# Dynamic Methods in the General Theory of Cauchy Type Functional Equations

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## 1 Introduction. Definitions and Discussions.

In the last 30-40 years functional equations have grown to be a large, independent branch of mathematics with its own methods, circle of problems and, what is of great importance, abounding in applications. If on an early stage of development (XVIII - XIX centuries) functional equations plaied some auxiliary, may be even decorative role, describing in an abstract form various fundamental functions from Analysis, then nowadays functional equations turn out to be a powerful tool when solving *analytical problems* in quite different fields of mathematics. Moreover, sometimes these equations arise as an adequate description of such problems. Then having solved the corresponding functional equation we solve thus an original problem. In this paper we deal with precisely the latter situation.

One of the first functional equations has been studied by Cauchy. He formulated the following problem: find a continuous functions $F(z)$ on $\mathbb{R}$ such that the equality

$$F(x+y) = F(x) + F(y) \tag{1}$$

holds for all points $(x, y) \in \mathbb{R}^2$. A solution of this problem is not complicated. From (1) it follows that

$$F(\sum_{j=1}^{k} z_j) = \sum_{j=1}^{k} F(z_j) \tag{2}$$

for arbitrary $k \geq 2$ and $z_j \in \mathbb{R}$. Setting $F(1) = \lambda$ and substituting $z_1 = \ldots = z_k = 1$ we get from (2)

$$F(k) = \lambda k.$$

If $z_1 = \ldots = z_k = 1/k$ in (2), then we arrive at the equality

$$F(1/k) = \lambda/k.$$

From the last two equalities it immediately follows that for all integers $m$ and $n$

$$F(m/n) = \lambda m/n,$$

and by continuity we find that

$$F(z) = \lambda z, \qquad z \in \mathbb{R}.$$

If we are interested in a function $F(z)$ not on the whole line but only on the interval $I = \{z \mid -1 \leq z \leq 1\}$, then it suffices that the equality (1) be valid at all points of the square

$$K = \{(x, y) \mid |x \pm y| \leq 1\}.$$

The previous arguments lead then to the unique solution $F(z) = \lambda z, \quad z \in I$.

The other Cauchy equation closely connected to (1) is that determining an exponential function, namely

$$F(x+y) = F(x)F(y), \quad (x, y) \in \mathbb{R}^2. \tag{3}$$

If $y = x$ in (3) then $F(2x) = (F(x))^2$ and hence $F(z) \geq 0$. If $F(y_0) = 0$ for some $y_0$ then $F(x + y_0) = 0$ for *all* $x$. Consequently $F(z) > 0$. But then the function

$$G(z) = \ln F(z), \quad z \in \mathbb{R}$$

satisfies equation (1). Therefore by above

$$G(z) = \lambda \quad \text{and} \quad F(z) = e^{\lambda z}.$$

Consider now a pair of continuous maps $\delta_1$ and $\delta_2$ in $I = \{t \mid -1 \leq t \leq 1\}$ and let

$$\mathcal{D} = I \cup \mathcal{R}[\delta_1 + \delta_2]$$

where $\mathcal{R}[f]$ denotes the range of a map $f$.

**Definition** Given a function $H : \mathcal{D} \to \mathbb{R}$ the equation

$$F \circ (\delta_1 + \delta_2) - F \circ \delta_1 - F \circ \delta_2 = H \tag{4}$$

with $F$ an unknown continuous function: $\mathcal{D} \to \mathbb{R}$ we call a *Cauchy type functional equation.*

Let $\Gamma$ be a continuous nonsingular curve in the plane $\mathbb{R}^2$ with a parametric representation

$$\Gamma = \{x = (x_1, x_2) \mid x_1 = \delta_1(t), \ x_2 = \delta_2(t); \ t \in I\},$$

where

$$\delta_1(-1) = \delta_2(-1) = 0. \tag{5}$$

If $H = 0$ then equation (4) under the name "The Cauchy equation on the curve $\Gamma$" was carefully investigated in the 70s - 80s in a series of works (starting with the pioneer paper of Zdun [11]) (see [1] and [2] with respect to references). The main goal of these works was to prove that (as in the case of equation (1)) linear function $F(z) = \lambda z$ is the only solution of homogeneous equation (4). Nonhomogeneous equation (4) has never been investigated.

Note that condition (5) means from a geometrical point of view that the origin $(0, 0)$ in $(t, z)$ - plane is an end point of all the three curves

$$z = \delta_1(t), \qquad z = \delta_2(t), \qquad z = \delta_1(t) + \delta_2(t). \tag{6}$$

Quite recently it was clarified (see [4] - [9]) that several problems in such different fields as Integral Geometry and boundary problems for Partial Differential Equations can be reduced (sometimes in equivalent manner) to some Cauchy type functional equations. What is characteristic is that in this case the corresponding curves (6) in the $(t, z)$ - plane form a configuration not satisfying hypothesis (5). For convenience we fix this difference in configurations in the following definition.

**Definition** Let

$$I = \{t \mid -1 \le t \le 1\} \qquad \text{and} \qquad \mathring{I} = \{t \mid -1 < t < 1\}.$$

We say that maps $\beta_1, \beta_2, \ldots, \beta_n$ in $I$ form a $Z$ - *configuration* if all the functions $\beta_1, \ldots, \beta_n$ do not decrease and

$$\beta_1(-1) = \ldots = \beta_n(-1) = 0.$$

We say that two maps $\beta_1$ and $\beta_2$ in $I$ form a $\mathcal{P}$ - *configuration* if both functions do not decrease,

$$\beta_1\beta_2(t) \ne 0 \quad \text{in} \quad \mathring{I}$$

and

$$\beta_1(-1) = \beta_2(1) = 0, \quad \beta_1(1) = 1, \quad \beta_2(-1) = -1. \tag{7}$$

The Figures 1 and 2 represent typical examples of $Z$- and $\mathcal{P}$ - configurations, respectively, $n = 2$. Dotted lines in both figures represent the graphs of functions $z = \beta_1(t) + \beta_2(t)$.

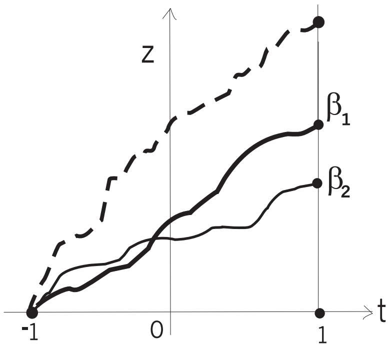


Figure 1

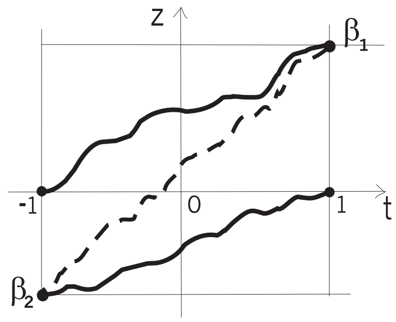


Figure 2

The Figures 1′ and 2′ represent the curves $\Gamma = \{(x_1, x_2) \mid \ x_1 = \beta_1(t),\ x_2 = \beta_2(t),\ t \in I\}$, corresponding to $Z$ - and $\mathcal{P}$ - configurations of pairs $\beta_1, \beta_2$, respectively.

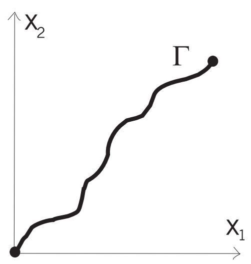


Figure 1′

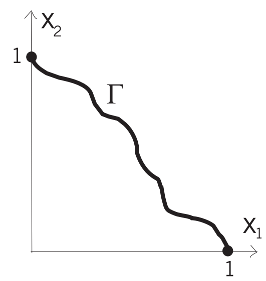


Figure 2′

It should be noted that when dealing with equation (4) a transition from a $Z$ - to a $\mathcal{P}$ - configuration leads to a significant complication of the proofs even in the homogeneous situation. In particular some new dynamic methods were worked out to overcome appearing difficulties. A new Maximum Principle for functional equations plays a significant role when studying a homogeneous Cauchy equation. All these results and methods are the subject of this survey. It is worth mentioning that the new approach leads to completely new results also when curves (6) form a $Z$ - configuration. It is interesting that the Cauchy type functional equation in the latter situation also turns out to be equivalent to some problem in Integral Geometry.

Note in conclusion that our methods are admissible when dealing with sufficiently smooth functions (both given and unknown). It would be interesting to weaken restrictions of such type.

# 2 Solvability of the Cauchy Type Functional Equations

## 2.1 The Case of a $\mathcal{P}$ -configuration

In this subsection we describe the results related to the unique solvability of the functional equation

$$F(\beta_1(t) + \beta_2(t)) - F(\beta_1(t)) - F(\beta_2(t)) = H(t), \quad t \in I \tag{8}$$

in the case when the functions $\beta_1(t)$ and $\beta_2(t)$ form a $\mathcal{P}$ - configuration and satisfy a nondegeneracy condition

$$\beta_1'(t) + \beta_2'(t) > 0. \tag{9}$$

Due to hypotheses (7) and (9) the map $\beta = \beta_1 + \beta_2$ in $I$ is a diffeomorfism preserving the boundary $\partial I$. Therefore the maps

$$\delta_1 = \beta_1 \circ \beta^{-1} \qquad \text{and} \qquad \delta_2 = \beta_2 \circ \beta^{-1}$$

form a $\mathcal{P}$ - configuration on $I$ and in addition satisfy the condition

$$\delta_1(t) + \delta_2(t) = t, \qquad t \in I. \tag{10}$$

Thus the change of variable $t \to \beta^{-1}(t)$ reduces equation (1) in equivalent manner to the form

$$\mathcal{B}F := F(t) - F \circ \delta_1(t) - F \circ \delta_2(t) = H(t), \qquad t \in I. \tag{11}$$

It turned out that an essential part of information connected with solvability of this equation may be derived by means of new dynamical methods, introduced in the author's papers [4] - [6],[8],[9]. The application of these methods becomes possible if we associate equation (11) with a semigroup $\Phi_\delta$ of maps in $I$, generated by $\delta_1$ and $\delta_2$. On the one hand in term of orbits of this semigroup a necessary and sufficient condition for the existence of a unique solution to equation (11) is easily formulated. On the other hand an essential part of the proof of both existence and uniqueness of a solution is based on the existence of very specific attractors of a noncommutative dynamic system generated by semigroup $\Phi_\delta$.

Turn now to the exact formulations. We denote by $\Phi_\delta$ a noncommutative semigroup of maps in $I$ generated by ($\delta_1$ and $\delta_2$). The elements of $\Phi_\delta$ are maps in $I$ of the form

$$\delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1},$$

where $J = (j_1, \dots, j_n)$ is an arbitrary $n$-tuple with all $j_k = 1$ or 2. The semigroup $\Phi_\delta$ naturally generates a dynamic system. In what follows we make use of the following geometric terminology, connected to $\Phi_\delta$.

(i) Given a map $\delta_J \in \Phi_\delta$ an ordered set $(t_1, \dots, t_{n+1})$ of points in $I$ is called an $J$ - *orbit* if

$$t_{k+1} = \delta_{j_k} t_k \quad \text{for} \quad 1 \le k \le n \le \infty. \tag{12}$$

(ii) If the end points of an orbit coincide, i.e. $t_{n+1} = t_1$, then the orbit $(t_1, \dots, t_{n+1})$ is called *cyclic* or, in short, *cycle*.

Introduce the *critical* sets

$$\mathcal{T}_j = \{t \in I \mid \delta_j'(t) = 0\}, \quad j = 1, 2, \qquad \text{and} \qquad \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2.$$

(iii) If all the points of a $J$-orbit belong to the set $\mathcal{T}$ then the orbit is called *critical*.

The last definition unlike the previous ones is specific in the circle of problems in question.

(iv) A $J$ - orbit $(t_1, \dots, t_{n+1})$, $n = 1, 2, \dots$, is called $\mathcal{T}$-*proper* if in (12)

$$\delta_{j_k} = \delta_1 \quad \text{when } t_k \in \mathcal{T}_1 \quad \text{and} \quad \delta_{j_k} = \delta_2 \quad \text{when } t_k \in \mathcal{T}_2.$$

**Definition** We denote by $\mathfrak{N}_\delta$ the set of all $\mathcal{T}$ - proper critical cycles in $I$.

Before formulating the main result related to the solvability of equation (11) we note that the kernel of the operator $\mathcal{B}$ contains all linear functions and as will be seen later on the cokernel of $\mathcal{B}$ is also nonempty. This makes us look for some necessary conditions for solvability. To this end substitute in (11) successively $t = -1$ and $t = 1$. As equalities (7) remain true for functions $\delta_1$ and $\delta_2$, we find that for an arbitrary function $F \in C(I)$

$$\mathcal{B}F(-1) = \mathcal{B}F(1) = -F(0). \tag{13}$$

It follows that the range of the operator $\mathcal{B}$ consists of functions $H$ whose values at boundary points of $I$ are equal. Moreover, the unknown function $F$ has to be connected with a given function $H$ by the equality

$$F(0) = -H(-1) = -H(1).$$

Taking into account that solutions $F(z) = \lambda z$ of homogeneous equation (11) are uniquely determined by the value $F(1)$ we arrive at the following natural problem:

> given numbers $\lambda, \mu \in \mathbb{R}$ and an arbitrary function $H \in C^2(I)$ satisfying the condition
>
> $$H(1) = H(-1) = -\mu \tag{14}$$
>
> find a function $F \in C^2(I)$ such that
>
> $$\mathcal{B}F = H \quad \text{on } I, \qquad F(0) = \mu, \quad F(1) = \lambda. \tag{15}$$

Denote by $\mathcal{T}_j'$ the sets of limit points of the sets $\mathcal{T}_j$, $j = 1, 2$. Now we are ready to formulate the first result.

**Theorem 1** *Suppose that*

$$\min_{\mathcal{T}_1'} t > \max_{\mathcal{T}_2'} t. \tag{16}$$

*If the hypothesis*

$$\mathfrak{N}_\delta = \emptyset \tag{17}$$

*is valid, then given arbitrary constants* $\lambda, \mu$ *and function* $H \in C^2(I)$ *satisfying condition* (14) *there is a unique solution* $F \in C^2(I)$ *of problem* (15). *The inverse operator* $H \mapsto F$ *is continuous:* $C^2(I) \to C^2(I)$.

**Remark** The assertion remains valid if we replace the boundary condition $F(1) = \lambda$ by the condition $F'(0) = \lambda$.

We now draw a reader's attention to the following interesting fact closely connected with the Cauchy equation (1).

In the plane $\mathbb{R}^2$ of variables $(x, y)$ consider an arbitrary nondecreasing twice differentiable curve $\Gamma$ which connects points $(0,1)$ and $(1,0)$ and admits a parametric representation of the form

$$x = \delta_1(t), \qquad y = -\delta_2(t), \qquad -1 \le t \le 1,$$

(see Fig.2$'$). If these functions $\delta_1$ and $\delta_2$ satisfy conditions (16) and (17) then by Theorem 1 the homogeneous equation (11) has no solutions except for $F(z) = \lambda z$, $|z| \le 1$.

Thus in order to determine a linear function $F$ on the interval $I$ the Cauchy equality (1) has not to be valid for *all* points $(x, y)$ of the square $K = \{(x, y) \mid |x + y| \le 1\}$. It suffices that the equality $F(x + y) = F(x) + F(y)$ is valid for all points of some abovementioned *curve* $\Gamma$. For instance a side

$$x = (t - 1)/2, \quad y = (t + 1)/2, \quad -1 \le t \le 1$$

of the square $K$ can play the role of the curve $\Gamma$. Thus the original Cauchy problem (1) turns out to be overdetermined.

When proving the uniqueness in Theorem 1 a Maximum Principle for functional equations of a rather general form plays a crucial role. This principle has been obtained for the first time in the author's papers [6],[8],[9]. In the situation under consideration the corresponding assertion looks as follows.

**Theorem 2** *If* $G \in C^1(I)$ *is a solution of equation*

$$G(t) - \delta_1'(t) G \circ \delta_1(t) - \delta_2'(t) G \circ \delta_2(t) = 0,$$

(*the differentiated homogeneous equation* (11)) *and the functions* $\delta_1, \delta_2$ *satisfy hypotheses* (16) *and* (17), *then* $G$ *takes its maximum and minimum at the boundary* $\partial I$.

## 2.2 The Case of a $Z$-configuration

In this subsection we deal with the unique solvability of the functional equation

$$\mathcal{B}F := F \circ \sum_{j=1}^{n} \beta_j(t) - \sum_{j=1}^{n} F \circ \beta_j(t) = H(t), \quad t \in I, \;\; n \ge 2, \tag{18}$$

where the functions $\beta_1, \dots, \beta_n$ form a $Z$ - configuration.

It is easily seen that the homogeneous equation $\mathcal{B}F = 0$ has nontrivial solutions $F = \lambda z$, $\lambda \in \mathbb{R}$. Furthermore, substituting zero for $t$ in (18) we arrive at a necessary condition for the solvability of equation (18):

$$(1-n)F(0) = H(0), \quad n \geq 2.$$

On the other hand the same procedure with the differentiated equation (18) leads to the equality (independently of $F$!)

$$H'(0) = 0.$$

These observations enable us to formulate the following well-posed problem for equation (18).

Given numbers $\mu$, $\lambda \in \mathbb{R}$ and a function $H \in C^2(I)$, satisfying conditions

$$H(0) = \lambda, \qquad H'(0) = 0, \tag{19}$$

find a function $F \in C^2(I)$ such that

$$\mathcal{B}F = H \quad \text{on } I, \qquad F(0) = \lambda/(1-n), \qquad F'(0) = \mu. \tag{20}$$

**Theorem 3** *Let $\beta_1, \ldots, \beta_n$ be twice continuously differentiable functions on $I$ satisfying the only condition*

$$\sum_{\substack{j,k=1\\ j\neq k}}^{n} \beta_j'\beta_k' > 0 \quad \textit{in a deleted neighborhood of the point } t = -1. \tag{21}$$

*Then for an arbitrary function $H \in C^2(I)$ satisfying condition* (19) *there exists a unique solution $F \in C^2(I)$ of problem* (20).

The proof of the existence in this theorem is based (as in Theorem 1) on a dynamical approach and cannot be given in the framework of this short survey. But the proof of uniqueness is easier then in the case of $\mathcal{P}$ - configuration. Indeed, let $H = 0$ and $\mu = \lambda = 0$ in (20). Then $\mathcal{B}F(0) = 0$ and consequently *the equalities*

$$\mathcal{B}F = 0 \qquad \textit{and} \qquad \frac{d}{dt}\mathcal{B}F = 0 \tag{22}$$

*are equivalent.* Introduce a new variable

$$Z = \sum_{j=1}^{n} \beta_j(t) := \beta(t)$$

and let

$$\delta_j = \beta_j \circ \beta^{-1}$$

be new maps in $I_z = [0, b]$ where $b = \beta(1)$. Then $\sum \delta_j(t) = t$ and the second equation in (22) can be written in the form

$$G(z) - \sum_{j=1}^{n} \delta_j'(z) G \circ \delta_j(z) = 0, \quad z \in I_z \tag{23}$$

with $G = F'$. It is clear that $\delta'_j(z) \geq 0,\ 1 \leq j \leq n$, and

$$\sum_{j=1}^{n} \delta'_j(z) = 1. \tag{24}$$

Therefore any constant solves equation (23). Let us show that there are no other solutions. Take an arbitrary solution $G(z)$ of (23) and denote

$$\max_I G(z) = M.$$

Let $\mathcal{M} = \{z \in I \mid G(z) = M\}$. Prove that $-1 \in \mathcal{M}$. Indeed, denote

$$T = \min_z \{z \mid z \in \mathcal{M}\}.$$

It is clear that $T \in \mathcal{M}$. If $T \neq -1$, substitute $T$ for $z$ in (23). Making use of (24) we find that $G(\delta_{j'}(T)) = M$ for some indices $j'$ (those for which $\delta'_{j'}(T) \neq 0$). If $\delta_{j'}(T) = T$, then $\delta_j(T) = 0$ for $j \neq j'$. It follows that $\delta_j(t) \equiv 0$ for $t \leq T$ and hence $\sum\limits_{j \neq k} (\delta'_j \delta'_k)(t) = 0$ in a neighborhood of $t = -1$. But this contradicts hypothesis (21). Therefore, $\delta_{j'}(T) < T$ in contradiction with the definition of $T$.

Repeating the same arguments with minimum instead of maximum results in equality

$$\min_I G(z) = G(-1).$$

Thus $\min G = \max G$ and therefore $G \equiv \text{const}$. As $F'(0) = 0$ it follows that $G = 0$. This proves the uniqueness in Theorem 3.

## 2.3 Multiplicative Cauchy type functional equations

In this subsection we will briefly concern with "nonhomogeneous" *multiplicative Cauchy functional equations*. We name like this any functional equation of the form

$$F \circ \sum_{j=1}^{n} \beta_j = \left(\prod_{j=1}^{n} F \circ \beta_j\right) H, \tag{25}$$

where $H > 0$ is a given function and functions $\beta_1, \dots, \beta_n$ form one of the above-mentioned configurations. To the best of the author's knowledge such a problem has never been investigated. When $H \equiv 1$ the only case of $Z$ - configuration and $n = 2$ has been studied (see [11]).

Let functions $\beta_1, \dots, \beta_n$ on $I$ form a $Z$-configuration and satisfy hypothesis (21). Repeating word for word the concluding arguments in Sec. 2.2 we find that *only positive functions $F$ may solve equation* (25). But then this equation is equivalent to equation

$$(\ln F) \circ \sum_j \beta_j = \sum_j (\ln F) \circ \beta_j + \ln H$$

which is nothing but equation (18). Consequently, applying Theorem 3 leads to the following result.

**Theorem 4** *Given real numbers $\mu$ and $\lambda > 0$ and an arbitrary positive function $H \in C^2(I)$ with $H(0) = \lambda,\ H'(0) = 0$ equation* (25) *has a unique solution $F \in C^2(I)$ satisfying conditions*

$$F(0) = \lambda^{1/(1-n)} \qquad \text{and} \qquad F'(0) = \mu\lambda^{1/(1-n)}.$$

Turn now to equation (25) with $n = 2$ and functions $\beta_1, \beta_2$ forming a $\mathcal{P}$ - configuration. The main novelty here compared with a $Z$ - configuration is that even a simplest equation

$$F = (F \circ \delta_1)(F \circ \delta_2) \qquad \text{in } I$$

with $\delta_1(t) + \delta_2(t) = t$ may have oscillating solutions. Nevertheless the following assertion is valid.

**Theorem 5** *Given positive numbers $\lambda, \mu \in \mathbb{R}$ and an arbitrary positive function $H \in C^2(I)$ satisfying the condition*

$$H(1) = H(-1) = \mu$$

*there is a unique positive solution $F \in C^2(I)$ of equation* (25) *such that*

$$F(0) = 1/\mu \qquad \textit{and} \qquad F(1) = e^{\lambda}.$$

# 3 Problems in Analysis Reducing to Cauchy Type Functional Equations

In this section we discuss several problems in classical Analysis which can be reduced (sometime in an equivalent manner) to a Cauchy type functional equation. What is interesting is that all these problems at first sight do not give even a merest hint about a connection with functional equations. But having solved a corresponding functional equation we automatically solve the original problem. We will trace this connection by considering two problems in Integral Geometry and in Partial Differential Equations which were studied for the first time in the author's papers [4] - [7].

## 3.1 Some problems in Integral Geometry and Cauchy Functional Equations

One of the typical problems in Integral Geometry is to reconstruct a function in a domain $D$ of $\mathbb{R}^n$ knowing its integrals over a family of subdomains in $D$. A peculiarity of the problem we deal with is that we consider bounded domains $D$ with the boundary $\partial D$. The statement of this problem and the corresponding results turn out to be intimately connected with both local and global properties of $\partial D$. This connection is realized by means of a semigroup of maps in $\partial D$ that we associate with a problem in question. But exactly the same situation arises when studying a Cauchy type functional equation (see Subsec. 2.1). It is no wonder: it will be proved below that *every Cauchy type functional equation is equivalent to* (*at least*) *two different problems in Integral Geometry.* These problems correspond to two different configurations formed by the functions $\beta_1$ and $\beta_2$.

### 3.1.1 The Case of a $\mathcal{P}$ - configuration

#### 3.1.1.1 Statement of the Problem

Let $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ be smooth nonsingular transversal vector fields in a disk $B \subset \mathbb{R}^2$. Introduce a curvilinear triangle $D = OA_1A_2$ whose sides $OA_1$ and $OA_2$ are trajectories of vector fields $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, respectively. As to the side $\Gamma = A_1A_2$ it is assumed to be an arbitrary smooth curve without singularities which is transversal at its ends to $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$. In addition the closure $\overline{D}$ of a domain $D$ is supposed to satisfy the following hypotheses.

$1^\circ$ For any point $p \in \overline{D}$ a trajectory of $\boldsymbol{l}_j$ passing through $p$ meets $OA_k$, $k, j = 1, 2$, $k \neq j$ at a point $\pi_j p$.

$2^\circ$ The set $\overline{D}$ is $\boldsymbol{l}_j$ - convex, $j = 1, 2$. This means that given points $p$ and $q$ on any trajectory $\gamma_j$ of the field $\boldsymbol{l}_j$ all the points $r \in \gamma_j$ between $p$ and $q$ belong to $\overline{D}$.

Given an arbitrary point $q \in \Gamma$ let $D_q$ be a curvilinear parallelogram $qq_1Oq_2$, where $q_j = \pi_j q$, $j = 1, 2$. The above conditions $1^\circ$ and $2^\circ$ guarantee an inclusion $\overline{D}_q \subset \overline{D}$ for all $q \in \Gamma$ (see Fig.3).

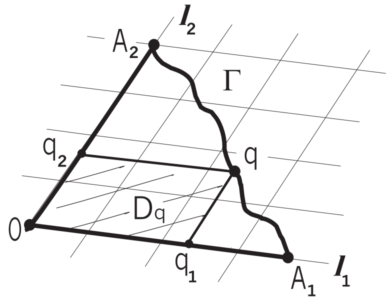


Figure 3

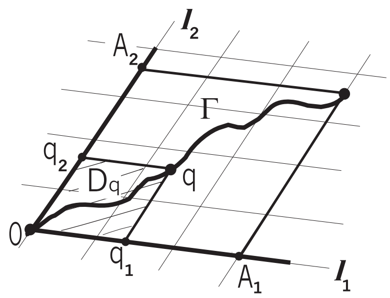


Figure 4

*In this subsection we will deal with a solvability of an integral equation of the following form*:

$$\int\limits_{D_q} f d\sigma = h(q), \quad q \in \Gamma. \tag{26}$$

Here $d\sigma$ is a measure in $B$, $h(q) \in C(\Gamma)$ is a given function and $f \in C(\overline{D})$ is an unknown function.

The general problem formulated in [3] in connection with this equation looks as follows: for which spaces of functions $f$ and $h$ is the map $f \mapsto h$ one-to-one, and which functions $h(q)$ can be represented by the integral (26).

As to the second question it follows from (26) that any such function $h$ belongs to the space $\mathcal{H}(\Gamma) = (C^2 \cap C_0)(\Gamma)$ of all twice continuously differentiable functions vanishing on the boundary $\partial\Gamma$. Therefore the best possible solution of the problem consists in a description of spaces $\mathcal{F}(D) \in C(\overline{D})$ such that the map

$$\mathcal{A} : \mathcal{F}(D) \to \mathcal{H}(\Gamma)$$

is one-to-one. One of the possible classes of such spaces is introduced below.

**Definition** Given a smooth nonsingular vector field $\boldsymbol{l}$ in $B$ we denote by $C_{\langle \boldsymbol{l} \rangle}(D)$ the set of all functions in $C(\overline{D})$ which remain constant along any trajectory of the field $\boldsymbol{l}$.

In this paper we consider the only case of vector fields

$$\boldsymbol{l} = r_1 \boldsymbol{l}_1 + r_2 \boldsymbol{l}_2, \quad r_1 r_2 > 0$$

with constant coefficients $r_1$ and $r_2$. But in this situation we obtain an exhaustive solution of the problem in question by formulating the necessary and sufficient condition for the curve $\Gamma$ to ensure the above mentioned property of the operator

$$\mathcal{A} : C_{\langle \boldsymbol{l} \rangle}(D) \to \mathcal{H}(\Gamma).$$

Much more general vector fields $\boldsymbol{l}$ are considered in the author's papers [5] and [9]. To formulate the abovementioned condition we introduce one more projection $\pi_{\boldsymbol{l}}$ : $\overline{D} \to \Gamma$ along trajectories of the vector field $\boldsymbol{l}$. Let

$$\zeta_1 = \pi_{\boldsymbol{l}} \circ \pi_1 \qquad \text{and} \qquad \zeta_2 = \pi_{\boldsymbol{l}} \circ \pi_2$$

be two maps in $\Gamma$. Denote by $\Phi_\zeta$ a noncommutative semigroup of maps in $\Gamma$, generated by $\zeta_1$ and $\zeta_2$. The analogy of $\Phi_\zeta$ with semigroup $\Phi_\delta$ considered in Subsec. 2.1 is obvious. As in the case of $\Phi_\delta$ we define an orbit in $\Gamma$ as a sequence of points $(q_1, \dots, q_n, \dots)$ in $\Gamma$ such that

$$q_{k+1} = \zeta_{j_k} q_k, \quad k = 1, 2, \dots$$

and all $q_{j_k}$ are equal $\zeta_1$ or $\zeta_2$. As above the *critical* sets

$$\mathcal{T}_j = \{q \in \Gamma \mid \boldsymbol{l}_j(q) \in T_q(\Gamma)\}, \quad j = 1, 2,$$

and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ are introduced and condition (16) is supposed to be fulfilled. Repeating word for word what has been said in Subsec. 2.1 we define *cyclic, critical* and $\mathcal{T}$ *-proper orbits* corresponding to the semigroup $\Phi_\zeta$. Finally, we introduce the set $\mathfrak{N}_\zeta$ of all $\mathcal{T}$ - proper critical cycling orbits generated by $\Phi_\zeta$ (see Fig.5 - Fig.7). Now everything is ready for formulation of the main result of this subsection.

#### 3.1.1.2 Main Result and an Outline of the Proof

**Theorem 6** *If all above hypotheses concerning to domain $D$, curve $\Gamma$ and vector fields $\mathbf{l}_1, \mathbf{l}_2$ and $\boldsymbol{l}$ are fulfilled then given an arbitrary function $h \in \mathcal{H}(\Gamma)$ there is a unique solution $f \in C_{\langle \mathbf{l} \rangle}(D)$ of equation (26) if and only if $\mathfrak{N}_\zeta = \emptyset$. The inverse operator $h \mapsto f$ is continuous: $\mathcal{H}(\Gamma) \to C_{\langle \mathbf{l} \rangle}(D)$.*

Let us outline the proof of this theorem. First of all choosing a special coordinate system $(x_1, x_2)$ in disk $B$ we reduce integral equation (26) to the form

$$\int_0^{\alpha_1(z)} \int_0^{\alpha_2(z)} f(\omega(x)) dx_1 dx_2 = h(z), \qquad z \in I_z \tag{27}$$

with $f$ an unknown continuous function on the interval $I_t = \{t \mid -r_1 \le t \le r_2\}$. Here $I_z = \{z \mid -1 \le z \le 1\}$, $\alpha(z) = (\alpha_1(z), \alpha_2(z))$ and equalities $x_1 = \alpha_1(z)$, $x_2 = \alpha_2(z)$, $z \in I_z$ define a parametric representation of the curve $\Gamma$. It is clear that

$$\alpha_1'(z) \ge 0, \quad \alpha_2'(z) \le 0, \quad \text{and} \quad (\alpha_1' - \alpha_2')(z) > 0, \quad z \in I_z.$$

As to $\omega$, this is a function

$$\omega(x) = r_2 x_1 - r_1 x_2$$

which does not change its values along trajectories of the vector field $\boldsymbol{l}$. Denote $\omega_1 = \omega(x_1, 0)$, $\omega_2 = \omega(0, x_2)$ and let

$$\sigma = \omega_\Gamma \circ \alpha \, : \, I_z \to I_t,$$

where $\omega_\Gamma$ is a restriction of $\omega$ to $\Gamma$. By the above the function $\sigma$ is invertible. Introducing a new unknown function

$$F(t) = -\int_0^t f(s)(t-s) ds / r_1 r_2, \quad t \in I_t,$$

we arrive at the following functional equation

$$F(\omega \circ \alpha) - F(\omega_1 \circ \alpha) - F(\omega_2 \circ \alpha) = h.$$

By setting

$$\delta_1 = \omega_1 \circ \alpha \circ \sigma^{-1}, \qquad \delta_2 = \omega_2 \circ \alpha \circ \sigma^{-1}$$

we rewrite this equation in the final form

$$F - F \circ \delta_1 - F \circ \delta_2 = h \circ \sigma \qquad \text{on } \; I_t. \tag{28}$$

It is remarkable that the functions $\delta_1$ and $\delta_2$ which are connected one-to-one with a *geometric* problem (26) form a $\mathcal{P}$ - configuration. Moreover as $\delta_1(t) + \delta_2(t) = t$ for all $t \in I_t$ equation (28) is nothing but a Cauchy type functional equation on $I_t$. As we know from Sec. 2.1 the solvability of this equation depends on whether the corresponding set $\mathfrak{N}_\delta$ is empty or not. Therefore to prove Theorem 2 it only remains to show that the sets $\mathfrak{N}_\delta$ and $\mathfrak{N}_\zeta$ are empty or nonempty simultaneously. The corresponding proof is given in the author's paper [8]. This completes the proof of Theorem 3.

**Remark** We would like to emphasize that by Theorem 3 *each Cauchy type functional equation* (4) *with the functions* $\delta_1$ *and* $\delta_2$ *forming a* $\mathcal{P}$ *- configuration is equivalent to some problem in Integral Geometry of the described type.*

To illustrate this result consider domains $D$ represented by figures 5,6 and 7.

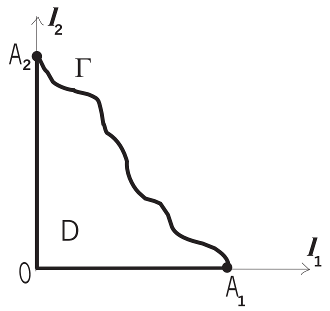

Figure 5

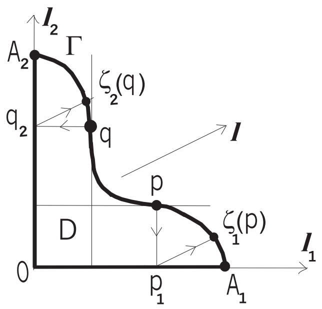

Figure 6

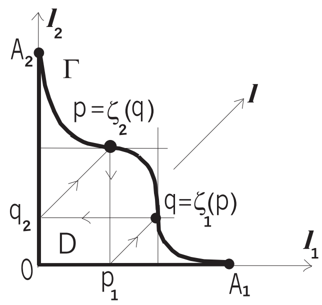

Figure 7

On these figures $p \in \mathcal{T}_1,\ q \in \mathcal{T}_2$ are the only points from $\mathcal{T}$. The curve $\Gamma$ on Fig.5 has no points in $\mathcal{T}$. It follows that $\mathfrak{N}_\zeta = \emptyset$, and by Theorem 3 problem (26) is uniquely solvable for all $h \in \mathcal{H}(\Gamma)$. On Fig.6 the orbits $(p, \zeta_1(p))$ and $(q, \zeta_2(q))$ are the only $\mathcal{T}$ - proper orbits corresponding to the semigroup $\Phi_\zeta$ with a beginning at points $p$ and $q$, respectively. As both orbits are not critical, we have $\mathfrak{N}_\zeta = \emptyset$ in the case, and problem (26) is also uniquely solvable for all $h \in \mathcal{H}(\Gamma)$. On Fig.7, as is easily seen, the orbit $(p, q, p)$ is a critical $\mathcal{T}$ - proper cycle (as well as the orbit $(q, p, q)$). Therefore by Theorem 3 the operator $\mathcal{A} : C_{\langle \boldsymbol{l} \rangle}(D) \to \mathcal{H}(\Gamma)$ is not one-to-one.

In the author's papers [5],[9] some necessary conditions for right hand sides $h$ are given to provide a solvability of equation (26). It is interesting that the number of these conditions coincides with the number of elements in $\mathfrak{N}_\zeta$.

### 3.1.2 The Case of a $\mathcal{Z}$ - configuration

Let $\boldsymbol{l}, \boldsymbol{l}_1$ and $\boldsymbol{l}_2$ be a triple of smooth nonsingular vector fields in a disk $B \subset \mathbb{R}^2$ such that $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are transversal and

$$\boldsymbol{l} = b_1 \boldsymbol{l}_1 + b_2 \boldsymbol{l}_2, \quad b_1 b_2 \geq 0,$$

with $b_1$ and $b_2$ some functions on $\Gamma$. Let $D = OA_1O'A_2$ be a (two-dimensional) curvilinear parallelogram in $B$ whose sides $OA_1$, $O'A_2$ and $OA_2$, $O'A_1$ are trajectories of the vector fields $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, respectively, and the "diagonal" $\Gamma = OO'$ is a trajectory of the vector field $\boldsymbol{l}$. Given an arbitrary point $q \in \Gamma$ denote by $D_q$ a curvilinear parallelogram $qq_1Oq_2$ with $q_j = \pi_j q$, $j = 1, 2$, the same projections of the point $q$ as in Subsec. 3.1.1.1 (see Fig. 4). In this Subsection we briefly discuss a solvability of the integral equation

$$\int_{D_q} f d\sigma = h(q), \qquad q \in \Gamma, \tag{29}$$

that is nothing other than equation (3), if $d\sigma$, $h$ and $f$ in (29) have the same meaning as in (3). All the discussions around the formulation of problem (3) completely relate to problem (29). The space $\mathcal{H}_0(\Gamma)$ of the right hand sides in (29) has to include all functions $h \in C_0^2(\Gamma)$ which are twice differentiable in $\Gamma$ and vanish at the point $O$ together with their first derivative $\partial_{\boldsymbol{l}} h$. This leads to the following result.

**Theorem 7** *Let*

$$\mathbf{r} = r_1 \boldsymbol{l}_1 + r_2 \boldsymbol{l}_2, \qquad r_1 r_2 < 0$$

*be a vector field with $r_1$ and $r_2$ some constants. Then for an arbitrary function $h \in \mathcal{H}_0(\Gamma)$ there is a unique solution $f \in C_{\langle \mathbf{r} \rangle}(D)$ of equation (29). The inverse operator $h \mapsto f$ is continuous: $\mathcal{H}_0(\Gamma) \to C_{\langle \mathbf{r} \rangle}(D)$.*

Repeating word for word what has been said in Subsec. 3.1.1.2 we reduce equation (29) in an equivalent manner to a Cauchy type functional equation

$$F - F \circ \delta_1 - F \circ \delta_2 = H \quad \text{on} \quad I_t$$

(see (28)). But in contrast to (28) this time the functions $\delta_1$ and $\delta_2$ form a $Z$ - configuration. The needed result follows, therefore, immediately from Theorem 3.

## 3.2 First Boundary Problem for Hyperbolic Differential Equations and Cauchy Type Functional Equations

To begin with we mention that in the framework of the classical theory of Partial Differential Equations boundary problems for hyperbolic equations (all needed definitions are given below) are usually studied in domains closely connected with the equation under consideration. What was typical to these equations (and more generally, to any evolution equation) is that if a domain is bounded, then a part of the boundary is usually free of a priori information about an unknown solution. In the presence of characteristics boundary conditions on the *whole* boundary of a *bounded* domain usually treated as prohibited. Nevertheless, it turned out (see [4] -[7]) that for a wide class of hyperbolic differential equations this taboo can be lifted. In other words any equation of such kind defines (a wide class of ) bounded domains $D \subset \mathbb{R}^2$ such that the problem in which boundary values of an unknown function are given on the whole boundary $\partial D$ is well posed. As will be shown later on this new boundary problem turns out to be *equivalent* to a Cauchy type functional equation. Maybe this explains why such a problem has never been investigated.

### 3.2.1 Statement of the Problem

For the sake of brevity we restrict ourselves to a homogeneous differential operator with constant coefficients.

In the $(x, y)$-plane $\mathbb{R}^2$ we consider an arbitrary homogeneous $x$-strictly hyperbolic operator $P(\partial_x, \partial_y)$ of the 3rd order. The $x$-strictly hyperbolicity means that the characteristic polynomial $P(\tau, \lambda)$ has, for any $\lambda \neq 0$, three distinct real roots in $\tau$. It follows that the operator $P = P(\partial_x, \partial_y)$ can be uniquely represented in the form

$$P(\partial_x, \partial_y) = a(\partial_x - a_1\partial_y)(\partial_x - a_2\partial_y)(\partial_x - a_3\partial_y) \tag{30}$$

with some constants $a, a_1, a_2, a_3$, where $a_j \neq a_k$ for $j \neq k$. The characteristics of the operator $P$ are straight lines

$$y + a_1 x = \text{const}, \quad y + a_2 x = \text{const}, \quad y + a_3 x = \text{const}.$$

Let $\boldsymbol{l}_1, \boldsymbol{l}_2$ and $\boldsymbol{l}_3$ be vector fields in $\mathbb{R}^2$ parallel to these lines, respectively. Denote by $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_6$ characteristic rays beginning at some point 0. Choose any triple of neighboring rays $\mathcal{R}_j$, say, $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$. Let $\mathcal{R}_3$ be the ray lying between $\mathcal{R}_1$ and $\mathcal{R}_2$. Consider a curvilinear triangle $D = OA_1A_2$ with sides $OA_1 \subset \mathcal{R}_1$, $OA_2 \subset \mathcal{R}_2$. As to the side $\Gamma = A_1A_2$ it is assumed to be an arbitrary smooth curve without singularities which is transversal to $OA_1$ and $OA_2$ (cf. Subsec 3.1.1.1). We suppose the closure $\overline{D}$ to satisfy the hypotheses $1^\circ$ and $2^\circ$ of Subsec. 3.1.1.1. It follows in particular that $\Gamma$ is transversal to the vector field $\boldsymbol{l}_3$.

The *First Boundary Problem* for the above operator $P(\partial_x, \partial_y)$ and domain $D$ is as follows.

*Given functions $F \in C(\overline{D})$ and $h \in C(\partial D)$ find a solution of the boundary problem*

$$Pu = F \quad \textit{in } D, \qquad u = h \quad \textit{on } \partial D. \tag{31}$$

We call a function $u$ in $\overline{D}$ a *generalized solution* of the problem (31) if $u \in C^2(D)$, $u = h$ on $\partial D$, and for all functions $\varphi \in C_0^\infty(D)$

$$\int_{\mathbb{R}^2} u\, {}^tP\varphi dxdy = \int_{\mathbb{R}^2} F\varphi dxdy,$$

where ${}^tP$ is the formally adjoint differential operator.

### 3.2.2 The Formulation of the Result and a Sketch of the Proof

To formulate the main result concerning a solvability of problem (31) let us consider the semigroup $\Phi_\zeta$ of maps in $\Gamma$ introduced in Subsec. 3.1.1.1 with $\boldsymbol{l} = \boldsymbol{l}_3$. The critical sets $\mathcal{T}_j$ in the theory of Partial Differential Equations are usually called *characteristic* sets. Similarly, we introduce $J$-orbits and accompanying notions of cyclic, characteristic and $\mathcal{T}$ - proper orbits. Finally we introduce the set $\mathfrak{N}_\zeta$ whose elements are all the $\mathcal{T}_j$ - proper cyclic orbits, consisting of only characteristic points in $\Gamma$.

Denote by $C^k(\partial D)$ the space of continuous functions on $\partial D$ whose restrictions to all sides of the triangle $D$ are $k$ times continuously differentiable functions. The main result concerning the problem (31) is as follows.

**Theorem 8 (see [10])** *Assume that the characteristic sets $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy condition* (16)*. Then for any functions $F \in C(\overline{D})$ and $h \in C^2(\partial D)$ there exists a unique* *generalized solution* $u(x,y)$ *of the problem* (31) *if and only if the set* $\mathfrak{N}_\zeta$ *is empty. The inverse operator* $(F,h)\mapsto u$ *is continuous:* $C(\overline{D})\times C^2(\partial\Omega)\to C^2(\overline{D})$. *If* $F\in C^k(D)$ *and* $h\in C^{k+2}(\partial D)$, $k\geq 1$ *is an integer, then* $u\in C^{k+2}(D)$ *is a classical solution of the problem in question.*

*Proof*: We restrict ourselves to the proof of the existence of a unique generalized solution to problem (31) with $F=0$. Let us write down the operator $P$ in the form (30). It is obvious that there exists a linear transformation in $\mathbb{R}^2$ reducing the problem under consideration to the problem

$$(r_1\partial_x+r_2\partial_y)\partial_x\partial_y u=0 \quad \text{in } D, \qquad u=h \quad \text{on } \partial D, \tag{32}$$

where $r_1r_2>0$. (For convenience we preserve the previous notations for the domain and functions). Here $D$ is a domain in $\mathbb{R}^2$ whose boundary $\partial D$ consists of three parts $\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where

$$\Gamma_1=\{(x,y)\mid y=0,\ \ 0\leq x\leq 1\}, \qquad \Gamma_2=\{(x,y)\mid x=0,\ \ 0\leq y\leq 1\},$$
$$\Gamma_3=\{(x,y)\mid x=\alpha_1(t), y=\alpha_2(t);\ \ 0\leq t\leq 1\},$$

and

$$\alpha_1(0)=0,\ \ \alpha_1(1)=1; \qquad \alpha_2(0)=1,\ \ \alpha_2(1)=0.$$

Note that the functions $\alpha_1(t)$ and $-\alpha_2(t)$ form a $\mathcal{P}$ - configuration.

Let

$$h=h_1(x) \quad \text{on } \Gamma_1, \qquad h=h_2(y) \quad \text{on } \Gamma_2, \quad \text{and} \quad h=h_3(x,y) \quad \text{on } \Gamma_3.$$

The continuity of the function $h$ leads to the natural compatibility conditions

$$h_1(0)=h_2(0), \qquad h_1(1)=h_3(1,0), \qquad h_2(1)=h_3(0,1). \tag{33}$$

Due to assumptions about the domain $D$ an arbitrary generalized solution $u$ of the equation in (32), satisfying boundary condition only on $\Gamma_1\cup\Gamma_2$, can be represented in the form

$$u(x,y)=\int_0^x\Big(\int_0^y F(r_2s-r_1t)dt\Big)ds+h_1(x)+h_2(y)-h_1(0), \quad 0\leq x,y\leq 1. \tag{34}$$

As to the function $F$, this is an arbitrary continuous function on the interval $I=(-r_2,r_1)$. The necessity of satisfying the boundary condition $u=h_3$ on $\Gamma_3$ leads to the following integral equation for an unknown function $F\in C(I)$:

$$\int_0^{\alpha_1(t)}\Big(\int_0^{\alpha_2(t)} F(r_2x-r_1y)dy\Big)dx=H(t), \quad 0\leq t\leq 1. \tag{35}$$

Here

$$H(t)=-h_1\Big(\alpha_1(t)\Big)-h_2\Big(\alpha_2(t)\Big)+h_3\Big(\alpha_1(t),\alpha_2(t)\Big)+h_1(0)$$

is a given function. What is important is that the function $H(t)$, generated by an arbitrary continuous and twice piecewise differentiable function $h$ in (32), belongs to the space $\mathcal{H}(I)=C^2\cap C_0(I)$ (see Subsec. 3.1.1.1). This follows from the compatibility conditions (33). Conversely, the function $u(x,y)$ which is defined by (34) with $F$ a solution of equation (35) solves the problem (32).

Thus the problem (32) turns out to be equivalent to the equation (35) which is nothing but the equation (27). The existence of a unique solution to the problem (32) provided that $\mathfrak{N}_\zeta=\emptyset$ follows immediately from Theorem 1.

# 4 Functional Equations Determining Polynomials

This section is devoted to a class of functional equations whose solutions are only polynomials. This class contains the Cauchy equation (1) and for many reasons can be considered as a natural generalization of this equation.

**Definition** Given a function $F$ in $\mathbb{R}$ and an integer $n \geq 2$ we denote by $\mathcal{P}_n$ the operator

$$F \to (\mathcal{P}_n F)(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

of the form

$$\mathcal{P}_n F(x) = F\Big(\sum_{j=1}^{n} x_j\Big) - \sum_{k=1}^{n} F\Big(\sum_{j \neq k} x_j\Big) + \cdots + (-1)^{n-1} \sum_{k=1}^{n} F(x_k).$$

With this notation equation (1) has a form

$$\mathcal{P}_2 F(x) = 0, \quad x \in \mathbb{R}^2,$$

and as we know all solutions $F$ of this equation are polynomials of degree 1 vanishing at the origin. The first assertion of this section treats the general case of the equation $\mathcal{P}_n F = 0$.

**Theorem 9** *If $n \geq 2$, then any continuous solution of the equation*

$$\mathcal{P}_n F = 0 \tag{37}$$

*is a polynomial of degree $n-1$ vanishing at the origin.*

*Proof*: We restrict ourselves to sufficiently smooth solutions $F$. Substituting $x = 0$ in (37) results in $F(0) = 0$. Denote $F_k = F^{(k)}$, $k = 1, \ldots, n$, and let

$$x_{(k)} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad \partial_k = \partial/\partial x_k, \quad \tau_k F(x) = F(x_{(k)}).$$

Then, as can be easily verified, we have

$$\tau_k \partial_k \mathcal{P}_n F = \mathcal{P}_{n-1}\Big(F_1(x_{(k)}) - F_1(0)\Big), \quad k = 1, \ldots, n. \tag{38}$$

Using successively this relation for $k = n, n-1, \ldots, 3$ we arrive at the equality

$$\mathcal{P}_2\Big(F_{n-2} - F_{n-2}(0)\Big)(x_1, x_2) = 0.$$

It follows by the Cauchy result (see Sec. 1) that

$$F_{n-2}(z) = F_{n-2}(0) + F_{n-1}(0)z.$$

Integrating this equality $n-2$ times leads to the desired result.

The following result considerably sharpens the previous theorem. It shows that as in the case of the original Cauchy equation (1) problem (37) is overdetermined (see Subsec. 2.1). In order to determine a polynomial of degree $(n-1)$ the equality $(\mathcal{P}_n F)(x) = 0$ has not to be valid for *all* points in $\mathbb{R}^n$. It suffices that it is valid at points of some hypersurface $\Gamma$.

**Theorem 10** *Let* $z = z(x_{(n)})$ *be a smooth function with* $z(0) = 0$ *and*

$$\partial_j z(x_{(n)}) > 0, \quad \textit{if} \quad x_{(n)} \geq 0, \quad j = 1, \dots, n-1.$$

*Then any solution* $F \in C^{n-2}(\mathbb{R}_+)$ *of the equation*

$$\mathcal{P}_n F\Big(x_{(n)}, z(x_{(n)})\Big) = 0$$

*is a polynomial of degree* $n-1$*, vanishing at the origin.*

The proof of this theorem is reduced to the case $n = 2$ with the help of some variant of equality (38), and making use of Theorem 3 completes the proof of Theorem 10.

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