On the calculus of pseudodifferential operators with an anisotropic analytic parameter

Thomas Krainer[1]

Institut für Mathematik
Universität Potsdam
Postfach 60 15 53
D–14415 Potsdam
Germany

krainer@math.uni-potsdam.de

January 4, 2002

Abstract

We introduce the Volterra calculus of pseudodifferential operators with an anisotropic analytic parameter based on "twisted" operator-valued Volterra symbols. We establish the properties of the symbolic and operational calculi, and we give and make use of explicit oscillatory integral formulas on the symbolic side. In particular, we investigate the kernel cut-off operator via direct oscillatory integral techniques purely on symbolic level.

We discuss the notion of parabolicity for the calculus of Volterra operators, and construct Volterra parametrices for parabolic operators within the calculus.

Mathematics Subject Classification (2000): 35S05 (Primary)
35S15, 35K40, 35K50, 35K65 (Secondary)

Contents

[1] Supported by the Deutsche Forschungsgemeinschaft

Introduction

The present article contributes to the axiomatic framework of pseudodifferential analysis of degenerate partial differential equations, e. g., partial differential equations on manifolds with singularities.

Substantial progress has been achieved in recent years, in particular, in the theory of degenerate elliptic equations. The general concept is to embed the natural systems of elliptic partial differential equations into a suitable calculus of pseudodifferential operators that admits the construction of parametrices of elliptic elements within, and to study the qualitative properties of the equations such as regularity and asymptotics of solutions and the Fredholm property via algebraic methods on side of the operator algebra, see, e. g., Schulze [37], [39], [40].

Recently, this concept has been further developed also in the study of parabolic equations, see, e. g., Buchholz and Schulze [6], and Krainer [23], Krainer and Schulze [25], [26]. More precisely, the natural systems of parabolic partial differential equations are embedded in a suitable calculus of pseudodifferential operators that admits the construction of inverses of parabolic elements. In particular, in addition to the elliptic theory, the existence and uniqueness of solutions follow, and, via analyzing the operator and symbolic structure of the calculus, insights about the qualitative properties of the equations and the structure of solution operators are obtained.

A typical feature of the parabolic theory is that the pseudodifferential operators have the so-called Volterra property with respect to time, i. e., the (anisotropic) symbols extend holomorphically in the time covariable to a complex half-plane, including the symbol estimates. The classical calculus of such operators was introduced by Piriou [29], [30] in the study of parabolic pseudodifferential equations on a finite time interval, and a closed (compact) spatial manifold.

The analysis of parabolic equations, as well as heat equation methods related to spectral problems, is characterized by parameter-dependent approaches, i. e., the time covariable is considered as a (spectral) parameter for the operators acting in space, and the resulting parameter-dependent behaviour is studied first. Afterwards, in a second step, the analysis and quantization with respect to time is performed built upon these structures. In particular, investigations of parameter-dependent theories with an anisotropic and analytic parameter constitute a necessary step in the investigation of parabolic problems. More information about this classical approach in various contexts is to be found, e. g., in Agranovich and Vishik [1], Seeley [41], [42], Melrose [28], Gilkey [15], Shubin [45], Grubb [16], Grubb and Seeley [17], or Gil [13].

The (pseudodifferential) analysis of (degenerate) partial differential equations on manifolds with singularities and boundary value problems encompasses, in particular, the crucial task to describe the behaviour of the equations close to the singular sets and the boundary, where, typically, extra conditions of trace and potential type are involved that are associated with the operators in a natural way. In this context, the abstract theory of pseudodifferential calculus with "twisted" operator-valued symbols was introduced by Schulze (see, e. g., [36], [37]) in order to describe the general structure of these conditions, as well as the structure of the operators and the singular Green remainders as they have to be (re)formulated with respect to a given splitting of coordinates on and transversal to wedges or boundaries.

Several authors have contributed since to the pseudodifferential calculus with operator-valued symbols for it provides a general axiomatic framework for the pseudodifferential analysis of degenerate partial differential equations and boundary value problems, i. e., in concrete situations such as the calculus on manifolds with conical singularities, edges, and corners, many functional analytic properties can be traced back to the calculus of operators with operator-valued symbols; see, e. g., Behm [2], Dorschfeldt [8], Dorschfeldt, Grieme, and Schulze [9], Krainer [23], and Seiler [44]. Material about pseudodifferential calculus with anisotropic operator-valued symbols can be found in Buchholz and Schulze [6], Gil [13], and Krainer [23], and the theory of operators with operator-valued Volterra symbols has so far been considered in Buchholz [4], Buchholz and Schulze [6], and Krainer [23], [24].

The purpose of the present paper is to give a unified approach to the calculus of pseudodifferential operators with operator-valued symbols and an anisotropic analytic parameter (Volterra calculus) in order to provide necessary fundamentals in the axiomatic framework of pseudodifferential analysis of parabolic equations and boundary value problems on manifolds with singularities. To this end, we employ explicit oscillatory integral formulas and symbols that satisfy global estimates in the variables ("Kumano-go's technique"). This direct approach via oscillatory integral techniques enables us to establish the symbolic and operational calculi in a transparent form, where, in particular, manipulations on both sides are considered separated from each other.

The text is organized as follows: In Section 1 we give an account on the basic notation and general conventions that are freely used throughout this work.

Sections 2 and 3 are concerned with the symbolic calculus of the classes of general and Volterra operator-valued symbols. We recall the basic definitions and properties of general anisotropic symbols as well as the concept of homogeneity and classical symbols in Section 2. The analyticity in the parameter represents the major difficulty in the Volterra symbolic calculus that is discussed in Section 3, because arguments involving excision functions cannot be employed. Usually, excision functions are used, e. g., to establish the asymptotic completeness, i. e., that it is possible to find symbols having a prescribed asymptotic expansion, as well as to prove that the principal symbol sequence is exact; both aspects are important for the construction of (parameter-dependent) parametrices of parabolic elements (see Section 5). The analysis of the translation operator in Volterra symbols, and the kernel cut-off operator (see Section 3.1), provide the appropriate tools to overcome these difficulties.

We give a definition of the kernel cut-off operator on symbolic level via a direct oscillatory integral formula that enables us to present its functional analytic properties in a transparent form. Note that kernel cut-off is a necessary and widely applied technique in the pseudodifferential analysis of degenerate partial differential equations, see, e. g., Schulze [37], [39], and our approach within the axiomatic framework provides the fundamentals for various generalizations, in particular, more complicated singularities.

In Section 4 we study the operational calculi of pseudodifferential operators that are built upon operator-valued parameter-dependent (Volterra) symbols, and we analyze how the manipulations on side of the operators are reflected on the symbolic side.

Finally, in Section 5, we recall the notion of parameter-dependent ellipticity and discuss the notion of parabolicity for the Volterra symbol class, and we give a proof of the existence of parameter-dependent (Volterra) parametrices within the calculi. The parametrix construction is performed via symbolic inversion and the classical formal Neumann series argument, and the algebraic properties and results from the preceding sections are used, in particular, in the discussion of symbolic invertibility and asymptotic expansions.

Acknowledgement: The author expresses his gratitude to Professor B.–W. Schulze from the University of Potsdam for numerous scientific discussions, and for his encouragement to prepare the present article.

1 Basic notation and general conventions

1.1 Sets of real and complex numbers

- We denote:

$\mathbb{C}$	the complex numbers,
$\mathbb{R}$	the reals,
$\mathbb{R}_+, \mathbb{R}_-$	the positive (negative) reals,
$\overline{\mathbb{R}}_+, \overline{\mathbb{R}}_-$	the non-negative (non-positive) reals,
$\mathbb{Z}$	the integers,
$\mathbb{N}$	the positive integers,
$\mathbb{N}_0$	the non-negative integers.

- Let $\mathbb{C}^N$ and $\mathbb{R}^N$ denote the complex N-space, respectively the Euclidean N-space, in the variables $(z_1, \ldots, z_N) \in \mathbb{C}^N$ or $(x_1, \ldots, x_N) \in \mathbb{R}^N$, respectively. In general, we allow N to be zero, and in this case these spaces degenerate to the set containing a single point only.
- The upper half-plane in $\mathbb{C}$ will be denoted as

$$\mathbb{H} := \{z \in \mathbb{C};\ \mathrm{Im}(z) \geq 0\}.$$

- The Euclidean norm of $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ is denoted as $|x| = \Big(\sum_{j=1}^{N} x_j^2\Big)^{\frac{1}{2}}$.

  Moreover, let $\langle x \rangle = \big(1 + |x|^2\big)^{\frac{1}{2}}$ be the standard regularized distance in $\mathbb{R}^N$.

  The inner product in $\mathbb{R}^N$ is denoted as $\langle x, \xi \rangle = x\xi = \sum_{j=1}^{N} x_j \xi_j$.

1.2 Multi-index notation

We employ the standard multi-index notation.

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N), \beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}_0^N$ we denote

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_N}{\beta_N} \qquad \alpha! = \prod_{j=1}^{N} \alpha_j! \qquad |\alpha| = \sum_{j=1}^{N} \alpha_j.$$

We write $\alpha \leq \beta$ if the inequality holds componentwise. Moreover, (normalized) partial derivatives with respect to the variables $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ are written as

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}} \qquad D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha.$$

In case a function $f(x,\lambda)$ depends on the group of complex variables $\lambda \in \mathbb{C}^M$ we also use the notations

$$\partial_\lambda^\beta f = \frac{\partial^{|\beta|}}{\partial_{\lambda_1}^{\beta_1} \ldots \partial_{\lambda_M}^{\beta_M}} f \qquad D_\lambda^\beta f = (-i)^{|\beta|} \partial_\lambda^\beta f,$$
$$\partial_{\overline{\lambda}}^\beta f = \frac{\partial^{|\beta|}}{\partial_{\overline{\lambda}_1}^{\beta_1} \ldots \partial_{\overline{\lambda}_M}^{\beta_M}} f \qquad D_{\overline{\lambda}}^\beta f = (-i)^{|\beta|} \partial_{\overline{\lambda}}^\beta f.$$

For $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ we write $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_N^{\alpha_N}$.

1.3 Functional analysis and basic function spaces

Unless stated explicitly otherwise, the spaces in this work are always assumed to be complex. For topological vector spaces E and F we denote the space of continuous linear operators $E \longrightarrow F$ as $\mathcal{L}(E,F)$. Moreover, the topological dual of E is denoted as E'. We write $E \otimes F$ for the algebraical tensor product of E and F. The projective topology on $E \otimes F$ is indicated by the subscript $E \otimes_\pi F$, while $E \widehat{\otimes}_\pi F$ denotes the completion. We employ the notation $\langle \cdot, \cdot \rangle_{E,F}$, or just $\langle \cdot, \cdot \rangle$, when we deal with a duality $E \times F \longrightarrow \mathbb{C}$. The inner product in a Hilbert space E is also denoted as $\langle \cdot, \cdot \rangle_E$, or simply as $\langle \cdot, \cdot \rangle$.

Moreover, we have the following spaces of E-valued functions on M (where M and E are appropriate):

$L^p(M,E)$	measurable functions u with $\int\limits_M \|u(x)\|_E^p \, dx < \infty$ (with respect to Lebesgue measure, $1 \le p < \infty$),
$C(M,E)$	continuous functions,
$\mathcal{A}(M,E)$	analytic functions,
$C^k(M,E)$	k-times continuously differentiable functions,
$C^\infty(M,E)$	smooth functions,
$C_0^\infty(M,E)$	smooth functions with compact support,
$C_b^\infty(M,E)$	smooth functions with bounded derivatives,
$\mathcal{S}(M,E)$	rapidly decreasing functions,
$\mathcal{D}'(M,E) = \mathcal{L}(C_0^\infty(M),E)$	distributions,
$\mathcal{E}'(M,E) = \mathcal{L}(C^\infty(M),E)$	distributions with compact support,
$\mathcal{S}'(M,E) = \mathcal{L}(\mathcal{S}(M),E)$	tempered distributions.

If $E = \mathbb{C}$ we drop it from the notation.

1.4 Tempered distributions and the Fourier transform

Let E be a Hilbert space. Partial derivatives of a distribution $u \in \mathcal{S}'(\mathbb{R}^n, E)$ are defined as $\langle \partial_x^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial_x^\alpha \varphi \rangle$, while multiplication with a function ψ of tempered growth is given as $\langle \psi u, \varphi \rangle = \langle u, \psi\varphi \rangle$. A distribution $u \in \mathcal{S}'(\mathbb{R}^n, E)$ is called regular, if u is a Bochner measurable function, and there exists $N \in \mathbb{N}_0$ with $\int_{\mathbb{R}^n} \langle x \rangle^{-N} \|u(x)\|_E \, dx < \infty$. Note that we identify regular distributions with their densities. In this sense we in particular have $L^p(\mathbb{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E)$.

We employ the normalized Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, i. e.,

$$(\mathcal{F}u)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x) \, dx,$$

$$(\mathcal{F}^{-1}u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} u(\xi) \, d\xi,$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. For Fréchet spaces E the Fourier transform extends to an isomorphism $\mathcal{S}(\mathbb{R}^n, E) \longrightarrow \mathcal{S}(\mathbb{R}^n, E)$ via $\mathcal{F} = \mathcal{F} \widehat{\otimes}_\pi \mathrm{id}_E$, noting that $\mathcal{S}(\mathbb{R}^n, E) \cong \mathcal{S}(\mathbb{R}^n) \widehat{\otimes}_\pi E$. If E is a Hilbert space we have $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n, E) \longrightarrow \mathcal{S}'(\mathbb{R}^n, E)$ via $\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle$.

In oscillatory integral formulas, however, we shall follow the tradition and employ the normalized measure $đ\xi = \big(2\pi\big)^{-n} d\xi$ on the side of the covariables.

2 General parameter-dependent symbols

2.1 Definition. Let $\ell \in \mathbb{N}$ be a given anisotropy.

a) For $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ define

$$|\xi, \lambda|_\ell := (|\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}},$$
$$\langle \xi, \lambda \rangle_\ell := (1 + |\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}},$$

where $|\cdot|$ denotes the Euclidean norm.

b) For a multi-index $\beta = (\alpha, \alpha') \in \mathbb{N}_0^{n+q}$ let

$$|\beta|_\ell := |\alpha| + \ell \cdot |\alpha'|,$$

where $|\cdot|$ denotes the usual length of a multi-index as the sum of its components.

2.2 Lemma. *There exists a constant $c > 0$ such that for all $s \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^n$, $\lambda_1, \lambda_2 \in \mathbb{R}^q$ the following inequality is fulfilled* (Peetre's inequality)*:*

$$\langle \xi_1 + \xi_2, \lambda_1 + \lambda_2 \rangle_\ell^s \leq c^{|s|} \langle \xi_1, \lambda_1 \rangle_\ell^{|s|} \langle \xi_2, \lambda_2 \rangle_\ell^s. \tag{2.i}$$

Moreover, we can compare the regularized "anisotropic distance" $\langle \cdot, \cdot \rangle_\ell$ with the "isotropic distance", i. e., there exist constants $c_1, c_2 > 0$ such that

$$c_1 \langle \xi, \lambda \rangle_\ell \leq \langle \xi, \lambda \rangle \leq c_2 \langle \xi, \lambda \rangle_\ell^\ell. \tag{2.ii}$$

2.3 Definition. Let E be a Hilbert space. A *strongly continuous group-action* on E is a strongly continuous group-representation

$$\kappa : (\mathbb{R}_+, \cdot) \longrightarrow \mathcal{L}(E). \tag{2.iii}$$

From the uniform boundedness principle we obtain the existence of constants $c, M \geq 0$ such that

$$\|\kappa_\varrho\|_{\mathcal{L}(E)} \leq c \max\{\varrho, \frac{1}{\varrho}\}^M \text{ for } \varrho \in \mathbb{R}_+. \tag{2.iv}$$

By the trivial group-action we mean the trivial representation, i. e., $\kappa_\varrho = \mathrm{Id}_E$ for all $\varrho \in \mathbb{R}_+$.

2.4 Definition. Let E and $\tilde{E}$ be Hilbert spaces endowed with strongly continuous group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. For $\mu \in \mathbb{R}$ we define

$$\begin{aligned} S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := \{&a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}));\ \text{for all } k \in \mathbb{N}_0 : \\ &p_k(a) := \sup_{\substack{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{R}^q \\ |\beta|_\ell \leq k}} \|\tilde{\kappa}^{-1}_{\langle \xi,\lambda\rangle_\ell} \partial^\beta_{(\xi,\lambda)} a(\xi,\lambda) \kappa_{\langle \xi,\lambda\rangle_\ell}\| \, \langle \xi,\lambda\rangle_\ell^{-\mu+|\beta|_\ell} < \infty\}. \end{aligned}$$

This is a Fréchet space with the topology induced by the seminorm-system $\{p_k;\ k \in \mathbb{N}_0\}$. Define

$$S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := \bigcap_{\mu\in\mathbb{R}} S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}).$$

By (2.ii) and Definition 2.3 this space does not depend on $\ell \in \mathbb{N}$ and the group-actions involved on E and $\tilde{E}$, and we have $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$. Moreover, for $\mu \in \mathbb{R}$ the spaces of x- (resp. x'-) and (x, x')-dependent symbols are defined as

$$\begin{aligned} S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) &:= C_b^\infty(\mathbb{R}^n, S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})), \\ S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) &:= C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n, S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})). \end{aligned}$$

Analogously, we obtain the spaces of order $-\infty$. If $E = \tilde{E} = \mathbb{C}$ with the trivial group-action involved we suppress the Hilbert spaces from the notation.

More generally, let $\{E_j\}_{j\in\mathbb{N}}$ and $\{\tilde{E}_j\}_{j\in\mathbb{N}}$ be scales of Hilbert spaces such that $E_j \hookrightarrow E_{j+1}$ and $\tilde{E}_{j+1} \hookrightarrow \tilde{E}_j$ for $j \in \mathbb{N}$. Moreover, let $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$ be defined on the unions of the $\{E_j\}$ and $\{\tilde{E}_j\}$, respectively, such that the restrictions on each E_j and $\tilde{E}_j$ are strongly continuous group-actions. Define

$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \operatorname*{ind-lim}_{j\in\mathbb{N}} E_j, \operatorname*{proj-lim}_{k\in\mathbb{N}} \tilde{E}_k) := \bigcap_{j,k\in\mathbb{N}} S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E_j, \tilde{E}_k)$$

with the natural Fréchet topologies induced. The spaces of order $-\infty$ are defined in an analogous manner, as well as the symbol spaces with dependence on $x, x' \in \mathbb{R}^n$. With this notion the case of single Hilbert spaces E and $\tilde{E}$ corresponds to the constant scales.

2.5 Notation. Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, in the sense of Definition 2.4. For short, we set

$$\mathcal{E} := \operatorname*{ind-lim}_{j\in\mathbb{N}} E_j \quad \text{and} \quad \tilde{\mathcal{E}} := \operatorname*{proj-lim}_{j\in\mathbb{N}} \tilde{E}_j.$$

2.6 Lemma. *Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert-spaces with strongly continuous group-actions $\{\kappa_\varrho\}$, $\{\tilde{\kappa}_\varrho\}$ and $\{\hat{\kappa}_\varrho\}$.*

a) *For $\mu \geq \mu'$ the embedding $S^{\mu';\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \hookrightarrow S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ is well-defined and continuous.*

b) *The embeddings $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \hookrightarrow S_{\mathbb{1}}^{\mu+M+\tilde{M};\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ and $S_{\mathbb{1}}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \hookrightarrow S^{\mu+M+\tilde{M};\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ are well-defined and continuous, where the subscript $\mathbb{1}$ indicates that the trivial group-actions are involved on the spaces E and $\tilde{E}$. Here M and $\tilde{M}$ are the constants in the estimates for the operator-norms of the group-actions from (2.iv).*

c) *For $\beta \in \mathbb{N}_0^{n+q}$ the operator of differentiation $\partial^\beta_{(\xi,\lambda)} : S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \to S^{\mu-|\beta|_\ell;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ is continuous.*

d) *For $\mu, \mu' \in \mathbb{R}$ pointwise multiplication (composition of operators) induces a continuous bilinear mapping*

$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \hat{E}) \times S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \to S^{\mu+\mu';\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \hat{E}).$$

2.1 Asymptotic expansion

2.7 Definition. Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be associated to scales of Hilbert spaces according to Notation 2.5.

Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. A symbol $a \in S^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ is called the *asymptotic expansion* of the a_k, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$a - \sum_{j=1}^{k} a_j \in S^{R;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}}).$$

The symbol a is uniquely determined modulo $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$.

For short we write $a \sim \sum_{j=1}^{\infty} a_j$.

2.8 Lemma. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Notation 2.5. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k > \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. Furthermore, for each $k \in \mathbb{N}$ let $(A_{k_j})_{j\in\mathbb{N}} \subseteq S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q;\mathcal{E},\tilde{\mathcal{E}})$ be a countable system of bounded sets. Let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ be a 0-excision function. Then there is a sequence $(c_i) \subseteq \mathbb{R}_+$ with $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$ such that for each $k \in \mathbb{N}$*

$$\sum_{i=k}^{\infty} \sup_{a\in A_{i_j}} p\big(\chi\Big(\frac{\xi}{d_i},\frac{\lambda}{d_i^\ell}\Big)a(\xi,\lambda)\big) < \infty \tag{2.v}$$

for all continuous seminorms p on $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q;\mathcal{E},\tilde{\mathcal{E}})$ and every $j \in \mathbb{N}$, and for all sequences $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_i$.

Proof. The proof of this lemma is a variant of the standard Borel argument. Note first that the set

$$\big\{\chi\Big(\frac{\xi}{\theta},\frac{\lambda}{\theta^\ell}\Big);\ \theta \in [1,\infty)\big\} \subseteq S^{0;\ell}(\mathbb{R}^n \times \mathbb{R}^q) \tag{1}$$

is bounded. To see this, assume that

$$\chi(\xi,\lambda) \equiv \begin{cases} 0 & \text{for } |\xi,\lambda|_\ell < c \\ 1 & \text{for } |\xi,\lambda|_\ell > \frac{1}{c} \end{cases}$$

for a sufficiently small $0 < c \ll 1$. Consequently, we see for all $0 \neq \beta \in \mathbb{N}_0^{n+q}$ that

$$\partial^\beta_{(\xi,\lambda)}\chi\Big(\frac{\xi}{\theta},\frac{\lambda}{\theta^\ell}\Big) = \big(\partial^\beta_{(\xi,\lambda)}\chi\big)\Big(\frac{\xi}{\theta},\frac{\lambda}{\theta^\ell}\Big)\theta^{-|\beta|_\ell} \neq 0$$

at most for $c|\xi,\lambda|_\ell \leq \theta \leq \frac{1}{c}|\xi,\lambda|_\ell$ which gives the boundedness of (1).

Now let $\mu \in \mathbb{R}$ and $A \subseteq S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q;\mathcal{E},\tilde{\mathcal{E}})$ be bounded. Then we get from Lemma 2.6 that the set

$$\big\{\chi\Big(\frac{\xi}{\theta},\frac{\lambda}{\theta^\ell}\Big)a(\xi,\lambda);\ \theta \in [1,\infty),\ a \in A\big\} \subseteq S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q;\mathcal{E},\tilde{\mathcal{E}})$$

is also bounded. Let $\mu' > \mu$. Then, given $\varepsilon > 0$, we see that for $\langle\xi,\lambda\rangle_\ell^{\mu'-\mu} \geq \frac{1}{\varepsilon}$

$$\langle\xi,\lambda\rangle_\ell^{\mu-|\beta|_\ell} \leq \varepsilon\,\langle\xi,\lambda\rangle_\ell^{\mu'-|\beta|_\ell}, \quad \beta \in \mathbb{N}_0^{n+q},$$

and consequently, by the defintion of the symbol spaces,

$$\sup_{a\in A} p\big(\chi\Big(\frac{\xi}{\theta},\frac{\lambda}{\theta^\ell}\Big)a(\xi,\lambda)\big) \underset{\theta\to\infty}{\longrightarrow} 0 \tag{2}$$

for all continuous seminorms p on $S^{\mu';\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$.

For the proof of the assertion we may assume without loss of generality that $A_{k_j} \subseteq A_{k_{j+1}}$ holds for the bounded sets (otherwise we pass to unions). For each $k \in \mathbb{N}$ let $p_1^k \leq p_2^k \leq \ldots$ be a fundamental system of seminorms for the topology of $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. From (2) we get a sequence $(c_i^1)_{i\in\mathbb{N}} \in \mathbb{R}_+$ with $c_i^1 < c_{i+1}^1 \underset{i\to\infty}{\longrightarrow} \infty$ such that

$$\sup\bigl\{p_i^1\bigl(\chi\Bigl(\frac{\xi}{\theta}, \frac{\lambda}{\theta^\ell}\Bigr)a(\xi,\lambda)\bigr);\ a \in A_{i_i},\ \theta \geq c_i^1\bigr\} < 2^{-i}$$

for $i > 1$. Iterating the argument, we obtain a system of sequences $(c_i^k)_{i\in\mathbb{N}} \in \mathbb{R}_+$ such that (c_i^{k+1}) is a subsequence of (c_i^k) and

$$\sup\bigl\{p_i^k\bigl(\chi\Bigl(\frac{\xi}{\theta}, \frac{\lambda}{\theta^\ell}\Bigr)a(\xi,\lambda)\bigr);\ a \in A_{i_i},\ \theta \geq c_i^k\bigr\} < 2^{-i}$$

for each $k \in \mathbb{N}$ and $i > k$. Set $c_i := c_i^i$. Let $k \in \mathbb{N}$ be given and p a continuous seminorm on $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. Moreover, let $j \in \mathbb{N}$ be fixed. Since $\{p_i^k;\ i \in \mathbb{N}\}$ is an increasing fundamental system of seminorms for the topology of $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ we find $i_0 \in \mathbb{N}$ and $C > 0$ such that $p \leq C p_i^k$ for $i > i_0$. For $i \geq j$ we have $A_{i_j} \subseteq A_{i_i}$. Note that by construction $c_i \geq c_i^k$ holds for $i \geq k$. Thus, for i sufficiently large,

$$\begin{aligned}
&\sup\bigl\{p\bigl(\chi\Bigl(\frac{\xi}{d_i}, \frac{\lambda}{d_i^\ell}\Bigr)a(\xi,\lambda)\bigr);\ a \in A_{i_j}\bigr\}\\
&\qquad\qquad \leq C \sup\bigl\{p_i^k\bigl(\chi\Bigl(\frac{\xi}{\theta}, \frac{\lambda}{\theta^\ell}\Bigr)a(\xi,\lambda)\bigr);\ a \in A_{i_i},\ \theta \geq c_i^k\bigr\}\\
&\qquad\qquad < C2^{-i}
\end{aligned}$$

whenever $(d_i) \subseteq \mathbb{R}_+$ is a sequence with $d_i \geq c_i$. This proves the lemma. □

2.9 Theorem. *Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ associated to scales of Hilbert spaces as in Notation 2.5. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. Then there exists $a \in S^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \sim \sum_{j=1}^{\infty} a_j$, and a is uniquely determined modulo $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$.*

Proof. Without loss of generality we may assume that $\mu_k > \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. For $k, j \in \mathbb{N}$ let

$$A_{k_j} := \bigl\{\partial_x^\alpha a_k(x);\ x \in \mathbb{R}^n,\ |\alpha| \leq j\bigr\}.$$

Then $A_{k_j} \subseteq S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ is bounded. Let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ be a 0-excision function. Now we may apply Lemma 2.8. With a suitable sequence $(c_i) \subseteq \mathbb{R}_+$

formula (2.v) becomes

$$\sum_{i=k}^{\infty} \sup\{p\big(\chi\Big(\frac{\xi}{c_i}, \frac{\lambda}{c_i^{\ell}}\Big)(\partial_x^{\alpha} a_i(x))(\xi,\lambda)\big);\ x \in \mathbb{R}^n,\ |\alpha| \leq j\} < \infty$$

for all continuous seminorms p on $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$, which shows that for every $k \in \mathbb{N}$

$$\sum_{i=k}^{\infty} \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^{\ell}}\Big) a_i$$

is unconditionally convergent in $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. Now define

$$a := \sum_{i=1}^{\infty} \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^{\ell}}\Big) a_i \in S^{\mu_1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}}).$$

We thus see

$$a - \sum_{i=1}^{k} a_i = \sum_{i=k+1}^{\infty} \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^{\ell}}\Big) a_i - \sum_{i=1}^{k} \big(1 - \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^{\ell}}\Big)\big) a_i$$

where

$$\sum_{i=1}^{k} \big(1 - \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^{\ell}}\Big)\big) a_i \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}}).$$

This yields the desired result, since the uniqueness assertion is clear. □

2.2 Homogeneity and classical symbols

2.10 Definition. Let E and $\tilde{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. A function $f : (\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\} \to \mathcal{L}(E, \tilde{E})$ is called *(anisotropic) homogeneous* of degree $\mu \in \mathbb{R}$, if for $(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\}$ and $\varrho > 0$

$$f(\varrho\xi, \varrho^{\ell}\lambda) = \varrho^{\mu} \tilde{\kappa}_\varrho f(\xi, \lambda) \kappa_\varrho^{-1}. \tag{2.vi}$$

A function $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathcal{L}(E, \tilde{E})$ is called *(anisotropic) homogeneous* of degree $\mu \in \mathbb{R}$ for large (ξ, λ), if relation (2.vi) holds for $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ with $|(\xi, \lambda)|$ sufficiently large and $\varrho \geq 1$.

In this work, homogeneity always is meant in this anisotropic sense.

2.11 Lemma. *Let $a \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$ be homogeneous of degree $\mu \in \mathbb{R}$ for large (ξ, λ). Then $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$.*

Proof. By differentiating relation (2.vi) we see, that for $\beta \in \mathbb{N}_0^{n+q}$ the function $\partial^\beta_{(\xi,\lambda)} a$ is homogeneous of degree $\mu - |\beta|_\ell$ for large (ξ,λ). Thus it suffices to show that $\|\tilde\kappa^{-1}_{\langle\xi,\lambda\rangle_\ell} a(\xi,\lambda)\kappa_{\langle\xi,\lambda\rangle_\ell}\| = O(\langle\xi,\lambda\rangle_\ell^\mu)$ for $|(\xi,\lambda)| \to \infty$. Extension by homogeneity shows that there is a homogeneous function f of degree μ such that $a = f$ for large $|(\xi,\lambda)|$. Note that f is continuous on $(\mathbb{R}^n \times \mathbb{R}^q)\setminus\{0\}$ with values in $\mathcal{L}_\sigma(E,\tilde E)$ for the group-actions are strongly continuous. In particular, f maps compact sets to bounded sets in $\mathcal{L}(E,\tilde E)$ by the uniform boundedness principle. Employing the identity

$$\begin{aligned}\tilde\kappa^{-1}_{\langle\xi,\lambda\rangle_\ell} f(\xi,\lambda)\kappa_{\langle\xi,\lambda\rangle_\ell} &= \tilde\kappa^{-1}_{\langle\xi,\lambda\rangle_\ell} f\Big(|\xi,\lambda|_\ell\frac{\xi}{|\xi,\lambda|_\ell}, |\xi,\lambda|_\ell^\ell\frac{\lambda}{|\xi,\lambda|_\ell^\ell}\Big)\kappa_{\langle\xi,\lambda\rangle_\ell}\\ &= |\xi,\lambda|_\ell^\mu\,\tilde\kappa_{\langle\xi,\lambda\rangle_\ell^{-1}|\xi,\lambda|_\ell} f\Big(\frac{\xi}{|\xi,\lambda|_\ell},\frac{\lambda}{|\xi,\lambda|_\ell^\ell}\Big)\kappa_{|\xi,\lambda|_\ell^{-1}\langle\xi,\lambda\rangle_\ell}\end{aligned}$$

for $(\xi,\lambda)\neq 0$ yields that $\|\tilde\kappa^{-1}_{\langle\xi,\lambda\rangle_\ell} f(\xi,\lambda)\kappa_{\langle\xi,\lambda\rangle_\ell}\| = O(\langle\xi,\lambda\rangle_\ell^\mu)$ for $|(\xi,\lambda)|\to\infty$. This proves the lemma. ☐

2.12 Corollary. *For $\mu\in\mathbb{R}$ the function $\langle\cdot,\cdot\rangle_\ell^\mu$ belongs to $S^{\mu;\ell}(\mathbb{R}^n\times\mathbb{R}^q)$.*

Proof. Induction over the length $|\beta|$ of $\beta\in\mathbb{N}_0^{n+q}$ shows

$$\partial^\beta_{(\xi,\lambda)}\langle\xi,\lambda\rangle_\ell^\mu = \sum_{k=0}^{|\beta|} p_{\beta,k}(\xi,\lambda)\,\langle\xi,\lambda\rangle_\ell^{\mu-2kl}$$

with suitable polynomials $p_{\beta,k}$ that are (anisotropic) homogeneous of degree $2kl - |\beta|_\ell$. From Lemma 2.11 we obtain the assertion. ☐

2.13 Definition. Let E and $\tilde E$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde\kappa_\varrho\}$, respectively. For $\mu\in\mathbb{R}$ define

$$S^{\mu;\ell}_{cl}(\mathbb{R}^n\times\mathbb{R}^q;E,\tilde E) := \Big\{a\in S^{\mu;\ell}(\mathbb{R}^n\times\mathbb{R}^q;E,\tilde E);\ a\sim\sum_{k=0}^\infty \chi a_{(\mu-k)}\Big\}$$

where $\chi\in C^\infty(\mathbb{R}^n\times\mathbb{R}^q)$ is a 0-excision function, and $a_{(\mu-k)}\in C^\infty((\mathbb{R}^n\times\mathbb{R}^q)\setminus\{0\},\mathcal{L}(E,\tilde E))$ are (anisotropic) homogeneous functions of degree $\mu-k$, the so called *homogeneous components* of a.

2.14 Remark. By Lemma 2.11 the space $S^{\mu;\ell}_{cl}(\mathbb{R}^n\times\mathbb{R}^q;E,\tilde E)$ is well-defined.

The homogeneous components of $a\in S^{\mu;\ell}_{cl}(\mathbb{R}^n\times\mathbb{R}^q;E,\tilde E)$ are uniquely determined by a. They can iteratively be recovered from the relation

$$\frac{1}{\varrho^{\mu-k}}\tilde\kappa_\varrho^{-1}\Big(a(\varrho\xi,\varrho^\ell\lambda) - \sum_{j=0}^{k-1} a_{(\mu-j)}(\varrho\xi,\varrho^\ell\lambda)\Big)\kappa_\varrho \underset{\varrho\to\infty}{\longrightarrow} a_{(\mu-k)}(\xi,\lambda) \tag{2.vii}$$

with convergence in $\mathcal{L}(E,\tilde{E})$, which holds locally uniformly for $0 \neq (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$.

Note that $S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E})$ is a Fréchet space with respect to the projective topology of the mappings

$$S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E}) \ni a \longmapsto \begin{cases} a - \sum\limits_{j=0}^{k-1} \chi a_{(\mu-j)} & \in S^{\mu-k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E}) \\ \qquad\qquad a_{(\mu-k)} & \in C^\infty((\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\}, \mathcal{L}(E,\tilde{E})) \end{cases}$$

for $k \in \mathbb{N}_0$.

The spaces of x- (resp. x'-) and (x,x')-dependent classical symbols are defined as

$$S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E}) := C_b^\infty(\mathbb{R}^n, S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E})),$$
$$S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E}) := C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n, S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E})).$$

From the definition of the topology on classical symbols we see, that the analogue of Lemma 2.8 holds within classical symbols where the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ now is given as $\mu_k := \mu - k$. This implies that also the analogue of Theorem 2.9 is valid:

Let $\mu \in \mathbb{R}$ and $a^{\mu-j} \in S_{cl}^{\mu-j;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E})$ for $j \in \mathbb{N}_0$. Then there is a symbol $a \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E})$ such that $a \sim \sum\limits_{j=0}^{\infty} a^{\mu-j}$. In fact, the homogeneous components of a are given by

$$a_{(\mu-k)} = \sum_{j\leq k} a^{\mu-j}_{(\mu-k)}$$

for $k \in \mathbb{N}_0$.

Analogously, we define the spaces of classical symbols when we start from scales of Hilbert spaces $\{E_j\}$ and $\{\tilde{E}_j\}$ instead of single spaces.

Proof. Because of its importance we prove relation (2.vii):

Note first, that for $(\xi,\lambda) \in K \Subset (\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\}$ we have $\chi(\varrho\xi, \varrho^\ell\lambda) \equiv 1$ for sufficiently large ϱ since χ is a 0-excision function. Therefore, we see for ϱ sufficiently large on K:

$$\begin{aligned}
&\Big\| \frac{1}{\varrho^{\mu-k}} \tilde{\kappa}_\varrho^{-1} \Big(a(\varrho\xi,\varrho^\ell\lambda) - \sum_{j=0}^{k-1} a_{(\mu-j)}(\varrho\xi,\varrho^\ell\lambda) \Big) \kappa_\varrho - a_{(\mu-k)}(\xi,\lambda) \Big\| \\
&\qquad = \Big\| \frac{1}{\varrho^{\mu-k}} \tilde{\kappa}_\varrho^{-1} \Big(a(\varrho\xi,\varrho^\ell\lambda) - \sum_{j=0}^{k} \chi(\varrho\xi,\varrho^\ell\lambda) a_{(\mu-j)}(\varrho\xi,\varrho^\ell\lambda) \Big) \kappa_\varrho \Big\| \\
&\qquad \leq \mathrm{Const} \cdot \big\| \tilde{\kappa}_{\varrho^{-1}\langle \varrho\xi,\varrho^\ell\lambda\rangle_\ell} \big\| \cdot \big\| \kappa_{\langle \varrho\xi,\varrho^\ell\lambda\rangle_\ell^{-1}\varrho} \big\| \cdot \frac{1}{\varrho^{\mu-k}} \left\langle \varrho\xi,\varrho^\ell\lambda \right\rangle_\ell^{\mu-k-1} \\
&\qquad \leq \mathrm{Const} \cdot \frac{1}{\varrho^{\mu-k}} \left\langle \varrho\xi,\varrho^\ell\lambda \right\rangle_\ell^{\mu-k-1} \underset{\varrho\to\infty}{\longrightarrow} 0
\end{aligned}$$

uniformly for $(\xi,\lambda) \in K$. □

2.15 Remark. The considerations about general anisotropic symbols carry over to the case where the parameter-space $\mathbb{R}^q$ is replaced by a conical subset $\emptyset \neq \Lambda \subseteq \mathbb{R}^q$ which is the closure of its interior. There only arise notational modifications. In this work, we will mainly make use of symbols and operators with either $\Lambda = \mathbb{R}^q$, or with the (upper) half-plane $\Lambda = \mathbb{H} \subseteq \mathbb{C} \cong \mathbb{R}^2$.

3 Symbols with an analytic parameter

3.1 Remark. Let

$$\mathbb{H} := \{z \in \mathbb{C};\ \mathrm{Im}(z) \geq 0\} \subseteq \mathbb{C} \cong \mathbb{R}^2$$

be the upper half-plane in $\mathbb{C}$. We shall employ anisotropic symbols with parameter-space $\mathbb{H}$, where in addition to the symbol estimates we require the analyticity in the interior of $\mathbb{H}$. Due to the connection to the pseudodifferential theory of parabolic equations such symbols are called Volterra symbols, or symbols with the Volterra property, see also Buchholz and Schulze [6], Krainer [24], Krainer and Schulze [26], Piriou [29], [30].

3.2 Definition. Let E and $\tilde{E}$ be Hilbert spaces endowed with strongly continuous group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. For $\mu \in \mathbb{R}$ we define

$$S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) := S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \cap \mathcal{A}(\mathring{\mathbb{H}}, C^\infty(\mathbb{R}^n, \mathcal{L}(E,\tilde{E}))),$$

which is a closed subspace of $S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. Analogously, we define

$$S_V^{-\infty}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) := \bigcap_{\mu\in\mathbb{R}} S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}),$$

as well as the spaces of x- (resp. x'-) and (x,x')-dependent symbols

$$S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) := C^\infty_b(\mathbb{R}^n, S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})),$$
$$S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) := C^\infty_b(\mathbb{R}^n \times \mathbb{R}^n, S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})).$$

These symbols are called symbols with the *Volterra property* — or simply Volterra symbols — which is indicated by the subscript V.

This notion also applies to the case of scales of Hilbert spaces involved instead of the single spaces only, and we shall employ the same conventions as in the case without the extra analyticity condition, cf. Definition 2.4.

From the definition we obtain, that the properties in Lemma 2.6 apply to symbols with the Volterra property, i. e., the analyticity condition remains preserved.

3.3 Proposition. *a) The restriction of the parameter to the real line induces a continuous embedding $S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \hookrightarrow S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$.*

b) The homogeneous components of a symbol $a \in S^{\mu;\ell}_{V\,cl}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ are analytic in $\overset{\circ}{\mathbb{H}}$.

Proof. The first assertion is immediate. Note that the homogeneous components of the "restricted" symbol in the classical case originate from the restrictions of the homogeneous components to the real line. The second assertion follows by induction from relation (2.vii) in Remark 2.14 together with the Weierstrass approximation theorem. $\square$

3.1 Kernel cut-off and asymptotic expansion

3.4 Definition. Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Notation 2.5.

Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max\limits_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S^{\mu_k;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$. A symbol $a \in S^{\overline{\mu};\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ is called the *asymptotic expansion* of the a_k, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$a - \sum_{j=1}^{k} a_j \in S^{R;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

The symbol a is uniquely determined modulo $S^{-\infty}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.

For short we write $a \underset{V}{\sim} \sum\limits_{j=1}^{\infty} a_j$.

3.5 Remark. Note that the notion of asymptotic expansion for Volterra symbols from Definition 3.4 is more refined than that for general symbols in Definition 2.7. If a Volterra symbol a has an asymptotic expansion in the sense just defined, then it has of course also this asymptotic expansion in the sense of general symbols.

What makes things more complicated is the analyticity condition. In particular, the arguments used to prove the existence of symbols having a prescribed asymptotic expansion from Lemma 2.8 (and Theorem 2.9) cannot be applied to obtain corresponding existence results for Volterra symbols, since they involve excision functions in the parameter which destroy the analyticity. We shall prove in Proposition 3.14 a substitute for Lemma 2.8 using kernel cut-off techniques.

3.6 Definition. Let E and $\tilde{E}$ be Hilbert spaces endowed with strongly continuous group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. Moreover, let $\varphi \in C_b^\infty(\mathbb{R})$. On $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$ define the *kernel cut-off operator* $H(\varphi)$ by means of the following oscillatory integral:

$$\big(H(\varphi)a\big)(\xi,\lambda) := \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} e^{-it\tau}\varphi(t)a(\xi,\lambda-\tau)\,dt\,đ\tau \tag{3.i}$$

for $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$.

Note that the integrand $\varphi(t)a(\xi,\lambda-\tau)$ may be regarded as a smooth function in the variables (ξ,λ) taking values in the $\mathcal{L}(E,\tilde{E})$-valued amplitude functions in (t,τ). This follows from the symbol estimates for a, keeping in mind the inequalities (2.i), (2.ii) and (2.iv) (see also Lemma 2.6). Consequently, $H(\varphi)a$ is well-defined as a function belonging to $C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathcal{L}(E,\tilde{E}))$.

3.7 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Notation 2.5. Then the mapping $H : (\varphi, a) \longmapsto H(\varphi)a$ is bilinear and continuous in the spaces*

$$H : \begin{cases} C_b^\infty(\mathbb{R}) \times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}) \\ C_b^\infty(\mathbb{R}) \times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}). \end{cases}$$

The following asymptotic expansion holds for $H(\varphi)a$ in terms of φ and a:

$$H(\varphi)a \underset{(V)}{\sim} \sum_{k=0}^{\infty} \Big(\frac{(-1)^k}{k!} D_t^k \varphi(0)\Big) \cdot \partial_\lambda^k a \tag{3.ii}$$

where ∂_λ denotes the complex derivative with respect to $\lambda \in \mathbb{H}$ in case of Volterra symbols.

Proof. For the proof we may without loss of generality restrict ourselves to the case of single Hilbert spaces E and $\tilde{E}$. We only have to check the following:

i) $H : C_b^\infty(\mathbb{R}) \times S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E}) \to S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$ is well-defined and continuous.

ii) For $\varphi \in C_b^\infty(\mathbb{R})$ we have $H(\varphi)\big(S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})\big) \subseteq S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.

iii) The validity of the asymptotic expansion (3.ii).

If this is proven, the complete assertions about the continuity of H in the corresponding spaces follows from the closed graph theorem. Recall that since we deal with Fréchet spaces the continuity of H is equivalent to separate continuity.

Note that for $a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ the definition (3.i) of $H(\varphi)a$ does not only make sense for $\lambda \in \mathbb{R}$, but also for $\lambda \in \mathbb{H}$, which consequently provides an extension to the half-plane. Let $\mathrm{Ampl}(\mathbb{R}\times\mathbb{R}, \mathcal{L}(E, \tilde{E}))$ denote the $\mathcal{L}(E, \tilde{E})$-valued amplitude functions in the variables $(t, \tau) \in \mathbb{R}\times\mathbb{R}$. For the mapping $(\varphi, a) \longmapsto \varphi(t)a(\xi, \lambda - \tau)$ is continuous in the spaces

$$\begin{aligned}
C_b^\infty(\mathbb{R}) \times S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E}) &\longrightarrow C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathrm{Ampl}(\mathbb{R}\times\mathbb{R}, \mathcal{L}(E, \tilde{E}))), \\
C_b^\infty(\mathbb{R}) \times S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) &\longrightarrow C^\infty(\mathbb{R}^n \times \mathbb{H}, \mathrm{Ampl}(\mathbb{R}\times\mathbb{R}, \mathcal{L}(E, \tilde{E}))) \\
&\qquad \cap \mathcal{A}(\mathring{\mathbb{H}}, C^\infty(\mathbb{R}^n, \mathrm{Ampl}(\mathbb{R}\times\mathbb{R}, \mathcal{L}(E, \tilde{E})))),
\end{aligned}$$

and for the oscillatory integral acts as a continuous linear mapping between $\mathrm{Ampl}(\mathbb{R}\times\mathbb{R}, \mathcal{L}(E, \tilde{E})) \longrightarrow \mathcal{L}(E, \tilde{E})$, we only have to check for the proof of i) and ii) that for each $\varphi \in C_b^\infty(\mathbb{R})$ and each $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$, respectively $a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$, we have $H(\varphi)a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$, respectively $H(\varphi)a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. Then the closed graph theorem gives the continuity in the corresponding spaces (in case of Volterra symbols the analyticity in the interior of $\mathbb{H}$ is already obtained).

First observe that $\partial_{(\xi,\lambda)}^\beta H(\varphi)a = H(\varphi)\big(\partial_{(\xi,\lambda)}^\beta a\big)$ for $\beta \in \mathbb{N}_0^{n+1}$. Thus for i) and ii) it remains to show

$$\|\tilde{\kappa}^{-1}_{\langle\xi,\lambda\rangle_\ell} H(\varphi)a(\xi,\lambda)\kappa_{\langle\xi,\lambda\rangle_\ell}\|_{\mathcal{L}(E,\tilde{E})} = O(\langle\xi,\lambda\rangle_\ell^\mu) \quad \text{for} \quad |(\xi,\lambda)| \to \infty.$$

For $M_1 \in \mathbb{N}_0$ sufficiently large we regularize the oscillatory integral $\big(H(\varphi)a\big)(\xi,\lambda)$ as

$$\iint e^{-it\tau}\langle t\rangle^{-2}\Big(\big(1-\partial_t^2\big)^{M_1}\varphi(t)\Big)\underbrace{\big(1-\partial_\tau^2\big)\Big[\langle\tau\rangle^{-2M_1}a(\xi,\lambda-\tau)\Big]}_{(*)}\,dt\,đ\tau.$$

$(*)$ is a linear combination of the terms $\{\big(\partial_\tau^j\langle\tau\rangle^{-2M_1}\big)\big(\partial_\lambda^k a\big)(\xi,\lambda-\tau);\ 0 \le j,k \le 2\}$. Let M and $\tilde{M}$ be the constants from the norm estimates for the group-actions in (2.iv). In view of (2.i) we have with a suitable constant $C > 0$ for all ξ, λ, τ:

$\|\tilde\kappa_{\langle\xi,\lambda-\tau\rangle_\ell\langle\xi,\lambda\rangle_\ell^{-1}}\|_{\mathcal{L}(\tilde E)} \le C\langle\tau\rangle^{\tilde M}$ and $\|\kappa_{\langle\xi,\lambda-\tau\rangle_\ell^{-1}\langle\xi,\lambda\rangle_\ell}\|_{\mathcal{L}(E)} \le C\langle\tau\rangle^{M}$. Consequently, we obtain for each of the terms $\big(\partial_\tau^j\langle\tau\rangle^{-2M_1}\big)\big(\partial_\lambda^k a\big)(\xi,\lambda-\tau)$ $(0\le j,k\le 2)$ the following estimate in the norm:

$$\begin{aligned}\|\tilde\kappa^{-1}_{\langle\xi,\lambda\rangle_\ell}\big(\partial_\tau^j\langle\tau\rangle^{-2M_1}\big)\big(\partial_\lambda^k a\big)(\xi,\lambda-\tau)\kappa_{\langle\xi,\lambda\rangle_\ell}\| &\le |\partial_\tau^j\langle\tau\rangle^{-2M_1}|\cdot\|\tilde\kappa_{\langle\xi,\lambda-\tau\rangle_\ell\langle\xi,\lambda\rangle_\ell^{-1}}\|\cdot\\ &\quad\cdot\|\tilde\kappa^{-1}_{\langle\xi,\lambda-\tau\rangle_\ell}\big(\partial_\lambda^k a\big)(\xi,\lambda-\tau)\kappa_{\langle\xi,\lambda-\tau\rangle_\ell}\|\cdot\|\kappa_{\langle\xi,\lambda-\tau\rangle_\ell^{-1}\langle\xi,\lambda\rangle_\ell}\|\\ &\le \text{Const}\,\langle\tau\rangle^{M+\tilde M+|\mu|+2\ell-2M_1}\cdot\langle\xi,\lambda\rangle_\ell^{\mu}.\end{aligned}$$

Hence also $(*)$ satisfies this estimate. If we now choose $M_1 > \frac{M+\tilde M+|\mu|+2\ell+2}{2}$ we get the desired assertion.

It remains to show the asymptotic expansion (3.ii). Carrying out a Taylor expansion in $t=0$ we obtain for each $N\in\mathbb{N}$

$$\varphi(t) = \sum_{k=0}^{N-1}\frac{1}{k!}\partial_t^k\varphi(0)t^k + \frac{t^N}{(N-1)!}\int_0^1(1-\theta)^{N-1}\big(\partial_t^N\varphi\big)(t\theta)\,d\theta.$$

The function $\psi_{(N)}:\mathbb{R}\ni t\mapsto\int_0^1(1-\theta)^{N-1}\big(\partial_t^N\varphi\big)(t\theta)\,d\theta$ belongs to $C_b^\infty(\mathbb{R})$. Now we obtain using integration by parts in the oscillatory integrals:

$$\begin{aligned}H(\varphi)a(\xi,\lambda) &= \sum_{k=0}^{N-1}\Big(\frac{1}{k!}\partial_t^k\varphi(0)\Big)\iint e^{-it\tau}t^k a(\xi,\lambda-\tau)\,dt\,đ\tau\\ &\quad+\frac{1}{(N-1)!}\iint e^{-it\tau}t^N\psi_{(N)}(t)a(\xi,\lambda-\tau)\,dt\,đ\tau\\ &= \sum_{k=0}^{N-1}\Big(\frac{(-1)^k}{k!}D_t^k\varphi(0)\Big)\underbrace{\iint e^{-it\tau}\big(\partial_\lambda^k a\big)(\xi,\lambda-\tau)\,dt\,đ\tau}_{=\big(\partial_\lambda^k a\big)(\xi,\lambda)}\\ &\quad+\frac{i^N}{(N-1)!}\underbrace{\iint e^{-it\tau}\psi_{(N)}(t)\big(\partial_\lambda^N a\big)(\xi,\lambda-\tau)\,dt\,đ\tau.}_{=H(\psi_{(N)})(\partial_\lambda^N a)(\xi,\lambda)}\end{aligned}$$

From the already proven results about the kernel cut-off operator we now conclude that the asymptotic expansion (3.ii) holds. This finishes the proof of the theorem. □

3.8 Corollary. *Let $\varphi\in C_0^\infty(\mathbb{R})$ with $\varphi\equiv 1$ near $t=0$. Then the operator $I-H(\varphi)$ is coninuous in the spaces*

$$\begin{aligned}&S^{\mu;\ell}(\mathbb{R}^n\times\mathbb{R};\mathcal{E},\tilde{\mathcal{E}})\to S^{-\infty}(\mathbb{R}^n\times\mathbb{R};\mathcal{E},\tilde{\mathcal{E}}),\\ &S_V^{\mu;\ell}(\mathbb{R}^n\times\mathbb{H};\mathcal{E},\tilde{\mathcal{E}})\to S_V^{-\infty}(\mathbb{R}^n\times\mathbb{H};\mathcal{E},\tilde{\mathcal{E}}).\end{aligned}$$

Proof. Since $\big(I - H(\varphi)\big)a = H(1-\varphi)a$ we obtain the assertion from Theorem 3.7, for $1-\varphi$ vanishes to infinite order at $t=0$. $\square$

3.9 Notation. Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Notation 2.5.

Moreover, let $z = \lambda + i\tau \in \mathbb{C}$ be the splitting of $z \in \mathbb{C}$ in real and imaginary part. For $\mu \in \mathbb{R}$ define the Fréchet spaces

$$\begin{aligned} S^{\mu;\ell}_{iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) &:= \mathcal{A}(\mathbb{C}, S^{\mu}(\mathbb{R}^n; \mathcal{E}, \tilde{\mathcal{E}})) \cap C^\infty(\mathbb{R}_\tau, S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}_\lambda; \mathcal{E}, \tilde{\mathcal{E}})), \\ S^{\mu;\ell}_{V,iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) &:= S^{\mu;\ell}_{iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) \cap S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \end{aligned}$$

with the induced topologies.

These holomorphic symbol spaces play a substantial role in the calculus of Mellin pseudodifferential operators, or, more generally, in the analysis of (degenerate) parabolic partial differential equations and boundary value problems with pseudodifferential methods, see, e. g., Krainer [24], Krainer and Schulze [26]. In this work, they are just used to describe the target spaces of the kernel cut-off operator when restricted to $C_0^\infty(\mathbb{R})$, see Theorem 3.10 below.

3.10 Theorem. *The kernel cut-off operator H restricts to continuous bilinear mappings*

$$H : \begin{cases} C_0^\infty(\mathbb{R}) \times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) \\ C_0^\infty(\mathbb{R}) \times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{V,iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}). \end{cases}$$

Given $\varphi \in C_0^\infty(\mathbb{R})$ and $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}})$, the following asymptotic expansion holds for $\big(H(\varphi)a\big)(\cdot + i\tau) \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}})$ in terms of φ and a for every $\tau \in \mathbb{R}$:

$$\big(H(\varphi)a\big)(\cdot + i\tau) \sim \sum_{k=0}^{\infty} \Big(\frac{(-1)^k}{k!} D_t^k \big(e^{t\tau}\varphi(t)\big)|_{t=0} \Big) \cdot \partial_\lambda^k a.$$

Proof. Without loss of generality we may restrict to single Hilbert spaces E and $\tilde{E}$. According to Theorem 3.7 and the closed graph theorem we only have to check for the first claim that the image of H restricted to the corresponding spaces is indeed as asserted. More precisely, it suffices to show that $H(\varphi)a \in S^{\mu;\ell}_{iO(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$ for $\varphi \in C_0^\infty(\mathbb{R})$ and $a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$.

For $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ we may rewrite $H(\varphi)a(\xi,\lambda)$ as

$$H(\varphi)a(\xi,\lambda) = H(e^{-it\lambda}\varphi(t))a(\xi,0).$$

Observe that pointwise multiplication of functions acts bilinear and continuous in $C^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$. The function $\mathbb{C} \ni z \mapsto \{\mathbb{R} \ni t \mapsto e^{-itz}\}$ belongs to $\mathcal{A}(\mathbb{C}, C^\infty(\mathbb{R}))$. Consequently, for $\varphi \in C_0^\infty(\mathbb{R})$ the function $\mathbb{C} \ni z \mapsto \{\mathbb{R} \ni t \mapsto e^{-itz}\varphi(t)\}$ belongs to $\mathcal{A}(\mathbb{C}, C_b^\infty(\mathbb{R}))$. From Theorem 3.7 we now obtain, that the function $\mathbb{C} \ni z \mapsto H(e^{-itz}\varphi(t))a$ belongs to $\mathcal{A}(\mathbb{C}, S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E}))$. This implies that $\mathbb{C} \ni z \mapsto H(e^{-itz}\varphi(t))a(\cdot, 0)$ provides an analytic function with values in $S^\mu(\mathbb{R}^n; E, \tilde{E})$ which coincides with $H(\varphi)a$ on the real line (via the obvious identifications). Writing $z = \lambda + i\tau$ we see $H(e^{-itz}\varphi(t))a(\xi, 0) = H(e^{t\tau}\varphi(t))a(\xi, \lambda)$ which depends smoothly on $\tau \in \mathbb{R}$ with values in $S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$. Summing up we now obtain the first assertion of the theorem. The claim about the asymptotic expansion follows from (3.ii) in view of $\big(H(\varphi)a\big)(\cdot + i\tau) = H(e^{t\tau}\varphi(t))a$ for $\tau \in \mathbb{R}$. □

3.11 Remark. For $\varphi \in C_0^\infty(\mathbb{R})$ and $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}})$ with $\mu \in \mathbb{R}$ sufficiently negative we can rewrite the kernel cut-off operator $H(\varphi)a$ as $H(\varphi)a = \mathcal{F}_{t\to\lambda}\varphi\mathcal{F}^{-1}_{\lambda\to t}a$ with the Fourier transform $\mathcal{F}$. The latter identity in fact motivates the name "kernel cut-off operator". To explain this assume $n = 0$ and $\ell = 1$. Then the associated pseudodifferential operator to a may be viewed as convolution operator with convolution kernel given by the conormal distribution $k(a) := \mathcal{F}^{-1}_{\lambda\to t}a$. Consequently, on the level of kernels, kernel cut-off with a function $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \equiv 1$ near $t = 0$, corresponds to localizing the kernel close to the origin, i. e., close to the singular support of $k(a)$. Kernel cut-off operators in that sense were introduced by Schulze in order to deal with meromorphic Mellin symbols and Mellin operator conventions in pseudodifferential calculi adapted to degenerate operators, which arise naturally in the investigation of non-smooth geometries such as manifolds with conical singularities, edges, corners etc. (see [37], [38], [39]).

3.12 Lemma. *Let $\beta \in \mathbb{N}_0$ and $\varphi \in C_0^\infty(\mathbb{R})$. Moreover, let E and $\tilde{E}$ be Hilbert spaces and $a \in \mathcal{A}(\overset{\circ}{\mathbb{H}}, \mathcal{L}(E, \tilde{E})) \cap C^\infty(\mathbb{H}, \mathcal{L}(E, \tilde{E}))$ such that $\sup_{\lambda\in\mathbb{H}} \|\lambda^{\beta+4}a(\lambda)\| < \infty$. Let $\varphi_c \in C_0^\infty(\mathbb{R})$ be defined as $\varphi_c(t) := \varphi(ct)$ for $c \in [1, \infty)$. Then we have for $H(D_x^\beta\varphi_c)a = (\mathcal{F}_{t\to\lambda}(D_x^\beta\varphi_c)\mathcal{F}^{-1}_{\lambda\to t})a$:*

$$\sup_{\lambda\in\mathbb{H}} \big\| \big(H(D_x^\beta\varphi_c)a\big)(\lambda)\big\| \leq k(\beta, \varphi) \cdot \frac{1}{c} \cdot \Big\{ \sup_{\lambda\in\mathbb{H}} \|\lambda^{\beta+2}a(\lambda)\| + \sup_{\lambda\in\mathbb{H}} \|\lambda^{\beta+3}a(\lambda)\| \Big\}$$

for $c \in [1, \infty)$ with a constant $k(\beta, \varphi) > 0$ depending neither on E and $\tilde{E}$ nor on a, but only on β and φ.

Proof. First we shall prove the following auxiliary estimates:

$$\Big\{ \int_{\mathbb{R}} \|\tau^{\beta+2}a(\tau)\|^2 \, d\tau \Big\}^{\frac{1}{2}} \leq \sqrt{\pi} \cdot \Big\{ \sup_{\lambda\in\mathbb{H}} \|\lambda^{\beta+2}a(\lambda)\| + \sup_{\lambda\in\mathbb{H}} \|\lambda^{\beta+3}a(\lambda)\| \Big\} \tag{1}$$

$$\Big\{ \int_{\mathbb{R}} |t|\langle t\rangle^2 \Big| (D_x^\beta\varphi_c)(t) \frac{t^{\beta+1}}{(\beta+1)!} \Big|^2 \, dt \Big\}^{\frac{1}{2}} \leq C(\beta, \varphi) \cdot \frac{1}{c} \tag{2}$$

with a constant $C(\beta,\varphi)\geq 0$ depending only on β and φ.

Let us prove (1):

$$\begin{aligned}\int\limits_{\mathbb{R}} \|\tau^{\beta+2}a(\tau)\|^2\,d\tau &= \int\limits_{\mathbb{R}} \frac{1}{\langle\tau\rangle^2}(1+|\tau|^2)\|\tau^{\beta+2}a(\tau)\|^2\,d\tau \\ &= \int\limits_{\mathbb{R}} \frac{1}{\langle\tau\rangle^2}\cdot\left[\|\tau^{\beta+2}a(\tau)\|^2+\|\tau^{\beta+3}a(\tau)\|^2\right]d\tau \\ &\leq \pi\cdot\Big\{\Big(\sup_{\lambda\in\mathbb{H}}\|\lambda^{\beta+2}a(\lambda)\|\Big)^2+\Big(\sup_{\lambda\in\mathbb{H}}\|\lambda^{\beta+3}a(\lambda)\|\Big)^2\Big\} \\ &\leq \pi\cdot\Big\{\sup_{\lambda\in\mathbb{H}}\|\lambda^{\beta+2}a(\lambda)\|+\sup_{\lambda\in\mathbb{H}}\|\lambda^{\beta+3}a(\lambda)\|\Big\}^2.\end{aligned}$$

This shows (1). Now let us prove (2):

$$\begin{aligned}&\int\limits_{\mathbb{R}} |t|\langle t\rangle^2\Big|(D_x^\beta\varphi_c)(t)\frac{t^{\beta+1}}{(\beta+1)!}\Big|^2\,dt \leq \int\limits_{\mathbb{R}}\langle t\rangle^4\Big|(D_x^\beta\varphi_c)(t)t^{\beta+1}\Big|^2\,dt \\ &= \int\limits_{\mathbb{R}}\big(1+2|t|^2+|t|^4\big)\cdot\Big|(D_x^\beta\varphi_c)(t)t^{\beta+1}\Big|^2\,dt \\ &= \int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi_c)(t)t^{\beta+1}\Big|^2\,dt+2\int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi_c)(t)t^{\beta+2}\Big|^2\,dt+\int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi_c)(t)t^{\beta+3}\Big|^2\,dt \\ &\leq 2c^{2\beta}\Big\{\sum_{j=1}^3\int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi)(c\cdot t)t^{\beta+j}\Big|^2\,dt\Big\} = 2c^{2\beta}\Big\{\sum_{j=1}^3\int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi)(c\cdot t)\cdot\frac{(ct)^{\beta+j}}{c^{\beta+j}}\Big|^2\frac{c}{c}\,dt\Big\} \\ &= 2c^{2\beta}\Big\{\sum_{j=1}^3\frac{1}{c^{2(\beta+j)+1}}\int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi)(t)t^{\beta+j}\Big|^2\,dt\Big\} \\ &\leq \underbrace{\Big\{2\cdot\sum_{j=1}^3\int\limits_{\mathbb{R}}\Big|(D_x^\beta\varphi)(t)t^{\beta+j}\Big|^2\,dt\Big\}}_{=:\,C(\beta,\varphi)^2}\cdot\big(\frac{1}{c}\big)^2.\end{aligned}$$

This shows the estimate (2).

By assumption we have $\mathcal{F}^{-1}a\in C^{\beta+2}(\mathbb{R},\mathcal{L}(E,\tilde{E}))$ and $\mathcal{F}^{-1}a\equiv 0$ on $\mathbb{R}_+$ due to Cauchy's theorem. Employing a Taylor expansion in $t=0$ up to order $\beta+1$ we see

$$\begin{aligned}(\mathcal{F}^{-1}a)(t) &= \Big\{\int\limits_0^1(1-\theta)^{\beta+1}\Big[\Big(\frac{\partial}{\partial t}\Big)^{\beta+2}(\mathcal{F}^{-1}a)\Big](\theta t)\,d\theta\Big\}\cdot\frac{t^{\beta+2}}{(\beta+1)!} \\ &= \Big\{i^{\beta+2}\int\limits_0^1(1-\theta)^{\beta+1}\mathcal{F}^{-1}\big(\lambda^{\beta+2}a(\lambda)\big)(\theta t)\,d\theta\Big\}\cdot\frac{t^{\beta+2}}{(\beta+1)!}\end{aligned}$$

for all $t \in \mathbb{R}$. Now let $e \in E$ be arbitrary. Then we may write

$$\sup_{\lambda\in\mathbb{H}}\big\|\big(H(D_x^\beta\varphi_c)a\big)(\lambda)e\big\|_{\tilde E} \le \Big\{\frac{1}{2\pi}\int_{-\infty}^{0}\frac{1}{\langle t\rangle^2}\,dt\Big\}^{\frac12}\cdot\Big\{\int_{-\infty}^{0}\Big\|(D_x^\beta\varphi_c)(t)(\mathcal{F}^{-1}a)(t)e\Big\|_{\tilde E}^2\langle t\rangle^2\,dt\Big\}^{\frac12}$$

$$\le\Big\{\int_{-\infty}^{0}\langle t\rangle^2\Big\|(D_x^\beta\varphi_c)(t)\Big(\frac{t^{\beta+2}}{(\beta+1)!}\Big)\cdot\Big(\int_0^1(1-\theta)^{\beta+1}\mathcal{F}^{-1}\big(\lambda^{\beta+2}a(\lambda)e\big)(\theta t)\,d\theta\Big)\Big\|_{\tilde E}^2\,dt\Big\}^{\frac12}$$

$$\le\Big\{\int_0^1|1-\theta|^{2(\beta+1)}\,d\theta\Big\}^{\frac12}\Big\{\int_{-\infty}^{0}\Big[\langle t\rangle^2\Big|(D_x^\beta\varphi_c)(t)\Big(\frac{t^{\beta+2}}{(\beta+1)!}\Big)\Big|^2\Big]\cdot$$

$$\cdot\Big[\int_0^1\Big\|\mathcal{F}^{-1}\big(\lambda^{\beta+2}a(\lambda)e\big)(\theta t)\Big\|_{\tilde E}^2\,d\theta\Big]\,dt\Big\}^{\frac12}=\Big(\frac{1}{\sqrt{2\beta+3}}\Big)\cdot$$

$$\cdot\Big\{\int_{-\infty}^{0}\Big[|t|\langle t\rangle^2\Big|(D_x^\beta\varphi_c)(t)\Big(\frac{t^{\beta+1}}{(\beta+1)!}\Big)\Big|^2\Big]\Big[\int_0^1\Big\|\mathcal{F}^{-1}\big(\lambda^{\beta+2}a(\lambda)e\big)(\theta t)\Big\|_{\tilde E}^2|t|\,d\theta\Big]\,dt\Big\}^{\frac12}$$

$$=\Big(\frac{1}{\sqrt{2\beta+3}}\Big)\Big\{\int_{-\infty}^{0}\Big[|t|\langle t\rangle^2\Big|(D_x^\beta\varphi_c)(t)\Big(\frac{t^{\beta+1}}{(\beta+1)!}\Big)\Big|^2\Big]\Big[\int_t^0\Big\|\mathcal{F}^{-1}\big(\lambda^{\beta+2}a(\lambda)e\big)(\theta)\Big\|_{\tilde E}^2\,d\theta\Big]\,dt\Big\}^{\frac12}$$

$$\le\Big\{\int_{-\infty}^{0}|t|\langle t\rangle^2\Big|(D_x^\beta\varphi_c)(t)\Big(\frac{t^{\beta+1}}{(\beta+1)!}\Big)\Big|^2\,dt\Big\}^{\frac12}\Big\{\int_{\mathbb{R}}\Big\|\mathcal{F}^{-1}\big(\lambda^{\beta+2}a(\lambda)e\big)(\theta)\Big\|_{\tilde E}^2\,d\theta\Big\}^{\frac12}$$

$$\underset{\text{Plancherel}}{=}\Big\{\int_{-\infty}^{0}|t|\langle t\rangle^2\Big|(D_x^\beta\varphi_c)(t)\Big(\frac{t^{\beta+1}}{(\beta+1)!}\Big)\Big|^2\,dt\Big\}^{\frac12}\Big\{\int_{\mathbb{R}}\Big\|\tau^{\beta+2}a(\tau)e\Big\|_{\tilde E}^2\,d\tau\Big\}^{\frac12}$$

$$\underset{(1),\,(2)}{\le}\underbrace{\Big(\sqrt{\pi}\cdot C(\beta,\varphi)\Big)}_{=:\,k(\beta,\varphi)}\cdot\frac{1}{c}\cdot\Big\{\sup_{\lambda\in\mathbb{H}}\|\lambda^{\beta+2}a(\lambda)\|_{\mathcal{L}(E,\tilde E)}+\sup_{\lambda\in\mathbb{H}}\|\lambda^{\beta+3}a(\lambda)\|_{\mathcal{L}(E,\tilde E)}\Big\}\cdot\|e\|_E.$$

This finishes the proof of the lemma. □

3.13 Lemma. *Let $N, M_1, M_2 \in \mathbb{N}_0$ and $\varphi \in C_0^\infty(\mathbb{R})$. Moreover, let E and $\tilde E$ be Hilbert spaces and $a \in \mathcal{A}(\overset{\circ}{\mathbb{H}}, C^\infty(\mathbb{R}^n,\mathcal{L}(E,\tilde E)))\cap C^\infty(\mathbb{H}, C^\infty(\mathbb{R}^n,\mathcal{L}(E,\tilde E)))$ such that*

$$\sup_{\substack{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}\\|\alpha|+|\beta|\le N}}\langle\lambda\rangle^{M_1+4}\langle\xi\rangle^{M_2}\|\partial_\lambda^\alpha\partial_\xi^\beta a(\xi,\lambda)\|<\infty.$$

Let $\varphi_c \in C_0^\infty(\mathbb{R})$ be defined as $\varphi_c(t) := \varphi(ct)$ for $c \in [1,\infty)$. Then we have for

$H(\varphi_c)a = (\mathcal{F}_{t\to\lambda}\varphi_c\mathcal{F}^{-1}_{\lambda\to t})a$:

$$\begin{aligned}\sup_{\substack{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}\\|\alpha|+|\beta|\le N}} &\langle\lambda\rangle^{M_1}\langle\xi\rangle^{M_2}\big\|\partial_\lambda^\alpha\partial_\xi^\beta\big(H(\varphi_c)a\big)(\xi,\lambda)\big\| \\ &\le \tilde{k}(M_1,\varphi)\cdot\frac{1}{c}\cdot\sup_{\substack{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}\\|\alpha|+|\beta|\le N}}\langle\lambda\rangle^{M_1+4}\langle\xi\rangle^{M_2}\|\partial_\lambda^\alpha\partial_\xi^\beta a(\xi,\lambda)\|\end{aligned}$$

for $c\in[1,\infty)$ with a constant $\tilde{k}(M_1,\varphi)>0$ depending neither on E and $\tilde{E}$ nor on a, but only on M_1 and φ.

Proof. First observe the following simple relationships for the kernel cut-off operator $H(\varphi_c)$:

$$\partial_\lambda^\alpha\partial_\xi^\beta\big(H(\varphi_c)a\big)(\xi,\lambda) = H(\varphi_c)\big(\partial_\lambda^\alpha\partial_\xi^\beta a\big)(\xi,\lambda) \tag{1}$$

$$\lambda^M\big(H(\varphi_c)a\big)(\xi,\lambda) = \sum_{j=0}^{M}\binom{M}{j}H(D_x^j\varphi_c)\big(\lambda^{M-j}a\big)(\xi,\lambda) \tag{2}$$

Employing Lemma 3.12 we thus may write for every $0\le M\le M_1$ and $|\alpha|+|\beta|\le N$:

$$\begin{aligned}&\sup_{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}\langle\xi\rangle^{M_2}\big\|\lambda^M\partial_\lambda^\alpha\partial_\xi^\beta\big(H(\varphi_c)a\big)(\xi,\lambda)\big\| \\ &= \sup_{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}\Big\|\lambda^M\Big(H(\varphi_c)\Big[\langle\xi\rangle^{M_2}\cdot\partial_\lambda^\alpha\partial_\xi^\beta a\Big]\Big)(\xi,\lambda)\Big\| \\ &= \sup_{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}\Big\|\sum_{j=0}^{M}\binom{M}{j}\Big(H(D_x^j\varphi_c)\Big[\lambda^{M-j}\langle\xi\rangle^{M_2}\cdot\partial_\lambda^\alpha\partial_\xi^\beta a\Big]\Big)(\xi,\lambda)\Big\| \\ &\le \sum_{j=0}^{M}\binom{M}{j}k(j,\varphi)\cdot\frac{1}{c}\cdot\Big\{\sup_{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}\Big\|\langle\xi\rangle^{M_2}\lambda^{M-j+2}\partial_\lambda^\alpha\partial_\xi^\beta a(\xi,\lambda)\Big\| \\ &\qquad\qquad + \sup_{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}\Big\|\langle\xi\rangle^{M_2}\lambda^{M-j+3}\partial_\lambda^\alpha\partial_\xi^\beta a(\xi,\lambda)\Big\|\Big\} \\ &\le \Big(2^{M_1+1}\cdot\max_{j=0}^{M_1}k(j,\varphi)\Big)\cdot\frac{1}{c}\cdot\sup_{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}\langle\lambda\rangle^{M_1+4}\langle\xi\rangle^{M_2}\|\partial_\lambda^\alpha\partial_\xi^\beta a(\xi,\lambda)\|.\end{aligned}$$

We have $\langle\lambda\rangle^{M_1}\le\big(\sqrt{2}\big)^{M_1}\cdot\big(1+|\lambda|\big)^{M_1} = \big(\sqrt{2}\big)^{M_1}\sum_{j=0}^{M_1}\binom{M_1}{j}|\lambda|^j$. Consequently, the assertion is fulfilled with the constant $\tilde{k}(M_1,\varphi) := \Big(2^{\frac{5M_1+2}{2}}\cdot\max_{j=0}^{M_1}k(j,\varphi)\Big)$. □

3.14 Proposition. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Notation 2.5. Let $(\mu_k)\subseteq\mathbb{R}$ such that $\mu_k\ge\mu_{k+1}\underset{k\to\infty}{\longrightarrow}-\infty$. Furthermore, for each $k\in\mathbb{N}$ let $(A_{k_j})_{j\in\mathbb{N}}\subseteq$*

$S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ be a countable system of bounded sets. Let $\varphi \in C_0^\infty(\mathbb{R})$, and for $c \in [1,\infty)$ let $\varphi_c \in C_0^\infty(\mathbb{R})$ be defined as in Lemma 3.12 and Lemma 3.13. Then there is a sequence $(c_i) \subseteq [1,\infty)$ with $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$ such that for each $k \in \mathbb{N}$

$$\sum_{i=k}^{\infty} \sup_{a \in A_{i_j}} p\big(H(\varphi_{d_i})a\big) < \infty \tag{3.iii}$$

for all continuous seminorms p on $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ and every $j \in \mathbb{N}$, and for all sequences $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_i$.

Proof. First consider the case of single Hilbert spaces E and $\tilde{E}$. Without loss of generality we may assume $(\mu_k) \subseteq \mathbb{R}_-$ and that $A_{k_j} \subseteq A_{k_{j+1}}$ holds for the bounded sets (otherwise we pass to unions). Define a system of seminorms $q_1 \leq q_2 \leq \dots$ via

$$q_\nu(a) := \sup_{\substack{\ell|\alpha|+|\beta|\leq\nu \\ (\xi,\lambda)\in\mathbb{R}^n\times\mathbb{H}}} \langle\xi,\lambda\rangle_\ell^{-\mu_\nu+\ell|\alpha|+|\beta|} \, \|\tilde{\kappa}^{-1}_{\langle\xi,\lambda\rangle_\ell} \partial_\lambda^\alpha \partial_\xi^\beta a(\xi,\lambda) \kappa_{\langle\xi,\lambda\rangle_\ell}\|_{\mathcal{L}(E,\tilde{E})}.$$

Let M and $\tilde{M}$ be the constants from the norm estimates (2.iv) for the group-actions on E and $\tilde{E}$, respectively. Employing Peetre's inequality (2.i) we see

$$c_1 \, \langle\xi,\lambda\rangle_\ell^R \leq \langle\xi\rangle^R \langle\lambda\rangle^{\frac{R}{\ell}} \leq c_2 \, \langle\xi,\lambda\rangle_\ell^{2R}$$

for every $R \geq 0$ and all $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$ with suitable constants $c_1, c_2 > 0$ (depending on R). Thus, given $\nu \in \mathbb{N}$, we see that for all $i \in \mathbb{N}$ such that $\mu_i < -2\ell\big(\mathrm{entier}(-\mu_\nu) + 2(\mathrm{entier}(M) + \mathrm{entier}(\tilde{M})) + \nu + 9\big)$ using Lemma 3.13:

$$q_\nu\big(H(\varphi_c)a\big) \leq \big(\mathrm{Const}\big) \cdot \frac{1}{c} \cdot q_i\big(a\big) \tag{1}$$

for $a \in S_V^{\mu_i;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ and $c \in [1,\infty)$.

By induction we construct for $\nu \in \mathbb{N}$ sequences $(c_{\nu_i})_{i\in\mathbb{N}} \subseteq [1,\infty)$ as follows: Employing (1) we find a sequence $(c_{1_i}) \subseteq [1,\infty)$ with $c_{1_i} < c_{1_{i+1}} \underset{i\to\infty}{\longrightarrow} \infty$, such that for all $i \in \mathbb{N}$ satisfying $\mu_i < -2\ell\big(\mathrm{entier}(-\mu_1) + 2(\mathrm{entier}(M) + \mathrm{entier}(\tilde{M})) + 10\big)$ we have $\sup_{a\in A_{i_1}} q_1\big(H(\varphi_{d_i})a\big) < 2^{-i}$ for all $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_{1_i}$. Assume that for some $\nu \in \mathbb{N}$ we have constructed the sequence (c_{ν_i}). Employing (1) we find a subsequence $(c_{\nu+1_i})$ of (c_{ν_i}) having the property that for all $i \in \mathbb{N}$ satisfying $\mu_i < -2\ell\big(\mathrm{entier}(-\mu_{\nu+1}) + 2(\mathrm{entier}(M) + \mathrm{entier}(\tilde{M})) + \nu + 10\big)$ we have $\sup_{a\in A_{i_{\nu+1}}} q_{\nu+1}\big(H(\varphi_{d_i})a\big) < 2^{-i}$ for all $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_{\nu+1_i}$.

Now define $c_i := c_{i_i}$ for $i \in \mathbb{N}$. Then we have $(c_i) \subseteq [1,\infty)$ with $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$ satisfying $c_i \geq c_{\nu_i}$ for $i \geq \nu$ by construction. Let $j,k \in \mathbb{N}$ be arbitrary and p a continuous seminorm on $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. There exists $\nu_0 \in \mathbb{N}$ such that for almost all $i \in \mathbb{N}$ the restriction of p to $S_V^{\mu_i;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is dominated by q_{ν_0} with a universal constant not depending on i, and we also have $A_{i_j} \subseteq A_{i_{\nu_0}}$ for almost all $i \in \mathbb{N}$. Thus we conclude, using Theorem 3.7, that the series (3.iii) is convergent for every sequence $(d_i) \subseteq \mathbb{R}_+$ satisfying $d_i \geq c_i$ with the given data $j, k \in \mathbb{N}$ and p. This shows the assertion in the case of single Hilbert spaces E and $\tilde{E}$.

Now consider the general case. For every $\nu \in \mathbb{N}$ we may apply the assertion for single Hilbert spaces to the pair E_ν and $\tilde{E}_\nu$. This gives a system of sequences $(c_{\nu_i})_{i\in\mathbb{N}} \subseteq [1,\infty)$ with $(c_{\nu+1_i})$ as a subsequence of (c_{ν_i}) such that the series (3.iii) is convergent for every $k, j \in \mathbb{N}$ and every continuous seminorm p on $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; E_\nu, \tilde{E}_\nu)$, for all sequences $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_{\nu_i}$. Consequently we obtain the desired result if we pass to the diagonal sequence $c_i := c_{i_i}$. $\square$

3.15 Remark. Proposition 3.14 together with the properties of the kernel cut-off operator from Theorem 3.7 and Corollary 3.8 now provides the tool to obtain existence results of Volterra symbols having a prescribed asymptotic expansion in the same spirit as Lemma 2.8 is used to achieve corresponding existence results in the case without the Volterra property (see Theorem 2.9). In order to do this we choose in Proposition 3.14 a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ near $t = 0$, see the proof of Theorem 3.16 below.

3.16 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Notation 2.5. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$. Then there exists $a \in S_V^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \underset{V}{\sim} \sum_{j=1}^{\infty} a_j$. The asymptotic sum a is uniquely determined modulo $S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.*

If the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ is given as $\mu_k = \overline{\mu} - k$ and $a_k \in S_{V\,cl}^{\overline{\mu}-k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$, then also $a \in S_{V\,cl}^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.

Proof. For the proof we may without loss of generality assume that $\mu_k \geq \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. For $k, j \in \mathbb{N}$ let

$$A_{k_j} := \{\partial_x^\alpha a_k(x);\ x \in \mathbb{R}^n,\ |\alpha| \leq j\}.$$

Then $A_{k_j} \subseteq S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ is bounded. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ near $t = 0$. Now we may apply Proposition 3.14. With a suitable sequence $(c_i) \subseteq [1,\infty)$ formula (3.iii) becomes

$$\sum_{i=k}^{\infty} \sup\{p\big(H(\varphi_{c_i})(\partial_x^\alpha a_i(x))\big);\ x \in \mathbb{R}^n,\ |\alpha| \leq j\} < \infty$$

for all continuous seminorms p on $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$, which shows that the series $\sum_{i=k}^{\infty} H(\varphi_{c_i})a_i$ is unconditionally convergent in $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ for every $k \in \mathbb{N}$. Now define

$$a := \sum_{i=1}^{\infty} H(\varphi_{c_i})a_i \in S_V^{\mu_1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

We thus see

$$a - \sum_{i=1}^{k} a_i = \sum_{i=k+1}^{\infty} H(\varphi_{c_i})a_i - \sum_{i=1}^{k}\big(I - H(\varphi_{c_i})\big)a_i$$

where

$$\sum_{i=1}^{k}\big(I - H(\varphi_{c_i})\big)a_i \in S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$$

in view of Corollary 3.8. This yields the desired result, since the uniqueness assertion is clear. That asymptotic expansions can be carried out within classical symbols now follows from the corresponding results in the case without the Volterra property (see Remark 2.14). $\square$

3.2 The translation operator in holomorphic symbols

3.17 Definition. For $z = i\tau \in i\mathbb{R} \subseteq \mathbb{C}$, $\tau \geq 0$, define the *translation operator* $T_{i\tau}$ on $S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ via

$$\big(T_{i\tau}a\big)(\xi, \lambda) := a(\xi, \lambda + i\tau).$$

3.18 Proposition. *For every $\tau \geq 0$ the translation operator $T_{i\tau}$ acts linear and continuous in the spaces*

$$T_{i\tau} : S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

Moreover, $T_{i\tau}a$ has the following asymptotic expansion in terms of τ and a:

$$T_{i\tau}a \underset{V}{\sim} \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \partial_\lambda^k a.$$

In particular, the operator $I - T_{i\tau}$ is continuous in the spaces

$$I - T_{i\tau} : S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S_{V(cl)}^{\mu-\ell;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

Proof. Without loss of generality we may restrict to single Hilbert spaces E and $\tilde{E}$. For $T_{i\tau}$ acts continuously in $\mathcal{A}(\overset{\circ}{\mathbb{H}}, C^\infty(\mathbb{R}^n, \mathcal{L}(E,\tilde{E}))) \cap C^\infty(\mathbb{R}^n \times \mathbb{H}, \mathcal{L}(E,\tilde{E}))$ we only have to check that $T_{i\tau}a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ for $a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$, which is trivially fulfilled in view of (2.i) and (2.iv), as well as the asymptotic expansion of $T_{i\tau}a$ in terms of τ and a. Then the closed graph theorem implies the remaining assertions.

Carrying out a Taylor expansion in $\tau = 0$ implies that for each $N \in \mathbb{N}$ we may write

$$a(\xi, \lambda + i\tau) = \sum_{k=0}^{N-1} \frac{(i\tau)^k}{k!} \cdot \partial_\lambda^k a(\xi,\lambda) + \frac{(i\tau)^N}{(N-1)!} \int_0^1 (1-\theta)^{N-1} \big(\partial_\lambda^N a\big)(\xi, \lambda + i\theta\tau)\, d\theta.$$

For the integrand in the remainder may be regarded as a continuous function in $\theta \in [0,1]$ with values in $S_V^{\mu - N\ell;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ we obtain the desired asymptotic expansion for $T_{i\tau}a$. $\square$

3.19 Notation. Let E and $\tilde{E}$ be Hilbert spaces endowed with group-actions as before. For $\mu \in \mathbb{R}$ let $S^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big)$ denote the closed subspace of $C^\infty\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}, \mathcal{L}(E,\tilde{E})\big)$ consisting of all anisotropic homogeneous functions of degree μ. Moreover, let

$$S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) := S^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) \cap \mathcal{A}(\overset{\circ}{\mathbb{H}}, C^\infty(\mathbb{R}^n, \mathcal{L}(E,\tilde{E}))),$$

which is a closed subspace of $S^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big)$.

3.20 Theorem. *For every $\tau > 0$ the mapping $T_{i\tau} : a(\xi,\lambda) \mapsto a(\xi, \lambda + i\tau)$ is continuous in the spaces*

$$T_{i\tau} : S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) \longrightarrow S_{V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}).$$

Moreover, for every 0-excision function $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{H})$, the following asymptotic expansion holds for $T_{i\tau}a$:

$$T_{i\tau}a \sim \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \chi\big(\partial_\lambda^k a\big).$$

This shows in particular, that for the homogeneous component of order μ we have the identity $\big(T_{i\tau}a\big)_{(\mu)} = a$.

In other words, the "principal symbol sequence" for Volterra symbols is topologically exact and splits:

$$0 \longrightarrow S_{V\,cl}^{\mu-1;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow S_{V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow$$
$$S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) \longrightarrow 0.$$

The operator $T_{i\tau}$ provides a splitting of this sequence. Analogous assertions hold in case of scales of Hilbert spaces involved.

Proof. For $T_{i\tau}$ acts continuously from $S_V^{(\mu;\ell)}\big((\mathbb{R}^n\times\mathbb{H})\setminus\{0\};E,\tilde{E}\big)$ into the space $C^\infty(\mathbb{R}^n\times\mathbb{H},\mathcal{L}(E,\tilde{E}))\cap\mathcal{A}(\overset{\circ}{\mathbb{H}},C^\infty(\mathbb{R}^n,\mathcal{L}(E,\tilde{E})))$ we only have to check that $T_{i\tau}a\in S_{cl}^{\mu;\ell}(\mathbb{R}^n\times\mathbb{H};E,\tilde{E})$ in view of the closed graph theorem, as well as the asymptotic expansion for $T_{i\tau}a$. For the assertion concerning the principal symbol sequence recall that the homogeneous components of a Volterra symbol are analytic in the interior of $\mathbb{H}$ according to Proposition 3.3.

Let $\alpha',\alpha''\in\mathbb{N}_0$, $\alpha:=\alpha'+\alpha''$, and $\beta\in\mathbb{N}_0^n$. For every $N\in\mathbb{N}$ we have for $|\xi,\lambda|_\ell$ sufficiently large using Taylor expansion

$$\begin{aligned}
&\big(\partial_\lambda+\partial_{\overline{\lambda}}\big)^{\alpha'}\big(\partial_\lambda-\partial_{\overline{\lambda}}\big)^{\alpha''}\partial_\xi^\beta\Big[a(\xi,\lambda+i\tau)-\sum_{k=0}^{N-1}\frac{(i\tau)^k}{k!}\cdot\chi(\xi,\lambda)\big(\partial_\lambda^k a\big)(\xi,\lambda)\Big]\\
&=\partial_\lambda^\alpha\partial_\xi^\beta\Big[a(\xi,\lambda+i\tau)-\sum_{k=0}^{N-1}\frac{(i\tau)^k}{k!}\cdot\big(\partial_\lambda^k a\big)(\xi,\lambda)\Big]\\
&=\frac{(i\tau)^N}{(N-1)!}\int_0^1(1-\theta)^{N-1}\big(\partial_\lambda^{N+\alpha}\partial_\xi^\beta a\big)(\xi,\lambda+i\tau\theta)\,d\theta.
\end{aligned}$$

The function $\big(\partial_\lambda^{N+\alpha}\partial_\xi^\beta a\big)$ is anisotropic homogeneous of degree $\mu-\ell(N+\alpha)-|\beta|$. Consequently we are done if we show that for a smooth anisotropic homogeneous function a of degree $\mu\in\mathbb{R}$ which is analytic in the interior of $\mathbb{H}$ we have

$$\|\tilde{\kappa}^{-1}_{\langle\xi,\lambda\rangle_\ell}a(\xi,\lambda+i\tau\theta)\kappa_{\langle\xi,\lambda\rangle_\ell}\|=O(\langle\xi,\lambda\rangle_\ell^\mu)$$

for $|\xi,\lambda|_\ell\to\infty$ uniformly for $\theta\in[0,1]$. Let M and $\tilde{M}$ be the constants in the norm estimates for the group-actions from (2.iv). Then we conclude using (2.i) for $(\xi,\lambda)\neq0$:

$$\begin{aligned}
&\|\tilde{\kappa}^{-1}_{\langle\xi,\lambda\rangle_\ell}a(\xi,\lambda+i\tau\theta)\kappa_{\langle\xi,\lambda\rangle_\ell}\|\le\mathrm{Const}\langle\tau\theta\rangle^{M+\tilde{M}}\cdot\\
&\cdot\Big\|\tilde{\kappa}^{-1}_{\langle\xi,\lambda+i\tau\theta\rangle_\ell}a\Big(\langle\xi,\lambda+i\tau\theta\rangle_\ell\frac{\xi}{\langle\xi,\lambda+i\tau\theta\rangle_\ell},\langle\xi,\lambda+i\tau\theta\rangle_\ell^\ell\frac{\lambda+i\tau\theta}{\langle\xi,\lambda+i\tau\theta\rangle_\ell^\ell}\Big)\kappa_{\langle\xi,\lambda+i\tau\theta\rangle_\ell}\Big\|\\
&=\mathrm{Const}\langle\tau\theta\rangle^{M+\tilde{M}}\,\langle\xi,\lambda+i\tau\theta\rangle_\ell^\mu\,\Big\|a\Big(\frac{\xi}{\langle\xi,\lambda+i\tau\theta\rangle_\ell},\frac{\lambda+i\tau\theta}{\langle\xi,\lambda+i\tau\theta\rangle_\ell^\ell}\Big)\Big\|\\
&\le\mathrm{Const}\langle\tau\theta\rangle^{M+\tilde{M}+|\mu|}\,\langle\xi,\lambda\rangle_\ell^\mu\,\Big\|a\Big(\frac{\xi}{\langle\xi,\lambda+i\tau\theta\rangle_\ell},\frac{\lambda+i\tau\theta}{\langle\xi,\lambda+i\tau\theta\rangle_\ell^\ell}\Big)\Big\|.
\end{aligned}$$

Observe that for $|\xi,\lambda|_\ell\ge1$ and $\theta\in[0,1]$ we have using (2.i) with a suitable constant $c > 0$

$$\begin{aligned} 1 &\geq \Big| \frac{\xi}{\langle \xi, \lambda + i\tau\theta\rangle_\ell}, \frac{\lambda + i\tau\theta}{\langle \xi, \lambda + i\tau\theta\rangle_\ell^\ell} \Big|_\ell = \frac{1}{\langle \xi, \lambda + i\tau\theta\rangle_\ell} \cdot |\xi, \lambda + i\tau\theta|_\ell \\ &\geq c \cdot \frac{1}{\langle \xi, \lambda\rangle_\ell \langle \tau\theta\rangle} \cdot |\xi, \lambda + i\tau\theta|_\ell = c \cdot \frac{1}{\big(1 + |\xi, \lambda|_\ell^{2\ell}\big)^{\frac{1}{2\ell}} \langle \tau\theta\rangle} \cdot |\xi, \lambda + i\tau\theta|_\ell \\ &\geq c \cdot \frac{1}{\big(2 \cdot |\xi, \lambda|_\ell^{2\ell}\big)^{\frac{1}{2\ell}} \cdot \langle \tau\theta\rangle} \cdot |\xi, \lambda|_\ell \geq c \cdot 2^{-\frac{1}{2\ell}} \cdot \langle \tau\rangle^{-1} =: \tilde{c} > 0. \end{aligned}$$

Summing up we thus obtain for $|\xi, \lambda|_\ell \geq 1$ for all $\theta \in [0,1]$

$$\big\| \tilde{\kappa}^{-1}_{\langle \xi, \lambda\rangle_\ell} a(\xi, \lambda + i\tau\theta) \kappa_{\langle \xi, \lambda\rangle_\ell} \big\| \leq \Big(\mathrm{Const} \cdot \langle \tau\rangle^{M + \tilde{M} + |\mu|} \cdot \sup_{\tilde{c} \leq |\tilde{\xi}, \tilde{\lambda}|_\ell \leq 1} \|a(\tilde{\xi}, \tilde{\lambda})\| \Big) \cdot \langle \xi, \lambda\rangle_\ell^\mu .$$

This finishes the proof of the theorem. □

4 The calculus of pseudodifferential operators

4.1 Definition. Let E and $\tilde{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. Let $\mu \in \mathbb{R}$. With a double-symbol $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ we associate a family of *pseudodifferential operators* $\mathrm{op}_x(a)(\lambda) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n, E), \mathcal{S}(\mathbb{R}^n, \tilde{E}))$ for $\lambda \in \mathbb{R}^q$ by means of the following oscillatory integral:

$$\begin{aligned} (\mathrm{op}_x(a)(\lambda)u)(x) :&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i(x-x')\xi} a(x, x', \xi, \lambda) u(x')\, dx'\, đ\xi \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-ix'\xi} a(x, x + x', \xi, \lambda) u(x + x')\, dx'\, đ\xi \end{aligned}$$

where as usually $đ\xi := (2\pi)^{-n} d\xi$ (Kohn–Nirenberg quantization).

Note that in view of Definition 2.3 (2.iv) and (2.i) the integrand is indeed a $\tilde{E}$-valued amplitude function. Regularizing the integral yields the asserted continuity of $\mathrm{op}_x(a)(\lambda)$ in the spaces of rapidly decreasing functions. The space of these operators is denoted by

$$L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) := \{\mathrm{op}_x(a)(\lambda);\ a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})\}.$$

Moreover, the space of *Volterra pseudodifferential operators*, respectively *operators with the Volterra property*, is defined as

$$\begin{aligned} L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) &:= \{\mathrm{op}_x(a)(\lambda);\ a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})\} \\ &\subseteq L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}). \end{aligned}$$

In the case of $E = \tilde{E} = \mathbb{C}$ with trivial group-actions the Hilbert spaces are suppressed from the notation.

4.2 Remark. From (2.i) and (2.iv) we see that for each fixed parameter $\lambda \in \mathbb{R}^q, \mathbb{H}$ the operator $\mathrm{op}_x(a)(\lambda)$ belongs to the space of pseudodifferential operators $L_1^{\mu+M+\tilde{M}}(\mathbb{R}^n; E, \tilde{E})$ with global symbols and trivial group-actions. In the scalar case these operators are discussed, e. g., in Kumano-go [27], see also Cordes [7], Shubin [45]. The theory of pseudodifferential operators with global operator-valued symbols without parameters is worked out to some extent also in Dorschfeldt, Grieme, and Schulze [9], and Seiler [43]. We therefore are able to make use of the theory of these operators to obtain the desired results for the parameter-dependent pseudodifferential (Volterra) calculus.

4.1 Elements of the calculus

4.3 Theorem. *Let $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then there exist unique left- and right-symbols $a_L(x,\xi,\lambda), a_R(x',\xi,\lambda) \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ such that $op_x(a)(\lambda) = op_x(a_L)(\lambda) = op_x(a_R)(\lambda)$ as operators on $\mathcal{S}(\mathbb{R}^n, E)$.*

These symbols are given by means of the following oscillatory integrals:

$$a_L(x,\xi,\lambda) = \iint e^{-iy\eta} a(x, y+x, \xi+\eta, \lambda)\, dy\, đ\eta,$$
$$a_R(x',\xi,\lambda) = \iint e^{-iy\eta} a(x'+y, x', \xi-\eta, \lambda)\, dy\, đ\eta.$$

The mappings $a \mapsto a_L$ and $a \mapsto a_R$ are continuous. Moreover, we have the asymptotic expansions

$$a_L(x,\xi,\lambda) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_{x'}^\alpha a(x,x',\xi,\lambda)|_{x'=x},$$
$$a_R(x',\xi,\lambda) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} (-1)^{|\alpha|} \partial_\xi^\alpha D_x^\alpha a(x,x',\xi,\lambda)|_{x=x'}.$$

If a is classical, so are a_L and a_R, and the mappings $a \mapsto a_L$ and $a \mapsto a_R$ are continuous with respect to the (stronger) topology of classical symbols.

If $a \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$, then the unique left- and right-symbol associated to the operator $op_x(a)(\lambda) \in L_{V(cl)}^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ are Volterra symbols, i. e., $a_L(x,\xi,\lambda), a_R(x',\xi,\lambda) \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$, and the asymptotic *expansions for a_L and a_R in terms of a are valid in the Volterra sense:*

$$a_L(x,\xi,\lambda) \underset{V}{\sim} \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha D_{x'}^\alpha a(x,x',\xi,\lambda)|_{x'=x},$$

$$a_R(x',\xi,\lambda) \underset{V}{\sim} \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}(-1)^{|\alpha|}\partial_\xi^\alpha D_{x}^\alpha a(x,x',\xi,\lambda)|_{x=x'}.$$

Proof. For $\lambda \in \Lambda = \mathbb{R}^q, \mathbb{H}$ fixed, we apply the corresponding result about the existence of unique left- and right-symbols for pseudodifferential operators with global symbols in $\mathbb{R}^n$ (see, e. g., Kumano-go [27]). This gives the asserted oscillatory integral-formulas for a_L and a_R, and at once yields the uniqueness assertion of the theorem. Differentiation under the integral sign and plugging in the group-actions show, that the so obtained functions belong to $S_{(V)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E})$, where we employ the symbol estimates for a and Peetre's inequality (2.i). Moreover, this yields the continuity of the mappings $a \mapsto a_L$ and $a \mapsto a_R$.

The assertions about the asymptotic expansions for a_L and a_R are obtained analogously to the case without parameters. We shortly sketch the proof for the left-symbol a_L:

Employing a Taylor expansion in $\eta = 0$ we may write for each $N \in \mathbb{N}$

$$a(x,y+x,\xi+\eta,\lambda) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,y+x,\xi,\lambda)\eta^\alpha + N\sum_{|\alpha|=N}\eta^\alpha\int_0^1 \frac{(1-\theta)^{N-1}}{\alpha!}\partial_\xi^\alpha a(x,y+x,\xi+\theta\eta,\lambda)\,d\theta.$$

The terms of the Taylor polynomial are amplitude functions in the variables (y,η). Moreover, from (2.i) and (2.ii) we conclude that the integrand in the Taylor remainder can be viewed as a continuous function of $\theta \in [0,1]$ with values in the amplitude functions in (y,η). Integrating by parts in the oscillatory integral formula for a_L and interchanging integrals of the remainder gives

$$a_L(x,\xi,\lambda) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha D_{x'}^\alpha a(x,x',\xi,\lambda)|_{x'=x} + r_{L,N}(x,\xi,\lambda),$$

where

$$r_{L,N}(x,\xi,\lambda) = N\sum_{|\alpha|=N}\int_0^1 \frac{(1-\theta)^{N-1}}{\alpha!}\iint e^{-iy\eta}\big(\partial_\xi^\alpha D_{x'}^\alpha a\big)(x,y+x,\xi+\theta\eta,\lambda)\,dy\,đ\eta\,d\theta.$$

Now we see

$$\frac{1}{\alpha!}\partial_\xi^\alpha D_{x'}^\alpha a(x,x',\xi,\lambda)|_{x'=x} \in S_{(V)}^{\mu-|\alpha|;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E}),$$
$$r_{L,N}(x,\xi,\lambda) \in S_{(V)}^{\mu-N;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E}),$$

which yields the desired asymptotic expansion.

These expansions imply that a_L and a_R are classical symbols if a is a classical symbol. The continuity of $a \mapsto a_L$ and $a \mapsto a_R$ with respect to the topology of classical symbols follows from the closed graph theorem. $\square$

4.4 Remark. From Theorem 4.3 we obtain, that the mapping op_x provides an isomorphism between the space of x-dependent symbols ("left-symbols") and pseudodifferential operators:

$$\left.\begin{matrix} S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \\ S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \end{matrix}\right\} \underset{\cong}{\xrightarrow{\mathrm{op}_x}} \left\{\begin{matrix} L_{(cl)}^{\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) \\ L_{V(cl)}^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}). \end{matrix}\right.$$

Via op_x we carry over the topologies, which turns the spaces of operators into Fréchet spaces.

Moreover, we have the spaces of parameter-dependent operators of order $-\infty$ which is independent of $\ell \in \mathbb{N}$ and the group-actions:

$$\begin{aligned} L^{-\infty}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) &= \bigcap_{\mu\in\mathbb{R}} L^{\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) = \mathcal{S}(\mathbb{R}^q)\widehat{\otimes}_\pi L^{-\infty}(\mathbb{R}^n; E, \tilde{E}) \\ &= \{\mathrm{op}_x(a)(\lambda);\ a \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})\}, \\ L_V^{-\infty}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) &= \bigcap_{\mu\in\mathbb{R}} L_V^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) = \mathcal{F}\big(\mathcal{S}_0(\mathbb{R}_-)\big)\widehat{\otimes}_\pi L^{-\infty}(\mathbb{R}^n; E, \tilde{E}) \\ &= \{\mathrm{op}_x(a)(\lambda);\ a \in S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})\}. \end{aligned}$$

Note that $\mathcal{S}_0(\mathbb{R}_-)$ denotes the space of rapidly decreasing functions on $\mathbb{R}$ supported by $\overline{\mathbb{R}}_-$, and $\mathcal{F}$ is the Fourier transform (Paley–Wiener theorem, see, e. g., Eskin [12]).

From Proposition 3.3 we see that the restriction of the parameter to the real line induces a continuous embedding

$$L_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \hookrightarrow L_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E}).$$

4.5 Definition. Let $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L_{(V)cl}^{\mu;\ell}(\mathbb{R}^n; \Lambda; E, \tilde{E})$, where $a \in S_{(V)cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E})$ $(\Lambda = \mathbb{R}^q, \mathbb{H})$. By Theorem 4.3 the symbol a is uniquely determined by $A(\lambda)$, and so are the homogeneous components of a by (2.vii).

We define $\sigma_\wedge^{\mu;\ell}(A)(x,\xi,\lambda) := a_{(\mu)}(x,\xi,\lambda)$ as the homogeneous component of highest order and call $\sigma_\wedge^{\mu;\ell}(A)$ the parameter-dependent homogeneous principal (edge) symbol of $A(\lambda)$ or simply *principal symbol*. The mapping $A(\lambda) \mapsto \sigma_\wedge^{\mu;\ell}(A)$ is continuous.

In case of $E = \mathbb{C}^{N_-}$ and $\tilde{E} = \mathbb{C}^{N_+}$ with trivial group-actions we write as usual $\sigma_\psi^{\mu;\ell}(A)$ instead of $\sigma_\wedge^{\mu;\ell}(A)$.

4.6 Theorem. *Let E, $\tilde{E}$, and $\hat{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$, $\{\tilde{\kappa}_\varrho\}$, and $\{\hat{\kappa}_\varrho\}$, respectively. Let $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\hat{E})$ and $B(\lambda) = op_x(b)(\lambda) \in L^{\mu';\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ with $a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^q;\tilde{E},\hat{E})$ and $b \in S^{\mu';\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^q;E,\tilde{E})$. Then the composition as operators on $\mathcal{S}(\mathbb{R}^n,E)$ belongs to $L^{\mu+\mu';\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;E,\hat{E})$. More precisely, we have $A(\lambda)B(\lambda) = C(\lambda) = op_x(a\#b)(\lambda)$ with the symbol $a\#b \in S^{\mu+\mu';\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^q;E,\hat{E})$ given by the oscillatory integral formula*

$$a\#b(x,\xi,\lambda) = \iint e^{-iy\eta}a(x,\xi+\eta,\lambda)b(x+y,\xi,\lambda)\,dy\,đ\eta. \tag{4.i}$$

Moreover, the following asymptotic expansion holds for $a\#b$:

$$a\#b \sim \sum_{\alpha\in\mathbb{N}_0^n}\frac{1}{\alpha!}(\partial_\xi^\alpha a)(D_x^\alpha b). \tag{4.ii}$$

The mapping $(a,b)\mapsto a\#b$ is bilinear and continuous. The symbol $a\#b$ is called the Leibniz-product *of a and b.*

If even $a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{H};\tilde{E},\hat{E})$ and $b \in S^{\mu';\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E})$, then the Leibniz-product $a\#b$ belongs to $S^{\mu+\mu';\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{H};E,\hat{E})$, and the asymptotic expansion

$$a\#b \underset{V}{\sim} \sum_{\alpha\in\mathbb{N}_0^n}\frac{1}{\alpha!}(\partial_\xi^\alpha a)(D_x^\alpha b) \tag{4.iii}$$

holds in the sense of Volterra symbols.

From the asymptotic expansions (4.ii) and (4.iii) we see that in the classical case the following relation holds for the principal symbol of the composition:

$$\sigma_\wedge^{\mu+\mu';\ell}(AB) = \sigma_\wedge^{\mu;\ell}(A)\sigma_\wedge^{\mu';\ell}(B). \tag{4.iv}$$

Proof. We associate to the operator $B(\lambda)$ the right-symbol $b_R(x',\xi,\lambda)$ according to Theorem 4.3. Then the composition $A(\lambda)B(\lambda)$ has the double-symbol $c(x,x',\xi,\lambda) = a(x,\xi,\lambda)b_R(x',\xi,\lambda) \in S^{\mu+\mu';\ell}_{(V)(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\Lambda;E,\hat{E})$, where

$\Lambda = \mathbb{R}^q, \mathbb{H}$. Employing again Theorem 4.3, we obtain $a\#b$ as the corresponding left-symbol associated to c. This also implies the continuity of the bilinear mapping $(a, b) \mapsto a\#b$. The oscillatory integral formula (4.i) for the Leibniz-product apriori holds within the symbol classes with trivial group-actions (without parameters) and necessarily restricts to the preceding situations. The asymptotic expansions (4.ii), (4.iii) follow from (4.i) via Taylor expansion analogously to the proof of Theorem 4.3. □

4.7 Proposition. *Let $\Lambda = \mathbb{R}^q, \mathbb{H}$, and let $A(\lambda) \in L^{\mu;\ell}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$ be given by $A(\lambda) = op_x(a)(\lambda)$ with a double-symbol $a(x, x', \xi, \lambda) \in S^{\mu;\ell}_{(V)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E})$, such that $a(x, x', \xi, \lambda) \equiv 0$ for $|x - x'| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then $A(\lambda) \in L^{-\infty}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$.*

Proof. We associate to $A(\lambda)$ the left-symbol a_L according to Theorem 4.3. All ingredients in the asymptotic expansion for a_L in terms of a vanish by assumption, from which we conclude that $a_L \in S^{-\infty}_{(V)}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E})$. □

4.8 Corollary. *Let $A(\lambda) \in L^{\mu;\ell}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$, where $\Lambda = \mathbb{R}^q, \mathbb{H}$. Moreover, let $\varphi, \psi \in C^\infty_b(\mathbb{R}^n)$ such that $dist(supp\varphi, supp\psi) > 0$. Then $\varphi A(\lambda)\psi \in L^{-\infty}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$. The mapping $L^{\mu;\ell}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E}) \ni A(\lambda) \longmapsto \varphi A(\lambda)\psi \in L^{-\infty}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$ is continuous.*

Proof. Let $a(x, \xi, \lambda)$ be the left-symbol for $A(\lambda)$. Then the operator $\varphi A(\lambda)\psi$ is given by the double-symbol $\varphi(x)a(x, \xi, \lambda)\psi(x')$, which vanishes identically for $|x - x'| < \varepsilon$ when $\varepsilon > 0$ is sufficiently small. Consequently, $\varphi A(\lambda)\psi \in L^{-\infty}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$ by Proposition 4.7. The continuity of $L^{\mu;\ell}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E}) \ni A(\lambda) \longmapsto \varphi A(\lambda)\psi \in L^{-\infty}_{(V)}(\mathbb{R}^n; \Lambda; E, \tilde{E})$ follows from Theorem 4.6 and the closed graph theorem. □

4.9 Remark. Proposition 4.7 and Corollary 4.8 yield the result of *pseudolocality* of the calculi of parameter-dependent pseudodifferential operators.

4.10 Theorem. *Let E and $\tilde{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. For every $\mu \in \mathbb{R}$ the principal symbol sequence in Volterra pseudodifferential operators is topologically exact and splits:*

$$\begin{aligned} 0 \longrightarrow & L^{\mu-1;\ell}_{V\,cl}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) \xrightarrow{\imath} L^{\mu;\ell}_{V\,cl}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) \xrightarrow{\sigma^{\mu;\ell}_\wedge} \\ & C^\infty_b\big(\mathbb{R}^n, S^{(\mu;\ell)}_V\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big)\big) \longrightarrow 0. \end{aligned}$$

The translation operator $T_{i\tau}$ for $\tau > 0$ gives rise to a splitting of this sequence.

Proof. This follows from Theorem 3.20. □

4.2 The formal adjoint operator

4.11 Definition. A triple $\{E_0, E, E_1; \kappa\}$ is called a *Hilbert triple*, if the following conditions are fulfilled:

a) There exists a Hausdorff topological vector space X such that E_0, E and E_1 are embedded in X.

b) $\kappa : (\mathbb{R}_+, \cdot) \to \mathcal{L}(X)$ is a representation, which restricts to strongly continuous group-actions on E_0, E and E_1. On E the action is assumed to be unitary.

c) $E_0 \cap E \cap E_1$ is dense in E_0, E and E_1.

d) The inner product on E induces a non-degenerate sesquilinear pairing $\langle \cdot, \cdot \rangle : E_0 \times E_1 \to \mathbb{C}$, that provides antilinear isomorphisms $E_0' \cong E_1$ and $E_1' \cong E_0$.

4.12 Remark. Let $\{E_0, E, E_1; \kappa\}$ and $\{\tilde{E}_0, \tilde{E}, \tilde{E}_1; \tilde{\kappa}\}$ be Hilbert triples.

a) The scalar product on $L^2(\mathbb{R}^n, E)$ induces a non-degenerate sesquilinear pairing

$$\langle \cdot, \cdot \rangle : \mathcal{S}(\mathbb{R}^n, E_0) \times \mathcal{S}(\mathbb{R}^n, E_1) \to \mathbb{C}.$$

b) To each $A \in \mathcal{L}(E_0, \tilde{E}_0)$ there is a unique operator $A^* \in \mathcal{L}(\tilde{E}_1, E_1)$ such that $\langle Ae_0, \tilde{e}_1 \rangle_{\tilde{E}} = \langle e_0, A^* \tilde{e}_1 \rangle_E$ for all $e_0 \in E_0$ and $\tilde{e}_1 \in \tilde{E}_1$. The mapping $A \mapsto A^*$ provides an antilinear isomorphism $\mathcal{L}(E_0, \tilde{E}_0) \to \mathcal{L}(\tilde{E}_1, E_1)$.

c) The mapping * from b) induces an antilinear isomorphism

$$S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E_0, \tilde{E}_0) \to S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}_1, E_1)$$

defined as $a^*(x, x', \xi, \lambda) := (a(x', x, \xi, \lambda))^*$. Note in particular, that left-symbols are mapped to right-symbols and vice versa.

4.13 Theorem. *Let $\{E_0, E, E_1; \kappa\}$ and $\{\tilde{E}_0, \tilde{E}, \tilde{E}_1; \tilde{\kappa}\}$ be Hilbert triples and $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E_0, \tilde{E}_0)$ with $a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E_0, \tilde{E}_0)$. Then the formal adjoint operator belongs to $L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; \tilde{E}_1, E_1)$. More precisely, for $u \in \mathcal{S}(\mathbb{R}^n, E_0)$ and $v \in \mathcal{S}(\mathbb{R}^n, \tilde{E}_1)$ we have $\langle A(\lambda)u, v \rangle_{L^2(\mathbb{R}^n, \tilde{E})} = \langle u, A(\lambda)^{(*)} v \rangle_{L^2(\mathbb{R}^n, E)}$ with $A(\lambda)^{(*)} = op_x(a^{(*)})(\lambda)$, where $a^{(*)} \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}_1, E_1)$ is given by means of the oscillatory integral*

$$a^{(*)}(x, \xi, \lambda) = \iint e^{-iy\eta} a^*(x + y, \xi + \eta, \lambda) \, dy \, đ\eta, \tag{4.v}$$

and the following asymptotic expansion is valid:

$$a^{(*)} \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha a^*. \tag{4.vi}$$

The mapping $a \mapsto a^{(*)}$ is antilinear and continuous. The symbol $a^{(*)}$ is called the adjoint symbol *to a.*

From the asymptotic expansion (4.v) we see that in the classical case the following relation holds for the principal symbol of the formal adjoint operator:

$$\sigma_\wedge^{\mu;\ell}(A^{(*)}) = \sigma_\wedge^{\mu;\ell}(A)^*. \tag{4.vii}$$

Proof. The assertion follows from Theorem 4.3, noting that a^* is the right-symbol for $A(\lambda)^{(*)}$. □

4.3 Sobolev spaces and continuity

4.14 Definition. Let E be a Hilbert space with group-action $\{\kappa_\varrho\}$. For $s \in \mathbb{R}$ define the space $\mathcal{W}^s(\mathbb{R}^n, E)$ to consist of all $u \in \mathcal{S}'(\mathbb{R}^n, E)$ such that $\mathcal{F}u$ is a regular distribution and

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^n,E)} := \Big(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}\|\kappa_{\langle\xi\rangle}^{-1}\mathcal{F}u(\xi)\|_E^2\,d\xi\Big)^{\frac{1}{2}} < \infty.$$

If the group-action is trivial we write $H^s(\mathbb{R}^n, E)$. In case of $E = \mathbb{C}$ and trivial group-action the space is suppressed from the notation.

4.15 Remark. Let E be a Hilbert space with group-action $\{\kappa_\varrho\}$.

a) $\mathcal{W}^s(\mathbb{R}^n, E)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{\mathcal{W}^s(\mathbb{R}^n,E)} := \int_{\mathbb{R}^n} \langle\xi\rangle^{2s}\langle\kappa_{\langle\xi\rangle}^{-1}\mathcal{F}u(\xi), \kappa_{\langle\xi\rangle}^{-1}\mathcal{F}v(\xi)\rangle_E\,d\xi.$$

b) The embedding $\mathcal{S}(\mathbb{R}^n, E) \hookrightarrow \mathcal{W}^s(\mathbb{R}^n, E)$ is continuous and dense.

c) If $E \hookrightarrow \tilde{E}$ and the restriction of the group-action $\{\tilde{\kappa}_\varrho\}$ of $\tilde{E}$ on E equals $\{\kappa_\varrho\}$, then the embedding $\mathcal{W}^s(\mathbb{R}^n, E) \hookrightarrow \mathcal{W}^{s'}(\mathbb{R}^n, \tilde{E})$ is well-defined and continuous for $s \geq s'$.

d) Let M be the constant in the norm-estimate (2.iv) of the group-action from Definition 2.3. Then

$$H^{s+M}(\mathbb{R}^n, E) \hookrightarrow \mathcal{W}^s(\mathbb{R}^n, E) \hookrightarrow H^{s-M}(\mathbb{R}^n, E).$$

For proofs of these assertions see, e. g., Hirschmann [18], and Schulze [37].

4.16 Theorem. *Let* $a \in S^{0;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda; E, \tilde{E})$. *Then* $op_x(a)(\lambda)$ *extends for* $\lambda \in \Lambda = \mathbb{R}^q, \mathbb{H}$ *by continuity to a continuous operator* $op_x(a)(\lambda) \in \mathcal{L}(\mathcal{W}^0(\mathbb{R}^n, E), \mathcal{W}^0(\mathbb{R}^n, \tilde{E}))$. *More precisely, there exists a constant* $c > 0$ *independent of* a*, such that with*

$$\pi(a)(\lambda) := \sup\{\|\tilde{\kappa}_{\langle\xi\rangle}^{-1}(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi,\lambda)\kappa_{\langle\xi\rangle}\|;\ x,\xi \in \mathbb{R}^n, \alpha \leq (1,\dots,1),\\ \beta \leq (\tilde{M}+1,\dots,\tilde{M}+1)\},$$

where $\tilde{M} \in \mathbb{N}_0$ *corresponds to* $\{\tilde{\kappa}_\varrho\}$ *via (2.iv), the following norm-estimate holds:*

$$\|op_x(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^0(\mathbb{R}^n,E),\mathcal{W}^0(\mathbb{R}^n,\tilde{E}))} \leq c\pi(a)(\lambda).$$

Proof. The assertion follows from the boundedness result of pseudodifferential operators with operator-valued symbols in Seiler [44]. ☐

4.17 Theorem. *Let* E *and* $\tilde{E}$ *be Hilbert spaces with group-actions* $\{\kappa_\varrho\}$ *and* $\{\tilde{\kappa}_\varrho\}$*, respectively. Moreover, let* M *and* $\tilde{M}$ *be the constants in the norm-estimates for* $\{\kappa_\varrho\}$ *and* $\{\tilde{\kappa}_\varrho\}$ *from (2.iv). Let* $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ *and* $s, \nu \in \mathbb{R}$ *where* $\nu \geq \mu$. *Then* $op_x(a)(\lambda)$ *extends for* $\lambda \in \mathbb{R}^q$ *by continuity to an operator* $op_x(a)(\lambda) \in \mathcal{L}(\mathcal{W}^s(\mathbb{R}^n, E), \mathcal{W}^{s-\nu}(\mathbb{R}^n, \tilde{E}))$*, and we have the following estimate for the norm:*

$$\|op_x(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^n,E),\mathcal{W}^{s-\nu}(\mathbb{R}^n,\tilde{E}))} \leq \begin{cases} C_{s,\nu}\langle\lambda\rangle^{\frac{\mu+M+\tilde{M}}{\ell}} & \nu \geq 0 \\ C_{s,\nu}\langle\lambda\rangle^{\frac{\mu-\nu+M+\tilde{M}}{\ell}} & \nu \leq 0, \end{cases} \tag{4.viii}$$

where $C_{s,\nu} > 0$ *is a constant depending on* s, ν *and* a*, which may be chosen uniformly for* a *in bounded subsets of* $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. *More precisely, this induces a continuous embedding*

$$L^{\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) \hookrightarrow \begin{cases} S_{\mathbb{1}}^{\frac{\mu+M+\tilde{M}}{\ell}}(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^n, E), \mathcal{W}^{s-\nu}(\mathbb{R}^n, \tilde{E})) & \nu \geq 0 \\ S_{\mathbb{1}}^{\frac{\mu-\nu+M+\tilde{M}}{\ell}}(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^n, E), \mathcal{W}^{s-\nu}(\mathbb{R}^n, \tilde{E})) & \nu \leq 0 \end{cases} \tag{4.ix}$$

into the space of operator-valued symbols with the trivial group-action involved on the Sobolev spaces (which is indicated by the subscript $\mathbb{1}$*).*

Moreover, for Volterra symbols, we find the embedding

$$L_V^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) \hookrightarrow \begin{cases} S_{V\,\mathbb{1}}^{\frac{\mu+M+\tilde{M}}{\ell}}(\mathbb{H}; \mathcal{W}^s(\mathbb{R}^n, E), \mathcal{W}^{s-\nu}(\mathbb{R}^n, \tilde{E})) & \nu \geq 0 \\ S_{V\,\mathbb{1}}^{\frac{\mu-\nu+M+\tilde{M}}{\ell}}(\mathbb{H}; \mathcal{W}^s(\mathbb{R}^n, E), \mathcal{W}^{s-\nu}(\mathbb{R}^n, \tilde{E})) & \nu \leq 0 \end{cases} \tag{4.x}$$

into the space of operator-valued Volterra symbols.

Proof. First consider the case $\mu = \nu = 0$: Then, if A is a bounded subset of $S^{0;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$, we get from (2.iv) and (2.i) that the set $\mathcal{A} := \{\langle\lambda\rangle^{-\frac{M+\tilde{M}}{\ell}} a(\cdot,\cdot,\lambda);\ a \in A,\ \lambda \in \mathbb{R}^q\}$ is bounded in $S^0(\mathbb{R}^n \times \mathbb{R}^n; E, \tilde{E})$, and consequently, also the set $\langle\xi\rangle^s I_{\tilde{E}} \# \mathcal{A} \# \langle\xi\rangle^{-s} I_E$ is bounded in $S^0(\mathbb{R}^n \times \mathbb{R}^n; E, \tilde{E})$. Now we get from Theorem 4.16 that the pseudodifferential operators with symbols in the latter set are bounded in $\mathcal{L}(\mathcal{W}^0(\mathbb{R}^n, E), \mathcal{W}^0(\mathbb{R}^n, \tilde{E}))$, which implies that $\mathrm{op}_x(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{W}^s(\mathbb{R}^n, E), \mathcal{W}^s(\mathbb{R}^n, \tilde{E}))$ is bounded. This gives (4.viii) for $\mu = \nu = 0$.

Now consider the general case: For $\mu' \in \mathbb{R}$ let $\langle D_x, \lambda\rangle_\ell^{\mu'} \in L^{\mu';\ell}(\mathbb{R}^n; \mathbb{R}^q; \tilde{E}, \tilde{E})$ denote the operator with symbol $\langle\xi,\lambda\rangle_\ell^{\mu'} I_{\tilde{E}}$. For $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ we may write

$$\mathrm{op}_x(a)(\lambda) = \langle D_x, \lambda\rangle_\ell^{\mu} \cdot \underbrace{\left(\langle D_x, \lambda\rangle_\ell^{-\mu}\, \mathrm{op}_x(a)(\lambda)\right)}_{\in L^{0;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E})},$$

and thus

$$\begin{aligned}\|\mathrm{op}_x(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^n,E),\mathcal{W}^{s-\nu}(\mathbb{R}^n,\tilde{E}))} &\le \|\,\langle D_x,\lambda\rangle_\ell^{\mu}\,\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^n,\tilde{E}),\mathcal{W}^{s-\nu}(\mathbb{R}^n,\tilde{E}))} \cdot \\ &\cdot \|\,\langle D_x,\lambda\rangle_\ell^{-\mu}\,\mathrm{op}_x(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^n,E),\mathcal{W}^s(\mathbb{R}^n,\tilde{E}))}.\end{aligned}$$

The first part of the proof allows us to reduce the general case to the operator $\langle D_x, \lambda\rangle_\ell^{\mu}$.

With the definition

$$\psi_{\mu,\nu}(\lambda) := \sup_{\xi\in\mathbb{R}^n} \langle\xi,\lambda\rangle_\ell^{\mu} \cdot \langle\xi\rangle^{-\nu} \quad (<\infty)$$

we have $\|\,\langle D_x,\lambda\rangle_\ell\,\|_{\mathcal{L}(\mathcal{W}^s,\mathcal{W}^{s-\nu})} \le \psi_{\mu,\nu}(\lambda)$ (note that $\nu \ge \mu$ by assumption). If $\nu \ge 0$ and $\mu \ge 0$ we obtain $\psi_{\mu,\nu}(\lambda) \le C\langle\lambda\rangle^{\frac{\mu}{\ell}}$ from Peetre's inequality (2.i). If $\nu \ge 0$ and $\mu \le 0$ we see $\langle\xi,\lambda\rangle_\ell^{\mu} \le \langle\lambda\rangle^{\frac{\mu}{\ell}}$ which gives the desired estimate for $\psi_{\mu,\nu}(\lambda)$. For $\nu \le 0$ the estimate $\langle\xi\rangle^{-\nu} \le C\,\langle\xi,\lambda\rangle_\ell^{-\nu}$ holds. By virtue of $\nu \ge \mu$ we conclude $\langle\xi,\lambda\rangle_\ell^{\mu-\nu} \le \langle\lambda\rangle^{\frac{\mu-\nu}{\ell}}$ which implies the assertion (4.viii).

From (4.viii) we now obtain, that the embeddings in (4.ix), (4.x) is well-defined, and moreover that bounded subsets are mapped into bounded subsets. But since we deal with Fréchet spaces (in particular with bornological spaces) this already gives the asserted continuity (see Schaefer [33], II.8). $\square$

4.4 Coordinate invariance

4.18 Definition. Let $U \subseteq \mathbb{R}^n$ be an open set. Then $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n; \Lambda; E, \tilde{E})$, where $\Lambda = \mathbb{R}^q, \mathbb{H}$, is said to be *compactly supported in* U, if for some $\varphi, \psi \in C_0^\infty(U)$ we have $A(\lambda) = \varphi B(\lambda)\psi$ with $B(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n; \Lambda; E, \tilde{E})$.

In other words: $A(\lambda)$ is compactly supported in U if and only if there is a compact set $K \subseteq U \times U$ such that

$$\operatorname{supp} K_{A(\lambda)} \subseteq K \text{ for all } \lambda \in \Lambda \tag{4.xi}$$

where $K_{A(\lambda)} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(E,\tilde{E}))$ denotes the operator-valued Schwartz kernel of the operator $A(\lambda)$.

For each compact set $K \subseteq U \times U$ the space of compactly supported (Volterra) operators $A(\lambda) \in L^{\mu;\ell}_{(V)(cl)}(\mathbb{R}^n;\Lambda;E,\tilde{E})$ satisfying (4.xi) is a closed subspace of $L^{\mu;\ell}_{(V)(cl)}(\mathbb{R}^n;\Lambda;E,\tilde{E})$.

Let $L^{\mu;\ell}_{\mathrm{comp}\,(V)(cl)}(U;\Lambda;E,\tilde{E})$ denote the space of all (classical) parameter-dependent pseudodifferential (Volterra) operators that are compactly supported in U. We endow this space with the inductive limit topology of the subspaces of operators with Schwartz kernels satisfying (4.xi) (taken over all compact sets $K \subseteq U \times U$). Thus it becomes a strict countable inductive limit of Fréchet spaces.

Note that $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L^{\mu;\ell}_{\mathrm{comp}\,(V)}(U;\Lambda;E,\tilde{E})$ acts as a family of continuous operators $A(\lambda) : C_0^\infty(U,E) \to C_0^\infty(U,\tilde{E})$, and its symbol $a(x,\xi,\lambda)$ is uniquely determined by this action.

4.19 Theorem. *Let U, $V \subseteq \mathbb{R}^n$ be open subsets and $\chi : U \to V$ a diffeomorphism. Then the operator pull-back $\chi^* A(\lambda)$ defined as*

$$(\chi^* A(\lambda))u := \chi^*(A(\lambda)(\chi_* u)) \tag{4.xii}$$

for $u \in C_0^\infty(U,E)$ and $A(\lambda) \in L^{\mu;\ell}_{comp\,(V)}(V;\Lambda;E,\tilde{E})$, with the pull-back χ^* and push-forward χ_* for C_0^∞-functions, defines a topological isomorphism $\chi^* : L^{\mu;\ell}_{comp\,(V)(cl)}(V;\Lambda;E,\tilde{E}) \to L^{\mu;\ell}_{comp\,(V)(cl)}(U;\Lambda;E,\tilde{E})$.

Moreover, given $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{comp\,(V)(cl)}(V;\Lambda;E,\tilde{E})$, then $\chi^* A(\lambda) = op_x(b)(\lambda)$ with a symbol $b \in S^{\mu;\ell}_{(V)(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda;E,\tilde{E})$ having the following asymptotic expansion in terms of a and χ:

$$b(x,\xi,\lambda) \underset{(V)}{\sim} \sum_{\alpha \in \mathbb{N}_0^n} (\partial_\xi^\alpha a)(\chi(x),[D\chi(x)^{-1}]^t\xi,\lambda)\varphi_\alpha(x,\xi) \tag{4.xiii}$$

with polynomials $\varphi_\alpha(x,\xi)$ in ξ of degree less or equal to $\frac{|\alpha|}{2}$ and $\varphi_0 \equiv 1$, that are given completely in terms of the diffeomorphism χ.

Note that the symbol a vanishes identically outside a compact set in V which gives this asymptotic expansion a meaning.

In particular, we obtain $b(x,\xi,\lambda) - a(\chi(x),[D\chi(x)^{-1}]^t\xi,\lambda) \in S^{\mu-1;\ell}_{(V)}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda;E,\tilde{E})$. This yields in the classical case to the following relation for the principal symbols:

$$\sigma_\wedge^{\mu;\ell}(\chi^* A)(x,\xi,\lambda) = \sigma_\wedge^{\mu;\ell}(A)(\chi(x),[D\chi(x)^{-1}]^t\xi,\lambda). \tag{4.xiv}$$

Proof. First consider the case of trivial group-actions. Then the proof makes use of the corresponding results for pseudodifferential operators with global symbols (without parameters) in the operator-valued case. As in the proof of Theorem 4.3 we may apply these results for any fixed $\lambda \in \Lambda$. Following the outlines of the proofs of coordinate-invariance, e. g., in Kumano-go [27], or Schulze [10], Shubin [45], we get the assertion of the theorem in this case.

Now consider the general case. Following Lemma 2.6 we then first apply the result in the case of trivial group-actions, since the symbol space with non-trivial group-actions embeds into a symbol space with trivial group-actions and enlarged order. The terms in the asymptotic expansion for the symbol b in (4.xiii) are operator-valued (Volterra) symbols in the spaces with non-trivial group-actions. For these symbol classes are asymptotically complete, i. e., asymptotic expansions can be carried out within these classes, we see that b in fact belongs to the space with non-trivial group-actions and order μ – recall that the spaces of order $-\infty$ are independent of the group-actions. $\square$

4.20 Remark. Theorem 4.19 and the pseudolocality properties from Proposition 4.7 and Corollary 4.8 provide the tools to define the calculus of pseudodifferential operators with anisotropic (analytic) parameter and operator-valued symbols on manifolds.

More precisely, let $\mathbb{M}$ be an n-dimensional manifold with empty boundary, and E and $\tilde{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively, and let either denote $\Lambda = \mathbb{R}^q$ or $\Lambda = \mathbb{H}$ as before.

We define

$$\begin{aligned} L^{-\infty}(\mathbb{M};\Lambda;E,\tilde{E}) &:= \mathcal{S}(\Lambda, L^{-\infty}(\mathbb{M};E,\tilde{E})), \\ L_V^{-\infty}(\mathbb{M};\mathbb{H};E,\tilde{E}) &:= L^{-\infty}(\mathbb{M};\mathbb{H};E,\tilde{E}) \cap \mathcal{A}(\mathring{\mathbb{H}}, L^{-\infty}(\mathbb{M};E,\tilde{E})), \end{aligned}$$

where $L^{-\infty}(\mathbb{M};E,\tilde{E})$ denotes the Fréchet space of all operators $C : C_0^\infty(\mathbb{M},E) \to C^\infty(\mathbb{M},\tilde{E})$ that are locally given as integral operators with smooth $\mathcal{L}(E,\tilde{E})$-valued integral kernels.

A family of operators $A(\lambda) : C_0^\infty(\mathbb{M},E) \to C^\infty(\mathbb{M},\tilde{E})$ for $\lambda \in \Lambda$ belongs to the space $L^{\mu;\ell}_{(V)(cl)}(\mathbb{M};\Lambda;E,\tilde{E})$ if and only if the following is fulfilled:

- For all $\varphi, \psi \in C^\infty(\mathbb{M})$ with disjoint supports we have
$$\varphi A(\lambda)\psi \in L^{-\infty}_{(V)}(\mathbb{M};\Lambda;E,\tilde{E}).$$
- For all $\varphi, \psi \in C_0^\infty(\mathbb{M})$ that are supported in the same coordinate chart $(U,\chi) \subseteq \mathbb{M}$ we have
$$\chi_*\big(\varphi A(\lambda)\psi\big) \in L^{\mu;\ell}_{\mathrm{comp}\,(V)(cl)}(\chi(U);\Lambda;E,\tilde{E}).$$

Notice that some ambiguity in the notation is involved, i. e., if we specialize to the case $\mathbb{M} = \mathbb{R}^n$ we do *not* recover the spaces of operators with global symbol estimates as they are discussed earlier in this section.

For classical operators $A(\lambda) \in L^{\mu;\ell}_{(V)\,cl}(\mathbb{M}; \Lambda; E, \tilde{E})$ the homogeneous principal symbol $\sigma_\wedge^{\mu;\ell}(A)(x,\xi,\lambda)$ is well-defined on $\big(T^*\mathbb{M}\times\Lambda\big) \setminus 0$ according to (4.xiv), and it satisfies the homogeneity relation

$$\sigma_\wedge^{\mu;\ell}(A)(x,\varrho\xi,\varrho^\ell\lambda) = \varrho^\mu \tilde{\kappa}_\varrho \sigma_\wedge^{\mu;\ell}(A)(x,\xi,\lambda)\kappa_\varrho^{-1}$$

for $\varrho > 0$ in the fibres. For classical Volterra operators the homogeneous principal symbol is analytic in the interior of the half-plane $\mathbb{H}$.

From the material in this section one easily deduces that the parameter-dependent calculi on the manifold $\mathbb{M}$ are well-behaved with respect to the algebraic operations, e. g., compositions can be carried out within the class (if one of the factors is properly supported), and an operator $A(\lambda) \in L^{\mu;\ell}(\mathbb{M}; \Lambda; E, \tilde{E})$ acts continuously in the spaces

$$A(\lambda) : \mathcal{W}^s_{\mathrm{comp}}(\mathbb{M}, E) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\mathbb{M}, \tilde{E}).$$

More precisely, if M and $\tilde{M}$ denote the constants in the norm-estimates for $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$ from (2.iv), we have for $s, \nu \in \mathbb{R}$ and $\nu \geq \mu$ the embedding

$$L^{\mu;\ell}_{(V)}(\mathbb{M}; \Lambda; E, \tilde{E}) \hookrightarrow \begin{cases} S_{(V)}^{\frac{\mu+M+\tilde{M}}{\ell}}(\Lambda; \mathcal{W}^s_{\mathrm{comp}}(\mathbb{M}, E), \mathcal{W}^{s-\nu}_{\mathrm{loc}}(\mathbb{M}, \tilde{E})) & \nu \geq 0 \\ S_{(V)}^{\frac{\mu-\nu+M+\tilde{M}}{\ell}}(\Lambda; \mathcal{W}^s_{\mathrm{comp}}(\mathbb{M}, E), \mathcal{W}^{s-\nu}_{\mathrm{loc}}(\mathbb{M}, \tilde{E})) & \nu \leq 0, \end{cases} \tag{4.xv}$$

which follows from Theorem 4.17.

Notice that the spaces of compactly supported and local $\mathcal{W}^s$-distributions are well-defined on the manifold $\mathbb{M}$ in an evident way, and they provide typical examples for spaces that are represented as countable inductive limits ($\mathcal{W}^s_{\mathrm{comp}}$-spaces) and projective limits ($\mathcal{W}^s_{\mathrm{loc}}$-spaces) of Hilbert spaces, in this case endowed with the trivial group-action, such as considered in Definition 2.4 (see also Notation 2.5, and subsequent considerations). Furthermore, the spaces of invariantly defined parameter-dependent pseudodifferential operators serve as typical examples for operator-valued symbols according to (4.xv).

5 Ellipticity and parabolicity

5.1 Ellipticity in the general calculus

5.1 Definition. Let E and $\tilde{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. A symbol $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ is called *parameter-*

dependent elliptic, if there is a symbol $b \in S^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ such that

$$a{\cdot}b - 1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E}),$$
$$b{\cdot}a - 1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$$

for some $\varepsilon > 0$. In particular, the condition of parameter-dependent ellipticity is not affected by perturbations of lower-order terms.

An operator $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E})$ is called *parameter-dependent elliptic*, if a is parameter-dependent elliptic.

5.2 Lemma. *a) Let $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then a is parameter-dependent elliptic if and only if for some $R > 0$ there exists $(a(x,\xi,\lambda))^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in \mathbb{R}^n$, $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ with $|\xi,\lambda|_\ell \geq R$, and*

$$\sup\{\|\kappa^{-1}_{\langle\xi,\lambda\rangle_\ell}(a(x,\xi,\lambda))^{-1}\tilde{\kappa}_{\langle\xi,\lambda\rangle_\ell}\|\,\langle\xi,\lambda\rangle_\ell^{\mu}\,;\ x \in \mathbb{R}^n,\ |\xi,\lambda|_\ell \geq R\} < \infty.$$

If $a \in S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$, then a is parameter-dependent elliptic if and only if the homogeneous principal component $a_{(\mu)}(x,\xi,\lambda) \in \mathcal{L}(E, \tilde{E})$ is invertible for all $x \in \mathbb{R}^n$ and $0 \neq (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$, and

$$\sup\{\|(a_{(\mu)}(x,\xi,\lambda))^{-1}\|;\ x \in \mathbb{R}^n,\ |\xi,\lambda|_\ell = 1\} < \infty.$$

b) Let $a \in S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then a is parameter-dependent elliptic if and only if there exists $b \in S^{-\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ such that $ab - 1 \in S^{-1;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ and $ba - 1 \in S^{-1;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$.

Proof. Note first that in view of Definition 5.1 the conditions in a) are clearly necessary for parameter-dependent ellipticity. To prove the sufficiency let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ such that $\chi \equiv 0$ for $|\xi,\lambda|_\ell \leq R+1$ and $\chi \equiv 1$ for $|\xi,\lambda|_\ell \geq R+2$. For $(x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q$ define

$$b(x,\xi,\lambda) := \begin{cases} \chi(\xi,\lambda)(a(x,\xi,\lambda))^{-1} & \text{in the general case} \\ \chi(\xi,\lambda)(a_{(\mu)}(x,\xi,\lambda))^{-1} & \text{in the classical case.} \end{cases}$$

Thus we see that $b \in S^{-\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$, and moreover $ab - 1 \in S^{-1;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ and $ba - 1 \in S^{-1;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$. This proves a) and b). □

5.3 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E})$. Then the following are equivalent:*

a) $A(\lambda)$ is parameter-dependent elliptic.

b) There exists an operator $P(\lambda) \in L^{-\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; \tilde{E}, E)$, such that $A(\lambda)P(\lambda) - 1 \in L^{-\varepsilon;\ell}(\mathbb{R}^n; \mathbb{R}^q; \tilde{E}, \tilde{E})$ and $P(\lambda)A(\lambda) - 1 \in L^{-\varepsilon;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, E)$ for some $\varepsilon > 0$.

c) *There exists an operator* $P(\lambda) \in L^{-\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$, *such that* $A(\lambda)P(\lambda)-1 \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\tilde{E})$ *and* $P(\lambda)A(\lambda)-1 \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;E,E)$.

If $A(\lambda) \in L^{\mu;\ell}_{cl}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ *is parameter-dependent elliptic, then every* $P(\lambda)$ *satisfying c) belongs to* $L^{-\mu;\ell}_{cl}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$. *Every* $P(\lambda) \in L^{-\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$ *satisfying c) is called a* (parameter-dependent) parametrix *of* $A(\lambda)$.

Proof. Assume that a) holds. Let $A(\lambda) = \mathrm{op}_x(a)(\lambda)$ with $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$. Let $b \in S^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},E)$ satisfying the condition of Definition 5.1. The asymptotic expansion (4.ii) of the Leibniz-product in Theorem 4.6 implies that $b\#a-1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,E)$ and $a\#b-1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},\tilde{E})$ for some $\varepsilon > 0$ which implies b). If $a \in S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$ we choose $b \in S^{-\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},E)$ satisfying condition b) of Lemma 5.2. We then even obtain $b\#a-1 \in S^{-1;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,E)$ and $a\#b-1 \in S^{-1;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},\tilde{E})$.

Now assume that b) is fulfilled. Let $P(\lambda) = \mathrm{op}_x(b)(\lambda)$ and $A(\lambda)P(\lambda) = 1 - \mathrm{op}_x(r)(\lambda)$ with $r \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},\tilde{E})$. From Theorem 2.9 and Theorem 4.6 we see that there is a symbol $c \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},\tilde{E})$ such that

$$c \sim \sum_{j=1}^{\infty} \underbrace{r\#\ldots\#r}_{j\text{-times}}.$$

Now define $P_{(R)}(\lambda) := \mathrm{op}_x(b\#(1+c))(\lambda)$. Then we have $A(\lambda)P_{(R)}(\lambda) - 1 \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\tilde{E})$ as desired. Analogously, we obtain a parametrix $P_{(L)}(\lambda)$ from the left. But both the left- and the right-parametrix differ only by a term in $L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$ which immediately follows from considering the product $P_{(L)}(\lambda)A(\lambda)P_{(R)}(\lambda)$. This implies c).

Note that if we had started with the case $\varepsilon = 1$ and $P(\lambda)$ as well as the remainder being classical, we would have obtained also a classical parametrix which proves the second assertion of the theorem (see also Remark 2.14).

c) implies a) follows at once from Theorem 4.6. □

5.4 Remark. Clearly, the considerations about parameter-dependent ellipticity carry over to the case where the parameter-space $\mathbb{R}^q$ is replaced by a conical subset $\emptyset \neq \Lambda \subseteq \mathbb{R}^q$ which is the closure of its interior. There only arise notational modifications.

5.2 Parabolicity in the calculus with an analytic parameter

5.5 Remark. In this section we study the *parabolicity* of Volterra operators which is defined by requiring the parameter-dependent ellipticity of the symbols. The main point here is that we are in need to construct a parametrix which itself has again the Volterra property. The latter cannot be obtained from Theorem 5.3 (or its proof) because there are arguments with excision functions involved, which destroy the analyticity in the interior of the half-plane (see, in particular, the characterization of parameter-dependent ellipticity from Lemma 5.2). The asymptotic expansion result from Theorem 3.16, which makes use of kernel cut-off techniques, and the translation operator from Definition 3.17 and its analysis in Volterra symbol spaces from Proposition 3.18 and Theorem 3.20 provide the tools to handle these difficulties, and we are in the position to construct Volterra parametrices via symbolic inversion and the formal Neumann series argument.

5.6 Definition. Let E and $\tilde{E}$ be Hilbert spaces with group-actions $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, respectively. A symbol $a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is called *parabolic*, if it is parameter-dependent elliptic as an element in $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.

An operator $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L_V^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ is called *parabolic*, if a is parabolic.

5.7 Proposition. *Let $a \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. Then a is parabolic if and only if there exists an element $b \in S_{V(cl)}^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ such that*

$$\begin{aligned} a{\cdot}b - 1 &\in S_{V(cl)}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, \tilde{E}), \\ b{\cdot}a - 1 &\in S_{V(cl)}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, E). \end{aligned}$$

Proof. We only have to prove the necessity of the condition, for the sufficiency follows immediately from the definition of parabolicity as parameter-dependent ellipticity (see Definition 5.1, or Lemma 5.2). Assume that $a \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is parabolic. Employing Lemma 5.2 we see that for some sufficiently large $R > 0$ there exists $(a(x,\xi,\lambda))^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in \mathbb{R}^n$, $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$ with $|\xi,\lambda|_\ell \geq R$, and

$$\sup\{\|\kappa_{\langle\xi,\lambda\rangle_\ell}^{-1} (a(x,\xi,\lambda))^{-1} \tilde{\kappa}_{\langle\xi,\lambda\rangle_\ell}\| \, \langle\xi,\lambda\rangle_\ell^\mu \,;\; x \in \mathbb{R}^n,\; |\xi,\lambda|_\ell \geq R\} < \infty.$$

Consequently, if we choose $\tau \in \mathbb{R}_+$ sufficiently large, we conclude that for all $x \in \mathbb{R}^n$ and all $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$ there exists $\big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} \in \mathcal{L}(\tilde{E}, E)$ with

$$\sup\{\|\kappa_{\langle\xi,\lambda\rangle_\ell}^{-1} \big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} \tilde{\kappa}_{\langle\xi,\lambda\rangle_\ell}\| \, \langle\xi,\lambda\rangle_\ell^\mu \,;\; x \in \mathbb{R}^n,\; (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}\} < \infty$$

for the symbol $T_{i\tau}a \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ (see Proposition 3.18). Recall that $a - T_{i\tau}a \in S_{V(cl)}^{\mu-\ell;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. Consequently we see, using Theorem 3.20, that the function

$$b(x,\xi,\lambda) := \begin{cases} \big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} & \text{in the general case} \\ T_{i\tau}\big(a_{(\mu)}\big)^{-1}(x,\xi,\lambda) & \text{in the classical case} \end{cases}$$

belongs to $S^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$, and satisfies the asserted condition. $\square$

5.8 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$. Then the following are equivalent:*

a) $A(\lambda)$ is parabolic.

b) There exists an operator $P(\lambda) \in L^{-\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$, such that $A(\lambda)P(\lambda) - 1 \in L^{-\varepsilon;\ell}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, \tilde{E})$ and $P(\lambda)A(\lambda) - 1 \in L^{-\varepsilon;\ell}_V(\mathbb{R}^n; \mathbb{H}; E, E)$ for some $\varepsilon > 0$.

c) There exists an operator $P(\lambda) \in L^{-\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$, such that $A(\lambda)P(\lambda) - 1 \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, \tilde{E})$ and $P(\lambda)A(\lambda) - 1 \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; E, E)$.

If $A(\lambda) \in L^{\mu;\ell}_{V\,cl}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ is parabolic then every $P(\lambda)$ satisfying c) belongs to $L^{-\mu;\ell}_{V\,cl}(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$. Every $P(\lambda) \in L^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$ satisfying c) is called a Volterra parametrix *of $A(\lambda)$.*

Proof. In view of Definition 5.6 of parabolicity for Volterra pseudodifferential operators and Theorem 5.3 it suffices to show that a) implies b), and b) implies c).

Assume that a) holds. Let $A(\lambda) = \mathrm{op}_x(a)(\lambda)$ with $a \in S^{\mu;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. Let $b \in S^{-\mu;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ satisfying the condition in Proposition 5.7. Now the asymptotic expansion (4.iii) in the Volterra sense of the Leibniz-product in Theorem 4.6 implies that $b\#a - 1 \in S^{-1;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, E)$ and $a\#b - 1 \in S^{-1;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, \tilde{E})$ which yields b).

Now assume that b) is fulfilled. Let $P(\lambda) = \mathrm{op}_x(b)(\lambda)$ and $A(\lambda)P(\lambda) = 1 - \mathrm{op}_x(r)(\lambda)$ with $r \in S^{-\varepsilon;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, \tilde{E})$. From Theorem 3.16 and Theorem 4.6 we see that there is a symbol $c \in S^{-\varepsilon;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, \tilde{E})$ such that

$$c \underset{V}{\sim} \sum_{j=1}^{\infty} \underbrace{r\#\ldots\#r}_{j\text{-times}}.$$

Now define $P_{(R)}(\lambda) := \mathrm{op}_x(b\#(1 + c))(\lambda)$. Then we have $A(\lambda)P_{(R)}(\lambda) - 1 \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, \tilde{E})$ as desired. Analogously, we obtain a Volterra parametrix $P_{(L)}(\lambda)$ from the left. But both the left- and the right-parametrix differ only by a term in $L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$ which follows from considering the product $P_{(L)}(\lambda)A(\lambda)P_{(R)}(\lambda)$. This implies c). $\square$

References

[1] M. S. Agranovich and M. I. Vishik. Elliptic problems with a parameter and parabolic problems of general form. *Russ. Math. Surv.*, 19(3):53–157, 1964. Engl. transl. of Usp. Mat. Nauk. 19(3):53–161, 1964.

[2] S. Behm. *Pseudo-differential Operators with Parameters on Manifolds with Edges.* PhD thesis, Universität Potsdam, 1995.

[3] L. Boutet de Monvel. Boundary problems for pseudo-differential operators. *Acta Math.*, 126:11–51, 1971.

[4] Th. Buchholz. *Parabolische Pseudodifferentialoperatoren mit operatorwertigen Symbolen.* PhD thesis, Universität Potsdam, 1996.

[5] Th. Buchholz and B.-W. Schulze. Anisotropic edge pseudo-differential operators with discrete asymptotics. *Math. Nachr.*, 184:73–125, 1997.

[6] Th. Buchholz and B.-W. Schulze. Volterra operators and parabolicity. Anisotropic pseudo-differential operators. Preprint 98/11, Institute for Mathematics, Potsdam, 1998.

[7] H. O. Cordes. *The Technique of Pseudodifferential Operators.* Cambridge University Press, 1995.

[8] Ch. Dorschfeldt. *Algebras of Pseudo-differential Operators near Edge and Corner Singularities*, volume 102 of *Math. Research.* Wiley-VCH, Berlin, 1998.

[9] Ch. Dorschfeldt, U. Grieme, and B.-W. Schulze. Pseudo-differential calculus in the Fourier-Edge approach on non-compact manifolds. In M. Demuth, E. Schrohe, B.-W. Schulze, and J. Sjöstrand, editors, *Spectral Theory, Microlocal Analysis, Singular Manifolds*, volume 14 of *Math. Topics, Advances in Partial Differential Equations*, pages 249–299, Akademie Verlag, Berlin, 1997.

[10] Ju. V. Egorov and B.-W. Schulze. *Pseudo-differential Operators, Singularities, Applications*, volume 93 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel-Boston-Berlin, 1997.

[11] S. D. Ejdel'man and N. V. Zhitarashu. *Parabolic Boundary-Value Problems*, volume 101 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel-Boston-Berlin, 1998.

[12] G. I. Eskin. *Boundary Value Problems for Elliptic Pseudodifferential Equations*, volume 52 of *Math. Monographs.* Amer. Math. Soc., Providence, Rhode Island, 1980. Engl. transl. of Nauka, Moskva, 1973.

[13] J. B. Gil. *Heat Trace Asymptotics for Cone Differential Operators*. PhD thesis, Universität Potsdam, 1998.

[14] J. B. Gil, B.-W. Schulze, and J. Seiler. Cone pseudodifferential operators in the edge symbolic calculus. *Osaka J. Math.*, 37:221–260, 2000.

[15] P. B. Gilkey. *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*. CRC Press, Boca Raton, Ann Arbor, Boston, London, second edition, 1995.

[16] G. Grubb. *Functional Calculus of Pseudodifferential Boundary Problems*, volume 65 of *Progress in Mathematics*. Birkhäuser Verlag, Basel-Boston-Berlin, second edition, 1996.

[17] G. Grubb and R. T. Seeley. Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems. *Invent. Math.*, 121:481–529, 1995.

[18] T. Hirschmann. Functional analysis in cone and edge Sobolev spaces. *Ann. Glob. Anal. Geom.*, 8(2):167–192, 1990.

[19] L. Hörmander. Pseudo-differential operators. *Comm. Pure Appl. Math.*, 18:501–517, 1965.

[20] L. Hörmander. *The Analysis of Linear Partial Differential Operators*, volume I–IV. Springer Verlag, Berlin-Heidelberg-New York, 1983–1985.

[21] J. J. Kohn and L. Nirenberg. An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.*, 18:269–305, 1965.

[22] V. A. Kondrat'ev. Boundary problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.*, 16:227–313, 1967.

[23] T. Krainer. *Parabolic pseudodifferential operators and long-time asymptotics of solutions*. PhD thesis, Universität Potsdam, 2000.

[24] T. Krainer. The calculus of Volterra Mellin pseudodifferential operators with operator-valued symbols. Preprint 2001/35, Institute of Mathematics, Potsdam, 2001. Submitted to Advances in Partial Differential Equations.

[25] T. Krainer and B.-W. Schulze. Long–time asymptotics with geometric singularities in the spatial variables. Preprint 2000/17, Institute of Mathematics, Potsdam, 2000.

[26] T. Krainer and B.-W. Schulze. On the inverse of parabolic systems of partial differential equations of general form in an infinite space–time cylinder. Preprint 2001/14–16, Institute of Mathematics, Potsdam, 2001. To appear in Advances in Partial Differential Equations.

[27] H. Kumano-go. *Pseudo-differential Operators.* The MIT Press, Cambridge (MA), London, 1981.

[28] R. B. Melrose. *The Atiyah-Patodi-Singer Index Theorem.* Research Notes in Mathematics. A. K. Peters, Wellesley, 1993.

[29] A. Piriou. Une classe d'opérateurs pseudo-différentiels du type de Volterra. *Ann. Inst. Fourier, Grenoble*, 20(1):77–94, 1970.

[30] A. Piriou. Problèmes aux limites généraux pour des opérateurs différentiels paraboliques dans un domaine borné. *Ann. Inst. Fourier, Grenoble*, 21(1):59–78, 1971.

[31] S. Rempel and B.-W. Schulze. *Index Theory of Elliptic Boundary Problems.* Akademie Verlag, Berlin, 1982.

[32] S. Rempel and B.-W. Schulze. Parametrices and boundary symbolic calculus for elliptic boundary problems without the transmission property. *Math. Nachr.*, 105:45–149, 1982.

[33] H. H. Schaefer. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics.* Springer Verlag, Berlin-Heidelberg-New York, 1971.

[34] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities I. In *Pseudo-differential Calculus and Mathematical Physics*, Advances in Partial Differential Equations, pages 97–209, Akademie Verlag, Berlin, 1994.

[35] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II. In *Boundary Value Problems, Schrödinger Operators, Deformation Quantization*, Advances in Partial Differential Equations, pages 70–205, Akademie Verlag, Berlin, 1995.

[36] B.-W. Schulze. Pseudo-differential operators on manifolds with edges. In H. Triebel et al., editors, *Symposium 'Partial Differential Equations' 1988, Holzhau (Germany)*, volume 112 of *Teubner Texte zur Mathematik*, pages 259–287, B. G. Teubner, Stuttgart, 1989.

[37] B.-W. Schulze. *Pseudo-differential Operators on Manifolds with Singularities.* North Holland, Amsterdam, 1991.

[38] B.-W. Schulze. *Pseudo-differential Boundary Value Problems, Conical Singularities, and Asymptotics*, volume 4 of *Math. Topics.* Akademie Verlag, Berlin, 1994.

[39] B.-W. Schulze. *Boundary Value Problems and Singular Pseudo-differential Operators.* J. Wiley, Chichester, 1998.

[40] B.-W. Schulze. Operator algebras with symbol hierarchies on manifolds with singularities. In J. Gil, D. Grieser, and M. Lesch, editors, *Approaches to Singular Analysis*, Advances in Partial Differential Equations, Oper. Theory Adv. Appl., pages 167–207, Birkhäuser Verlag, Basel-Boston-Berlin, 2001.

[41] R. T. Seeley. *Complex powers of an elliptic operator*, volume 10 of *Amer. Math. Soc. Proc. Symp. Pure Math.*, pages 288–307. Amer. Math. Soc., Providence, Rhode Island, 1967.

[42] R. T. Seeley. The resolvent of an elliptic boundary problem. *Amer. J. Math.*, 91:889–919, 1969.

[43] J. Seiler. *Pseudodifferential Calculus on Manifolds with Non-Compact Edges.* PhD thesis, Universität Potsdam, 1997.

[44] J. Seiler. Continuity of edge and corner pseudodifferential operators. *Math. Nachr.*, 205:163–182, 1999.

[45] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory.* Springer Verlag, Berlin-Heidelberg-New York, 1987.

[46] V. A. Solonnikov. Boundary value problems for linear parabolic systems of differential equations in general form. *Proc. Steklov Inst. Math.*, 83, 1965. Engl. transl. of Tr. Mat. Inst. Steklova 83, 1965.