Formation and construction of a shock wave for 3-D compressible Euler equations with spherical initial data*

Yin Huicheng

(Department of Mathematics, Nanjing University, Nanjing 210093, P.R.China)

(Institute of Mathematical Sciences, CUHK, Shatin, N.T., HongKong)

Email Address: huicheng@nju.edu.cn

Abstract

In this paper, the problem on formation and construction of a shock wave for three dimensional compressible Euler equations with the small perturbed spherical initial data is studied. If the given smooth initial data satisfies certain nondegenerate condition, then from the results in [20], we know that there exists a unique blowup point at the blowup time such that the first order derivatives of smooth solution blow up meanwhile the solution itself is still continuous at the blowup point. From the blowup point, we construct a weak entropy solution which is not uniformly Lipschitz continuous on two sides of shock curve, moreover the strength of the constructed shock is zero at the blowup point and then gradually increases. Additionally, some detailed and precise estimates on the solution are obtained in the neighbourhood of the blowup point.

Keywords: Compressible Euler equations, lifespan, nondegenerate condition, shock wave

Mathematics Subject Classification: 35L70, 35L65

§1. Introduction

In this paper, we are concerned with the development of singularities of solution to the following three dimensional isentropic compressible Euler equations with the smooth spherical initial data :

$$\begin{cases} \partial_t\rho + div(\rho u) = 0 \\ \partial_t(\rho u) + div(\rho u\otimes u + pI) = 0 \\ \partial_t(\rho e + \frac{1}{2}\rho u^2) + div((\rho e + \frac{1}{2}\rho u^2 + p)u) = 0 \\ \rho|_{t=0} = \bar\rho + \varepsilon\rho_0(x), u|_{t=0} = \varepsilon u_0(x), S|_{t=0} = \bar S \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity, ρ the density, p the pressure, e the internal energy, I the 3×3 unit matrix, and S the specific entropy. Moreover, the pressure function $p = p(\rho, S)$ and the internal energy function $e = e(\rho, S)$ are smooth on their arguments, in particular, $\partial_\rho p(\rho, S)$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$. With respect to the initial data in (1.1), $\bar\rho > 0$ and $\bar S$ are constants, $\varepsilon > 0$ is a sufficiently small parameter, $\rho_0(x), u_0(x) \in C^\infty(\mathbb{R}^3)$ and have compact supports in the ball $B(0, M)$. In what follows, we assume that $u_0(x) = w_0(x)x$, where $w_0(x)$ is a smooth function in $\mathbb{R}^3$, and $w_0(x)$, $\rho_0(x)$ depend only on r, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Under the above assumptions, by the results in [1] or [2], we know that the lifespan T_ε of

* The author was supported by the National Natural Science Foundation of China.

smooth solution to (1.1) satisfies:

$$\lim_{\varepsilon\to 0}\varepsilon lnT_\varepsilon=\tau_0\equiv-\frac{2\bar{c}}{(\bar{\rho}c'(\bar{\rho},\bar{S})+\bar{c})\min_{|q|\leq M}[q^2\partial_q w_0(q)+3qw_0(q)+\frac{\bar{c}}{\bar{\rho}}(q\partial_q\rho_0(q)+\rho_0(q))]} \tag{1.2}$$

where $c(\rho,S)=\sqrt{\partial_\rho p(\rho,S)}$, $\bar{c}=c(\bar{\rho},\bar{S})$, moreover $\tau_0>0$ as long as $\rho_0(x)\not\equiv 0$ or $u_0(x)\not\equiv 0$. Therefore, (1.2) implies that the nontrivial smooth solution of (1.1) must blow up in finite time. To better understand the physical process of development of singularities from smooth flow and the evolution of singularities starting from the blowup point, we are motivated to give a precise description on the estimates of solution and its derivatives in the neighbourhood of the blowup point.

Now we briefly mention some remarkable works on the hyperbolic conservation laws in one space dimension, maybe this will be helpful to understand our motivation better in this paper. With respect to the global existence and uniqueness of weak solution with the small value in the BV spaces for 1-D hyperbolic conservation laws (not contain the source terms), many important literatures have treated this problem and make the great success(see [3-7] and the references therein), for instance, the results in [5] and [7] yield the uniqueness, continuous dependence and global stability of weak, entropy-admissible solutions of the Cauchy problem for general $n\times n$ systems of conservation laws with small initial data. For the system (1.1), if it has a spherical structure, then by use of the polar coordinate transformation, one can change it into a 3×3 conservation law with the source terms which contain the singularities on $r=0$. Because of the appearance of the singular source terms, many essential difficulties can be met when one wants to study the global existence and the uniqueness of weak solution. It seems that the problem on the global existence of spherical weak entropy solution to (1.1) is still open. Here we should point out that if the system (1.1) does not contain the third equation on the entropy, the authors in [8] have proved the global existence of weak entropy solution for $p(\rho)=A\rho^\gamma$ $(1\leq\gamma\leq\frac{5}{3})$ outside a core with the center at the origin. Anyway, whether the global weak solution exists or not, the all above results don't give the detailed properties of solution near the blowup point when the classical solution breaks down. Corresponding to the understanding of physical process, an interesting problem is to give a clear picture on the appearance of singularities starting from the blowup point, particularly, the singularity with the type of shock.

As in [9], the above problem is still called as formation and construction of shock. For the scalar equation, this problem has been completely solved early(see [10-12] and so on). It is well known that in this case the formation of shock is caused by the squeeze of characteristics. For 2×2 p-system of gas dynamics the same fact is also true (see [13-14]) under the nondegenerate conditions on the initial data or the blowup point. One of the basic ideas in [13] and [14] is to introduce the Riemann invariants so that the p-system can be diagonalized and subsequently analyze the singularity property of blowup point as in 1-D scalar equation.

For the $n\times n$ $(n\geq 3)$ quasilinear strictly hyperbolic system in one space dimension with the small initial data, if it is genuinely nonlinear with respect to a characteristic family and the given smooth initial data satisfy the nondegenerate condition, then it is well known that the corresponding smooth solution blows up at only one point at the blowup time(see [15-18]). Near the blowup point, in [9] we have constructed a weak entropy solution. In contrast to the case of 2×2 p-system in [13] and [14], where the existence of Riemann invariants play the crucial role, one of the new ideas in [9] is to find a new transformation such that the solution is more singular along one direction than other directions for the $n \times n$ ($n \geq 3$) hyperbolic conservation law, which generally can not be diagonalized by the Riemann invariants. By use of this new form of $n \times n$ hyperbolic system and S.Alinhac's result on the analysis of blowup system (see [18]), in [9] we complete the construction of a shock starting from the blowup point through the delicate analysis.

A natural problem is: If only the first order derivatives of smooth solution to 1-D conservation law system with the source terms blow up meanwhile the solution itself is bounded, will the shock starting from the blowup point be formed and propagated as in [9] and [13-14]? In particular, for 1-D compressible isentropic Euler equations, [9] has shown that the new shock will be formed and propagated from the blowup point, does the similar phenomena happen for the multidimensional system (1.1) with the spherical structure? For the latter question, we will give an affirmative answer in this paper. To prove our conclusion, as in [9] and [13], one of the main difficulties is that the derivatives of solution blow up with the ratio of $\frac{1}{T_\varepsilon - t}$ and are not integrable with respect to the space variable or the time variable although the solution itself is bounded, so here problem is different from the usual Riemann problem on the hyperbolic conservation laws because the usual Riemann problem generally has the discontinuous and piecewise smooth initial data. However, comparing with the methods in [9], we find that some new ingredients are needed. The first one is that we need a result on the extension of solution across the blowup time so that we can analyze the blowup mechanism of smooth solution at the blowup point and give the detailed descriptions on the derivatives of solution. More precisely, we will show that the 3-shock will be formed from the blowup point. The second one is that we need to give a more complicated computation induced by the appearance of source terms in order to prove the convergence of iterative scheme when the shock is constructed, particularly, we should note that the constructed approximate solution sequence is not uniformly Lipschitzian in the corresponding domain.

Our paper is organized as follows. In §2, we firstly prove that the solution of (1.1) blows up with respect to the third eigenvalue, moreover there only exists a unique blowup point under the nondegenerate assumption on the initial data. Secondly, we transform the system (1.1) to a new form and give a precise description of result on the formation and construction of a shock. In §3, by constructing an iterative sequence of approximate solutions near the blowup point, we prove the existence of solution with a shock starting from the blowup point.

§2. Analysis on the blowup mechanism of (1.1) and main theorem

Now we study the blowup mechanism of smooth solution to (1.1) and extend the solution of (1.1) across the blowup time meanwhile we will distinguish along which direction the first order derivatives of ρ and u will blow up.

Let's give a reduction on the system (1.1). Firstly, it is easy to know that (1.1) is actually equivalent to the following system for the smooth solution:

$$\begin{cases} \partial_t \rho + div(\rho u) = 0 \\ \partial_t u + u\nabla u + \frac{\nabla p}{\rho} = 0 \\ \partial_t S + u\nabla S = 0 \\ \rho|_{t=0} = \bar{\rho} + \varepsilon \rho_0(r), u|_{t=0} = \varepsilon w_0(r)x, S|_{t=0} = \bar{S} \end{cases} \tag{2.1}$$

where ∇ denotes $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

Here we emphasize that the system (1.1) and (2.1) are not equivalent for the weak solution. When the smooth solution of (1.1) blows up and the shock wave is formed, then across the shock the entropy S will become a function of (t, x) other than a constant.

Noting the initial velocity in (2.1) is irrotational, then the smooth solution to (2.1) has such a form in $t < T_\varepsilon$: $\rho(t,x) = \rho(t,r)$, $u(t,x) = \nabla\omega(t,r)$ and $S(t,x) = \bar{S}$, where $\omega(t,r)$ is a potential function of velocity.

From the second equation in (2.1), we get

$$\partial_t \nabla\omega + \nabla(\frac{1}{2}|\nabla\omega|^2) = -\nabla h(\rho) \tag{2.2}$$

where $h'(\rho) = \frac{c^2(\rho,\bar{S})}{\rho}$, $h(\bar{\rho}) = 0$. Hence $\partial_t\omega + \frac{1}{2}|\nabla\omega|^2 = -h(\rho)$.

Notice $h'(\rho) > 0$ for $\rho > 0$, then by the implicit function theorem we know that

$$\rho = h^{-1}(-(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2)), \bar{\rho} = h^{-1}(0) \tag{2.3}$$

Substituting (2.3) into the first equation in (2.1) and noting the variance of initial data, we have

$$\begin{cases} \partial_t^2\omega - c^2(h^{-1}(-(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2)), \bar{S})\triangle\omega + 2\sum_{k=1}^{3}\partial_k\omega\partial_t\partial_k\omega + \sum_{i,k=1}^{3}\partial_i\omega\partial_k\omega\partial_{ik}^2\omega = 0 \\ \omega(0,r) = \varepsilon\int_M^r sw_0(s)ds \\ \partial_t\omega(0,r) = -\varepsilon\frac{\bar{c}^2}{\bar{\rho}}\rho_0(r) + \varepsilon^2 g(r,\varepsilon) \end{cases} \tag{2.4}$$

where $g(r,\varepsilon) = -\int_M^r[\int_0^1 \frac{d}{d\rho}(\frac{c^2(\rho,\bar{S})}{\rho})|_{\rho=\bar{\rho}+\theta\varepsilon\rho_0(s)}d\theta]\rho_0(s){\rho'}_0(s)ds - \frac{1}{2}r^2(w_0(r))^2$.

For the convenience to be written, without loss of generality we assume $\bar{c} = 1$ in this paper.

According to the results in [2] and [20], we know that the solution of (2.4) only blows up for $t \leq T_\varepsilon$ in the domain $D = \{(t,r) : e^{\frac{\tau_0}{2\varepsilon}} \leq t \leq T_\varepsilon, -4M \leq r - t \leq M\}$, which is near the surface of forward light cone (in fact, the blowup point lies in $t = T_\varepsilon$). In light of the standard process to dispose the problem on the nonlinear wave equations with small initial data, as in [16] or [21-22], we introduce the normal transformation $\sigma = r - t$ and the slow time variable $\tau = \varepsilon lnt$ to rewrite the equation (2.4). Denoting by $\omega(t,r) = \frac{\varepsilon}{r}G(\tau,\sigma)$, then a direct computation yields an equation on G in the domain D

$$\partial_{\sigma\tau}^2 G + p(G,\nabla G)\partial_\sigma^2 G + \varepsilon e^{-\frac{\tau}{\varepsilon}}q(G,\nabla G)\partial_\tau^2 G + e^{-\frac{\tau}{\varepsilon}}r(G,\nabla G) = 0 \tag{2.5}$$

where

$$\begin{aligned} p(G,\nabla G) &= \frac{t(c^2(\rho,\bar{S}) - (1-\partial_r\omega)^2)}{2\varepsilon(1-\partial_r\omega)} \\ &= (1 + \bar{\rho}c'(\bar{\rho},\bar{S}))\partial_\sigma G + e^{-\frac{\tau}{\varepsilon}}O(\sigma, e^{-\frac{\tau}{\varepsilon}}, G, \nabla_{\sigma,\tau}G) \\ q(G,\nabla G) &= -\frac{1}{2(1-\partial_r\omega)} \\ &= -\frac{1}{2} + e^{-\frac{\tau}{\varepsilon}}O(\sigma, e^{-\frac{\tau}{\varepsilon}}, G, \nabla_{\sigma,\tau}G) \\ r(G,\nabla G) &= \frac{1}{2}(\partial_\tau G - 2(\partial_\sigma G)^2) + e^{-\frac{\tau}{\varepsilon}}O(\sigma, e^{-\frac{\tau}{\varepsilon}}, G, \nabla_{\sigma,\tau}G) \end{aligned}$$

Here the notation "$O(\sigma, e^{-\frac{\tau}{\varepsilon}}, G, \nabla_{\sigma,\tau}G)$" denotes the generic smooth function on its arguments.

To study the blowup mechanism of solution to (2.5), as in [19] and [20], we introduce a transformation:

$$\tau = \tau, \sigma = \varphi(\tau, y) \tag{2.6}$$

which is only singular at the blowup point. Of course, this kind of function $\varphi(\tau, y)$ is unknown and will be determined together with the solution G of (2.5). Writing $G(\tau, \varphi(\tau, y)) = m(\tau, y)$, $(\partial_\sigma G)(\tau, \varphi(\tau, y)) = v(\tau, y)$. Then (2.5) is reformulated as follows

$$\frac{\partial_y v}{\partial_y \varphi} I_1 + I_2 = 0 \tag{2.7}$$

where

$$\begin{aligned} I_1 &= \frac{t(c^2(\rho, \bar{S}) - (\frac{\varepsilon}{t}\partial_\tau\varphi - \partial_r\omega + 1)^2)}{2\varepsilon(1 - \partial_r\omega)} \\ &= 2\partial_\tau\varphi - 2(1 + \bar{\rho}c'(\bar{\rho}, \bar{S}))v + e^{-\frac{\tau}{\varepsilon}} q_1(e^{-\frac{\tau}{\varepsilon}}, \varphi, m, v, \partial_\tau\varphi, \partial_\tau m) \\ I_2 &= -2\partial_\tau v + e^{-\frac{\tau}{\varepsilon}} q_2(e^{-\frac{\tau}{\varepsilon}}, \varphi, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v, \partial_\tau^2\varphi, \partial_\tau^2 m) \end{aligned}$$

here the functions $q_i (i = 1, 2)$ are smooth.

By the definition of blowup system corresponding to the quasilinear wave equation in [19], the blowup system of (2.5) is defined as

$$\begin{cases} I_1 = 0, I_2 = 0, I_3 = \partial_y m - v\partial_y\varphi = 0 \\ \varphi(\frac{\tau_0}{2}, y) = y, m(\frac{\tau_0}{2}, y) = G(\frac{\tau_0}{2}, y), v(\frac{\tau_0}{2}, y) = (\partial_\sigma G)(\frac{\tau_0}{2}, y) \end{cases} \tag{2.8}$$

Obviously, if (2.8) is solved in the class of smooth functions, then (2.5) is also solved in the domain where the transformation (2.6) is inverse. In particular, when the function $\varphi(\tau, y)$ satisfies the following nondegenerate properties at some point $(\tau_\varepsilon = \varepsilon ln T_\varepsilon, y_\varepsilon)$:

$$\partial_y\varphi(\tau_\varepsilon, y_\varepsilon) = 0, \partial_y^2\varphi(\tau_\varepsilon, y_\varepsilon) = 0, \partial_y^3\varphi(\tau_\varepsilon, y_\varepsilon) > 0, \partial_{y\tau}^2\varphi(\tau_\varepsilon, y_\varepsilon) < 0$$

and the function v has the property $\partial_y v(\tau_\varepsilon, y_\varepsilon) \neq 0$, then one can get a complete description on the blowup mechanism of smooth solution to (2.4) at the blowup point $(T_\varepsilon, r_\varepsilon = T_\varepsilon + \varphi(\tau_\varepsilon, y_\varepsilon))$. Indeed, a simple computation implies that the solution $\omega(t, r)$ and its first order derivatives are continuous at the blowup point meanwhile the second order derivatives of $\omega(t, r)$ blow up with the ratio of $\frac{1}{T_\varepsilon - t}$. Furthermore, we can give an extension property of solution to (2.8).

Lemma 2.1. In the domain $\bar{D} = \{(\tau, y) : \frac{\tau_0}{2} \le \tau \le 2\tau_0, -4M \le y \le M\}$, the blowup system (2.8) has a smooth solution (φ, m, v) for small ε, moreover $|\varphi|_{C^k(\bar{D})} + |m|_{C^k(\bar{D})} + |v|_{C^k(\bar{D})} \le C_k$, here $k \in \mathbb{N}$ and C_k is a constant independent of ε. Particularly, if the function $F(q) = q^2\partial_q w_0(q) + 3qw_0(q) + \frac{\bar{c}}{\bar{\rho}}(q\partial_q\rho_0(q) + \rho_0(q))$ satisfies the nondegenerate condition at a unique minimum point, that is, $F(q)$ has a unique strictly negative quadratic minimum point, then in the subdomain $\bar{D}_1 = \{(\tau, y) : \frac{\tau_0}{2} \le \tau \le \tau_\varepsilon, -4M \le y \le M\}$ of $\bar{D}$ we have

$$\partial_y\varphi(\tau, y) \ge 0$$

moreover there exists a unique point $(\tau_\varepsilon, y_\varepsilon)$ such that

$$\partial_y \varphi(\tau, y) = 0 \Leftrightarrow (\tau, y) = (\tau_\varepsilon, y_\varepsilon), \partial_y^2 \varphi(\tau_\varepsilon, y_\varepsilon) = 0, \partial_y^3 \varphi(\tau_\varepsilon, y_\varepsilon) > 0, \partial_{y\tau}^2 \varphi(\tau_\varepsilon, y_\varepsilon) < 0$$

and $\partial_y v(\tau_\varepsilon, y_\varepsilon) \neq 0$.

Remark 2.1. By (1.2), we know that τ_ε actually satisfies $\lim_{\varepsilon \to 0} \tau_\varepsilon = -\frac{2}{(\bar\rho c'(\bar\rho, \bar S)+1) \min_q F(q)}$. Hence the action of $F(q)$ is very silimar to that of initial data for Burger's equation.

Proof. The proof has been given in Theorem 2 of [20], so we omit it.

Based on Lemma 2.1, we can determine the blowup direction of (ρ, u, S) and construct a 3-shock starting from the blowup point $(T_\varepsilon, r_\varepsilon = T_\varepsilon + \varphi(\tau_\varepsilon, y_\varepsilon))$. Motivated by the physical background we set $u(t,x) = \tilde u(t,r)\frac{x}{r}$. From the system (1.1), we can get a conservation law system on $(\rho(t,r), \tilde u(t,r), S(t,r))$ with the source terms

$$\begin{cases} \partial_t \rho + \partial_r(\rho \tilde u) = -\frac{2\rho\tilde u}{r} \\ \partial_t(\rho\tilde u) + \partial_r(\rho \tilde u^2 + p) = -\frac{2\rho\tilde u^2}{r} \\ \partial_t(\rho e + \frac12 \rho\tilde u^2) + \partial_r((\rho e + \frac12\rho\tilde u^2 + p)\tilde u) = -\frac{2}{r}(\rho e + \frac12 \rho \tilde u^2 + p)\tilde u \\ \rho(0,r) = \bar\rho + \varepsilon\rho_0(r), \tilde u(0,r) = \varepsilon w_0(r), S(0,r) = \bar S \end{cases} \tag{2.9}$$

Here we should notice that the blowup point of (1.1) is far away from $r = 0$ and the new shock will be constructed in the domain near the blowup point, hence the factor $\frac{1}{r}$ is not singular in our study.

A simple computation yields that (2.9) has three distinct eigenvalues $\lambda_1(t,r) = \tilde u - c(\rho, S)$, $\lambda_2(t,r) = \tilde u$ and $\lambda_3(t,r) = \tilde u + c(\rho, S)$. The corresponding left eigenvectors are $l_1 = (1, -\frac{\rho}{c(\rho,S)}, 0)$, $l_2 = (0,0,1)$ and $l_3 = (1, \frac{\rho}{c(\rho,S)}, 0)$ respectively. Now we give a detailed information on the blowup direction of $(\rho, \tilde u, S)$ at the blowup point.

Lemma 2.2. Under the nondegenerate condition on $F(q)$ in Lemma 2.1, then $l_3\partial_r \begin{pmatrix} \rho \\ \tilde u \\ S \end{pmatrix}$ blows up at the blowup point $(T_\varepsilon, r_\varepsilon)$ meanwhile $l_1\partial_r \begin{pmatrix} \rho \\ \tilde u \\ S \end{pmatrix}$ and $l_2\partial_r \begin{pmatrix} \rho \\ \tilde u \\ S \end{pmatrix}$ are still continuous and bounded.

Proof. By (2.3) and $u = \nabla\omega, S = \bar S$ for $t \le T_\varepsilon$, we can get

$$l_3\partial_r \begin{pmatrix} \rho \\ \tilde u \\ S \end{pmatrix} = \frac{\rho}{c(\rho, \bar S)}(\partial_r^2\omega - \frac{1}{c(\rho,\bar S)}(\partial_{tr}^2\omega + \partial_r\omega\partial_r^2\omega))$$

Noting $\omega = \frac{\varepsilon}{r} G(\varepsilon lnt, r-t)$, then one has

$$\begin{aligned} l_3\partial_r \begin{pmatrix} \rho \\ \tilde u \\ S \end{pmatrix} &= \frac{\varepsilon\rho}{rc(\rho,\bar S)}\Big\{[1 + \frac{1}{c(\rho,\bar S)} - \frac{\varepsilon}{rc(\rho,\bar S)}(\partial_\sigma G - \frac{G}{r})]\partial_\sigma^2 G - \frac{\varepsilon}{tc(\rho,\bar S)}\partial_{\sigma\tau}^2 G\Big\} \\ &+ h_1(\varepsilon, r, t, G, \partial_\sigma G, \partial_\tau G) \end{aligned}$$

Where h_1 is a smooth function on its arguments.

Additionally,

$$(\partial_\tau G)(\tau,\varphi(\tau,y)) = \partial_\tau m - v\partial_\tau\varphi, (\partial^2_\sigma G)(\tau,\varphi(\tau,y)) = \frac{\partial_y v}{\partial_y\varphi},$$
$$(\partial^2_{\sigma\tau} G)(\tau,\varphi(\tau,y)) = \partial_\tau v - \frac{\partial_y v}{\partial_y\varphi}\partial_\tau\varphi \tag{2.10}$$

then

$$l_3\partial_r \begin{pmatrix} \rho \\ \tilde{u} \\ S \end{pmatrix} = \frac{\varepsilon\rho}{rc(\rho,\bar{S})}[1 + \frac{1}{c(\rho,\bar{S})} - \frac{\varepsilon}{rc(\rho,\bar{S})}(v - \frac{m}{r}) + \frac{\varepsilon}{tc(\rho,\bar{S})}\partial_\tau\varphi]\frac{\partial_y v}{\partial_y\varphi}$$
$$+ \tilde{h}_1(\varepsilon, r, t, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v)$$

where $\tilde{h}_1$ is smooth.

Since $I_1 = 0$ yields

$$\partial_\tau\varphi = \frac{(c(\rho,\bar{S})-1)t}{\varepsilon} + \frac{t}{r}(v - \frac{m}{r})$$

Hence

$$l_3\partial_r \begin{pmatrix} \rho \\ \tilde{u} \\ S \end{pmatrix} = \frac{2\varepsilon\rho}{rc(\rho,\bar{S})}\frac{\partial_y v}{\partial_y\varphi} + \tilde{h}_1$$

Then by Lemma 2.1, we know that $l_3\partial_r \begin{pmatrix} \rho \\ \tilde{u} \\ S \end{pmatrix}$ blows up at the point $(T_\varepsilon, r_\varepsilon)$.

Similarly, by a direct computation we have

$$l_1\partial_r \begin{pmatrix} \rho \\ \tilde{u} \\ S \end{pmatrix} = \frac{\varepsilon\rho}{rc(\rho,\bar{S})}\left[-1 + \frac{1}{c(\rho,\bar{S})} - \frac{\varepsilon}{rc(\rho,\bar{S})}(v - \frac{m}{r}) + \frac{\varepsilon}{tc(\rho,\bar{S})}\partial_\tau\varphi\right]\frac{\partial_y v}{\partial_y\varphi}$$
$$+ h_2(\varepsilon, r, t, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v)$$
$$= \tilde{h}_2(r, t, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v)$$
$$l_2\partial_r \begin{pmatrix} \rho \\ \tilde{u} \\ S \end{pmatrix} = 0$$

where h_2 and $\tilde{h}_2$ are smooth.

Therefore Lemma 2.2 is proved.

Now we give a reduction on (2.9) so that each equation in new system only contains the differentiation along the same direction. This reduction will bring us much convenience in order to obtain the convergence of iterative scheme in the process of shock construction.

Lemma 2.3. The system (2.9) can be reduced to the following form by an invertible transform in the neighborhood of $(\bar{\rho}, 0, \bar{S})$:

$$\begin{cases} \partial_t w + A(w)\partial_r w = \frac{B(w)}{r} \\ w(0,r) = \varepsilon \bar{w}_0(r,\varepsilon) \end{cases} \tag{2.11}$$

here $A(w) = \begin{pmatrix} \lambda_1(w) & a(w) & 0 \\ 0 & \lambda_2(w) & 0 \\ 0 & -a(w) & \lambda_3(w) \end{pmatrix}$ and $a(0) = 0$, $B(w) = \begin{pmatrix} b_1(w) \\ 0 \\ b_3(w) \end{pmatrix}$ is a smooth vector function. Moreover (2.11) can be rewritten as

$$\begin{cases} \partial_t w_1 + \lambda_1(w)\partial_r w_1 + \bar{a}(w)(\partial_t w_2 + \lambda_1(w)\partial_r w_2) = \frac{\bar{b}_1(w)}{r} \\ \partial_t w_2 + \lambda_2(w)\partial_r w_2 = 0 \\ \partial_t w_3 + \lambda_3(w)\partial_r w_3 - \bar{a}(w)(\partial_t w_2 + \lambda_3(w)\partial_r w_2) = \frac{\bar{b}_3(w)}{r} \end{cases} \tag{2.12}$$

where $\bar{a}(w)$ and $\bar{b}_i(w)$ $(i = 1, 2)$ are smooth, moreover, $\bar{a}(0) = \bar{b}_i(0) = 0$.

Proof. As in 2×2 system, we introduce the Riemann invariants as follows

$$\begin{cases} \alpha_1 = \tilde{u} - F(\rho, S) \\ \alpha_2 = S - \bar{S} \\ \alpha_3 = \tilde{u} + F(\rho, S) \end{cases} \tag{2.13}$$

where $\partial_\rho F(\rho, S) = \frac{c(\rho,S)}{\rho}$ and $F(\bar{\rho}, \bar{S}) = 0$.

Obviously, (2.13) is invertible as long as $\rho > 0$(in our discussion, ρ is only a small perturbation of $\bar{\rho}$, hence $\rho > 0$ is fulfilled).

Under the transformation (2.13), a direct computation from (2.9) yields

$$\begin{cases} \partial_t \alpha_1 + \lambda_1 \partial_r \alpha_1 + q(\rho, S)\partial_r \alpha_2 = -\frac{2\tilde{u}^2}{r} + \frac{2\tilde{u}c(\rho,S)}{r} \\ \partial_t \alpha_2 + \lambda_2 \partial_r \alpha_2 = 0 \\ \partial_t \alpha_3 + \lambda_1 \partial_r \alpha_3 + q(\rho, S)\partial_r \alpha_2 = -\frac{2\tilde{u}^2}{r} - \frac{2\tilde{u}c(\rho,S)}{r} \end{cases}$$

where $q(\rho, S) = \frac{\partial_S p(\rho,S)}{\rho} - c(\rho, S)\partial_S F(\rho, S)$.

By a linear transformation as follows

$$\begin{cases} w_1 = \alpha_1 - q(\bar{\rho}, \bar{S})\alpha_2 \\ w_2 = \alpha_2 \\ w_3 = \alpha_3 + q(\bar{\rho}, \bar{S})\alpha_2 \end{cases} \tag{2.14}$$

then one can get

$$\begin{cases} \partial_t w_1 + \lambda_1(w)\partial_r w_1 + a(w)\partial_r w_2 = -\frac{2\tilde{u}^2}{r} + \frac{2\tilde{u}c(\rho,S)}{r} \\ \partial_t w_2 + \lambda_2(w)\partial_r w_2 = 0 \\ \partial_t w_3 + \lambda_3(w)\partial_r w_3 - a(w)\partial_r w_3 = -\frac{2\tilde{u}^2}{r} - \frac{2\tilde{u}c(\rho,S)}{r} \\ w_1(0,r) = \varepsilon w_0(r) - F(\bar{\rho} + \varepsilon\rho_0(r), \bar{S}), w_2(0,r) = 0 \\ w_3(0,r) = \varepsilon w_0(r) + F(\bar{\rho} + \varepsilon\rho_0(r), \bar{S}) \end{cases}$$

where $a(w) = -q(\bar\rho, \bar S)c(\rho, S) + q(\rho, S)$.

Obviously, (2.13) and (2.14) transform the point $(\rho, \tilde u, S) = (\bar\rho, 0, \bar S)$ to the point $(w_1, w_2, w_3) = (0, 0, 0)$. Moreover $a(0) = 0$.

In addition, by a simple algebraic computation, (2.12) comes from (2.11) directly, hence Lemma 2.3 is proved.

By Lemma 2.2, it is easy to know that $\partial_r w_3$ blows up at the blowup point $(T_\varepsilon, r_\varepsilon)$, meanwhile $\partial_r w_1$ and $\partial_r w_2$ are continuous at $(T_\varepsilon, r_\varepsilon)$. Hence we expect that a 3-shock will be formed and constructed from the blowup point. Our result can be stated as

Theorem 2.1. For the system (2.9), suppose that $F(q)$ satisfies the nondegenerate condition at a unique point, then for small ε (2.11) admits a weak entropy solution with a continuously differentiable shock curve Γ: $r = \phi(t)$ starting from the blowup point $(T_\varepsilon, r_\varepsilon)$ in $[T_\varepsilon, T_\varepsilon + 1]$. In $[T_\varepsilon, T_\varepsilon + 1] \times \mathbb{R}\backslash\Gamma$, the solution w is also continuously differentiable. Besides the solution w satisfies the Rankine-Hugoniot condition and entropy condition on Γ and the following estimates hold in the neighbourhood Ω of $(T_\varepsilon, r_\varepsilon)$:

$$
\begin{aligned}
&\phi(t) = r_\varepsilon + \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon) + O((t - T_\varepsilon)^2)\\
&w_1(t, r) = w_1(T_\varepsilon, r_\varepsilon) + O((t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon))^2)^{\frac{1}{3}}\\
&w_2(t, r) = O((t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon))^2)^{\frac{1}{2}}\\
&w_3(t, r) = w_3(T_\varepsilon, r_\varepsilon) + O((t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon))^2)^{\frac{1}{6}}
\end{aligned}
$$

Therefore, returning to the system (2.9) we have near $(T_\varepsilon, r_\varepsilon)$:

$$
\begin{aligned}
&\rho(t, r) = \rho(T_\varepsilon, r_\varepsilon) + O((t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t - T_\varepsilon))^2)^{\frac{1}{6}},\\
&\tilde u(t, r) = \tilde u(T_\varepsilon, r_\varepsilon) + O((t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t - T_\varepsilon))^2)^{\frac{1}{6}},\\
&S(t, r) = \bar S + O((t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t - T_\varepsilon))^2)^{\frac{1}{2}}
\end{aligned}
$$

here "O'' stands for a uniformly bounded quantity independent of ε.

Remark 2.2. Some weaker singularities of the solution of (2.9) may propagate into the domain $[T_\varepsilon, T_\varepsilon + 1] \times \mathbb{R}$ along the 1-characteristics and 2-characteristics through $(T_\varepsilon, r_\varepsilon)$ although the solution itself is continuous there.

Remark 2.3. By [2] and [20], under the assumptions of Theorem 2.1 we know that the solution of (1.1) or (2.9) doesn't blow up away from the small neighbourhood of r_ε for $t \in [T_\varepsilon, T_\varepsilon + 1]$. Hence in order to complete the construction of shock wave in $t \in [T_\varepsilon, T_\varepsilon + 1]$, we only study that problem in the neighbourhood Ω of $(T_\varepsilon, r_\varepsilon)$, here $\Omega = \{(t, r) : T_\varepsilon < t \le T_\varepsilon + 1, r_\varepsilon - 2(T_\varepsilon + 1 - t) \le r \le r_\varepsilon + 2(T_\varepsilon + 1 - t)\}$.

Remark 2.4. For 2-D compressible Euler equations (1.1) with the axisymmetric initial data, which satisfies the nondegenerate condition, by the same method in this paper we can obtain a similar conclusion as in Theorem 2.1.

Remark 2.5. From the proof of Theorem 2.1 below, we can extend our result in the time interval $[T_\varepsilon, T_\varepsilon + \frac{A}{\varepsilon}]$, where $A > 0$ is an appropriate constant depending on the initial data of (1.1).

§3. The proof of Theorem 2.1

As remarked in the last section we only need to do analysis in the neighbourhood Ω of $(T_\varepsilon, r_\varepsilon)$. The solution w of (2.11) in $t \geq T_\varepsilon$ will be constructed by an iterative procedure. To this aim, we will construct a sequence of approximate solutions $\{w^{(n)}(t,r)\}$ and a corresponding sequence $\{\phi^{(n)}(t)\}$ standing for the location of the approximate shock, and show the convergence of these sequences. Here we choose the solution of blowup system (2.8) as the first approximation $w^{(0)}(t,r)$, while $\phi^{(0)}(t)$ is determined by an ordinary differential equation, which is derived from the Rankine-Huginiot conditions. The advantage of this choice is that we can get a piecewise continuous solution of (2.11) which satisfies the entropy condition on $\phi^{(0)}(t)$ and a "good" estimate near the point $(T_\varepsilon, r_\varepsilon)$. Subsequently the whole sequence $\{w^{(n)}(t,r)\}$ can be successively determined by the characteristics method, and $\{\phi^{(n)}(t)\}$ can be determined by the R-H conditions correspondingly.

This section is arranged as follows: In Step 1, we give a first approximation of system (2.11) and some precise descriptions of the approximation as a preparation for further discussion. In Step 2, we will give an iterative scheme to construct the sequence $\{w^{(n)}(t,r)\}$ of approximate solutions, and establish the estimates on $\{w^{(n)}\}, \{\partial_t w^{(n)}\}$ and $\{\partial_r w^{(n)}\}$. Step 3 is devoted to the proof on the convergence of all these sequences.

Step 1. First Approximation

Denoting by $H(t,y) = t + \varphi(\varepsilon lnt, y)$, then by Lemma 2.1, we know that $H(t,y)$ satisfies the following properties

$$\partial_y H(t,y) = 0 \Longleftrightarrow (t,y) = (T_\varepsilon, y_\varepsilon), \partial_y^2 H(T_\varepsilon, y_\varepsilon) = 0, \partial_y^3 H(T_\varepsilon, y_\varepsilon) > 0, \partial_{yt}^2 H(T_\varepsilon, y_\varepsilon) < 0 \quad (3.1)$$

More precisely, by a similar treatment in [9] and [13], we can show the following two lemmas which describe some subtle properties of $H(t,y)$.

Lemma 3.1. 1) For $t \in (T_\varepsilon, T_\varepsilon + 1]$ and in the small neighbourhood of y_ε, $\partial_y H(t,y) = 0$ has two distinct real function roots $\eta_-^\varepsilon(t)$ and $\eta_+^\varepsilon(t)$, moreover $\eta_+^\varepsilon(t) < y_\varepsilon < \eta_-^\varepsilon(t)$ and $\eta_-^\varepsilon(t)$, $\eta_+^\varepsilon(t) \in C^\infty(T_\varepsilon, T_\varepsilon + 1]$.

2) Set $r_-^\varepsilon(t) = H(t, \eta_-^\varepsilon(t))$ and $r_+^\varepsilon(t) = H(t, \eta_+^\varepsilon(t))$, then

$r = H(t,y)$ has three distinct real roots $y_-^\varepsilon(t,r) < y_c^\varepsilon(t,r) < y_+^\varepsilon(t,r)$ if $r \in (r_+^\varepsilon(t), r_-^\varepsilon(t))$.

$r = H(t,y)$ has a unique real root $y_+^\varepsilon(t,r)$ if $r \geq r_-^\varepsilon(t)$.

$r = H(t,y)$ has a unique real root $y_-^\varepsilon(t,r)$ if $r \leq r_+^\varepsilon(t)$.

3) Denote $\Omega_+ = \{(t,r) \in \Omega : T_\varepsilon < t \leq T_\varepsilon + 1, r > r_+^\varepsilon(t)\}$ and $\Omega_- = \{(t,r) \in \Omega : T_\varepsilon < t \leq T_\varepsilon + 1, r < r_-^\varepsilon(t)\}$, then $y_\pm^\varepsilon(t,r) \in C^\infty(\Omega_\pm)$ and $y_\pm^\varepsilon(t,r) \in C(\bar{\Omega}_\pm)$.

Lemma 3.2. Denoting $d_\varepsilon = (t - T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon))^2$, then

$$|y_\pm^\varepsilon(t,r) - y_\varepsilon| < Cd_\varepsilon^{\frac{1}{6}}, |\partial_r y_\pm^\varepsilon(t,r)| \leq Cd_\varepsilon^{-\frac{1}{3}}$$

$$|\partial_\ell y_\pm^\varepsilon(t,r)| \leq Cd_\varepsilon^{-\frac{1}{6}}, |\partial_x^2 y_\pm^\varepsilon(t,r)| \leq Cd_\varepsilon^{-\frac{5}{6}}$$

where ℓ is the direction of third characteristics passing $(T_\varepsilon, r_\varepsilon)$, and the generic constant C is independent of ε.

Based on Lemma 3.1 and Lemma 2.1, we can get two extensions of solution of (2.11) across the blowup time T_ε.

In fact, we denote by $\rho^*(t,y) = \rho(t,H(t,y))$ and $u^*(t,y) = \tilde{u}(t,H(t,y))$, then by use of the definitions of $G(\tau,\sigma)$, $m(\tau,y)$, and $v(\tau,y)$ they are defined as follows

$$\left\{\begin{array}{l} \rho^*(t,y) = h^{-1}(g(t,y)) \\ u^*(t,y) = \frac{\varepsilon}{H(t,y)}(v(\varepsilon lnt,y) - \frac{m(\varepsilon lnt,y)}{H(t,y)}) \end{array}\right. \tag{3.2}$$

where $g(t,y) = \frac{\varepsilon}{H(t,y)}\Bigg\{v(\varepsilon lnt,y) - \frac{\varepsilon}{t}\Bigg[\partial_\tau m(\varepsilon lnt,y) - v(\varepsilon lnt,y)\partial_\tau\varphi(\varepsilon lnt,y) - \frac{\varepsilon}{2H(t,y)}(v(\varepsilon lnt,y) - \frac{m(\varepsilon lnt,y)}{H(t,y)})^2\Bigg]\Bigg\}$.

From (2.13), (2.14) and (3.2), we can define two vector value functions $w^0_\pm(t,r) = (w^0_{1,\pm}(t,r), w^0_{2,\pm}(t,r), w^0_{3,\pm}(t,r))$ as follows

$$\left\{\begin{array}{l} w^0_{1,\pm}(t,r) = u^*(t,y^\varepsilon_\pm(t,r)) - F(\rho^*(t,y^\varepsilon_\pm(t,r)),\bar{S}) \\ w^0_{2,\pm}(t,r) = 0 \\ w^0_{3,\pm}(t,r) = u^*(t,y^\varepsilon_\pm(t,r)) + F(\rho^*(t,y^\varepsilon_\pm(t,r)),\bar{S}) \end{array}\right. \tag{3.3}$$

Note that $w^0_\pm(t,r)$ are the smooth solutions of (2.11) in $\Omega_\pm$ respectively. Therefore, they both are the extension solutions of (2.11).

Now we begin to define the first approximate shock curve $\phi^0(t)$ starting from the point $(T_\varepsilon, r_\varepsilon)$. Since we have chosen the entropy $S \equiv \bar{S}$, then we hope that $\phi^0(t)$ can be determined by the corresponding Rankine- Hugoniot conditions for the first two equations in (2.9), that is,

$$\left\{\begin{array}{l} [\rho](\phi^0(t))' = [\rho\tilde{u}] \\ [\rho\tilde{u}](\phi^0(t))' = [\rho\tilde{u}^2 + p(\rho,\bar{S})] \end{array}\right. \tag{3.4}$$

Hence $\phi^0(t)$ should satisfy an ordinary differential equation as follows

$$\left\{\begin{array}{l} \frac{d\phi^0(t)}{dt} = \tilde{\lambda}_3(t,\phi^0(t)) \\ \phi^0(T_\varepsilon) = r_\varepsilon \end{array}\right. \tag{3.5}$$

where

$$\begin{aligned}\tilde{\lambda}_3(t,r) = \Bigg\{\int_0^1 c^2(\theta G(\frac{w^0_{3,+} - w^0_{1,+}}{2},\bar{S}) + (1-\theta)G(\frac{w^0_{3,-} - w^0_{1,-}}{2},\bar{S}))d\theta \\ - \frac{(w^0_{1,+} + w^0_{3,+} - w^0_{1,-} - w^0_{3,-})^2}{48}\Bigg\}^{\frac{1}{2}} + \frac{w^0_{1,+} + w^0_{3,+} + w^0_{1,-} + w^0_{3,-}}{4}\end{aligned}$$

and $G(\frac{w_3 - w_1}{2},\bar{S})$ is the inverse function of ρ in the transform (2.13) and (2.14) for the case of $S = \bar{S}$.

As in [9] Lemma3.2, we have

Lemma 3.3. The equation (3.5) has a solution $\phi^0(t) \in C^\infty[T_\varepsilon, T_\varepsilon + 1]$, moreover $\phi^0(t)$ satisfies $r^\varepsilon_+(t) < \phi^0(t) < r^\varepsilon_-(t)$, and

$$\phi^0(t) = r_\varepsilon + \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon) + O((t-T_\varepsilon)^2), t \in [T_\varepsilon, T_\varepsilon + 1]$$

Here the notation "O'' still represents a generic quantity independent of ε.

Define the function

$$w_i^0(t,r) = \begin{cases} w_{i,+}^0(t,r), r > \phi^0(t) \\ w_{i,-}^0(t,r), r < \phi^0(t) \qquad i = 1,2 \end{cases}$$

$$w_2^0(t,r) \equiv 0$$

in Ω. Obviously, $w^0(t,x)$ is the solution of (2.11) in $\Omega_\pm$ respectively. But it isn't a weak solution of (2.11) because it doesn't satisfy the Rankine-Hugoniot condition along the curve γ: $r = \phi^0(t)$. We will use an iterative scheme to construct the shock starting from the point $(T_\varepsilon, r_\varepsilon)$ for the system (2.11) through modifying the location of curve γ as well as the solution on both sides of γ. In the process of forthcoming iteration, $(w^0(t,r), \phi^0(t))$ will be chosen as the first approximation of iterative scheme. For the further requirements, now we give two lemmas on $w^0(t,r)$.

Lemma 3.4. In the domain $\Omega\backslash\gamma$, we have

1) $w_3^0(t,r)$ satisfies the estimates:

$$\begin{cases} |w_3^0(t,r) - w_3^0(T_\varepsilon, r_\varepsilon)| \le C\varepsilon d_\varepsilon^{\frac{1}{6}} \\ |\partial_\ell w_3^0(t,r)| \le C\varepsilon d_\varepsilon^{-\frac{1}{6}} \\ |\partial_r w_3^0(t,r)| \le C\varepsilon d_\varepsilon^{-\frac{1}{3}} \\ |\partial_r^2 w_3^0(t,r)| \le C\varepsilon d_\varepsilon^{-\frac{5}{6}} \end{cases} \tag{3.6}$$

2) $w_1^0(t,r)$ satisfies the estimates:

$$\begin{cases} |w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon)| \le C\varepsilon d_\varepsilon^{\frac{1}{3}} \\ |\partial_t w_1^0(t,r)| \le C\varepsilon \\ |\partial_r w_1^0(t,r)| \le C\varepsilon \\ |\partial_r^2 w_1^0(t,r)| \le C\varepsilon d_\varepsilon^{-\frac{1}{2}} \end{cases} \tag{3.7}$$

Proof. It is enough to prove the lemma in the domain Ω_+.

For the simplicity to write, from (2.13) and (2.14) we denote by $w_1^*(t,y) = u^*(t,y) - F(\rho^*(t,y), \bar{S})$ and $w_3^*(t,y) = u^*(t,y) + F(\rho^*(t,y), \bar{S})$. Then $w_{i,\pm}^0(t,r) = w_i^*(t, y_\pm^\varepsilon(t,r))$ for $i = 1,3$.

1) Thanks to the existence and regularity in Lemma 2.1, one has

$$\begin{aligned} w_3^0(t,r) - w_3^0(T_\varepsilon, r_\varepsilon) &= w_3^*(t, y_+^\varepsilon(t,r)) - w_3^*(T_\varepsilon, y_\varepsilon) \\ &= \partial_t w_3^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + \partial_y w_3^*(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon(t,r) - y_\varepsilon) + O(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r) - y_\varepsilon)^2) \end{aligned}$$

$$\partial_\ell w_3^0(t,r) = \partial_t w_3^*(t, y_+^\varepsilon(t,r)) + \partial_y w_3^*(t, y_+^\varepsilon(t,r))\partial_\ell y_+^\varepsilon(t,r)$$

$$\partial_r w_3^0(t,r) = \partial_y w_3^*(t, y_+^\varepsilon(t,r))\partial_r y_+^\varepsilon(t,r)$$

$$\partial_r^2 w_3^0(t,r) = \partial_y^2 w_3^*(t, y_+^\varepsilon(t,r))(\partial_r y_+^\varepsilon(t,r))^2 + \partial_y w_3^*(t, y_+^\varepsilon(t,r))\partial_r^2 y_+^\varepsilon(t,r)$$

Hence (3.6) follows from Lemma 2.1 and Lemma 3.2.

2) Firstly we claim that

$$\partial_y w_1^*(T_\varepsilon, y_\varepsilon) = 0 \tag{3.8}$$

In fact, by a direct computation from (3.2) one has

$$\partial_y u^*(t,y) = \partial_y H(-\frac{u^*}{H} + \frac{\varepsilon m}{H^3}) + \frac{\varepsilon}{H}(\partial_y v - \frac{\partial_y m}{H})$$
$$\partial_y \rho^*(t,y) = \frac{\rho^*}{c^2(\rho^*, \bar{S})}\partial_y g$$

and $\partial_y g(t,y) = \partial_y H\{-\frac{g}{H} + \frac{\varepsilon^2}{2H^3}(v - \frac{m}{H})^2 - \frac{\varepsilon^2 m}{H^4}(v - \frac{m}{H})\} + \frac{\varepsilon}{H}\{\partial_y v - \frac{\varepsilon}{t}(\partial^2_{y\tau} m - \partial_y v \partial_\tau \varphi - v\partial^2_{y\tau}\varphi) - \frac{\varepsilon}{H}(v - \frac{m}{H})(\partial_y v - \frac{\partial_y m}{H})\}$.

Since $\partial_y H(T_\varepsilon, y_\varepsilon) = 0, \partial_y \varphi(\tau_\varepsilon, y_\varepsilon) = 0$ and $\partial_y m(\tau_\varepsilon, y_\varepsilon) = 0$, then

$$\partial_y w_1^*(T_\varepsilon, y_\varepsilon) = \frac{\varepsilon}{H(T_\varepsilon, y_\varepsilon)}(I + II)$$

where

$$I = \frac{\varepsilon}{T_\varepsilon c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}(\partial^2_{y\tau} m(\tau_\varepsilon, y_\varepsilon) - v(\tau_\varepsilon, y_\varepsilon)\partial^2_{y\tau}\varphi(\tau_\varepsilon, y_\varepsilon))$$
$$II = \frac{\partial_y v(\tau_\varepsilon, y_\varepsilon)}{c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}\Big(c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S}) - 1 + \frac{\varepsilon}{H(T_\varepsilon, y_\varepsilon)}(v(\tau_\varepsilon, y_\varepsilon) - \frac{m(\tau_\varepsilon, y_\varepsilon)}{H(T_\varepsilon, y_\varepsilon)}) - \frac{\varepsilon}{T_\varepsilon}\partial_\tau \varphi(\tau_\varepsilon, y_\varepsilon)\Big)$$

Taking the first order derivative on τ on two sides of $I_3 = 0$ in (2.8), and using the property of $\partial_y \varphi(\tau_\varepsilon, y_\varepsilon) = 0$, one has

$$\partial^2_{y\tau} m(\tau_\varepsilon, y_\varepsilon) - v(\tau_\varepsilon, y_\varepsilon)\partial^2_{y\tau}\varphi(\tau_\varepsilon, y_\varepsilon) = 0$$

that is, $I = 0$.

Additionally, $I_1 = 0$ in (2.8) implies

$$\frac{\varepsilon}{T_\varepsilon}\partial_\tau \varphi(\tau_\varepsilon, y_\varepsilon) = c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S}) - 1 + \frac{\varepsilon}{H(T_\varepsilon, y_\varepsilon)}(v(\tau_\varepsilon, y_\varepsilon) - \frac{m(\tau_\varepsilon, y_\varepsilon)}{H(T_\varepsilon, y_\varepsilon)})$$

This leads to $II = 0$. Hence (3.8) is proved.

Secondly, we claim that

$$\partial_y^2 w_1^*(T_\varepsilon, y_\varepsilon) = 0 \tag{3.9}$$

Indeed, by use of $\partial_y H(T_\varepsilon, y_\varepsilon) = \partial_y^2 H(T_\varepsilon, y_\varepsilon) = 0$ and $\partial_y \varphi(\tau_\varepsilon, y_\varepsilon) = \partial_y w(\tau_\varepsilon, y_\varepsilon) = \partial_y^2 m(\tau_\varepsilon, y_\varepsilon) = 0$, we have

$$\partial_y^2 u^*(T_\varepsilon, y_\varepsilon) = \frac{\varepsilon}{H(T_\varepsilon, y_\varepsilon)}\partial_y^2 v(\tau_\varepsilon, y_\varepsilon)$$

Using $I_1 = 0$ and $I_3 = 0$ again in (2.8), one has

$$\partial_y g(T_\varepsilon, y_\varepsilon) = \frac{\varepsilon^2 c^2(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}{H^2(T_\varepsilon, y_\varepsilon)}(\partial_y v(\tau_\varepsilon, y_\varepsilon))^2$$
$$\partial_y^2 g(T_\varepsilon, y_\varepsilon) = \frac{\varepsilon c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}{H(T_\varepsilon, y_\varepsilon)}\partial_y^2 v(\tau_\varepsilon, y_\varepsilon) + \frac{\varepsilon^2 \partial_\rho c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})\rho^*(T_\varepsilon, y_\varepsilon)}{H^2(T_\varepsilon, y_\varepsilon)c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}(\partial_y v(\tau_\varepsilon, y_\varepsilon))^2$$

Hence

$$\partial_y^2 w_1^*(T_\varepsilon, y_\varepsilon) = \partial_y^2 u^*(T_\varepsilon, y_\varepsilon) + \frac{\partial_\rho c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})\rho^*(T_\varepsilon, y_\varepsilon)}{c^4(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}(\partial_y g(\tau_\varepsilon, y_\varepsilon))^2 - \frac{\partial_y^2 g(T_\varepsilon, y_\varepsilon)}{c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}$$
$$= 0$$

Now we start to prove (3.7).

Since (3.8), (3.9) and Taylor's formula yield

$$\partial_y w_1^*(t, y_+^\varepsilon(t,r)) = \partial_{ty}^2 w_1^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + O(\varepsilon(t - T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r) - y_\varepsilon)^2)$$
$$\partial_y^2 w_1^*(t, y_+^\varepsilon(t,r)) = \partial_t \partial_y^2 w_1^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + \partial_y^3 w_1^*(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon(t,r) - y_\varepsilon)$$
$$+ O(\varepsilon(t - T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r) - y_\varepsilon)^2)$$

then by Lemma 2.1 and Lemma 3.2 one has

$$|\partial_y w_1^*(t, y_+^\varepsilon(t,r))| \le C\varepsilon d_\varepsilon^{\frac{1}{3}}, |\partial_y^2 w_1^*(t, y_+^\varepsilon(t,r))| \le C\varepsilon d_\varepsilon^{\frac{1}{3}} \tag{3.10}$$

Additionally,

$$w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon) = \partial_t w_1^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + \partial_y w_1^*(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon(t,r) - y_\varepsilon) + O(\varepsilon(t - T_\varepsilon)^2 +$$
$$+ \varepsilon(t - T_\varepsilon)(y_+^\varepsilon(t,r) - y_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r) - y_\varepsilon)^3)$$
$$\partial_t w_1^0(t,r) = \partial_t w_1^*(t, y_+^\varepsilon(t,r)) + \partial_y w_1^*(t, y_+^\varepsilon(t,r))\partial_t y_+^\varepsilon(t,r)$$
$$\partial_r w_1^*(t,r) = \partial_y w_1^*(t, y_+^\varepsilon(t,r))\partial_r y_+^\varepsilon(t,r)$$
$$\partial_r^2 w_1^0(t,r) = \partial_y^2 w_1^*(t, y_+^\varepsilon(t,r))(\partial_r y_+^\varepsilon(t,r))^2 + \partial_y w_1^*(t, y_+^\varepsilon(t,y))\partial_r^2 y_+^\varepsilon(t,r)$$

Combine this with (3.10) and Lemma 3.2, we show that (3.7) holds.

Denoting the jump of $w_i^0(t,x)$ on γ by $[w_i^0]$, which equals $w_i^0(t, \phi^0(t) + 0) - w_i^0(t, \phi^0(t) - 0)$, we have

Lemma 3.5. The jump of $w_i^0 (i = 1, 3)$ satisfies the estimates

$$|[w_1^0]| \le C_0\varepsilon(t - T_\varepsilon)^{\frac{1}{2}}, |[w_3^0]| \le C_0\varepsilon(t - T_\varepsilon)^{\frac{3}{2}},$$

Proof. By using the estimates of $\phi^0(t)$ on γ, we have $d_\varepsilon = (t - T_\varepsilon)^3 + (\phi^0(t) - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t - T_\varepsilon))^2 \sim (t - T_\varepsilon)^3$. Therefore Lemma 3.4 1) implies

$$|[w_3^0]| \le |w_3^0(t, \phi^0(t) + 0) - w_3^0(T_\varepsilon, r_\varepsilon)| + |w_3^0(t, \phi^0(t) - 0) - w_3^0(T_\varepsilon, r_\varepsilon)| \le C_0\varepsilon(t - T_\varepsilon)^{\frac{1}{2}}$$

.

Now we show Lemma 3.5 holds for $[w_1^0]$.

Since $w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon) = \partial_t w_1^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + O(\varepsilon(t - T_\varepsilon)^2 + \varepsilon(t - T_\varepsilon)(y_+^\varepsilon - y_\varepsilon)^2 + \varepsilon(y_+^\varepsilon - y_\varepsilon)^3)$ in Ω_+ and $w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon) = \partial_t w_1^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + O(\varepsilon(t - T_\varepsilon)^2 + \varepsilon(t - T_\varepsilon)(y_-^\varepsilon - y_\varepsilon)^2 + \varepsilon(y_-^\varepsilon - y_\varepsilon)^3)$ in Ω_-, then $|[w_1^0]| = |w_1^0(t, \phi(t) + 0) - w_1^0(T_\varepsilon, r_\varepsilon) - \{w_1^0(t, \phi(t) - 0) - w_1^0(T_\varepsilon, r_\varepsilon)\}| = |O(\varepsilon d_\varepsilon^{\frac{1}{2}})| \le C_0\varepsilon(t - T_\varepsilon)^{\frac{3}{2}}$.

Step 2. Iterative Scheme

Next we are going to improve the approximation sequence successively. Denote the unknown shock curve by $r = \phi(t)$. Then the slope of shock $\sigma(t) = \phi'(t)$ must satisfy the Rankine-Hugoniot conditions:

$$\begin{cases} \sigma[\rho] = [\rho\tilde{u}] \\ \sigma[\rho\tilde{u}] = [\rho\tilde{u}^2 + P(\rho, S)] \\ \sigma[\rho e(\rho, S) + \frac{1}{2}\rho\tilde{u}^2] = [(\rho e(\rho, S) + \frac{1}{2}\rho\tilde{u}^2 + P(\rho, S))\tilde{u}] \end{cases} \tag{3.11}$$

and the entropy condition for 3-shock.

In light of the transform in (2.13) and (2.14), if we denote their inverse transform as $(\rho, \tilde{u}, S) = (q(w), \frac{w_1+w_3}{2}, \bar{S} + w_2)$, then (3.11) is equivalent to the following conditions

$$\begin{cases} \sigma[\rho_1(w)] - [F_1(w)] = 0 \\ \sigma[\rho_2(w)] - [F_2(w)] = 0 \\ \sigma[\rho_3(w)] - [F_3(w)] = 0 \end{cases} \tag{3.12}$$

where $\rho_1(w) = q(w), \rho_2(w) = \frac{w_1+w_3}{2}\rho_1(w), \rho_3(w) = \rho_1(w)(e(\rho_1(w), \bar{S} + w_2) + \frac{1}{2}(\frac{w_1+w_3}{2})^2)$, and $(F_1(w), F_2(w), F_3(w)) = (\rho_2(w), (\frac{w_1+w_3}{2})^2\rho_1(w) + P(\rho_1(w), \bar{S} + w_2), (e(\rho_1(w), \bar{S} + w_2) + \frac{\rho_2^2(w)}{2\rho_1^2(w)} + \frac{P(\rho_1(w), \bar{S}+w_2)}{\rho_1(w)})\rho_2(w))$.

The entropy condition for 3-shock can be written as

$$\lambda_3(w_-(t)) < \sigma(t) < \lambda_3(w_+(t)), \lambda_2(w_-(t)) < \sigma(t) \tag{3.13}$$

here $w_\pm(t) = (w_{1,\pm}(t), w_{2,\pm}(t), w_3(t, \pm)) = (w_1(t, \phi(t) \pm 0), w_2(t, \phi(t) \pm 0), w_3(t, \phi(t) \pm 0))$.

Now we claim that for small ε, $(w_{1,-}(t), w_{2,-}(t))$ can be uniquely determined from $(w_{1,+}(t), w_{2,+}(t), w_{3,\pm}(t), \sigma(t))$ by two of three equalities in (3.12). This assertion is important because by the entropy condition (3.13) we need the boundary value $(w_{1,-}(t), w_{2,-}(t))$ in order to solve $w_{1,-}(t, r)$ and $w_{2,-}(t, r)$ in the domain Ω_-.

Lemma 3.6 $(w_{1,-}(t), w_{2,-}(t))$ can be solved from the equations $\sigma[\rho_1(w)] - [F_1(w)] = 0$ and $\sigma[\rho_2(w)] - [F_2(w)] = 0$.

Proof. By Lemma 2.3 and the assumption on $\bar{c} = 1$, we know that

$$\left(\frac{\partial(\rho_1, \rho_2, \rho_3)}{\partial(w_1, w_2, w_3)}(0)\right)^{-1} \left(\frac{\partial(F_1, F_2, F_3)}{\partial(w_1, w_2, w_3)}(0)\right) = diag\{-1, 0, 1\}$$

that is,

$$\left(\frac{\partial(F_1, F_2, F_3)}{\partial(\rho_1, \rho_2, \rho_3)}(0) - I\right) \begin{pmatrix} \frac{\partial\rho_1}{\partial w_1}(0) & \frac{\partial\rho_1}{\partial w_2}(0) \\ \frac{\partial\rho_2}{\partial w_1}(0) & \frac{\partial\rho_2}{\partial w_2}(0) \\ \frac{\partial\rho_3}{\partial w_1}(0) & \frac{\partial\rho_3}{\partial w_2}(0) \end{pmatrix} = \begin{pmatrix} -2\frac{\partial\rho_1}{\partial w_1}(0) & -\frac{\partial\rho_1}{\partial w_2}(0) \\ -2\frac{\partial\rho_2}{\partial w_1}(0) & -\frac{\partial\rho_2}{\partial w_2}(0) \\ -2\frac{\partial\rho_3}{\partial w_1}(0) & -\frac{\partial\rho_3}{\partial w_2}(0) \end{pmatrix}$$

Additionally, a direct computation shows that

$$\frac{\partial(\rho_1,\rho_2)}{\partial(w_1,w_2,w_3)}(0) = \begin{pmatrix} -\frac{\bar\rho}{2} & -\partial_S p(\bar\rho,\bar S) \\ -\frac{\bar\rho}{2} & 0 \\ -\frac{\bar\rho}{2}(e(\bar\rho,\bar S)+\bar\rho\partial_\rho e(\bar\rho,\bar S)) & \bar\rho\partial_S e(\bar\rho,\bar S)-\partial_S p(\bar\rho,\bar S)(e(\bar\rho,\bar S)+\bar\rho\partial_\rho e(\bar\rho,\bar S)) \end{pmatrix}$$

has the rank 2, hence by the implicit function theorem we know that $(w_{1,-}(t), w_{2,-}(t))$ can be determined by the two equalities $\sigma[\rho_1(w)]-[F_1(w)]=0$ and $\sigma[\rho_2(w)]-[F_2(w)]=0$.

Consequently, from Lemma 3.6, (3.12) is equivalent to:

$$\begin{cases} \sigma[\rho_1(w)]-[F_1(w)]=0 \\ \sigma[\rho_2(w)]-[F_2(w)]=0 \\ \sigma=\tilde\lambda_3(\int_0^1(\partial_{\rho_i}F_j)(\theta\rho(w_+(t))+(1-\theta)\rho(w_-(t)))d\theta) \end{cases} \tag{3.14}$$

where $\tilde\lambda_3$ is the third eigenvalue of matrix $(\int_0^1(\partial_{\rho_i}F_j)(\theta\rho(w_+(t))+(1-\theta)\rho(w_-(t)))d\theta)_{i,j=1}^3$, and $\rho(w_\pm(t))=(\rho_1(w_\pm(t)),\rho_2(w_\pm(t)),\rho_3(w_\pm(t)))$

Based on the above preparations we are going to construct the weak entropy solution of (2.11) by using an approximate procedure. To avoid the difficulty caused by the unknown shock curve, which may change its location in the process of iteration, we introduce a coordinate transform to fix the shock on the $t-$axis:

$$\begin{cases} z=r-\phi(t) \\ t=t \end{cases} \tag{3.15}$$

Under the new coordinates, the blowup point becomes $(T_\varepsilon,0)$ and the system (2.12) can be changed into the following form:

$$\begin{cases} \partial_t w_1+(\lambda_1-\sigma(t))\partial_z w_1+\bar a(w)(\partial_t w_2+(\lambda_1-\sigma(t))\partial_z w_2)=\frac{\bar b_1(w)}{z+\phi(t)} \\ \partial_t w_2+(\lambda_2-\sigma(t))\partial_z w_2=0 \\ \partial_t w_3+(\lambda_3-\sigma(t))\partial_z w_3-\bar a(w)(\partial_t w_2+(\lambda_3-\sigma(t))\partial_z w_2)=\frac{\bar b_3(w)}{z+\phi(t)} \\ w_i(t,z)|_{t=T_\varepsilon}=w_i^0(T_\varepsilon,z+r_\varepsilon), i=1,2,3 \end{cases} \tag{3.16}$$

Denoting $\tilde\Omega_-=\{(t,z): T_\varepsilon\le t\le T_\varepsilon+1, -2(T_\varepsilon+1-t)\le z<0\}$ and $\tilde\Omega_+=\{(t,z): T_\varepsilon\le t\le T_\varepsilon+1, 0<z\le 2(T_\varepsilon+1-t)\}$. Obviously, $\tilde\Omega_-\cup\tilde\Omega_+$ for small ε locates in the determinate region of $\{(T_\varepsilon,z): -K\le z\le K\}$. In order to construct the weak entropy solution of (2.11) in the domain $\tilde\Omega_-\cup\tilde\Omega_+$ and prove Theorem 2.1, we take the following iterative scheme:

$$\begin{cases} \partial_t w_{1,+}^{n+1}+(\lambda_1(w_+^n)-\sigma^n(t))\partial_z w_{1,+}^{n+1}+\bar a(w_+^n)(\partial_t w_{2,+}^n+(\lambda_1(w_+^n)-\sigma^n(t))\partial_z w_{2,+}^n)=\frac{\bar b_1(w_+^n)}{z+\phi^n(t)} \\ \partial_t w_{2,+}^{n+1}+(\lambda_2(w_+^n)-\sigma^n(t))\partial_z w_{2,+}^{n+1}=0 \\ \partial_t w_{3,\pm}^{n+1}+(\lambda_3(w_\pm^n)-\sigma^n(t))\partial_z w_{3,\pm}^{n+1}-\bar a(w_\pm^n)(\partial_t w_{2,\pm}^n+(\lambda_3(w_\pm^n)-\sigma^n(t))\partial_z w_{2,\pm}^n)=\frac{\bar b_3(w_\pm^n)}{z+\phi^n(t)} \\ w_{1,+}^{n+1}(t,z)|_{t=T_\varepsilon}=w_{1,+}^0(T_\varepsilon,z+r_\varepsilon), w_{2,+}^{n+1}(t,z)|_{t=T_\varepsilon}=0, \\ w_{3,\pm}^{n+1}(t,z)|_{t=T_\varepsilon}=w_{3,\pm}^0(T_\varepsilon,z+r_\varepsilon), \end{cases} \tag{3.17}$$

and

$$\begin{cases} \partial_t w_{1,-}^{n+1} + (\lambda_1(w_-^n) - \sigma^n(t))\partial_z w_{1,-}^{n+1} + \bar{a}(w_-^n)(\partial_t w_{2,-}^n + (\lambda_1(w_-^n) - \sigma^n(t))\partial_z w_{2,-}^n) = \frac{\bar{b}_1(w_-^n)}{z+\phi^n(t)} \\ \partial_t w_{2,-}^{n+1} + (\lambda_2(w_-^n) - \sigma^n(t))\partial_z w_{2,-}^{n+1} = 0 \\ \sigma^n(t) = \tilde{\lambda}_3(\int_0^1(\partial_{\rho_i}F_j)(\theta\rho(w_+^n(t,0+)) + (1-\theta)\rho(w_-^n(t,0-)))d\theta) \\ w_{1,-}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{1,-}^0(T_\varepsilon, z+r_\varepsilon), w_{2,-}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{2,-}^0(T_\varepsilon, z+r_\varepsilon), \\ w_{1,-}^{n+1}(t,z)|_{z=0} = w_{1,-}^{n+1}(t,0-), w_{2,-}^{n+1}(t,z)|_{z=0} = w_{2,-}^{n+1}(t,0-) \end{cases} \tag{3.18}$$

where $w_{1,-}^{n+1}(t,0-)$ and $w_{2,-}^{n+1}(t,0-)$ are determined by the equalities:

$$\begin{cases} \sigma^n[\rho_1(w^{n+1})] = [F_1(w^{n+1})] \\ \sigma^n[\rho_3(w^{n+1})] = [F_3(w^{n+1})] \end{cases} \tag{3.19}$$

Additionally, $\phi^n(t) = T_\varepsilon + \int_{T_\varepsilon}^t \sigma^n(t)dt$.

By the entropy condition (3.13), (3.17) and (3.18) can be solved by the characteristics method. Since $w_{2,+}^{n+1} \equiv \bar{S}$, then (3.17) becomes

$$\begin{cases} \partial_t w_{1,+}^{n+1} + (\lambda_1(w_+^n) - \sigma^n(t))\partial_z w_{1,+}^{n+1} = \frac{\bar{b}_1(w_+^n)}{z+\phi^n(t)} \\ \partial_t w_{3,+}^{n+1} + (\lambda_3(w_+^n) - \sigma^n(t))\partial_z w_{3,+}^{n+1} = \frac{\bar{b}_3(w_\pm^n)}{z+\phi^n(t)} \\ \partial_t w_{3,-}^{n+1} + (\lambda_3(w_-^n) - \sigma^n(t))\partial_z w_{3,-}^{n+1} - \bar{a}(w_-^n)(\partial_t w_{2,-}^n + (\lambda_3(w_-^n) - \sigma^n(t))\partial_z w_{2,-}^n) = \frac{\bar{b}_3(w_-^n)}{z+\phi^n(t)} \\ w_{1,+}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{1,+}^0(T_\varepsilon, z+r_\varepsilon), w_{3,\pm}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{3,\pm}^0(T_\varepsilon, z+r_\varepsilon), \end{cases} \tag{3.20}$$

For the requirements to estimate $\{w_\pm^n\}$ and $\{\sigma^n(t)\}$ below, we need a relation between $[w_i](i = 1,2)$ and $[w_3]$, that is

Lemma 3.7. There exist two smooth functions $G_i(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))$ $(i = 1,2)$ such that

$$[w_i] = G_i(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))[w_3]^3, i = 1,2 \tag{3.21}$$

Proof. The equality on $[w_2]$ is well known, since the change of entropy across a shock is a small quantity of third order of the strength of the shock (for example, see [23] and [24]).

To prove the first equality (3.21), we can rewrite (3.12) as

$$\begin{aligned} &\left(\left(\frac{\partial(F_1,F_2,F_3)}{\partial(\rho_1,\rho_2,\rho_3)}(w_-(t,0-)) - \sigma I\right)\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\right)(w_-(t,0-))\begin{pmatrix}[w_1]\\ [w_2]\\ [w_3]\end{pmatrix}\right) \\ &= \tilde{B}\begin{pmatrix}[w_1]^2 & [w_1][w_2] & [w_1][w_3] \\ [w_1][w_2] & [w_2]^2 & [w_2][w_3] \\ [w_1][w_3] & [w_2][w_3] & [w_3]^2\end{pmatrix} \end{aligned} \tag{3.22}$$

here $\tilde{B} = (\tilde{b}_{ij}(w_-(t,0-), w_+(t,0+)))_{i,j=1}^3$ is a 3×3 smooth function matrix.

Since $\sigma(t) = \lambda_3(w_-(t,0-)) + \sum_{i=1}^{3} g_i(w_-(t,0-))[w_i] + \sum_{i,j=1}^{3} g_{ij}(w_-(t,0-), w_+(t,0+))[w_i][w_j]$ and by Lemma 2.3

$$\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\right)^{-1}\Bigg|_{w=w_-(t,0-)} \left(\frac{\partial(F_1,F_2,F_3)}{\partial(\rho_1,\rho_2,\rho_3)}(w_-(t,0-)) - \sigma I\right) \frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\Bigg|_{w=w_-(t,0-)}$$
$$= \begin{pmatrix} \lambda_1(w_-(t,0-)) - \sigma & a(w_-(t,0-)) & 0 \\ 0 & \lambda_2(w_-(t,0-)) - \sigma & 0 \\ 0 & -a(w_-(t,0-)) & \lambda_3(w_-(t,0-)) - \sigma \end{pmatrix}$$

then multiplying $\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\right)^{-1}\Big|_{w=w_-(t,0-)}$ on two sides of (3.22) we can get

$$[w_1] = \sum_{i,j=1}^{3} Q_{ij}(w_-(t,0-))[w_i][w_j] + \sum_{i,j,k=1}^{3} Q_{ijk}(w_-(t,0-), w_+(t,0+))[w_i][w_j][w_k] \qquad (3.23)$$

here Q_{ij} and Q_{ijk} are smooth.

Exchange the position of $w_-(t,0-)$ and $w_+(t,0+)$ in (3.23), one gets

$$[w_1] = -\sum_{i,j=1}^{3} Q_{ij}(w_+(t,0+))[w_i][w_j] + \sum_{i,j,k=1}^{3} Q_{ijk}(w_+(t,0+), w_-(t,0-))[w_i][w_j][w_k] \qquad (3.24)$$

Summing up (3.23) and (3.24), we have

$$[w_1] = \sum_{i,j,k=1}^{3} \tilde{Q}_{ijk}(w_-(t,0-), w_+(t,0+))[w_i][w_j][w_k]$$

where $\tilde{Q}_{ijk}$ are smooth.

Set $[w_1] = \mu[w_2]^3$, and note that $w_{1,-}(t,0-) = w_{1,+}(t,0+) - [w_1]$, $[w_2] = G_2(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))[w_3]^3$, then applying the implicit function theorem, one can obtain for small $[w_3]$, $\mu = G_1(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))$, here the function G_1 is smooth. Hence Lemma 3.7 is proved.

Step 3. The estimates on $\{w_\pm^{n+1}(t,z)\}$ and $\{\sigma^n(t)\}$

In this section, we are going to give the estimates on $\{w_\pm^{n+1}(t,z)\}$ and $\{\sigma^n(t)\}$. In the following discussion, "N'' represents a constant independent of n and ε, which may be take different values in different inequality.

Lemma 3.8. For small ε, there exists a constant $N > C_0$ independent of ε, such that in $\tilde{\Omega}_-$

or $\tilde{\Omega}_+$

$$w_\pm^n \in C^1(\tilde{\Omega}_\pm \setminus (T_\varepsilon, 0)) \tag{3.25}$$
$$|w_{3,\pm}^n - w_{3,\pm}^0| \leq N\varepsilon(t - T_\varepsilon) \tag{3.26}$$
$$|\partial_z(w_{3,\pm}^n - w_{3,\pm}^0)| \leq N\varepsilon((t - T_\varepsilon)^3 + z^2)^{-\frac{1}{6}} \tag{3.27}$$
$$|\partial_t(w_{3,\pm}^n - w_{3,\pm}^0)| \leq N\varepsilon((t - T_\varepsilon)^3 + z^2)^{-\frac{1}{6}} \tag{3.28}$$
$$|w_{i,\pm}^n - w_{i,\pm}^0| \leq N\varepsilon(t - T_\varepsilon)^{\frac{3}{2}}, i = 1, 2 \tag{3.29}$$
$$|\partial_z(w_{i,\pm}^n - w_{i,\pm}^0)| \leq N\varepsilon(t - T_\varepsilon)^{\frac{1}{2}}, i = 1, 2, \tag{3.30}$$
$$|\partial_t(w_{i,\pm}^n - w_{i,\pm}^0)| \leq N\varepsilon(t - T_\varepsilon)^{\frac{1}{2}}, i = 1, 2, \tag{3.31}$$

hold for all n, here C_0 is the constant appeared in Lemma 3.5.

Proof. Obviously, (3.25)-(3.31) hold for $n = 0$. Now we prove the conclusion by induction. Assume these estimates are valid for n, we are going to prove they are still valid for $n + 1$. The proof is proceeded as the following six parts.

Part 1. The estimate of $\sigma^n(t)$

If (3.25)-(3.31) are true, by using the expression of $\sigma^n(t)$ in (3.18) and the mean value theorem we have

$$|\sigma^n(t) - \sigma^0(t)| \leq C_N\varepsilon(t - T_\varepsilon)$$

in $[T_\varepsilon, T_\varepsilon + 1]$. here C_N is a constant depending only on N.

Part 2. Estimates of $w_{3,\pm}^{n+1}$, $w_{1,+}^{n+1}$ and $w_{2,+}^{n+1}$

We only give the estimate on $w_{3,-}^{n+1}$, the estimates of others are completely similar or even simpler, in particularly, $w_{2,+}^{n+1} \equiv \bar{S}$.

Set $v(t, z) = w_{3,-}^{n+1} - w_{3,-}^0$, then $v(t, z)$ satisfies the equation:

$$\begin{cases} \partial_t v + (\lambda_3(w_-^n) - \sigma^n)\partial_z v = (\lambda_3(w_-^0) - \lambda_3(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{3,-}^0 + \bar{a}(w_-^n)\{\partial_t(w_{2,-}^n - w_{2,-}^0) \\ +(\lambda_3(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^0) - (\lambda_3(w_-^0) - \lambda_3(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{2,-}^0\} \\ +(\bar{a}(w_-^n) - \bar{a}(w_-^0))(\partial_t w_{2,-}^0 + (\lambda_3(w_-^0) - \sigma^0)\partial_z w_{2,-}^0) + \frac{\bar{b}_3(w_-^n)}{z+\phi^n(t)} - \frac{\bar{b}_3(w_-^0)}{z+\phi^0(t)} \\ v(T_\varepsilon, z) = 0 \end{cases} \tag{3.32}$$

Noting

$$\bar{a}(w_-^n)\{\partial_t(w_{2,-}^n - w_{2,-}^0) + (\lambda_3(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^0)\} = (\partial_t + (\lambda_3(w_-^n)$$
$$- \sigma^n)\partial_z)(\bar{a}(w_-^n)(w_{2,-}^n - w_{2,-}^0)) - \{\sum_{j=1}^{3}(\partial_{w_j}\bar{a})(w_-^n)(\partial_t w_{j,-}^n + (\lambda_3(w_-^n)$$
$$- \sigma^n)\partial_z w_{j,-}^n)\}(w_{2,-}^n - w_{2,-}^0)$$

and

$$\frac{\bar{b}_3(w_-^n)}{z + \phi^n(t)} - \frac{\bar{b}_3(w_-^0)}{z + \phi^0(t)} = \frac{\bar{b}_3(w_-^n) - \bar{b}_3(w_-^0)}{z + \phi^n(t)} + \frac{\bar{b}_3(w_-^0)}{(z + \phi^n(t))(z + \phi^0(t))}\int_{T_\varepsilon}^t (\sigma^n(t) - \sigma^0(t))dt$$

in view of the inductive hypothesis and $\bar{a}(0)=0$ and Lemma 3.4 we can use the characteristics method to derive

$$|v(t,y)| \le |\bar{a}(w_-^n)(w_{2,-}^n - w_{2,-}^0)| + C_N\varepsilon^2\int_{T_\varepsilon}^t (1+\sqrt{s-T_\varepsilon})ds \le C_N\varepsilon^2(t-T_\varepsilon)$$

here and below C_N denotes a generic constant depending only on N. Hence for small ε (3.26) holds.

Similarly, we can show

$$|w_{1,+}^{n+1} - w_{1,+}^0| \le C_N\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}}$$

Part 3. Estimates of $w_{1,-}^{n+1}$ and $w_{2,-}^{n+1}$

It is enough to give the estimate on $w_{1,-}^{n+1}$. Set $v(t,z) = w_{1,-}^{n+1} - w_{1,-}^0$, then $v(t,z)$ satisfies the equation:

$$\begin{cases} \partial_t v + (\lambda_1(w_-^n) - \sigma^n)\partial_z v = (\lambda_1(w_-^0) - \lambda_1(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{1,-}^0 - \bar{a}(w_-^n)\{\partial_t(w_{2,-}^n \\ -w_{2,-}^0) + (\lambda_1(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^0) - (\lambda_1(w_-^0) - \lambda_1(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{2,-}^0\} \\ -(\bar{a}(w_-^n) - \bar{a}(w_-^0))(\partial_t w_{2,-}^0 + (\lambda_1(w_-^0) - \sigma^0)\partial_z w_{2,-}^0) + \frac{\bar{b}_1(w_-^n)}{z+\phi^n(t)} - \frac{\bar{b}_1(w_-^0)}{z+\phi^0(t)} \\ v(T_\varepsilon, z) = 0, v(t,z)|_{z=0} = w_{1,-}^{n+1}(t,0-) - w_{1,-}^0(t,0-) \end{cases} \tag{3.33}$$

Let $\xi = \xi(t,z,s)$ be the back characteristics of (3.33) through the point (t,z) in the domain $\tilde{\Omega}_-$. If the characteristics $\xi = \xi(t,z,s)$ intersects with $z-$axis, then similar to the estimate in Part 2, we have $|v(t,z)| \le C_M\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}}$. If the characteristics $\xi = \xi(t,z,s)$ intersects with $t-$axis at $(s,0)$ with $s > T_\varepsilon$, then we have to estimate the value of $w_{1,-}^{n+1}(t,0-)$. Firstly, by using the inductive hypothesis and characteristics method we have

$$|v(t,z)| \le |w_{1,-}^{n+1}(s,0-) - w_{1,-}^0(s,0-)| + C_N\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}} \tag{3.34}$$

Secondly, by Lemma 3.7 we know

$$[w_1^{n+1}] = G_1(w_{1,+}^{n+1}(s,0+), w_{3,+}^{n+1}(s,0+), w_{3,-}^{n+1}(s,0-)[w_3^{n+1}]^3 \tag{3.35}$$

Since

$$|w_{1,-}^{n+1}(s,0-) - w_{1,-}^0(s,0-)| \le |[w_1^{n+1}]| + |w_{1,+}^{n+1}(s,0+) - w_{1,+}^0(s,0+)| + |[w_1^0]|$$

and

$$|[w_3^{n+1}]| \le |w_{3,+}^{n+1}(s,0+) - w_{3,+}^0(s,0+)| + |w_{3,-}^{n+1}(s,0-) - w_{3,-}^0(s,0-)| + |[w_3^0]|$$

Hence by (3.34) and (3.35) and Part 2, one has for small ε

$$|v(t,z)| \le C_0\varepsilon(t-T_\varepsilon)^{\frac{3}{2}} + C_N\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}} \le N\varepsilon(t-T_\varepsilon)^{\frac{3}{2}} \tag{3.36}$$

Part 4. Estimates of $|\nabla(w_{3,\pm}^{n+1} - w_{3,\pm}^0)|$

Set $v(t,z)=\partial_z(w^{n+1}_{3,-}-w^0_{3,-})$, then $v(t,z)$ satisfies the following equation

$$\begin{cases}\partial_t v+(\lambda_3(w^n_-)-\sigma^n(t))\partial_z v+\partial_z(\lambda_3(w^n_-))v=\bar{a}(w^n_-)\{\partial^2_{tz}(w^n_{2,-}-w^0_{2,-}) \\ \quad +(\lambda_3(w^n_-)-\sigma^n)\partial^2_z(w^n_{2,-}-w^0_{2,-})\}+(\lambda_3(w^n_-)-\lambda_3(w^0_-)-\sigma^n+\sigma^0)\partial^2_z w^0_{2,-}\} \\ \quad +(\lambda_3(w^0_-)-\lambda_3(w^n_-)+\sigma^n-\sigma^0)\partial^2_z w^0_{3,-}-\sum_{j=1}^{3}\{(\partial_{w_j}\lambda_3)(w^n_-)\partial_z w^n_{j,-} \\ \quad -(\partial_{w_j}\lambda_3)(w^0_-)\partial_z w^0_{j,-}\}\partial_z w^0_{3,-}+\sum_{j=1}^{3}\{(\partial_{w_j}\bar{a})(w^n_-)\partial_z w^n_{j,-}\{\partial_t(w^n_{2,-}-w^0_{2,-}) \\ \quad +(\lambda_3(w^n_-)-\sigma^n)\partial_z(w^n_{2,-}-w^0_{2,-})-(\lambda_3(w^0_-)-\lambda_3(w^n_-)+\sigma^n-\sigma^0)\partial_z w^0_{2,-}\} \\ \quad -((\partial_{w_j}\bar{a})(w^n_-)\partial_z w^n_{j,-}-(\partial_{w_j}\bar{a})(w^0_-)\partial_z w^0_{j,-})(\partial_t w^0_{2,-}+(\lambda_3(w^0_-)-\sigma^0)\partial_z w^0_{2,-})\} \\ \quad +\sum_{j=1}^{3}\bar{a}(w^n_-)\{\partial_{w_j}\lambda_3(w^n_-)\partial_z w^n_{j,-}\partial_z w^n_{2,-}-\partial_{w_j}\lambda_3(w^0_-)\partial_z w^0_{j,-}\partial_z w^0_{2,-}\} \\ \quad +\partial_z(\frac{\bar{b}_3(w^n_-)}{z+\phi^n(t)})-\partial_z(\frac{\bar{b}_3(w^0_-)}{z+\phi^0(t)}) \\ v(T_\varepsilon,z)=0\end{cases} \tag{3.37}$$

Let $\xi^{n+1}=\xi^{n+1}(t,z,s)$ be the back characteristics of (3.37) through the point (t,z), that is, ξ^{n+1} satisfies the equation

$$\begin{cases}\frac{d\xi^{n+1}}{ds}=\lambda_3(w^n_-(s,\xi^{n+1}))-\sigma^n(s), \qquad T_\varepsilon\le s\le t \\ \xi^{n+1}|_{s=t}=z\end{cases}$$

Similar to the treatments in lemma 8.1 and lemma 8.3 in [13] (the main reasons which we can use the methods in [13] are: firstly, $\partial_{w_3}\lambda_3(0)\neq 0$ holds. Secondly, λ_3 and σ has a similar relation as in [13] Lemma 3.1, see the formula (3.50) below. The two conditions are just only the keys to prove Lemma 8.1 and Lemma 8.3 of [13] for the p-system of 2×2 gas dynamics), we can show that there exists a constant C independent of n and ε such that

$$(s-T_\varepsilon)^3+(\xi^{n+1})^2\ge C((t-T_\varepsilon)^3+z^2) \tag{3.38}$$

and

$$|\int_{T_\varepsilon}^{t}(\partial_z(\lambda_3(w^n_-)))(s,\xi^{n+1})ds|\le ln\frac{3}{2}+C_M\varepsilon\sqrt{t-T_\varepsilon}<\frac{1}{2} \tag{3.39}$$

Combining the above inequalities with Lemma 3.4, $\bar{b}_1(0)=0$ and inductive hypothesis, and integrating (3.37) we have

$$\begin{aligned}&|v(t,z)|\le|\bar{a}(w^n_-)\partial_z(w^n_{2,-}-w^0_{2,-})(t,z)|+\int_{T_\varepsilon}^{t}|(\partial_z(\lambda_3(w^n_-)))(s,\xi^{n+1})||v(s,y)|ds \\ &+C_M\varepsilon^2\int_{T_\varepsilon}^{t}\{\frac{s-T_\varepsilon}{((t-T_\varepsilon)^3+z^2)^{\frac{5}{6}}}+\frac{\sqrt{s-T_\varepsilon}}{((t-T_\varepsilon)^3+z^2)^{\frac{1}{3}}}+\frac{1}{((t-T_\varepsilon)^3+z^2)^{\frac{1}{2}}}\}ds \\ &\le C_M\varepsilon^2((t-T_\varepsilon)^3+z^2)^{-\frac{1}{6}}+\int_{T_\varepsilon}^{t}|(\partial_z(\lambda_3(w^n_-)))(s,\xi^{n+1})||v(s,y)|ds\end{aligned}$$

Then in view of (3.39) and Gronwall's inequality, for small ε we know (3.27) holds. Moreover in light of (3.33) we obtain (3.28).

Part 5. Estimates on $|\nabla(w_{1,+}^{n+1} - w_{1,+}^0)|$

Set $v(t,z) = \partial_z(w_{1,+}^{n+1} - w_{1,+}^0)$, then $v(t,z)$ satisfies the equation:

$$\begin{cases} \partial_t v + (\lambda_1(w_+^n) - \sigma^n(t))\partial_z v + \partial_z(\lambda_1(w_+^n))v + \sum_{j=1}^3 \{\partial_{w_j}\lambda_1(w_+^n)\partial_z w_{j,+}^n \\ \quad -\partial_{w_j}\lambda_1(w_+^0)\partial_z w_{j,+}^0)\partial_z w_{1,+}^0\} = \partial_z(\frac{\bar b_1(w_+^n)}{z+\phi^n(t)}) - \partial_z(\frac{\bar b_1(w_+^0)}{z+\phi^0(t)}) \\ v(T_\varepsilon, z) = 0 \end{cases} \tag{3.40}$$

Let $\xi_1^{n+1} = \xi_1^{n+1}(t,z,s)$ be the back characteristics through the point (t,z), for small ε we have

$$\xi_1^{n+1} \geq z + \frac{t-s}{2} \tag{3.41}$$

By the characteristics method and $\bar b_1(0) = 0$ we get

$$\begin{aligned} |v(t,z)| \leq\ & C_N\varepsilon \int_{T_\varepsilon}^t \frac{|v(s,\xi_1^{n+1})|}{((s-T_\varepsilon)^3 + (\xi_1^{n+1})^2)^{\frac{1}{3}}} ds \\ & + C_N\varepsilon^2 \int_{T_\varepsilon}^t \{\frac{\sqrt{s-T_\varepsilon}}{((s-T_\varepsilon)^3 + (\xi_1^{n+1})^2)^{\frac{1}{3}}} + \frac{s-T_\varepsilon}{((s-T_\varepsilon)^3 + (\xi_1^{n+1})^2)^{\frac{1}{2}}}\} ds \end{aligned}$$

Substituting (3.41) into the above inequality, one has

$$|v(t,z)| \leq C_N\varepsilon^2\sqrt{t-T_\varepsilon} + C_N\varepsilon \int_{T_\varepsilon}^t \frac{|v(s,\xi_1^{n+1})|}{(t-s)^{\frac{2}{3}}} ds$$

Hence Gronwall's inequality implies

$$|v(t,y)| \leq C_N\varepsilon^2\sqrt{t-T_\varepsilon}$$

Part 6. Estimates on $|\nabla(w_{1,-}^{n+1} - w_{1,-}^0)|$ **and** $|\nabla(w_{2,-}^{n+1} - w_{2,-}^0)|$

We only compute $|\partial_t(w_{1,-}^{n+1} - w_{1,-}^0)|$ and $|\partial_z(w_{1,-}^{n+1} - w_{1,-}^0)|$.

It is convenient to estimate $\partial_t(w_{1,-}^{n+1} - w_{1,-}^0)$ firstly since we can take the derivative on t for the boundary value $w_{1,-}^n(t,0-) - w_{1,-}^0(t,0-)$. Now we set $v(t,z) = \partial_t(w_{1,-}^{n+1} - w_{1,-}^0)$, then $v(t,z)$ satisfies the equation:

$$\begin{cases} \partial_t v + (\lambda_1(w_-^n) - \sigma^n(t))\partial_z v + \partial_t(\lambda_1(w_-^n) - \sigma^n(t))v + \bar a(w_-^n)\{\partial_t^2(w_{2,-}^n - w_{2,-}^0) \\ \quad +(\lambda_1(w_-^n) - \sigma^n)\partial_{tz}^2(w_{2,-}^n - w_{2,-}^0)\} + (\lambda_1(w_-^n) - \lambda_1(w_-^0) - \sigma^n + \sigma^0)\partial_{tz}^2 w_{2,-}^0\} \\ \quad +(\bar a(w_-^n) - \bar a(w_-^0))\{\partial_t^2 w_{2,-}^0 + (\lambda_1(w_-^0) - \sigma^0)\partial_{tz}^2 w_{2,-}^0 - (\lambda_1(w_-^0) - \lambda_1(w_-^n) \\ \quad +\sigma^n - \sigma^0)\partial_{tz}^2 w_{1,-}^0 + \sum_{j=1}^3 \{(\partial_{w_j}\bar a)(w_-^n)\partial_t w_{j,-}^n(\partial_t w_{2,-}^n + (\lambda_1)(w_-^n) - \sigma^n)\partial_z w_{2,-}^n) \\ \quad -(\partial_{w_j}\bar a)(w_-^0)\partial_t w_{j,-}^0(\partial_t w_{2,-}^0 + (\lambda_1(w_-^0)) - \sigma^0)\partial_z w_{2,-}^0\} + \bar a(w_-^n)\partial_t(\lambda_1(w_-^n) \\ \quad -\sigma^n)\partial_z w_{2,-}^n - \bar a(w_-^0)\partial_t(\lambda_1(w_-^0) - \sigma^0)\partial_z w_{2,-}^0 = \partial_t(\frac{\bar b_1(w_-^n)}{z+\phi^n(t)}) - \partial_t(\frac{\bar b_1(w_-^0)}{z+\phi^0(t)}) \\ v(T_\varepsilon, z) = 0 \end{cases} \tag{3.42}$$

It follows from Lemma 3.4, $\bar{b}_1(0)=0$ and inductive hypothesis that $|w^n_{i,-}(t,z)-w^n_{i,+}(t,0+)| \le C_N\varepsilon((t-T_\varepsilon)^3+z^2)^{\frac{1}{6}}$ and $|w^n_{i,-}(t,z)-w^0_{i,-}(t,0-)| \le C_N\varepsilon((t-T_\varepsilon)^3+z^2)^{\frac{1}{6}}$ for $i=1,2,3$.

Hence by the expressions of λ_3 and $\sigma^n(t)$, we have

$$|\lambda_3(w^n_-)-\sigma^n(t)| \le C_N\varepsilon((t-T_\varepsilon)^3+z^2)^{\frac{1}{6}} \tag{3.43}$$

Furthermore, by Lemma 3.4 and the inductive assumption and (3.43) one has

$$|\partial_t w^n_{3,-}(t,z)| \le \frac{C_N\varepsilon}{\sqrt{t-T_\varepsilon}} \tag{3.44}$$

Let $\xi=\xi(t,z,s)$ be the back characteristics of (3.42) through the point (t,z) in the domain $\tilde{\Omega}_-$. If $\xi=\xi(t,z,s)$ intersects with $z-$axis before it meets t-axis, then integrating (3.42) along characteristics and using the result in Part 1, inductive hypothesis and $\bar{a}(0)=\bar{b}_1(0)=0$, we have

$$|v(t,z)| \le C_N\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}}+\int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s))-(\partial_t\sigma^n)(s)||v(s,\xi(t,z,s))|ds$$

In view of (3.44) and the inductive hypothesis and the expression of $\sigma^n(t)$,one has

$$\int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s))-(\partial_t\sigma^n)(s)|ds \le C_N\varepsilon\sqrt{t-T_\varepsilon}$$

So by Gronwall's inequality, for small ε we obtain:

$$|v(t,z)| \le N\varepsilon(t-T_\varepsilon)^{\frac{1}{2}}$$

When $\xi=\xi(t,z,s)$ intersects with $t-$axis at $(s,0)$ with $s>T_\varepsilon$, then we have to use the boundary conditions on t-axis. Indeed, by integration along characteristics we get:

$$\begin{aligned}|v(t,z)| \le\ & C_N\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}}+|(\partial_s(w^{n+1}_{1,-}-w^0_{1,-}))(s,0-)| \\ &+\int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s))-(\partial_t\sigma^n)(s)||v(s,\xi(t,z,s))|ds\end{aligned} \tag{3.45}$$

Below we estimate the additional boundary condition $|(\partial_s(w^{n+1}_{1,-}-w^0_{1,-}))(s,0-)|$.

Since

$$\begin{aligned}|[w^{n+1}_3]| &\le |w^{n+1}_{3,+}(s,0+)-w^0_{3,+}(s,0+)|+|w^{n+1}_{3,-}(s,0-)-w^0_{3,-}(s,0+)|+|[w^0_3]| \\ &\le C_N\varepsilon(s-T_\varepsilon)^{\frac{1}{2}} \\ |[\partial_s w^{n+1}_3]-[\partial_s w^0_3]| &\le \frac{C_N\varepsilon}{\sqrt{s-T_\varepsilon}} \\ |[\partial_s w^{n+1}_3]| &\le \frac{C_N\varepsilon}{s-T_\varepsilon}\end{aligned}$$

So by (3.35)and Lemma 3.4 we can get $|\partial_s(w_{1,-}^{n+1}(s,0-)-w_{1,-}^0(s,0-))| \le C_N\varepsilon^2(s-T_\varepsilon)^{\frac{1}{2}}$ for small ε. Substituting this into (3.45) and using Gronwall's inequality one has $|v(t,z)| \le C_N\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}}$ for small ε.

Finally, since $|\lambda_1(w_-^n)-\sigma^n| \ge \frac{1}{4}$ for small ε, then it follows from equation (3.33) that $|\partial_z(w_{1,-}^{n+1}-w_{1,-}^0)(t,z)| \le C_N\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}}$.

Summarizing all estimates obtained above we complete the lemma by induction.

Step 4. Convergence

To prove the contractivity of the sequences $\{\sigma^n\}$ in $[T_\varepsilon, T_\varepsilon+1]$ and $\{w_{i,\pm}^n\}$ in $\tilde{\Omega}_\pm$, we will prove the contractivity of these sequences.

Lemma 3.9 For small ε, there exists a constant C_N independent of ε and n such that

$$\|\sigma^n-\sigma^{n-1}\|_{L^\infty([T_\varepsilon,T_\varepsilon+1])} \le C_N\sum_{i=1}^{3}\|w_{i,\pm}^n-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \tag{3.46}$$

$$\begin{aligned}\|w_{3,\pm}^{n+1}-w_{3,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} + C_N\sum_{i=1,2}\|w_{i,\pm}^n-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} &\le (1-\varepsilon)\{\|w_{3,\pm}^n-w_{3,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}\\ &+C_N\sum_{i=1,3}\|w_{i,\pm}^n-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}\}\end{aligned} \tag{3.47}$$

where $\|w_{i,\pm}^n-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} = \|w_{i,+}^n-w_{i,+}^{n-1}\|_{L^\infty(\tilde{\Omega}_+)} + \|w_{i,-}^n-w_{i,-}^{n-1}\|_{L^\infty(\tilde{\Omega}_-)}$

Proof. Firstly, in terms of the expression of $\sigma^n(t)$ and Lemma 3.8, (3.46) is obvious.

In order to prove (3.47), now we give an estimate on $w_{3,-}^{n+1}-w_{3,-}^n$. Set $v(t,z)=w_{3,-}^{n+1}-w_{3,-}^n$, then $v(t,z)$ satisfies the following equation:

$$\begin{cases}\partial_t v+(\lambda_3(w_-^n)-\sigma^n)\partial_z v=(\lambda_3(w_-^{n-1})-\lambda_3(w_-^n)+\sigma^n-\sigma^{n-1})\partial_z w_{3,-}^{n-1}\\ +\bar{a}(w_-^n)\{\partial_t(w_{2,-}^n-w_{2,-}^{n-1})+(\lambda_3(w_-^n)-\sigma^n)\partial_z(w_{2,-}^n-w_{2,-}^{n-1})-(\lambda_3(w_-^{n-1})\\ -\lambda_3(w_-^n)+\sigma^n-\sigma^{n-1})\partial_z w_{2,-}^n\}+(\bar{a}(w_-^n)-\bar{a}(w_-^{n-1}))(\partial_t w_{2,-}^{n-1}+(\lambda_3(w_-^{n-1})\\ -\sigma^{n-1})\partial_z w_{2,-}^{n-1})+\frac{\bar{b}_3(w_-^n)}{z+\phi^n(t)}-\frac{\bar{b}_3(w_-^{n-1})}{z+\phi^{n-1}(t)}\\ v(T_\varepsilon,z)=0\end{cases} \tag{3.48}$$

The most trouble term in the right side of (3.48) is the first term $(\lambda_3(w_-^{n-1})-\lambda_3(w_-^n)+\sigma^n-\sigma^{n-1})\partial_z w_{3,-}^{n-1}$, because it contains the unbounded term $\partial_z w_{3,-}^{n-1}$, moreover $\partial_z w_{3,-}^{n-1}$ isn't integrable.

To estimate $(\lambda_3(w_-^{n-1})-\lambda_3(w_-^n))\partial_z w_{3,-}^{n-1}$, we rewrite $(\partial_{w_k}\lambda_3)(w_-^n)\partial_z w_{3,-}^n = \frac{(\partial_{w_k}\lambda_3)(w_-^n)}{(\partial_{w_3}\lambda_3)(w_-^n)}\times \{\partial_z(\lambda_3(w_-^n))-\sum_{j=1,2}(\partial_{w_j}\lambda_3)(w_-^n)\partial_z w_{j,-}^n\}$ for $k=1,2$, which is plausible due to $\partial_{w_3}\lambda_3(0)\ne 0$. Note that $\partial_z w_{j,-}^n(j=1,2)$ and $\partial_z(\lambda_3(w_-^n))$ can be estimated due to Lemma 3.8 and (3.9). We now set

$$(\lambda_3(w_-^n)-\lambda_3(w_-^{n-1}))\partial_z w_{3,-}^{n-1}=\sum_{i=1}^{7}J_i$$

where

$$J_1 = \sum_{j,k=1}^{3}\{\int_0^1\int_0^1(\partial^2_{w_jw_k}\lambda_3)(\theta_1(\theta w_-^n + (1-\theta)w_-^{n-1}) + (1-\theta_1)w_-^{n-1})\theta d\theta d\theta_1$$
$$\times (w_{k,-}^n - w_{k,-}^{n-1})(w_{j,-}^n - w_{j,-}^{n-1})\}\partial_z w_{3,-}^{n-1}$$
$$J_2 = \sum_{j=1}^{3}\{(\partial_{w_j}\lambda_3)(w_-^{n-1}) - (\partial_{w_j}\lambda_3)(w_-^n)\}\partial_z w_{3,-}^{n-1}(w_{j,-}^n - w_{j,-}^{n-1})$$
$$J_3 = \sum_{j=1}^{3}(\partial_{w_j}\lambda_3)(w_-^n)(\partial_z w_{3,-}^{n-1} - \partial_z w_{3,-}^n)(w_{j,-}^n - w_{j,-}^{n-1})$$
$$J_4 = \partial_z(\lambda_3(w_-^n))(w_{3,-}^n - w_{3,-}^{n-1})$$
$$J_5 = -\sum_{k=1,2}(\partial_{w_k}\lambda_3)(w_-^n)\partial_z w_{k,-}^n(w_{3,-}^n - w_{3,-}^{n-1})$$
$$J_6 = \sum_{k=1,2}\frac{(\partial_{w_k}\lambda_3)(w_-^n)}{(\partial_{w_3}\lambda_3)(w_-^n)}\partial_z(\lambda_3(w_-^n))(w_{k,-}^n - w_{k,-}^{n-1})$$
$$J_7 = -\sum_{k,j=1,3}\frac{(\partial_{w_k}\lambda_3)(w_-^n)}{(\partial_{w_3}\lambda_3)(w_-^n)}\{(\partial_{w_j}\lambda_3)(w_-^n)\partial_z w_{j,-}^n\}(w_{k,-}^n - w_{k,-}^{n-1})$$

Hence one can get

$$|(\lambda_3(w_-^n) - \lambda_3(w_-^{n-1}))\partial_z w_{3,-}^{n-1}| \le (|\partial_z(\lambda_3(w_-^n))| + \frac{C_N\varepsilon}{\sqrt{t-T_\varepsilon}})|w_{3,-}^n - w_{3,-}^{n-1}|$$
$$+ C_N\sum_{i=1,2}|w_{i,\pm}^n - w_{i,\pm}^{n-1}| \tag{3.49}$$

Next we dispose the term $(\sigma^n - \sigma^{n-1})\partial_z w_{3,-}^{n-1}$. Note that the relation between σ^n and $\lambda_3(w_\pm^n(t,0\pm))$ can be derived in a similar way as in [23] or [24] as follows:

$$\sigma^n = \lambda_3(w_-(t,0-)) + \frac{1}{2}\sum_{k=1}^{3}(\partial_{w_k}\lambda_3)(w_-(t,0-))[w_k^n] + O([w^n]^2) \tag{3.50}$$

Similar to the estimate for (3.49), we can obtain:

$$|(\sigma^n - \sigma^{n-1})\partial_z w_{3,-}^{n-1}| \le (\frac{1}{2}|\partial_z(\lambda_3(w_-^n))| + \frac{C_N\varepsilon}{\sqrt{t-T_\varepsilon}})|w_{3,-}^n - w_{3,-}^{n-1}|$$
$$+ C_N\sum_{i=1,2}|w_{i,+}^n - w_{i,+}^{n-1}| \tag{3.51}$$

Based on the estimates (3.49) and (3.51), by a similar way as in the proof of Lemma 3.8 (in particular, the Part 4) we can establish

$$\|w_{3,-}^{n+1} - w_{3,-}^n\|_{L^\infty(\tilde\Omega_-)} \le (ln\frac{3}{2} + C_N\varepsilon\sqrt{t-T_\varepsilon})\|w_{3,-}^n - w_{3,-}^{n-1}\|_{L^\infty(\tilde\Omega_-)} + (\frac{1}{2}ln\frac{3}{2} +$$
$$+ C_N\varepsilon\sqrt{t-T_\varepsilon})\|w_{3,\pm}^n - w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}) + C_N\sum_{i=1,2}\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}$$

Similarly, we also have

$$\|w_{3,+}^{n+1}-w_{3,+}^{n}\|_{L^\infty(\tilde\Omega_+)}\le (ln\frac{3}{2}+C_N\varepsilon\sqrt{t-T_\varepsilon})\|w_{3,+}^{n}-w_{3,+}^{n-1}\|_{L^\infty(\tilde\Omega_+)}+(\frac{1}{2}ln\frac{3}{2}+$$
$$+C_N\varepsilon\sqrt{t-T_\varepsilon})\|w_{3,\pm}^{n}-w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)})+C_N\sum_{i=1,2}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}$$

The addition of two above inequalities derives

$$\|w_{3,\pm}^{n+1}-w_{3,\pm}^{n}\|_{L^\infty(\tilde\Omega_\pm)}\le (2ln\frac{3}{2}+C_N\varepsilon\sqrt{t-T_\varepsilon})\|w_{3,\pm}^{n}-w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}+$$
$$+C_N\sum_{i=1,2}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} \tag{3.52}$$

where the coefficient before the summation in the right side is less than 1 provided ε is very small.

Similar to the analysis of Part 3 in Lemma 3.8, we can also establish the estimate

$$\|w_{1,+}^{n+1}-w_{1,+}^{n}\|_{L^\infty(\tilde\Omega_+)}\le C_N\varepsilon(t-T_\varepsilon)\sum_{i=1}^{3}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} \tag{3.53}$$

Finally, we give the estimate on $w_{1,-}^{n+1}-w_{1,-}^{n}$. At this time, we should use the boundary condition on $z=0$. Set $v(t,z)=w_{1,-}^{n+1}-w_{1,-}^{n}$, then $v(t,z)$ satisfies the equation:

$$\begin{cases}\partial_t v+(\lambda_1(w_-^n)-\sigma^n)\partial_z v=(\lambda_1(w_-^{n-1})-\lambda_1(w_-^n)+\sigma^n-\sigma^{n-1})\partial_z w_{1,-}^n-\bar a(w_-^n)\times\\ \times\{\partial_t(w_{2,-}^n-w_{2,-}^{n-1})+(\lambda_1(w_-^n)-\sigma^n)\partial_z(w_{2,-}^n-w_{2,-}^{n-1})+(\lambda_1(w_-^{n-1})-\lambda_1(w_-^n)\\ +\sigma^n-\sigma^{n-1})\partial_z w_{2,-}^n\}-(\bar a(w_-^n)-\bar a(w_-^{n-1}))(\partial_t w_{2,-}^n+(\lambda_1(w_-^n)-\sigma^n)\partial_z w_{2,-}^n)\\ +\frac{\bar b_1(w_-^n)}{z+\phi^n(t)}-\frac{\bar b_1(w_-^{n-1})}{z+\phi^{n-1}(t)}\\ v(T_\varepsilon,z)=0, v(t,z)|_{z=0}=w_{1,-}^{n+1}(t,0-)-w_{1,-}^n(t,0-)\end{cases}$$

If the back characteristics $\xi=\xi(t,z,s)$ through the point (t,z) intersects with $z-$axis before it meets the t-axis, we have

$$|v(t,z)|\le|\bar a(w_-^n)(w_{2,-}^n-w_{2,-}^{n-1})|+C_N\varepsilon\sum_{i=1}^{3}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\int_{T_\varepsilon}^{t}(1+\frac{1}{\sqrt{s-T_\varepsilon}})ds$$
$$\le C_N\varepsilon\sum_{i=1}^{3}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} \tag{3.54}$$

While if $\xi=\xi(t,z,s)$ intersects with t-axis at $(s,0)$ with $s\ge T_\varepsilon$, then similarly we have

$$|v(t,z)|\le|w_{1,-}^{n+1}(s,0-)-w_{1,-}^{n}(s,0-)|+C_N\varepsilon\sum_{i=1}^{3}\|w_{i,\pm}^{n+1}-w_{i,\pm}^{n}\|_{L^\infty(\tilde\Omega_\pm)} \tag{3.55}$$

By (3.35) and Lemma 3.7 and the above estimates, we get

$$
\begin{aligned}
|w_{1,-}^{n+1}(s,0-)-w_{1,-}^{n}(s,0-)| &\le |w_{1,+}^{n+1}(s,0+)-w_{1,+}^{n}(s,0+)|+C_N\varepsilon(|w_{1,-}^{n+1}(s,0-)\\
&\quad -w_{1,-}^{n}(s,0-)|+|w_{2,-}^{n+1}(s,0-)-w_{2,-}^{n}(s,0-)|+|w_{3,\pm}^{n+1}(s,0\pm)-w_{3,\pm}^{n}(s,0\pm)|)\\
&\le C_N\varepsilon\sum_{i=1}^{3}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\Omega_\pm)}
\end{aligned}
$$

Hence $|v(t,z)|\le C_N\varepsilon\sum_{i=1}^{3}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}$

The estimate for $|w_{2,-}^{n+1}(t,z)-w_{2,-}^{n}(t,z)|$ is similar and even simpler.

Synthesizing the above estimates, we have

$$
\begin{aligned}
&\|w_{3,\pm}^{n+1}-w_{3,\pm}^{n}\|_{L^\infty(\tilde\Omega_\pm)}+\sum_{i=1,2}(C_N+1)\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\le(2ln\frac{3}{2}+C_N\varepsilon\sqrt{t-T_\varepsilon}\\
&\quad +C_N(C_N+1)\varepsilon)\|w_{3,\pm}^{n}-w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}+\frac{C_N+C_N(C_N+1)\varepsilon}{C_N+1}\sum_{i=1,2}(C_N+1)\|w_{i,\pm}^{n}\\
&\quad -w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}
\end{aligned}
$$

Using $2ln\frac{3}{2}<1$ and $\frac{C_N}{C_N+1}<1$, and replacing C_N+1 by C_N, then we know that Lemma 3.9 hold for small ε.

The proof of Theorem 2.1.

From the convergence of Lemma 3.9, we know that there exist functions $\sigma(t)\in C[T_\varepsilon,T_\varepsilon+1]$ and $w_{i,\pm}(t,z)\in C(\tilde\Omega_\pm)$ such that $\sigma^n(t)$ uniformly converge to $\sigma(t)$ in $[T_\varepsilon,T_\varepsilon+1]$ and $w_{i,\pm}^{n}(t,z)$ uniformly converge to $w_{i,\pm}(t,z)$ in $\tilde\Omega_\pm$ respectively. Similarly, we can prove $\nabla_{t,z}w_{i,\pm}^{n}(t,z)$ uniformly converge to $\nabla_{t,z}w_{i,\pm}(t,z)$ in the any fixed closed subset of $\tilde\Omega_\pm$ respectively. Moreover, by Lemma 3.6 and the systems (3.17) and (3.18) we know that $w_{i,\pm}^{n}(t,z)$ are equicontinuous on z for the fixed $t\in(T_\varepsilon,T_\varepsilon+1)$ in $\Omega_\pm$ respectively. Hence $w_{i,\pm}(t,0\pm)$ exist in $(T_\varepsilon,T_\varepsilon+1)$, moreover we can conclude that the Rankine-Hugonit conditions hold on the shock curve $\Gamma: r=\phi(t)=T_\varepsilon+\int_{T_\varepsilon}^{t}\sigma(t)dt$ by the equivalence of (3.12) and (3.14). Additionally, the entropy condition is also guaranteed by Lemma 3.1 and the estimates in Lemma 3.8. So the functions $w_i(t,z)=\begin{cases} w_{i,-}(t,z), z<\phi(t)\\ w_{i,+}(t,z), z>\phi(t)\end{cases}$ is the weak entropy solution of (2.11). Finally, the estimates in Theorem 2.1 are the direct conclusion of Lemma 3.6 and Lemma 3.8 combining with the convergence of the sequence of approximate solutions.

Acknowledgments. I'm very happy to acknowledge the guidance and encouragement of Prof.Chen Shuxing and Prof.Xin Zhouping, with whom the work on this problem was jointly planned. Their support is greatly appreciated. I also deeply thank the Institute of Mathematics in Potsdam University, where I worked on this paper during my visiting period.

References

1. Yin Huicheng, Qiu Qingjiu, *The blowup of solutions for three dimensional spherically symmetric compressible Euler equations, Nagoya M.J., Vol.154, 157-169 (1999).*
2. Yin Huicheng, *The blowup mechanism of small data solutions for the quasilinear wave equations in three space dimensions, Acta Math. Sinica, English Series, Vol.17, No.1, 35-76 (2001).*
3. J.Glimm, *Solution in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math., 18, 697-715 (1965).*
4. P.D.Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shocks waves, Conf. Board Math.Sci. SIAM, 11, 1973.*
5. Alberto, Bressan; Graziano, Crasta; Benedetto, Piccoli, *Well-posedness of the Cauchy problem for* $n \times n$ *systems of conservation laws, Mem.Amer.Math.Soc.146, no.694 (2000).*
6. R.Diperna, *Uniqueness of solutions to hyperbolic conservation laws, Indiana Univ. Math.J.,28, 224-257 (1979).*
7. Liu T.P., Yong T., *Well-posedness theory for hyperbolic conservation laws, Comm. Pure Appl. Math., 52, no.12, 1553-1586 (1999).*
8. Chen G.Q., J.Glimm, *Global solutions to the compressible Euler equations with geometrical structure, Comm. Math. Phys.180, 153-193 (1996).*
9. Chen Shuxing, Xin Zhouping, Yin Huicheng, *Formation and construction of shock wave for quasilinear hyperbolic system and its application to 1-D inviscid compressible flow, Preprint (1999).*
10. Chen Shuxing, Zhang Z.B., *On the generation of shock waves of first order quasilinear equations, Fudan Journal(Natural Science), 13-22 (1963).*
11. A.Majda, *Compressible fluid flow and systems of conservation laws in several space variables, Springer - Verlag, 1984.*
12. C.M.Dafermos, *Generalized characteristics in hyperbolic system of conservation laws, Arch.Rat. Mech. Anal. 107, 127-155(1989).*
13. M.P.Lebaud, *Description de la formation d'un choc dans le p-systems, J.Math.Pures Appl. 73, 523-565 (1994).*
14. Chen Shuxing, Dong Liming, *Formation of shock for the p-system with general smooth initial data, Sci.in China,Ser A, 44, no.9, 1139-1147 (2001).*
15. F.John, *Formation of singularities in one-dimensional nonlinear wave propagation, Comm. Pure Appl. Math., 27, 377-405 (1974).*
16. L.Hömander, *Lectures on nonlinear hyperbolic differential equations, Mathematics and Applications 26, Springer-Verlag, Berlin, 1997.*
17. Liu, Taiping, *Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations, J.D.E., 33, 92-111(1979).*
18. S.Alinhac, *Temps de vie precise et explosion geometrique pour des systemes hyperboliques quasilineaires en dimension un d'espace, Ann.Scoula Norm.Sup.Pisa. Serie IV Vol. XXII(3), 493-515(1995).*
19. S.Alinhac, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions, Ann.of Math., 149(1), 97-127(1999).*
20. Yin Huicheng, *The blowup mechanism of axisymmetric solutions for three dimensional quasilinear wave equations with small data, Science in China (Series A), Vol.43, No.3,252-266(2000).*
21. F.John, S.Klainerman, *Almost global existence to nonlinear wave equations in three space, Comm. Pure Appl. Math., 37, no.4, 443-455 (1984).*
22. F.John, *Existence for large times of strict solutions of nonlinear wave equations in three space for small initial data, Comm. Pure Appl. Math., 40, no.1, 79-109 (1987).*
23. P.D.Lax, *Hyperbolic systems of conservation laws II, Comm.Pure Appl. Math., X, 537-566 (1957).*
24. J.A.Smoller, *Shock waves and reaction-diffusion equations, Berlin-Heiderberg-New York, Springer-Verlag, New York, 1984.*