Kernel Spikes of Singular Problems *†

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Abstract

Function spaces with asymptotics is a usual tool in the analysis on manifolds with singularities. The asymptotics are singular ingredients of the kernels of pseudodifferential operators in the calculus. They correspond to potentials supported by the singularities of the manifold, and in this form asymptotics can be treated already on smooth configurations. This paper is aimed at describing refined asymtotics in the Dirichlet problem in a ball. The beauty of explicit formulas actually highlights the structure of asymptotic expansions in the calculi on singular varieties.

Contents

Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$ with centre at the origin, and y_0 be a fixed point on the boundary of $\mathbb{B}$. Consider the Dirichlet problem of finding a harmonic function u in $\mathbb{B}$ with prescribed limit values u_0 on $\partial\mathbb{B} = \mathbb{S}$. Were y_0 a singular point of $\mathbb{S}$, one had to take care of interpreting the equality $u = u_0$ at y_0. In the analysis on manifolds with singularities one copes with this problem in an evident and soft way, cf. [Kon67, Sch98]. Namely, one requires the condition $u = u_0$ away from the point y_0 on the sphere. This results in an infinite dimensional null-space of the problem, which consists of all harmonic functions in $\mathbb{B}$ vanishing on $\mathbb{S} \setminus \{y_0\}$. From the structure theorem for distributions with a point support it follows that every such harmonic function is a finite linear combination of the derivatives of the Poisson kernel $\wp(x, y)$ for $\mathbb{B}$. One can thus control the null-space by considering solutions in weighted spaces with weight functions being powers of the distance $|x-y_0|$. The data of the problem are forced to be in weighted functions spaces, too, which makes difficult the use of the Poisson kernel. To apply this latter, the Dirichlet data should be regularised at y_0 to define a distribution on $\partial\mathbb{B}$ coinciding with u_0 outside of y_0. In fact the regularisation consists of subtracting a finite number of terms of the Taylor expansion of $\wp(x, y)$ at $y = y_0$. Hence the cokernel of the problem is also spanned by harmonic functions in $\mathbb{B}$ which vanish on $\mathbb{S} \setminus \{y_0\}$. Summarising we deduce that the variation of the index of the Dirichlet problem in weighted spaces under changing the weight exponent is easily evaluated from the structure theorem for harmonic functions in $\mathbb{B}$ vanishing outside y_0 on $\mathbb{S}$. The purpose of this paper is to derive a canonical representation for harmonic functions in $\mathbb{B}$ vanishing on $\mathbb{S} \setminus \{y_0\}$. It can be thought of as an analogue of the Laurent series for harmonic functions, cf. [Tar91], although the construction of the latter is much easier.

1 Soft expansions

Suppose u is a harmonic function in $\mathbb{B}$ of finite order of growth near the boundary $\mathbb{S}$. Then u has weak limit values $u_0 \in \mathcal{D}'(\mathbb{S})$ on the sphere in the sense that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{S}} u((1-\varepsilon)y)v(y)\, d\sigma = \langle u_0, v \rangle$$

for each $v \in C^\infty(\mathbb{S})$, where $d\sigma$ is the Lebesgue measure on $\mathbb{S}$. Moreover, u can be reconstructed from its weak boundary values by the Poisson formula $u(x) = \langle u_0, \wp(x, \cdot) \rangle$ for $x \in \mathbb{B}$. This formula shows in particular that the behaviour of u close to a boundary point $y \in \mathbb{S}$ in $\mathbb{B}$ is completely determined by the behaviour of u_0 near y on $\mathbb{S}$. Conversely, given any distribution u_0 on $\mathbb{S}$, the function $u(x) = \langle u_0, \wp(x, \cdot) \rangle$, $x \in \mathbb{B}$, is harmonic in $\mathbb{B}$ and its weak limit values on $\mathbb{S}$ coincide with u_0. For these and more general results we refer the reader to [Roi71].

What happens when u_0 fails to be a distribution near a compact set K of zero measure on $\mathbb{S}$? Consider the simplest case where $K = \{y_0\}$ is a fixed point of $\mathbb{S}$ and $u_0(y) = |y - y_0|^z$, with $\Re z \le -(n-1)$. This function u_0 can be extended to a distribution on all of $\mathbb{S}$ in many ways. Pick e.g. a local chart U around y_0 on $\mathbb{S}$ with coordinates $y' = (y_1, \ldots, y_{n-1})$ and a function $\chi \in C^\infty(\mathbb{S})$ supported in U, which is equal to 1 near y_0. If N is a natural number with $\Re z + N > -(n-1)$, then

$$\langle u_{0,N}, v\rangle = \langle u_0, \chi\Big(v(y') - \sum_{|\beta|<N} \frac{\partial_{y'}^\beta v(y_0)}{\beta!}(y'-y_0)^\beta\Big)\rangle + \langle u_0, (1-\chi)v\rangle$$

is well defined for all $v \in C^\infty(\mathbb{S})$. Hence $u_{0,N}$ is a distribution on $\mathbb{S}$ which is equal to u_0 outside of y_0. By the above, the function $u_N(x) = \langle u_{0,N}, \wp(x,\cdot)\rangle$, $x \in \mathbb{B}$, is harmonic in $\mathbb{B}$ and its weak boundary values on $\mathbb{S}$ are $u_{0,N}$. In particular, u_N coincides with u_0 on $\mathbb{S}\setminus\{y_0\}$. Were u another harmonic function in $\mathbb{B}$ of finite order of growth near $\mathbb{S}$, such that $u = u_0$ on $\mathbb{S}\setminus\{y_0\}$, then the weak boundary values of u would have the form $u_{0,N} + \sum_{|\beta|\le B} c_\beta\, \partial_{y'}^\beta \delta_{y_0}(y')$ whence

$$u(x) = u_N(x) + \sum_{|\beta|\le B} c_\beta\, (-\partial_{y'})^\beta \wp(x, y_0)$$

for all $x \in \mathbb{B}$.

For any $\beta \in \mathbb{Z}_+^{n-1}$, the potential $(-\partial_{y'})^\beta \wp(x,y_0)$ is a harmonic functions in $\mathbb{B}$ vanishing on $\mathbb{S}\setminus\{y_0\}$. Using hyperfunctions allows one to get rid of the condition of finite order of growth near $\mathbb{S}$. Recall that any harmonic function u in $\mathbb{B}$ has boundary values u_0 on $\mathbb{S}$ which is a hyperfunction. Moreover, u can be restored from u_0 by the Poisson formula $u(x) = \langle u_0, \wp(x,\cdot)\rangle$ for $x \in \mathbb{B}$. Conversely, given any hyperfunction u_0 on $\mathbb{S}$, the function $u(x) = \langle u_0, \wp(x,\cdot)\rangle$, $x \in \mathbb{B}$, is harmonic in $\mathbb{B}$ and its limit values on the sphere $\mathbb{S}$ coincide with u_0, cf. [SKK73].

Lemma 1.1 *Let u_0 be a hyperfunction on $\mathbb{S}\setminus\{y_0\}$, and $u_{0,R}$ any extension of u_0 to a hyperfunction on all of $\mathbb{S}$. Any harmonic function u in $\mathbb{B}$ equal to u_0 on $\mathbb{S}\setminus\{y_0\}$ has the form*

$$u(x) = \langle u_{0,R}, \wp(x,\cdot)\rangle + \sum_{\beta\in\mathbb{Z}_+^{n-1}} c_\beta\, (-\partial_{y'})^\beta \wp(x,y_0), \quad x \in \mathbb{B}, \tag{1.1}$$

where $(c_\beta)_{\beta\in\mathbb{Z}_+^{n-1}}$ is a sequence of complex numbers with $\lim\limits_{|\beta|\to\infty} \sqrt[|\beta|]{|\beta! c_\beta|} = 0$.

We mention that any hyperfunction u_0 on $\mathbb{S}\setminus\{y_0\}$ can be extended to a hyperfunction on $\mathbb{S}$, for the sheaf of hyperfunctions is flabby.

Proof. By assumption, the limit values of u on $\mathbb{S}$ differ from $u_{0,R}$ by a hyperfunction supported at y_0. From the structure theorem for hyperfunctions with a point support [SKK73] it follows that the boundary values of u on $\mathbb{S}$ have the form

$$u_{0,R} + \sum_{\beta \in \mathbb{Z}_+^{n-1}} c_\beta \, \partial_{y'}^\beta \delta_{y_0}(y'),$$

with $(c_\beta)_{\beta \in \mathbb{Z}_+^{n-1}}$ a sequence of complex numbers satisfying

$$\lim_{|\beta| \to \infty} \sqrt[|\beta|]{|\beta! c_\beta|} = 0. \tag{1.2}$$

Applying this decomposition to the Poisson kernel $\wp(x, \cdot)$ for a fixed $x \in \mathbb{B}$ completes the proof.

□

Note that the condition (1.2) on the coefficients guarantees that the series on the right-hand side of (1.1) converges uniformly in x on compact subsets of $\mathbb{B}$. This follows from the real analyticity of the Poisson kernel $\wp(x, y)$ in $(x, y) \in \mathbb{B} \times U$.

Lemma 1.2 *For each $\beta \in \mathbb{Z}_+^{n-1}$, the function $(-\partial_{y'})^\beta \wp(x, y_0)$ is harmonic in $x \in \mathbb{B}$ and vanishes on $\mathbb{S} \setminus \{y_0\}$.*

Proof. Indeed, since the Poisson kernel $\wp(x, y_0)$ is harmonic in $x \in \mathbb{B}$ we get

$$\begin{aligned} \Delta(D_x) \left((-\partial_{y'})^\beta \wp \right)(x, y_0) &= \left((-\partial_{y'})^\beta \Delta(D_x) \wp \right)(x, y_0) \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{B}$. Moreover, since $\wp(x, y_0)$ vanishes for $x \in \mathbb{S} \setminus \{y_0\}$, the lemma follows.

□

2 Harmonic extension

The following lemma seems to be a classical result, although the authors have not been able to provide any proper reference.

Lemma 2.1 *Every function u harmonic in a ball $\mathbb{B}$ and vanishing on a non-empty open subset O of $\mathbb{S} = \partial \mathbb{B}$ extends harmonically across O to $\mathbb{R}^n \setminus \overline{\mathbb{B}}$.*

Proof. Denote by $\mathbb{K}u$ the Kelvin transform of u, i.e., the harmonic function on $\mathbb{R}^n \setminus \overline{\mathbb{B}}$ given by

$$\mathbb{K}u(x) = \frac{1}{|x|^{n-2}} \, u\left(\frac{x}{|x|^2} \right)$$

for $|x| > 2$. Set

$$U(x) = \begin{cases} u(x) & \text{if} \quad x \in \mathbb{B} \cup O, \\ -\mathbb{K}u(x) & \text{if} \quad x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}. \end{cases}$$

This function is continuous away from the closed set $\mathbb{S} \setminus O$ in $\mathbb{R}^n$ because u vanishes on O. Moreover,

$$\begin{aligned} \frac{\partial}{\partial |x|} \mathbb{K}u(x) &= \sum_{j=1}^{n} \frac{x_j}{|x|} \frac{\partial}{\partial x_j} \left(\frac{1}{|x|^{n-2}} u \left(\frac{x}{|x|^2} \right) \right) \\ &= \frac{2-n}{|x|^{n-1}} u \left(\frac{x}{|x|^2} \right) - \frac{1}{|x|^n} \frac{\partial u}{\partial |x|} \left(\frac{x}{|x|^2} \right) \end{aligned}$$

for any $|x| > 2$. Hence it follows that the derivatives of u and $-\mathbb{K}u$ in the outward normal vector to $\mathbb{S}$ coincide on O. By Theorem 3.2 of [Tar91] we conclude that $U(x)$ is a harmonic function on $\mathbb{B} \cup O \cup (\mathbb{R}^n \setminus \overline{\mathbb{B}})$, which completes the proof.

□

Given any harmonic function u in $\mathbb{B}$ vanishing on $\mathbb{S} \setminus \{y_0\}$, Lemma 2.1 shows that u extends to a harmonic function U on $\mathbb{R}^n \setminus \{y_0\}$, which is given by

$$U(x) = \begin{cases} u(x) & \text{for} \quad x \in \overline{\mathbb{B}} \setminus \{y_0\}, \\ -\mathbb{K}u(x) & \text{for} \quad x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}. \end{cases} \tag{2.1}$$

Recall that the Poisson kernel for the ball $\mathbb{B}$ is

$$\wp(x, y) = \frac{1}{\sigma_n} \frac{1 - |x|^2}{|x - y|^n},$$

where σ_n is the area of the unit sphere in $\mathbb{R}^n$. It is actually defined away from the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Lemma 1.1 prompts that $\wp(x, y_0)$ survives under the transformation (2.1).

Lemma 2.2 *Given any $\beta \in \mathbb{Z}_+^{n-1}$, the equality*

$$-\mathbb{K} \left((-\partial_{y'})^\beta \wp \right) (x, y_0) = (-\partial_{y'})^\beta \wp(x, y_0)$$

holds for all $x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}$.

Proof. Since $\mathbb{K}$ is applied in x one can assume without restriction of generality that $\beta = 0$. In this particular case the verification is fairly straightforward.

□

3 Auxiliary results

Expansion (1.1) is not invariant because it contains the derivatives $(-\partial_{y'})^\beta$ in local coordinates near y_0 on $\mathbb{S}$. To get rid of the non-invariance we can express the local derivatives on $\mathbb{S}$ through the derivatives in the coordinates of the surrounding space $\mathbb{R}^n$. Assume, e.g., that y_0 lies in the upper half-space $x_n > 0$. Then $\mathbb{S}$ near y_0 is the graph of $y_n = \sqrt{1-|y'|^2}$ where $y' = (y_1, \ldots, y_{n-1})$, hence we may take y' as local coordinates on $\mathbb{S}$. For $j = 1, \ldots, n-1$, the full derivative in y_j is $\partial_j - (y_j/y_n)\partial_n$. The tangential space to $\mathbb{S}$ is also spanned by the system of vector fields

$$\frac{\partial}{\partial y_j} - \frac{y_j}{|y|}\frac{\partial}{\partial |y|},$$

$j = 1, \ldots, n$, although these latter are not independent. Substituting this into (1.1) leads to

$$u(x) = \langle u_{0,R}, \wp(x,\cdot)\rangle + \left(\sum_{j=0}^{\infty} \frac{h_j(x-y_0)}{|x-y_0|^{2j}}\right) \wp(x,y_0), \quad x \in \mathbb{B}, \tag{3.1}$$

where $h_j(z)$ are homogeneous polynomials of degree j whose coefficients satisfy growth estimates

$$\lim_{j\to\infty} \sqrt[2j]{\frac{1}{j!}\, |h_j(D)^* h_j(z)|} = 0, \tag{3.2}$$

cf. Corollary 8.11 in [Tar91].

Lemma 3.1 *Let $h(z)$ be a homogeneous polynomial of degree j on $\mathbb{R}^n$. In order that*

$$\frac{h(x-y_0)}{|x-y_0|^{2j}}\wp(x,y_0)$$

be harmonic in $x \in \mathbb{B}$ it is necessary and sufficient that

$$(1-|x|^2)\,\Delta h(x-y_0) - 4\,\langle \nabla h(x-y_0), y_0\rangle \equiv 0. \tag{3.3}$$

Proof. Indeed, a trivial verification shows that

$$\Delta(D_x)\left(\frac{h(x-y_0)}{|x-y_0|^{2j}}\wp(x,y_0)\right) = \frac{1}{\sigma_n}\frac{(1-|x|^2)\Delta h(x-y_0) - 4\langle \nabla h(x-y_0), y_0\rangle}{|x-y_0|^{n+2j}}$$

for all $x \neq y_0$. Since the numerator on the right-hand side is a polynomial in x, it vanishes for all $x \in \mathbb{B}$ if and only if it vanishes identically on $\mathbb{R}^n$. Hence the lemma follows.

□

Replacing $x - y_0$ by z in (3.3) yields

$$(|z|^2 + 2\langle z, y_0\rangle)\Delta h(z) + 4\,\langle \nabla h(z), y_0\rangle = 0$$

for all $z \in \mathbb{R}^n$. As the left-hand side is the sum of two homogeneous polynomials of orders j and $j-1$, respectively, we readily conclude that the identity (3.3) is equivalent to

$$\left\{ \begin{array}{rcl} |z|^2 \, \Delta h(z) & = & 0, \\ \langle \Delta h(z) \, z + 2 \, \nabla h(z), y_0 \rangle & = & 0, \end{array} \right.$$

or

$$\left\{ \begin{array}{rcl} \Delta h(z) & = & 0, \\ \langle \nabla h(z), y_0 \rangle & = & 0 \end{array} \right. \tag{3.4}$$

for all $z \in \mathbb{R}^n$.

Our next goal is to show that the terms of (3.1) are invariant under the transformation (2.1).

Lemma 3.2 *Given any $j \in \mathbb{Z}_+$, the equality*

$$-\mathbb{K} \left(\frac{h_j(x-y_0)}{|x-y_0|^{2j}} \wp(x,y_0) \right) = \frac{h_j(x-y_0)}{|x-y_0|^{2j}} \wp(x,y_0)$$

holds for all $x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}$.

Proof. By Lemma 2.2 it suffices to show that

$$\frac{h_j \Big(\dfrac{x}{|x|^2} - y_0 \Big)}{\Big| \dfrac{x}{|x|^2} - y_0 \Big|^{2j}} = \frac{h_j(x-y_0)}{|x-y_0|^{2j}}$$

for all $x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}$. This equality easily reduces to

$$h_j(x - |x|^2 y_0) = h_j(x-y_0). \tag{3.5}$$

To prove this latter we observe that the polynomial on the left-hand side is harmonic, which is a consequence of (3.4). The polynomial on the right-hand side of (3.5) is harmonic, too. These polynomials coincide on the unit sphere $\mathbb{S}$, hence they are equal on all of $\mathbb{R}^n$, as desired.

□

4 Formulas for coefficients

Suppose u is a harmonic function in $\mathbb{B}$ vanishing on $\mathbb{S} \setminus \{y_0\}$. By Lemma 2.1, u extends to a harmonic function U on $\mathbb{R}^n \setminus \{y_0\}$. The harmonic continuation U

is given by (2.1). Combining (3.1) and Lemma 3.2 we deduce that U represents by the formula

$$U(x) = \left(\sum_{j=0}^{\infty} \frac{h_j(x-y_0)}{|x-y_0|^{2j}} \right) \wp(x,y_0) \tag{4.1}$$

for all $x \in \mathbb{R}^n \setminus \{y_0\}$. Here $h_j(z)$ are homogeneous polynomials of degree j satisfying (3.2) and (3.4). The first of these two conditions implies that series (4.1) converges uniformly in x on compact subsets of $\mathbb{R}^n \setminus \{y_0\}$. Since the summands are harmonic functions the series actually converges in the space $C^\infty(\mathbb{R}^n \setminus \{y_0\})$. Hence we may integrate it termwise over each cycle away from y_0 in $\mathbb{R}^n$.

Let $G_\Delta(g,u)$ stand for the standard Green operator of the Laplace operator in $\mathbb{R}^n$, i.e.,

$$G_\Delta(g,u) = g \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial u}{\partial x_k} dx[k] - u \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial x_k} dx[k]$$

where $dx[k]$ is the wedge product of the differentials $dx_1, \ldots, dx_n$ excepting dx_k.

Lemma 4.1 *Let $\mathcal{D}$ be a bounded domain with smooth boundary, such that $y_0 \in \mathcal{D}$, and F a harmonic function on $\overline{\mathcal{D}}$. For any sufficiently small $\varepsilon > 0$ holds*

$$\begin{aligned} \int_{\partial\mathcal{D}} G_\Delta(F,U) &= \sum_{j=0}^{\infty} \frac{-2(n+2j)}{\sigma_n \varepsilon^j} \int_{\mathbb{B}} F(y_0+\varepsilon z) h_j(z)\, dz \\ &+ \sum_{j=0}^{\infty} \frac{n+2j}{\sigma_n \varepsilon^j} \int_{\mathbb{S}} F(y_0+\varepsilon z) h_j(z)\, d\sigma \\ &+ \sum_{j=0}^{\infty} \frac{2(n+2j)}{\sigma_n \varepsilon^{j+1}} \int_{\mathbb{S}} F(y_0+\varepsilon z) \langle z, y_0 \rangle h_j(z)\, d\sigma. \end{aligned} \tag{4.2}$$

Proof. Choose any $\varepsilon > 0$ with the property that the ball $B(y_0,\varepsilon)$ lies in $\mathcal{D}$, i.e., $\varepsilon \le \operatorname{dist}(y_0, \partial\mathcal{D})$. Since U is harmonic outside of y_0 the Stokes formula gives

$$\begin{aligned} \int_{\partial\mathcal{D}} G_\Delta(F,U) &= \int_{\partial B(y_0,\varepsilon)} G_\Delta(F,U) \\ &= \sum_{j=0}^{\infty} \int_{\partial B(y_0,\varepsilon)} G_\Delta\Big(F(x), \frac{h_j(x-y_0)}{|x-y_0|^{2j}} \wp(x,y_0)\Big). \end{aligned}$$

An easy computation shows that the integrands, if restricted to the sphere $\partial B(y_0,\varepsilon)$, are equal to

$$\begin{aligned} & G_\Delta\Big(F(x), \frac{h_j(x-y_0)}{|x-y_0|^{2j}}\wp(x,y_0)\Big) \\ = & \frac{1}{\sigma_n \varepsilon^{n+2j}} \sum_{k=1}^{n} (-1)^{k-1} \Big(F(x)\frac{\partial}{\partial x_k} g_j(x) - \frac{\partial}{\partial x_k} F(x)\, g_j(x)\Big) dx[k] \\ - & \frac{n+2j}{\sigma_n \varepsilon^{n+2j+1}} F(x) g_j(x)\, d\sigma \end{aligned}$$

where $g_j(x) = (1-|x|^2) h_j(x-y_0)$.

We now apply Stokes' formula for $B(y_0,\varepsilon)$ to the first term on the right-hand side. Since F is harmonic and $h_j(x-y_0)$ satisfies (3.3) the exterior derivative is

$$\frac{1}{\sigma_n \varepsilon^{n+2j}}\, F(x)\, \Delta g_j(x)\, dx = \frac{-2(n+2j)}{\sigma_n \varepsilon^{n+2j}}\, F(x) h_j(x-y_0)\, dx.$$

It follows that

$$\begin{aligned} \int_{\partial \mathcal{D}} G_\Delta(F,U) &= \sum_{j=0}^{\infty} \frac{-2(n+2j)}{\sigma_n \varepsilon^{n+2j}} \int_{B(y_0,\varepsilon)} F(x) h_j(x-y_0)\, dx \\ &- \sum_{j=0}^{\infty} \frac{n+2j}{\sigma_n \varepsilon^{n+2j+1}} \int_{\partial B(y_0,\varepsilon)} F(x)(1-|x|^2) h_j(x-y_0)\, d\sigma. \end{aligned}$$

Changing the variables by $x = y_0 + \varepsilon z$, with $z \in \mathbb{B}$, and taking into account the homogeneity of $h_j(z)$ we arrive at (4.2), as desired.

□

The equations (3.4) make it obvious that

$$\begin{aligned} \Delta\left(\langle z, y_0\rangle h_j(z)\right) &= \langle z, y_0\rangle\, \Delta h_j(z) + 2\, \langle \nabla h_j(z), y_0\rangle \\ &= 0, \end{aligned}$$

i.e., $\langle z, y_0\rangle h_j(z)$ is a homogeneous harmonic polynomial of degree $j+1$ for all j.

Lemma 4.2 *Let $\mathcal{D}$ be a bounded domain with smooth boundary, such that $y_0 \in \mathcal{D}$. Then for every homogeneous harmonic polynomial $H(z)$ of degree k we have*

$$\begin{aligned} & \int_{\partial \mathcal{D}} G_\Delta(H(x-y_0), U(x)) \\ = & \frac{n+2k-2}{\sigma_n} \int_{\mathbb{S}} H(z) h_k(z)\, d\sigma + \frac{2(n+2k-2)}{\sigma_n} \int_{\mathbb{S}} H(z) \langle z, y_0\rangle h_{k-1}(z)\, d\sigma. \end{aligned} \quad (4.3)$$

Proof. Let us apply (4.2) for $F(x) = H(x - y_0)$. Then

$$\begin{aligned}\int_{\mathbb{B}} F(y_0 + \varepsilon z) h_j(z) dz &= \varepsilon^k \int_{\mathbb{B}} H(z) h_j(z)\, dz \\ &= \varepsilon^k \int_0^1 r^{k+j+n-1} dr \int_{\mathbb{S}} H(z) h_j(z)\, d\sigma \\ &= \frac{\varepsilon^k}{k+j+n} \int_{\mathbb{S}} H(z) h_j(z)\, d\sigma\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{S}} F(y_0 + \varepsilon z) h_j(z) d\sigma &= \varepsilon^k \int_{\mathbb{S}} H(z) h_j(z)\, d\sigma, \\ \int_{\mathbb{S}} F(y_0 + \varepsilon z) \langle z, y_0 \rangle h_j(z)\, d\sigma &= \varepsilon^k \int_{\mathbb{S}} H(z) \langle z, y_0 \rangle h_j(z)\, d\sigma\end{aligned}$$

for all $j = 0, 1, \ldots$. As the homogeneous harmonic polynomials of different degrees are orthogonal under integration over the sphere $\mathbb{S}$, the equality (4.3) follows from (4.2).

□

Formula (4.3) uniquely determines the polynomials $h_j(z)$ through the function u in $\mathbb{B}$.

5 Laurent series

Let $(Y_{k,l}(z))$ be a set of homogeneous harmonic polynomials in $\mathbb{R}^n$ whose restrictions to $\mathbb{S}$ form an orthonormal basis in $L^2(\mathbb{S})$ (spherical harmonics). Here k is the degree of $Y_{k,l}$ and, given any $k \in \mathbb{Z}_+$, the index l varies from 1 to $\sigma(n,k)$. For example,

$$\sigma(n,k) = \frac{(n + 2k - 2)(n + k - 3)!}{k!(n-2)!}$$

if $n > 2$, cf. [SW71, Sob74].

For $k = 0$ the only homogeneous harmonic polynomial of degree k and of $L^2(\mathbb{S})$-norm 1 is $Y_{0,1} = 1/\sqrt{\sigma_n}$. Hence

$$h_0 = \int_{\partial\mathcal{D}} G_\Delta\Big(\frac{1}{n-2}, U(x)\Big). \tag{5.1}$$

We now proceed by induction. Given any $k \geq 1$, suppose $h_0, h_1, \ldots, h_{k-1}$ have already been defined. Write

$$h_k(z) = \sum_{l=1}^{\sigma(n,k)} c_{k,l} Y_{k,l}(z),$$

then the coefficients $c_{k,l}$ are uniquely determined from (4.3). More precisely, we get

$$c_{k,l} = \int_{\partial\mathcal{D}} G_{\Delta}\Big(\frac{\sigma_n}{n+2k-2}\overline{Y_{k,l}(x-y_0)}, U(x)\Big) - 2\int_{\mathbb{S}} \overline{Y_{k,l}(z)}\langle z, y_0\rangle h_{k-1}(z)\, d\sigma \tag{5.2}$$

for $l = 1, \ldots, \sigma(n,k)$.

We are now in a position to formulate the main result of this paper. It specifies Laurent series expansions for harmonic functions in $\mathbb{B}$ vanishing on $\partial\mathbb{B} \setminus \{y_0\}$, cf. [Tar91].

Theorem 5.1 *Let u_0 be a hyperfunction on $\mathbb{S}\setminus\{y_0\}$, and $u_{0,R}$ any extension of u_0 to a hyperfunction on all of $\mathbb{S}$. Every harmonic function u in $\mathbb{B}$ equal to u_0 on $\mathbb{S} \setminus \{y_0\}$ has the form*

$$u(x) = \langle u_{0,R}, \wp(x,\cdot)\rangle + \sum_{j=0}^{\infty} \frac{h_j(x-y_0)}{|x-y_0|^{2j}}\, \wp(x,y_0), \quad x \in \mathbb{B}, \tag{5.3}$$

where $(h_j(z))_{j=0,1,\ldots}$ is a sequence of homogeneous harmonic polynomials of degree j in $\mathbb{R}^n$, which are uniquely determined by formulas (5.1) and (5.2) with $u - \langle u_{0,R}, \wp(x,\cdot)\rangle$ in place of u.

Proof. The theorem follows from Lemma 1.1 completed by the recurrence relation (5.2). What is left is to show that the polynomials h_j are actually independent of the particular choice of the system $(Y_{k,l})$. To this end we assume that

$$\sum_{j=0}^{\infty} \frac{h_j(x-y_0)}{|x-y_0|^{2j}}\, \wp(x,y_0) = 0$$

for all $x \in \mathbb{B}$. Since $\wp(x,y_0) \neq 0$ in $\mathbb{B}$ it follows that the series

$$\sum_{j=0}^{\infty} \frac{h_j(x-y_0)}{|x-y_0|^{2j}}$$

vanishes in $\mathbb{B}$. A familiar homogeneity argument now shows that all the h_j are identically zero, as desired.

□

6 Expansion of the Poisson kernel

In this section we sketch a direct approach to formula (5.3), which is based on the expansion of the Poisson kernel $\wp(x,y)$ in spherical harmonics. A similar expansion for the standard fundamental solution of the Laplace operator goes back as far as [Den49].

Lemma 6.1 *In the cone* $\mathrm{C}_{y_0} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \ |x-y_0| > |y-y_0|\}$ *the equality holds*

$$\wp(x,y) = \sum_{k=0}^{\infty} \sum_{l=1}^{\sigma(n,k)} \frac{\sigma_n}{1-\left(\dfrac{|y-y_0|}{|x-y_0|}\right)^2} \overline{Y_{k,l}(y-y_0)} \, \frac{Y_{k,l}(x-y_0)}{|x-y_0|^{2k}} \, \wp(x,y_0)$$

where the series converges absolutely together with all derivatives uniformly on compact subsets of C_{y_0}.

Proof. Setting $x-y_0 = w$ and $y-y_0 = z$ and cancelling the factor $1-|x|^2$ on both sides of the equality, we reduce the expansion to

$$\frac{1}{|w-z|^n} = \sum_{k=0}^{\infty} \sum_{l=1}^{\sigma(n,k)} \frac{\sigma_n}{1-(|z|/|w|)^2} \overline{Y_{k,l}(z)} \, \frac{Y_{k,l}(w)}{|w|^{2k}} \, \frac{1}{|w|^n} \tag{6.1}$$

for all $w, z \in \mathbb{R}^n$ with $|w| > |z|$.

Let $z \in \mathbb{B}$ be fixed. We represent $|w-z|^{-n}$ by the Fourier series in $L^2(\mathbb{S})$. Namely,

$$\frac{1}{|w-z|^n} = \sum_{k=0}^{\infty} \sum_{l=1}^{\sigma(n,k)} c_{k,l}(z) Y_{k,l}(w)$$

where $c_{k,l}(z)$ are the Fourier coefficients of $|w-z|^{-n}$ with respect to the system $(Y_{k,l})$, i.e.,

$$c_{k,l}(z) = \int_{\mathbb{S}} \frac{1}{|w-z|^n} \overline{Y_{k,l}(w)} \, d\sigma.$$

These integrals can be easily evaluated by the Poisson formula, for $\overline{Y_{k,l}(w)}$ are harmonic. Namely, we get

$$\begin{aligned} c_{k,l}(z) &= \frac{\sigma_n}{1-|z|^2} \int_{\mathbb{S}} \wp(z,w) \, \overline{Y_{k,l}(w)} \, d\sigma \\ &= \frac{\sigma_n}{1-|z|^2} \overline{Y_{k,l}(z)} \end{aligned}$$

whence

$$\frac{1}{|w-z|^n} = \sum_{k=0}^{\infty} \sum_{l=1}^{\sigma(n,k)} \frac{\sigma_n}{1-|z|^2} \overline{Y_{k,l}(z)} Y_{k,l}(w),$$

the series converging in the norm of $L^2(\mathbb{S})$ uniformly in z on compact subsets of $\mathbb{B}$.

The harmonic extension with respect to w leads us to the equality

$$\frac{1}{\sigma_n} \, \frac{1}{|w|^{n-2}} \, \frac{\left|\dfrac{w}{|w|^2}\right|^2 - |z|^2}{\left|\dfrac{w}{|w|^2} - z\right|^n} = \sum_{k=0}^{\infty} \sum_{l=1}^{\sigma(n,k)} \overline{Y_{k,l}(z)} Y_{k,l}(w)$$

for all $w \in \mathbb{B}$, where the series converges absolutely and uniformly with respect to w and z on compact subsets of $\mathbb{B}$. Applying to this the Kelvin transformation in w yields

$$\frac{1}{|w-z|^n} = \sum_{k=0}^{\infty} \sum_{l=1}^{\sigma(n,k)} \frac{\sigma_n}{1-(|z|/|w|)^2}\, \overline{Y_{k,l}(z)}\, \frac{Y_{k,l}(w)}{|w|^{2k}}\, \frac{1}{|w|^n},$$

which proves (6.1).

□

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