Anisotropic Edge Problems *

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April 11, 2002

AMS subject classification: primary: 35S15, 47G05, 58G15; secondary: 35H05.

Abstract

We investigate elliptic pseudodifferential operators which degenerate in an anisotropic way on a submanifold of arbitrary codimension. To find Fredholm problems for such operators we adjoint to them boundary and coboundary conditions on the submanifold. The algebra obtained this way is a far reaching generalisation of Boutet de Monvel's algebra of boundary value problems with transmission property. We construct left and right regularisers and prove theorems on hypoellipticity and local solvability.

Contents

Introduction

In the analysis of boundary value problems on domains with singularities the methods apply which are typical for elliptic theory. These are freezing of coefficients, parametrix constructions by the Fourier transform, and so on. However, the investigation of boundary value problems on smooth domains relies on the study of ordinary differential equations with constant coefficients on the semi-axis $r > 0$, while in the analysis on singular domains these ordinary differential equations have variable coefficients. The ordinary differential equations under study depend on additional large parameters, and the type of the equations may change for $r = 0$. Hence singular boundary value problems have actually double degeneration, for the parameters tend to infinity and the variable r tends to zero.

Elliptic theory on singular spaces has many common features with the theory of equations having small parameter by the highest derivative, cf. Vishik and Lyusternik [VL57, VL60], Denk and Volevich [DV99, DV00], et al. In much the same way as for equations with a small parameter by the highest derivative there is a boundary layer zone and a zone where the solution depends regularly on the parameter, any degenerate boundary value problem has a zone close to the singular surface $r = 0$, and a complementary zone where the equation has quite different properties.

The boundary itself, even if being smooth, is a singular set for the analysis of pseudodifferential operators. Only those having the transmission property with respect to the boundary still behave properly in the classical Sobolev-Slobodetskii spaces. Hence elliptic regularity and the local solvability near the boundary can no longer rely on these spaces. Enriching the function tools enables one in turn to enlarge Boutet de Monvel's algebra of pseudodifferential operators with transmission property, cf. [BdM71]. Let us explain this in more details.

Suppose $\mathcal{D}$ is a domain with a C^∞ boundary in $\mathbb{R}^N$. By choosing suitable coordinates in $\mathbb{R}^N$ we can identify $\mathcal{D}$ locally with the half-space $\mathbb{R}^N_+$. This rectifies $\partial\mathcal{D}$ locally to the hyperplane $\mathbb{R}^{N-1} = \{v \in \mathbb{R}^N : \ v_1 = 0\}$. We split coordinates $v = (v_1, \ldots, v_N)$ into $v = (x, y)$ where $x = v_1$ is the normal coordinate and $y = (v_2, \ldots, v_N)$ are local coordinates on $\partial\mathcal{D}$. Given any differential operator A of order m with smooth coefficients near the closure of $\mathcal{D}$, we can write

$$\begin{aligned} A &= \sum_{\alpha+|\beta|\leq m} A_{\alpha\beta}(x,y)\, D_x^\alpha D_y^\beta \\ &= \frac{1}{x^{pm}} \sum_{\alpha+|\beta|\leq m} a_{\alpha\beta}(x,y)\, x^{p(\alpha+|\beta|)} D_x^\alpha D_y^\beta \end{aligned}$$

where $p \geq 0$ is any real number, and the coefficients $a_{\alpha\beta} = x^{p(m-\alpha-|\beta|)} A_{\alpha\beta}$ are smooth for $x > 0$ and continuous up to $x = 0$. Since $[x^p D_x, x^p] = (p/\imath) x^{2p-1}$ the latter differential operators are typical on manifolds with cuspidal edges provided $p \geq 1$, cf. [RST98].

To avoid singular coefficients at $x = 0$ we multiply A by an inessential scalar factor x^{s_0}, with $s^0 > 0$ large enough. The new operator

$$x^{s_0} A = \sum_{\alpha + |\beta| \leq m} a_{\alpha\beta}(x, y)\, x^{s_0 - p(m-\alpha-|\beta|)} D_x^\alpha D_y^\beta$$

has the advantage of being much more insusceptible to the singularities of $\partial\mathcal{D}$, which is due to positive powers of x by the derivatives.

More generally, let the variables $v \in \mathbb{R}^N$ be separated into two groups $v = (x, y)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^q$ and $n + q = N$. Let moreover there be given an integer $m > 0$ and a rational number $\delta > 0$ with the property that δm is an integer.

Denote by $\mathcal{I}$ the set of all triples (α, β, γ) of multi-indices of dimension n, q and n, respectively, with nonnegative components, such that

$$\begin{array}{rcccl} |\alpha| + |\beta| & \leq & m, & & \\ \delta|\beta| - (m - |\alpha| - |\beta|) & \leq & |\gamma| & \leq & \delta m. \end{array} \tag{0.1}$$

Let us observe that $|\gamma| = \delta m - p(m - |\alpha| - |\beta|)$ satisfies the second line of (0.1) for all multi-indices $(\alpha, \beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^q$ with $|\alpha| + |\beta| \leq m$, provided that $p - 1 \leq \delta$.

Consider the pseudodifferential operator

$$A(v, D) = \sum_{(\alpha,\beta,\gamma) \in \mathcal{I}} a_{\alpha\beta\gamma}(v, D) x^\gamma D_x^\alpha D_y^\beta \tag{0.2}$$

where $a_{\alpha\beta\gamma} \in \Psi^0(\mathbb{R}^N)$ are classical pseudodifferential operators of zero order on all of $\mathbb{R}^N$. Their symbols $a_{\alpha\beta\gamma}(v, \theta)$ are C^∞ functions on $\mathbb{R}^N \times \mathbb{R}^N$ bearing asymptotic expansions in homogeneous functions of orders $0, -1, -2, \ldots$ in $\theta \neq 0$. We will assume that $a_{\alpha 0 \gamma}$ is independent of D for $|\alpha| = m$, i.e., $a_{\alpha 0 \gamma} = a_{\alpha 0 \gamma}(v)$ is a C^∞ function for $|\alpha| = m$.

The dual variables for $v = (x, y)$ also separate naturally into two groups $\theta = (\xi, \eta)$, where $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^q$. For a fixed point $y^0 \in \mathbb{R}^q$, we construct an auxiliary operator

$$\sigma_{\mathrm{edge}}(A)(y^0, \eta) = \sum_{(\alpha,\beta,\gamma) \in \mathcal{I}_0} a_{\alpha\beta\gamma,0}(0, y^0; 0, \eta) x^\gamma D_x^\alpha \eta^\beta \tag{0.3}$$

where $\mathcal{I}_0 = \{(\alpha, \beta, \gamma) \in \mathcal{I} : \ |\gamma| = \delta|\beta| - (m - |\alpha| - |\beta|)\}$, and $a_{\alpha\beta\gamma,0}(v, \theta)$ is the homogeneous component of $a_{\alpha\beta\gamma}(v, \theta)$ of order 0 in θ. This is a family of differential operators with polynomial coefficients on $\mathbb{R}^n$, parametrised by $\eta \in \mathbb{R}^q$.

We shall make a standing assumption on the operators A under consideration. Namely, we require that (0.3) be elliptic with parameter η if $x \neq 0$, i.e.,

$$\begin{aligned} \sigma^m \left(\sigma_{\mathrm{edge}}(A)\right)(x, \xi, \eta) & := \sum_{\substack{(\alpha,\beta,\gamma)\in\mathcal{I}_0 \\ |\alpha|+|\beta|=m}} a_{\alpha\beta\gamma}(0, y^0; 0, \eta) x^\gamma D_x^\alpha \eta^\beta \\ & \neq 0 \end{aligned} \tag{0.4}$$

for all $(\xi, \eta) \in \mathbb{R}^N$ with $(\xi, \eta) \neq 0$, cf. [AV64].

Operators of the form (0.2) for which (0.4) holds were studied by Grushin in [Gru70b], where sufficient conditions for them to be hypoelliptic were found. In [Gru71] these operators are treated under boundary and coboundary conditions.

In the present paper we develop these results in the context of manifolds with smooth edges.

In Section 1 the basic definition related to families of spaces depending on an index $-\infty < s < +\infty$ are given. We then discuss weighted Sobolev spaces to be used, and we give also a brief exposition of the properties of boundary and coboundary operators. At the end of this section we recall certain theorems on differential operators with polynomial coefficients on $\mathbb{R}^n$ proved in [Gru70b, Gru71].

Section 2 presents local properties of the operator $\mathcal{A}$ which is obtained by adjoining boundary and coboundary conditions to $A(v, D)$. Under certain further assumptions, left and right regularisers are constructed and theorems on the hypoellipticity and local solvability of $\mathcal{A}$ are proved. Thus a theory is constructed which is analogous to the local theory of elliptic boundary value problems. However, while conditions of ellipticity of boundary value problems are related to certain simple properties of ordinary differential equations with constant coefficients on the half-line, the study of operators of the form (0.2) is based, in turn, on the analysis of properties of the partial differential operator $\sigma_{\mathrm{edge}}(A)(y^0, \eta)$ in $\mathbb{R}^n$.

As shown in [Gru70b], even very simple examples of such operators reduce to the problem of determining eigenvalues and eingenfunctions. This is also seen from the examples of Section 3.

In the first example we study the operator with symbol $ax^p\sqrt{|\xi|^2 + |\eta|^2} + \imath\xi$, $n = 1$, which arises in the study of the problem with oblique derivative for the Laplace equation. For this example the theorems proved in Section 3 overlaps with certain results of Hörmander [Hoe66], Maz'ya and Paneyakh [MP74], and Eskin [Esk70] obtained by other methods.

In the second example, following Grushin [Gru71] we consider the elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} + \imath\lambda \frac{\partial u}{\partial y} = f(x, y) \tag{0.5}$$

in $\mathbb{R}^2$, which degenerates at the y-axis. It is proved that this equation is hypoelliptic and is locally solvable only for $\lambda \neq \pm(2k+1)$ where $k = 0, 1, 2, \ldots$. If $\lambda = \pm(2k+1)$ then one can impose on the equation (0.5) boundary and coboundary conditions, such that the corresponding problem $\mathcal{A}$ is hypoelliptic and locally solvable. The reason that the values $\lambda = \pm(2k+1)$ are of exceptional character is that these are eigenvalues of the ordinary differential operator $-d^2/dx^2 + x^2$.

Note that for the hypoellipticity of operators with polynomial coefficients of the form

$$A(v, D) = \sum_{(\alpha,\beta,\gamma)\in\mathcal{I}_0} a_{\alpha\beta\gamma} x^\gamma D_x^\alpha D_y^\beta \tag{0.6}$$

satisfying (0.3) necessary and sufficient conditions were formulated in [Gru70b]. They read that for $\eta \neq 0$ the operator $\sigma_{\mathrm{edge}}(A)(\eta)$ in $L^2(\mathbb{R}^n)$ must not have zero eigenvalues. Obviously, even for operators of this particular form there is no algebraic criterion for hypoellipticity in terms of coefficients, but the global spectral condition.

At the end of Section 3 we study elliptic operators on a smooth compact manifold M of dimension N which degenerate on a submanifold $\mathcal{S}$ of codimension n. Assume that in a neighbourhood of any point $y^0 \in \mathcal{S}$ the operator A has the form (0.2), $(x, y) \in \mathbb{R}^n \times \mathbb{R}^q$ being local coordinates. The operator is framed by a collection of boundary and coboundary conditions on $\mathcal{S}$. A condition is shown under which the operator $\mathcal{A}$ is Fredholm in a special weighted space of Sobolev type. For $n = 1$ this result specifies to the Fredholm property of boundary value problems for degenerate elliptic equations.

In Section 4 we prove that for a certain class of operators of the form (0.2) the problem of determining the hypoellipticity reduces to the same problem for some auxiliary pseudodifferential operator $B(y, D_y)$ in $\mathbb{R}^q$. Thus, it is shown, e.g., that the operator (0.5) for $\lambda = -1$ becomes hypoelliptic if we adjoin $ay^2(\partial/\partial y)$, where $a \neq 0$.

At the end of Section 4 we discuss the problem of the analyticity of solutions to equations of the form

$$\sum_{(\alpha,\beta,\gamma)\in\mathcal{I}} a_{\alpha\beta\gamma}(x, y) x^\gamma D_x^\alpha D_y^\beta u(x, y) = f(x, y)$$

with analytic coefficients and right-hand side $f(x, y)$. If the ellipticity condition (0.4) holds then in some neighbourhood U of $(0, y^0)$ the equation is elliptic if $x \neq 0$. Hence a solution $u(x, y)$ for $(x, y) \in U$ and $x \neq 0$ is analytic by a theorem of Petrovskii, cf. [Pet39]. However, the ellipticity of an operator is not necessary for the analyticity of solutions. Thus, Mizohata proved in [Miz61] that this property holds for the equation

$$\frac{\partial u}{\partial x} + \imath x^{2p} \frac{\partial u}{\partial y} = f(x, y)$$

for any integer p. We prove that if the condition of [Gru70b] holds for our equation then any its solution is analytic in some neighbourhood of the point $(0, y^0)$.

1 Auxiliary propositions

1.1 Families of function spaces

In the space $\mathbb{R}^N$ let us fix a positive measurable function $w(\theta)$ with the property that

$$C_1 (1 + |\theta|)^{s_1} \le w(\theta) \le C_2 (1 + |\theta|)^{s_2}$$

for all $\theta \in \mathbb{R}^N$, where s_1, s_2 and C_1, C_2 are some constants. Let H^w denote the Hilbert spaces of generalised functions $u \in \mathcal{S}'(\mathbb{R}^N)$ such that the Fourier transform $\hat{u}(\theta)$ is an ordinary function on $\mathbb{R}^N$ and it satisfies

$$\begin{array}{rcl} \|u\|_{H^w} & := & \left(\displaystyle\int |w(\theta)\hat{u}(\theta)|^2 d\theta \right)^{\frac{1}{2}} \\ & < & \infty. \end{array}$$

If $w(\theta) = (1 + |\theta|)^s$ then H^w is denoted H^s and the norm $\|u\|_{H^w}$ is denoted $\|u\|_s$.

For each real number s let there be given a Banach space $\mathcal{U}^s$. We will say that the $\mathcal{U}^s$ form a family, if for any $s_1 < s_2$ the space $\mathcal{U}^{s_2}$ embeds continuously in $\mathcal{U}^{s_1}$ and is a dense subset there. The embedding is denoted $\imath$.

The union of all the $\mathcal{U}^s$ is denoted $\mathcal{U}^{-\infty}$, and their intersection is denoted $\mathcal{U}^{\infty}$.

Let there be two families $\mathcal{U}^s$ and $\mathcal{F}^s$. We will say that a linear operator A from $\mathcal{U}^{-\infty}$ into $\mathcal{F}^{-\infty}$ has order m if $A \in \mathcal{L}(\mathcal{U}^s, \mathcal{F}^{s-m})$ for all s.

In the sequel we will restrict our discussion only to families which can be realised as subspaces of $\mathcal{D}'(U)$ for some domain $U \subset \mathbb{R}^N$. In such a case, for any subdomain $O \subset U$, one can form new families

$$\begin{array}{rcl} \overset{\circ}{\mathcal{U}}{}^s(O) & = & \{u \in \mathcal{U}^s : \ \operatorname{supp} u \subset \bar{O}\}, \\ \mathcal{U}^s(O) & = & \{r_O u : \ u \in \mathcal{U}^s\}, \end{array}$$

where r_O is the operation of restriction of functions to O.

If the spaces $\mathcal{U}^s$ are semilocal, then by a familiar procedure one can form the family $\mathcal{U}^s_{\mathrm{loc}}(O)$.

We will say that A is pseudolocal if from $u \in \mathcal{U}^{-\infty}$ and $u = 0$ in O it follows that $Au \in \mathcal{F}^{\infty}(O)$ for any domain $O \subset U$. Clearly, we have

Lemma 1.1 *If $A : \ \mathcal{U}^{-\infty} \to \mathcal{F}^{-\infty}$ is pseudolocal of order m then for any s it follows that*

$$A : \ \mathcal{U}^{-\infty} \cap \mathcal{U}^s_{\mathrm{loc}}(O) \to \mathcal{F}^{-\infty} \cap \mathcal{F}^{s-m}_{\mathrm{loc}}(O).$$

In subsequent applications the index s will be related to the smoothness of the function u. When investigating the smoothness of solutions of the operator equation $Au = f$, one often uses the following definition. An operator $R: \mathcal{F}^{-\infty} \to \mathcal{U}^{-\infty}$ is said to be a left *regulariser* of A in O if the order of R is $-m$ and $RA = I - S$ where I is the identity operator and $r_O S$, as an operator from $\mathcal{U}^{-\infty}$ to $\mathcal{U}^{-\infty}(O)$, has order < 0. The right regulariser is defined in a similar way.

The following lemma is easily proved by means of sequential commutations of the multiplication operator by functions in $C^\infty_{\mathrm{comp}}(O)$ and A.

Lemma 1.2 *Let A be a pseudolocal operator of order m and let $[A, \varphi]$ have order $< m$ for all $\varphi \in C^\infty_{\mathrm{comp}}(O)$. If there is a left regulariser R for A in O then*

$$\begin{array}{rcl} u & \in & \mathcal{U}^{-\infty}, \\ (A+K)u & \in & \mathcal{F}^{s-m}_{\mathrm{loc}}(O) \end{array} \quad \Rightarrow \quad u \in \mathcal{U}^{-\infty} \cap \mathcal{U}^s_{\mathrm{loc}}(O),$$

where K is any pseudolocal operator of order $< m$.

Of course, in studying various concrete operators the main difficulty is the construction of the regulariser. One can in much the same way set forth questions related to local solvability.

We will say that the equation $Au = f$ is locally solvable at a point $v \in U$ if for any domain $O \subset U$ with $v \in O$ one can find a function $u \in \mathring{\mathcal{U}}^{-\infty}(O)$ such that $Au = f$ is satisfied in some smaller domain O' containing v, i.e., $r_{O'} Au = r_{O'} f$.

Let us impose further restrictions on the family $\mathcal{F}^s$. We will assume that $\mathcal{F}^s$ is Hilbert and the operator norm of

$$r_{O'} : \ \mathcal{F}^{s+0}(O) \to \mathcal{F}^s(O') \tag{1.1}$$

tends to zero as the diameter of O' tends to zero, $s+0$ meaning any real number exceeding s. These properties hold, for example, for the spaces $H^s(\mathbb{R}^N)$ with $s \le N/2$. In fact, it is known that for $s \le N/2$ the norm of the embedding $\imath : \ \mathring{H}^{-s}(O') \to \mathring{H}^{-s-0}(O)$ tends to zero as the diameter of O' tends to zero. Since the restriction operator $r_{O'}$ is the transpose of $\imath$, the norm of $r_{O'}$ also tends to zero.

Lemma 1.3 *Assume that the commutator $[A, \varphi]$ has order $< m$ for all $\varphi \in C^\infty_{\mathrm{comp}}(O)$. If there is a right regulariser R for A in O then the equation $Au = f$ is locally solvable at any point $v^0 \in O$ for all $f \in \mathcal{F}^s$. Moreover, one can find a solution u in $\mathring{\mathcal{U}}^{s+m}(O)$.*

Proof. Since $\mathcal{F}^s$ is semilocal, by the closed graph theorem the multiplication operator by a function $\varphi \in C^\infty_{\mathrm{comp}}(U)$ is continuous in $\mathcal{F}^s$. Hence, if $O' \Subset O$ then $\mathcal{F}^s(O')$ is the quotient space

$$\mathring{\mathcal{F}}^s(O)/\mathring{\mathcal{F}}^s(O \setminus \bar{O}').$$

Let e denote a continuation operator $\mathcal{F}^s(O') \to \mathring{\mathcal{F}}^s(O)$, i.e., any operator with the property $r_{O'}e = I$. In Hilbert spaces there always exists a continuation operator whose norm is 1. Fix $\varphi \in C^\infty_{\mathrm{comp}}(O)$ such that $\varphi(v) = 1$ in some neighbourhood of v^0. If the diameter of O' is small enough, then $\varphi(v) = 1$ in O'. Hence

$$r_{O'} A\varphi ReF = F - r_{O'}\left(SeF - [A,\varphi]ReF\right) \tag{1.2}$$

for any $F \in \mathcal{F}^s(O')$. By the assumptions on the orders of S and $[A,\varphi]$ we obtain that the operator to which $r_{O'}$ is applied on the right-hand side of (1.2) has order < 0. Since the norm of the operator (1.1) tends to zero as the diameter of O' tends to zero, we can choose O' so that the operator (1.2) is invertible. Hence we can find an F such that $r_{O'}A\varphi ReF = r_{O'}f$. Then the function $u = \varphi ReF$ fulfills $Au = f$ in O'. □

From the proof it is seen that if the conditions of Lemma 1.3 hold then O' can be chosen to be the same for all $f \in \mathcal{F}^s$.

1.2 Weighted Sobolev spaces

Let $\delta > 0$ be defined by the structure of A, as above. Following [Gru70b], given any $s \in \mathbb{Z}_+$ and $t \in \mathbb{R}$, we denote by $H^{s,t,\delta}$ the space of generalised functions $u(x,y)$ in $\mathbb{R}^N$, such that the derivatives $D^\alpha_x \hat{u}(x,\eta)$, $|\alpha| \le s$, are ordinary functions and

$$\|u\|_{s,t,\delta} = \left(\int_{\mathbb{R}^q} (1+|\eta|^2)^t \Big(\sum_{|\alpha|\le s} \|(1+|x|^\delta|\eta| + |\eta|^{\frac{1}{1+\delta}})^{s-|\alpha|} D^\alpha_x \hat{u}(x,\eta)\|^2 \Big) d\eta \right)^{\frac{1}{2}} \tag{1.3}$$

is finite, where $\hat{u}(x,\eta)$ is the Fourier transform of $u(x,y)$ in y and $\|U(x)\|$ is the norm in $L^2(\mathbb{R}^n_x)$.

Set

$$\begin{aligned} H^{s,t} &:= H^{s,t,0}, \\ H^{s,-\infty,\delta} &:= \textstyle\bigcup_t H^{s,t,\delta}. \end{aligned}$$

It is easy to see that $H^{s,t} = H^w$ if $w(\theta) = (1+|\eta|)^t(1+|\xi|+|\eta|)^s$. The space H^w will be denoted $\mathcal{H}^{s,t,\delta}$ if

$$w(\theta) = (1+|\eta|)^t(1+|\xi|+|\eta|^{\frac{1}{1+\delta}})^s,$$

the number s here being not necessarily an integer. It is easy to see that $H^{s,t,\delta}$ embeds into $\mathcal{H}^{s,t,\delta}$.

Write B_ϱ for the ball $|x| < \varrho$ in $\mathbb{R}^n$. Denote by $T_\varrho = B_\varrho \times \mathbb{R}^q$ the tube around $\mathbb{R}^q$. There are embeddings

$$\mathring{H}^{s,t}(T_\varrho) \hookrightarrow H^{s,t,\delta}(T_\varrho) \hookrightarrow H^{s,t-s}(T_\varrho)$$

whence

$$\mathring{H}^{s,-\infty}(T_\varrho) = \mathring{H}^{s,-\infty,\delta}(T_\varrho). \tag{1.4}$$

Lemma 1.4 *The functions of $C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$ are dense in $H^{s,t,\delta}$.*

Proof. Choose an arbitrary function $\varphi(x)$ in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with the property that $\varphi(0) = 1$. From (1.3) it is clear that $\varphi(\varepsilon x)u(x,y) \to u(x,y)$ as $\varepsilon \to 0$ in the sense of convergence in $H^{s,t,\delta}$ for all $u \in H^{s,t,\delta}$. Hence $\bigcup_\varrho \mathring{H}^{s,t,\delta}(T_\varrho)$ is dense in $H^{s,t,\delta}$. Since the functions in $C^\infty_{\mathrm{comp}}(T_\varrho)$ are dense in $\mathring{H}^{s,t}(T_\varrho)$, cf. [VP65], it remains to show that $\mathring{H}^{s,t}(T_\varrho)$ is dense in $\mathring{H}^{s,t,\delta}(T_\varrho)$. To this end pick any function $\psi(\eta)$ in $C^\infty_{\mathrm{comp}}(\mathbb{R}^q)$ with $\psi(0) = 1$. Let $u_\varepsilon(x,y)$ denote the inverse Fourier transform of the product $\psi(\varepsilon\eta)\hat{u}(x,\eta)$ where $u \in \mathring{H}^{s,t,\delta}(T_\varrho)$. As is seen in (1.3), u_ε belongs to $\mathring{H}^{s,t}(T_\varrho)$ and converges to u in the topology of $\mathring{H}^{s,t,\delta}(T_\varrho)$, as desired. □

1.3 Boundary operators

Let now $b(v,\theta)$ be a function on $\mathbb{R}^N \times \mathbb{R}^N$ of exponential growth in θ. As usual, we assign a canonical pseudodifferential operator $B(v,D)$ to $b(v,\theta)$, defined by

$$B(v,D)u = \frac{1}{(2\pi)^N} \int e^{\imath\langle v,\theta\rangle} b(v,\theta)\hat{u}(\theta)\,d\theta$$

for $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$. Recall that $\theta = (\xi,\eta)$.

Suppose the function $b(v;\xi,\eta)$ lies in C^∞ if $(\xi,\eta) \neq 0$, and it is quasihomogeneous in (ξ,η) of order r in the sense that

$$b(v;\lambda\xi,\lambda^{1+\delta}\eta) = \lambda^r b(v;\xi,\eta) \tag{1.5}$$

for all $\lambda > 0$. For simplicity we ignore the behaviour of $b(v;\xi,\eta)$ for large v and assume that $b(v;\xi,\eta) = 0$ if $|v| \geq C$ for some $C > 0$. If $r < 0$, then $b(v;\theta)$ in the definition of $B(v,D)$ is replaced by the product $\psi(\theta)b(v;\theta)$, where $\psi \in C^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ vanishes in a neighbourhood of 0 and is equal to 1 for sufficiently large θ.

We will say that the condition of smoothness of order $k = 1, 2, \ldots$ holds for $b(v; \xi, \eta)$ if

$$D_\eta^\beta b(v, \xi, 0) = 0$$

for all $\beta \in \mathbb{Z}_+^q$ with $|\beta| \leq k - 1$. In this case $B(v, D)$ can be written in the form

$$B(v, D) = \sum_{|\beta|=k} B_\beta(v, D) D_y^\beta, \tag{1.6}$$

the symbols $b_\beta(v; \xi, \eta)$ being quasihomogeneous in (ξ, η) of order $r - (1 + \delta)k$ and possessing the properties indicated above.

Denote by $r_\mathcal{S}$ the operator of restriction to the submanifold $\mathcal{S} := \mathbb{R}^q$ of $\mathbb{R}^N$ given by $x = 0$. Initially $r_\mathcal{S}$ is determined for functions in $C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$.

Lemma 1.5 *The operator $r_\mathcal{S}$ extends continuously to a mapping in*

$$\mathcal{L}\left(\mathcal{H}^{s,t,\delta}, H^{t+\frac{s-n/2}{1+\delta}}(\mathbb{R}^q_y)\right)$$

for all $s > n/2$ and $-\infty < t < \infty$.

Lemma 1.6 *Given any $s, t \in \mathbb{R}$, the operator $B(x, D)$ extends continuously to a mapping in $\mathcal{L}\left(\mathcal{H}^{s,t,\delta}, \mathcal{H}^{s-r,t,\delta}\right)$. The norm of this operator is majorised by the maximum of a certain (depending on s and t) number of derivatives of $b(v, \theta)$ in v for $|\theta| = 1$. For any $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$ the commutator $[B(v, D), \varphi]$ is in $\mathcal{L}\left(\mathcal{H}^{s,t,\delta}, \mathcal{H}^{s-r+1,t,\delta}\right)$.*

For the proof of these assertions we refer the reader to [Vol67]. By means of the embedding $H^{s,t,\delta} \hookrightarrow \mathcal{H}^{s,t,\delta}$ and representation (1.6) we obtain, from Lemmas 1.5 and 1.6.

Theorem 1.7 *Suppose $r - (1 + \delta)k < s - n/2$, then*

$$r_\mathcal{S} B(v, D) \ \in \ \mathcal{L}\left(H^{s,t,\delta}, H^{t+\frac{s-r-n/2}{1+\delta}}(\mathbb{R}^q_y)\right),$$

the norm of this operator being majorised by the maximum of a certain (depending on s and t) number of derivatives of $b(v, \theta)$ in v for $|\theta| = 1$. For any $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$

$$r_\mathcal{S} \left[B(v, D), \varphi\right] \ \in \ \mathcal{L}\left(H^{s,t,\delta}, H^{t+\frac{s-r-n/2}{1+\delta}+o}(\mathbb{R}^q_y)\right),$$

where $o = (1 + \delta)^{-1}$

1.4 Coboundary operators

We now turn to *coboundary operators.* Let $b(v,\theta)$ have the same properties as above. The coboundary operator is the mapping

$$u(y) \mapsto B(x,D)\,(\delta(x) \otimes u(y)) \tag{1.7}$$

defined first for $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}^q)$.

Theorem 1.8 *If $r-(1+\delta)k < -n/2$, then the operator (1.7) extends continuously to a mapping in*

$$\mathcal{L}\left(H^{t+\frac{r+n/2}{1+\delta}}(\mathbb{R}^q_y), H^{0,t}\right),$$

and the norm of this operator is majorised by the maximum of a certain (depending on t) number of derivatives of $b(v,\theta)$ in v for $|\theta| = 1$. For any function $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$ the operator

$$u(y) \mapsto [B(v,D),\varphi]\,(\delta(x) \otimes u(y))$$

is in $\mathcal{L}\left(H^{t+\frac{r+n/2}{1+\delta}}(\mathbb{R}^q_y), H^{0,t+o}\right)$.

Proof. By means of the Fourier transformation it is easy to check that if $s < -n/2$ then

$$\delta(x)\otimes \;\in\; \mathcal{L}\left(H^{t+\frac{r+n/2}{1+\delta}}(\mathbb{R}^q_y), \mathcal{H}^{s,t+\frac{r-s}{1+\delta},\delta}\right),$$

where $\delta(x)\otimes$ is the mapping $\delta(x)\otimes u(y)$. Hence from (1.6) and Lemma 1.6 it follows that (1.7) lies in

$$\mathcal{L}\left(H^{t+\frac{r+n/2}{1+\delta}}(\mathbb{R}^q_y), \mathcal{H}^{s-r+(1+\delta)k,t-k+\frac{r-s}{1+\delta},\delta}\right).$$

Taking $s = r-(1+\delta)k$ we obtain the first assertion of Theorem 1.8, because $\mathcal{H}^{0,t,\delta} = H^{0,t}$. Similarly one shows the second assertion.

□

Corollary 1.9 *If $t \geq 0$ and $r-(1+\delta)(k-t) < -n/2$, then the operator (1.7) lies in $\mathcal{L}\left(H^{t+\frac{r+n/2}{1+\delta}}(\mathbb{R}^q_y), H^t\right)$.*

Proof. The proof if similar to that of Theorem 1.8, the only difference being in replacing $s = r-(1+\delta)k$ by $s = r-(1+\delta)(k-t)$. We moreover make use of the fact that for each $t \geq 0$ the embedding $\mathcal{H}^{(1+\delta)t,0,\delta} \hookrightarrow H^t$ takes place.

□

Theorems 1.7 and 1.8 are valid not only for symbols which are compactly supported in v. They are valid, in particular, if the symbol does not depend on v. The norm of the corresponding operators is dominated by the maximum of the symbol $b(\theta)$ itself for $|\theta| = 1$. Thus these results still hold if the symbol $b(v,\theta)$ is the sum of two symbols one of which does not depend on v while the other is compactly supported in v.

1.5 Operators with polynomial coefficients

Consider an operator

$$A(x, D_x) = \sum_{\substack{|\alpha| \le m \\ |\gamma| \le \delta(m-|\alpha|)}} a_{\alpha\gamma} x^\gamma D_x^\alpha$$

where $a_{\alpha\gamma}$ are complex numbers. We assume that A is elliptic, i.e.,

$$\begin{aligned} \sigma^m(A)(\xi) &= \sum_{|\alpha|=m} a_{\alpha 0}\, \xi^\alpha \\ &\neq 0 \end{aligned} \tag{1.8}$$

for all $\xi \neq 0$. Moreover, let

$$\sum_{\substack{|\alpha| \le m \\ |\gamma| = \delta(m-|\alpha|)}} a_{\alpha\gamma} x^\gamma \xi^\alpha \neq 0 \tag{1.9}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq 0$.

On $\mathbb{R}^n$ we introduce the weighted space $H^{m,\delta}(\mathbb{R}^n)$ consisting of all functions $u(x)$ with finite norm

$$\|u\|_{m,\delta} = \left(\int \sum_{|\alpha| \le m} (1 + |x|^2)^{\delta(m-|\alpha|)} |D^\alpha u(x)|^2 dx \right)^{\frac{1}{2}}.$$

The following assertions go back at least as far as [Gru70b].

Lemma 1.10 *If (1.8), (1.9) are satisfied, then $A(x, D_x)$ induces a Fredholm operator $H^{m,\delta}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Lemma 1.11 *If (1.8), (1.9) hold, then any solution u of the homogeneous equation $A(x, D_x)u(x) = 0$ in $L^2(\mathbb{R}^n)$ actually lies in $\mathcal{S}(\mathbb{R}^n)$.*

Lemma 1.11 easily follows from the properties of the parametrix of A constructed in the proof of Lemma 1.10. The existence of such a parametrix actually implies the following results.

Lemma 1.12 *Let (1.8), (1.9) be fulfilled, and let u be a solution to the inhomogeneous equation $A(x, D_x)u(x) = f(x)$ in the space $\mathcal{S}'(\mathbb{R}^n)$. If $f \in \mathcal{S}(\mathbb{R}^n)$ then $u \in \mathcal{S}(\mathbb{R}^n)$, and if $f \in L^2(\mathbb{R}^n)$ then $u \in H^{m,\delta}(\mathbb{R}^n)$.*

Lemma 1.13 *Let (1.8), (1.9) hold, and let $\{u_\nu\}$ be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ converging to u in $\mathcal{S}'(\mathbb{R}^n)$. If $f_\nu(x) = A(x, D_x)u_\nu(x)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and converges in this space, then $u_\nu \to u$ in $\mathcal{S}(\mathbb{R}^n)$.*

Corollary 1.14 *If $A(x,D)$ is viewed as an unbounded operator in $L^2(\mathbb{R}^n)$ with domain $H^{m,\delta}(\mathbb{R}^n)$, then $A(x,D)$ is closed.*

Corollary 1.15 *The range of $A(x,D)$ consists precisely of those functions $f \in L^2(\mathbb{R}^n)$ which are orthogonal to solutions in $\mathcal{S}(\mathbb{R}^n)$ of the formally adjoint equation $A^*(x,D)g(x) = 0$.*

Recall that the formally adjoint operator $A^*(x,D)$ for $A(x,D)$ is defined by the formula

$$A^*(x,D_x) = \sum_{\substack{|\alpha|\le m \\ |\gamma|\le\delta(m-|\alpha|)}} \bar{a}_{\alpha\gamma}\, D_x^\alpha x^\gamma.$$

Corollary 1.16 *If the operator $A(x,D)$ on $C^\infty_{\text{comp}}(\mathbb{R}^n)$ is symmetric (i.e. $A^*(x,D) = A(x,D)$), then the operator $A(x,D)$ with domain $H^{m,\delta}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ is selfadjoint.*

Theorem 1.17 *Suppose (1.8), (1.9) are satisfied. If $A(x,D)$ is symmetric then it has a discrete spectrum.*

Proof. If $A(x,D)$ is symmetric, the equation $A(x,D)u+\lambda u = 0$ for $\Im\lambda \neq 0$ cannot have solutions in $\mathcal{S}(\mathbb{R}^n)$. By Lemmas 1.10 and 1.12 and Corollary 1.15 we deduce that $A(x,D)+\lambda$ determines an isomorphism between $H^{m,\delta}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. Since $A(x,D)+\lambda$ depends analytically on λ, it follows that the inverse operator R_λ is a finitely meromorphic function of λ, cf. [Ble69]. Hence there exists at most a sequence $\lambda_\nu \to \infty$ of real values of λ such that $A(x,D)+\lambda$ has no inverse. Let us pick any real λ with the property that the inverse R_λ exists for $A(x,D)+\lambda$. Since the embedding of $H^{m,\delta}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ is compact, R_λ, if viewed as an operator on $L^2(\mathbb{R}^n)$, is compact and selfadjoint. By the spectral theorem we can assert that in $L^2(\mathbb{R}^n)$ there exists for R_λ, and hence also for $A(x,D)$, an orthogonal basis of eigenfunctions. This establishes the theorem.

□

Corollary 1.18 *Suppose (1.8), (1.9) are satisfied. If $A(x,D)$ is symmetric then in $L^2(\mathbb{R}^n)$ there is an orthonormal basis of eigenfunctions of $A(x,D)$. The eigenfunctions lie in $\mathcal{S}(\mathbb{R}^n)$.*

All these results are also valid for systems, i.e., in case the coefficients $a_{\alpha\gamma}$ of $A(x,D)$ are square matrices. Conditions (1.8), (1.9) now mean that the corresponding matrices have trivial null-spaces. Several results carry over also to the case of rectangular matrices. For example, these are Lemmas 1.11, 1.12 and 1.13. But then from the conditions of Lemma 1.10 it follows merely that $A(x,D)$ defines a homomorphism with finite-dimensional kernel. If the conditions of Lemma 1.10 are fulfilled for $A^*(x,D)$ then the cokernel of $A(x,D)$ is finite dimensional.

2 Local solvability and smoothness

2.1 Boundary value problems

Let $A(v,D)$ be an operator of the form (0.2) which fulfills condition (0.4). We will study the operator $A(v,D)$ under boundary and coboundary conditions.

For $i=1,\ldots,k$ and $j=1,\ldots,l$, let there be given symbols $b_i(v,\theta)$, $p_j(v,\theta)$ and $e_{ij}(y,\eta)$ of class C^∞ admitting asymptotic expansions

$$\begin{aligned} b_i(v,\theta) &\sim \sum_{\nu=0}^{\infty} b_{i,\nu}(v,\theta), \\ p_j(v,\theta) &\sim \sum_{\nu=0}^{\infty} p_{j,\nu}(v,\theta), \\ e_{ij}(y,\eta) &\sim \sum_{\nu=0}^{\infty} e_{ij,\nu}(y,\eta) \end{aligned} \tag{2.1}$$

for large values of the covariables.

We assume that all the functions $b_{i,\nu}(v,\theta)$ and $p_{j,\nu}(v,\theta)$ satisfy the hypotheses of Theorems 1.7 and 1.8, respectively. The order of $b_{i,0}$ is m_i, that of $p_{j,0}$ is r_j, and the orders of the terms of the expansions decrease and tend to $-\infty$ when $\nu\to\infty$. The symbols $e_{ij,\nu}(y,\eta)$ lie in C^∞ if $\eta\neq 0$, and are homogeneous in η of order $t_j-s_i-\nu$, where

$$\begin{aligned} s_i &= \frac{m-m_i-(n/2)}{1+\delta}, \\ t_j &= \frac{r_j+(n/2)}{1+\delta}. \end{aligned}$$

Suppose expansions (2.1) can be differentiated. We will think of all the symbols in (2.1) as being compactly supported in v (e_{ij} in y). This is not an essential restriction, since in this section we study only local properties of the corresponding operators.

Consider the operator

$$\mathcal{A}: \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} f \\ g \end{pmatrix}$$

given by

$$\left\{ \begin{aligned} A(v,D)u(v) + \sum_{j=1}^{l} P_j(v,D)\,(\delta(x)\otimes v_j(y)) &= f(v), \\ r_{\mathcal{S}}\, B_i(v,D)u(v) + \sum_{j=1}^{l} E_{ij}(y,D_y)v_j(y) &= g_i(y) \end{aligned} \right. \tag{2.2}$$

for $i = 1, \ldots, k$, where v and g are regarded as columns of functions $v_j(y)$ and $g_i(y)$, respectively.

From Theorems 1.7 and 1.8 it follows that $\mathcal{A}$ induces a continuous linear operator

$$\mathcal{A}: \begin{array}{c} H^{m,t,\delta} \\ \oplus \\ \oplus_{j=1}^{l} H^{t+t_j}(\mathbb{R}^q) \end{array} \to \begin{array}{c} H^{0,t} \\ \oplus \\ \oplus_{i=1}^{k} H^{t+s_i}(\mathbb{R}^q) \end{array} \qquad (2.3)$$

for all $t \in \mathbb{R}$.

Define the auxiliary operator

$$\left\{ \begin{array}{rcl} \sigma_{\mathrm{edge}}(A)(y^0, D_y)u(v) + \displaystyle\sum_{j=1}^{l} P_{j,0}(0, y^0; D)\,(\delta(x) \otimes v_j(y)) & = & f(v), \\ r_{\mathcal{S}}\, B_{i,0}(0, y^0; D)u(v) + \displaystyle\sum_{j=1}^{l} E_{ij,0}(y^0; D_y)v_j(y) & = & g_i(y), \end{array} \right. \qquad (2.4)$$

$i = 1, \ldots, k$. Passing to the Fourier transform in $y \in \mathbb{R}^q$, for each $\eta \neq 0$ we obtain the operator

$$\sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = \begin{pmatrix} \sigma_{\mathrm{edge}}(A)(y^0, \eta) & \sigma_{\mathrm{edge}}(P)(y^0, \eta) \\ \sigma_{\mathrm{edge}}(B)(y^0, \eta) & \sigma_{\mathrm{edge}}(E)(y^0, \eta) \end{pmatrix} \qquad (2.5)$$

where

$$\begin{array}{rcl} \sigma_{\mathrm{edge}}(B)(y^0, \eta) & = & \big(r_0 b_{i,0}(0, y^0; D_x, \eta)\big)\,, \\ \sigma_{\mathrm{edge}}(P)(y^0, \eta) & = & \big(\mathcal{F}^{-1}_{\xi \mapsto x} p_{j,0}(0, y^0; \xi, \eta)\big)\,, \\ \sigma_{\mathrm{edge}}(E)(y^0, \eta) & = & \big(e_{ij,0}(y^0, \eta)\big)\,, \end{array}$$

r_0 being the operator that assigns the value at $x = 0$ to each function $u(x)$ on $\mathbb{R}^n$.

The symbol mapping (2.5) is thought of as a family of continuous linear operators

$$\sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta): \begin{array}{c} H^{m,\delta}(\mathbb{R}^n) \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} L^2(\mathbb{R}^n) \\ \oplus \\ \mathbb{C}^k \end{array} \qquad (2.6)$$

parametrised by $\eta \in \mathbb{R}^q \setminus \{0\}$. From Lemma 1.10 it follows that (2.6) is Fredholm for every $\eta \neq 0$.

Note that $\sigma_{\mathrm{edge}}(A)(y^0, \eta)$ is quasihomogeneous in the sense that

$$\sigma_{\mathrm{edge}}(A)\left(y^0, \lambda\eta\right) = \lambda^{\frac{m}{1+\delta}}\, \kappa_\lambda\, \sigma_{\mathrm{edge}}(A)(y^0, \eta)\, \kappa_\lambda^{-1} \qquad (2.7)$$

for all $\lambda > 0$, where

$$(\kappa_\lambda u)(x) = c\, u\left(\lambda^{\frac{1}{1+\delta}} x\right)$$

with c an arbitrary constant depending on λ. In fact this just amounts to saying that

$$\sigma_{\text{edge}}(A)\left(y^0, \lambda^{1+\delta}\eta; \frac{x}{\lambda}, \lambda\xi\right) = \lambda^m \, \sigma_{\text{edge}}(A)\left(y^0, \eta; x, \xi\right),$$

which is due to (0.3). From (2.7) we obtain at once the following assertion, cf. [Gru70b, GV69a].

Lemma 2.1 *Suppose the condition (0.4) hold and* $\ker \sigma_{\text{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $\eta \in \mathbb{R}^q$ *with* $|\eta| = 1$. *Then for each* $\eta \neq 0$ *one can construct for* $\sigma_{\text{edge}}(\mathcal{A})(y^0, \eta)$ *a left inverse operator* $P(y^0, \eta)$ *which is piecewise continuous in* η *and*

$$\sum_{|\alpha| \le m} \| \left(|\eta|^{\frac{1}{1+\delta}} + |x|^\delta |\eta| \right)^{m-|\alpha|} D^\alpha u(x) \| + \sum_{j=1}^{l} |\eta|^{t_j} |v_j| \le C \left(\|f\| + \sum_{i=1}^{k} |\eta|^{s_i} |g_i| \right)$$

where $(u, v) = P(y^0, \eta)(f, g)$.

We will say that $R(y^0, \eta)$ is a left almost inverse of $\sigma_{\text{edge}}(A)(y^0, \eta)$ if there is an $\varrho > 0$ such that $R(y^0, \eta) \in \mathcal{L}(L^2(B_\varrho), H^m(B_\varrho))$ and

$$R(y^0, \eta)\sigma_{\text{edge}}(A)(y^0, \eta)u(x) = u(x)$$

for all $u \in \overset{\circ}{H}{}^m(B_\varrho)$, where $B_\varrho = \{x \in \mathbb{R}^n : |x| < \varrho\}$. The following was proved in [Gru70b].

Lemma 2.2 *Let (0.4) hold. If* ϱ *is sufficiently small then one can construct a family of left almost inverses* $R(y^0, \eta)$ *for* $\sigma_{\text{edge}}(A)(y^0, \eta)$ *for all* $|\eta| \le 1$, *depending continuously on* η.

2.2 Construction of the regularisers

Denote by $\mathcal{U}^t$ and $\mathcal{F}^t$ the families

$$\begin{aligned} \mathcal{U}^t &= H^{m,t,\delta} \times \prod_{j=1}^{l} H^{t+t_j}(\mathbb{R}^q), \\ \mathcal{F}^t &= H^{0,t} \times \prod_{i=1}^{k} H^{t+s_i}(\mathbb{R}^q) \end{aligned}$$

entering into (2.3). We will view $\mathcal{A}$ and $\mathcal{A}_{0,y^0}$, the latter being the auxiliary operator (2.4), as operators of zero order from $\mathcal{U}^t$ to $\mathcal{F}^t$. Set $T_\varrho = B_\varrho \times \mathbb{R}^q$.

By a left (or right) regulariser we will mean an operator satisfying the conditions of Section 1.1. By a familiar construction (see e.g. [GV69a]) we deduce from Lemmas 2.1 and 2.2

Lemma 2.3 *If condition (0.4) holds and* $\ker \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$*, then there is an* $\varrho > 0$ *with the property that* $\mathcal{A}_{0,y^0}$ *has a left regulariser* $\mathcal{R}_{0,y^0}$ *in the tube* T_ϱ*.*

Fix a function $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$ such that $\varphi(v) = 0$ for $|v - (0, y^0)| > \varrho_1$ and $\varphi(v) = 1$ for $|v - (0, y^0)| < \varrho_1/2$. The number $0 < \varrho_1 < 1$ will be chosen later. Set

$$\varphi_\varepsilon(v) = \varphi\Big(\frac{x}{\varepsilon}, \frac{y - y^0}{\varepsilon^{1+\delta}}\Big),$$

for $\varepsilon > 0$.

Consider the operator

$$\Delta_\varepsilon = \varphi_\varepsilon \left(\mathcal{A}^0 - \mathcal{A}_{0,y^0}\right) \varphi_\varepsilon \tag{2.8}$$

where $\mathcal{A}^0$ is defined by formula (2.2) if we replace all the symbols in it (except $A(v, D)$) by their principal parts and the symbols of the coefficients of $A(v, D)$ are changed to $a_{\alpha\beta\gamma,0}(v, \theta)$. The operator thus obtained from $A(v, D)$ is denoted by $A^0(v, D)$.

Theorem 2.4 *Let condition (0.4) hold and* $\ker \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$*. Then there are positive numbers* ε *and* ϱ *with* $\varepsilon < \varrho$*, such that the operator* $\mathcal{A}_{0,y^0} + \Delta_\varepsilon$ *has a left regulariser* $\mathcal{R}$ *in the tube* T_ϱ*.*

Proof. For the case when there are no boundary and coboundary conditions a similar assertion was shown in [Gru70b]. In the general case the proofs follows the same scheme. The principal feature of the proof is that if ε is chosen small enough and one puts $\varrho = \varepsilon\varrho_1$, then in $\mathcal{U}^t(T_\varrho)$ and $\mathcal{F}^t(T_\varrho)$ one can introduce new norms equivalent to the initial ones, so that the norm of the operator $\varphi_\varepsilon \left(A^0 - A_{0,y^0}\right) \varphi_\varepsilon$ will be as small as desired, while that of $\mathcal{R}_{0,y^0}$ does not change.

This fact is proved in Propositions 5.1 and 5.2 in [Gru70b]. If the orders of other operators in (2.2) are nonnegative then by repeating the arguments in [Gru70b] relative to boundary and coboundary operators it is easy to verify that the norm of the operator Δ_ε can thus be made as small as desired. Then it suffices to set

$$\mathcal{R} = \left(I + \Delta_\varepsilon \mathcal{R}_{0,y^0}\right)^{-1} \mathcal{R}_{0,y^0}.$$

In [Gru70b] renormings of this kind are arranged by first making the change of variables $(x, y) \mapsto (\varepsilon x, \varepsilon^{1+\delta}(y + y^0))$, carrying $\varphi_\varepsilon(v)$ to $\varphi(v)$. Since (2.7) holds and the other symbols in (2.4) are quasihomogeneous, the form of $\mathcal{A}_{0,y^0}$ is not altered by such a substitution if one raises the functions in (2.4) by the corresponding power of the parameter ε. We choose ϱ_1 so that Lemma 2.3 holds for it. After this, $(1 + |\eta|^2)^t$ in (1.3) is replaced by $(1 + |\vartheta\eta|^2)^t$ where ϑ is some positive number depending on t.

Note that the above substitution is also equivalent to a certain renorming in the initial spaces. If, for example, m_i is negative then under such transformations the operator $B_i(0, y^0; D)$ is not invariant, since for pseudodifferential operators of negative order it is still necessary to use an excision function. But in varying this factor the corresponding operators differ by a smoothing operator, i.e., an operator of order $-\infty$. The same is also true for the other operators in (2.2) and (2.4). Hence we conclude that after our transformations $\Delta_\varepsilon = \Delta'_\varepsilon + \Delta''_\varepsilon$, where the norm of Δ'_ε is as small as desired, while Δ''_ε is a smoothing operator. Then we set

$$\mathcal{R} = \left(I + \Delta'_\varepsilon \mathcal{R}_{0,y^0}\right)^{-1} \mathcal{R}_{0,y^0}$$

and obtain a left regulariser for $\mathcal{A}_{0,y^0} + \Delta_\varepsilon$.

□

Corollary 2.5 *If the conditions of Theorem 2.4 are satisfied then for some* $0 < \varepsilon < \varrho$ *there exists in the tube* T_ϱ *a left regulariser* $\mathcal{R}$ *for the operator*

$$\mathfrak{A}_\varepsilon = \mathcal{A}_{0,y^0} + \varphi_\varepsilon \left(\mathcal{A} - \mathcal{A}_{0,y^0}\right) \varphi_\varepsilon.$$

Proof. For $\mathbb{R}$ one can take the same operator as in Theorem 2.4 because $\varphi_\varepsilon \left(\mathcal{A} - \mathcal{A}^0\right) \varphi_\varepsilon$ is an operator of negative order.

□

We now fix ε and $\varphi_\varepsilon(v)$ so that the assertion of the corollary takes place. Let U denote some neighbourhood of the point $(0, y^0)$ on which $\varphi_\varepsilon(v) = 1$. We will write $\operatorname{supp}(u, v) \subset U$ provided both $\operatorname{supp} u$ and $\operatorname{supp} \delta(x) \otimes v(y)$ lie in U. Note that for such (u, v) the expressions $\mathcal{A}(u, v)$ and $\mathfrak{A}(u, v)$ coincide on U, $\mathfrak{A}$ standing for $\mathfrak{A}_\varepsilon$. This allows one to use the regulariser constructed above for $\mathfrak{A}$ in studying local properties of $\mathcal{A}$. We will take $\varphi_\varepsilon = 1$ in a neighbourhood of the closure of U.

Theorem 2.6 *The operator* $\mathcal{R}$ *constructed in Theorem 2.4 is a left regulariser for* $\mathcal{A}$ *in the domain* U.

Proof. Let $\tilde{\varphi}(v)$ be a function in C^∞_{comp} such that $\varphi_\varepsilon \tilde{\varphi} = \tilde{\varphi}$ and $\tilde{\varphi}(v) = 1$ in U. By Corollary 2.5 $\mathcal{R}$ is a left regulariser for $\mathfrak{A}$ in the tube T_ϱ. Hence it suffices to show that $(\mathcal{A} - \mathfrak{A})\tilde{\varphi}$ is an operator of negative order. This follows from the fact that

$$\begin{aligned} (\mathcal{A} - \mathfrak{A})\, \tilde{\varphi} &= (\mathcal{A} - \mathcal{A}_{0,y^0})\, \tilde{\varphi} - \varphi_\varepsilon \left(\mathcal{A} - \mathcal{A}_{0,y^0}\right) \tilde{\varphi} \\ &= -[\mathcal{A} - \mathcal{A}_{0,y^0}, \tilde{\varphi}], \end{aligned}$$

for the commutators $[\mathcal{A}, \tilde{\varphi}]$ and $[\mathcal{A}_{0,y^0}, \tilde{\varphi}]$ have negative order.

□

In much the same way one proves the theorem on the existence of a right regularisers.

Theorem 2.7 *Let condition (0.4) hold and* $\operatorname{coker} \sigma_{\text{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$. *Then in some neighbourhood* U *of* $(0, y^0)$ *there exists a right regulariser for* $\mathcal{A}$.

2.3 Theorems on smoothness

Using Lemma 1.2 we now deduce the corresponding theorem on the local smoothness of solutions for $\mathcal{A}$.

Theorem 2.8 *Let condition (0.4) hold and* $\ker \sigma_{\text{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$. *Then there exists a neighbourhood* U *of* $(0, y^0)$ *such that for any* $(u, v) \in \mathcal{U}^{-\infty}$

$$\mathcal{A}\,(u, v) \in \mathcal{F}^t_{\text{loc}}(O) \;\Rightarrow\; (u, v) \in \mathcal{U}^t_{\text{loc}}(O) \tag{2.9}$$

whenever O *is an open subset of* U.

Since $\mathcal{A}$ is pseudolocal it would be sufficient to assume in this theorem that $(u, v) \in \overset{\circ}{\mathcal{U}}{}^{-\infty}(U)$.

This assumption can be further weakened by requiring only that

$$\begin{aligned} u &\in \mathcal{E}'(\mathbb{R}^N), \\ v &\in \prod_{j=1}^{l} H^{-\infty}(\mathbb{R}^q). \end{aligned}$$

In order to prove this, note that the function

$$F(v) = \sum_{j=1}^{l} p_j(v, D)\,(\delta(x) \otimes v_j(y))$$

lies in $H^{0,-\infty}$. Hence $A(v, D)u(v) = f(v) - F(v)$ where $f - F \in H^{0,-\infty}$. Consider the differential operator

$$\Delta(v, D_x) = \sum_{|\alpha|=m} a_\alpha(v)\, D_x^\alpha$$

on U, where

$$a_\alpha(v) = \sum_{|\gamma| \le \delta m} a_{\alpha 0 \gamma}(v) x^\gamma$$

are C^∞ functions on $\mathbb{R}^N$. Then

$$\Delta(v, D_x)u(v) = f(v) - F(v) - \sum_{\substack{(\alpha,\beta,\gamma) \in \mathcal{I} \\ |\alpha| < m}} a_{\alpha\beta\gamma}(v, D)\, x^\gamma D_x^\alpha D_y^\beta u(v). \tag{2.10}$$

From condition (0.4) it clearly follows that Δ is elliptic with respect to x at the point $(0, y^0)$. We choose a neighbourhood U of $(0, y^0)$ sufficiently small, so that Δ is elliptic with respect to x for any $v \in U$.

We next invoke the equality (2.10) to study the smoothness of generalised solutions of the equation $A(v, D)u = f$ in all variables of $\mathbb{R}^N$. The following lemma is proved by the same arguments as in Lemma 4.1 of [Gru70b].

Lemma 2.9 *Suppose Δ is elliptic with respect to x for any $v \in U$. If $u \in \mathcal{E}'(\mathbb{R}^N)$ then*

$$A(v, D)u \in H^{0,-\infty}_{\mathrm{loc}}(O) \;\Rightarrow\; u \in H^{m,-\infty,\delta}_{\mathrm{loc}}(O)$$

for any open set $O \subset U$.

Combining Theorem 2.8 with Lemma 2.9 and the pseudolocality property yields the desired sharpening of the theorem.

Theorem 2.10 *If the hypotheses of Theorem 2.8 are fulfilled then there exists a neighbourhood U of $(0, y^0)$ such that (2.9) holds for any $u \in \mathcal{E}'(\mathbb{R}^N)$ and $v \in \prod_{j=1}^{l} H^{-\infty}(\mathbb{R}^q)$.*

2.4 Local solvability

Using Lemma 1.3 we can analogously obtain the corresponding theorem on local solvability for $\mathcal{A}$.

Theorem 2.11 *Let condition (0.4) hold and* $\operatorname{coker} \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all $|\eta| = 1$. Then the equation*

$$\mathcal{A}\,(u, v) = (f, g) \tag{2.11}$$

is locally solvable at $(0, y^0)$ for all $(f, g) \in \mathcal{F}^t$ with $\max\{t, t + s_i\} \leq q/2$. *Moreover, for any neighbourhood U of $(0, y^0)$ one can find $(u, v) \in \mathring{\mathcal{U}}^t(U)$ such that (2.11) holds on O, a smaller neighbourhood of $(0, y^0)$ depending only on t and U.*

2.5 Hypoellipticity

Theorem 2.10 gives full information on what smoothness is enjoyed by the functions $u(v)$ and $v(y)$ in y, provided the smoothness of the functions $f(x, y)$ and $g(y)$ is known. But we still do not know what can be concluded about a solution $u(x, y)$ to (2.2) if, for example, $f \in H^s_{\mathrm{loc}}(O)$, $s \geq 0$. In the remainder of this section we assume that the conditions of smoothness of any order hold for the coboundary operators $P_j(v, D)$, i.e., all the symbols $p_{j,\nu}(v; \xi, \eta)$ vanish up to the infinite order at the subspace $\eta = 0$. It would indeed be sufficient that the smoothness condition of suitable finite order depending on ν and s be fulfilled.

Lemma 2.12 *If* $v \in H^{-\infty}(\mathbb{R}^q)$ *then*

$$v \in H^{t+t_j}_{\mathrm{loc}}(\mathbb{R}^q \cap O) \;\Rightarrow\; P_j(v, D)\,(\delta(x) \otimes v(y)) \in H^t_{\mathrm{loc}}(O).$$

Proof. The assertion follows from Corollary 1.9 and the pseudolocal property of pseudodifferential operators.

□

As a corollary of Theorem 2.10 and Lemma 2.12 we obtain the following technical lemma.

Lemma 2.13 *Let condition (0.4) hold and* $\ker \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$. *If* $\mathcal{A}(u, v) = (f, g)$ *lies in* $\mathcal{F}^s_{\mathrm{loc}}(O)$ *then*

$$f \in H^s_{\mathrm{loc}}(O) \;\Rightarrow\; A(v, D)u \in H^s_{\mathrm{loc}}(O)$$

for all $s \geq 0$.

Before turning to the study of the smoothness in x of generalised solutions to (2.2), we establish two properties of the operator Δ in (2.10). Suppose that Δ is elliptic in x for each fixed $y \in \mathbb{R}^q$.

Lemma 2.14 *If both* u *and* $D^\alpha_x \Delta u$, *for* $|\alpha| \leq A$, *belong to* $H^{0,t}_{\mathrm{loc}}(U)$, *then* $D^\alpha_x u \in H^{0,t}_{\mathrm{loc}}(U)$ *for any* $|\alpha| \leq m + A$.

Lemma 2.15 *If both* u *and* $D^\alpha_x \Delta u$, *for* $|\alpha| \leq A$, *belong to* $H^s_{\mathrm{loc}}(U)$, *then* $D^\alpha_x u \in H^s_{\mathrm{loc}}(U)$ *for any* $|\alpha| \leq m + A$.

Since for any elliptic operator $\Delta(v, D_x)$ one can always construct a regulariser in the form of a pseudodifferential operator, the assertions of Lemmas 2.14 and 2.15 follow from Lemma 1.2 and the theorem on the boundedness of pseudodifferential operators in [Vol67], if for the families $\mathcal{U}^s$, $-\infty < s < +\infty$, we take the spaces H^w, where

$$\begin{aligned} w &= (1 + |\xi|)^s (1 + |\eta|)^t, \\ w &= (1 + |\xi|)^s (1 + |\xi| + |\eta|)^t \end{aligned}$$

in the first and second cases, respectively.

Lemma 2.16 *Suppose* Δ *is elliptic in* U *with respect to* x. *If* $A(v, D)u = f$ *holds for* $u \in \mathcal{E}'(\mathbb{R}^N)$ *then for any open set* $O \subset U$ *and any* s

$$\left\{ \begin{array}{lcl} u & \in & H^{m,s,\delta}_{\mathrm{loc}}(O), \\ f & \in & H^s_{\mathrm{loc}}(O) \end{array} \right. \;\Rightarrow\; D^\alpha_x u \in H^{0,s+m-|\alpha|}_{\mathrm{loc}}(O) \cap H^{s+m-|\alpha|}_{\mathrm{loc}}(O)$$

whenever $|\alpha| \geq m$.

Proof. Let $s \geq 1$ and $|\alpha| = m+1$. By Lemma 2.14 it suffices to check in this case that

$$\frac{\partial}{\partial x_j}\, \Delta(v, D_x) u \in H^{0,s-1}_{\mathrm{loc}}(O) \tag{2.12}$$

for all $1 \leq j \leq n$. If we differentiate equality (2.10) in x_j and use the fact that the operator $[\partial/\partial x_j, a_{\alpha\beta\gamma}(v, D)]$ has zero order, then we see that (2.12) follows from

$$\frac{\partial}{\partial x_j} \left(x^{\gamma} D_x^{\alpha} D_y^{\beta} u \right) \in H^{0,s-1}_{\mathrm{loc}}(O)$$

for all $(\alpha, \beta, \gamma) \in \mathcal{I}$ with $|\alpha| < m$. This in turn is a consequence of the condition $u \in H^{m,s,\delta}_{\mathrm{loc}}(O)$. In the general case Lemma 2.16 is proved by induction on $|\alpha|$ by the same arguments, both Lemmas 2.14 and 2.15 being required.

□

Having disposed of these preliminary steps we are in a position to prove the main result of this section.

Theorem 2.17 *Let condition (0.4) hold and* $\ker \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$. *Then there exists a neighbourhood* U *of the point* $(0, y^0)$ *with the property that for all* $u \in \mathcal{E}'(\mathbb{R}^N)$ *and* $v \in \prod_{j=1}^{l} H^{-\infty}(\mathbb{R}^q)$, *any open subset* O *of* U, *and each* $s \geq 0$

$$\left\{ \begin{array}{rcl} \mathcal{A}(u,v) & \in & \mathcal{F}^s_{\mathrm{loc}}(O), \\ A(v,D)u & \in & H^s_{\mathrm{loc}}(O) \end{array} \right. \;\Rightarrow\; D_x^{\alpha} u \in H^{0,s+m-|\alpha|}_{\mathrm{loc}}(O) \cap H^{s+m-|\alpha|}_{\mathrm{loc}}(O)$$

whenever $|\alpha| \geq m$.

Proof. From Lemma 2.13 it follows that $f - F \in H^s_{\mathrm{loc}}(O)$, where F is defined in (2.10). Moreover, Theorem 2.10 implies $u \in H^{m,s,\delta}(O)$. Hence Theorem 2.17 follows from Lemma 2.16.

□

Corollary 2.18 *Suppose condition (0.4) holds and* $\ker \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ *for all* $|\eta| = 1$. *Then the operator* $\mathcal{A}$ *is hypoelliptic in some neighbourhood of* $(0, y^0)$, *i.e., for all* $u \in \mathcal{E}'(\mathbb{R}^N)$ *and* $v \in \prod_{j=1}^{l} H^{-\infty}(\mathbb{R}^q)$ *and any open subset* O *of* U

$$\mathcal{A}(u,v) \in C^{\infty}(O) \;\Rightarrow\; u \in C^{\infty}(O).$$

Here by the inclusion $\mathcal{A}(u,v) \in C^{\infty}(O)$ is meant that $f \in C^{\infty}(O)$ and $g \in C^{\infty}(\Omega)$, where $\mathcal{A}(u,v) = (f,g)$ and Ω is the intersection of O with $x = 0$.

Theorems 2.8 and 2.17 give a rather exact answer to the question raised above on the smoothness of a solution with respect to any variables. However, in many cases it is more convenient to use the following simpler relations.

Theorem 2.19 *If the hypotheses of Theorem 2.17 hold then*

$$x^\gamma D_x^\alpha D_y^\beta u \in H^s_{\mathrm{loc}}(O)$$

for all $(\alpha, \beta, \gamma) \in \mathcal{I}$.

Proof. It is sufficient to check that

$$\begin{array}{rcl} x^\gamma D_x^{\alpha+\alpha'} D_y^\beta u & \in & H^{s-A}_{\mathrm{loc}}(O), \\ x^\gamma D_x^\alpha D_y^{\beta+\beta'} u & \in & H^{s-A}_{\mathrm{loc}}(O) \end{array} \tag{2.13}$$

for all α' and β' with $|\alpha'| = |\beta'| = A$ and $A \geq s$.

If $|\alpha + \alpha'| \leq m$ then the first relation (2.13) follows from Theorem 2.8. Indeed, if $|\beta| \leq |\alpha'|$ then $x^\gamma D_x^{\alpha+\alpha'} D_y^\beta u = D_y^\beta x^\gamma D_x^{\alpha+\alpha'} u$ lies in $H^{0,s-|\beta|}_{\mathrm{loc}}(O)$, for $D_x^{\alpha+\alpha'} u \in H^{0,s}_{\mathrm{loc}}(O)$ by Theorem 2.8. The first relation (2.13) now follows from the inclusion

$$H^{0,s-|\beta|}_{\mathrm{loc}}(O) \hookrightarrow H^{0,s-A}_{\mathrm{loc}}(O) \hookrightarrow H^{s-A}_{\mathrm{loc}}(O).$$

If now $|\beta| \geq |\alpha'|$ then we write β as $\beta = \beta' + \beta''$, where $|\beta'| = A$. Then $x^\gamma D_x^{\alpha+\alpha'} D_y^\beta u = D_y^{\beta'} x^\gamma D_x^{\alpha+\alpha'} D_y^{\beta''} u$ lies in $H^{0,s-A}_{\mathrm{loc}}(O)$, since Theorem 2.8 yields $x^\gamma D_x^{\alpha+\alpha'} D_y^{\beta''} u \in H^{0,s}_{\mathrm{loc}}(O)$.

It remains to treat the case $|\alpha + \alpha'| \geq m$. But it this case we get by Theorem 2.17

$$x^\gamma D_x^{\alpha+\alpha'} u \in H^{0,s+m-|\alpha|-A}_{\mathrm{loc}}(O) \cap H^{s+m-|\alpha|-A}_{\mathrm{loc}}(O)$$

whence

$$\begin{array}{rcl} x^\gamma D_x^{\alpha+\alpha'} D_y^\beta u & = & D_y^\beta x^\gamma D_x^{\alpha+\alpha'} u \\ & \in & H^{0,s-A}_{\mathrm{loc}}(O) \cap H^{s-A}_{\mathrm{loc}}(O) \\ & \hookrightarrow & H^{s-A}_{\mathrm{loc}}(O). \end{array}$$

The second of the relations (2.13) follows immediately from Theorem 2.8, which finishes the proof.

□

3 Some applications

3.1 Symbol index

Denote by $\operatorname{ind} \sigma_{\mathrm{edge}}(A)(y^0, \eta)$ the index of the operator defined in (0.3), i.e.,

$$\operatorname{ind} \sigma_{\mathrm{edge}}(A)(y^0, \eta) = \dim \ker \sigma_{\mathrm{edge}}(A)(y^0, \eta) - \dim \operatorname{coker} \sigma_{\mathrm{edge}}(A)(y^0, \eta),$$

where $\sigma_{\text{edge}}(A)(y^0,\eta)$ is viewed as an operator from $H^{m,\delta}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. If condition (0.4) holds then $\sigma_{\text{edge}}(A)(y^0,\eta)$ is, by Lemma 1.10, a Fredholm operator for all $\eta \neq 0$. From the homotopic invariance of the index it follows that $\operatorname{ind}\sigma_{\text{edge}}(A)(y^0,\eta)$ does not depend on η for $q>1$, while for $q=1$ it depends only on the sign of η. In the case $q=1$ we will also require in this section that $\operatorname{ind}\sigma_{\text{edge}}(A)(y^0,\eta)$ not depend on the sign of η. Then $\operatorname{ind}\sigma_{\text{edge}}(\mathcal{A})(y^0,\eta)$ is also independent of η, for

$$\operatorname{ind}\sigma_{\text{edge}}(\mathcal{A})(y^0,\eta) = \operatorname{ind}\sigma_{\text{edge}}(A)(y^0,\eta) + l - k.$$

Suppose that l and k are chosen so that

$$k - l = \operatorname{ind}\sigma_{\text{edge}}(A)(y^0,\eta) \tag{3.1}$$

for all $\eta \in \mathbb{R}^q$ with $|\eta| = 1$. Hence it follows that $\operatorname{ind}\sigma_{\text{edge}}(\mathcal{A})(y^0,\eta) = 0$ for all $\eta \neq 0$.

As was proved in [Gru70b], in the scalar case $\operatorname{ind}\sigma_{\text{edge}}(A)(y^0,\eta) = 0$ provided that $n > 1$. Hence for $n > 1$ the equality (3.1) reduces to $k = l$. For $n = 1$ this is no longer so. For example, the index of $d/dx + ax^{2p+1}$, $\Re a \neq 0$, is equal to 1 if $\Re a > 0$, and -1 if $\Re a < 0$.

In the case $n = 1$ there is the following simple rule for computing the index of an operator with polynomial coefficients $A(x, D_x)$ for which (1.8) and (1.9) hold.

Lemma 3.1 *Let $\nu_\pm$ be the number of roots of the equation*

$$\sum_{\substack{|\alpha|\le m \\ |\gamma| = \delta(m-|\alpha|)}} a_{\alpha\gamma}(\pm 1)^\gamma \zeta^\alpha = 0$$

for which $\pm\Im\zeta > 0$. Then

$$\operatorname{ind} A(x, D_x) = \nu_+ + \nu_- - m.$$

Proof. The proof of Lemma 3.1 can be carried out in the same way as that of Theorem 5.1 in [GV69b].

□

Assume that for the symbols $p_j(v,\theta)$ the smoothness condition of Section 1.3 is fulfilled for any order. Then $p_{j,0}(0,y^0;\xi,\eta)$ lies in $\mathcal{S}(\mathbb{R}^n)$ for each $\eta \neq 0$. Since in this case $\mathcal{F}^{-1}_{\xi\mapsto x} p_{j,0}(0,y^0;\xi,\eta) \in \mathcal{S}(\mathbb{R}^n)$, we deduce from Lemma 1.12 that

$$(u(x), v) \in \ker \sigma_{\text{edge}}(\mathcal{A})(y^0,\eta) \ \Rightarrow\ u(x) \in \mathcal{S}(\mathbb{R}^n)$$

for any $\eta \neq 0$. Hence the following four assertion are equivalent:

1) $\ker \sigma_{\text{edge}}(\mathcal{A})(y^0,\eta) = 0$ for $|\eta| = 1$;

2) $\operatorname{coker} \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta) = 0$ for $|\eta| = 1$;
3) the mapping $\sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta)$ is an isomorphism for $|\eta| = 1$;
4) the null-space of (2.6) contains no nontrivial $(u(x), v)$ with $u(x) \in \mathcal{S}(\mathbb{R}^n)$.

Under these assumptions all the theorems of the preceding section are valid for the operator $\mathcal{A}$. In particular, $\mathcal{A}$ is hypoelliptic and locally solvable. Let us illustrate these assertions by two examples.

3.2 Oblique derivative problem

Let

$$A(v, D) = \frac{\partial}{\partial x} + a(y) x^p |D| \tag{3.2}$$

where p is an integer and x is a variable. Such pseudodifferential operators arise in the study of oblique derivative problem for the Laplace equation, cf. [MP74, PP97].

The symbol of the operator (3.2) is defined by the formula

$$a(v, \theta) = \imath \xi + a(y) x^p \sqrt{|\xi|^2 + |\eta|^2}$$

for $(\xi, \eta) \in \mathbb{R}^1 \times \mathbb{R}^q$. Since

$$|\theta| = \sum_{j=1}^{N} \frac{\theta_j}{|\theta|} \theta_j$$

and $\theta_j / |\theta|$ are homogeneous zero order functions, it follows that $A(v, D)$ has the form (0.2) for $\delta = p$.

Condition (0.4) holds here if $\Re a(y^0) \neq 0$. Furthermore, the operator (0.3) has the form

$$\sigma_{\mathrm{edge}}(A)(y^0, \eta) = \frac{\partial}{\partial x} + a(y^0) x^p |\eta|.$$

The other requirements formulated above hold for $A(v, D)$ without boundary and coboundary conditions if p is even.

If p is odd and $\Re a(y^0) > 0$ then the operator is locally solvable but not hypoelliptic. However, if we adjoin a boundary operator to $A(v, D)$, for example, $u(0, y) = g(y)$, then the equation

$$\mathcal{A} u := \begin{pmatrix} A(v, D) u \\ u(0, y) \end{pmatrix} = \begin{pmatrix} f(v) \\ g(y) \end{pmatrix}$$

is hypoelliptic and locally solvable.

If p is odd and $\Re a(y^0) < 0$ then the operator is hypoelliptic but is not locally solvable. In this case one can obtain from $A(v, D)$ an operator possessing these two properties if we add a coboundary operator to $A(v, D)$.

3.3 Unsolvable equations

The operator

$$A(v,D) = \frac{\partial^2}{\partial v_1^2} + v_1^2 \frac{\partial^2}{\partial v_2^2} + \imath\lambda \frac{\partial}{\partial v_2} \tag{3.3}$$

has the form (0.2) if we set $v_1 = x$, $v_2 = y$ and $\delta = 1$. Here the operator (0.3) has the form

$$\sigma_{\text{edge}}(A)(y^0, \eta) = \frac{\partial^2}{\partial x^2} - x^2 \, |\eta|^2 - \lambda \, \eta.$$

Hence (3.3) is hypoelliptic and locally solvable when $\pm\lambda$ does not coincide with eigenvalues of the ordinary differential operator $d^2/dx^2 - x^2$ on $L^2(\mathbb{R}^1)$. The eigenvalues and eigenfunctions of this latter operator are well known, namely

$$\begin{array}{rcl} \lambda_\nu & = & -(2\nu+1), \\ u_\nu(x) & = & H_\nu(x) \, \exp(-x^2/2) \end{array} \tag{3.4}$$

where $H_\nu(x)$ are Chebyshev-Hermite polynomials,

$$H_\nu(x) = (-1)^\nu e^{x^2} \frac{d^\nu}{dx^\nu} e^{-x^2}$$

for $\nu = 0, 1, \ldots$.

The first eigenvalue is -1, corresponding to the eigenfunction $\exp(-x^2/2)$. Hence (3.3) is hypoelliptic and locally solvable for all $\lambda \neq \pm(2\nu+1)$.

If λ coincides with $\pm(2\nu+1)$ then one can obtain an operator having these properties by adding to $A(v,D)$ one boundary and one coboundary condition. For example, if $\lambda = \pm 1$ this will hold for the operator

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A(v,D) + P(v,D)\,(\delta(x) \otimes v(y)) \\ u(0,y) \end{pmatrix},$$

if P satisfies the requirements imposed above on the coboundary operators, and if $\mathcal{F}^{-1}_{\xi\mapsto x} p_0(0, y^0; \xi, \eta)$ in (2.5) for $|\eta| = 1$ is not orthogonal to the function $\exp(-x^2/2)$.

As was shown in [Gru70b], the operator (3.3) itself is not hypoelliptic for $\lambda = \pm(2\nu+1)$. Let us show that it is not even locally solvable. Consider first the case $\lambda = -1$.

Theorem 3.2 *For any neighbourhood U of the point $(0, y^0)$ in $\mathbb{R}^2$ there exists a function $f \in C^\infty_{\text{comp}}(U)$ such that the equation*

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} - \imath \frac{\partial}{\partial y} = f \tag{3.5}$$

has no solutions $u \in \mathcal{D}'(U)$.

Proof. Denote by $A(v, D)$ the operator (3.5). It is formally selfadjoint on functions in $C^\infty_{\mathrm{comp}}(U)$. The local solvability of a differential equation is known to imply a certain a priori estimate for the adjoint operator. Let $O \Subset U$. Then this estimate has the form

$$\|g\|_{s_1} \leq C \; \|A(v, D)g\|_{s_2} \tag{3.6}$$

for all $g \in C^\infty_{\mathrm{comp}}(O)$, where s_1, s_2 and C are certain fixed constants. Let us show that (3.6) does not hold for our operator. To this end, note that for any $\varepsilon > 0$ and any integer N the function

$$u_{N,\varepsilon}(x, y) = \left(x + \imath \left(\frac{(y - y^0)^2}{2} + \varepsilon \right) \right)^{-N}$$

is a solution to $A(v, D)u_{N,\varepsilon} = 0$. Set $g_{N,\varepsilon} = \varphi \, u_{N,\varepsilon}$ where $\varphi \in C^\infty_{\mathrm{comp}}(O)$ and $\varphi(v) = 1$ in a smaller neighbourhood of $(0, y^0)$. Let ε tend to zero. If N is sufficiently large, the left-hand side of (3.6) tends to ∞, while the right-hand side is bounded. This contradiction proves the theorem.

□

Theorem 3.3 *If $\lambda = \pm(2\nu + 1)$ then for any neighbourhood U of the point $(0, y^0)$ in $\mathbb{R}^2$ there is a function $f \in C^\infty_{\mathrm{comp}}(U)$ such that the equation $A(v, D)u = f$, where $A(v, D)$ is defined by formula (3.3), has no solutions $u \in \mathcal{D}'(U)$.*

Proof. Set

$$\mathcal{F}_{y\mapsto\eta} u_{N,\varepsilon}(x, y) = |\eta|^N \, \theta(-\lambda\eta) \, e^{-\varepsilon|\eta|} u_\nu(x\sqrt{|\eta|})$$

for $\varepsilon \geq 0$, where u_ν is defined by (3.4), and $\theta(y)$ is the Heaviside function. If we write $u_{N,\varepsilon}(x, y)$ for the inverse Fourier transform of $\mathcal{F}_{y\mapsto\eta} u_{N,\varepsilon}(x, y)$ with respect to η, then for $\varepsilon > 0$ we obtain a smooth solution of the equation $A(v, D)u_{N,\varepsilon} = 0$. Using the explicit form of the function u_ν, it is easy to verify that $u_{N,\varepsilon} \to u_{N,0}$in $\mathcal{D}'(\mathbb{R}^2)$ and $C^\infty(\mathbb{R}^2 \setminus \{0\})$ as $\varepsilon \to \infty$, where the order of singularity of generalised functions $u_{N,0}$ is unbounded as $N \to \infty$. We now can repeat the proof of Theorem 3.2, substituting the new functions $u_{N,\varepsilon}$ for the old ones throughout.

□

3.4 Problems on a compact manifold

Let M be a smooth compact closed manifold of dimension N, and let $\mathcal{S}$ be a smooth submanifold in M of dimension q. Fix a covering $\{U_\nu\}$ of some neighbourhood U of $\mathcal{S}$ by coordinate patches on M, and a local coordinate system in each U_ν. Suppose that the coordinates split into $v = (x, y)$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_q)$, so that $\mathcal{S} \cap U_\nu$ is defined by the equation $x = 0$. We furthermore assume that for any U_ν and U_μ the variables y in the new local coordinate system are expressed in terms of the same coordinates in the old one.

Consider a pseudodifferential operator $A(v, D)$ of order m on M, which is elliptic off $\mathcal{S}$. We assume that in each neighbourhood U_ν the symbol of $A(v, D)$ coincides with that of some operator of the form (0.2), i.e., all the terms of the corresponding asymptotic expansions coincide.

Let y^0 be an arbitrary point on $\mathcal{S} \cap U_\nu$. We identify it with $(0, y^0)$ in the local coordinate system in U_ν and require that, having done so, condition (0.4) of the Introduction hold.

Similarly, let us define boundary and coboundary operators $B_i(v, D)$ and $P_j(v, D)$ and pseudodifferential operators $E_{ij}(y, D_y)$ on $\mathcal{S}$. There is no loss of generality in assuming that the symbols of $B_i(v, D)$ and $P_j(v, D)$ are identically zero away from U. In each neighbourhood U_ν we require asymptotic expansions of the form (2.1) in the corresponding local coordinate system to hold. We will study the operator $\mathcal{A}$ defined by formula (2.2).

Given any $(0, y^0) \in \mathcal{S} \cap U_\nu$, we consider the symbol $\sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta)$ which is defined by formula (2.5) in the local coordinate system. Suppose that both $\ker \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta)$ and $\operatorname{coker} \sigma_{\mathrm{edge}}(\mathcal{A})(y^0, \eta)$ are trivial for all $|\eta| = 1$.

Finally, let us define a family of spaces $H^{s,\mathcal{I}}(M)$ for $s \geq 0$. A function $u \in L^2(M)$ is said to lie in $H^{s,\mathcal{I}}(M)$ if for any neighbourhood U_ν we have

$$x^\gamma D_x^\alpha D_y^\beta u \in H^s_{\mathrm{loc}}(U_\nu)$$

for all $(\alpha, \beta, \gamma) \in \mathcal{I}$ and, moreover, $u \in H^{s+m}_{\mathrm{loc}}(M \setminus \mathcal{S})$. In the standard way we introduce a structure of Hilbert space on $H^{s,\mathcal{I}}(M)$.

Theorem 3.4 *Under the conditions formulated above, the operator*

$$\mathcal{A} : \begin{array}{c} H^{s,\mathcal{I}}(M) \\ \oplus \\ \oplus_{j=1}^{l} H^{s+t_j}(\mathcal{S}) \end{array} \to \begin{array}{c} H^{s}(M) \\ \oplus \\ \oplus_{i=1}^{k} H^{s+s_i}(\mathcal{S}) \end{array}$$

is Fredholm for all $s \geq 0$, where s_i and t_j are defined in Section 2.1.

The proof of this theorem is carried out in the same way as the usual proof of the Fredholm property of elliptic boundary value problems (cf., for example, [GV69a]) if the local theorems of Section 2 are used.

3.5 Edge problems

A singular space with edges looks locally like $\mathcal{C} \times \Omega$ where $\mathcal{C}$ is a cone in $\mathbb{R}^n$ and Ω an open set in $\mathbb{R}^q$, cf. Fig. 1.

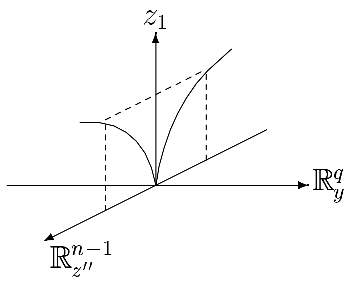

Fig. 1: Splitting of coordinates

Suppose the cone $\mathcal{C}$ is given by

$$\left\{ \begin{array}{rcl} z_1 & = & x_1, \\ z_2 & = & f_2(x_1)\, x_2, \\ \dots & = & \dots \\ z_n & = & f_n(x_1)\, x_n, \end{array} \right. \tag{3.7}$$

where $x'' = (x_2, \dots, x_n)$ varies over a compact domain B with a C^∞ boundary in $\mathbb{R}^{n-1}$, and $f_2, \dots, f_n$ are C^∞ functions of $x_1 > 0$, tending to 0 when $x_1 \to 0$.

Using the coordinates $x = (x_1, x'')$ actually blows up the singularity along the edge, thus resulting in operators on $\mathbb{R}^n$ which are degenerate on the subspace $\mathbb{R}^q$. More precisely, under the change of variables (3.7) vector fields transform by

$$\begin{array}{rcl} D_{z_1} & = & D_{x_1} + \dfrac{f_2'(x_1)}{f_2(x_1)}\, x_2 D_{x_2} + \dots \dfrac{f_n'(x_1)}{f_n(x_1)}\, x_n D_{x_n}, \\ D_{z_2} & = & \dfrac{1}{f_2(x_1)}\, D_{x_2}, \\ \dots & \dots & \dots \\ D_{z_n} & = & \dfrac{1}{f_n(x_1)}\, D_{x_n}. \end{array}$$

The first derivative is actually the Euler operator over $\mathbb{R}^n$ in the case of power-like cuspidal edges, i.e., $f_j(x_1) = x_1^{p_j}$ with $p_j > 0$. In this case any differential operator

$$A = \sum_{|\alpha|+|\beta| \le m} A_{\alpha\beta}(z,y) D_z^\alpha D_y^\beta$$

with smooth coefficients in a neighbourhood of the wedge $\mathcal{C} \times \Omega$ takes in the coordinates (x, y) the form

$$\pi^* A = x_1^{-s_0} \sum_{(\alpha,\beta,\gamma) \in \mathcal{I}} a_{\alpha\beta\gamma}(x,y)\, x^\gamma\, (x_1 D_{x_1})^{\alpha_1}\, D_{x''}^{\alpha''} D_y^\beta, \tag{3.8}$$

where $\mathcal{I}$ is the set of all triples (α, β, γ) of multi-indices which satisfy the conditions

$$\begin{array}{rcccl} & & |\alpha| + |\beta| & \leq & m, \\ 0 & \leq & \gamma'' & \leq & \alpha'', \\ s_0 - \langle p'', \alpha'' \rangle - (m - |\alpha''| - |\beta|) & \leq & |\gamma| - \langle p'', \gamma'' \rangle & \leq & s_0 - \langle p'', \alpha'' \rangle - \alpha_1, \end{array}$$

cf. (0.1). Here

$$\begin{array}{rcl} \alpha'' & = & (\alpha_2, \dots, \alpha_n), \\ \gamma'' & = & (\gamma_2, \dots, \gamma_n), \\ p'' & = & (p_2, \dots, p_n). \end{array}$$

We thus arrive at the class of degenerate operators (0.2) on $\mathbb{R}^N$, the only difference being in the analysis of such operators on the half-space $v_1 \geq 0$. Such problems can be treated by combining two particular cases of the theory developed in Section 2. Namely, we adjoin to A boundary conditions on the hyperplane $x_1 = 0$, to be fulfilled away from the boundary submanifold $\mathbb{R}^q_y$. This corresponds to the case $n = 1$, the bijectivity condition for the relevant edge symbol (2.5) being simply the Lopatinskii condition. Having constructed a local parametrix for the boundary value problem close to the points of $\{(x, y) \in \mathbb{R}^N : \ x_1 = 0, \ x \neq 0\}$, we can proceed by posing conditions along the edge $\mathbb{R}^q_y$ defined by the equation $x = 0$. We are thus led to considering block matrices

$$\mathcal{A} = \begin{pmatrix} A_{11} & P_{12} & P_{13} \\ B_{21} & A_{22} & P_{23} \\ B_{31} & B_{32} & A_{33} \end{pmatrix} \tag{3.9}$$

whose entries are pseudodifferential operators. These are typical for the analysis on stratified spaces, cf. [ST00].

The upper left corner

$$\begin{pmatrix} A_{11} & P_{12} \\ B_{21} & A_{22} \end{pmatrix}$$

defines a boundary value problem in the half-space $x_1 > 0$ with boundary data away from the edge $x = 0$. Using a parametrix of this problem, if available, we may reduce the equation $\mathcal{A}U = F$ to an equation on the edge $\mathbb{R}^q_y$ which is a smooth closed manifold.

Instead we make use of a parametrix of A_{11} for $x_1 > 0$ to reduce the problem $\mathcal{A}U = F$ to a problem on the boundary $x_1 = 0$ with data on the subspace $\mathbb{R}^q_y$. To such a problem the theory of Section 2 applies to give conditions of hypoellipticity and local solvability.

More precisely, write A_{11}^{-1} for a left regulariser of A_{11} on the half-space $x_1 > 0$. The existence of an appropriate regulariser of A_{11} in the interior is an essential part of the ellipticity condition of $\mathcal{A}$. By Green's formula we have $A^{-1}A + GB = 1$ up to a smoothing operator on the closed half-space $x_1 \geq 0$, where B is a boundary operator (Cauchy data) and G a coboundary operator (double layer potential) relative to the hyperplane $x_1 = 0$. Writing u_j and f_i for the components of U and F, respectively, we get from the first equality in $\mathcal{A}U = F$

$$u_1 = GBu_1 + A_{11}^{-1}(f_1 - P_{12}u_2 - P_{13}u_3)$$

modulo smoothing terms for $x_1 \geq 0$. Substituting this to the last two equalities of $\mathcal{A}U = F$ we obtain

$$\begin{cases} B_{21}GBu_1 + (A_{22} - B_{21}A_{11}^{-1}P_{12})u_2 + (P_{23} - B_{21}A_{11}^{-1}P_{13})u_3 = f_2 - B_{21}A_{11}^{-1}f_1, \\ B_{31}GBu_1 + (B_{32} - B_{31}A_{11}^{-1}P_{12})u_2 + (A_{33} - B_{31}A_{11}^{-1}P_{13})u_3 = f_3 - B_{31}A_{11}^{-1}f_1 \end{cases} \tag{3.10}$$

up to smoothing terms for $x_1 \geq 0$.

The unknown functions in (3.10) are Bu_1, u_2 and u_3, the first two being functions on $x_1 = 0$, and the last a function on $x = 0$. The problem (3.10) is a particular case of (2.2).

3.6 Edge algebra

The reduction to the boundary of the proceeding section shows that everything goes on much easier if one controls the operators within an algebra. In this section we present such an algebra. This is actually a slight modification of the edge algebra of [Sch98] to the anisotropic case, cf. [BS97]. To obtain such an algebra one reformulates pseudodifferential operators over $\mathbb{R}^N$ as those over $\mathbb{R}^q$ with symbols taking their values in an algebra of pseudodifferential operators over the fiber $\mathbb{R}^N/\mathbb{R}^q \cong \mathbb{R}^n$. Then it is a good luck if the function spaces under consideration in $\mathbb{R}^N$ admit a "twisted" reformulation over $\mathbb{R}^q$ including an appropriate group action in the fiber. The symbols are required to satisfy the corresponding "twisted" estimates. Any reasonable Fredholm theory for parameter-dependent operators over the fiber results then in a Fredholm theory on all of $\mathbb{R}^N$.

It remains to specify these data for our algebra. As starting operators on $\mathbb{R}^N$ we take (0.2). The group action is already fixed by (2.7) up to a constant factor c. We choose

$$c = \lambda^{\frac{n}{2}-m},$$

m standing for the smoothness in the fiber $\mathbb{R}^n$. A Fredholm theory for operators on $\mathbb{R}^n$ is discussed in Section 1.5, whence the weighted Sobolev spaces $H^{m,\delta}(\mathbb{R}^n)$ over fibers.

We now recall the definition of abstract edge Sobolev spaces. Let V be a Hilbert space, and $(\kappa_\lambda)_{\lambda\in\mathbb{R}_+}$ be a strongly continuous group of isomorphisms on V.

Definition 3.5 *Given any $t \in \mathbb{R}$, by $H^t(\mathbb{R}^q, \pi^* V)$ is meant the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm*

$$\|u\|_{H^t(\mathbb{R}^q, \pi^* V)} = \left(\int \langle \eta \rangle^{2t} \| \kappa_{\langle \eta \rangle}^{-1} \mathcal{F}_{y \to \eta} u \|_V^2 d\eta \right)^{\frac{1}{2}}.$$

Equivalent norms in V give rise to equivalent norms in $H^s(\mathbb{R}^q, \pi^* V)$. Instead of $\langle \eta \rangle = (1 + |\eta|^2)^{1/2}$ we may equivalently use any C^∞ function $\eta \mapsto \langle \eta \rangle$ on $\mathbb{R}^q$, such that $\langle \eta \rangle > 0$ and $\langle \eta \rangle = |\eta|$ for all $|\eta| > c$, where c is a positive constant.

Lemma 3.6 *For each $t \in \mathbb{R}$, the Banach space $H^{m,t,\delta}$ is isomorphic to the space $H^t(\mathbb{R}^q, \pi^* H^{m,\delta}(\mathbb{R}^n))$.*

Proof. This immediately follows from

$$\left(1 + \langle \eta \rangle^{\frac{2}{1+\delta}} |x|^2 \right)^{\delta} \langle \eta \rangle^{\frac{2}{1+\delta}} \sim \left(1 + |x|^{\delta} |\eta| + \langle \eta \rangle^{\frac{1}{1+\delta}} \right)^2.$$

□

4 Reduction to a smaller number of variables

4.1 Motivation

Consider a differential operator of the form

$$A(v, D) = \sum_{(\alpha,\beta,\gamma) \in \mathcal{I}} a_{\alpha\beta\gamma}(v)\, x^{\gamma} D_x^{\alpha} D_y^{\beta} \tag{4.1}$$

with C^∞ coefficients, for which condition (0.4) holds. If $\ker \sigma_{\mathrm{edge}}(A)(y^0, \eta) = 0$ for all $|\eta| = 1$ then the operator (4.1) is hypoelliptic in some neighbourhood of $(0, y^0)$.

We now suppose that $\ker \sigma_{\mathrm{edge}}(A)(y^0, \eta) \neq 0$ for some $|\eta| = 1$. If $A(v, D)$ has the form (0.6), it is not hypoelliptic, cf. [Gru70b]. For an operator of the form (4.1) the situation is more involved. For example, we show that the equation

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} - \imath \frac{\partial u}{\partial y} + a x^2 \frac{\partial u}{\partial y} = f(x, y) \tag{4.2}$$

is hypoelliptic for $a \neq 0$.

Let us adjoin to $A(v, D)$ boundary and coboundary operators such that the conditions of Theorem 2.17 are satisfied for the operator $\mathcal{A}$ defined by formulas (2.2). Then the operator (4.1) will be hypoelliptic in U if and only if

$$\left\{ \begin{array}{rcl} f & \in & C^\infty(O), \\ v(y) & = & 0 \end{array} \right. \quad \Rightarrow \quad g \in C^\infty(\Omega)$$

where O is any open subset of U and Ω is the intersection of O with the manifold $x = 0$. The main goal of this section is to show that under certain conditions one can construct a pseudodifferential operator $Q(y, D_y)$ on $\mathbb{R}^q$, such that

$$f \in C^\infty(O) \;\Rightarrow\; v - Q(y, D_y)g \in C^\infty(\Omega). \tag{4.3}$$

In this case it follows by Lemma 2.12 and Corollary 2.18 that the operator $A(v, D)$ is hypoelliptic if and only if $Q(y, D_y)$ is hypoelliptic.

4.2 Parametrisation of boundary value problems

Thus, let boundary and coboundary operators be given, $B_i(v, D)$ satisfying the smoothness conditions of Section 2.1, $P_j(v, D)$ bearing smoothness conditions of any order.

For simplicity we will assume that the symbols of $B_i(v, D)$ and $P_j(v, D)$ do not depend on v and are quasihomogeneous in θ, i.e.

$$\begin{aligned} b_i &= b_{i,0}(\theta), \\ p_j &= p_{j,0}(\theta), \end{aligned}$$

while the operators $E_{ij}(v, D)$ are equal to zero. We suppose that

$$\begin{aligned} \ker \sigma_{\text{edge}}(\mathcal{A})(y^0, \eta) &= 0, \\ \operatorname{coker} \sigma_{\text{edge}}(\mathcal{A})(y^0, \eta) &= 0 \end{aligned} \tag{4.4}$$

for all $|\eta| = 1$.

Theorem 4.1 *Under the above assumptions one can construct a pseudodifferential operator $Q(y, D_y)$ in $\mathbb{R}^q$, such that (4.3) holds for some neighbourhood of the point $(0, y^0)$.*

Proof. Note that conditions (0.4) and (4.4) imposed on the operator (4.1) are stable in the sense that they are preserved under a small variation of the coefficients $a_{\alpha\beta\gamma}$ in (4.1). In particular, they are still preserved in some neighbourhood of $(0, y^0)$, i.e., at any point $(0, v) \in \mathcal{S}$ with $|v - y^0| < \varrho$, where ϱ is sufficiently small. Let us first formally describe the construction of the operator $Q(y, D_y)$.

A function $F(x, y, \xi, \eta)$ is said to be a generalised homogeneous function of order ν provided

$$F\left(\frac{x}{\lambda}, \frac{y}{\lambda^{1+\delta}}, \lambda\xi, \lambda^{1+\delta}\eta\right) = \lambda^\nu F(x, y, \xi, \eta)$$

for all $\lambda > 0$. Given any $v \in \mathbb{R}^q$ and $N > 0$, the operator (4.1) can be written in the form

$$A(v, D) = \sum_{\nu=0}^{N} A_\nu(x, v, y - v; D_x, D_y) + A_N, \tag{4.5}$$

where $A_\nu(x,\upsilon,y;\xi,\eta)$ are polynomials in $(x,y;\xi,\eta)$ which are generalised homogeneous functions of their arguments, while the coefficients of the operator A_N vanish at the point $(0,\upsilon)$ for large N, the order of the zero growing as $N\to\infty$. The variables υ here play the role of parameters, and the orders of the A_ν (in the sense of the definition given above) are equal to $m-\nu/m$. The expansion (4.5) is obtained if we substitute for the coefficients $a_{\alpha\beta\gamma}(x,y)$ their Taylor expansions at the point $(0,\upsilon)$ and group those terms which have identical generalised order. In particular, the component A_0 is defined by the expression

$$\begin{aligned} A_0(x,\upsilon,y;\xi,\eta) &= \sum_{(\alpha,\beta,\gamma)\in\mathcal{I}_0} a_{\alpha\beta\gamma}(0,\upsilon)\,x^\gamma D_x^\alpha D_y^\beta \\ &= \sigma_{\mathrm{edge}}(A)(\upsilon,\eta). \end{aligned}$$

Set $\hat{A}_\nu = A_\nu(x,\upsilon,-D_\eta,D_x,\eta)$. Consider the following infinite system of partial differential equations in $\mathbb{R}^n$

$$\left\{\begin{array}{rcl} \hat{A}_0E_0(x,\upsilon;\eta)+\hat{p}(x,\eta)Q_0(\upsilon;\eta) &=& 0, \\ \hat{A}_0E_1(x,\upsilon;\eta)+\hat{p}(x,\eta)Q_1(\upsilon;\eta) &=& -\hat{A}_1E_0(x,\upsilon;\eta), \\ \dots & \dots & \dots \\ \hat{A}_0E_N(x,\upsilon;\eta)+\hat{p}(x,\eta)Q_N(\upsilon;\eta) &=& -\sum\limits_{\nu=1}^{N}\hat{A}_\nu E_{N-\nu}(x,\upsilon;\eta), \\ \dots & \dots & \dots \end{array}\right. \tag{4.6}$$

where $\hat{p}(x,\eta)=\mathcal{F}^{-1}_{\xi\mapsto x}p(\xi,\eta)$. The unknowns $E_\nu(x,\upsilon;\eta)$ in this system are vector-valued functions written in the form of a row. The vector $\hat{p}(x,\eta)$ is written in the form of a row, too, and $Q_\nu(\upsilon;\eta)$ is an $(l\times k)$-matrix. We assume that all the components of $E_\nu(x,\upsilon;\eta)$ lie in $\mathcal{S}(\mathbb{R}^n_x)$ in the variables x, and

$$\left\{\begin{array}{rcl} r_0\,B(\eta,D_x)E_0(x,\upsilon;\eta) &=& I, \\ r_0\,B(\eta,D_x)E_\nu(x,\upsilon;\eta) &=& 0 \end{array}\right.$$

for each $\nu>0$, where BE_ν is the matrix with rows B_iE_ν, $1\le i\le k$, and I is the unit matrix.

The theorems proved in Section 1.5 and the conditions imposed on the operator $\mathcal{A}$ guarantee the unique solvability of (4.6), and the solution is infinitely differentiable in υ and η for $|\upsilon-y^0|<\varrho$ and $\eta\neq 0$. Here it is important to keep in mind that $\hat{p}(x,\eta)\in\mathcal{S}(\mathbb{R}^n_x)$ for $\eta\neq 0$, since smoothness conditions of any order hold for p_j.

It suffices to solve (4.6) only on the unit sphere $|\eta|=1$. Then for the other η the solution is easily determined if we make use of the property of quasihomogeneity of the operators $\hat{A}_\nu$, b_i and p_j. Hence it is clear that the components of E_ν are quasihomogeneous of order $-m_i-\nu/m$, $1\le i\le k$, while the entries $Q_{ji,\nu}$ of the matrix Q_ν are usual homogeneous functions of order $(m - m_i - r_j - n - \nu/m)/(1+\delta)$.

The procedure indicated above for the construction of matrices E_ν and Q_ν remind of the construction of the refined parametrisation of elliptic boundary value problems in [VG67b].

4.3 Reduction

We now define an operator G_N which maps generalised functions $g(y)$ with a support in the ball $|y - y^0| < \varrho$ to generalised functions on $\mathbb{R}^N$ by the formula

$$G_N g = \mathcal{F}^{-1}_{\eta \mapsto y} \mathcal{F}_{\upsilon \mapsto \eta} \psi(\eta) \sum_{\nu=0}^{N} E_\nu(x, \upsilon; \eta) g(\upsilon), \tag{4.7}$$

and the operator $Q_N(y, D_y)$ by the formula

$$Q_N(y, D_y) g = \mathcal{F}^{-1}_{\eta \mapsto y} \mathcal{F}_{\upsilon \mapsto \eta} \psi(\eta) \sum_{\nu=0}^{N} Q_\nu(\upsilon; \eta) g(\upsilon), \tag{4.8}$$

where $\psi \in C^\infty(\mathbb{R}^q)$ is an excision function, i.e., $\psi(\eta) = 0$ if $|\eta| < 1$ and $\psi(\eta) = 1$ if $|\eta| > 2$. Analysis similar to that in [VG67b] shows that

$$\begin{aligned} A(v, D) G_N g + P(D) \left(\delta(x) \otimes Q_N(y, D_y) g \right) &= S_N g, \\ r_0 \, B(D) G_N g &= g + S_N g, \end{aligned}$$

where S_N in the first equality denotes the operator mapping g to a generalised function on $\{v \in \mathbb{R}^N : |v - (0, y^0)| < \varrho\}$, and S_N in the second equality denotes the operator mapping g to a generalised function on $\{y \in \mathbb{R}^q : |y - y^0| < \varrho\}$. For sufficiently large N these operators are smoothing and the smoothing order grows with N.

Formula (4.8) shows that $Q_N(y, D_y)$ is a pseudodifferential operator with double symbol $\sum_{\nu=0}^{N} Q_\nu(\upsilon; \eta)$. Any such operator coincides, up to an arbitrarily strongly smoothing operator, with some pseudodifferential operator of the form

$$g \mapsto \frac{1}{(2\pi)^q} \int e^{\imath \langle y, \eta \rangle} Q_N(y, \eta) \hat{g}(\eta) \, d\eta.$$

Since the orders of the operators Q_ν tend to $-\infty$, one can construct a pseudodifferential operator $Q(y, D_y)$ such that the difference $Q(y, D_y) - Q_N(y, D_y)$ is an arbitrarily strongly smoothing operator for sufficiently large N.

In order to show that $Q(y, D_y)$ possesses property (4.3) for any open set O in the ball $\{v \in \mathbb{R}^N : |v - (0, y^0)| < \varrho\}$, where ϱ is small enough, it suffices now to use Theorem 2.17. Indeed, since

$$\mathcal{A}(u, v) - \mathcal{A}(G_N g, Qg) = (f - S_N g, -S_N g)$$

the difference $v - Qg$ is, by Theorem 2.17, an arbitrarily smooth function in O for large N. This establishes (4.3).

□

As already mentioned, Theorem 4.1 implies

Corollary 4.2 *Let the conditions of Theorem 4.1 be fulfilled. The operator $A(v, D)$ is hypoelliptic in O if and only if the operator $Q(y, D_y)$ is hypoelliptic on Ω.*

4.4 An example

Invoking the construction which was used in the proof of Theorem 4.1 one can compute the symbol of the operator Q in concrete examples. Let us compute the principal part of the symbol of Q for the operator (4.2) when we adjoin to it the boundary operator $B = 1$ and the coboundary operator for which $\hat{p} = \exp(-x^2|\eta|/2)$. Then

$$\begin{array}{rcl} A_0(x; D_x, \eta) & = & \dfrac{d^2}{dx^2} - (x\eta)^2 + \eta, \\ A_4(x; D_x, \eta) & = & \imath a\, x^2 \eta, \end{array}$$

while $A_1 = A_2 = A_3 = 0$.

The first equation in (4.6) is

$$\left(\frac{d^2}{dx^2} - (x\eta)^2 + \eta \right) E_0(x, \eta) + \exp\left(-x^2|\eta|/2\right) Q_0(\eta) = 0$$

where $E_0(0, \eta) = 1$. Since the factor at $Q_0(\eta)$ is an eigenfunction of the operator $A_0(x; D_x, \eta)$, there exists here a unique solution

$$\begin{array}{rcl} E_0(x, \eta) & = & \exp\left(-x^2|\eta|/2\right), \\ Q_0(\eta) & = & \begin{cases} 0 & \text{if } \eta > 0, \\ 2|\eta| & \text{if } \eta < 0. \end{cases} \end{array}$$

Since $Q_0(\eta) \neq 0$ for $\eta < 0$, it remains to find the $Q_\nu(\eta)$ for $\eta > 0$. It is easy to see that $E_\nu = Q_\nu = 0$ for $\nu = 1, 2, 3$. Let us look at the fourth equation in (4.6),

$$\left(\frac{d^2}{dx^2} - (x\eta)^2 + \eta \right) E_4(x, \eta) + \exp\left(-x^2|\eta|/2\right) Q_4(\eta) = -\imath a\, x^2 \eta \exp\left(-x^2|\eta|/2\right)$$

and $E_4(0, \eta) = 0$. Since the eigenfunctions of the operator $A_0(x; D_x, \eta)$ form an orthogonal basis and $\exp\left(-x^2|\eta|/2\right)$ is the first function of this basis, we have

$$-Q_4(\eta) = \imath a \eta \, \frac{\int x^2 \exp\left(-x^2|\eta|/2\right) dx}{\int \exp\left(-x^2|\eta|/2\right) dx}$$

for $\eta > 0$, so that $Q_4(\eta) = C\, a$ where C is a nonzero constant.

Theorem 4.3 *The equation (4.2) is hypoelliptic for $a \neq 0$.*

Proof. Since the principal part of the symbol of Q is equal to $2|\eta|$ for $\eta < 0$ and $C\,a$ for $\eta > 0$, Q belongs to the class of $(0,1)$-elliptic operators defined in [VG67a]. As was shown in [VG67a], such operators are hypoelliptic (a pseudodifferential regulariser for them is constructed in [VG67a]). Hence Theorem 4.3 follows from Corollary 4.2.

□

4.5 Analyticity of solutions

Let

$$A(v,D) = \sum_{(\alpha,\beta,\gamma)\in\mathcal{I}} a_{\alpha\beta\gamma}(v)\, x^{\gamma} D_x^{\alpha} D_y^{\beta}$$

and let the coefficients $a_{\alpha\beta\gamma}(v)$ be real analytic in some neighbourhood of a point $(0,y^0)$. If condition (0.4) holds then it follows from Lemma 1.10 that for each $\eta \neq 0$ the equation

$$\sigma_{\mathrm{edge}}(A)(y^0,\eta)u(x) = 0 \tag{4.9}$$

has only a finite number of linearly independent solutions in the space $\mathcal{S}(\mathbb{R}^n)$. We will impose a stronger condition, namely that the equation (4.9) has no nontrivial solutions in $\mathcal{S}(\mathbb{R}^n)$ for all $|\eta| = 1$.

Under these assumptions it was shown in [Gru70b] that $A(v,D)u = f$ is hypoelliptic in some neighbourhood of the point $(0,y^0)$. As a special case this assertion is contained in Corollary 2.18, for Lemma 1.11 implies that $\ker \sigma_{\mathrm{edge}}(A)(y^0,\eta) = 0$ if $\sigma_{\mathrm{edge}}(A)(y^0,\eta)$ is thought of as an operator from $H^{m,\delta}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

In this section we prove the following theorem on the analyticity of solutions to $A(v,D)u = f$.

Theorem 4.4 *Let (0.4) hold and let (4.9) have no nontrivial solutions in $\mathcal{S}(\mathbb{R}^n)$ for all $|\eta| = 1$. Then there is a neighbourhood U of $(0,y^0)$ such that every (generalised) solution of $A(v,D)u = f$ with function f analytic in U is itself an analytic function in U.*

Proof. The proof uses a priori estimates which follow from the results of Section 2, and in its idea reminds one of the proof of the analyticity of solutions to elliptic equations put forth by Morrey and Nirenberg [MN57]. Cf. [Gru71] for more details.

□

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