ASYMPTOTIC BEHAVIOUR OF THE TRACE FOR SCHRÖDINGER OPERATOR ON FRACTAL DRUMS

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ABSTRACT. Let $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ be an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ and finite valume $|\Omega|_n$, where Ω_m are bounded, connected open domains with piecewise smooth boundaries, and $\Omega_i \neq \Omega_j$, if $i \neq j$. In particular, Ω is an open subset with fractal boundary $\partial\Omega$, which we also say that Ω is a fractal drum. We consider the following eigenvalue problem of Schrödinger operator

$$\text{(P)} \qquad \begin{cases} Lu \equiv -\Delta u + q(x)u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $q(x)$ is a bounded differentiable function in Ω. If we denote λ_j as the j-th eigenvalue of (P), and μ_j denotes the j-th eigenvalue of the Dirichlet Laplacian, and v_j denotes the unit eigenfunction associated with μ_j, we can prove that

$$\frac{1}{A(R)} \sum_{\mu_j \le R} [\lambda_j - \mu_j - (qv_j, v_j)] = O\left(\frac{\log R}{R}\right), \text{ as } R \to +\infty.$$

where $A(R)$ represents the "counting function" of Dirichlet Laplacian, i.e. $A(R) = \#\{k \geq 1; \mu_k \leq R\}$.

1. Introduction

It is well-known that an eigenvalue of differentiable operator is usually difficult to evaluate, and the "counting function" of the eigenvalues is hard to estimate too. So, we wish the sum of all eigenvalues with the same order could discribe the eigenvalue problem of the differential operator. The simplest case is the trace of the operator in [2]. In this aspect, L. M. Gelfand and B. M. Levitan were pioneers to study the trace of the Sturm-Liouville problem [3]

$$(1.1) \qquad \begin{cases} -y'' + q(x)y = \lambda y, \ x \in (0, \pi), \\ y(0) = y(\pi) = 0, \end{cases}$$

where $q(x)$ is bounded and differentiable on $[0, \pi]$. They succeeded in obtaining the following trace identity

$$(1.2) \qquad \sum_{j=1}^{\infty} [\lambda_j - j^2 - \frac{1}{\pi}\int_0^{\pi} q(x)dx] = -\frac{q(0) + q(\pi)}{4} + \frac{1}{2\pi}\int_0^{\pi} q(x)dx.$$

This result reveals a direct relation between the eigenvalues and the operator quantities. From then onwards, the research focus is on the trace identity of Sturm-Liouville equation with general boundary conditions and of singular Sturm-Liouville equation.

The trace identity in n-dimensional case was discussed by Cao Cewen [1] in 1979. He considered the eigenvalue problem

$$\text{(1.3)} \qquad \begin{cases} -\Delta u + q(x)u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded, connected domain with piecewise smooth boundary $\partial\Omega$, $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x^2}$ denotes the Dirichlet Laplacian on Ω, $q(x)$ is bounded and differentiable in Ω. Let λ_j be the j-th eigenvalue of (1.3), μ_j be the j-th eigenvalue of the Dirichlet Laplacian when $q(x) \equiv 0$. He proved the following average displacement formula

$$\text{(1.4)} \qquad \lim_{N\to N} \frac{1}{N} \sum_{j=1}^{\infty} (\lambda_j - \mu_j) = \frac{1}{|\Omega|_n} \int_\Omega q(x)dx,$$

where $|\Omega|_n$ represents the n-dimensional Lebesgue measure of Ω. And in a special case, when Ω is an n-dimensional open cube, we have more sharp result (see [1])

$$\text{(1.5)} \quad \lim_{R\to+\infty} R^{-\frac{n-1}{2}} \sum_{\mu_j \le R} \left(\lambda_j - \mu_j - \frac{1}{|\Omega|_n} \int_\Omega q(x)dx \right) = - \frac{|\partial\Omega|_{n-1}}{4\omega_{n-1}} \left(\frac{1}{|\partial\Omega|_{n-1}} \int_{\partial\Omega} q d\sigma - \frac{1}{|\Omega|_n} \int_\Omega q(x)dx \right)$$

where $|\partial\Omega|_{n-1}$ denotes the $(n-1)$-dimensional Lebesgue measure of $\partial\Omega$, $\omega_{n-1} = (2\sqrt{\pi})^{n-1}\Gamma(\frac{n-1}{2}+1)$. The second result is the extension of Gelfand-Levitan's trace identity in case of Ω is a cube.

In this paper, we study the asymptotic behaviour of the trace for Schrödinger operator. Let us consider the eigenvalue problem again

$$\text{(P)} \qquad \begin{cases} Lu \equiv -\Delta u + q(x)u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ is an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$, and the connected regions Ω_m with piecewise smooth boundaries are bounded and pairwise disjoint. We assume throughout this paper that $|\Omega_1|_n \ge |\Omega_2|_n \ge \cdots \ge |\Omega_m|_n \ge \cdots$, and $|\Omega_m|_n \to 0$ as $m \to +\infty$, and $|\Omega|_n = \sum_{m=1}^{\infty} |\Omega_m|_n < +\infty$. As is well-known that the scalar λ is said to be an eigenvalue of (P) if there exists a nonzero $u \in H_0^1(\Omega)$ which satisfies (P) in the distributional sense. The problem (P) has discrete eigenvalues which can be written as a sequence $(\lambda_j)_{j=1}^{\infty}$ with $\lambda_j \to +\infty$ as $j \to +\infty$. (But here, they are maybe not arranged in non-decreasing order. Their permutation can be discussed in section 3).

When $q(x) \equiv 0$, the eigenvalue problem of Dirichlet Laplacian associated with (P) is as follows:

$$\text{(D)} \qquad \begin{cases} L_0 u \equiv -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The spectrum of (D) is discrete and consists of a sequence $(\mu_j)_{j=1}^{\infty}$ and can be written in non-decreasing order according to their finite multiplicity :

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_j \le \cdots < +\infty \text{ with } \mu_j \to +\infty \text{ as } j \to +\infty.$$

Here we denote $v_j \in H_0^1(\Omega)$ as unit eigenfunctions associated with μ_j.

Now, we can state our main results as follows:

Theorem 1.1. *The following formula*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}[\lambda_j-\mu_j-(qv_j,v_j)]=0$$

is true, where $(\cdot,\cdot)$ *denotes the inner product in* $L^2(\Omega)$.

Theorem 1.2. *There exists* $R\to+\infty$ *such that*

$$\frac{1}{A(R)}\sum_{\mu_j\le R}[\lambda_j-\mu_j-(qv_j,v_j)]=O(\frac{\log R}{R}),$$

where $A(R)=\#\{k\ge 1;\mu_k\le R\}$.

The similar results can be found in [1].

Remark 1. The results can be extended to the case of that the domain Ω is a fractal drum, i.e., Ω is an open subset with fractal boundary $\partial\Omega$, even if Ω is a connected fractal drum. This is because if Ω is connected, we can use tessellation technique of Ω, i.e., Whitney covering of $\mathbb{R}^n$ (see [7] in details), from interior of Ω and exterior of Ω, to exhause the domain Ω. In this case, if $\delta\in(n-1,n)$ is the Minkowski dimension of the boundary $\partial\Omega$, then we know that $A(R)=(2\pi)^{-n}B_n|\Omega|_nR^{n/2}+O(R^{\delta/2})$ as $R\to\infty$.

2. Preparation

In this section we shall mainly discuss the problems on a single domain Ω_j, namely,

$$(P_j)\qquad\begin{cases} Lu\equiv-\Delta u+q(x)u=\lambda u, & \text{in } \Omega_j,\\ u=0, & \text{on } \partial\Omega_j,\end{cases}$$

and

$$(D_j)\qquad\begin{cases} L_0u\equiv-\Delta u=\lambda u, & \text{in } \Omega_j,\\ u=0, & \text{on } \partial\Omega_j.\end{cases}$$

It is convenient to assume that $q(x)$ is nonnegative temporarily, and at the end of this paper, we will indicate a possible way to remove this restriction. Suppose that Q_j is the upper bound of $q(x)$ in Ω_j and Q the upper bound of $q(x)$ in Ω. So, $0\le Q_j\le Q$.

We know that L_0 and L are unbounded self-adjoint operator in the real Hilbert space $H_0^1(\Omega_j)$. $L=L_0+q$ is nonnegative since L_0 is nonnegative. We write the eigenvalues of L_0 and L in Q_j as $0<\mu_1^{(j)}\le\mu_2^{(j)}\le\cdots$ and $0<\lambda_1^{(j)}\le\lambda_2^{(j)}\le\cdots$, and denote by $v_k^{(j)}$and $u_k^{(j)}$ unit eigenfunctions associated with $\mu_k^{(j)}$and $\lambda_k^{(j)}$, respectively, and $\{v_k^{(j)}\}$ and $\{u_k^{(j)}\}$ are both complete orthonormal bases in $H_0^1(\Omega_j)$ (cf. [4]). We put all $\mu_k^{(j)}$ and $\lambda_k^{(j)}$ together, and rearrange them and denote them by $\{\sigma_k^{(j)}\}$ in non-decreasing order. For a positive number R , let

$$\begin{aligned} A_j(R) &= \#\{n\ge 1;\mu_k^{(j)}\le R\}\\ B_j(R) &= \#\{n\ge 1;\lambda_k^{(j)}\le R\}\\ C_j(R) &= \#\{n\ge 1;\sigma_k^{(j)}\le R\}.\end{aligned}$$

Thus we have following result:

Lemma 2.1. *For any positive number R, we have*

1) $0 \le \mu_k^{(j)} \le \lambda_k^{(j)} \le \mu_k^{(j)} + Q_j$;
2) $A_j(R - Q_j) \le B_j(R) \le A_j(R)$;
3) $C_j(R) = A_j(R) + B_j(R)$, *and* $2A_j(R - Q_j) \le C_j(R) \le 2A_j(R)$.

Proof. According to the Lourant-Weinstein principle (see [4]), let T be a self-adjoint, continuous, linear and compact operator from a real Hilbert space H into H. Suppose $(Tx, x) > 0$ for nonzero $x \in H$, then all eigenvalues can be obtained by the formula

$$\nu_k = \min_{h_1,\cdots,h_{k-1}\in H} \max_{\substack{x\neq 0\\ (x,h_k)}} \frac{(Tx,x)}{(x,x)}, \tag{2.1}$$

where h_k is the eigenvector corresponding to the eigenvalue ν_k. Thus result 1) is the straight consequence of the principle. 2) and 3) are trival since 1) holds. □

Lemma 2.2. *Let H be a real Hilbert space, T be an operator as in* Lemma 2.1, *h_k be the k-th unit eigenvector corresponding to the k-th eigenvalue ν_k. Suppose that $f_1, \cdots, f_N \in H$ are pairwise orthogonal unit vectors, then*

$$\sum_{k=1}^{N} (Tf_k, f_k) \ge \sum_{k=1}^{N} \nu_k. \tag{2.2}$$

Proof. We know that $\{h_k\}$ is a complete orthonormal bases in H. By the Parseval's formula

$$\begin{aligned}
& \textstyle\sum_{k=1}^N (Tf_k, f_k) - \sum_{m=1}^N \nu_m \\
= & \textstyle\sum_{k=1}^N (Tf_k, f_k) - \sum_{m=1}^N \nu_m \|h_m\|^2 \\
= & \textstyle\sum_{k=1}^N \sum_{m=1}^\infty (Tf_k, h_m)(f_k, h_m) - \sum_{m=1}^N \sum_{j=1}^\infty \nu_m (h_m, f_j)^2 \\
\ge & \textstyle\nu_{N+1}[\sum_{k=1}^N \sum_{m=N+1}^\infty (f_k, h_m)^2 - \sum_{m=1}^N \sum_{j=N+1}^\infty (f_j, h_m)^2] \\
= & \textstyle\nu_{N+1}[\sum_{k=1}^N \|f_k\|^2 - \sum_{m=1}^N \|h_m\|^2] \\
= & 0
\end{aligned}$$

The lemma is proved. □

Specially, when $T = L$ or L_0, as a result of Lemma 2.2, we get

$$\sum_{k=1}^{N} (Lf_k, f_k) \ge \sum_{k=1}^{N} \lambda_k^{(j)}, \quad \sum_{k=1}^{N} (L_0 f_k, f_k) \ge \sum_{k=1}^{N} \mu_k^{(j)}$$

Lemma 2.3. *For any positive number R,*

$$0 \le -\sum_{\mu_k^{(j)} \le R} [\lambda_k^{(j)} - \mu_k^{(j)} - (qv_k^{(j)}, v_k^{(j)})] \le F_j(R), \tag{2.3}$$

where

$$F_j(R) = \sum_{\mu_k^{(j)} \le R} (qv_k^{(j)}, v_k^{(j)}) - \sum_{\lambda_k^{(j)} \le R} (qu_k^{(j)}, u_k^{(j)}). \tag{2.4}$$

Proof. We use Lemma 2.2, and let $f_k = v_k^{(j)}$ or $f_k = u_k^{(j)}$ respectively, and note that $(Lu_k^{(j)}, u_k^{(j)}) = \lambda_k^{(j)}$, $(L_0 v_k^{(j)}, v_k^{(j)}) = \mu_k^{(j)}$, thus

$$\begin{aligned}\sum_{\mu_n^{(j)} \le R} (qv_k^{(j)}, v_k^{(j)}) &= \sum_{k=1}^{A_j(R)} (Lv_k^{(j)}, v_k^{(j)}) - \sum_{k=1}^{A_j(R)} (L_0 v_k^{(j)}, v_k^{(j)}) \\ &\ge \sum_{k=1}^{A_j(R)} (\lambda_k^{(j)} - \mu_k^{(j)}) = \sum_{\mu_k^{(j)} \le R} (\lambda_k^{(j)} - \mu_k^{(j)})\end{aligned}$$

and

$$\begin{aligned}\sum_{\lambda_k^{(j)} \le R} (qu_k^{(j)}, u_k^{(j)}) &= \sum_{k=1}^{B_j(R)} (Lu_k^{(j)}, u_k^{(j)}) - \sum_{k=1}^{B_j(R)} (L_0 u_k^{(j)}, u_k^{(j)}) \\ &\le \sum_{k=1}^{B_j(R)} (\lambda_k^{(j)} - \mu_k^{(j)}) = \sum_{\lambda_k^{(j)} \le R} (\lambda_k^{(j)} - \mu_k^{(j)}).\end{aligned}$$

By Lemma 2.1, $\lambda_k^{(j)} - \mu_k^{(j)} \ge 0$, $B_j(R) \le A_j(R)$, hence

$$\begin{aligned}0 &\le -\sum_{\mu_k^{(j)} \le R} [\lambda_k^{(j)} - \mu_k^{(j)} - (qv_k^{(j)}, v_k^{(j)})] \\ &\le \sum_{\mu_k^{(j)} \le R} (qv_k^{(j)}, v_k^{(j)}) - \sum_{\lambda_k^{(j)} \le R} (\lambda_k^{(j)} - \mu_k^{(j)}) \\ &\le \sum_{\mu_k^{(j)} \le R} (qv_k^{(j)}, v_k^{(j)}) - \sum_{\lambda_k^{(j)} \le R} (qu_k^{(j)}, u_k^{(j)}).\end{aligned}$$

□

Lemma 2.4. *Let R, S be any two real numbers satisfying $0 < S < R$, then*

$$0 \le F_j(R) \le \sum_{\sigma_k^{(j)} \le S} \frac{Q_j^2}{R - \sigma_k^{(j)}} + \sum_{S < \sigma_k^{(j)} \le R} Q_j \tag{2.5}$$

Proof. Notice that $\{u_k^{(j)}\}$ and $\{v_k^{(j)}\}$ are both complete orthonormal bases in $H_0^1(\Omega_j)$, we deduce from Lemma 2.3 and the Parseval's formula

$$\begin{aligned}F_j(R) &= \sum_{\mu_k^{(j)} \le R} (qv_k^{(j)}, v_k^{(j)}) - \sum_{\lambda_k^{(j)} \le R} (qu_k^{(j)}, u_k^{(j)}) \\ &= \sum_{\mu_k^{(j)} \le R} \sum_{m=1}^{\infty} (qv_k^{(j)}, u_m^{(j)})(v_k^{(j)}, u_m^{(j)}) - \sum_{\lambda_k^{(j)} \le R} \sum_{i=1}^{\infty} (qu_k^{(j)}, v_i^{(j)})(u_k^{(j)}, v_i^{(j)}) \\ &= \sum_{\mu_k^{(j)} \le R} \sum_{\lambda_m^{(j)} > R}^{\infty} (qv_k^{(j)}, u_m^{(j)})(v_k^{(j)}, u_m^{(j)}) - \sum_{\lambda_k^{(j)} \le R} \sum_{\mu_i^{(j)} > R} (qu_k^{(j)}, v_i^{(j)})(u_k^{(j)}, v_i^{(j)}) \\ &= I - II.\end{aligned}$$

In order to estimate I, we write $\eta_k = \sum_{\lambda_m^{(j)}>R}^{\infty}(qv_k^{(j)}, u_m^{(j)})(v_k^{(j)}, u_m^{(j)})$, then $|\eta_k| \leq \left\|qv_k^{(j)}\right\| \left\|v_k^{(j)}\right\| \leq Q_j$. On the other hand $(\lambda_m^{(j)} - \mu_k^{(j)})(u_m^{(j)}, v_k^{(j)}) = \lambda_m^{(j)}(u_m^{(j)}, v_k^{(j)}) - \mu_k^{(j)}(u_m^{(j)}, v_k^{(j)}) = (Lu_m^{(j)}, v_k^{(j)}) - (u_m^{(j)}, L_0 v_k^{(j)}) = (qu_m^{(j)}, v_k^{(j)})$. Thus, for $\mu_k^{(j)} < R$,

$$\begin{aligned} |\eta_k| &= \sum_{\lambda_m^{(j)}>R} \frac{(qv_k^{(j)}, u_m^{(j)})^2}{\lambda_m^{(j)} - \mu_k^{(j)}} \\ &< \sum_{\lambda_m^{(j)}>R} \frac{(qv_k^{(j)}, u_m^{(j)})^2}{R - \mu_k^{(j)}} \leq \frac{Q_j^2}{R - \mu_k^{(j)}} \end{aligned}$$

So

$$\begin{aligned} |I| &= \sum_{\mu_k^{(j)}\leq R} |\eta_k| \\ &= \sum_{\mu_k^{(j)}\leq S} |\eta_k| + \sum_{S<\mu_k^{(j)}\leq R} |\eta_k| \\ &\leq \sum_{\mu_k^{(j)}\leq S} \frac{Q_j^2}{R - \mu_k^{(j)}} + \sum_{S<\mu_k^{(j)}\leq R} Q_j. \end{aligned}$$

Similarly, we have

$$|II| \leq \sum_{\lambda_k^{(j)}\leq S} \frac{Q_j^2}{R - \lambda_k^{(j)}} + \sum_{S<\lambda_k^{(j)}\leq R} Q_j.$$

Finally, $0 \leq F_j(R) \leq |I| + |II|$, Lemma 2.4 is proved. □

The following two lemmas are the uniform estimates on the increasing rate of $A_j(R)$.

Lemma 2.5. *For arbitrary $\varepsilon > 0$, there exists sufficiently small $\delta > 0$, such that for the operator L_0, we have following uniformly estimate on j*

$$\lim_{R\to+\infty} \frac{A_j(R) - A_j(R - \delta R)}{A_j(R)} \leq \varepsilon. \tag{2.6}$$

Proof. From [5-7], we know that $A_j(R) = (2\pi)^{-n} B_n \left|\Omega_j\right|_n R^{\frac{n}{2}} + o(R^{\frac{n}{2}})$ as $R \to +\infty$, where B_n denotes the volume of the unit ball in $\mathbb{R}^n$, then

$$\begin{aligned} &\frac{A_j(R) - A_j(R - \delta R)}{A_j(R)} \\ = &\frac{(2\pi)^{-n}\omega_n \left|\Omega_j\right|_n R^{\frac{n}{2}} + \circ(R^{\frac{n}{2}}) - (2\pi)^{-n}\omega_n \left|\Omega_j\right|_n (R - \delta R)^{\frac{n}{2}} + \circ((R - \delta R)^{\frac{n}{2}})}{(2\pi)^{-n}\omega_n \left|\Omega_j\right|_n R^{\frac{n}{2}} + \circ(R^{\frac{n}{2}})} \\ = &1 - (1 - \delta)^{\frac{n}{2}}, \text{as } R \to +\infty \end{aligned}$$

Thus, for arbitrary $\varepsilon > 0$, we take $0 < \delta < 1 - (1 - \varepsilon)^{\frac{2}{n}}$, the lemma holds. □

Lemma 2.6. *For operator L_0, we have uniformly estimate on j,*

$$\frac{A_j(2R)}{A_j(R)} = O(1), \text{ as } R \to +\infty. \tag{2.7}$$

Proof. It is deduced directly by the process of the proof for Lemma 2.5. □

3. Proof of the main results

In section 2, we considered asymptotic behaviour of the trace for the operators L and L_0 in a single domain Ω_j. Next, we shall consider the same problem in the case of $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. First, we confirm that the set of all the eigenvalues of (P) will be the set of all eigenvalues of $(P_j)_{j=1}^{\infty}$.

Lemma 3.1. *Suppose that $\lambda_m^{(j)}$ is an eigenvalue of (P_j), then $\lambda_m^{(j)}$is certainly the eigenvalue of (P); on the other hand, for an eigenvalue λ_m of (P), there exists j such that λ_k is the eigenvalue of (P_j).*

Proof. If $\lambda_m^{(j)}$ is an eigenvalue of (P_j), $u_m^{(j)} \in H_0^1(\Omega_j)$ the eigenfunction corresponding to $\lambda_m^{(j)}$, then $u_m^{(j)} \neq 0$ and satisfies

$$-\Delta u_m^{(j)} + q(x) u_m^{(j)} = \lambda_m^{(j)} u_m^{(j)}, \text{ in } \Omega_j. \tag{3.1}$$

Let u_m be the zero-extension of $u_m^{(j)}$, i.e., we define $u_m = 0$ in other Ω_k $(k \neq j)$, then $u_m \in H_0^1(\Omega)$ and satisfies

$$-\Delta u_m + q(x) u_m = \lambda_m^{(j)} u_m, \text{ in } \Omega,$$

hence $\lambda_m^{(j)}$ is the eigenvalue of (P), u_m is the eigenfunction associated with $\lambda_m^{(j)}$.

On the other hand, if λ is an eigenvalue of (P), $u \in H_0^1(\Omega)$ is the eigenfunction corresponding to λ, and for any test function $\varphi \in C_0^{\infty}(\Omega)$, we have

$$\int_{\Omega} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} q \cdot u \cdot \varphi dx = \lambda \int_{\Omega} u \cdot \varphi dx. \tag{3.2}$$

Note that $u \neq 0$ in $H_0^1(\Omega)$, and Ω_j are pairwise disjoint, therefore, there exists j such that $u \in H_0^1(\Omega_j)$. Since $\varphi \in C_0^{\infty}(\Omega_j)$ implies $\varphi \in C_0^{\infty}(\Omega)$, we can rewrite the formula (3.2) for $\varphi \in C_0^{\infty}(\Omega_j)$, i.e.,

$$\int_{\Omega_j} \nabla u \cdot \nabla \varphi dx + \int_{\Omega_j} q \cdot u \cdot \varphi dx = \lambda \int_{\Omega_j} u \cdot \varphi dx$$

which implies that λ is the eigenvalue of (P_j). □

From the process above, it is easy to see that we shall have the same result for the eigenvalues between problem (D) and $(D_j)_{j=1}^{\infty}$. Thus to consider the eigenvalue problems (P) and (D) would be equivalent to consider eigenvalues $(P_j)_{j=1}^{\infty}$ and $(D_j)_{j=1}^{\infty}$. Now, we rerrange the all eigenvalues $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \{\mu_k^{(j)}\}$ of (D) which are rewritten by $\{\mu_k\}$ in non-decreasing order. Denote that v_k are the unit eigenfunctions associated with μ_k. For any μ_k, there exist j and m such that $\mu_k = \mu_m^{(j)}$, Coresponding to μ_k, we rewrite $\lambda_m^{(j)}$ by λ_k. Then by Lemma 3.1, $\bigcup_{k=1}^{\infty} \{\lambda_k\}$ is the whole eigenvalues $\bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \{\lambda_m^{(j)}\}$. Perhaps, λ_k are not arranged in non-decreasing order, but $\lambda_k \to +\infty$ as $k \to \infty$. We denote $A(R) = \#\{k \mid \mu_k \leq R\}$, then $A(R) = \sum_{j=1}^{\infty} A_j(R)$ and the sum is finite for any fixed R.

Lemma 3.2.

$$\lim_{R\to+\infty}\frac{1}{A(R)}\sum_{j=1}^{\infty}\sum_{\mu_m^{(j)}\le R}[\lambda_m^{(j)}-\mu_m^{(j)}-(qv_m^{(j)},v_m^{(j)})]=0,$$

or

$$\lim_{R\to+\infty}\frac{1}{A(R)}\sum_{m=1}^{A(R)}[\lambda_m-\mu_m-(qv_m,v_m)]=0.$$

Proof. By Lemma 2.3, it is necessary to prove that

$$\lim_{R\to+\infty}\frac{1}{A(R)}\sum_{j=1}^{\infty}F_j(R)=0,$$

namely,

$$\lim_{R\to+\infty}\sum_{j=1}^{\infty}\frac{F_j(R)}{A_j(R)}\cdot\frac{A_j(R)}{A(R)}=0.$$

For any given $\varepsilon>0$, we take $\delta>0$ which satisfies the conditions in Lemma 2.5, and $\theta=\frac{\delta}{2}$, $S=R-\theta R$ as mentioned in Lemma 2.4. Then we have

$$\begin{aligned}
0 &\le \frac{F_j(R)}{A_j(R)}\\
&\le \frac{1}{A_j(R)}\sum_{\sigma_m^{(j)}\le S}\frac{Q_j^2}{R-\sigma_m^{(j)}}+\frac{1}{A_j(R)}\sum_{S<\sigma_m^{(j)}\le R}Q_j\\
&\le \frac{Q_j^2}{R\theta}\cdot\frac{C_j(R-\theta R)}{A_j(R)}+Q_j\cdot\frac{C_j(R)-C_j(R-\theta R)}{A_j(R)}\\
&\le \frac{Q^2}{R\theta}\cdot\frac{C_j(R)}{A_j(R)}-Q\cdot\frac{C_j(R)-C_j(R-\theta R)}{A_j(R)}.
\end{aligned}$$

From Lemma 2.1, we obtain

$$\begin{aligned}
0 &\le \frac{F_j(R)}{A_j(R)}\\
&\le \frac{2Q^2}{R\theta}+2Q\cdot\frac{A_j(R)-A_j(R-\theta R-Q_j)}{A_j(R)}\\
&\le \frac{2Q^2}{R\theta}+2Q\cdot\frac{A_j(R)-A_j(R-\delta R)}{A_j(R)},
\end{aligned}$$

where we take R sufficiently large such that $Q_j\le Q<\theta R$. Thus for large R, Lemma 2.5 gives that, for j uniformly,

$$\frac{A_j(R)-A_j(R-\delta R)}{A_j(R)}\le\varepsilon.$$

On the other hand, $\frac{2Q^2}{R\theta}\le Q\varepsilon$ for sufficiently large R, thus

$$0\le\frac{F_j(R)}{A_j(R)}\le 3Q\varepsilon,$$

which implies that for R large enough, we have

$$\begin{aligned}0 &\le \frac{1}{A(R)}\sum_{j=1}^{\infty}F_j(R)\\ &\le 3Q\varepsilon\sum_{j=1}^{\infty}\frac{A_j(R)}{A(R)}\\ &= 3Q\varepsilon.\end{aligned}$$

Since ε is arbitrary, Lemma 3.2 holds.

□

Now we can prove our main results.

The proof of Theorem 1.1 Since $A(R)$ is the right continuous function and we let $t_1 < t_2 < \cdots < t_k < \cdots$ be the leap points, where $t_k \to +\infty$ as $k \to +\infty$. It is trivial to see that $A(t_{k+1}-1) \le A(t_k)$. For given $\varepsilon > 0$, by Lemma 2.5, there exists $\delta > 0$ such that uniformly for j,

$$\lim_{R\to+\infty}\frac{A_j(R)}{A_j(R-\delta R)} \le \frac{1}{1-\varepsilon}.$$

For such δ, we take k sufficiently large, so that $\delta t_k > 1$, and $t_{k+1} - \delta t_{k+1} \le t_k$, then $A(t_{k+1} - \delta t_{k+1}) \le A(t_{k+1}-1) \le A(t_k)$, since $A(R)$ and $A_j(R)$ are both non-decreasing. Let $A(t_k) = \alpha_k$, then for k large enough, we have

$$\begin{aligned}1 &< \frac{\alpha_{k+1}}{\alpha_k} = \sum_{j=1}^{\infty}\frac{A_j(t_{k+1})}{A(t_k)}\\ &\le \sum_{j=1}^{\infty}\frac{A_j(t_{k+1})}{A_j(t_{k+1}-\delta t_{k+1})}\cdot\frac{A_j(t_k)}{A(t_k)}\\ &\le \frac{1}{1-\varepsilon}\sum_{j=1}^{\infty}\frac{A_j(t_k)}{A(t_k)} = \frac{1}{1-\varepsilon},\end{aligned}$$

so $\lim_{k\to\infty}\frac{\alpha_{k+1}}{\alpha_k} = 1$ was proved by taking $\varepsilon \to 0^+$.

Notice that $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \cdots$, and $\alpha_k \to +\infty$ as $k \to \infty$. Thus for any positive integer N, there exists a natural number k such that $\alpha_k < N \le \alpha_{k+1}$. Since $\lambda_j - \mu_j \ge 0$, we have

$$\begin{aligned}&-\sum_{j=1}^{N}[\lambda_j - \mu_j - (qv_j, v_j)]\\ \le\ &-\sum_{j=1}^{\alpha_k}[\lambda_j - \mu_j - (qv_j, v_j)] + \sum_{\alpha_k < j \le N}(qv_j, v_j)\\ \le\ &-\sum_{j=1}^{\alpha_k}[\lambda_j - \mu_j - (qv_j, v_j)] + Q(\alpha_{k+1} - \alpha_k),\end{aligned}$$

therefore,

$$\begin{aligned} 0 &\leq -\frac{1}{N}\sum_{j=1}^{N}[\lambda_j - \mu_j - (qv_j, v_j)] \\ &\leq -\frac{1}{\alpha_k}\sum_{j=1}^{\alpha_k}[\lambda_j - \mu_j - (qv_j, v_j)] + Q(\frac{\alpha_{k+1}}{\alpha_k} - 1). \end{aligned}$$

Here $N \to \infty$ implies $k \to \infty$, we obtain the following as a result of Lemma 3.2

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}[\lambda_j - \mu_j - (qv_j, v_j)] = 0.$$

The proof of Theorem 1.2 Let $C_k^{(j)}$ be the number of $\sigma_k^{(j)}$ in $I_k = (k-1, k]$, then $C^{(j)}(m) = C_1^{(j)} + \cdots + C_m^{(j)}$. Put $R = m$ and $S = m-1$ as that we have in Lemma 2.4, then

$$\begin{aligned} 0 &\leq F_j(m) \\ &\leq \sum_{\sigma_k^{(j)} \leq m-1} \frac{Q_j^2}{m - \sigma_k^{(j)}} + \sum_{m-1 < \sigma_k^{(j)} \leq m} Q_j \\ &\leq \sum_{k=1}^{m-1} \sum_{\sigma_k^{(j)} \in I_k} \frac{Q^2}{m - \sigma_k^{(j)}} + QC_m^{(j)} \\ &\leq Q^2 \sum_{k=1}^{m-1} \frac{C_k^{(j)}}{m-k} + QC_m^{(j)}, \end{aligned}$$

hence

$$F(m) = \sum_{j=1}^{\infty} F_j(m) \leq \sum_{j=1}^{\infty}(Q^2 \sum_{k=1}^{m-1} \frac{C_k^{(j)}}{m-k} + QC_m^{(j)}),$$

and

$$\begin{aligned}
& \sum_{m=N+1}^{2N} F(m) \\
\leq & \sum_{m=N+1}^{2N} [\sum_{j=1}^{\infty}(Q^2 \sum_{k=1}^{m-1} \frac{C_k^{(j)}}{m-k} + QC_m^{(j)})] \\
= & \sum_{j=1}^{\infty}(Q^2 \sum_{m=N+1}^{2N} \sum_{k=1}^{m-1} \frac{C_k^{(j)}}{m-k} + Q \sum_{m=N+1}^{2N} C_m^{(j)}) \\
\leq & \sum_{j=1}^{\infty}[Q^2 \sum_{k=1}^{N} \sum_{m=N+1}^{2N} + \sum_{k=N+1}^{2N-1} \sum_{m=N+1}^{2N}) \frac{C_k^{(j)}}{m-k} + QC^{(j)}(2N)] \\
\leq & Q^2 \sum_{j=1}^{\infty} C^{(j)}(2N)(1+\log N) + QC(2N) \\
= & C(2N)[Q^2(1+\log N)+Q].
\end{aligned}$$

So

$$\frac{1}{N} \sum_{m=N+1}^{2N} F(m) \leq C(2N) \cdot \frac{Q^2(1+\log N)+Q}{N} \equiv \Gamma.$$

Now, we have N nonnegative numbers $F(m)$, in which the arithmetic mean-value does not exceed Γ. It is clear that there exists at least one $m = R$, $N < R \leq 2N$ such that $F(R) \leq \Gamma$.

Finally, we have from Lemma 2.3, Lemma 2.1 and Lemma 2.6

$$\begin{aligned}
0 \leq & -\frac{1}{A(R)} \sum_{\mu_j \leq R} [\lambda_j - \mu_j - (qv_j, v_j)] \\
\leq & \frac{1}{A(R)} \sum_{j=1}^{\infty} F_j(R) = \frac{F(R)}{A(R)} \\
\leq & \frac{2A(2N)}{A(N)} \cdot \frac{Q^2(1+\log N)+Q}{N} \\
= & 2\sum_{j=1}^{\infty} \frac{A_j(2N)}{A_j(N)} \cdot \frac{A_j(N)}{A(N)} \cdot \frac{Q^2(1+\log N)+Q}{N} \\
= & O(\frac{\log N}{N}) = O(\frac{\log R}{R}).
\end{aligned}$$

At the end of this paper, the restriction on $q(x)$, i.e., to ask $q(x)$ is nonnegative, can be removed. Since q is bounded, let $E > 0$ such that $|q| \leq E$ in Ω, then $q+E$ is nonnegative. Theorem 1.1 and Theorem 1.2 hold for operator $L+E$ with eigenvalues $\lambda_j + E$. The following identity

$$\sum_{j=1}^{\infty}[\lambda_j - \mu_j - (qv_j, v_j)] = \sum_{j=1}^{\infty}[(\lambda_j + E) - \mu_j - ((q+E)v_j, v_j)]$$

implies that Theorem 1.1 and Theorem 1.2 are also true for the operator $L = L_0 + q$ in which q is bounded.

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