B}_{cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \mathbb{H})$.

Proposition 2.24. Consider a principal boundary symbol

$$a_0 = \begin{pmatrix} op_{x_n}^+(a^0) + \sum_{j=0}^d g_j \partial_+^j & k \\ \sum_{j=0}^d t_j \partial_+^j & s \end{pmatrix}$$

of order $\mu \in \mathbb{Z}$ and type $d \leq \mu_+$, i.e.,

- $a^0 \equiv a^0((x', 0), \xi, \lambda)$ is (anisotropic) homogeneous of order μ and independent of the variable x_n , and it satisfies the symmetry relation (2.2),
- $g_j(x', \xi', \lambda)$ is (anisotropic) homogeneous of order $\mu - j$,
- k is (anisotropic) homogeneous of order μ ,
- $t_j(x', \xi', \lambda)$ is (anisotropic) homogeneous of order $\mu - j$,
- s is (anisotropic) homogeneous of order μ .

We assume that a_0 is parameter-dependent elliptic:

- There exists the inverse $p^0((x', 0), \xi, \lambda) = a^0((x', 0), \xi, \lambda)^{-1}$ for $x' \in \mathbb{R}^{n-1}$ and $0 \neq (\xi, \lambda) \in \mathbb{R}^n \times \Lambda$, and we have $\|p^0((x', 0), \xi, \lambda)\| = O(1)$ as $|x'| \rightarrow \infty$, uniformly for $(|\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}} = 1$.

ii) There exists some $s_0 > d - \frac{1}{2}$ such that $a_0(x', \xi', \lambda)$ acts as an isomorphism in the spaces

$$a_0(x', \xi', \lambda) : \begin{array}{c} H^{s_0}(\mathbb{R}_+, \mathbb{C}^{N_-}) \\ \oplus \\ \mathbb{C}^{M_-} \end{array} \longrightarrow \begin{array}{c} H^{s_0-\mu}(\mathbb{R}_+, \mathbb{C}^{N_+}) \\ \oplus \\ \mathbb{C}^{M_+} \end{array}$$

for $(x', \xi', \lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda$, $(\xi', \lambda) \neq 0$, and for the inverse we have $\|a_0(x', \xi', \lambda)^{-1}\| = O(1)$ as $|x'| \rightarrow \infty$, uniformly for $(|\xi'|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}} = 1$.

Then the inverse a_0^{-1} is a principal boundary symbol of order $-\mu$ and type $d' \leq (-\mu)_+$. More precisely, we have

$$a_0^{-1} = \begin{pmatrix} op_{x_n}^+(p^0) + \sum_{j=0}^{d'} \tilde{g}_j \partial_+^j & \tilde{k} \\ \sum_{j=0}^{d'} \tilde{t}_j \partial_+^j & \tilde{s} \end{pmatrix},$$

where

- p^0 is the inverse of a^0 ,
- $\tilde{g}_j(x', \xi', \lambda)$ is (anisotropic) homogeneous of order $-\mu - j$,
- \tilde{k} is (anisotropic) homogeneous of order $-\mu$,
- $\tilde{t}_j(x', \xi', \lambda)$ is (anisotropic) homogeneous of order $-\mu - j$,
- \tilde{s} is (anisotropic) homogeneous of order $-\mu$.

Moreover, if a_0 is a Volterra principal boundary symbol, i.e., all components a^0 , g_j , t_j , k , and s are analytic in the interior of the half-plane \mathbb{H} , then also a_0^{-1} is a Volterra principal boundary symbol.

Proof. In the case of general principal boundary symbols, i.e., without the analyticity in the interior of the half-plane, the assertion is subject to the classical analysis in Boutet de Monvel's algebra (see Grubb [26], Schulze [65], or Schrohe [58]).

It remains to prove that the inverse a_0^{-1} is again a Volterra principal boundary symbol, provided that a_0 is a Volterra principal boundary symbol. First notice that a_0^{-1} is analytic in the interior of \mathbb{H} , and so are \tilde{k} and \tilde{s} . Moreover, as being the inverse of a^0 , we conclude that p^0 is a Volterra principal symbol. Hence we just have to show that

$$\tilde{g} = \sum_{j=0}^{d'} \tilde{g}_j \partial_+^j \quad \text{and} \quad \tilde{t} = \sum_{j=0}^{d'} \tilde{t}_j \partial_+^j \tag{1}$$

can be represented as asserted, i.e., \tilde{g}_j and \tilde{t}_j can be arranged to be analytic in the interior of \mathbb{H} . Note that \tilde{g} and \tilde{t} are analytic since a_0^{-1} is analytic. Now apply

Proposition 2.12 (adapted to principal symbols) to obtain

$$\tilde{g} = \sum_{j=0}^{d'-1} \tilde{k}_j \gamma_j + g', \quad \tilde{t} = \sum_{j=0}^{d'-1} \tilde{s}_j \gamma_j + t' \quad (2)$$

with unique principal potential symbols \tilde{k}_j , a unique principal singular Green symbol g' of type zero, unique principal pseudodifferential symbols \tilde{s}_j , and a unique principal trace symbol t' of type zero. For these representations hold topologically, we see that \tilde{k}_j , g' , \tilde{s}_j and t' are analytic in the interior of \mathbb{H} . Employing Example 2.9 we now conclude that, starting from (2), we indeed can arrange all “coefficients” in (1) to be analytic, which finishes the proof of the proposition. \square

Theorem 2.25. *Let $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \Lambda)$ be a classical parameter-dependent (Volterra) boundary operator of order $\mu \in \mathbb{Z}$ and type $d \leq \mu_+$ with boundary symbol a_0 . The following are equivalent:*

- i) $\mathcal{A}(\lambda)$ is parameter-dependent elliptic (parabolic).
- ii) There exists a classical (Volterra) boundary symbol p_0 of order $-\mu$ and type $d' \leq (-\mu)_+$ such that $a_0 p_0 - 1$ and $p_0 a_0 - 1$ are classical (Volterra) boundary symbols of order -1 .
- iii) There exists $\mathcal{P}(\lambda) \in \mathcal{B}_{(V)cl}^{-\mu, d'; \ell}(\mathbb{R}_+^n; \Lambda)$, $d' \leq (-\mu)_+$, such that

$$\begin{aligned} \mathcal{A}(\lambda) \mathcal{P}(\lambda) - 1 &\in \mathcal{B}_{(V)}^{-\infty, d_1}(\mathbb{R}_+^n; \Lambda), \\ \mathcal{P}(\lambda) \mathcal{A}(\lambda) - 1 &\in \mathcal{B}_{(V)}^{-\infty, d_2}(\mathbb{R}_+^n; \Lambda), \end{aligned}$$

where $d_1 = \max\{-\mu + d, d'\}$ and $d_2 = \max\{\mu + d', d\}$.

Proof. Note first that we may pass from the pointwise product in ii) to the Leibniz-product in view of Theorem 2.14. Hence the equivalence of ii) and iii) is evident: iii) implies ii) follows immediately from Theorem 2.14, while ii) implies iii) is a consequence of the usual formal Neumann series argument which is applicable since asymptotic expansions can be carried out within classical (Volterra) boundary symbols of fixed type.

iii) implies i) follows from the multiplicativity of the tuple of principal symbols under forming compositions on the level of operators. Hence it remains to show that i) implies ii): Let $p(x, \xi, \lambda) = \sigma_{\psi}^{\mu; \ell}(\mathcal{A})(x, \xi, \lambda)^{-1}$, and denote $p^0(x', \xi, \lambda) = p|_{\{x_n=0\}}$. According to Proposition 2.24 the inverse of the principal boundary symbol of $\mathcal{A}(\lambda)$ is a principal (Volterra) boundary symbol of order $-\mu$ and type $d' \leq (-\mu)_+$, where more precisely

$$\sigma_{\partial}^{\mu; \ell}(\mathcal{A})^{-1} = \begin{pmatrix} \text{op}_{x_n}^+(p^0) + \tilde{g} & \tilde{k} \\ \tilde{t} & \tilde{s} \end{pmatrix}$$

Now we define p_0 as follows:

- In case of general boundary symbols choose a 0-excision function $\chi \in C^\infty(\mathbb{R}^n \times \Lambda)$. With $\chi' := \chi|_{\{\xi_n=0\}}$ set $p'(x, \xi, \lambda) := \chi(\xi, \lambda)p(x, \xi, \lambda)$, and

$$p_0 := \begin{pmatrix} \text{op}_{x_n}^+(p') + \chi' \tilde{g} & \chi' \tilde{k} \\ \chi' \tilde{t} & \chi' \tilde{s} \end{pmatrix}.$$

- In case of Volterra boundary symbols we set

$$p_0(x', \xi', \zeta) := \begin{pmatrix} \text{op}_{x_n}^+(p(x, \xi, \zeta + i)) + \tilde{g}(x', \xi', \zeta + i) & \tilde{k}(x', \xi', \zeta + i) \\ \tilde{t}(x', \xi', \zeta + i) & \tilde{s}(x', \xi', \zeta + i) \end{pmatrix}.$$

Thus p_0 is a classical (Volterra) boundary symbol of order $-\mu$ and type $d' \leq (-\mu)_+$, and the tuple of principal symbols associated with p_0 equals the inverted tuple of principal symbols associated with a_0 . Consequently, $a_0 p_0 - 1$ and $p_0 a_0 - 1$ are classical (Volterra) boundary symbols of order -1 , and the proof of the theorem is complete. \square

3. VOLTERRA BOUNDARY VALUE PROBLEMS ON A MANIFOLD

The present section is devoted to establish the parameter-dependent Volterra calculus of boundary value problems on a compact manifold with boundary, i.e., we shall give the globalized definitions and properties of the calculus from Section 2. In fact, these operators constitute the elementary building blocks in what follows in the remaining parts of this work.

To this end, let \overline{X} be a compact smooth manifold of dimension $\dim \overline{X} = n$ with boundary $Y = \partial X$, $\dim Y = n - 1$. We set $X := \overline{X} \setminus Y$. Let E and F be complex vector bundles over \overline{X} of dimensions N_- and N_+ , respectively, and let J_- and J_+ be complex vector bundles over the boundary Y of dimensions M_- and M_+ . Note that the dimensions of the bundles are allowed to be zero. With \overline{X} we associate its double $2\overline{X}$, which is a closed manifold and contains a positive and negative copy \overline{X}_\pm of \overline{X} ; we identify \overline{X} with the positive copy \overline{X}_+ in $2\overline{X}$. Recall that vector bundles on \overline{X} can be lifted to $2\overline{X}$ via the projection $\pi^+ : 2\overline{X} \rightarrow \overline{X}$, and in general we make no difference between a vector bundle and its lifting to the double. For convenience, we fix Riemannian metrics on \overline{X} and on Y , as well as Hermitean inner products on the bundles E , F , J_- and J_+ . In particular, these data determine Hilbert space structures on the L^2 -spaces of sections in the vector bundles.

A local chart will be denoted as a tuple (κ, Ω, U) , or simply κ , where $\Omega \subseteq \overline{X}$ is an open subset and $\kappa : \Omega \rightarrow U$ is a diffeomorphism. We distinguish between two kinds of local charts:

- (κ, Ω, U) is called a *local interior chart* if $\Omega \subseteq X$ is contained in the interior X , and $U \subseteq \mathbb{R}^n$ is some open subset.
- (κ, Ω, U) is called a *local boundary chart* if $\Omega \cap Y \neq \emptyset$, i.e., Ω covers some part of the boundary, and $U \subseteq \overline{\mathbb{R}}_+^n$ is open.

In any case, we will assume that local charts are chosen in such a way that the involved vector bundles are trivial over Ω , respectively over $\Omega \cap Y$. The transition matrices of the fibres that arise in changing local frames of the bundles are suppressed from the notation.

The pull-back of a vector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of functions u_1 in \mathbb{R}^n and u_2 in \mathbb{R}^{n-1} with respect to a chart κ will be denoted as κ^*u , where

$$\kappa^*u = \begin{pmatrix} \kappa^*u_1 \\ (\kappa|_Y)^*u_2 \end{pmatrix} = \begin{pmatrix} \kappa^* & 0 \\ 0 & (\kappa|_Y)^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

and similarly we handle the push-forward κ_*v of vectors $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of functions v_1 in \overline{X} and v_2 in Y . Hence also the operator pull-back and push-forward with respect to κ is well-defined with the usual formula.

For $\varphi \in C^\infty(\overline{X})$ we denote the multiplication of φ with a vector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of functions u_1 in \overline{X} and u_2 in Y as φu , where

$$\varphi u = \begin{pmatrix} \varphi u_1 \\ \varphi|_Y u_2 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi|_Y \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Recall that we may find finite open coverings of \overline{X} that consist of some interior and some boundary charts, and to each such covering there exist subordinated partitions of unity. Consequently, most constructions can be carried out first locally, and afterwards patched together globally on the manifold.

3.1. Sobolev spaces on a manifold with boundary. For $s \in \mathbb{R}$ let $H_0^s(\overline{X}, E)$ be the closed subspace of all $u \in H^s(2\overline{X}, E)$ with $\text{supp}(u) \subseteq \overline{X}$, which equals the closure of $C_0^\infty(X, E)$ in $H^s(2\overline{X}, E)$.

With the operator $r^+ : \mathcal{D}'(2\overline{X}, E) \rightarrow \mathcal{D}'(X_+, E)$ of restriction we set $H^s(X, E) = r^+ H^s(2\overline{X}, E)$, endowed with the quotient topology. The kernel of r^+ in $H^s(2\overline{X}, E)$ is given as $H_0^s(\overline{X}_-, E)$, and there are continuous extension operators $e_s : H^s(X, E) \rightarrow H^s(2\overline{X}, E)$, i.e., the sequence $0 \rightarrow H_0^s(\overline{X}_-, E) \rightarrow H^s(2\overline{X}, E) \rightarrow H^s(X, E) \rightarrow 0$ is topologically exact and splits.

The $L^2(X, E)$ -inner product extends to a sesquilinear pairing $H^s(X, E) \times H_0^{-s}(\overline{X}, E) \rightarrow \mathbb{C}$ and provides an identification of the dual spaces $H_0^s(\overline{X}, E)' \cong H^{-s}(X, E)$ and $H^s(X, E)' \cong H_0^{-s}(\overline{X}, E)$. The space $C^\infty(\overline{X}, E) = r^+ C^\infty(2\overline{X}, E)$ of smooth sections up to the boundary is isomorphic to $\text{proj-lim}_{s \rightarrow \infty} H^s(X, E)$ and is dense in $H^s(X, E)$ for all $s \in \mathbb{R}$. For the dual space we have $C^\infty(\overline{X}, E)' \cong \text{ind-lim}_{s \rightarrow -\infty} H_0^s(\overline{X}, E)$.

Let e^+ denote the operator of extension by zero for sections defined on X_+ to $2\overline{X}$. Hence e^+ makes sense as an operator $H_0^s(\overline{X}, E) \rightarrow H^s(2\overline{X}, E)$ for every $s \in \mathbb{R}$, which coincides with the inclusion. For $-\frac{1}{2} < s < \frac{1}{2}$ we have $H_0^s(\overline{X}, E) \cong H^s(X, E)$, and thus the operator e^+ is well-defined in $H^s(X, E) \rightarrow \mathcal{D}'(2\overline{X}, E)$ for $s > -\frac{1}{2}$.

3.2. Operator calculus and symbolic structure.

Notation 3.1. We denote the operator $\partial_+ := r^+ \partial_\nu e^+$, where ∂_ν is a vector field on \overline{X} supported near the boundary which coincides close to Y with the inward unit vector field with respect to some Riemannian metric on \overline{X} .

Moreover, we shall also write $\partial_+ = \begin{pmatrix} \partial_+ & 0 \\ 0 & 1 \end{pmatrix}$ as an operator acting on vectors $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of sections u_1 defined on \overline{X} , and u_2 defined on Y .

We intend to define spaces of operator families

$$\mathcal{A}(\lambda) : \begin{array}{ccc} C^\infty(\overline{X}, E) & & C^\infty(\overline{X}, F) \\ \oplus & \longrightarrow & \oplus \\ C^\infty(Y, J_-) & & C^\infty(Y, J_+) \end{array} \quad (3.1)$$

depending on the parameter $\lambda \in \Lambda$. For better readability we will suppress the vector bundles from the notation.

Definition 3.2. The space $\mathcal{B}^{-\infty,0}(X; \Lambda)$ consists of all operator families (3.1) which are given (componentwise) as integral operators with smooth kernel sections, depending rapidly decreasing on the parameter $\lambda \in \Lambda$. In other words, the space $\mathcal{B}^{-\infty,0}(X; \Lambda)$ consists of all those operator families that act continuously in the spaces

$$\mathcal{A}(\lambda) : \begin{array}{ccc} H_0^s(\overline{X}, E) & & H^t(X, F) \\ \oplus & \longrightarrow & \oplus \\ H^s(Y, J_-) & & H^t(Y, J_+) \end{array}$$

for all $s, t \in \mathbb{R}$, which induces a topological isomorphism

$$\mathcal{B}^{-\infty,0}(X; \Lambda) = \mathcal{S}\left(\Lambda, \mathcal{L}\left(\begin{array}{cc} C^\infty(\overline{X}, E)' & C^\infty(\overline{X}, F) \\ \oplus & \oplus \\ \mathcal{D}'(Y, J_-) & C^\infty(Y, J_+) \end{array}\right)\right).$$

The space $\mathcal{B}_V^{-\infty,0}(X; \mathbb{H})$ is by definition the closed subspace of all $\mathcal{A}(\lambda) \in \mathcal{B}^{-\infty,0}(X; \mathbb{H})$ that are analytic in the interior of \mathbb{H} .

For $d \in \mathbb{N}_0$ we define the spaces $\mathcal{B}_{(V)}^{-\infty,d}(X; \Lambda)$ to consist of all operator families (3.1) having a representation $\mathcal{A}(\lambda) = \sum_{j=0}^d \mathcal{G}(\lambda) \partial_+^j$ with $\mathcal{G}(\lambda) \in \mathcal{B}_{(V)}^{-\infty,0}(X; \Lambda)$. We may regard these spaces as Fréchet subspaces

$$\mathcal{B}_{(V)}^{-\infty,d}(X; \Lambda) \subseteq \mathcal{S}_{(V)}^{-\infty}\left(\Lambda; \begin{array}{cc} H^s(X, E) & H^t(X, F) \\ \oplus & \oplus \\ H^s(Y, J_-) & H^t(Y, J_+) \end{array}\right)$$

for $s > d - \frac{1}{2}$ and $t \in \mathbb{R}$, endowed with the topology of the non-direct sum (with respect to the spaces of type zero above).

For $\mu \in \mathbb{Z}$ and $d \in \mathbb{N}_0$ the spaces $\mathcal{B}_{(V)cl}^{\mu,d;\ell}(X; \Lambda)$ consist of all operator families (3.1) that satisfy the following conditions:

- i) For all $\varphi, \psi \in C^\infty(\overline{X})$ having disjoint support we require $\varphi \mathcal{A}(\lambda) \psi \in \mathcal{B}_{(V)}^{-\infty,d}(X; \Lambda)$.

- ii) For all $\varphi, \psi \in C^\infty(\overline{X})$ supported in a local boundary chart (κ, Ω, U) we have $\kappa_*(\varphi \mathcal{A}(\lambda) \psi) \in \mathcal{B}_{(V)cl}^{\mu, d; \ell}(\mathbb{R}_+^n; \Lambda)$.
- iii) For all $\varphi, \psi \in C^\infty(\overline{X})$ supported in a local interior chart (κ, Ω, U) we have

$$\kappa_*(\varphi \mathcal{A}(\lambda) \psi) = \begin{pmatrix} \text{op}_x(a)(\lambda) + g(\lambda) & k(\lambda) \\ t(\lambda) & s(\lambda) \end{pmatrix},$$

where a is a classical parameter-dependent (Volterra) pseudodifferential symbol of order $\mu \in \mathbb{Z}$, and $g(\lambda)$, $k(\lambda)$, $t(\lambda)$ and $s(\lambda)$ are integral operators with rapidly decreasing kernels that depend rapidly decreasing on the parameter $\lambda \in \Lambda$, and in the Volterra case they are additionally analytic in the interior of the half-plane $\Lambda = \mathbb{H}$.

We endow these spaces with the projective topology with respect to the mappings induced by i)–iii).

We call the operators in $\mathcal{B}_{(V)cl}^{\mu, d; \ell}(X; \Lambda)$ (*Volterra boundary operators*) of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}_0$. The operators of order $-\infty$ are called *regularizing*.

Definition 3.3. With an operator $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu, d; \ell}(X; \Lambda)$ we associate the following tuple of principal symbols (see also Theorem 2.22):

- Let $\pi : (T^*\overline{X} \times \Lambda) \setminus 0 \rightarrow \overline{X}$ be the canonical projection. Then the *principal pseudodifferential symbol* of $\mathcal{A}(\lambda)$ is well-defined as a smooth (anisotropic) homogeneous section

$$\sigma_\psi^{\mu; \ell}(\mathcal{A}) : (T^*\overline{X} \times \Lambda) \setminus 0 \rightarrow \text{Hom}(\pi^*E, \pi^*F), \quad (3.2)$$

i.e., we have $\sigma_\psi^{\mu; \ell}(\mathcal{A})(\varrho \xi_x, \varrho^\ell \lambda) = \varrho^\mu \sigma_\psi^{\mu; \ell}(\mathcal{A})(\xi_x, \lambda)$ for $\varrho > 0$. For Volterra operators the principal pseudodifferential symbol is analytic in the interior of the half-plane $\Lambda = \mathbb{H}$.

- The *principal boundary symbol* of $\mathcal{A}(\lambda)$ is given as a smooth section

$$\sigma_\partial^{\mu; \ell}(\mathcal{A}) : (T^*Y \times \Lambda) \setminus 0 \rightarrow \text{Hom} \left(\begin{array}{c} H^s(\mathbb{R}_+) \otimes \pi^*E|_Y \\ \oplus \\ \pi^*J_- \end{array}, \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \otimes \pi^*F|_Y \\ \oplus \\ \pi^*J_+ \end{array} \right) \quad (3.3)$$

for $s > d - \frac{1}{2}$, where $\pi : (T^*Y \times \Lambda) \setminus 0 \rightarrow Y$ is the canonical projection. We may also replace the Sobolev spaces $H^s(\mathbb{R}_+)$ and $H^{s-\mu}(\mathbb{R}_+)$ by the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}_+)$. The principal boundary symbol is (anisotropic) homogeneous in the sense

$$\sigma_\partial^{\mu; \ell}(\mathcal{A})(\varrho \xi_{x'}, \varrho^\ell \lambda) = \varrho^\mu \begin{pmatrix} \kappa_\varrho \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial^{\mu; \ell}(\mathcal{A})(\xi_{x'}, \lambda) \begin{pmatrix} \kappa_\varrho^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $\varrho > 0$ with the group-action $\{\kappa_\varrho\}$ from (2.1). For Volterra operators the principal boundary symbol is analytic in the interior of $\Lambda = \mathbb{H}$.

The mapping $\mathcal{A}(\lambda) \mapsto (\sigma_\psi^{\mu; \ell}(\mathcal{A}), \sigma_\partial^{\mu; \ell}(\mathcal{A}))$ is continuous, and the tuple of principal symbols determines the operator $\mathcal{A}(\lambda)$ up to classical parameter-dependent (Volterra) boundary operators of order $\mu - 1$ and type d .

Theorem 3.4. *a) Every $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu,d;\ell}(X; \Lambda)$ extends from the spaces (3.1) to a family of continuous operators*

$$\mathcal{A}(\lambda) : \begin{array}{ccc} H^s(X, E) & & H^{s-\mu}(X, F) \\ \oplus & \longrightarrow & \oplus \\ H^s(Y, J_-) & & H^{s-\mu}(Y, J_+) \end{array} \quad (3.4)$$

for $s > d - \frac{1}{2}$, which induces an embedding of the boundary operators into spaces of operator-valued (Volterra) symbols within these Sobolev spaces.

b) The composition as operators on (3.1) gives rise to continuous bilinear mappings $\mathcal{B}_{(V)cl}^{\mu_1,d_1;\ell}(X; \Lambda) \times \mathcal{B}_{(V)cl}^{\mu_2,d_2;\ell}(X; \Lambda) \longrightarrow \mathcal{B}_{(V)cl}^{\mu_1+\mu_2,d;\ell}(X; \Lambda)$, where $d = \max\{\mu_2 + d_1, d_2\}$. Note that we assume that the bundles fit together in order to be able to carry out the composition.

For $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu_1,d_1;\ell}(X; \Lambda)$ and $\mathcal{B}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu_2,d_2;\ell}(X; \Lambda)$ we obtain the following formulas for the principal symbols of the composition:

$$\sigma_{\psi}^{\mu_1+\mu_2;\ell}(\mathcal{A}\mathcal{B}) = \sigma_{\psi}^{\mu_1;\ell}(\mathcal{A})\sigma_{\psi}^{\mu_2;\ell}(\mathcal{B}), \quad \sigma_{\partial}^{\mu_1+\mu_2;\ell}(\mathcal{A}\mathcal{B}) = \sigma_{\partial}^{\mu_1;\ell}(\mathcal{A})\sigma_{\partial}^{\mu_2;\ell}(\mathcal{B}).$$

c) Taking the formal adjoint operator with respect to the L^2 -inner product(s) gives rise to an antilinear continuous mapping $*$: $\mathcal{B}_{cl}^{\mu,0;\ell}(X; \Lambda) \longrightarrow \mathcal{B}_{cl}^{\mu,0;\ell}(X; \Lambda)$ for $\mu \leq 0$. For $\mathcal{A}(\lambda) \in \mathcal{B}_{cl}^{\mu,0;\ell}(X; \Lambda)$ the principal symbols of the formal adjoint operator are given as

$$\sigma_{\psi}^{\mu;\ell}(\mathcal{A}^*) = \sigma_{\psi}^{\mu;\ell}(\mathcal{A})^*, \quad \sigma_{\partial}^{\mu;\ell}(\mathcal{A}^*) = \sigma_{\partial}^{\mu;\ell}(\mathcal{A})^*.$$

Note that the space of Volterra boundary operators is not preserved under this operation.

Proof. The global result a) on the boundedness of operators in the Sobolev spaces follows from the corresponding local results.

For the proof of b) note first that the spaces of regularizing operators remain invariant under composition with arbitrary boundary operators, including the formula for the types, which is immediate in view of a) and Definition 3.2. Consequently, the proof is reduced to consider the composition of operators supported in a local chart. But for these we may apply the local results: If we deal with local boundary charts we obtain the desired assertion from Theorem 2.18, while for local interior charts it follows directly from the general calculus of (Volterra) pseudodifferential operators.

By Definition 3.2 assertion c) holds for regularizing operators. Hence the proof reduces to consider operators supported in a local chart, and for these we may apply the local results from Theorem 2.18 for local boundary charts, and the general theory for local interior charts. \square

Remark 3.5. The global spaces of (Volterra) boundary operators of fixed type $d \in \mathbb{N}_0$ are invariant with respect to taking asymptotic expansions:

Given $\mathcal{A}_j(\lambda) \in \mathcal{B}_{(V)cl}^{\mu-j, d; \ell}(X; \Lambda)$ for $j \in \mathbb{N}_0$ we find an operator $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu, d; \ell}(X; \Lambda)$ such that $\mathcal{A}(\lambda) \underset{(V)}{\sim} \sum_{j=0}^{\infty} \mathcal{A}_j(\lambda)$, i.e.,

$$\mathcal{A}(\lambda) - \sum_{j=0}^N \mathcal{A}_j(\lambda) \in \mathcal{B}_{(V)cl}^{\mu-N-1, d; \ell}(X; \Lambda)$$

for $N \in \mathbb{N}_0$. Clearly, $\mathcal{A}(\lambda)$ is uniquely determined up to regularizing (Volterra) boundary operators of type d .

To see this note that the spaces of boundary operators are built upon a hierarchy of operator-valued (Volterra) symbols, and the standard result about asymptotic completeness of these implies the asymptotic completeness of (Volterra) boundary operators of fixed type.

In case of Volterra boundary operators we may construct $\mathcal{A}(\zeta)$ as in Section 1.3 as $\mathcal{A}(\zeta) = \sum_{j=0}^{\infty} (H(\varphi(c_j t)) \mathcal{A}_j)(\zeta)$ with convergence in the space of Volterra boundary operators using the (Fourier) kernel cut-off operator H from (1.3).

3.3. Ellipticity, parabolicity, and global parametrices.

- Definition 3.6.** a) An operator $\mathcal{A}(\lambda) \in \mathcal{B}_{cl}^{\mu, d; \ell}(X; \Lambda)$, where $d \leq \mu_+$, is called *parameter-dependent elliptic* provided that both the principal pseudodifferential symbol and the principal boundary symbol of $\mathcal{A}(\lambda)$ are invertible, i.e., the families (3.2) and (3.3) are pointwise bijective.
- b) An operator $\mathcal{A}(\zeta) \in \mathcal{B}_{Vcl}^{\mu, d; \ell}(X; \mathbb{H})$, where $d \leq \mu_+$, is called *parabolic* if $\mathcal{A}(\zeta)$ is parameter-dependent elliptic in the sense of a) as an element in $\mathcal{B}_{cl}^{\mu, d; \ell}(X; \mathbb{H})$.

Theorem 3.7. Let $\mathcal{A}(\lambda) \in \mathcal{B}_{(V)cl}^{\mu, d; \ell}(X; \Lambda)$, where $d \leq \mu_+$. Then the following are equivalent:

- i) $\mathcal{A}(\lambda)$ is parameter-dependent elliptic (parabolic).
- ii) There exists $\mathcal{P}(\lambda) \in \mathcal{B}_{(V)cl}^{-\mu, d'; \ell}(X; \Lambda)$, $d' \leq (-\mu)_+$, such that $\mathcal{A}(\lambda)\mathcal{P}(\lambda) - 1 \in \mathcal{B}_{(V)}^{-\infty, d_1}(X; \Lambda)$ and $\mathcal{P}(\lambda)\mathcal{A}(\lambda) - 1 \in \mathcal{B}_{(V)}^{-\infty, d_2}(X; \Lambda)$, where $d_1 = \max\{-\mu + d, d'\}$ and $d_2 = \max\{\mu + d', d\}$.

Every $\mathcal{P}(\lambda)$ satisfying ii) is called a (Volterra) parametrix of $\mathcal{A}(\lambda)$.

Proof. Clearly, ii) implies i) in view of the multiplicativity of the tuple of principal symbols under composition.

Let us prove that i) implies ii): Using a covering of \overline{X} by local charts and a subordinated partition of unity we may pass from the global operator $\mathcal{A}(\lambda)$ to local representations on the half-space (for local boundary charts), respectively to operators on \mathbb{R}^n with regularizing entries except for the upper left corner (for local interior charts). In case of the half-space these operators are parameter-dependent elliptic (parabolic) in the sense of Definition 2.23, and in case of operators in the interior we obtain parameter-dependent elliptic (parabolic) elements in the upper left corner. More precisely, the property of being parameter-dependent elliptic

(parabolic) is localized on some compact subset. Using Theorem 2.25 for the involved (local) boundary operators as well as the standard parametrix construction for the operators in the interior (for Volterra operators see also Section 1.5) we obtain to each local representation a (Volterra) parametrix. Patching these together on the manifold we find the parametrix $\mathcal{P}(\lambda)$ as asserted. \square

Theorem 3.8. a) Let $\mathcal{A}(\lambda) \in \mathcal{B}_{cl}^{\mu, d; \ell}(X; \Lambda)$, $d \leq \mu_+$, be parameter-dependent elliptic. Then for $|\lambda| \gg 0$ sufficiently large the operator $\mathcal{A}(\lambda)$ is invertible, regarded either as an operator in the spaces of smooth functions (3.1), or alternatively as an operator in the Sobolev spaces (3.4) for sufficiently large regularity. More precisely, the set of all $\lambda \in \Lambda$ such that $\mathcal{A}(\lambda)$ is not invertible is compact in Λ , and for each neighbourhood $U \subseteq \Lambda$ of this set there exists a parametrix $\mathcal{P}(\lambda) \in \mathcal{B}_{cl}^{-\mu, d'; \ell}(X; \Lambda)$, $d' \leq (-\mu)_+$, such that $\mathcal{P}(\lambda) = \mathcal{A}(\lambda)^{-1}$ for $\lambda \notin U$.
b) Let $\mathcal{A}(\zeta) \in \mathcal{B}_{Vcl}^{\mu, d; \ell}(X; \mathbb{H})$, $d \leq \mu_+$, be parabolic. Moreover, assume that $\mathcal{A}(\zeta)$ acts as an isomorphism either in the spaces of smooth functions (3.1), or in the Sobolev spaces (3.4) for sufficiently large regularity. Then the inverse $\mathcal{P}(\zeta) = \mathcal{A}(\zeta)^{-1}$ belongs to $\mathcal{B}_{Vcl}^{-\mu, d'; \ell}(X; \mathbb{H})$ for some $d' \leq (-\mu)_+$.

Proof. Let us prove a): According to Theorem 3.7 we first choose a parametrix $\mathcal{P}'(\lambda)$ of the operator $\mathcal{A}(\lambda)$, and let $\mathcal{A}(\lambda)\mathcal{P}'(\lambda) = 1 + R_R(\lambda)$ and $\mathcal{P}'(\lambda)\mathcal{A}(\lambda) = 1 + R_L(\lambda)$. For $R_L(\lambda)$ and $R_R(\lambda)$ depend rapidly decreasing on the parameter $\lambda \in \Lambda$, we conclude that both $1 + R_L(\lambda)$ and $1 + R_R(\lambda)$ are invertible for $|\lambda| \gg 0$ sufficiently large. Hence the set of all $\lambda \in \Lambda$ such that $\mathcal{A}(\lambda)$ is not invertible is necessarily compact. Let $\chi \in C^\infty(\Lambda)$ be an excision function of the non-bijectivity points of the operator $\mathcal{A}(\lambda)$ with $\chi \equiv 1$ outside the given neighbourhood U . Now we define

$$\begin{aligned}\mathcal{P}_L(\lambda) &:= \mathcal{P}'(\lambda) - R_L(\lambda)\mathcal{P}'(\lambda) + R_L(\lambda)\chi(\lambda)\mathcal{A}(\lambda)^{-1}R_R(\lambda), \\ \mathcal{P}_R(\lambda) &:= \mathcal{P}'(\lambda) - \mathcal{P}'(\lambda)R_R(\lambda) + R_L(\lambda)\chi(\lambda)\mathcal{A}(\lambda)^{-1}R_R(\lambda).\end{aligned}$$

From the defining mapping property of regularizing elements in the calculus we conclude that both $\mathcal{P}_L(\lambda)$ and $\mathcal{P}_R(\lambda)$ belong to $\mathcal{B}_{cl}^{-\mu, d'; \ell}(X; \Lambda)$, $d' \leq (-\mu)_+$, and in fact are parametrices of $\mathcal{A}(\lambda)$. Moreover, a simple algebraic calculation shows that $\mathcal{P}_L(\lambda) = \mathcal{P}_R(\lambda) = \mathcal{A}(\lambda)^{-1}$ for $\lambda \notin U$, i.e., with either $\mathcal{P}(\lambda) = \mathcal{P}_R(\lambda)$ or $\mathcal{P}(\lambda) = \mathcal{P}_L(\lambda)$ assertion a) is fulfilled.

The proof of b) is even simpler for we need not argue with an excision function. \square

Proposition 3.9. Assume that the vector bundles are given as $E = F$ and $J_- = J_+$. Let $\mathcal{G}(\lambda) \in \mathcal{B}_{(V)}^{-\infty, d}(X; \Lambda)$ such that $1 + \mathcal{G}(\lambda)$ is invertible in the spaces (3.1), or in (3.4) for sufficiently large regularity. Then the inverse is given as $(1 + \mathcal{G}(\lambda))^{-1} = 1 + \mathcal{G}'(\lambda)$ with some $\mathcal{G}'(\lambda) \in \mathcal{B}_{(V)}^{-\infty, d}(X; \Lambda)$.

Proof. This is a simple consequence of the identity

$$(1 + \mathcal{G}(\lambda))^{-1} = 1 - \mathcal{G}(\lambda) + \mathcal{G}(\lambda)(1 + \mathcal{G}(\lambda))^{-1}\mathcal{G}(\lambda).$$

□

4. PARABOLICITY AND INVERTIBILITY IN AN INFINITE SPACE-TIME CYLINDER

The aim of the present section is to give a rough description of the inverse of parabolic boundary value problems in an infinite space-time cylinder $[t_0, \infty) \times X$. To this end, we consider boundary value problems in $\mathbb{R} \times X$ that are given in terms of Volterra pseudodifferential operators

$$\text{op}_t(a)u(t) = \mathcal{F}^{-1}a(t, \zeta)\mathcal{F}u = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(t-t')\tau} a(t, \tau)u(t') dt' d\tau$$

with symbols $a(t, \zeta) \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{Vcl}^{\mu, d; \ell}(X; \mathbb{H}_-))$, i.e., we assume that the “coefficients” depend on the time variable $t \in \mathbb{R}$ like a classical symbol of order 0. Clearly, $\text{op}_t(a)$ is well-defined as a continuous operator

$$\text{op}_t(a) : \begin{array}{c} C_0^\infty(\mathbb{R}, C^\infty(\overline{X}, E)) \\ \oplus \\ C_0^\infty(\mathbb{R}, C^\infty(Y, J_-)) \end{array} \longrightarrow \begin{array}{c} C^\infty(\mathbb{R}, C^\infty(\overline{X}, F)) \\ \oplus \\ C^\infty(\mathbb{R}, C^\infty(Y, J_+)) \end{array}. \quad (4.1)$$

Observe that we may write

$$\text{op}_t(a) = e^{\gamma t} \text{op}_t(T_{-i\gamma}a) e^{-\gamma t} : C_0^\infty\left(\mathbb{R}, \begin{array}{c} C^\infty(\overline{X}, E) \\ \oplus \\ C^\infty(Y, J_-) \end{array}\right) \longrightarrow C^\infty\left(\mathbb{R}, \begin{array}{c} C^\infty(\overline{X}, F) \\ \oplus \\ C^\infty(Y, J_+) \end{array}\right) \quad (4.2)$$

for $\gamma \geq 0$, where $(T_{-i\gamma}a)(t, \zeta) = a(t, \zeta - i\gamma) \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{Vcl}^{\mu, d; \ell}(X; \mathbb{H}_-))$.

With the symbol $a(t, \zeta)$ we associate the following tuple of principal symbols:

- i) The *principal pseudodifferential symbol* is given as a smooth section

$$\sigma_\psi^{\mu; \ell}(a) : \mathbb{R} \times ((T^*\overline{X} \times \mathbb{H}_-) \setminus 0) \longrightarrow \text{Hom}(\pi^*E, \pi^*F)$$

that is homogeneous in the sense $\sigma_\psi^{\mu; \ell}(a)(t, \varrho\xi_x, \varrho^\ell\zeta) = \varrho^\mu \sigma_\psi^{\mu; \ell}(a)(t, \xi_x, \zeta)$ for $\varrho > 0$.

- ii) The *principal boundary symbol* is well-defined as a smooth section

$$\sigma_\partial^{\mu; \ell}(a) : \mathbb{R} \times ((T^*Y \times \mathbb{H}_-) \setminus 0) \longrightarrow \text{Hom}\left(\begin{array}{c} H^s(\mathbb{R}_+) \otimes \pi^*E|_Y \\ \oplus \\ \pi^*J_- \end{array}, \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \otimes \pi^*F|_Y \\ \oplus \\ \pi^*J_+ \end{array}\right)$$

for $s > d - \frac{1}{2}$ that is homogeneous in the sense

$$\sigma_\partial^{\mu; \ell}(a)(t, \varrho\xi_{x'}, \varrho^\ell\zeta) = \varrho^\mu \begin{pmatrix} \kappa_\varrho \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial^{\mu; \ell}(a)(t, \xi_{x'}, \zeta) \begin{pmatrix} \kappa_\varrho^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $\varrho > 0$ with the group-action $\{\kappa_\varrho\}$ from (2.1).

iii) The *principal pseudodifferential-exit symbol*

$$\sigma_{\psi,e}^{\mu;\ell}(a) : (\mathbb{R} \setminus \{0\}) \times ((T^*\overline{X} \times \mathbb{H}_-) \setminus 0) \longrightarrow \text{Hom}(\pi^*E, \pi^*F)$$

is by definition the principal part with respect to $t \in \mathbb{R}$ of the principal pseudodifferential symbol associated with $a(t, \zeta)$. In particular, it is homogeneous in the sense $\sigma_{\psi,e}^{\mu;\ell}(a)(\varrho_1 t, \varrho_2 \xi_x, \varrho_2^\ell \zeta) = \varrho_2^\mu \sigma_{\psi,e}^{\mu;\ell}(a)(t, \xi_x, \zeta)$ for $\varrho_1, \varrho_2 > 0$.

iv) The *principal boundary-exit symbol*

$$\sigma_{\partial,e}^{\mu;\ell}(a) : (\mathbb{R} \setminus \{0\}) \times ((T^*Y \times \mathbb{H}_-) \setminus 0) \longrightarrow \text{Hom} \left(\begin{array}{c} H^s(\mathbb{R}_+) \otimes \pi^*E|_Y \\ \oplus \\ \pi^*J_- \end{array}, \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \otimes \pi^*F|_Y \\ \oplus \\ \pi^*J_+ \end{array} \right)$$

for $s > d - \frac{1}{2}$ is defined as the principal part with respect to $t \in \mathbb{R}$ of the principal boundary symbol. It is homogeneous in the sense

$$\sigma_{\partial,e}^{\mu;\ell}(a)(\varrho_1 t, \varrho_2 \xi_{x'}, \varrho_2^\ell \zeta) = \varrho_2^\mu \begin{pmatrix} \kappa_{\varrho_2} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_{\partial,e}^{\mu;\ell}(a)(t, \xi_{x'}, \zeta) \begin{pmatrix} \kappa_{\varrho_2}^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $\varrho_1, \varrho_2 > 0$ with the group-action $\{\kappa_\varrho\}$ from (2.1).

v) The *principal exit symbol* is by definition the homogeneous principal component of $a(t, \zeta)$ with respect to $t \in \mathbb{R}$. It is well-defined as a family of operators depending on $t \in \mathbb{R} \setminus \{0\}$ and $\zeta \in \mathbb{H}_-$ in the spaces

$$\sigma_e^0(a)(t, \zeta) : \begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(X, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{array}$$

for $s > d - \frac{1}{2}$ that is homogeneous in the sense $\sigma_e^0(a)(\varrho t, \zeta) = \sigma_e^0(a)(t, \zeta)$ for $\varrho > 0$.

Observe that also $\sigma_{\psi}^{\mu;\ell}(\sigma_e^0(a)) = \sigma_{\psi,e}^{\mu;\ell}(a)$ and $\sigma_{\partial}^{\mu;\ell}(\sigma_e^0(a)) = \sigma_{\partial,e}^{\mu;\ell}(a)$, and all principal symbols are holomorphic with respect to ζ in the interior of the lower half-plane \mathbb{H}_- .

Both the principal pseudodifferential-exit and the boundary-exit symbol are by definition constant on $\{t > 0\}$ and $\{t < 0\}$ with values $\sigma_{\psi,e}^{\mu;\ell}(a)(\pm\infty, \xi_x, \zeta)$ and $\sigma_{\partial,e}^{\mu;\ell}(a)(\pm\infty, \xi_{x'}, \zeta)$, respectively, as they are homogeneous with respect to $t \in \mathbb{R} \setminus \{0\}$ of order 0. Moreover, we have $\sigma_{\psi}^{\mu;\ell}(a)(t, \xi_x, \zeta) \xrightarrow{t \rightarrow \pm\infty} \sigma_{\psi,e}^{\mu;\ell}(a)(\pm\infty, \xi_x, \zeta)$ and $\sigma_{\partial}^{\mu;\ell}(a)(t, \xi_{x'}, \zeta) \xrightarrow{t \rightarrow \pm\infty} \sigma_{\partial,e}^{\mu;\ell}(a)(\pm\infty, \xi_{x'}, \zeta)$. Therefore, we may also consider the principal pseudodifferential symbol and the principal boundary symbol as given on the compactification $[-\infty, +\infty]$ of \mathbb{R} , and drop the separate control of the “mixed” principal symbols.

Notation 4.1. Let $\Sigma^{(\mu),d}$ be the set of triples $(a_{(\mu)}^\psi(t, \xi_x, \zeta), a_{(\mu)}^\partial(t, \xi_{x'}, \zeta), a_{(0)}^e(t, \zeta))$, where

- $a_{(\mu)}^\psi(t, \xi_x, \zeta)$ depends like a classical symbol of order 0 on $t \in \mathbb{R}$ taking values in the space of principal pseudodifferential symbols on $(T^*\overline{X} \times \mathbb{H}_-) \setminus 0$ of order

- $\mu \in \mathbb{Z}$ that satisfy the transmission condition with respect to the boundary ∂X ;
- $a_{(\mu)}^\partial(t, \xi_{x'}, \zeta)$ depends like a classical symbol of order 0 on $t \in \mathbb{R}$ taking values in the principal boundary symbols of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}_0$ on $(T^*Y \times \mathbb{H}_- \setminus 0)$, and the upper left corner of $a_{(\mu)}^\partial$ is compatible with $a_{(\mu)}^\psi$;
 - $a_{(0)}^e(t, \zeta)$ is a homogeneous function of order 0 with respect to $t \in \mathbb{R} \setminus \{0\}$ taking values in $\mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-)$;
 - $\sigma_\psi^{\mu; \ell}(a_{(0)}^e) = \sigma_e^0(a_{(\mu)}^\psi)$ and $\sigma_\partial^{\mu; \ell}(a_{(0)}^e) = \sigma_e^0(a_{(\mu)}^\partial)$ (compatibility condition);
 - all functions are holomorphic with respect to ζ in the interior of the lower half-plane \mathbb{H}_- .

Theorem 4.2. *The principal symbol sequence*

$$0 \longrightarrow S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu-1, d; \ell}(X; \mathbb{H}_-)) \longrightarrow S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-)) \xrightarrow{(\sigma_\psi, \sigma_\partial, \sigma_e)} \Sigma^{(\mu), d} \longrightarrow 0$$

is exact and splits.

Proof. Clearly, the tuple of principal symbols determines a Volterra symbol $a(t, \zeta)$ up to terms of lower order, both with respect to $t \in \mathbb{R}$ and with respect to the pseudodifferential order in the parameter-dependent Boutet de Monvel's calculus.

Let $(a_{(\mu)}^\psi, a_{(\mu)}^\partial, a_{(0)}^e) \in \Sigma^{(\mu), d}$ be given. For the principal symbol sequence for parameter-dependent Volterra boundary operators in Boutet de Monvel's calculus is topologically split exact (the splitting is induced by the translation operator in local coordinates, see also Section 1.4), we first find a symbol $\tilde{a}(t, \zeta) \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-))$ with $\sigma_\psi^{\mu; \ell}(\tilde{a}) = a_{(\mu)}^\psi$ and $\sigma_\partial^{\mu; \ell}(\tilde{a}) = a_{(\mu)}^\partial$. With a 0-excision function $\chi \in C^\infty(\mathbb{R})$, i.e., $\chi \equiv 1$ for $|t| \gg 0$ and $\chi \equiv 0$ near $t = 0$, we now define $a(t, \zeta) := \tilde{a}(t, \zeta) - \chi(t)(\sigma_e^0(\tilde{a})(t, \zeta) - a_{(0)}^e(t, \zeta))$. Thus $a \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-))$ with $(\sigma_\psi^{\mu; \ell}(a), \sigma_\partial^{\mu; \ell}(a), \sigma_e^0(a)) = (a_{(\mu)}^\psi, a_{(\mu)}^\partial, a_{(0)}^e)$, and the proof of the theorem is complete. \square

Definition 4.3. We define spaces of exponentially weighted Sobolev spaces as follows: Let Z be a closed manifold, and $V \in \text{Vect}(Z)$ a complex vector bundle over Z . For $s \in \mathbb{R}$ let

$$H^{s; \ell}(\mathbb{R} \times Z, V) := \begin{cases} L^2(\mathbb{R}, H^s(Z, V)) \cap H^{\frac{s}{\ell}}(\mathbb{R}, L^2(Z, V)) & s \geq 0, \\ L^2(\mathbb{R}, H^s(Z, V)) + H^{\frac{s}{\ell}}(\mathbb{R}, L^2(Z, V)) & s \leq 0, \end{cases}$$

and $H^{s, \gamma; \ell}(\mathbb{R} \times Z, V) := e^{\gamma t} H^{s; \ell}(\mathbb{R} \times Z, V)$ for $\gamma \in \mathbb{R}$.

Moreover, we define the weighted anisotropic Sobolev spaces on the infinite cylinder with boundary as $H^{s, \gamma; \ell}(\mathbb{R} \times X, E) := r^+ H^{s, \gamma; \ell}(\mathbb{R} \times (2\overline{X}), E)$, i.e., by restriction of the corresponding Sobolev distributions from the double.

The closed subspaces of distributions with support in $[t_0, \infty)$ are denoted by $H_0^{s, \gamma; \ell}([t_0, \infty) \times Z, V)$ and $H_0^{s, \gamma; \ell}([t_0, \infty) \times X, E)$, respectively.

Theorem 4.4. *For every $\gamma \geq 0$ the operator $op_t(a)$ extends from the spaces (4.1) by continuity to*

$$op_t(a) : \begin{array}{ccc} H^{s,\gamma;\ell}(\mathbb{R} \times X, E) & & H^{s-\mu,\gamma;\ell}(\mathbb{R} \times X, F) \\ \oplus & \longrightarrow & \oplus \\ H^{s,\gamma;\ell}(\mathbb{R} \times Y, J_-) & & H^{s-\mu,\gamma;\ell}(\mathbb{R} \times Y, J_+) \end{array} \quad (4.3)$$

for $s > d - \frac{1}{2}$. Moreover, it restricts to a continuous operator

$$op_t(a) : \begin{array}{ccc} H_0^{s,\gamma;\ell}([t_0, \infty) \times X, E) & & H_0^{s-\mu,\gamma;\ell}([t_0, \infty) \times X, F) \\ \oplus & \longrightarrow & \oplus \\ H_0^{s,\gamma;\ell}([t_0, \infty) \times Y, J_-) & & H_0^{s-\mu,\gamma;\ell}([t_0, \infty) \times Y, J_+) \end{array} \quad (4.4)$$

for every $t_0 \in \mathbb{R}$.

Proof. The continuity of $op_t(a)$ in the spaces (4.3) for $\gamma = 0$ follows from the general boundedness results of operators in Boutet de Monvel's calculus. The case of general $\gamma \geq 0$ then follows from the identity (4.2).

Clearly, the Volterra operator $op_t(a)$ respects the spaces (4.4). Recall that this follows from the Paley–Wiener characterization of the Fourier image of functions supported by $[t_0, \infty)$ as holomorphic functions in the interior of \mathbb{H}_- (see, e.g., Eskin [16], Rempel and Schulze [57]); these spaces of holomorphic functions remain invariant with respect to multiplication with the symbol $a(t, \zeta)$. \square

Theorem 4.5. *Let $a \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu_1, d_1; \ell}(X; \mathbb{H}_-))$ and $b \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu_2, d_2; \ell}(X; \mathbb{H}_-))$. Provided the vector bundles fit together, the composition as operators in the spaces (4.3) for each $\gamma \geq 0$ is given as $op_t(a) \circ op_t(b) = op_t(a \# b)$ with the Leibniz-product*

$$a \# b(t, \zeta) = \iint e^{-it'\tau} a(t, \zeta + \tau) b(t + t', \zeta) dt' d\tau \sim \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{\zeta}^k a) (D_t^k b).$$

The asymptotic expansion is to be understood in the following sense:

$$a \# b - \sum_{k=0}^{N-1} \frac{1}{k!} (\partial_{\zeta}^k a) (D_t^k b) \in S_{cl}^{-N}(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu_1 + \mu_2 - N, d; \ell}(X; \mathbb{H}_-))$$

for $N \in \mathbb{N}_0$, where $d = \max\{\mu_2 + d_1, d_2\}$. In particular, we have

$$\begin{aligned} \sigma_{\psi}^{\mu_1 + \mu_2; \ell}(a \# b)(t, \xi_x, \zeta) &= \sigma_{\psi}^{\mu_1; \ell}(a)(t, \xi_x, \zeta) \sigma_{\psi}^{\mu_2; \ell}(b)(t, \xi_x, \zeta), \\ \sigma_{\partial}^{\mu_1 + \mu_2; \ell}(a \# b)(t, \xi_{x'}, \zeta) &= \sigma_{\partial}^{\mu_1; \ell}(a)(t, \xi_{x'}, \zeta) \sigma_{\partial}^{\mu_2; \ell}(b)(t, \xi_{x'}, \zeta), \\ \sigma_e^0(a \# b)(t, \zeta) &= \sigma_e^0(a)(t, \zeta) \sigma_e^0(b)(t, \zeta). \end{aligned}$$

Proof. The general pseudodifferential calculus implies that the composition as operators in (4.3) for $\gamma = 0$ is indeed given as $op_t(a \# b)$, and the oscillatory integral formula for the Leibniz-product holds. From this formula it is easy to see that $a \# b$ is a Volterra symbol of boundary value problems, and the asymptotic expansion

holds as desired. The assertion for general $\gamma \geq 0$ now follows from the case $\gamma = 0$ and (4.2) via

$$\begin{aligned} \text{op}_t(a) \circ \text{op}_t(b) &= (e^{\gamma t} \text{op}_t(T_{-i\gamma} a) e^{-\gamma t}) (e^{\gamma t} \text{op}_t(T_{-i\gamma} b) e^{-\gamma t}) \\ &= e^{\gamma t} \text{op}_t(T_{-i\gamma}(a \# b)) e^{-\gamma t} = \text{op}_t(a \# b), \end{aligned}$$

and the proof of the theorem is complete. \square

4.1. Parabolicity and the construction of the inverse.

Definition 4.6. A Volterra symbol $a \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-))$, $d \leq \mu_+$, is called *parabolic with respect to the weight* $\gamma_0 \geq 0$, if the following conditions are fulfilled:

- i) Both the principal pseudodifferential symbol $\sigma_{\psi}^{\mu; \ell}(a)(t, \xi_x, \zeta)$ and the principal boundary symbol $\sigma_{\partial}^{\mu; \ell}(a)(t, \xi_{x'}, \zeta)$ are invertible for all $t \in [-\infty, +\infty]$ and all covectors $(\xi_x, \zeta) \neq 0$ and $(\xi_{x'}, \zeta) \neq 0$.

Recall that this actually means that the principal pseudodifferential symbol, the principal boundary symbol, the principal pseudodifferential–exit symbol, and the principal boundary–exit symbol are invertible.

- ii) There exists $s_0 > d - \frac{1}{2}$ such that the operator family

$$\sigma_e^0(a)(t, \zeta - i\gamma_0) : \begin{array}{c} H^{s_0}(X, E) \\ \oplus \\ H^{s_0}(Y, J_-) \end{array} \longrightarrow \begin{array}{c} H^{s_0-\mu}(X, F) \\ \oplus \\ H^{s_0-\mu}(Y, J_+) \end{array} \quad (4.5)$$

is invertible for all $\zeta \in \mathbb{H}_-$ and $t \neq 0$.

Proposition 4.7. Assume that $a(t, \zeta)$ fulfills condition i) in Definition 4.6. Then $a(t, \zeta)$ is parabolic with respect to some weight $\gamma_0 \geq 0$, i.e., condition ii) in Definition 4.6 is simply a weight condition.

Proof. For $\sigma_{\psi}^{\mu; \ell}(\sigma_e^0(a))$ and $\sigma_{\partial}^{\mu; \ell}(\sigma_e^0(a))$ are invertible by condition i) of Definition 4.6 we see that $\sigma_e^0(a)(t, \zeta) \in \mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-)$ is parabolic in the sense of Definition 3.6, and therefore the operator family (4.5) is invertible for $\gamma_0 \gg 0$ sufficiently large according to Theorem 3.8 — note that the principal exit symbol is constant on $\{t > 0\}$ and $\{t < 0\}$ as it is homogeneous of order 0 with respect to $t \in \mathbb{R} \setminus \{0\}$. \square

Theorem 4.8. For a Volterra symbol $a \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{\mu, d; \ell}(X; \mathbb{H}_-))$, $d \leq \mu_+$, the following are equivalent:

- i) $a(t, \zeta)$ is parabolic with respect to the weight $\gamma_0 \geq 0$.
- ii) There exists $b \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{-\mu, d'; \ell}(X; \mathbb{H}_-))$, $d' \leq (-\mu)_+$, such that $a(t, \zeta - i\gamma_0) \# b(t, \zeta) = 1$ and $b(t, \zeta) \# a(t, \zeta - i\gamma_0) = 1$, i.e., $T_{-i\gamma_0} a$ has the Leibniz-inverse b .

In particular, the parabolic operator $\text{op}_t(a)$ is invertible in the spaces (4.3), (4.4) for $\gamma \geq \gamma_0$, and the inverse is given as $\text{op}_t(a)^{-1} = e^{\gamma_0 t} \text{op}_t(b) e^{-\gamma_0 t}$.

Proof. For $a(t, \zeta - i\gamma_0)$ is parabolic with respect to the weight 0 if and only if $a(t, \zeta)$ is parabolic with respect to the weight γ_0 , we may assume for the proof of the equivalence of i) and ii) that $\gamma_0 = 0$.

Let us assume first that ii) holds. Theorem 4.5 implies that the tuple of principal symbols associated with $b(t, \zeta)$ invert the tuple of principal symbols associated with $a(t, \zeta)$, and consequently $a(t, \zeta)$ is parabolic.

Next assume that $a(t, \zeta)$ is parabolic with respect to the weight 0. Then the triple $(\sigma_\psi^{\mu; \ell}(a)^{-1}, \sigma_\theta^{\mu; \ell}(a)^{-1}, \sigma_e^0(a)^{-1})$ of inverted principal symbols defines an element in $\Sigma^{(-\mu), d'}$ with $d' \leq (-\mu)_+$ — note, in particular, that the inverted principal exit symbol is indeed a principal exit symbol due to Theorem 3.8. From Theorem 4.2 we conclude that there exists a Volterra symbol $c(t, \zeta) \in S_{cl}^0(\mathbb{R}, \mathcal{B}_{V_{cl}}^{-\mu, d'; \ell}(X; \mathbb{H}_-))$ having this tuple of principal symbols, and by Theorem 4.5 we have $a\#c = 1 - r_1$ and $c\#a = 1 - r_2$ with $r_j \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_{V_{cl}}^{-1, d_j; \ell}(X; \mathbb{H}_-))$ for $j = 1, 2$, where $d_1 = \max\{-\mu + d, d'\}$ and $d_2 = \max\{\mu + d', d\}$.

Hence the proof is reduced to show that for a Volterra symbol $r \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_{V_{cl}}^{-1, d; \ell}(X; \mathbb{H}_-))$, where $d \in \mathbb{N}_0$, there exists $r' \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_{V_{cl}}^{-1, d; \ell}(X; \mathbb{H}_-))$ such that $(1 - r)\#(1 + r') = 1$ and $(1 + r')\#(1 - r) = 1$. We apply a formal Neumann series argument twice to obtain further reductions.

The first application of the argument refers to the pseudodifferential order in the parameter-dependent Boutet de Monvel's calculus on X , i.e., let $r'(t, \zeta)$

such that $r' - \sum_{j=1}^{N-1} r^{j(\#)} \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_{V_{cl}}^{-N, d; \ell}(X; \mathbb{H}_-))$ for $N \in \mathbb{N}$. Then we have

$(1 - r)\#(1 + r') - 1 \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$ and $(1 + r')\#(1 - r) - 1 \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$, and consequently we may assume from the very beginning that $r \in S_{cl}^{-1}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$.

The second application now refers to the order in the variable $t \in \mathbb{R}$, i.e., let $r'(t, \zeta)$ such that $r' - \sum_{j=1}^{N-1} r^{j(\#)} \in S_{cl}^{-N}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$ for $N \in \mathbb{N}$.

Then $(1 - r)\#(1 + r') - 1 \in S^{-\infty}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$ and $(1 + r')\#(1 - r) - 1 \in S^{-\infty}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$, and the proof is reduced to show that for $r \in S^{-\infty}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$, where $d \in \mathbb{N}_0$ is arbitrary, there exists $r' \in S^{-\infty}(\mathbb{R}, \mathcal{B}_V^{-\infty, d}(X; \mathbb{H}_-))$ such that $(1 - r)\#(1 + r') = 1$ and $(1 + r')\#(1 - r) = 1$.

For the operator $G := \text{op}_t(r)$ is compact in the Sobolev spaces, $1 - G$ is invertible if and only if it is one-to-one. Note that

$$\ker(1 - G) \subseteq \begin{array}{c} \mathcal{S}(\mathbb{R}, C^\infty(\overline{X}, E)) \\ \oplus \\ \mathcal{S}(\mathbb{R}, C^\infty(Y, J_-)) \end{array},$$

and consequently it is sufficient to prove the injectivity of the continuous extension

$$1 - G : L^2\left(\mathbb{R}, \begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array}\right) \longrightarrow L^2\left(\mathbb{R}, \begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array}\right) \quad (1)$$

for some $s > d - \frac{1}{2}$. We may write $(Gu)(t) = \int_{\mathbb{R}} g(t, t')u(t') dt'$ with

$$g(t, t') \in \mathcal{S}\left(\mathbb{R} \times \mathbb{R}, \mathcal{L}\left(\begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array}\right)\right),$$

and $g(t, t') \equiv 0$ for $t < t'$. The general theory of Volterra integral operators in L^2 -spaces now implies that (1) is invertible, i.e., the Neumann series $(1 - G)^{-1} = \sum G^j$ is convergent. Thus $\ker(1 - G) = \{0\}$, and the operator $1 - G$ is invertible in the Sobolev spaces, too. Writing $G = \sum_{j=0}^d G_j \partial_+^j$ with $G_j = \text{op}_t(r_j)$, $r_j \in S^{-\infty}(\mathbb{R}, \mathcal{B}_V^{-\infty, 0}(X; \mathbb{H}_-))$, we conclude that $(1 - G)^{-1} = 1 + G + \sum_{j=0}^d (G(1 - G)^{-1}G_j)\partial_+^j$, and each of the Volterra operators $G(1 - G)^{-1}G_j$ is continuous in the spaces

$$G(1 - G)^{-1}G_j : \begin{array}{ccc} \mathcal{S}(\mathbb{R}, C^\infty(\overline{X}, E))' & & \mathcal{S}(\mathbb{R}, C^\infty(\overline{X}, E)) \\ \oplus & \longrightarrow & \oplus \\ \mathcal{S}(\mathbb{R}, C^\infty(Y, J_-))' & & \mathcal{S}(\mathbb{R}, C^\infty(Y, J_-)) \end{array},$$

i.e., $G(1 - G)^{-1}G_j = \text{op}_t(\hat{r}_j)$ with $\hat{r}_j \in S^{-\infty}(\mathbb{R}, \mathcal{B}_V^{-\infty, 0}(X; \mathbb{H}_-))$. Consequently, $(1 - G)^{-1} = 1 + G'$ with $G' = \text{op}_t(r')$ as asserted, and the proof of the equivalence of i) and ii) is complete.

The second assertion of the theorem about the structure of the inverse of a parabolic operator $\text{op}_t(a)$ now follows immediately from (4.2) and Theorem 4.4. \square

Remark 4.9. Usually we are just interested in the invertibility of a parabolic boundary value problem in the spaces (4.4) and the structure of the inverse operator therein. Actually, we can relax the parabolicity conditions from Definition 4.6 to be fulfilled only on the time interval $[t_0, \infty)$, and Theorem 4.8 localized to this interval is then valid, i.e., there exist Volterra symbols $b_1(t, \zeta)$ and $b_2(t, \zeta)$ such that $(T_{-i\gamma_0}a) \# b_1|_{[t_0, \infty)} \equiv 1$ and $b_2 \# (T_{-i\gamma_0}a)|_{[t_0, \infty)} \equiv 1$, and the operator $\text{op}_t(a)$ is bijective in (4.4) for $\gamma \geq \gamma_0$ with inverse $\text{op}_t(a)^{-1} = e^{\gamma_0 t} \text{op}_t(b) e^{-\gamma_0 t}$, where we can choose b either as b_1 or b_2 .

In the theory of parabolic partial differential equations and boundary value problems we are interested not only in existence, uniqueness, and smoothness, but also in the long-time behaviour of solutions. Thereby, it is most natural to ask whether the solution has exponential long-time asymptotics of the form $u(t) \sim \sum_j \sum_{k=0}^{m_j} c_{j,k} t^k e^{p_j t}$ as $t \rightarrow \infty$, where $p_j \in \mathbb{C}$ and the $c_{j,k}$ are smooth sections on X (and on Y), provided that the right hand side of the equation has an analogous behaviour. Clearly, such can only be expected if the dependence of the equation and the boundary conditions on the time variable is “moderate” as $t \rightarrow \infty$.

Under our present hypotheses, exponential long-time asymptotics cannot be observed in general. Moreover, the structure result about the inverse operator (solution operator) of a parabolic boundary value problem as given in Theorem 4.8 is too coarse to provide insights about the long-time behaviour in terms of exponential long-time asymptotics, even if the assumptions about the particular equation under study are sufficiently strong.

For this reason, the program of the remaining sections is to construct a subalgebra of pseudodifferential Volterra boundary value problems that also allows the control of exponential long-time asymptotics. This subalgebra is closed under inversion of parabolic elements, and from the more specific symbolic and operational structure of this smaller calculus we obtain much stronger results about the inverse operators (near $t = \infty$) than those given in Theorem 4.8 within the large calculus. In particular, the control of exponential long-time asymptotics is a regularity feature for parabolic elements in this calculus — this requires meromorphic symbols near $t = \infty$ in combination with Paley–Wiener characterizations of function spaces with asymptotics.

To this end, we make use of the substitution $r = e^{-t}$ which transports the relevant effects near $t = \infty$ to $r = 0$. We assume that the equations and boundary conditions under study are totally characteristic in the new coordinates with respect to the hypersurface $r = 0$. This clearly is a more restrictive time dependence than that considered in the present section. To illustrate this condition, let $A = \sum_{j=0}^M A_j(t) \partial_t^j$ be a differential boundary value problem, where the $A_j(t)$ are operator-valued coefficients taking values in differential boundary value problems on X . In the new coordinates we may write A as $\tilde{A} = \sum_{j=0}^M A_j(-\log r) (-r \partial_r)^j$, and our assumption means that the coefficients $A_j(-\log r)$ extend smoothly up to the origin $r = 0$. Note that exponential long-time asymptotics are transformed to conormal asymptotics $\tilde{u}(r) \sim \sum_j \sum_{k=0}^{m_j} \tilde{c}_{j,k} \log^k(r) r^{-p_j}$ as $r \rightarrow 0$, while the exponentially weighted Sobolev spaces are mapped to totally characteristic Sobolev spaces with power weights at the origin — the latter are expressed as weighted Mellin Sobolev spaces.

Summing up, we analytically find a situation of totally characteristic pseudodifferential operators. The elliptic theory of these has been investigated extensively in recent years, and we adopt certain structures that have been developed in this theory for our construction of the Volterra calculus of boundary value problems; in particular, the works of Schrohe and Schulze [59, 60] about elliptic boundary value problems on manifolds with conical singularities are to be mentioned in this context. For this reason, we will call the operators in our calculus *Volterra boundary cone operators*, even if there is just the analytic and not a geometric correspondence.

5. CONORMAL SYMBOLS OF BOUNDARY VALUE PROBLEMS

In the present section we are going to introduce certain spaces of meromorphic operator functions taking values in Boutet de Monvel's algebra that later will serve as Mellin symbols. Notice that meromorphic Mellin symbols naturally occur in the pseudodifferential calculus of totally characteristic operators, and in the elliptic theory of boundary value problems they were investigated by Schrohe and Schulze [59, 60]. Volterra conormal symbols were considered in the boundaryless case by Krainer and Schulze [39].

Definition 5.1. A *Mellin asymptotic type* (of boundary value problems) is a finite or countably infinite set $P = \{(p_j, m_j, L_j); j \in \mathbb{Z}\}$, where the $m_j \in \mathbb{N}_0$ are integers, the L_j are finite-dimensional subspaces of $\mathcal{B}^{-\infty, d}(X)$ consisting of finite-dimensional operators, and the $p_j \in \mathbb{C}$ are complex numbers such that only finitely many are located in each strip $\{z \in \mathbb{C}; c < \operatorname{Re}(z) < c'\}$ with $c, c' \in \mathbb{R}$.

For the empty asymptotic type we shall use the notation O , and we denote the “projection” of P to \mathbb{C} by $\pi_{\mathbb{C}}P := \{p_j; j \in \mathbb{Z}\}$. The collection of all Mellin asymptotic types is denoted by $\operatorname{As}(\mathcal{B}^{-\infty, d}(X))$. Note that we suppress the vector bundles E and F on \overline{X} as well as J_- and J_+ on the boundary Y from the notation for better readability.

Notation 5.2. For $\beta \in \mathbb{R}$ we denote $\Gamma_\beta := \{z \in \mathbb{C}; \operatorname{Re}(z) = \beta\}$, and $\mathbb{H}_\beta := \{z \in \mathbb{C}; \operatorname{Re}(z) \geq \beta\}$. We shall consider parameter-dependent operators with parameter running over Γ_β as well as \mathbb{H}_β which are then to be identified with Γ_0 and \mathbb{H}_0 via shifting the origin to β .

Definition 5.3. For $\mu \in \mathbb{Z}$ and $P \in \operatorname{As}(\mathcal{B}^{-\infty, d}(X))$ the space $M_{P, cl}^{\mu, d; \ell}(X)$ of (*anisotropic*) *meromorphic Mellin symbols* (of boundary value problems) of order μ and type $d \in \mathbb{N}_0$ with asymptotic type P consists of all functions $a \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}P, \mathcal{B}^{\mu, d}(X))$ with the following properties:

- For every $(p, m, L) \in P$ we may write in a neighbourhood $U(p) \setminus \{p\}$

$$a(z) = \sum_{k=0}^m \nu_k (z - p)^{-(k+1)} + a_0(z)$$

with $\nu_k \in L$, $k = 0, \dots, m$, and a_0 holomorphic in p taking values in $\mathcal{B}^{\mu, d}(X)$.

- For every compact interval $I \subseteq \mathbb{R}$ we have

$$a(\beta + i\tau) - \sum_{\{(p_j, m_j, L_j); \operatorname{Re}(p_j) \in I\}} \sum_{k=0}^{m_j} \sigma_{j_k} \psi_{p_j, k} \in \mathcal{B}_{cl}^{\mu, d; \ell}(X; \Gamma_\beta) \quad (5.1)$$

uniformly for $\beta \in I$ with suitable $\sigma_{j_k} \in L_j$, where the functions $\psi_{p_j, k}$ are analytic in $\mathbb{C} \setminus \{p_j\}$ and meromorphic in p_j with a pole of multiplicity $k+1$ such that for every p_j -excision function $\chi \in C^\infty(\mathbb{C})$ the function $\chi \cdot \psi_{p_j, k}$ belongs to $C^\infty(\mathbb{R}_\beta, \mathcal{S}(\Gamma_\beta))$.

Let \mathbb{H}_β be a right half-plane in \mathbb{C} , and $\pi_{\mathbb{C}}P \cap \mathbb{H}_\beta = \emptyset$. We define the space of *meromorphic Volterra Mellin symbols* (of boundary value problems) of order μ and type $d \in \mathbb{N}_0$ with asymptotic type P as

$$M_{V,P,cl}^{\mu,d;\ell}(X; \mathbb{H}_\beta) := M_{P,cl}^{\mu,d;\ell}(X) \cap \mathcal{B}_{V,cl}^{\mu,d;\ell}(X; \mathbb{H}_\beta). \quad (5.2)$$

Analogously, we define the spaces $M_P^{-\infty,d}(X)$ and $M_{V,P}^{-\infty,d}(X; \mathbb{H}_\beta)$ of regularizing meromorphic (Volterra) Mellin symbols of type $d \in \mathbb{N}_0$ with asymptotic type P .

If $P = O$ is the empty asymptotic type the spaces are called holomorphic Mellin symbols.

Remark 5.4. The topology on the space $M_{P,cl}^{\mu,d;\ell}(X)$ is determined by the following ingredients:

- The topology of $\mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}P, \mathcal{B}^{\mu,d}(X))$.
- Convergence of the Laurent coefficients ν_k in the corresponding coefficient spaces $L_j \subseteq \mathcal{B}^{-\infty,d}(X)$.
- Uniform convergence of (5.1) for $\beta \in I$ for every compact interval $I \subseteq \mathbb{R}$.

With this topology $M_{P,cl}^{\mu,d;\ell}(X)$ is a Fréchet space. Note that the topology does not depend on the particular choice of the functions $\psi_{p_j,k}$ involved in (5.1) and the coefficients σ_{j_k} determined by them.

The topology of meromorphic Volterra Mellin symbols is given as the intersection topology determined by (5.2). Notice that the spaces of meromorphic Volterra Mellin symbols are independent of the right half-plane \mathbb{H}_β as long as $\mathbb{H}_\beta \cap \pi_{\mathbb{C}}P = \emptyset$. In particular, holomorphic Volterra Mellin symbols are parameter-dependent Volterra boundary value problems with respect to any right half-plane $\mathbb{H}_\beta \subseteq \mathbb{C}$. Therefore, we suppress the half-plane from the notation when we deal with holomorphic Volterra Mellin symbols.

Remark 5.5. The Mellin kernel cut-off operator with respect to the weight $\gamma \in \mathbb{R}$

$$(H_\gamma(\varphi)a)(z) := \int_{\mathbb{R}} \int_{\mathbb{R}_+} r^{i\tau} \varphi(r) a(z - i\tau) \frac{dr}{r} d\tau \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!} (r\partial_r)^k \varphi(r)|_{r=1} \cdot \partial_z^k a$$

for $z \in \Gamma_{\frac{1}{2}-\gamma}$, respectively $z \in \mathbb{H}_{\frac{1}{2}-\gamma}$, actually gives rise to continuous bilinear mappings in the spaces

$$H_\gamma : \begin{cases} C_0^\infty(\mathbb{R}_+) \times \mathcal{B}_{cl}^{\mu,d;\ell}(X; \Gamma_{\frac{1}{2}-\gamma}) \longrightarrow M_{O,cl}^{\mu,d;\ell}(X), \\ C_0^\infty(\mathbb{R}_+) \times \mathcal{B}_{V,cl}^{\mu,d;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma}) \longrightarrow M_{V,O,cl}^{\mu,d;\ell}(X). \end{cases}$$

General statements about the (Mellin) kernel cut-off operator as an operator acting in spaces of (holomorphic) operator-valued (Volterra) symbols can also be found in [38], [37].

In particular, for $P \in \text{As}(\mathcal{B}^{-\infty,d}(X))$ we have

$$\begin{aligned} M_{P,cl}^{\mu,d;\ell}(X) &= M_{O,cl}^{\mu,d;\ell}(X) + M_P^{-\infty,d}(X), \\ M_{V,P,cl}^{\mu,d;\ell}(X; \mathbb{H}_\beta) &= M_{V,O,cl}^{\mu,d;\ell}(X) + M_{V,P}^{-\infty,d}(X; \mathbb{H}_\beta) \end{aligned}$$

algebraically and topologically with the topology of the non-direct sum of Fréchet spaces on the right hand sides.

These decompositions imply that with a meromorphic (Volterra) Mellin symbol $a(z)$ we can associate a unique tuple of principal symbols

$$\begin{aligned}\sigma_{\psi}^{\mu;\ell}(a) &= \sigma_{\psi}^{\mu;\ell}(a|_{\Gamma_{\beta}}) & \sigma_{\psi}^{\mu;\ell}(a) &= \sigma_{\psi}^{\mu;\ell}(a|_{\mathbb{H}_{\beta}}) \\ \sigma_{\partial}^{\mu;\ell}(a) &= \sigma_{\partial}^{\mu;\ell}(a|_{\Gamma_{\beta}}) & \sigma_{\partial}^{\mu;\ell}(a) &= \sigma_{\partial}^{\mu;\ell}(a|_{\mathbb{H}_{\beta}}),\end{aligned}$$

where $\beta \in \mathbb{R}$ is chosen such that the principal symbols make sense; actually they then do not depend on the particular choice of β .

Definition 5.6. a) Let $(\gamma, (-N, 0])$ be a weight datum, $N \in \mathbb{N}$. For $\mu \in \mathbb{Z}$ define the space of *conormal symbols* of type $d \in \mathbb{N}_0$ with respect to the weight datum $(\gamma, (-N, 0])$ as

$$\Sigma_M^{\mu,d;\ell}(X, (\gamma, (-N, 0])) := \{(h_0, \dots, h_{N-1}); h_j \in M_{P_j, cl}^{\mu,d;\ell}(X), \pi_{\mathbb{C}} P_0 \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset\}.$$

The subspace of *Volterra conormal symbols* of type $d \in \mathbb{N}_0$ with respect to the weight datum $(\gamma, (-N, 0])$ is defined as

$$\Sigma_{M,V}^{\mu,d;\ell}(X, (\gamma, (-N, 0])) := \{(h_0, \dots, h_{N-1}); h_j \in M_{V,P_j, cl}^{\mu,d;\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma+j})\}.$$

b) We define the *Mellin translation product*

$$\begin{aligned}\Sigma_M^{\mu_1,d_1;\ell}(X, (\gamma, (-N, 0])) \times \Sigma_M^{\mu_2,d_2;\ell}(X, (\gamma, (-N, 0])) &\longrightarrow \Sigma_M^{\mu_1+\mu_2,d;\ell}(X, (\gamma, (-N, 0])), \\ (g_0, \dots, g_{N-1}) \# (h_0, \dots, h_{N-1}) &:= (\tilde{h}_0, \dots, \tilde{h}_{N-1}), \\ \tilde{h}_k &:= \sum_{p+q=k} (T_{-q} g_p)(h_q),\end{aligned}$$

where $d = \max\{\mu_2 + d_1, d_2\}$. Here T denotes the translation operator, i.e., $((T_{-q} g_p)(h_q))(z) = g_p(z - q)h_q(z)$. Notice that the bundles have to fit together such that the product can be calculated.

c) For $\mu \leq 0$ we define a $*$ -operation

$$\begin{aligned}\Sigma_M^{\mu,0;\ell}(X, (\gamma, (-N, 0])) &\longrightarrow \Sigma_M^{\mu,0;\ell}(X, (-\gamma, (-N, 0])), \\ (h_0, \dots, h_{N-1})^* &:= (\tilde{h}_0, \dots, \tilde{h}_{N-1}), \\ \tilde{h}_k(z) &:= (h_k(n+1-k-\bar{z}))^{(*)},\end{aligned}$$

where $(*)$ denotes the formal adjoint with respect to the L^2 -inner products on the manifold and on the boundary.

Theorem 5.7. a) $\Sigma_{M(V)}^{\mu,d;\ell}(X, (\gamma, (-N, 0]))$ is a vector space with componentwise addition and scalar multiplication.

b) The Mellin translation product $\#$ induces an associative product, i.e., it is well-defined as a bilinear mapping

$$\Sigma_{M(V)}^{\mu_1,d_1;\ell}(X, (\gamma, (-N, 0])) \times \Sigma_{M(V)}^{\mu_2,d_2;\ell}(X, (\gamma, (-N, 0])) \longrightarrow \Sigma_{M(V)}^{\mu_1+\mu_2,d;\ell}(X, (\gamma, (-N, 0])),$$

where $d = \max\{\mu_2 + d_1, d_2\}$, and we have $(a\#b)\#c = a\#(b\#c)$ in the corresponding spaces.

c) The $*$ -operation is well-defined as an antilinear mapping

$$* : \Sigma_M^{\mu,0;\ell}(X, (\gamma, (-N, 0])) \longrightarrow \Sigma_M^{\mu,0;\ell}(X, (-\gamma, (-N, 0]))$$

for $\mu \leq 0$, and we have $(a\#b)^* = b^*\#a^*$, $(a^*)^* = a$.

5.1. Ellipticity and parabolicity for conormal symbols.

Definition 5.8. a) An element $a = (h_0, \dots, h_{N-1}) \in \Sigma_M^{\mu,d;\ell}(X, (\gamma, (-N, 0]))$, $d \leq \mu_+$, is called *elliptic* if

- $h_0|_{\Gamma_{\frac{n+1}{2}-\gamma}} \in \mathcal{B}_{cl}^{\mu,d;\ell}(X; \Gamma_{\frac{n+1}{2}-\gamma})$ is parameter-dependent elliptic,
- there exists $s_0 \in \mathbb{R}$, $s_0 \gg 0$ sufficiently large, such that

$$h_0(z) : \begin{array}{ccc} H^{s_0}(X, E) & & H^{s_0-\mu}(X, F) \\ \oplus & \longrightarrow & \oplus \\ H^{s_0}(Y, J_-) & & H^{s_0-\mu}(Y, J_+) \end{array}$$

is bijective for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

b) An element $a = (h_0, \dots, h_{N-1}) \in \Sigma_{M,V}^{\mu,d;\ell}(X, (\gamma, (-N, 0]))$, $d \leq \mu_+$, is called *parabolic* if

- $h_0|_{\mathbb{H}_{\frac{n+1}{2}-\gamma}} \in \mathcal{B}_{V,cl}^{\mu,d;\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma})$, is parabolic as an element in $\mathcal{B}_{V,cl}^{\mu,d;\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma})$,
- there exists $s_0 \in \mathbb{R}$, $s_0 \gg 0$ sufficiently large, such that

$$h_0(z) : \begin{array}{ccc} H^{s_0}(X, E) & & H^{s_0-\mu}(X, F) \\ \oplus & \longrightarrow & \oplus \\ H^{s_0}(Y, J_-) & & H^{s_0-\mu}(Y, J_+) \end{array}$$

is bijective for all $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$.

Notation 5.9. Let us denote the unit with respect to the Mellin translation product as

$$\mathbf{1} := (1, 0, \dots, 0) \in \Sigma_{M(V)}^{0,0;\ell}(X, (\gamma, (-N, 0])).$$

Theorem 5.10. Let $a \in \Sigma_{M(V)}^{\mu,d;\ell}(X, (\gamma, (-N, 0]))$, $d \leq \mu_+$. Then the following are equivalent:

- a is elliptic (parabolic) in the sense of Definition 5.8.
- a is invertible within the algebra of (Volterra) conormal symbols, i.e., there exists $b \in \Sigma_{M(V)}^{-\mu,d';\ell}(X, (\gamma, (-N, 0]))$, $d' \leq (-\mu)_+$, such that $a\#b = \mathbf{1}$ and $b\#a = \mathbf{1}$.

Proof. Without loss of generality we may assume $N = 1$; the general case is subject to simple algebra.

Let us assume that a is elliptic. In particular, there exists $\beta \in \mathbb{R}$ such that $a|_{\Gamma_\beta} \in \mathcal{B}_{cl}^{\mu,d;\ell}(X; \Gamma_\beta)$ is parameter-dependent elliptic. Let $p \in \mathcal{B}_{cl}^{-\mu,d';\ell}(X; \Gamma_\beta)$, $d' \leq (-\mu)_+$, be a parameter-dependent parametrix according to Theorem 3.7. Substituting p by $H_{\frac{1}{2}-\beta}(\psi)p$, where $\psi \in C_0^\infty(\mathbb{R}_+)$ with $\psi \equiv 1$ near $r = 1$, we may

assume that p is a holomorphic Mellin symbol of order $-\mu$ and type d' (see also Remark 5.5).

We have $ap = 1 + r$ with a regularizing meromorphic Mellin symbol r . Note that $1 + r(z)$ is a finitely meromorphic Fredholm family in the Sobolev spaces that is invertible for $|\operatorname{Im}(z)|$ sufficiently large, uniformly for $\operatorname{Re}(z)$ in compact intervals. The general invertibility result for such families (see Gohberg and Sigal [22]) implies the invertibility of $1 + r$ with inverse being a finitely meromorphic Fredholm family, and from the identity $(1 + r)^{-1} = 1 - r + r(1 + r)^{-1}r$ we derive that $(1 + r)^{-1} = 1 + r'$ with a regularizing meromorphic Mellin symbol r' . This shows that a is invertible from the right with inverse $p(1 + r')$; the invertibility from the left follows analogously, and thus a is invertible with inverse $b = p(1 + r')$ as desired. Note that in view of the bijectivity of $a(z)$ for $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ clearly b is free of poles on this weight line, and consequently belongs to the space of conormal symbols associated with the weight datum $(\gamma, (-1, 0])$ as asserted.

If a is a Volterra conormal symbol we first apply the invertibility of a as a general conormal symbol as proved above, and from Theorem 3.8 we conclude that the inverse in fact belongs to the space of Volterra conormal symbols as desired.

This finishes the proof of the theorem for the converse assertion is evident. \square

Notation 5.11. We denote the subspaces of regularizing (Volterra) conormal symbols of type $d \in \mathbb{N}_0$ as $\Sigma_{M(V)}^d(X, (\gamma, (-N, 0]))$.

Proposition 5.12. *Let $a = (h_0, \dots, h_{N-1}) \in \Sigma_{M(V)}^d(X, (\gamma, (-N, 0]))$. Then the following are equivalent:*

a) *There exists $s_0 \gg 0$ sufficiently large such that*

$$1 + h_0(z) : \begin{array}{ccc} H^{s_0}(X, E) & & H^{s_0}(X, E) \\ \oplus & \longrightarrow & \oplus \\ H^{s_0}(Y, J_-) & & H^{s_0}(Y, J_-) \end{array}$$

is bijective for $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ (or $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$).

b) *$1 + a$ is invertible with respect to the Mellin translation product with inverse $(1 + a)^{-1} \in 1 + \Sigma_{M(V)}^d(X, (\gamma, (-N, 0]))$.*

Proof. This follows from the proof of Theorem 5.10 (note that it suffices to consider the case $N = 1$ also for this proof). \square

6. SOBOLEV SPACES OF THE VOLTERRA CONE CALCULUS

The present section gives the basic definitions of the natural anisotropic weighted Sobolev spaces and their subspaces with asymptotics that are employed in the pseudodifferential calculus associated with parabolic boundary value problems. Recall that we make use of a variant of the cone calculus for boundary value problems as introduced by Schrohe and Schulze [59, 60]. Therefore, material about Sobolev spaces similar to those considered below can also be found in their works;

however, near infinity we will employ spaces that are strictly different even in the isotropic setting.

A discussion of the spaces in the boundaryless case can be found in [39]. Observe that on a manifold with boundary we obtain the desired Sobolev spaces on the one hand simply by restriction from the double, and on the other hand by taking the closure of the smooth functions with support in the interior (see Section 6.3 below).

We refer, e.g., to Lions and Magenes [44], Agranovich and Vishik [3], Grubb and Solonnikov [28] for introductory classical material about anisotropic Sobolev spaces adapted to parabolic equations in a cylindrical configuration; general statements about weighted Mellin Sobolev spaces and spaces with asymptotics in singular analysis can be found in the monographs of Schulze [63, 65], see also Dorschfeldt [12].

6.1. Abstract cone Sobolev spaces. Let E be a Hilbert space. For $s, \gamma \in \mathbb{R}$ the abstract weighted Mellin Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E)$ is defined as the closure of $C_0^\infty(\mathbb{R}_+, E)$ with respect to the norm

$$\|u\|^2 = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} \langle \text{Im}(z) \rangle^{2s} \|\mathcal{M}_\gamma u(z)\|_E^2 dz$$

with the weighted Mellin transform $\mathcal{M}_\gamma u(z) = \int_{\mathbb{R}_+} r^z u(r) \frac{dr}{r}$ for $z \in \Gamma_{\frac{1}{2}-\gamma}$.

Recall that for $s \in \mathbb{R}$ the standard abstract Sobolev space $H^s(\mathbb{R}, E)$ is the closure of $C_0^\infty(\mathbb{R}, E)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}} \langle \xi \rangle^{2s} \|\mathcal{F}u(\xi)\|_E^2 d\xi$$

with the (normalized) Fourier transform \mathcal{F} . Moreover, for $s, \delta \in \mathbb{R}$ let $H^s(\mathbb{R}, E)_\delta := \langle t \rangle^{-\delta} H^s(\mathbb{R}, E)$.

Observe that the mapping $u(r) \mapsto e^{(\gamma-\frac{1}{2})t} u(e^{-t})$ induces an isomorphism $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E) \cong H^s(\mathbb{R}, E)$. In particular, the control of exponential weights near $t = \infty$ in the space $H^s(\mathbb{R}, E)$ is transformed to the control of power weights in $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E)$ near $r = 0$ — the latter is reflected by the parameter $\gamma \in \mathbb{R}$.

For $s, \gamma, \delta \in \mathbb{R}$ the abstract cone Sobolev space is defined as

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+, E)_\delta := \{\omega(r)u_1(r) + (1-\omega(r))u_2(r); u_1 \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+, E), u_2 \in H^s(\mathbb{R}, E)_\delta\}$$

with an arbitrary cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$, i.e., $\omega \equiv 1$ near $r = 0$. Notice that $\mathcal{K}^{s,\gamma}(\mathbb{R}_+, E)_\delta$ is represented as a non-direct sum of Hilbert spaces, and we have $\mathcal{K}^{0,0}(\mathbb{R}_+, E)_0 = L^2(\mathbb{R}_+, E)$.

6.2. Sobolev spaces in the boundaryless case. Let Z be a closed manifold of dimension n , and $E \in \text{Vect}(Z)$ a complex vector bundle. As usual, we employ the notation $Z^\wedge = \mathbb{R}_+ \times Z$ for the cone over Z .

For $s, \gamma, \delta \in \mathbb{R}$ we define

$$\mathcal{K}^{s, \gamma; \ell}(Z^\wedge, E)_\delta := \begin{cases} \mathcal{K}^{0, \gamma - \frac{n}{2}}(\mathbb{R}_+, H^s(Z, E))_\delta \cap \mathcal{K}^{\frac{s}{\ell}, \gamma - \frac{n}{2}}(\mathbb{R}_+, L^2(Z, E))_\delta & s \geq 0, \\ \mathcal{K}^{0, \gamma - \frac{n}{2}}(\mathbb{R}_+, H^s(Z, E))_\delta + \mathcal{K}^{\frac{s}{\ell}, \gamma - \frac{n}{2}}(\mathbb{R}_+, L^2(Z, E))_\delta & s \leq 0. \end{cases}$$

The shift of the weight by one half of the dimension of Z is employed for traditional reasons, and is a tribute to the origin of the function spaces in the analysis on manifolds with conical singularities.

Moreover, with some cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ we define $\mathcal{S}^\gamma(Z^\wedge, E)$ to be the space of all $u(r) = \omega(r)u_1(r) + (1 - \omega(r))u_2(r)$ with $u_1 \in \mathcal{T}_{\gamma - \frac{n}{2}}(\mathbb{R}_+, C^\infty(Z, E))$ and $u_2 \in \mathcal{S}(\mathbb{R}, C^\infty(Z, E))$, where for any Fréchet space F and $\gamma \in \mathbb{R}$ we denote $\mathcal{T}_\gamma(\mathbb{R}_+, F) = \mathcal{M}_\gamma^{-1}(\mathcal{S}(\Gamma_{\frac{1}{2} - \gamma}, F))$. Note that $\mathcal{S}^\gamma(Z^\wedge, E)$ carries a nuclear Fréchet topology.

6.2.1. Spaces with asymptotics. An asymptotic type associated with the weight datum (γ, Θ) , $\Theta = (\theta, 0]$ with $\theta < 0$, is a finite set P of triples (p, m, L) , where $m \in \mathbb{N}_0$ is an integer, L a finite-dimensional subspace of $C^\infty(Z, E)$, and $p \in \mathbb{C}$ such that $\frac{n+1}{2} - \gamma + \theta < \operatorname{Re}(p) < \frac{n+1}{2} - \gamma$.

Let $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ be an arbitrary cut-off function near $r = 0$, and P an asymptotic type associated with (γ, Θ) . Define

$$\mathcal{E}_P(Z^\wedge, E) := \{\omega(r) \sum_{(p, m, L) \in P} \sum_{k=0}^m c_{p, k} r^{-p} \log^k(r); c_{p, k} \in L\}.$$

This is a finite-dimensional subspace of $C^\infty(Z^\wedge, E)$, and we endow this space with the norm topology. Moreover, we define

$$\begin{aligned} \mathcal{K}_\Theta^{s, \gamma; \ell}(Z^\wedge, E)_\delta &:= \bigcap_{\bar{\gamma} < \gamma - \theta} \mathcal{K}^{s, \bar{\gamma}; \ell}(Z^\wedge, E)_\delta, \\ \mathcal{K}_P^{s, \gamma; \ell}(Z^\wedge, E)_\delta &:= \mathcal{K}_\Theta^{s, \gamma; \ell}(Z^\wedge, E)_\delta + \mathcal{E}_P(Z^\wedge, E), \\ \mathcal{S}_P^\gamma(Z^\wedge, E) &:= \bigcap_{s, \delta \in \mathbb{R}} \mathcal{K}_P^{s, \gamma; \ell}(Z^\wedge, E)_\delta, \end{aligned}$$

which are Fréchet subspaces of $\mathcal{K}^{s, \gamma; \ell}(Z^\wedge, E)_\delta$ and $\mathcal{S}^\gamma(Z^\wedge, E)$, respectively.

6.3. Sobolev spaces in the case with boundary. We employ the notation and conventions from Section 3. With the restriction r^+ of distributions defined on $(2\overline{X})^\wedge$ to X^\wedge let

$$\begin{aligned} \mathcal{K}_{(P)}^{s, \gamma; \ell}(X^\wedge, E)_\delta &:= r^+ \mathcal{K}_{(P)}^{s, \gamma; \ell}((2\overline{X})^\wedge, E)_\delta, \\ \mathcal{S}_{(P)}^\gamma(X^\wedge, E) &:= r^+ \mathcal{S}_{(P)}^\gamma((2\overline{X})^\wedge, E), \end{aligned}$$

endowed with the corresponding quotient topology. Moreover, let $\mathcal{K}_0^{s, \gamma; \ell}(\overline{X}^\wedge, E)_\delta$ be the closure of $C_0^\infty(X^\wedge, E)$ in $\mathcal{K}^{s, \gamma; \ell}((2\overline{X})^\wedge, E)_\delta$.

Observe that the $r^{-\frac{n}{2}}L^2$ -inner product in the space $r^{-\frac{n}{2}}L^2(X^\wedge, E) \oplus r^{-\frac{n}{2}}L^2(Y^\wedge, J_-)$ extends to a non-degenerate sesquilinear pairing

$$\langle \cdot, \cdot \rangle : \begin{array}{ccc} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_{\frac{n}{2} + \delta} & \times & \mathcal{K}_0^{-s, -\gamma; \ell}(\overline{X}^\wedge, E)_{\frac{n}{2} - \delta} \\ \oplus & & \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_{\frac{n}{2} + \delta} & & \mathcal{K}^{-s, -\gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_{\frac{n}{2} - \delta} \end{array} \longrightarrow \mathbb{C}$$

for $s, \gamma, \delta \in \mathbb{R}$, which allows an identification of the dual spaces.

Following the tradition from the analysis on manifolds with conical singularities, we will take the $r^{-\frac{n}{2}}L^2$ -inner product as the reference inner product in the scale of Sobolev spaces.

7. ALGEBRAS OF SMOOTHING OPERATORS

Before being able to formulate the Volterra cone calculus associated with parabolic boundary value problems we need some preparations about the ideals of smoothing operators close to $r = 0$. They are given as Green operators — the smallest ideal in the calculus without any relevant symbolic information — and the larger ideal of smoothing Mellin and Green operators. The quotient of the latter ideal by the Green operators is isomorphic to the regularizing conormal symbols (with the Mellin translation product as multiplicative structure) as considered in Section 5.

Definition 7.1. a) An operator

$$G \in \mathcal{L} \left(\begin{array}{cc} \mathcal{K}_0^{s, \gamma; \ell}(\overline{X}^\wedge, E)_\delta & \mathcal{K}^{t, \gamma; \ell}(X^\wedge, F)_{\delta'} \\ \oplus & \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta & \mathcal{K}^{t, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta'} \end{array} \right)$$

for all $s, t, \delta, \delta' \in \mathbb{R}$ is called a *Green operator of type zero* with respect to the asymptotic types (P_1, P_2) and (Q_1, Q_2) , if G and its formal adjoint G^* with respect to the $r^{-\frac{n}{2}}L^2$ -inner product induce continuous operators

$$\begin{array}{ccc} \mathcal{K}_0^{s, \gamma; \ell}(\overline{X}^\wedge, E)_\delta & \longrightarrow & \mathcal{S}_{P_1}^\gamma(X^\wedge, F) \\ G : \quad \oplus & & \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta & \longrightarrow & \mathcal{S}_{P_2}^{\gamma - \frac{1}{2}}(Y^\wedge, J_+) \\ \\ \mathcal{K}_0^{s, -\gamma; \ell}(\overline{X}^\wedge, F)_\delta & \longrightarrow & \mathcal{S}_{Q_1}^{-\gamma}(X^\wedge, E) \\ G^* : \quad \oplus & & \oplus \\ \mathcal{K}^{s, -\gamma - \frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta} & \longrightarrow & \mathcal{S}_{Q_2}^{-\gamma - \frac{1}{2}}(Y^\wedge, J_-) \end{array}$$

for all $s, \delta \in \mathbb{R}$. The space of all Green operators of type zero is denoted by $C_G^0(X^\wedge, (\gamma, \Theta))$.

b) A Green operator $G \in C_G^0(X^\wedge, (\gamma, \Theta))$ of type zero is called a *Volterra Green operator of type zero* provided that the following condition is fulfilled:

For every $r_0 \in \mathbb{R}_+$ we have $(Gu)(r) \equiv 0$ for $r > r_0$ for all $u = (u_1, u_2) \in C_0^\infty(\mathbb{R}_+, C^\infty(\overline{X}, E)) \oplus C_0^\infty(\mathbb{R}_+, C^\infty(Y, J_-))$ such that $u(r) \equiv 0$ for $r > r_0$.

The space of all Volterra Green operators of type zero is denoted by $C_{G,V}^0(X^\wedge, (\gamma, \Theta))$.

- c) The space $C_{G(V)}^d(X^\wedge, (\gamma, \Theta))$ of (Volterra) Green operators of type $d \in \mathbb{N}_0$ consists of all operators G of the form $G = \sum_{j=0}^d G_j \partial_+^j$ with $G_j \in C_{G(V)}^0(X^\wedge, (\gamma, \Theta))$.

Recall that we also write ∂_+ for the operator matrix $\begin{pmatrix} \partial_+ & 0 \\ 0 & 1 \end{pmatrix}$.

In particular, the operators in $C_G^d(X^\wedge, (\gamma, \Theta))$ are well-defined as continuous operators in the spaces

$$G : \begin{array}{c} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{S}^\gamma(X^\wedge, F) \\ \oplus \\ \mathcal{S}^{\gamma - \frac{1}{2}}(Y^\wedge, J_+) \end{array}$$

for $s > d - \frac{1}{2}$ and $\delta \in \mathbb{R}$.

Remark 7.2. From Definition 7.1 we conclude that the class of (Volterra) Green operators is independent of the particular anisotropy $\ell \in 2\mathbb{N}$. Moreover, it forms an operator algebra, i.e., the composition induces a well-defined mapping

$$C_{G(V)}^{d_1}(X^\wedge, (\gamma, \Theta)) \times C_{G(V)}^{d_2}(X^\wedge, (\gamma, \Theta)) \longrightarrow C_{G(V)}^{d_2}(X^\wedge, (\gamma, \Theta))$$

for $d_1, d_2 \in \mathbb{N}_0$; note that the vector bundles are assumed to fit together to be able to calculate the composition.

Proposition 7.3. *An operator G belongs to $C_G^0(X^\wedge, (\gamma, \Theta))$ with respect to the asymptotic types (P_1, P_2) and (Q_1, Q_2) if and only if*

$$G \in \left(\begin{pmatrix} \mathcal{K}^{\infty, -\gamma}(X^\wedge, E)_\infty \\ \oplus \\ \mathcal{K}^{\infty, -\gamma - \frac{1}{2}}(Y^\wedge, J_-)_\infty \end{pmatrix} \widehat{\otimes}_\pi \begin{pmatrix} \mathcal{S}_{P_1}^\gamma(X^\wedge, F) \\ \oplus \\ \mathcal{S}_{P_2}^{\gamma - \frac{1}{2}}(Y^\wedge, J_+) \end{pmatrix} \right) \cap \left(\begin{pmatrix} \mathcal{S}_{Q_1}^{-\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{S}_{Q_2}^{-\gamma - \frac{1}{2}}(Y^\wedge, J_-) \end{pmatrix} \widehat{\otimes}_\pi \begin{pmatrix} \mathcal{K}^{\infty, \gamma}(X^\wedge, F)_\infty \\ \oplus \\ \mathcal{K}^{\infty, \gamma - \frac{1}{2}}(Y^\wedge, J_+)_\infty \end{pmatrix} \right).$$

The Sobolev spaces of orders ∞ are by notation the intersections of all spaces with finite orders. In particular, we have

$$C_G^0(X^\wedge, (\gamma, \Theta)) \hookrightarrow \begin{pmatrix} \mathcal{S}^{-\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{S}^{-\gamma - \frac{1}{2}}(Y^\wedge, J_-) \end{pmatrix} \widehat{\otimes}_\pi \begin{pmatrix} \mathcal{S}^\gamma(X^\wedge, F) \\ \oplus \\ \mathcal{S}^{\gamma - \frac{1}{2}}(Y^\wedge, J_+) \end{pmatrix},$$

and every Green operator G of type zero induces a nuclear operator

$$G \in \ell^1 \left(\begin{array}{c} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta \end{array} , \begin{array}{c} \mathcal{K}^{t, \gamma; \ell}(X^\wedge, F)_{\delta'} \\ \oplus \\ \mathcal{K}^{t, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta'} \end{array} \right)$$

for $s > -\frac{1}{2}$ and all $t, \delta, \delta' \in \mathbb{R}$.

The projective tensor products are to be understood in the sense that in the representation of the nuclear operator G as a projective series the functionals originate from the spaces in the first factor as these are considered as distribution spaces according to the Riesz representation induced by the $r^{-\frac{n}{2}}L^2$ -inner product.

Proof. The proof follows from the following general statement which is subject to elementary properties of tensor products (see, e.g., Jarchow [33]):

Let H_1 and H_2 be Hilbert spaces, and $E \hookrightarrow H_1'$ and $F \hookrightarrow H_2$ nuclear Fréchet spaces. Then an operator $G : H_1 \rightarrow H_2$ satisfies $G(H_1) \subseteq F$ and $G^t(H_2') \subseteq E$ if and only if $G \in E \widehat{\otimes}_\pi H_2 \cap H_1' \widehat{\otimes}_\pi F \subseteq H_1' \widehat{\otimes}_\pi H_2 = \ell^1(H_1, H_2)$. \square

Theorem 7.4. a) Let $G \in C_G^d(X^\wedge, (\gamma, \Theta))$ such that

$$1 + G : \begin{array}{c} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s, \gamma - \frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta \end{array} \quad (7.1)$$

is invertible for some $s > d - \frac{1}{2}$ and some $\delta \in \mathbb{R}$. Then $1 + G$ is invertible in the Sobolev spaces for all $s > d - \frac{1}{2}$ and all $\delta \in \mathbb{R}$, and the inverse is given as $(1 + G)^{-1} = 1 + G_1$ with a Green operator $G_1 \in C_G^d(X^\wedge, (\gamma, \Theta))$.

b) Let $G \in C_{G,V}^d(X^\wedge, (\gamma, \Theta))$. Then $1 + G$ is invertible as an operator in (7.1) for all $s > d - \frac{1}{2}$ and all $\delta \in \mathbb{R}$, and we have $(1 + G)^{-1} = 1 + G_1$ with a Volterra Green operator $G_1 \in C_{G,V}^d(X^\wedge, (\gamma, \Theta))$.

Proof. For the proof of a) note first that we may write

$$(1 + G)^{-1} = 1 - G + G(1 + G)^{-1}G.$$

Writing $G = \sum_{j=0}^d \tilde{G}_j \partial_+^j$ with Green operators \tilde{G}_j of type zero we see that the operator $G_1 := -G + G(1 + G)^{-1}G$ fulfills the conditions in Definition 7.1, and consequently belongs to $C_G^d(X^\wedge, (\gamma, \Theta))$.

Let us prove b). We first consider the weight $\gamma = \frac{n}{2}$. For Green operators are compact (even nuclear) in the Sobolev spaces, the operator $1 + G$ is Fredholm of index zero. Thus for the proof of the invertibility it suffices to check that $1 + G$ is one-to-one. From Definition 7.1 we see that

$$\ker(1 + G) \subseteq \begin{array}{c} \mathcal{S}^{\frac{n}{2}}(X^\wedge, E) \\ \oplus \\ \mathcal{S}^{\frac{n-1}{2}}(Y^\wedge, J_-) \end{array},$$

and consequently we may regard $1 + G$ as an operator

$$1 + G : \begin{array}{c} \mathcal{S}^{\frac{n}{2}}(X^\wedge, E) \\ \oplus \\ \mathcal{S}^{\frac{n-1}{2}}(Y^\wedge, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{S}^{\frac{n}{2}}(X^\wedge, E) \\ \oplus \\ \mathcal{S}^{\frac{n-1}{2}}(Y^\wedge, J_-) \end{array}. \quad (1)$$

From Proposition 7.3 we conclude that (1) extends by continuity to

$$1 + G : L^2\left(\mathbb{R}_+, \begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array}\right) \longrightarrow L^2\left(\mathbb{R}_+, \begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array}\right)$$

for $s > d - \frac{1}{2}$. Moreover, G is a Volterra integral operator in this L^2 -space with continuous Volterra integral kernel

$$k \in C\left(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}\left(\begin{array}{c} H^s(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array}\right)\right),$$

i.e., $k(r, r') \equiv 0$ for $r > r'$, that satisfies the estimate $\sup\{g(r)g(r')\|k(r, r')\|; r, r' \in \mathbb{R}_+\} < \infty$, where $g \in C(\mathbb{R}_+)$ is a function of the form $g(r) = \omega(r)r^{\frac{1}{2}}\langle \log(r) \rangle + (1 - \omega(r))r$ with a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ near $r = 0$. In particular, we have $\int_{\mathbb{R}_+} \frac{1}{g(r)^2} dr < \infty$, and the general theory now implies that G is quasinilpotent in L^2 — the properties of the kernel $k(r, r')$ as stated above imply that the series of iterated kernels associated with $(-G)^j$ for $j \in \mathbb{N}$ is convergent, and consequently the Neumann series $\sum (-G)^j$ converges.

In particular, $1 + G$ is one-to-one in the spaces (1), which implies the invertibility as an operator in the cone Sobolev spaces (7.1). In fact, the inverse is given as $(1 + G)^{-1} = 1 + G_1$ with a Volterra Green operator of type d , which follows in the same way as in the proof of a). This proves the assertion for the weight $\gamma = \frac{n}{2}$.

Next consider the case of general weights $\gamma \in \mathbb{R}$. We may write $1 + G = r^{\gamma - \frac{n}{2}}(1 + \tilde{G})r^{-(\gamma - \frac{n}{2})}$, where the operator $\tilde{G} := r^{-(\gamma - \frac{n}{2})}Gr^{\gamma - \frac{n}{2}}$ belongs to $C_{G,V}^d(X^\wedge, (\frac{n}{2}, \Theta))$. From the first part of the proof we conclude that $1 + \tilde{G}$ is invertible, and we have $(1 + \tilde{G})^{-1} = 1 + \tilde{G}_1$ with a Volterra Green operator $\tilde{G}_1 \in C_{G,V}^d(X^\wedge, (\frac{n}{2}, \Theta))$. Thus also $1 + G$ is invertible with inverse $(1 + G)^{-1} = 1 + G_1$, where $G_1 := r^{\gamma - \frac{n}{2}}\tilde{G}_1r^{-(\gamma - \frac{n}{2})}$ is a Volterra Green operator in the space $C_{G,V}^d(X^\wedge, (\gamma, \Theta))$. This completes the proof of the theorem. \square

7.1. Smoothing Mellin and Green operators.

Definition 7.5. The space $C_{M+G}^d(X^\wedge, (\gamma, (-N, 0]))$ of *smoothing Mellin and Green operators* of type $d \in \mathbb{N}_0$ with respect to the weight datum $(\gamma, (-N, 0])$, $N \in \mathbb{N}$, consists of all operators

$$A : \begin{array}{c} \mathcal{S}^\gamma(X^\wedge, E) \\ \oplus \\ \mathcal{S}^{\gamma - \frac{1}{2}}(Y^\wedge, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{S}^\gamma(X^\wedge, F) \\ \oplus \\ \mathcal{S}^{\gamma - \frac{1}{2}}(Y^\wedge, J_+) \end{array}$$

of the form

$$A = \sum_{j=0}^{N-1} \omega r^j \text{op}_M^{\gamma_j}(h_j)\omega + G \quad (7.2)$$

with a Green operator $G \in C_G^d(X^\wedge, (\gamma, (-N, 0]))$, and

- i) a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ near $r = 0$,
- ii) $\gamma - \frac{n}{2} - j \leq \gamma_j \leq \gamma - \frac{n}{2}$,
- iii) meromorphic Mellin symbols $h_j \in M_{P_j}^{-\infty, d}(X)$ such that $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2} - \gamma_j} = \emptyset$.

Recall that $\text{op}_M^{\gamma_j}(h_j)u = \mathcal{M}_{\gamma_j}^{-1}h_j(z)\mathcal{M}_{\gamma_j}u$ is the Mellin pseudodifferential operator with respect to the weight γ_j associated with the symbol h_j .

Moreover, we define the space $C_{M+G, V}^d(X^\wedge, (\gamma, (-N, 0]))$ of *smoothing Volterra Mellin and Green operators* of type $d \in \mathbb{N}_0$ with respect to the weight datum $(\gamma, (-N, 0])$, $N \in \mathbb{N}$, to consist of all operators

$$A = \sum_{j=0}^{N-1} \omega r^j \text{op}_M^{\gamma - \frac{n}{2} - j}(h_j)\omega + G \quad (7.3)$$

with a Volterra Green operator $G \in C_{G, V}^d(X^\wedge, (\gamma, (-N, 0]))$, and

- i) a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ near $r = 0$,
- ii) meromorphic Volterra Mellin symbols $h_j \in M_{V, P_j}^{-\infty, d}(X; \mathbb{H}_{\frac{n+1}{2} - \gamma + j})$.

Remark 7.6. The following properties follow analogously to the corresponding assertions about smoothing Mellin and Green operators in the calculus of boundary value problems on manifolds with conical singularities, see Schrohe and Schulze [59, 60]. A detailed discussion of Volterra Mellin and Green operators in the boundaryless case can be found in [39]; similar arguments also apply in the present situation. One basic ingredient for the proofs are characterizations of the Mellin images of Mellin Sobolev spaces and their subspaces with asymptotics as meromorphic functions (Paley–Wiener theorems; see also Dorschfeldt [12], Schulze [61, 63]). Hence the application of the operators is reduced to multiplication with the meromorphic symbols, and we may also shift from one weight line to another using the residue theorem to express the error term.

- a) $C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$ is indeed a linear space; any change of the cut-off functions involved in the representations (7.2) and (7.3) just results in a change of a (Volterra) Green operator of type d , and the same applies to changes of the weights γ_j according to ii) of (7.2).
- b) We have a well-defined *conormal symbol mapping*

$$\begin{aligned} \sigma_M : C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0])) &\longrightarrow \Sigma_{M(V)}^d(X, (\gamma, (-N, 0])), \\ A &\longmapsto (h_0, \dots, h_{N-1}) \end{aligned}$$

with the meromorphic (Volterra) Mellin symbols h_k from iii) of (7.2), respectively ii) of (7.3). We write $\sigma_M^{-k}(A) := h_k$ for $k = 0, \dots, N-1$, and call $\sigma_M^{-k}(A)$ the *conormal symbol of order $-k$* of the operator A . For simplicity, we refer to $\sigma_M^0(A)$ just as the *conormal symbol* of A .

c) The composition induces a bilinear mapping

$$\begin{aligned} C_{M+G(\cdot, V)}^{d_1}(X^\wedge, (\gamma, (-N, 0])) \times C_{M+G(\cdot, V)}^{d_2}(X^\wedge, (\gamma, (-N, 0])) \\ \longrightarrow C_{M+G(\cdot, V)}^{d_2}(X^\wedge, (\gamma, (-N, 0])). \end{aligned}$$

The conormal symbols $\sigma_M(AB)$ of the composition of operators A and B are given by the Mellin translation product $\sigma_M(AB) = \sigma_M(A) \# \sigma_M(B)$ of the conormal symbols associated with A and B .

d) The (Volterra) Green operators share the properties of a two-sided ideal in the smoothing (Volterra) Mellin and Green operators. In fact, the space of (Volterra) Green operators of type $d \in \mathbb{N}_0$ coincides with the kernel of the conormal symbol mapping in the space $C_{M+G(\cdot, V)}^d(X^\wedge, (\gamma, (-N, 0]))$, which induces an isomorphism

$$C_{M+G(\cdot, V)}^d(X^\wedge, (\gamma, (-N, 0])) / C_{G(\cdot, V)}^d(X^\wedge, (\gamma, (-N, 0])) \cong \Sigma_{M(\cdot, V)}^d(X, (\gamma, (-N, 0])).$$

e) Taking the formal adjoint with respect to the $r^{-\frac{n}{2}}L^2$ -inner product induces a well-defined antilinear mapping

$$* : C_{M+G}^0(X^\wedge, (\gamma, (-N, 0])) \longrightarrow C_{M+G}^0(X^\wedge, (-\gamma, (-N, 0])).$$

The conormal symbols $\sigma_M(A^*)$ of the formal adjoint A^* of a smoothing Mellin and Green operator A of type zero are given as $\sigma_M(A^*) = \sigma_M(A)^*$ with the $*$ -operation from Definition 5.6.

Definition 7.7. a) Let $A \in C_{M+G}^d(X^\wedge, (\gamma, (-N, 0]))$. Then the operator $1 + A$ is called *elliptic* if there exists $s_0 \in \mathbb{R}$, $s_0 > d - \frac{1}{2}$, such that the operator family

$$\begin{aligned} 1 + \sigma_M^0(A)(z) : \begin{array}{ccc} H^{s_0}(X, E) & & H^{s_0}(X, E) \\ \oplus & \longrightarrow & \oplus \\ H^{s_0}(Y, J_-) & & H^{s_0}(Y, J_-) \end{array} \end{aligned} \quad (7.4)$$

is bijective for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

b) Let $A \in C_{M+G, V}^d(X^\wedge, (\gamma, (-N, 0]))$. The operator $1 + A$ is called *parabolic* if there exists $s_0 \in \mathbb{R}$, $s_0 > d - \frac{1}{2}$, such that the operator family (7.4) is bijective for all $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$.

Theorem 7.8. a) Let $A \in C_{M+G}^d(X^\wedge, (\gamma, (-N, 0]))$. Then the following are equivalent:

- i) $1 + A$ is elliptic in the sense of Definition 7.7.
- ii) There exists $B \in C_{M+G}^d(X^\wedge, (\gamma, (-N, 0]))$ such that $(1 + A)(1 + B) = 1 + G_1$ and $(1 + B)(1 + A) = 1 + G_2$ with Green operators $G_1, G_2 \in C_G^d(X^\wedge, (\gamma, (-N, 0]))$.

b) Let $A \in C_{M+G, V}^d(X^\wedge, (\gamma, (-N, 0]))$. Then the following are equivalent:

- i) $1 + A$ is parabolic in the sense of Definition 7.7.
- ii) There exists $B \in C_{M+G, V}^d(X^\wedge, (\gamma, (-N, 0]))$ such that $(1 + A)(1 + B) = 1$ and $(1 + B)(1 + A) = 1$, i.e., $1 + A$ is invertible with inverse $(1 + A)^{-1} = 1 + B$.

Proof. According to Definition 7.7 the operator $1 + A$ is elliptic (parabolic) if and only if the conormal symbol $1 + \sigma_M(A)$ fulfills the conditions of Proposition 5.12. Consequently, the ellipticity (parabolicity) of $1 + A$ is equivalent to the invertibility modulo (Volterra) Green operators which proves a). Using Theorem 7.4 we conclude that also assertion b) holds. \square

8. THE ALGEBRA OF VOLTERRA BOUNDARY CONE OPERATORS

The aim of the present section is to establish the Volterra cone calculus associated with boundary value problems. As announced before, when localized near the origin this calculus forms a proper subset of the operators considered in Section 4 (up to weight-shifts) — note that we formulate the calculus in the new time coordinate $r = e^{-t}$ — and with the more involved symbolic and operational structure we are able to derive results about conormal asymptotics (which corresponds to exponential long-time asymptotics in the original coordinates) as a regularity feature. The discussion of parabolicity and invertibility, however, is postponed to the subsequent section.

Notation 8.1. For functions $\varphi, \psi : \Omega \rightarrow \mathbb{C}$ defined on a topological space Ω we write $\varphi \prec \psi$ if $\psi \equiv 1$ in a neighbourhood of $\text{supp}(\varphi)$.

Definition 8.2. We define the space $C_{(V)cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$ of classical (Volterra) boundary cone operators of order $(\mu, \nu) \in \mathbb{Z} \times \mathbb{R}$ and type $d \in \mathbb{N}_0$ associated with the weight datum $(\gamma, (-N, 0])$, $N \in \mathbb{N}$, as follows:

An operator

$$A : \begin{array}{ccc} S^\gamma(X^\wedge, E) & & S^\gamma(X^\wedge, F) \\ \oplus & \longrightarrow & \oplus \\ S^{\gamma-\frac{1}{2}}(Y^\wedge, J_-) & & S^{\gamma-\frac{1}{2}}(Y^\wedge, J_+) \end{array}$$

belongs to $C_{(V)cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$ if and only if

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we have

$$\omega A \tilde{\omega} = \text{op}_M^{\gamma-\frac{\nu}{2}}(h) + A_{M+G} \quad (8.1)$$

with some $h \in C^\infty(\overline{\mathbb{R}}_+, M_{(V)Ocl}^{\mu, d; \ell}(X))$, and a smoothing (Volterra) Mellin and Green operator $A_{M+G} \in C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$ of type d ,

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we may write

$$(1 - \omega)A(1 - \tilde{\omega}) = \text{op}_r(a) \quad (8.2)$$

with some $a \in S_{cl}^\nu(\mathbb{R}, \mathcal{B}_{cl}^{\mu, d; \ell}(X; \mathbb{R}))$ (resp. $a \in S_{cl}^\nu(\mathbb{R}, \mathcal{B}_{Vcl}^{\mu, d; \ell}(X; \mathbb{H}))$),

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ such that $\omega \prec \tilde{\omega}$ we have

$$\omega A(1 - \tilde{\omega}), (1 - \tilde{\omega})A\omega \in C_{G(V)}^d(X^\wedge, (\gamma, (-N, 0])). \quad (8.3)$$

Remark 8.3. According to Definition 8.2 a (Volterra) boundary cone operator is given near the origin $r = 0$ by a Mellin pseudodifferential operator up to a smoothing (Volterra) Mellin and Green operator. Recall that

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)u(r) = \mathcal{M}_{\gamma-\frac{n}{2}}^{-1}h(r, z)\mathcal{M}_{\gamma-\frac{n}{2}}u = \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \int_{\mathbb{R}_+} \left(\frac{r}{r'}\right)^{-z} h(r, z)u(r') \frac{dr'}{r'} dz$$

for smooth functions u with compact support. The symbol $h(r, z)$ is thereby a function depending smoothly on $r \in \mathbb{R}_+$ taking values in operator-valued symbols with respect to the variable $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. More precisely, these operator-valued symbols are themselves parameter-dependent boundary operators in Boutet de Monvel's calculus on X , and the parameter in fact runs over the whole complex plane with a specific control in terms of (locally) uniform estimates.

From the identity (8.1) we see that we are free to specify a certain behaviour of $h(r, z)$ with respect to $r \rightarrow \infty$. It is natural in the symbolic calculus of Mellin operators to impose global C_B^∞ -estimates, i.e., we additionally assume $(-r\partial_r)^k h(r, z)$ to be bounded on \mathbb{R}_+ for all $k \in \mathbb{N}_0$. Using such global estimates we automatically obtain explicit oscillatory integral formulas, e.g., for the Leibniz-product (symbol of composition) and the formal adjoint symbol of Mellin operators (Kumano-go's technique applied to Mellin operators), and from these explicit expressions it is easy to see that the operators we have in mind — such that are based upon Volterra symbols and holomorphic Mellin symbols of boundary value problems — are indeed well-behaved with respect to all manipulations.

For a discussion of general Mellin pseudodifferential techniques in the study of operators on manifolds with conical singularities we refer to the monographs of Schulze, see, e.g., [63]; Mellin operators of boundary value problems are considered in Schrohe and Schulze [59, 60]; Volterra Mellin operators (also with holomorphic Mellin symbols) in the general framework of operators with operator-valued symbols are studied, e.g., in [37], and in the more concrete setting of parabolicity in the boundaryless case in [39].

Analogously, a (Volterra) boundary cone operator is given near $r = \infty$ as an ordinary Kohn–Nirenberg quantized pseudodifferential operator with (Volterra) symbol $a(r, \zeta)$, i.e.,

$$\mathrm{op}_r(a)u(r) = \mathcal{F}^{-1}a(r, \zeta)\mathcal{F}u = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(r-r')\tau} a(r, \tau)u(r') dr' d\tau,$$

where we assume an additional behaviour as a classical symbol of order $\nu \in \mathbb{R}$ with respect to the variable $r \in \mathbb{R}$ as $r \rightarrow \infty$.

Observe that we have reversed orientation in the time coordinate. Therefore, the half-plane of Volterra symbols changes from the lower (as considered in Section 4) to the upper half-plane for standard pseudodifferential operators, and a right half-plane is involved for Mellin symbols in the Mellin pseudodifferential calculus.

Recall that we are mainly interested in the structure of the calculus near $r = 0$, which corresponds to time $t \rightarrow \infty$ in the original space–time coordinates.

Therefore, we could equally well study only Mellin operators restricted to some subinterval $(0, r_0]$ of \mathbb{R}_+ . Actually, this is precisely what we obtain when we restrict a boundary cone operator to such an interval. However, we want to emphasize that the operators away from $r = 0$ are just ordinary (Volterra) operators, which is an important structural information. Moreover, our constructions as $r \rightarrow \infty$ may also be regarded as a generalization of the operator calculus from Section 4 to arbitrary order $\nu \in \mathbb{R}$.

Theorem 8.4. *Let $A \in C_{cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$. Then A extends by continuity to an operator*

$$A : \begin{array}{ccc} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta & & \mathcal{K}^{s-\mu, \gamma; \ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus & \longrightarrow & \oplus \\ \mathcal{K}^{s, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta & & \mathcal{K}^{s-\mu, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

for all $s > d - \frac{1}{2}$ and all $\delta \in \mathbb{R}$. Moreover, for every pair (P_1, P_2) of asymptotic types there exists a pair (Q_1, Q_2) of asymptotic types such that A restricts to continuous operators

$$A : \begin{array}{ccc} \mathcal{K}_{P_1}^{s, \gamma; \ell}(X^\wedge, E)_\delta & & \mathcal{K}_{Q_1}^{s-\mu, \gamma; \ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus & \longrightarrow & \oplus \\ \mathcal{K}_{P_2}^{s, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta & & \mathcal{K}_{Q_2}^{s-\mu, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

for all $s > d - \frac{1}{2}$ and all $\delta \in \mathbb{R}$.

Proof. The first boundedness assertion is clear. Carrying out a Taylor expansion in $r = 0$ of the holomorphic Mellin symbol $h(r, z)$ from (8.1) reveals that we may write $A = \sum_{j=0}^{N-1} \omega r^j \text{op}_M^{\gamma_j}(h_j) \omega + G$ similar to (7.2) and (7.3), where $\sigma_M(A) = (h_0, \dots, h_{N-1})$ is a conormal symbol, and G is continuous in the spaces

$$G : \begin{array}{ccc} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta & & \mathcal{K}_{R_1}^{s-\mu, \gamma; \ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus & \longrightarrow & \oplus \\ \mathcal{K}^{s, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta & & \mathcal{K}_{R_2}^{s-\mu, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

for $s > d - \frac{1}{2}$ with some asymptotic types (R_1, R_2) . Hence the proof is reduced to consider the Mellin operators, but for these the desired assertion follows from the Paley–Wiener characterizations of the Mellin images of the weighted Mellin Sobolev spaces and their subspaces with asymptotics as meromorphic functions (see Dorschfeldt [12], Schulze [63]), which reduce the action of the operators to multiplication with the meromorphic symbols. \square

As a consequence of the Paley–Wiener characterizations of the Fourier and Mellin images of functions supported by a half-line as analytic functions in a half-plane, we conclude that the following mapping property holds for Volterra boundary cone operators A :

For every $r_0 \in \mathbb{R}_+$ we have $(Au)(r) \equiv 0$ for $r > r_0$ for all $u = (u_1, u_2) \in C_0^\infty(\mathbb{R}_+, C^\infty(\overline{X}, E)) \oplus C_0^\infty(\mathbb{R}_+, C^\infty(Y, J_-))$ such that $u(r) \equiv 0$ for $r > r_0$.

Using this property and a density argument, we obtain from Theorem 8.4 at once the boundedness of Volterra boundary cone operators in the subspaces consisting of those elements (distributions) that are supported by $(0, r_0]$ for each $r_0 \in \mathbb{R}_+$ (see also Theorem 4.4).

Theorem 8.5. *An operator A belongs to $C_{(V)cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$ if and only if for some (all) cut-off functions $\omega_3 \prec \omega_1 \prec \omega_2$ we may write*

$$A = \omega_1 \operatorname{op}_M^{\gamma - \frac{n}{2}}(h) \omega_2 + (1 - \omega_1) \operatorname{op}_r(a) (1 - \omega_3) + A_{M+G}, \quad (8.4)$$

where $A_{M+G} \in C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$, $h \in C^\infty(\overline{\mathbb{R}}_+, M_{(V)Ocl}^{\mu, d; \ell}(X))$, $a \in S_{cl}^\nu(\mathbb{R}, \mathcal{B}_{cl}^{\mu, d; \ell}(X; \mathbb{R}))$ (respectively $a \in S_{cl}^\nu(\mathbb{R}, \mathcal{B}_V^{\mu, d; \ell}(X; \mathbb{H}))$).

Theorem 8.5 says something about the richness of the (Volterra) cone calculus of boundary value problems. The proof is quite technical, and most of the calculations are rather straightforward.

The most difficult aspect of the proof is that we have to be able to rewrite Mellin operators as Kohn–Nirenberg quantized pseudodifferential operators and vice versa away from $r = 0$ and $r = \infty$. This is in fact subject to *Mellin quantization*, and there exist many proofs of this result in the literature about pseudodifferential calculus on manifolds with singularities (see, e.g., Schulze [63], Gil, Seiler, and Schulze [19, 20], Schrohe and Schulze [59, 60] in the framework of boundary value problems on manifolds with conical singularities, also [37], [39] for a proof in the framework of Volterra operators). Therefore, we restrict ourselves to give a precise statement below in that form as it is needed in this work:

Let $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$ with $\varphi \prec \psi$.

- i) For $h \in C^\infty(\mathbb{R}_+, M_{(V)Ocl}^{\mu, d; \ell}(X))$ there exists $a \in C^\infty(\mathbb{R}, \mathcal{B}_{cl}^{\mu, d; \ell}(X; \mathbb{R}))$ (respectively $a \in C^\infty(\mathbb{R}, \mathcal{B}_V^{\mu, d; \ell}(X; \mathbb{H}))$) such that $\varphi \operatorname{op}_M^{\gamma - \frac{n}{2}}(h) \psi \equiv \varphi \operatorname{op}_r(a) \psi$ modulo $C_{G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$.
- ii) Conversely, given $a \in C^\infty(\mathbb{R}, \mathcal{B}_{cl}^{\mu, d; \ell}(X; \mathbb{R}))$ or $a \in C^\infty(\mathbb{R}, \mathcal{B}_V^{\mu, d; \ell}(X; \mathbb{H}))$, there exists $h \in C^\infty(\mathbb{R}_+, M_{(V)Ocl}^{\mu, d; \ell}(X))$ such that $\varphi \operatorname{op}_r(a) \psi \equiv \varphi \operatorname{op}_M^{\gamma - \frac{n}{2}}(h) \psi$ modulo $C_{G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$.

The principal symbols associated with a and h in i) and ii) are related as follows:

$$\begin{aligned} \varphi(r) \sigma_\psi^{\mu; \ell}(h)(r, \xi_x, \frac{n+1}{2} - \gamma - i(r\lambda)) &= \varphi(r) \sigma_\psi^{\mu; \ell}(a)(r, \xi_x, \lambda), \\ \varphi(r) \sigma_\theta^{\mu; \ell}(h)(r, \xi_{x'}, \frac{n+1}{2} - \gamma - i(r\lambda)) &= \varphi(r) \sigma_\theta^{\mu; \ell}(a)(r, \xi_{x'}, \lambda) \end{aligned}$$

for $r \in \mathbb{R}_+$ and $(\xi_x, \lambda) \in (T^*\overline{X} \times \Lambda) \setminus 0$, respectively $(\xi_{x'}, \lambda) \in (T^*Y \times \Lambda) \setminus 0$, where either $\Lambda = \mathbb{R}$ or $\Lambda = \mathbb{H}$.

8.1. The symbolic structure. Let $A \in C_{(V)cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$, and $0 < T_2 < T_1 < \infty$. From Definition 8.2 we see that there exist

$$h \in C_B^\infty(\overline{\mathbb{R}}_+, M_{(V)Ocl}^{\mu, d; \ell}(X)), \quad a \in \begin{cases} S_{cl}^\nu(\mathbb{R}, \mathcal{B}_{cl}^{\mu, d; \ell}(X; \mathbb{R})) \\ S_{cl}^\nu(\mathbb{R}, \mathcal{B}_V^{\mu, d; \ell}(X; \mathbb{H})) \end{cases}$$

such that for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}_+})$ near $r = 0$ with $\chi_{[0, T_2]} \prec \omega, \tilde{\omega} \prec \chi_{[0, T_1]}$ we have

$$\begin{aligned} \omega A \tilde{\omega} - \omega \text{op}_M^{\gamma - \frac{n}{2}}(h) \tilde{\omega} &\in C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0])), \\ (1 - \omega)A(1 - \tilde{\omega}) - (1 - \omega)\text{op}_r(a)(1 - \tilde{\omega}) &\in C_{G(V)}^d(X^\wedge, (\gamma, (-N, 0])). \end{aligned} \quad (8.5)$$

In particular, if $\chi_{[0, T_2]} \prec \omega_3 \prec \omega_1 \prec \omega_2 \prec \chi_{[0, T_1]}$ are cut-off functions, we have $A = \omega_1 \text{op}_M^{\gamma - \frac{n}{2}}(h) \omega_2 + (1 - \omega_1) \text{op}_r(a)(1 - \omega_3) + A_{M+G}$ with $A_{M+G} \in C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$.

Any tuple (h, a) that satisfies (8.5) is called a *complete interior symbol* of the operator A , subordinated to the covering $\{[0, T_1], (T_2, \infty)\}$ of \mathbb{R}_+ . Note that the relation $A \mapsto (h, a)$ is non-canonical.

Definition 8.6. With $A \in C_{(V)cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$ we associate the following principal symbols — note that below Λ is either given as $\Lambda = \mathbb{R}$ in case of a (general) boundary cone operator A , or $\Lambda = \mathbb{H}$ in case of a Volterra boundary cone operator A :

i) The *principal pseudodifferential symbol* is well-defined as a smooth section

$$\sigma_\psi^{\mu; \ell}(A) : \mathbb{R}_+ \times ((T^*\overline{X} \times \Lambda) \setminus 0) \longrightarrow \text{Hom}(\pi^*E, \pi^*F) \quad (8.6)$$

that is homogeneous in the sense $\sigma_\psi^{\mu; \ell}(A)(r, \varrho \xi_x, \varrho^\ell \lambda) = \varrho^\mu \sigma_\psi^{\mu; \ell}(A)(r, \xi_x, \lambda)$ for $\varrho > 0$. For Volterra boundary cone operators the principal pseudodifferential symbol is analytic in the interior of the half-plane \mathbb{H} .

ii) The *principal boundary symbol* is given as a smooth section

$$\sigma_\partial^{\mu; \ell}(A) : \mathbb{R}_+ \times ((T^*Y \times \Lambda) \setminus 0) \longrightarrow \text{Hom} \left(\begin{array}{c} H^s(\mathbb{R}_+) \otimes \pi^*E|_Y \\ \oplus \\ \pi^*J_- \end{array}, \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \otimes \pi^*F|_Y \\ \oplus \\ \pi^*J_+ \end{array} \right) \quad (8.7)$$

for $s > d - \frac{1}{2}$, and it is homogeneous in the sense

$$\sigma_\partial^{\mu; \ell}(A)(r, \varrho \xi_{x'}, \varrho^\ell \lambda) = \varrho^\mu \begin{pmatrix} \kappa_\varrho \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial^{\mu; \ell}(A)(r, \xi_{x'}, \lambda) \begin{pmatrix} \kappa_\varrho^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $\varrho > 0$ with the group-action $\{\kappa_\varrho\}$ from (2.1). For Volterra boundary cone operators the principal boundary symbol is analytic in the interior of \mathbb{H} .

More precisely, let $A = \omega_1 \text{op}_M^{\gamma - \frac{n}{2}}(h) \omega_2 + (1 - \omega_1) \text{op}_r(a)(1 - \omega_3) + A_{M+G}$ be any representation of A according to (8.4). Then

$$\begin{aligned} \sigma_\psi^{\mu; \ell}(A)(r, \xi_x, \lambda) &= \omega_1(r) \sigma_\psi^{\mu; \ell}(h)(r, \xi_x, \frac{n+1}{2} - \gamma - i(r\lambda)) + (1 - \omega_1(r)) \sigma_\psi^{\mu; \ell}(a)(r, \xi_x, \lambda), \\ \sigma_\partial^{\mu; \ell}(A)(r, \xi_{x'}, \lambda) &= \omega_1(r) \sigma_\partial^{\mu; \ell}(h)(r, \xi_{x'}, \frac{n+1}{2} - \gamma - i(r\lambda)) + (1 - \omega_1(r)) \sigma_\partial^{\mu; \ell}(a)(r, \xi_{x'}, \lambda), \end{aligned} \quad (8.8)$$

for $r \in \mathbb{R}_+$ and $(\xi_x, \lambda) \in (T^*\overline{X} \times \Lambda) \setminus 0$, respectively $(\xi_{x'}, \lambda) \in (T^*Y \times \Lambda) \setminus 0$, with the associated principal symbols to h and a . Consequently, both $\sigma_\psi^{\mu; \ell}(A)(r, \xi_x, r^{-1}\lambda)$

and $\sigma_{\partial}^{\mu;\ell}(A)(r, \xi'_{x'}, r^{-1}\lambda)$ extend as smooth (anisotropic) homogeneous sections up to the origin $r = 0$, and $\sigma_{\psi}^{\mu;\ell}(A)$ and $\sigma_{\partial}^{\mu;\ell}(A)$ behave like classical symbols of order $\nu \in \mathbb{R}$ as $r \rightarrow \infty$.

- iii) Let A be written according to (8.4). We define the *conormal symbol of order $-k$* of A as $\sigma_M^{-k}(A)(z) = \frac{1}{k!}(\partial_r^k h)(0, z) + \sigma_M^{-k}(A_{M+G})(z)$ for $k = 0, \dots, N-1$. The conormal symbol of order 0 is also called the conormal symbol simply, and it is regarded as a family of operators

$$\sigma_M^0(A)(z) : \begin{array}{ccc} H^s(X, E) & & H^{s-\mu}(X, F) \\ \oplus & \longrightarrow & \oplus \\ H^s(Y, J_-) & & H^{s-\mu}(Y, J_+) \end{array} \quad (8.9)$$

for $s > d - \frac{1}{2}$, depending on $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ in case of general boundary cone operators, and $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$ in case of Volterra boundary cone operators.

The tuple $\sigma_M(A) = (\sigma_M^0(A), \dots, \sigma_M^{-(N-1)}(A))$ defines an element in the space $\Sigma_{M(V)}^{\mu, d; \ell}(X, (\gamma, (-N, 0]))$, see Definition 5.6.

- iv) The *principal pseudodifferential-exit symbol*

$$\sigma_{\psi, e}^{\mu, \nu; \ell}(A) : \mathbb{R}_+ \times ((T^*\overline{X} \times \Lambda) \setminus 0) \longrightarrow \text{Hom}(\pi^*E, \pi^*F) \quad (8.10)$$

is by definition the principal part with respect to $r \rightarrow \infty$ of the principal pseudodifferential symbol associated with A , i.e., we have $\sigma_{\psi, e}^{\mu, \nu; \ell}(A) = \sigma_{\psi, e}^{\mu, \nu; \ell}(a)$ with the symbol $a(r, \lambda)$ in (8.8) (restricted to \mathbb{R}_+). In particular, it is homogeneous in the sense $\sigma_{\psi, e}^{\mu, \nu; \ell}(A)(\varrho_1 r, \varrho_2 \xi_x, \varrho_2^\ell \lambda) = \varrho_1^\nu \varrho_2^\mu \sigma_{\psi, e}^{\mu, \nu; \ell}(A)(r, \xi_x, \lambda)$ for $\varrho_1, \varrho_2 > 0$, and for Volterra boundary cone operators the principal pseudodifferential-exit symbol is analytic in the interior of \mathbb{H} .

- v) The *principal boundary-exit symbol*

$$\sigma_{\partial, e}^{\mu, \nu; \ell}(A) : \mathbb{R}_+ \times ((T^*Y \times \mathbb{H}_-) \setminus 0) \rightarrow \text{Hom} \left(\begin{array}{ccc} H^s(\mathbb{R}_+) \otimes \pi^*E|_Y & & H^{s-\mu}(\mathbb{R}_+) \otimes \pi^*F|_Y \\ \oplus & & \oplus \\ \pi^*J_- & & \pi^*J_+ \end{array} \right) \quad (8.11)$$

for $s > d - \frac{1}{2}$ is defined as the principal part with respect to $r \rightarrow \infty$ of the principal boundary symbol, i.e., we may write $\sigma_{\partial, e}^{\mu, \nu; \ell}(A) = \sigma_{\partial, e}^{\mu, \nu; \ell}(a)$ with the symbol $a(r, \lambda)$ in (8.8) (restricted to \mathbb{R}_+). The principal boundary-exit symbol is homogeneous in the sense

$$\sigma_{\partial, e}^{\mu, \nu; \ell}(A)(\varrho_1 r, \varrho_2 \xi'_{x'}, \varrho_2^\ell \lambda) = \varrho_1^\nu \varrho_2^\mu \left(\begin{array}{cc} \kappa_{\varrho_2} \otimes 1 & 0 \\ 0 & 1 \end{array} \right) \sigma_{\partial, e}^{\mu, \nu; \ell}(A)(r, \xi'_{x'}, \lambda) \left(\begin{array}{cc} \kappa_{\varrho_2}^{-1} \otimes 1 & 0 \\ 0 & 1 \end{array} \right)$$

for $\varrho_1, \varrho_2 > 0$ with the group-action $\{\kappa_\varrho\}$ from (2.1), and for Volterra boundary cone operators it is analytic in the interior of \mathbb{H} .

- vi) The *principal exit symbol* is by definition the homogeneous principal component of the symbol $a(r, \lambda)$ from the representation (8.4) of A with respect to

$r \rightarrow \infty$. It is well-defined as a family of operators depending on $r \in \mathbb{R}_+$ and $\lambda \in \Lambda$ in the spaces

$$\sigma_e^\nu(A)(r, \lambda) : \begin{array}{ccc} H^s(X, E) & \longrightarrow & H^{s-\mu}(X, F) \\ \oplus & \longrightarrow & \oplus \\ H^s(Y, J_-) & \longrightarrow & H^{s-\mu}(Y, J_+) \end{array} \quad (8.12)$$

for $s > d - \frac{1}{2}$ that is homogeneous in the sense $\sigma_e^\nu(A)(\varrho r, \lambda) = \varrho^\nu \sigma_e^\nu(A)(r, \lambda)$ for $\varrho > 0$. For Volterra boundary cone operators the principal exit symbol is analytic in the interior of \mathbb{H} .

8.2. Compositions and adjoints. Let $A \in C_{(V)cl}^{\mu, \nu, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$. Carrying out a Taylor expansion in $r = 0$ of the holomorphic Mellin symbol $h(r, z)$ from (8.1) yields a representation $A = \sum_{j=0}^{N-1} \omega r^j \text{op}_M^{\gamma_j}(h_j) \omega + G$ similar to (7.2) and (7.3), where $\sigma_M(A) = (h_0, \dots, h_{N-1})$ is the tuple of conormal symbols associated with A , and the remainder G is an operator of order (μ, ν) and type d that generates asymptotics; in particular,

$$G : \begin{array}{ccc} \mathcal{K}^{s, \gamma; \ell}(X^\wedge, E)_\delta & \longrightarrow & \mathcal{K}_{R_1}^{s-\mu, \gamma; \ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus & \longrightarrow & \oplus \\ \mathcal{K}^{s, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_-)_\delta & \longrightarrow & \mathcal{K}_{R_2}^{s-\mu, \gamma-\frac{1}{2}; \ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

for $s > d - \frac{1}{2}$ with some asymptotic types (R_1, R_2) .

Using such representations it is easy to see that the composition AB of (Volterra) boundary cone operators $A \in C_{(V)cl}^{\mu_1, \nu_1, d_1; \ell}(X^\wedge, (\gamma, (-N, 0]))$ and $B \in C_{(V)cl}^{\mu_2, \nu_2, d_2; \ell}(X^\wedge, (\gamma, (-N, 0]))$ can be written as $AB = \sum_{j=0}^{N-1} \omega r^j \text{op}_M^{\tilde{\gamma}_j}(g_j) \omega + G'$, where the tuple (g_0, \dots, g_{N-1}) is given as the Mellin translation product $\sigma_M(A) \# \sigma_M(B)$ of the conormal symbols associated with A and B , and G' is an operator of order $(\mu_1 + \mu_2, \nu_1 + \nu_2)$ and type $d = \max\{\mu_2 + d_1, d_2\}$ that generates asymptotics. In particular, we obtain the following:

Proposition 8.7. *Provided the vector bundles fit together, the composition of operators is well-defined in the spaces*

$$\begin{aligned} & C_{(V)cl}^{\mu, \nu, d_1; \ell}(X^\wedge, (\gamma, (-N, 0])) \times C_{(M+)G(V)}^{d_2}(X^\wedge, (\gamma, (-N, 0])) \\ & \longrightarrow C_{(M+)G(V)}^{d_2}(X^\wedge, (\gamma, (-N, 0])), \\ & C_{(M+)G(V)}^{d_1}(X^\wedge, (\gamma, (-N, 0])) \times C_{(V)cl}^{\mu, \nu, d_2; \ell}(X^\wedge, (\gamma, (-N, 0])) \\ & \longrightarrow C_{(M+)G(V)}^{d_2}(X^\wedge, (\gamma, (-N, 0])). \end{aligned}$$

In other words, the smoothing (Volterra) Mellin and Green operators as well as the (Volterra) Green operators form two-sided ideals in the algebra of (Volterra) boundary cone operators (see Theorem 8.8 below).

Theorem 8.8. *Consider (Volterra) boundary cone operators*

$$\begin{aligned} A &\in C_{(V)cl}^{\mu_1, \nu_1, d_1; \ell}(X^\wedge, (\gamma, (-N, 0])), \\ B &\in C_{(V)cl}^{\mu_2, \nu_2, d_2; \ell}(X^\wedge, (\gamma, (-N, 0])), \end{aligned}$$

and assume that the vector bundles fit together in order to be able to form the composition.

Then $AB \in C_{(V)cl}^{\mu_1+\mu_2, \nu_1+\nu_2, d; \ell}(X^\wedge, (\gamma, (-N, 0]))$, where $d = \max\{\mu_2 + d_1, d_2\}$, and the symbols associated with the composition are given in terms of the symbols associated with A and B as follows:

- Let $0 < T_2 < T_1 < \infty$, and (h_1, a_1) and (h_2, a_2) be complete interior symbols of A and B subordinated to the covering $\{[0, T_1), (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$. Then the tuple of Leibniz-products $(h_1 \# h_2, a_1 \# a_2)$ is a complete interior symbol of the composition AB ; recall that

$$h_1 \# h_2(r, z) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} s^{in} h_1(r, z + i\eta) h_2(rs, z) \frac{ds}{s} d\eta \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_z^k h_1) ((-r\partial_r)^k h_2),$$

$$a_1 \# a_2(r, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} e^{-is\eta} a_1(r, \lambda + \eta) a_2(r + s, \lambda) ds d\eta \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_\lambda^k a_1) (D_r^k a_2).$$

- The principal pseudodifferential and boundary symbols are given as

$$\begin{aligned} \sigma_\psi^{\mu_1+\mu_2; \ell}(AB) &= \sigma_\psi^{\mu_1; \ell}(A) \sigma_\psi^{\mu_2; \ell}(B), \\ \sigma_\partial^{\mu_1+\mu_2; \ell}(AB) &= \sigma_\partial^{\mu_1; \ell}(A) \sigma_\partial^{\mu_2; \ell}(B). \end{aligned}$$

- We have the identity $\sigma_M(AB) = \sigma_M(A) \# \sigma_M(B)$ for the conormal symbols, where $\#$ denotes the Mellin translation product (see Definition 5.6).
- The principal symbolic structure as $r \rightarrow \infty$ is determined as follows:

$$\begin{aligned} \sigma_{\psi, e}^{\mu_1+\mu_2, \nu_1+\nu_2; \ell}(AB) &= \sigma_{\psi, e}^{\mu_1, \nu_1; \ell}(A) \sigma_{\psi, e}^{\mu_2, \nu_2; \ell}(B), \\ \sigma_{\partial, e}^{\mu_1+\mu_2, \nu_1+\nu_2; \ell}(AB) &= \sigma_{\partial, e}^{\mu_1, \nu_1; \ell}(A) \sigma_{\partial, e}^{\mu_2, \nu_2; \ell}(B), \\ \sigma_e^{\nu_1+\nu_2}(AB) &= \sigma_e^{\nu_1}(A) \sigma_e^{\nu_2}(B). \end{aligned}$$

Proof. Choose cut-off functions $\chi_{[0, T_2]} \prec \omega, \tilde{\omega} \prec \hat{\omega} \prec \tilde{\tilde{\omega}} \prec \chi_{[0, T_1]}$. We may write

$$\begin{aligned} \omega AB \tilde{\omega} &= (\omega A \hat{\omega}) (\tilde{\omega} B \tilde{\omega}) + \omega A (1 - \hat{\omega}) B \tilde{\omega} \equiv \omega \text{op}_M^{\gamma - \frac{n}{2}}(h_1) \hat{\omega} \text{op}_M^{\gamma - \frac{n}{2}}(h_2) \tilde{\omega} \\ &= \omega \text{op}_M^{\gamma - \frac{n}{2}}(h_1 \# h_2) \tilde{\omega} - \omega \text{op}_M^{\gamma - \frac{n}{2}}(h_1) (1 - \hat{\omega}) \text{op}_M^{\gamma - \frac{n}{2}}(h_2) \tilde{\omega} \equiv \omega \text{op}_M^{\gamma - \frac{n}{2}}(h_1 \# h_2) \tilde{\omega}, \end{aligned}$$

where \equiv means equivalence modulo $C_{M+G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$. Note that we have used, in particular, the considerations from Section 7, Proposition 8.7, and general properties of the symbolic and operational calculus of Mellin pseudodifferential operators (with holomorphic (Volterra) symbols).

Next let $\chi_{[0,T_2]} \prec \hat{\omega} \prec \tilde{\omega} \prec \omega, \tilde{\omega} \prec \chi_{[0,T_1]}$, and write

$$\begin{aligned} (1-\omega)AB(1-\tilde{\omega}) &= ((1-\omega)A(1-\hat{\omega}))((1-\tilde{\omega})B(1-\tilde{\omega})) + (1-\omega)A\tilde{\omega}B(1-\tilde{\omega}) \\ &\equiv (1-\omega)\text{op}_r(a_1)(1-\hat{\omega})(1-\tilde{\omega})\text{op}_r(a_2)(1-\tilde{\omega}) = (1-\omega)\text{op}_r(a_1\#a_2)(1-\tilde{\omega}) \\ &\quad - (1-\omega)\text{op}_r(a_1)\tilde{\omega}\text{op}_r(a_2)(1-\tilde{\omega}) \equiv (1-\omega)\text{op}_r(a_1\#a_2)(1-\tilde{\omega}), \end{aligned}$$

where in this case \equiv means equivalence modulo $C_{G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$. Also for this calculation we have used, in particular, the references given above, and general properties of (Volterra) pseudodifferential calculus (see also Section 4).

It remains to consider the operators $\omega AB(1-\tilde{\omega})$ and $(1-\tilde{\omega})AB\omega$ with cut-off functions $\omega \prec \tilde{\omega}$. Let $\hat{\omega} \in C_0^\infty(\mathbb{R}_+)$ such that $\omega \prec \hat{\omega} \prec \tilde{\omega}$, and write

$$\begin{aligned} \omega AB(1-\tilde{\omega}) &= (\omega A(1-\hat{\omega}))B(1-\tilde{\omega}) + \omega A(\hat{\omega}B(1-\tilde{\omega})), \\ (1-\tilde{\omega})AB\omega &= ((1-\tilde{\omega})A\hat{\omega})B\omega + (1-\tilde{\omega})A((1-\hat{\omega})B\omega). \end{aligned}$$

From Definition 8.2 and Proposition 8.7 we conclude $\omega AB(1-\tilde{\omega}), (1-\tilde{\omega})AB\omega \in C_{G(V)}^d(X^\wedge, (\gamma, (-N, 0]))$.

Summing up, we have shown that the composition AB is indeed a (Volterra) boundary cone operator of order $(\mu_1 + \mu_2, \nu_1 + \nu_2)$ and type $d = \max\{\mu_2 + d_1, d_2\}$ as desired, and the formula for the complete interior symbol holds. This formula implies the corresponding formulae for the principal pseudodifferential symbol, the principal boundary symbol, and the principal symbols as $r \rightarrow \infty$. The formula for the conormal symbol follows from the arguments at the beginning of this section. \square

Theorem 8.8 represents the main algebraic result about the calculus. Below we state a result about the formal adjoint operator associated with a boundary value problem which is given as a boundary cone operator of order less or equal to zero and type zero. However, as we are mainly interested in the class of Volterra boundary cone operators and the study of parabolicity, and as this class is not preserved under taking the formal adjoint, we restrict ourselves to give the statement and skip the proof.

Theorem 8.9. *Let $A \in C_{cl}^{\mu, \nu, 0; \ell}(X^\wedge, (\gamma, (-N, 0]))$, and $\mu \leq 0$. Then the formal adjoint operator A^* with respect to the $r^{-\frac{n}{2}}L^2$ -inner product is a boundary cone operator in the space $C_{cl}^{\mu, \nu, 0; \ell}(X^\wedge, (-\gamma, (-N, 0]))$, and the symbols of A^* are given as follows:*

- *Let $0 < T_2 < T_1 < \infty$, and let (h, a) be a complete interior symbol of A subordinated to the covering $\{[0, T_1), (T_2, \infty)\}$ of \mathbb{R}_+ . Then $(h^{(*)}, a^{(*)})$ is a*

complete interior symbol of A^* , where

$$\begin{aligned} h^{(*)}(r, z) &= \int \int_{\mathbb{R}\mathbb{R}_+} s^{i\eta} h(sr, n+1-\bar{z}+i\eta)^* \frac{ds}{s} d\eta \sim \sum_{k=0}^{\infty} \frac{1}{k!} \partial_z^k (-r\partial_r)^k h(r, n+1-\bar{z})^*, \\ a^{(*)}(r, \lambda) &= \iint e^{-is\eta} [r]^{-n} a(r+s, \lambda+\eta)^* [r+s]^n ds d\eta \\ &\sim \sum_{k=0}^{\infty} \sum_{p+q=k} \frac{1}{p!q!} ([r]^{-n} D_r^p [r]^n) (\partial_\lambda^k D_r^q a^*). \end{aligned}$$

Here $[\cdot]$ denotes a suitable smoothing of $|\cdot|$.

- The homogeneous principal symbols are given as

$$\begin{aligned} \sigma_\psi^{\mu;\ell}(A^*) &= \sigma_\psi^{\mu;\ell}(A)^*, & \sigma_{\psi,e}^{\mu,\nu;\ell}(A^*) &= \sigma_{\psi,e}^{\mu,\nu;\ell}(A)^*, \\ \sigma_\partial^{\mu;\ell}(A^*) &= \sigma_\partial^{\mu;\ell}(A)^*, & \sigma_{\partial,e}^{\mu,\nu;\ell}(A^*) &= \sigma_{\partial,e}^{\mu,\nu;\ell}(A)^*, \\ \sigma_e^\nu(A^*) &= \sigma_e^\nu(A)^*. \end{aligned}$$

- We have the identity $\sigma_M(A^*) = \sigma_M(A)^*$ for the conormal symbols with the $*$ -operation from Definition 5.6.

9. PARABOLICITY AND INVERTIBILITY IN THE VOLTERRA CONE CALCULUS

This section is devoted to study parabolicity and invertibility for Volterra boundary cone operators. The result about the equivalence of parabolicity and the existence of the inverse within the calculus was announced at the end of Section 4, and sharpens Theorem 4.8. As a Volterra boundary cone operator, the inverse of a parabolic boundary value problem encodes sufficiently much structural information such that also conormal asymptotics (exponential long-time asymptotics) of solutions can be observed and controlled.

In addition, we obtain as a by-product a Fredholm theory for general anisotropic elliptic boundary value problems, and construct a parametrix within the calculus of boundary cone operators.

Definition 9.1. Let $A \in C_{(V)cl}^{\mu,\nu,d;\ell}(X^\wedge, (\gamma, (-N, 0]))$, where $d \leq \mu_+$. A is *elliptic* (resp. *parabolic*), if and only if the following conditions are fulfilled — in i), ii), iv), v), and vi) below we denote either $\Lambda = \mathbb{R}$ (ellipticity) or $\Lambda = \mathbb{H}$ (parabolicity):

- i) The principal pseudodifferential symbol $\sigma_\psi^{\mu;\ell}(A)(r, \xi_x, r^{-1}\lambda)$ is invertible in $\text{Hom}(\pi^*E, \pi^*F)$ for all $(\xi_x, \lambda) \in (T^*\overline{X} \times \Lambda) \setminus 0$ and all $r \in \overline{\mathbb{R}}_+$.
- ii) The principal boundary symbol $\sigma_\psi^{\mu;\ell}(A)(r, \xi'_{x'}, r^{-1}\lambda)$ is invertible in (8.7) for all $(\xi'_{x'}, \lambda) \in (T^*Y \times \Lambda) \setminus 0$ and all $r \in \overline{\mathbb{R}}_+$.
- iii) The conormal symbol $\sigma_M^0(A)(z)$ is invertible in (8.9) for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ (ellipticity), or $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$ (parabolicity), respectively.
- iv) The principal pseudodifferential–exit symbol $\sigma_{\psi,e}^{\mu,\nu;\ell}(A)(r, \xi_x, \lambda)$ is invertible in $\text{Hom}(\pi^*E, \pi^*F)$ for all $(\xi_x, \lambda) \in (T^*\overline{X} \times \Lambda) \setminus 0$ and $r \in \mathbb{R}_+$.

- v) The principal boundary—exit symbol $\sigma_{\partial,e}^{\mu,\nu;\ell}(A)(r, \xi_{x'}, \lambda)$ is invertible in (8.11) for all $(\xi_{x'}, \lambda) \in (T^*Y \times \Lambda) \setminus 0$ and $r \in \mathbb{R}_+$.
- vi) The principal exit symbol $\sigma_e^\nu(A)(r, \lambda)$ is invertible in (8.12) for all $\lambda \in \Lambda$ and $r \in \mathbb{R}_+$.

Theorem 9.2. *For $A \in C_{(V)cl}^{\mu,\nu,d;\ell}(X^\wedge, (\gamma, (-N, 0]))$, where $d \leq \mu_+$, the following are equivalent:*

- i) *A is elliptic (parabolic).*
- ii) *There exists $P \in C_{(V)cl}^{-\mu,-\nu,d';\ell}(X^\wedge, (\gamma, (-N, 0]))$, $d' \leq (-\mu)_+$, such that $AP - 1 \in C_{G(V)}^{d_1}(X^\wedge, (\gamma, (-N, 0]))$ and $PA - 1 \in C_{G(V)}^{d_2}(X^\wedge, (\gamma, (-N, 0]))$, where $d_1 = \max\{-\mu + d, d'\}$ and $d_2 = \max\{\mu + d', d\}$.*

Moreover, if A is parabolic, then we even find a Volterra boundary cone operator P in ii) with $AP = 1$ and $PA = 1$, i.e., A is invertible within the algebra of Volterra boundary cone operators.

Proof. We carry out the proof for Volterra operators and parabolicity only — the case of elliptic operators is similar (even simpler).

Clearly, ii) implies i) for the principal symbols associated with P invert the principal symbols associated with A by Theorem 8.8.

Now assume that A is parabolic, and let (h, a) be a complete interior symbol of A subordinated to some covering $\{[0, T_1], (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$, and let $\{[0, \tilde{T}_1], (\tilde{T}_2, \infty)\}$ be a strictly refined covering.

Let us consider the holomorphic Volterra Mellin symbol $h(r, z)$ first: From Definition 9.1 we conclude that we may regard $h(r, z)$ as a parabolic C^∞ -family in $\mathcal{B}_{Vcl}^{\mu,d;\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma})$, depending on $r \in [0, T_1]$. By Theorem 3.7 there exists $h' \in C^\infty([0, T_1], \mathcal{B}_{Vcl}^{-\mu,d';\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma}))$ such that $h(r, z)h'(r, z) - 1$ and $h'(r, z)h(r, z) - 1$ are C^∞ -families of regularizing Volterra boundary operators in the parameter-dependent Boutet de Monvel's calculus (see Section 3). Employing a formal Neumann series argument, we find a modified $h'(r, z) \in C_B^\infty(\overline{\mathbb{R}}_+, \mathcal{B}_{Vcl}^{-\mu,d';\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma}))$ such that for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ with $\omega \prec \tilde{\omega} \prec \chi_{[0, \tilde{T}_1]}$ we have that $(\omega h)\#(\tilde{\omega} h') - \omega$ and $(\omega h')\#(\tilde{\omega} h) - \omega$ are again a C^∞ -families of regularizing Volterra boundary operators in the parameter-dependent Boutet de Monvel's calculus. Now define $\tilde{h}(r, z) := (H_{\gamma-\frac{n}{2}}(\varphi)h')(r, z)$ with the Mellin kernel cut-off operator $H_{\gamma-\frac{n}{2}}$, where $\varphi \in C_0^\infty(\mathbb{R}_+)$ with $\varphi \equiv 1$ near $r = 1$ (see Remark 5.5). We thus end up with $\omega(\text{op}_M^{\gamma-\frac{n}{2}}(h\#\tilde{h}) - 1)\tilde{\omega} \in C_{M+G,V}^{d_1}(X^\wedge, (\gamma, (-N, 0]))$ and $\omega(\text{op}_M^{\gamma-\frac{n}{2}}(\tilde{h}\#h) - 1)\tilde{\omega} \in C_{M+G,V}^{d_2}(X^\wedge, (\gamma, (-N, 0]))$ for all cut-off functions $\omega, \tilde{\omega} \prec \chi_{[0, \tilde{T}_1]}$; note that $\tilde{h} \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,Ocl}^{-\mu,d';\ell}(X))$.

It is much easier to prove that there exists $\tilde{a} \in S_{cl}^{-\nu}(\mathbb{R}, \mathcal{B}_{Vcl}^{-\mu,d';\ell}(X; \mathbb{H}))$ such that for all cut-off functions $\chi_{[0, \tilde{T}_2]} \prec \omega, \tilde{\omega}$ we have $(1 - \omega)(\text{op}_r(a\#\tilde{a}) - 1)(1 - \tilde{\omega}) \in C_{G,V}^{d_1}(X^\wedge, (\gamma, (-N, 0]))$ and $(1 - \omega)(\text{op}_r(\tilde{a}\#a) - 1)(1 - \tilde{\omega}) \in C_{G,V}^{d_2}(X^\wedge, (\gamma, (-N, 0]))$ (see also the proof of Theorem 4.8 for the case $\nu = 0$).

Choose cut-off functions $\chi_{[0,\tilde{T}_2]} \prec \omega_3 \prec \omega_1 \prec \omega_2 \prec \chi_{[0,\tilde{T}_1]}$, and define $P' := \omega_1 \text{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2 + (1-\omega_1)\text{op}_r(\tilde{a})(1-\omega_3) \in C_{Vcl}^{-\mu,-\nu,d';\ell}(X^\wedge, (\gamma, (-N, 0]))$. For the tuple (\tilde{h}, \tilde{a}) is a complete interior symbol of P' subordinated to the covering $\{[0, \tilde{T}_1), (\tilde{T}_2, \infty)\}$ of \mathbb{R}_+ , we conclude from Theorem 8.8 that $AP' - 1 \in C_{M+G,V}^{d_1}(X^\wedge, (\gamma, (-N, 0]))$ and $P'A - 1 \in C_{M+G,V}^{d_2}(X^\wedge, (\gamma, (-N, 0]))$.

By Theorem 5.10 we have $g := \sigma_M^0(A)^{-1} - \sigma_M^0(P') \in M_{V,Q}^{-\infty,d'}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma})$. With a cut-off function $\omega \in C_0^\infty(\mathbb{R}_+)$ define $\tilde{P} := P' + \omega \text{op}_M^{\gamma-\frac{n}{2}}(g)\omega \in C_{Vcl}^{-\mu,-\nu,d';\ell}(X^\wedge, (\gamma, (-N, 0]))$. Then $A\tilde{P} - 1 \in C_{M+G,V}^{d_1}(X^\wedge, (\gamma, (-N, 0]))$ and $\tilde{P}A - 1 \in C_{M+G,V}^{d_2}(X^\wedge, (\gamma, (-N, 0]))$ with $\sigma_M^0(A\tilde{P}) = 1$ and $\sigma_M^0(\tilde{P}A) = 1$.

According to Theorem 7.8 there exist $D_j \in C_{M+G,V}^{d_j}(X^\wedge, (\gamma, (-N, 0]))$, $j = 1, 2$, such that both $((1 + D_2)\tilde{P})A - 1$ and $A(\tilde{P}(1 + D_1)) - 1$ are Volterra Green operators, and thus either $P := \tilde{P}(1 + D_1)$ or $P := (1 + D_2)\tilde{P}$ fulfills the conditions in ii). The missing claim about the invertibility of A within the calculus follows from Theorem 7.4. \square

Corollary 9.3. *Let $A \in C_{cl}^{\mu,\nu,d;\ell}(X^\wedge, (\gamma, (-N, 0]))$, $d \leq \mu_+$, be elliptic. Then*

$$A : \begin{array}{c} \mathcal{K}^{s,\gamma;\ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma;\ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

is a Fredholm operator for all $s > \mu_+ - \frac{1}{2}$ and $\delta \in \mathbb{R}$.

Proof. This follows from Theorem 9.2, and from the fact that Green operators induce nuclear, in particular compact, operators in the cone Sobolev spaces by Proposition 7.3. \square

Corollary 9.4. *Let $A \in C_{Vcl}^{\mu,\nu,d;\ell}(X^\wedge, (\gamma, (-N, 0]))$, $d \leq \mu_+$, be parabolic. Then*

$$A : \begin{array}{c} \mathcal{K}^{s,\gamma;\ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma;\ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

is bijective for all $s > \mu_+ - \frac{1}{2}$ and $\delta \in \mathbb{R}$, i.e., the equation $Au = f$ is uniquely solvable in these spaces.

Moreover, if f has asymptotics of “length” N , i.e., if f belongs to the subspaces with asymptotics associated with the weight interval $(-N, 0]$, then so does the solution u .

Remark 9.5. Let $A \in C_{Vcl}^{\mu,\nu,d;\ell}(X^\wedge, (\gamma, (-N, 0]))$, $d \leq \mu_+$. If just the conditions i) — iii) of Definition 9.1 of parabolicity are fulfilled we still find $P \in C_{Vcl}^{-\mu,-\nu,d';\ell}(X^\wedge, (\gamma, (-N, 0]))$, $d' \leq (-\mu)_+$, that is an inverse of A when considered as operators in the subspaces of the cone Sobolev spaces that consist of all distributions having their support in $(0, r_0]$ for every fixed $r_0 \in \mathbb{R}_+$.

9.1. Reduction to the boundary. Classically, boundary value problems are often studied via reduction to the boundary. Given the solution operator to a particular boundary value problem associated with a fixed interior operator, we can judge about the solvability and regularity of other boundary conditions simply by studying an operator on the boundary that is determined from these conditions and the solution operator.

In this final section we demonstrate that reduction to the boundary of parabolic boundary value problems in infinite space-time naturally fits into the pseudodifferential calculi constructed in the present article and in [39]. More precisely, the reduced operators on the boundary are Volterra pseudodifferential operators in the boundaryless calculus from [39], and an interior operator is parabolic with some boundary conditions if and only if its reduction to the boundary by a fixed parabolic boundary value problem is parabolic on the boundary.

Assume that

$$\begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{array}{c} \mathcal{K}^{s,\gamma;\ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma;\ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_+)_{\delta-\nu} \end{array}$$

is a parabolic Volterra boundary cone operator in $C_{Vcl}^{\mu,\nu,d;\ell}(X^\wedge, (\gamma, (-N, 0]))$, where $d \leq \mu_+$, and let $\begin{pmatrix} \hat{P} & \hat{K} \\ \hat{T} & \hat{Q} \end{pmatrix} \in C_{Vcl}^{-\mu,-\nu,d';\ell}(X^\wedge, (\gamma, (-N, 0]))$ be the inverse according to Theorem 9.2.

Moreover, let

$$\begin{pmatrix} A & \tilde{K} \\ \tilde{T} & \tilde{Q} \end{pmatrix} : \begin{array}{c} \mathcal{K}^{s,\gamma;\ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2};\ell}(Y^\wedge, \tilde{J}_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma;\ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\frac{1}{2};\ell}(Y^\wedge, \tilde{J}_+)_{\delta-\nu} \end{array}$$

in $C_{Vcl}^{\mu,\nu,\tilde{d};\ell}(X^\wedge, (\gamma, (-N, 0]))$, $\tilde{d} \leq \mu_+$, be another set of boundary conditions associated with A .

With appropriate order reductions $R^{\mu,\nu}, \tilde{R}^{\mu,\nu} \in C_{Vcl}^{\mu,\nu;\ell}(Y^\wedge, (\gamma - \frac{1}{2}, (-N, 0]))$, i.e., $R^{\mu,\nu}$ and $\tilde{R}^{\mu,\nu}$ are parabolic Volterra cone operators in the boundaryless calculus with respect to the bundles J_- and \tilde{J}_- , respectively, and thus they are invertible with inverses $R^{-\mu,-\nu}$ and $\tilde{R}^{-\mu,-\nu}$ (see [39]), we may write

$$\begin{pmatrix} A & \tilde{K} & K \\ \tilde{T} & \tilde{Q} & 0 \\ 0 & 0 & R^{\mu,\nu} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{T}\hat{P} & \tilde{T}\hat{K} & (\tilde{Q} - \tilde{T}\hat{P}\hat{K})\tilde{R}^{-\mu,-\nu} \\ R^{\mu,\nu}\hat{T} & R^{\mu,\nu}\hat{Q} & -R^{\mu,\nu}\hat{T}\hat{K}\tilde{R}^{-\mu,-\nu} \end{pmatrix} \begin{pmatrix} A & \tilde{K} & K \\ T & 0 & Q \\ 0 & \tilde{R}^{\mu,\nu} & 0 \end{pmatrix}$$

$$: \begin{array}{c} \mathcal{K}^{s,\gamma;\ell}(X^\wedge, E)_\delta \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2};\ell}(Y^\wedge, \tilde{J}_-)_\delta \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_-)_\delta \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma;\ell}(X^\wedge, F)_{\delta-\nu} \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\frac{1}{2};\ell}(Y^\wedge, \tilde{J}_+)_{\delta-\nu} \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\frac{1}{2};\ell}(Y^\wedge, J_-)_{\delta-\nu} \end{array}.$$

Consequently, the Volterra boundary cone operator $\begin{pmatrix} A & \tilde{K} \\ \tilde{T} & \tilde{Q} \end{pmatrix}$ is parabolic if and only if the Volterra cone operator

$$S := \begin{pmatrix} \tilde{T}\tilde{K} & (\tilde{Q} - \tilde{T}\tilde{P}\tilde{K})\tilde{R}^{-\mu, -\nu} \\ R^{\mu, \nu}\tilde{Q} & -R^{\mu, \nu}\tilde{T}\tilde{K}\tilde{R}^{-\mu, -\nu} \end{pmatrix} \in C_{Vcl}^{0,0;\ell}(Y^\wedge, (\gamma - \frac{1}{2}, (-N, 0]))$$

is parabolic. Recall that S is the reduction to the boundary of the boundary value problem $\begin{pmatrix} A & \tilde{K} \\ \tilde{T} & \tilde{Q} \end{pmatrix}$ by $\begin{pmatrix} A & K \\ T & Q \end{pmatrix}$.

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