Index Defects in the Theory of Nonlocal Boundary Value Problems and the η-invariant

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Abstract

The paper deals with elliptic theory on manifolds with boundary represented as a covering space. We compute the index for a class of nonlocal boundary value problems. For a nontrivial covering, the index defect of the Atiyah–Patodi–Singer boundary value problem is computed. We obtain the Poincaré duality in the K-theory of the corresponding manifolds with singularities.

Keywords: elliptic operator, boundary value problem, finiteness theorem, nonlocal problem, covering, relative η-invariant, index, mod$n$-index, Poincaré duality.

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Contents

Introduction

1. Nonlocal boundary value problems. Let M be a smooth manifold such that the boundary ∂M is a finite-sheeted covering with projection

$$\pi : \partial M \longrightarrow X. \tag{1}$$

Then the space $C^\infty(\partial M)$ of smooth functions is isomorphic to the space of sections of the vector bundle $\pi_! 1 \in \operatorname{Vect}(X)$ on the base of the covering:

$$C^\infty(\partial M) \overset{\beta}{\simeq} C^\infty(X, \pi_! 1).$$

Here $\pi_! 1$ is the direct image of the trivial one-dimensional bundle under the projection π.

For an elliptic operator D on M, we consider boundary value problems of the form

$$\begin{cases} Du = f, \\ B\beta \left. u\right|_{\partial M} = g. \end{cases} \tag{2}$$

Here u and f are functions on M, g a function on X, and the operator B of boundary conditions also acts on X. In terms of the original manifold M, the boundary conditions in Eq. (2) are *nonlocal*, i.e. they relate values of functions at different points of the manifold.

We prove a finiteness theorem and obtain an index formula for this class of nonlocal boundary value problems. Without going into the detail at the moment, let us mention two essential features of the theory.

First, the analog of the Atiyah–Singer difference construction for a nonlocal boundary value problem is an element of the K-group of a *noncommutative C^*-algebra* associated with the cotangent bundle and the covering. Recall that in the classical index theorem it suffices to use topological K-theory.

Second, in the embedding proof of the index theorem, we embed our manifolds in a special space, defined in terms of the classifying space BS_n of the symmetric group S_n, while in the classical index theorem ([1]) this space is $\mathbb{R}^N$.

Nonlocal boundary value problems are not exhausted by those associated with coverings. Another interesting class of nonlocal boundary value problems can be defined for manifolds equipped with an action of a finite group on the boundary. Index theory of such boundary value problems is apparently related to the index theory on orbifolds ([2, 3]). Related classes of nonlocal boundary value problems are beyond the scope of the present paper. Here we only give a reference to the book [4].

2. Spectral problems on manifolds with a covering. A general approach to the boundary value problems, which is free of the Atiyah–Bott obstruction, was given by Atiyah, Patodi, and Singer [5]. For a class of first-order elliptic operators (which however includes all geometric operators) one has so-called *spectral boundary value problems* denoted by (D, Π_+). Spectral boundary value problems enjoy the Fredholm property. However, their index is not determined by the principal symbol of D.

Interesting invariants arise if the boundary has the structure of a covering and we consider the class of elliptic operators pulled back from the base of the covering in a neighborhood of the boundary. Then the principal symbol defines an element

$$[\sigma(D)] \in K^0\left(\overline{T^*M}^{\pi}\right)$$

in the K-group of the singular space $\overline{T^*M}^{\pi}$. This space is obtained from the cotangent bundle T^*M under the identification of points in the fibers of the covering (for details, see Section 5). The element $[\sigma(D)]$ has a topological index

$$\operatorname{ind}_t[\sigma(D)] \in \mathbb{Q}/n\mathbb{Z},$$

where n is the number of sheets. However, the equality of the analytical and topological indices is valid only for trivial coverings. For a general covering, we obtain the *index defect formula*

$$\operatorname{mod} n\text{-ind}(D, \Pi_+) - \operatorname{ind}_t[\sigma(D)] = \eta\left(D|_X \otimes 1_{n-\pi_!1}\right) \in \mathbb{Q}/n\mathbb{Z}. \tag{3}$$

The defect (difference between the analytical index modulo n and the topological index) is equal to the relative Atiyah–Patodi–Singer η-invariant of the restriction of D to the boundary with coefficients in the flat bundle $\pi_!1$. For a trivial covering, the relative η-invariant is zero, and the index defect formula is reduced to the index formula

$$\operatorname{mod} n\text{-ind}(D, \Pi_+) = \operatorname{ind}_t[\sigma(D)]$$

obtained by Melrose and Freed [6] (see also [7, 8, 9, 10]). However, our proof is new even in this case. It is interesting to note that the main step in the proof is a realization of the fractional analytic invariant

$$\operatorname{mod} n\text{-ind}\,(D,\Pi_+) - \eta\left(D|_X \otimes 1_{n-\pi_! 1}\right) \in \mathbb{Q}/n\mathbb{Z} \tag{4}$$

as an index (in a suitable elliptic theory).

There is a beautiful geometric link between the two elliptic theories described.

3. Poincaré isomorphism and duality. We establish *Poincaré isomorphisms* on the singular spaces $\overline{T^*M}^{\pi}$ and $\overline{M}^{\pi}$. For the identity covering $\pi = Id, X = \partial M$, they are reduced to the well-known isomorphisms

$$K^0(T^*M) \simeq K_0(M,\partial M), \quad K^0(T^*(M\setminus\partial M)) \simeq K_0(M). \tag{5}$$

In contrast with the smooth case, the isomorphism for singular spaces relates the K-groups of a commutative algebra of functions and some dual noncommutative algebra. The Poincaré isomorphisms are defined on the elements as *quantizations*, i.e. by defining an operator by the symbol. More precisely, an analog of the first isomorphism in (5) is defined in terms of the operators of Section 2, while in the second case one uses nonlocal problems of Section 1. It should be mentioned that in the C^*-algebra K-theory one knows an abstract form of duality (see [11, 12]).

Let us describe the contents of the paper. The first section contains the definition of the class of nonlocal boundary value problems on manifolds with a covering on the boundary and a proof of the Fredholm property. The index formula is obtained in the following two sections. In Section 2 we give the homotopy classification of nonlocal problems. Then the index theorem is proved. By way of example, we consider the Hirzebruch operator on a manifold with reflecting boundary. The index defect formula is obtained in the following three sections. The first of them contains necessary definitions. Then the index defect formula (3) is proved in Section 5. This is one of the central results of the paper. Section 6 contains applications to the problem of computing the fractional part of the η-invariant. It is also shown that the invariant (4) can be computed by the Lefschetz formula. The Poincaré duality and the isomorphisms in K-theory of singular spaces that correspond to manifolds with a covering on the boundary are constructed in the remaining three sections.

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1 Nonlocal boundary value problems

1. Coverings and nonlocal operators. Let Y be a finite covering over a manifold X with projection

$$\pi : Y \longrightarrow X.$$

The projection defines the direct image mapping

$$\pi_! : \mathrm{Vect}\,(Y) \longrightarrow \mathrm{Vect}\,(X)\,,$$

which takes a vector bundle $E \in \mathrm{Vect}\,(Y)$ to the following (finite dimensional) bundle

$$\pi_! E \in \mathrm{Vect}\,(X)\,, \quad (\pi_! E)_x = C^\infty\left(\pi^{-1}(x)\,, E\right), \quad x \in X$$

— the space of sections on the fibers.

This definition gives an isomorphism

$$\beta_E : C^\infty(Y, E) \xrightarrow{\simeq} C^\infty(X, \pi_! E)$$

of section spaces on the total space Y and on the base X of the covering.

The isomorphism makes it possible to transport operators between the total space and the base. More precisely, for a differential operator

$$D : C^\infty(Y, E) \longrightarrow C^\infty(Y, E)$$

on Y let $\pi_! D$ denote the direct image

$$\pi_! D = \beta_E D \beta_E^{-1} : C^\infty(X, \pi_! E) \longrightarrow C^\infty(X, \pi_! E)\,.$$

It is a differential operator. However, the inverse image

$$\pi^! D' = \beta_E^{-1} D' \beta_E : C^\infty(Y, E) \longrightarrow C^\infty(Y, E) \tag{6}$$

for operator D' on X turns out to be a *nonlocal operator* (moreover, it is not pseudolocal). This is shown in the following example.

Example 1 For a trivial covering

$$Y = \underbrace{X \sqcup X \sqcup ... \sqcup X}_{n \text{ copies}} \longrightarrow X,$$

consisting of n copies of the base X, and a trivial bundle $E = \mathbb{C}$ we have $\pi_! E = \mathbb{C}^n$. On the other hand, the direct image of a differential operator on Y is always a diagonal operator

$$\pi_! D = \mathrm{diag}\left(D|_{X_1}, ..., D|_{X_n}\right) : C^\infty(X, \mathbb{C}^n) \longrightarrow C^\infty(X, \mathbb{C}^n)\,,$$

while the inverse image can have an arbitrary matrix form. In these terms, the off-diagonal entries of the matrix operator (6) are nonlocal operators on Y, since they interchange the functions between different leafs of the covering.

2. Nonlocal boundary value problems. Let M be a smooth compact manifold with boundary ∂M. Suppose that the boundary is a covering space over a smooth closed manifold X with projection

$$\pi : \partial M \longrightarrow X.$$

We fix a collar neighborhood $\partial M \times [0,1)$ of the boundary. The normal coordinate is denoted by t.

For a smooth function $u \in C^\infty(M)$, denote the restriction to the boundary of order $(m-1)$ jet in the normal direction by $j^{m-1}_{\partial M} u$

$$j^{m-1}_{\partial M} u = \left(u|_{\partial M}, -i\frac{\partial}{\partial t} u\Big|_{\partial M}, \ldots, \left(-i\frac{\partial}{\partial t}\right)^{m-1} u\Big|_{\partial M} \right).$$

The operator $j^{m-1}_{\partial M}$ acts continuously in the following Sobolev spaces

$$j^{m-1}_{\partial M} : H^s(M) \longrightarrow \bigoplus_{k=0}^{m-1} H^{s-1/2-k}(\partial M), \qquad s > m - 1/2.$$

Definition 1 *Nonlocal boundary value problem* for a differential operator

$$D : C^\infty(M, E) \longrightarrow C^\infty(M, F)$$

of order m is a system of the form

$$\begin{cases} Du = f, & u \in H^s(M, E), f \in H^{s-m}(M, F), \\ B\beta_E j^{m-1}_{\partial M} u = g, & g \in H^\delta(X, G), \end{cases} \tag{7}$$

where the boundary conditions are defined by a pseudodifferential operator

$$B : \bigoplus_{k=0}^{m-1} H^{s-1/2-k}(X, \pi_!\, E|_{\partial M}) \longrightarrow H^\delta(X, G)$$

on X. We assume that the component

$$B_k : H^{s-1/2-k}(X, \pi_!\, E|_{\partial M}) \longrightarrow H^\delta(X, G)$$

of B has order $s - 1/2 - k - \delta$.

Note that for the identity covering $\pi = Id$, $X = \partial M$ the problem (7) is just a classical boundary value problem (e.g., see [13]).

3. Nonlocal and classical boundary value problems. Finiteness theorem. The collar neighborhood $\partial M \times [0,1)$ of the boundary is also a covering

$$\pi \times 1 : \partial M \times [0,1) \longrightarrow X \times [0,1).$$

The induced isomorphism of function spaces is denoted by

$$\beta'_E : C^\infty(\partial M \times [0,1), E) \longrightarrow C^\infty(X \times [0,1), \pi_! E).$$

The nonlocal problem (D,B) can be represented in the neighborhood of the boundary as inverse image of a classical boundary value problem

$$\begin{pmatrix} \beta'_F & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} D \\ B\beta_E j^{m-1}_{\partial M} \end{pmatrix} \circ (\beta'_E)^{-1} = \begin{pmatrix} \beta'_F D (\beta'_E)^{-1} \\ B j^{m-1}_X \end{pmatrix}.$$

In more detail, this is a boundary value problem

$$\begin{pmatrix} \beta'_F D (\beta'_E)^{-1} \\ B j^{m-1}_X \end{pmatrix} : C^\infty (X \times [0,1), \pi_! E) \longrightarrow \begin{array}{c} C^\infty (X \times [0,1), \pi_! F) \\ \oplus \\ C^\infty (X, G), \end{array} \tag{8}$$

for a differential operator $(\pi \times 1)_! D = \beta'_F D (\beta'_E)^{-1}$ on the cylinder $X \times [0,1)$. In local coordinates it is a diagonal matrix with elements acting on different leafs of the covering. The problem (8) is denoted by $((\pi \times 1)_! D, B)$.

Let us note that the *classical boundary value problem*(8) *is defined only in the neighborhood of the boundary* , since the covering is defined only near the boundary.

Definition 2 (D,B) is said to be *elliptic*, if D is elliptic and the classical boundary value problem $((\pi \times 1)_! D, B)$ is elliptic, i.e. it satisfies the Shapiro–Lopatinskii condition (e.g., see [13]).

The standard techniques give the following finiteness theorem.

Theorem 1 *An elliptic boundary value problem $\mathcal{D} = (D,B)$ defines a Fredholm operator.*

Proof. Denote by D^{-1} the parametrix in the interior. Similarly, the parametrix of the classical boundary value problem on the cylinder $X \times [0,1)$ is denoted by $[(\pi \times 1)_! D, B]^{-1}$. These are pasted together globally on M by the formula

$$\mathcal{D}^{-1} = \psi_1 D^{-1} \varphi_1 + \psi_2 (\pi \times 1)^! [(\pi \times 1)_! D, B]^{-1} \varphi_2.$$

Here the partition of unity $\varphi_{1,2}$ satisfies condition

$$\varphi_1 \equiv 0 \text{ near the boundary, } \varphi_2 \equiv 0 \text{ far from the boundary.} \tag{9}$$

Both $\psi_{1,2}$ satisfy the equality

$$\psi_i \equiv 1 \quad \text{on} \quad \text{supp}\varphi_i$$

and (9). The functions are shown on Fig. 1. Obviously, $\mathcal{D}^{-1}$ is a two-sided parametrix for $\mathcal{D}$.

The theorem is proved.

□

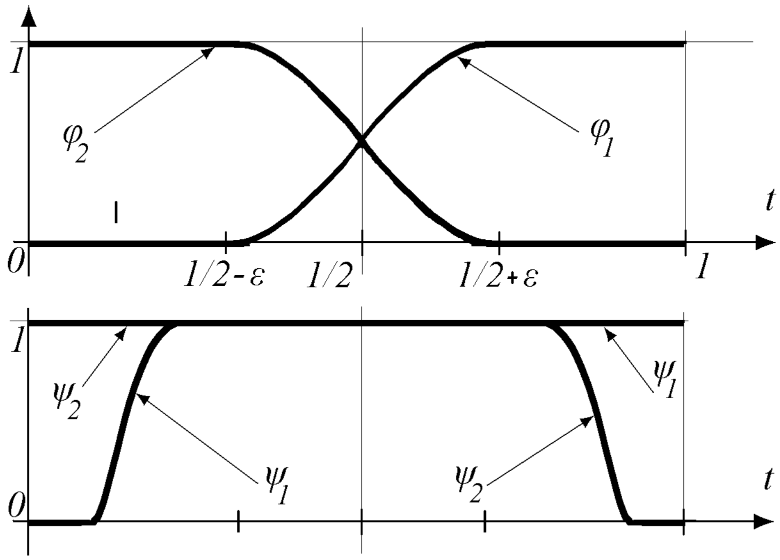

Figure 1: Cut-off functions

2 Homotopy classification

In the previous section, a nonlocal boundary value problem was represented near the boundary in terms of an equivalent classical boundary value problem. Therefore, we can apply the well-known topological methods (e.g., see [14, 26]) to the index problem for nonlocal boundary value problems.

1. Reduction to zero-order operators. We introduce a class of nonlocal operators. A linear operator

$$D : C^\infty(M,E) \longrightarrow C^\infty(M,F)$$

is said to be *admissible* if it can be represented in the form

$$D = \psi_1 D' \varphi_1 + \psi_2 (\pi \times 1)^! D'' \varphi_2, \tag{10}$$

for cut-off functions $\varphi_{1,2}, \psi_{1,2}$, as in the proof of Theorem 1. Here D' denotes a pseudodifferential operator on M

$$D' : C^\infty(M,E) \longrightarrow C^\infty(M,F),$$

while

$$D'' : C^\infty(X \times [0,1), \pi_! E) \longrightarrow C^\infty(X \times [0,1), \pi_! F)$$

is a pseudodifferential operator with continuous symbol (in the sense of [1]) on $X \times [0,1)$. Let us suppose additionally that near the boundary D'' is a differential operator in the normal direction to the boundary:

$$D''|_{t \le \varepsilon} = \sum_{k=0}^{m} D_k(t) \left(-i\frac{\partial}{\partial t}\right)^{m-k}, \tag{11}$$

where $D_k(t)$ stand for smooth families of pseudodifferential operators on X of order k. Moreover, $D_0(t)$ is a vector bundle homomorphism.

In other words, admissible operators have the following description.

1. Far from the boundary, an admissible operator is a usual pseudodifferential operator;
2. In the neighborhood of the boundary an admissible operator is represented as the inverse image of a differential operator in the normal direction on the cylinder $X \times [0,1)$;
3. In the domain $\partial M \times (0,1)$ an admissible operator is the inverse image of a pseudodifferential operator with a continuous symbol.

The statement of nonlocal boundary value problems, as well as the ellipticity condition extend straightforwardly to the class of admissible operators (cf. a similar generalization [13] in the classical case).

Example 2 Let $E \in \mathrm{Vect}(M)$ be a vector bundle. Suppose that its direct image is decomposed as a sum of subbundles

$$\pi_! \, E|_{\partial M} = E_+ \oplus E_-, \; E_\pm \in \mathrm{Vect}(X).$$

Let us define a nonlocal operator $D_\pm : C^\infty(M,E) \to C^\infty(M,E)$ by an expression

$$D_\pm = \psi_2 (\pi \times 1)^! \left[\left(-i\frac{\partial}{\partial t} + i\sqrt{\Delta_{X,E_+}} \right) \oplus \left(i\frac{\partial}{\partial t} + i\sqrt{\Delta_{X,E_-}} \right) \right] \varphi_2 + \psi_1 i \sqrt{\Delta_M} \varphi_1 \quad (12)$$

(here Δ denote nonnegative Laplacians of the corresponding manifolds, while the cut-off functions are chosen as before). Operator $D_\pm$ is an elliptic admissible operator. We equip it with the Dirichlet boundary condition

$$P_{E_-} \beta \; u|_{\partial M} = g \in C^\infty(X, E_-),$$

where $P_{E_-} : \pi_! \, E|_{\partial M} \to \pi_! \, E|_{\partial M}$ is a projection on the subbundle E_-. Denote this boundary value problem by $\mathcal{D}_\pm$. Similarly to the classical case (e.g., see [13]), one proves that this boundary value problem has index zero.

This class contains boundary value problems corresponding to the decomposition

$$\pi_! \, E|_{\partial M} = E_+, \qquad E_- = 0.$$

In this case, the operator, denoted by D_+, has the Fredholm property without the boundary condition.

Denote by $\mathrm{Ell}^m(M,\pi)$, $m \geq 1$ the Grothendieck group of homotopy classes of elliptic boundary value problems for admissible operators of order m, modulo boundary value problems $\mathcal{D}_\pm \circ D_+^{m-1}$. The group of stable homotopy classes of zero-order admissible elliptic operators is denoted by $\mathrm{Ell}^0(M,\pi)$ (recall that the stabilization is taken with respect to operators induced by vector bundle isomorphisms). It should be noted that elliptic operators of order zero do not require boundary conditions, since near the boundary they are induced by vector bundle isomorphisms.

Similarly to the classical theory (see [14] or [13]), the order of a nonlocal boundary value problem can be reduced to zero by means of stable homotopies. More precisely, one has

Theorem 2 (order reduction) *Composition with operator D_+ induces an isomorphism*

$$\begin{array}{rcl}\times D_+^m : \mathrm{Ell}^0(M,\pi) & \longrightarrow & \mathrm{Ell}^m(M,\pi),\\ [D] & \mapsto & \left[D\circ D_+^m\right].\end{array}$$

Proof of this result is a straightforward generalization of the proof in the classical case (see [14]). For nonlocal boundary value problems there are no significant changes in the proof. Therefore, we omit the proof in this paper.

□

Let us now give a homotopy classification for zero-order operators.

2. Difference construction. Let us cut M into two pieces

$$M' = M\backslash\{\partial M\times[0,1)\}\simeq M \qquad \text{and} \qquad \partial M\times[0,1].$$

Then, the symbol $\sigma(D)$ of an admissible elliptic operator D of order zero is naturally represented as a pair of usual symbols

$$\sigma(D)|_{M'} \text{ and } (\pi\times 1)_!\,\sigma(D)|_{\partial M\times[0,1]}. \tag{13}$$

Both have difference constructions

$$[\sigma(D)|_{M'}]\in K(T^*M'), \qquad \left[(\pi\times 1)_!\,\sigma(D)|_{\partial M\times[0,1]}\right]\in K(T^*(X\times(0,1]))$$

(here and in what follows $K(Y)$ denotes K-group with compact support). Note that in the latter case an elliptic symbol $(\pi\times 1)_!\,\sigma(D)$ of order zero is invertible over the base $X\times\{0\}$ of $X\times[0,1]$ (this follows from the decomposition (11)).

However, it is impossible to define a unique element of a single K-group, since the manifolds T^*M' and $T^*(X\times(0,1])$ can not be glued smoothly together (their boundaries are nondiffeomorphic). It turns out that the gluing can be realized for the function algebras, rather than for the manifolds themselves.

Indeed, on a disjoint union of the spaces, consider an algebra

$$C_0(T^*M')\oplus C_0(T^*(X\times(0,1]),\mathrm{End}\,p^*\pi_!1)$$

of continuous functions vanishing at infinity. Note that on the second space we consider functions with values in endomorphisms of the bundle $\pi_!1\in\mathrm{Vect}(X)$ (the direct image of the trivial line bundle), and

$$p: T^*(X\times(0,1])\to X$$

is the natural projection. Consider a subalgebra

$$\mathcal{A}_{T^*M,\pi} \subset C_0\left(T^*M'\right) \oplus C_0\left(T^*\left(X \times (0,1]\right), \mathrm{End} p^*\pi_!1\right),$$

defined by the condition of coincidence on the boundary

$$\mathcal{A}_{T^*M,\pi} = \left\{u \oplus v \mid \beta\, u|_{\partial M'} \beta^{-1} = v|_{t=1}\right\}$$

(here t is the coordinate along the generatrix of $X \times (0,1]$).

Remark 1 For the identity covering $\partial M \to \partial M = X$ this algebra coincides with the commutative algebra of continuous functions on $T^*\left(M \backslash \partial M\right)$ vanishing at infinity.

Let us define the *difference construction* for nonlocal boundary value problems. It is a mapping

$$\chi : \mathrm{Ell}^0\left(M, \pi\right) \longrightarrow K_0\left(\mathcal{A}_{T^*M,\pi}\right) \tag{14}$$

to the group K_0 (e.g., see [15]) of the C^*-algebra $\mathcal{A}_{T^*M,\pi}$. We now give an explicit formula for the element $\chi\left[D\right]$ corresponding to

$$D : C^\infty\left(M, E\right) \longrightarrow C^\infty\left(M, F\right).$$

To this end, let us fix embeddings of E, F into the trivial bundles of a sufficiently large dimension. Let

$$P_E, P_F : \mathbb{C}^{N+L} \longrightarrow \mathbb{C}^{N+L}$$

be the projections defining the corresponding subbundles

$$E \simeq \mathrm{Im}\, P_E \subset \mathbb{C}^N \oplus 0, \qquad F \simeq \mathrm{Im}\, P_F \subset 0 \oplus \mathbb{C}^L.$$

Denote the direct images of these projections by $P_{\pi_!E}, P_{\pi_!F}$. These define the direct images of the corresponding subbundles in the neighborhood of the boundary.

We define now the difference construction of D

$$\chi\left[D\right] \in K_0\left(\mathcal{A}_{T^*M,\pi}\right)$$

by the formula

$$\chi\left[D\right] = \left[P_1 \oplus P_2\right] - \left[P_F \oplus P_{\pi_!F}\right],$$

where the projection P_1 over M' is

$$P_1 = \begin{cases} P_E \cos^2|\xi| + P_F \sin^2|\xi| + P_F \sigma\left(D\right)\left(x,\xi\right) P_E \sin 2|\xi|, & |\xi| \leq \pi/2, \\ P_F & |\xi| > \pi/2 \end{cases} \tag{15}$$

(we suppose that the principal symbol $\sigma\left(D\right)$ is homogeneous of order zero in ξ), and the projection P_2 over $X \times [0,1]$ is defined as

$$\begin{cases} P_{\pi_!E} \cos^2|\xi| + P_{\pi_!F} \sin^2|\xi| + P_{\pi_!F}\widetilde{\sigma}\left(x',\xi\right) P_{\pi_!E} \sin 2|\xi|, & x' \in X \times [1/2,1], |\xi| \leq \pi/2 \\ P_{\pi_!E} \cos^2\varphi + P_{\pi_!F} \sin^2\varphi + P_{\pi_!F}\widetilde{\sigma}\left(x',0\right) P_{\pi_!E} \sin 2\varphi, & x' \in X \times [0,1/2], |\xi| < \pi t \\ P_{\pi_!F}, & \text{otherwise.} \end{cases}$$

Here we suppose for brevity

$$\varphi = |\xi| + \pi/2\,(1-2t)\,, \quad \widetilde{\sigma}\,(x',\xi) = (\pi\times 1)_!\,\sigma\,(D)\,(x',\xi)\,.$$

Geometrically these projections define a subbundle coinciding over the zero section (for $|\xi|=0$) with $E\subset\mathbb{C}^{N+L}$, for $|\xi|\geq\pi/2$ coinciding with the orthogonal bundle $F\subset\mathbb{C}^{N+L}$, while in the intermediate region the subbundle is a result of rotation of the first bundle towards the second by means of the symbol $\sigma\,(D)$ defining their isomorphism.

By definition, the projections P_1 and P_F coincide outside a compact set in T^*M', while P_2 and $P_{\pi_!F}$ coincide outside a compact set in $T^*\,(X\times(0,1])$. In other words, the difference $[P_1\oplus P_2]-[P_F\oplus P_{\pi_!F}]$ is indeed an element of the group $K_0\,(\mathcal{A}_{T^*M,\pi})$.

Remark 2 The constructed element of the K-group of a C^*-algebra can be equivalently defined by different expressions (cf. [16]).

Theorem 3 *The difference construction* (14) *is a well-defined group isomorphism.*

Proof. The mapping χ preserves the equivalence relations in the groups $\mathrm{Ell}^0\,(M,\pi)$ and $K\,(\mathcal{A}_{T^*M,\pi})$. Indeed, under an operator homotopy the symbols vary continuously. Therefore, the corresponding projections $P_{1,2}$ are joined by a continuous homotopy. Furthermore, the element $\chi\,[D]$ is independent of the choice of embedding in a trivial bundle, since all such embeddings are homotopic, while for a trivial D (i.e., one induced by a vector bundle isomorphism) $\chi[D]$ is equal to zero. This shows that χ is well defined. The proof that this mapping is one-to-one meets no essential difficulties and is left to the reader.

□

3. An exact sequence. The algebra $\mathcal{A}_{T^*M,\pi}$ has an ideal

$$I = C_0\,(T^*\,(X\times(0,1))\,,\mathrm{End}p^*\pi_!1)\,.$$

The short exact sequence of algebras

$$0\to I\to\mathcal{A}_{T^*M,\pi}\to\mathcal{A}_{T^*M,\pi}/I\to 0$$

induces a long exact sequence in K-theory

$$\ldots\longrightarrow K_0\,(I)\longrightarrow K_0\,(\mathcal{A}_{T^*M,\pi})\longrightarrow K_0\,(\mathcal{A}_{T^*M,\pi}/I)\longrightarrow K_1\,(I)\longrightarrow\ldots. \tag{16}$$

The quotient algebra is obviously isomorphic to

$$\mathcal{A}_{T^*M,\pi}/I\simeq C_0\,(T^*M)\,.$$

On the other hand, for a vector bundle $G\in\mathrm{Vect}\,(Y)$ one has isomorphisms

$$K_*\,(C_0\,(Y,\mathrm{End}G))\simeq K_*\,(C_0\,(Y))\simeq K^*\,(Y)$$

(the first isomorphism takes a projection in a direct sum of G to the subbundle it defines). Hence, (16) acquires the form

$$\to K(T^*X) \xrightarrow{\alpha} K_0(\mathcal{A}_{T^*M,\pi}) \to K(T^*M) \to K^1(T^*X) \to \dots \tag{17}$$

The first mapping of this sequence has a natural realization in terms of elliptic operators. Namely, let us define a commutative square

$$\begin{array}{ccc} \mathrm{Ell}(X) & \xrightarrow{\alpha'} & \mathrm{Ell}(M,\pi), \\ \simeq\downarrow & & \downarrow\simeq \\ K(T^*X) & \xrightarrow{\alpha} & K_0(\mathcal{A}_{T^*M,\pi}). \end{array} \tag{18}$$

Here $\mathrm{Ell}(X)$ denotes the group of stable homotopy classes of elliptic operators on a closed manifold, and the isomorphisms are defined by the difference constructions. The mapping α' in elliptic theory is defined as follows. Consider an elliptic operator

$$B : C^\infty(X, G) \longrightarrow C^\infty\left(X, \mathbb{C}^k\right)$$

and an embedding of $\mathbb{C}^k$ in a direct sum $N\pi_!1$ (for N large). The mapping takes B to a nonlocal boundary value problem $(D_\pm, B^{-1})$

$$\alpha'[B] = \left[D_\pm, B^{-1}\right], \tag{19}$$

where operator $D_\pm$ and the boundary value problem were defined by Eq. (12), corresponding to the bundle $E = \mathbb{C}^N$ and the decomposition

$$\pi_! E = E_+ \oplus E_-, \quad E_- \simeq \mathbb{C}^k, E_+ = E_-^\perp.$$

In other words, an operator is taken to a boundary value problem, where the initial operator enters as the boundary condition.

A computation shows that α' makes the diagram (18) commute.

3 Index theorem

1. Classifying space for coverings. Denote by BS_n the classifying space of the symmetric group S_n. The universal principal bundle $ES_n \to BS_n$ has an associated covering with n sheets

$$ES_n \underset{\rho}{\times} n \longrightarrow BS_n \tag{20}$$

corresponding to the tautological representation $\rho = Id : S_n \to S_n$ as permutations of n elements. For instance, for $n = 2$ we have $S_2 = \mathbb{Z}_2$ and the covering coincides with the universal bundle

$$ES_2 \underset{\rho}{\times} 2 = \mathbb{S}^\infty \longrightarrow \mathbb{RP}^\infty = BS_2.$$

Lemma 1 *The homotopy groups of the space $ES_n \underset{\rho}{\times} n$ are trivial.*

Proof. The classifying space of a finite group is an Eilenberg–MacLane complex: $BS_n = K(S_n, 1)$. Thus the homotopy groups $\pi_i(ES_n \times_\rho n)$ of the covering space are trivial for $i \geq 2$. The triviality of the group π_1 and the set π_0 follows from the exact sequence

$$\pi_1(n) \to \pi_1 \left(ES_n \underset{\rho}{\times} n \right) \to \pi_1(BS_n) \to \pi_0(n) \to \pi_0 \left(ES_n \underset{\rho}{\times} n \right) \to \pi_0(BS_n).$$

□

The contractibility of $ES_n \times_\rho n$ shows that the covering (20) is universal, i.e. an arbitrary n-sheeted covering $\pi : Y \to X$ is induced by (20) under a mapping $f : X \to BS_n$ unique up to homotopy. In other words, one has a bijective mapping

$$\mathrm{Cov}^n(X) \simeq [X, BS_n],$$

where $\mathrm{Cov}^n(X)$ denotes the set of isomorphism classes of coverings over X with n sheets.

2. Universal space for manifolds with a covering on the boundary. Consider a half-infinite cylinder

$$\left(ES_n \underset{\rho}{\times} n \right) \times [0, \infty). \tag{21}$$

Its boundary $ES_n \times_\rho n$ is a covering over the classifying space BS_n.

There are natural finite-dimensional models for our infinite-dimensional spaces (e.g., see [17]). Consider an embedding $S_n \subset U(n)$ of the symmetric group in the unitary group. It is induced by the permutations of the elements in the base. We define closed smooth manifolds

$$(ES_n)_N = V_{N,n}, \quad (BS_n)_N = V_{N,n}/\, S_n.$$

Here $V_{N,n}$ is the Stiefel manifold of n-frames in $\mathbb{C}^N$ with the free action of the symmetric group. The corresponding projection is denoted by π'

$$\pi' : V_{N,n} \longrightarrow V_{N,n}/\, S_n.$$

Finite-dimensional manifold $((ES_n)_N \times_\rho n) \times [0, \infty)$ is denoted for brevity by $\mathcal{M}_N$, while the original space (21) by $\mathcal{M}_\infty$.

Consider the sequence (17) on $\mathcal{M}_N$. The groups $K^i(T^*\mathcal{M}_N)$ are trivial, since $[0, \infty)$ is contractible. Thus, we obtain

$$K_0(\mathcal{A}_{T^*\mathcal{M}_N, \pi'}) \overset{\alpha}{\simeq} K^0(T^*(BS_n)_N) \tag{22}$$

(cf. diagram (18)). In terms of elliptic theory this isomorphism has the following sense: on a half-infinite cylinder one can always reduce a boundary value problem to the boundary.

3. Embeddings of manifolds and the direct image mapping. Consider two pairs (M, π) and (M_1, π_1) of manifolds with coverings on the boundary.

Definition 3 A smooth mapping $f : M \to M_1$ is said to be an *embedding* of (M, π) in (M_1, π_1), if f is an embedding of manifolds with boundary and its restriction $f|_{\partial M} : \partial M \to \partial M_1$ is induced by an embedding

$$\widetilde{f} : X \to Y$$

of the bases. In other words, we assume there is a commutative diagram

$$\begin{array}{ccc} M & \stackrel{f}{\longrightarrow} & M_1 \\ \cup & & \cup \\ \partial M & \stackrel{f|_{\partial M}}{\longrightarrow} & \partial M_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ X & \stackrel{\tilde{f}}{\longrightarrow} & Y, \end{array}$$

where the horizontal mappings are embeddings.

For an embedding $f : M \to M_1$, one has the *direct image mapping*

$$f_! : K_0 \left(\mathcal{A}_{T^*M,\pi}\right) \longrightarrow K_0 \left(\mathcal{A}_{T^*M_1,\pi_1}\right) \tag{23}$$

defined by the usual formulas. More precisely, making use of the difference construction (14), an element of $K_0 \left(\mathcal{A}_{T^*M,\pi}\right)$ can be defined as a pair of usual elliptic symbols

$$\sigma(D)|_{M'} \text{ and } (\pi \times 1)_! \, \sigma(D)|_{\partial M \times [0,1]}.$$

A direct check shows that the explicit formula for the direct image [1] maps such a pair to a similar pair on the target manifold M_1.

Let us now consider an arbitrary pair (M, π). The covering $\pi : \partial M \to X$ is induced by a mapping

$$\widetilde{f} : X \longrightarrow BS_n$$

of X to the classifying space BS_n. The corresponding mapping of the total spaces of the coverings

$$\partial M \longrightarrow ES_n \underset{\rho}{\times} n$$

extends to M, since $ES_n \times_\rho n$ is contractible. Hence, the pair (M, π) defines a mapping

$$M \longrightarrow \mathcal{M}_\infty$$

that on the boundary induces the covering π. This mapping is unique up to homotopy. In this sense the *space* $\mathcal{M}_\infty$ *is universal for manifolds with a fibered boundary* with the fiber consisting of n points. For N sufficiently large, the mapping to the universal space can be approximated by an embedding of M in a finite-dimensional submanifold $\mathcal{M}_N \subset \mathcal{M}_\infty$

$$f : M \longrightarrow \mathcal{M}_N. \tag{24}$$

Remark 3 This embedding result (24) is similar to the Whitney embedding theorem.

4. Index theorem. Embedding (24) makes it possible to prove the following index theorem.

Theorem 4 *There is a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Ell}\,(M,\pi) & & \xrightarrow{\quad\chi\quad} & & K_0\left(\mathcal{A}_{T^*M,\pi}\right) \\ \downarrow{\scriptstyle \mathrm{ind}} & & & & \downarrow{\scriptstyle f_!} \\ \mathbb{Z} & \xleftarrow{\ p_!\ } & K^0\left(T^*\left(BS_n\right)_N\right) & \overset{\alpha}{=\!=\!=} & K_0\left(\mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right) \end{array} \tag{25}$$

where the Atiyah–Singer direct image mapping $p_!$ *is induced by the projection*

$$p : \left(BS_n\right)_N \to pt.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathrm{Ell}\,(M,\pi) & & \xrightarrow{\quad\chi_M\quad} & & K_0\left(\mathcal{A}_{T^*M,\pi}\right) \\ \Big\downarrow & \searrow^{\mathrm{ind}} & & \swarrow & \Big\downarrow{\scriptstyle f_!} \\ & & \mathbb{Z} & & \\ & \nearrow_{\mathrm{ind}} & & \nwarrow_{\mathrm{ind}_t} & \\ \mathrm{Ell}\,(\mathcal{M}_N,\pi') & & \xrightarrow{\quad\chi_{\mathcal{M}_N}\quad} & & K_0\left(\mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right). \end{array} \tag{26}$$

Here the mappings ind stand for the analytic index, while the mapping ind_t is defined as the composition in the bottom row of diagram (25), and the direct image mapping in elliptic theory (the vertical map in the left part of the diagram) is defined in terms of the topological direct image $f_!$ as a composition

$$\left(\chi_{\mathcal{M}_N}\right)^{-1} f_!\chi_M : \mathrm{Ell}\,(M,\pi) \longrightarrow \mathrm{Ell}\,(\mathcal{M}_N,\pi').$$

In terms of this diagram, the theorem claims the commutativity of the big upper triangle

$$\begin{array}{ccc} \mathrm{Ell}\,(M,\pi) & \xrightarrow{\quad\chi_M\quad} & K_0\left(\mathcal{A}_{T^*M,\pi}\right) \\ & \searrow^{\mathrm{ind}} & \Big\downarrow{\scriptstyle f_!} \\ & \mathbb{Z} \xleftarrow{\ \mathrm{ind}_t\ } & K_0\left(\mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right). \end{array}$$

In its turn, the big triangle commutes, since two smaller triangles are commutative:

1) The commutativity of the triangle

$$
\begin{array}{ccc}
\mathrm{Ell}\,(M,\pi) & & \\
\Big\downarrow & \overset{\mathrm{ind}}{\searrow} & \\
 & & \mathbb{Z} \\
 & \overset{\mathrm{ind}}{\nearrow} & \\
\mathrm{Ell}\,(\mathcal{M}_N,\pi') & &
\end{array}
$$

is equivalent to the statement that the index is preserved under the direct image. The proof of this fact is the same as in the classical case, see [1].

2) To prove that the lower triangle in (26) is commutative, let us make use of the isomorphism α

$$K_0\left(\mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right) \overset{\alpha}{\simeq} K^0\left(T^*\left(BS_n\right)_N\right)$$

on $\mathcal{M}_N$. Then, we obtain the desired from the diagram

$$
\begin{array}{ccc}
\mathrm{Ell}\,(\mathcal{M}_N,\pi') & \xrightarrow{\chi_{\mathcal{M}_N}} & K_0\left(\mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right) \\
\simeq\Big\uparrow & & \Big\uparrow\alpha \\
\mathrm{Ell}\,((BS_n)_N) & \xrightarrow{\simeq} & K^0\left(T^*\left(BS_n\right)_N\right) \\
\mathrm{ind}\Big\downarrow & & \Big\downarrow p_! \\
\mathbb{Z} & = \!=\!=\!=\!= & \mathbb{Z},
\end{array}
$$

where the upper square coincides with the square (18) defined previously, while the lower square commutes according to the Atiyah–Singer index formula on $(BS_n)_N$.

The theorem is thereby proved.

$\square$

One can write the index formula in terms of an integral over the manifold.

Corollary 1 *One has*

$$\mathrm{ind} D = \int\limits_{T^*M'} \mathrm{ch}\,[\sigma(D)]\,\mathrm{Td}\,(T^*M\otimes\mathbb{C}) + \int\limits_{T^*(X\times[0,1])} \mathrm{ch}\,[(\pi\times 1)_!\,\sigma(D)]\,\mathrm{Td}\,(T^*X\otimes\mathbb{C}), \quad (27)$$

for an elliptic zero-order operator, where the characteristic classes ch *and* Td *are represented by differential forms, corresponding to the product metric*

$$dt^2 + \pi^* g_X \quad (28)$$

on a collar neighborhood of the boundary and g_X is a metric on the base X.

The right-hand side of the equality is preserved under homotopies and the direct image mapping on elliptic symbols. In addition, it coincides with the index for cylindrical manifolds. Indeed, an admissible operator on such manifold can be considered globally as the inverse image of an operator on the base of the covering. Thus, the first term in (27) is equal to zero and the sum reduces to the Atiyah–Singer formula on the cylinder $X \times (0, \infty)$. Similarly to the proof of the index formula, this implies that (27) is valid for an arbitrary manifold.

5. Embedding in the universal space. We can pass to the limit as $N \to \infty$ in the statement of the index theorem. To this end, consider an infinite sequence of embedded manifolds

$$\mathcal{M}_N \subset \mathcal{M}_{N+1} \subset \ldots \subset \mathcal{M}_\infty = \left(ES_n \underset{\rho}{\times} n \right) \times [0, \infty) .$$

There is a sequence of their algebras

$$\mathcal{A}_{T^*\mathcal{M}_N, \pi'}, \quad \mathcal{A}_{T^*\mathcal{M}_{N+1}, \pi'}, \quad \ldots$$

The direct image as $N \to \infty$ for the corresponding K-groups under the mapping (23) is denoted by

$$K_0 \left(\mathcal{A}_{T^*\mathcal{M}_\infty} \right) = \varinjlim K_0 \left(\mathcal{A}_{T^*\mathcal{M}_N, \pi'} \right) .$$

Lemma 2 *There is an isomorphism*

$$K_0 \left(\mathcal{A}_{T^*\mathcal{M}_\infty} \right) = \mathbb{Z}.$$

Moreover, the elements are classified by the index.

The proof is given in the Appendix.

Thus, we can write the index theorem in terms of a commutative square

$$\begin{array}{ccc} \mathrm{Ell}\,(M, \pi) & \overset{\chi}{\longrightarrow} & K_0 \left(\mathcal{A}_{T^*M, \pi} \right) \\ \mathrm{ind} \downarrow & & \downarrow f_! \\ \mathbb{Z} & \simeq & K_0 \left(\mathcal{A}_{T^*\mathcal{M}_\infty} \right) . \end{array}$$

6. Example. *Manifolds with reflecting boundary*, see [18]. Let M be a $4k$-dimensional compact oriented Riemannian manifold with boundary ∂M. Suppose that ∂M has an orientation-reversing smooth involution

$$G : \partial M \longrightarrow \partial M, \quad G^2 = Id$$

without fixed points. The involution defines a free action of the group $\mathbb{Z}_2$, giving a two-sheeted covering

$$\pi : \partial M \longrightarrow \partial M / \mathbb{Z}_2.$$

Consider the Hirzebruch operator

$$d + d^* : \Lambda^+ (M) \longrightarrow \Lambda^- (M) ,$$

where $\Lambda^{\pm}(M)$ are the ± 1-eigenspaces of the involution

$$\alpha : \Lambda^*(M) \longrightarrow \Lambda^*(M)$$

defined on the exterior forms as

$$\alpha|_{\Lambda^p(M)} = (-1)^{\frac{p(p-1)}{2}+k} * .$$

On the boundary, we have isomorphisms

$$\Lambda^{\pm}(M)\big|_{\partial M} \simeq \Lambda^*(\partial M).$$

Let us fix a metric in a neighborhood of the boundary such that it lifts from $[0,1]\times\partial M/\mathbb{Z}_2$. Then the Hirzebruch operator can be expressed as (see [5])

$$\frac{\partial}{\partial t} + A$$

(up to a bundle isomorphism) for an elliptic self-adjoint operator on the boundary

$$A : \Lambda^*(\partial M) \longrightarrow \Lambda^*(\partial M),$$

$$A\omega = (-1)^{k+p}(d * -\varepsilon * d)\,\omega,$$

where for an even degree form $\omega \in \Lambda^{2p}(\partial M)$ we put $\varepsilon = 1$, and $\varepsilon = -1$ otherwise. Since G reverses the orientation, we have anticommutativity

$$G^* A = -AG^*.$$

It is known that the Dirac operator never has well-posed boundary conditions. However, it admits the following nonlocal boundary value problem

$$\begin{cases} (d+d^*)\,\omega = f, \\ \frac{(1+G^*)}{2}\,\omega|_{\partial M} = g, \qquad g \in \Lambda^*(\partial M)^{\mathbb{Z}_2} \simeq \Lambda^*(\partial M/\mathbb{Z}_2) \end{cases} \tag{29}$$

on a manifold with reflecting boundary. Here $\Lambda^*(\partial M)^{\mathbb{Z}_2}$ denotes the subspace of G-invariant forms on the boundary.

Proposition 1 *The nonlocal boundary value problem* (29) *is elliptic.*

Proof. Indeed, consider an arbitrary point $x \in \partial M/\mathbb{Z}_2$. An explicit computation shows that in a neighborhood of the point the classical boundary value problem, corresponding to the nonlocal problem has the form

$$\begin{cases} \left(\frac{\partial}{\partial t} + A\right)\omega_1 = f_1, \quad \left(\frac{\partial}{\partial t} - A\right)\omega_2 = f_2, \\ \omega_1|_{\partial M/\mathbb{Z}_2} + \omega_2|_{\partial M/\mathbb{Z}_2} = g. \end{cases}$$

Its ellipticity (Shapiro–Lopatinskii condition) is fulfilled since the symbol of the operator of boundary conditions defines an isomorphism of spaces

$$\operatorname{Im}\sigma(\Pi_+)(x,\xi)\oplus\operatorname{Im}\sigma(\Pi_-)(x,\xi)\simeq\Lambda^*(\partial M)_x,$$

at an arbitrary point $(x,\xi)\in S^*\partial M/\mathbb{Z}_2$ of the cosphere bundle, where Π_+ stands for the nonnegative spectral projection of A

$$\Pi_+=\frac{A+|A|}{2|A|},$$

and $\Pi_-=1-\Pi_+$ is the negative projection.

Thus, we obtain the desired Fredholm property, since the corresponding classical boundary value problem is elliptic.

□

Theorem 5 *The following equality is valid*

$$\operatorname{ind}\left(d+d^*,\frac{(1+G^*)}{2}\right)=\operatorname{sign}M,$$

where $\operatorname{sign}M$ *is the signature of* M.

Proof. The symbol of the problem (29) coincides with the symbol of the composition of the spectral Atiyah–Patodi–Singer boundary value problem

$$\begin{cases}(d+d^*)\omega=f\\ \Pi_+\,\omega|_{\partial M}=\omega', \qquad \omega'\in\operatorname{Im}\Pi_+\subset\Lambda^*(\partial M),\end{cases}$$

and the Fredholm operator

$$\frac{(1+G^*)}{2}:\operatorname{Im}\Pi_+\longrightarrow\Lambda^*(\partial M)^{\mathbb{Z}_2}. \tag{30}$$

Let us compute both indices.

1) For the index of the spectral boundary value problem one has (see [5])

$$\operatorname{ind}(d+d^*,\Pi_+)=\operatorname{sign}M-\frac{\dim\ker A}{2}.$$

In addition, by Hodge–de Rham theory we obtain

$$\frac{\dim\ker A}{2}=\frac{\dim H^*(\partial M)}{2},$$

where $H^*(\partial M)$ denotes cohomology.

2) On the other hand, it is easy to check that the operator (30) is surjective, while its kernel coincides with the space of G-antiinvariant harmonic forms. The Hodge operator $*$ interchanges the anti- and invariant subspaces. Thus, we obtain

$$\dim\ker\left(1+G^*\right)|_{\operatorname{Im}\Pi_+(A)} = \frac{\dim\ker A}{2}.$$

Combining the index of the spectral problem and the index of $(1+G^*)$, we have the desired

$$\operatorname{ind}\left(d+d^*, \frac{(1+G^*)}{2}\right) = \operatorname{sign} M - \frac{\dim\ker A}{2} + \frac{\dim\ker A}{2} = \operatorname{sign} M.$$

The theorem is proved.

□

Remark 4 One can also obtain a local expression (see [18]) for the index

$$\operatorname{ind}\left(d+d^*, \frac{(1+G^*)}{2}\right) = \operatorname{sign} M = \int_M \widehat{L}\left(p_1,\ldots,p_k\right)(M),$$

as an integral over M of the Hirzebruch polynomial in Pontryagin forms, provided near the boundary we have a metric as in (28).

Later in Section 5 we will use a families index formula. Let us briefly state the corresponding results.

5. Index theorem for families. Let P be a compact space. Denote by $\operatorname{Ell}_P(M,\pi)$ the group of stable homotopy classes of elliptic families on M parametrized by P. One has a difference construction

$$\chi_P : \operatorname{Ell}_P(M,\pi) \longrightarrow K_0\left(C\left(P, \mathcal{A}_{T^*M,\pi}\right)\right),$$

where $C\left(P, \mathcal{A}_{T^*M,\pi}\right)$ is the algebra of continuous functions on P with values in the C^*-algebra $\mathcal{A}_{T^*M,\pi}$. One also defines the direct image mapping for an embedding $M \subset M'$ and an isomorphism on a half-infinite cylinder

$$K_0\left(C\left(P, \mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right)\right) \simeq K^0\left(P \times T^*\left(BS_n\right)_N\right),$$

generalizing (22).

The following index theorem is valid.

Theorem 6 (index for families of nonlocal operators) *There is a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Ell}_P(M,\pi) & \xrightarrow{\quad\chi\quad} & K_0\left(C\left(P, \mathcal{A}_{T^*M,\pi}\right)\right) \\
\downarrow{\scriptstyle \operatorname{ind}} & & \downarrow{\scriptstyle f_!} \\
K^0(P) \xleftarrow{\;p_!\;} & K^0\left(P \times T^*\left(BS_n\right)_N\right) \overset{\alpha}{=\!=} & K_0\left(C\left(P, \mathcal{A}_{T^*\mathcal{M}_N,\pi'}\right)\right).
\end{array}$$

4 Homotopy invariant for manifolds with covering on the boundary

1. Class of operators. On a manifold M with a covering π on the boundary, we consider elliptic differential operators

$$D : C^\infty(M,E) \longrightarrow C^\infty(M,F)$$

that in a neighborhood of the boundary are pull-backs from the base of the covering. Technically, we suppose that the following condition is satisfied.

Assumption 1. The restrictions of E and F to the boundary are pull-backs from the base. We fix isomorphisms

$$E|_{\partial M} \simeq \pi^* E_0, \quad F|_{\partial M} \simeq \pi^* F_0, \qquad E_0, F_0 \in \operatorname{Vect}(X).$$

Secondly, the direct image $(\pi\times 1)_! D$ of the initial operator D in a neighborhood of the boundary satisfies a commutative diagram

$$\begin{array}{ccc} C^\infty(X\times[0,1), \pi_! E) & \xrightarrow{(\pi\times 1)_! D} & C^\infty(X\times[0,1), \pi_! F) \\ \simeq\downarrow & & \downarrow\simeq \\ C^\infty(X\times[0,1), E_0\otimes\pi_! 1) & \xrightarrow{D_0\otimes 1} & C^\infty(X\times[0,1), F_0\otimes\pi_! 1) \end{array} \qquad (31)$$

for an operator

$$D_0 : C^\infty(X\times[0,1), E_0) \longrightarrow C^\infty(X\times[0,1), F_0)$$

on the cylinder with boundary X. In the diagram (31) $D_0\otimes 1$ stands for D_0 with coefficients in the flat bundle $\pi_! 1$ (e.g., see [19]).

We will also suppose that D has order one and the following assumption is satisfied.

Assumption 2. In the neighborhood $X\times[0,\varepsilon)$ of the boundary the operator has the form

$$D_0|_{X\times[0,\varepsilon)} = \Gamma\left(\frac{\partial}{\partial t} + A_0\right)$$

for a bundle isomorphism Γ, where A_0 is an elliptic self-adjoint first-order operator on X. This operator is called the *tangential operator* corresponding to D_0.

If D satisfies Assumptions 1 and 2 then near the boundary it has the form

$$D = \frac{\partial}{\partial t} + \pi^!(A_0\otimes 1)$$

(up to a vector bundle isomorphism). For brevity, self-adjoint operator $\pi^!(A_0\otimes 1)$ will be denoted by A.

2. Homotopy invariant. We consider the *spectral Atiyah–Patodi–Singer boundary value problem* [5]

$$\begin{cases} Du &= f, \\ \Pi_+ u|_{\partial M} &= g, \qquad g\in \operatorname{Im}\Pi_+ \subset H^{s-1/2}(\partial M, E), \end{cases}$$

where

$$\Pi_+ = \frac{A + |A|}{2|A|}$$

is the nonnegative spectral projection for the self-adjoint operator A. The spectral problem has the Fredholm property. However, its index $\operatorname{ind}(D, \Pi_+)$ is not homotopy invariant. Homotopy invariance can be obtained if the index is modified by suitable η-invariants.

Proposition 2 *The sum*

$$\widetilde{\operatorname{ind}} D \stackrel{def}{=} \bmod n\text{-} \left(\operatorname{ind}(D, \Pi_+) + \eta(A) - n\eta(A_0) \right) \in \mathbb{R}/n\mathbb{Z}, \tag{32}$$

is a homotopy invariant of D. Here n is the number of sheets of the covering, and $\eta(A), \eta(A_0)$ are spectral Atiyah–Patodi–Singer η-invariants of the tangential operators A and A_0.

Proof. Consider the unreduced invariant

$$\operatorname{ind}(D, \Pi_+) + \eta(A) - n\eta(A_0). \tag{33}$$

The results of [19] imply that for a smooth operator family D_t this expression is a piecewise smooth function of the parameter t (the corresponding families of tangential operators are denoted by $A_t, A_{0,t}$).

1) We claim that (33) is a piecewise constant function. The derivative of the η-invariant

$$\frac{d}{dt}\eta(A_t)$$

with respect to t is local, i.e. it is equal to an integral over the manifold of an expression determined by the complete symbol of the tangential family A_t. However, the complete symbols of A_t and $A_{0,t}$ coincide locally by the Assumption 1. Thus, we have

$$\frac{d}{dt}\eta(A_t) = n\frac{d}{dt}\eta(A_{0,t}).$$

Therefore, (33) is a piecewise constant function.

2) Let us show that the jumps of this function are multiples of the number of sheets of the covering. Indeed, for a homotopy D_t the index, as well as the η-invariants, change by the spectral flow of the corresponding families of tangential operators. Hence,

$$\left[\operatorname{ind}(D_t, \Pi_{+,t}) + \eta(A_t) - n\eta(A_{0.t})\right]\Big|_{t=0,1} = (-1+1)\operatorname{sf}(A_t)_{t\in[0,1]} - n\operatorname{sf}(A_{0,t})_{t\in[0,1]} \in n\mathbb{Z},$$

as desired.

□

Remark 5 For a trivial covering, our invariant reduces to the mod n-*index* of Freed-Melrose [6]

$$\mathrm{mod}\, n\text{-ind}(D, \Pi_+) \in \mathbb{Z}_n \subset \mathbb{R}/n\mathbb{Z}.$$

On the other hand, the fractional part of the invariant (32) is the so-called *relative Atiyah–Patodi–Singer* η-*invariant* [20, 19]

$$\{\eta\,(A_0 \otimes 1_{\pi_! 1}) - n\eta\,(A_0)\} \in \mathbb{R}/\mathbb{Z}$$

of A_0 with coefficients in the flat bundle $\pi_! 1 \in \mathrm{Vect}\,(X)$.

The invariant $\widetilde{\mathrm{ind}}$ has an interesting interpretation as an obstruction to extending the covering π to the entire manifold M simultaneously with the extension of D_0 to the base of the corresponding covering. In more detail, suppose that M is the total space of a covering $\widetilde{\pi}$ with base Y such that it induces the original covering π over the boundary

$$\begin{array}{ccc} \partial M & \subset & M \\ \pi \downarrow & & \downarrow \widetilde{\pi} \\ X & \subset & Y. \end{array}$$

Proposition 3 *If* $D : C^\infty(M, E) \to C^\infty(M, F)$ *is a pull-back of an operator* D_0 *from the base* Y*. Then one has*

$$\widetilde{\mathrm{ind}} D = 0.$$

Proof. We claim that in this case the invariant considered is expressed as a true index

$$\widetilde{\mathrm{ind}} D = n \mathrm{ind}\,(D_0, \Pi_{+,0}) = 0 \in \mathbb{R}/n\mathbb{Z}$$

of the Atiyah–Patodi–Singer problem on the base of the covering $\widetilde{\pi}$. Indeed, according to the Atiyah–Patodi–Singer formula (see [5]), the sum

$$\mathrm{ind}\,(D, \Pi_+) + \eta\,(A)$$

is equal to an integral over the manifold of a local expression defined by the complete symbol of D. Since D and D_0 coincide locally, one has

$$\mathrm{ind}\,(D, \Pi_+) + \eta\,(A) = n\,(\mathrm{ind}\,(D_0, \Pi_{+,0}) + \eta\,(A_0))\,.$$

If we place the term $n\eta\,(A_0)$ to the left-hand side of the equality, we get the desired formula.

□

The following section is devoted to the computation of the homotopy invariant (32) in topological terms.

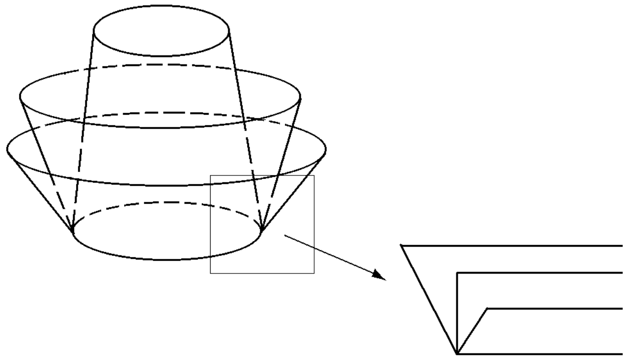

Figure 2: Singular space

5 Index defect formula

1. Difference construction. The pair (M, π) defines a corresponding singular space $\overline{M}^{\pi}$

$$\overline{M}^{\pi} = M/\{x \sim x',\ \text{if } x, x' \in \partial M \text{ and } \pi(x) = \pi(x')\},$$

obtained by identification of the points in the fibers of the covering $\pi : \partial M \to X$ (see Figure 2 in the case of a trivial covering). Similarly, the boundary of noncompact manifold T^*M is also a covering over the product $T^*X \times \mathbb{R}$ and a corresponding singular space $\overline{T^*M}^{\pi}$ is defined.

Consider an elliptic operator D that satisfies the Assumption 1. It is easy to show that its principal symbol defines a K-theory element

$$[\sigma(D)] \in K\left(\overline{T^*M}^{\pi}\right).$$

Thus, we have a homomorphism

$$\begin{array}{rccc} \chi: & \operatorname{Ell}\left(\overline{M}^{\pi}\right) & \longrightarrow & K\left(\overline{T^*M}^{\pi}\right), \\ & \chi[D] & = & [\sigma(D)]. \end{array}$$

Here $\operatorname{Ell}\left(\overline{M}^{\pi}\right)$ is the Grothendieck group of homotopy classes of elliptic operators D on M that satisfy the Assumptions 1 and 2.

The topological formula for the invariant $\widetilde{\operatorname{ind}}$ uses the Poincaré pairing on the manifold $\overline{T^*M}^{\pi}$ with singularities. Let us define this pairing.

2. A pairing in K-theory of a singular manifold. Similarly to the algebra $\mathcal{A}_{T^*M,\pi}$, corresponding to the cotangent bundle, one can define an algebra for M itself

$$\mathcal{A}_{M,\pi} \subset C_0(M') \oplus C_0(X \times (0,1], \operatorname{End}\pi_!1).$$

The subalgebra is specified by the condition

$$\mathcal{A}_{M,\pi} = \left\{ u \oplus v \mid \beta\, u|_{\partial M'} \beta^{-1} = v|_{t=1} \right\}.$$

The K-theory of this algebra has a nice geometric realization.

Lemma 3 *The group $K_0(\mathcal{A}_{M,\pi})$ is isomorphic to the group of stable homotopy classes of triples*

$$(E, F, \sigma), \qquad E, F \in \operatorname{Vect}(M),\ \sigma : \pi_!\, E|_{\partial M} \longrightarrow \pi_!\, F|_{\partial M}$$

and σ is a bundle isomorphism. The trivial triples are those, with σ induced by an isomorphism over M.

Proof. Note that this proposition is similar to the statement of Theorem 3, which can be also considered as giving a realization of the group $K_0(\mathcal{A}_{T^*M,\pi})$ in topological terms. Along the same lines, a triple (E, F, σ) defines the following element of the K_0–group

$$[P_E \oplus P_2] - [P_F \oplus P_{\pi_! F}] \in K_0(\mathcal{A}_{M,\pi}),$$

where the projection P_2 over $X \times [0,1]$ has the form

$$P_2 = P_{\pi_! E} \cos^2 \varphi + P_{\pi_! F} \sin^2 \varphi + P_{\pi_! F} \sigma(x) P_{\pi_! E} \sin 2\varphi, \quad \varphi = \frac{\pi}{2}(1-t).$$

Here $P_E, P_F \subset \mathbb{C}^N$ are projections on subbundles isomorphic to E and F. We also suppose that the subbundles are orthogonal to each other.

The proof that this mapping induces an isomorphism with the group $K_0(\mathcal{A}_{M,\pi})$ is similar to the previous proof and is omitted.

□

This realization makes it possible to define a product

$$K^0\left(\overline{T^*M}^{\pi}\right) \times K_0(\mathcal{A}_{M,\pi}) \longrightarrow K_0(\mathcal{A}_{T^*M,\pi})$$

by means of the construction of a symbol with coefficients in a vector bundle. More precisely, for elements

$$[\sigma] \in K\left(\overline{T^*M}^{\pi}\right), \qquad [E, F, \sigma'] \in K_0(\mathcal{A}_{M,\pi})$$

consider the symbol

$$\sigma \otimes 1_E \oplus \sigma^{-1} \otimes 1_F \tag{34}$$

on M. The direct image of its restriction to the boundary has the form

$$\begin{aligned} \pi_!\left(\sigma \otimes 1_E \oplus \sigma^{-1} \otimes 1_F|_{\partial M}\right) &= \pi_! \sigma \otimes 1_{\pi_! E|_{\partial M}} \oplus \pi_! \sigma^{-1} \otimes 1_{\pi_! F|_{\partial M}} \simeq \\ &\simeq \left(\pi_! \sigma \oplus \pi_! \sigma^{-1}\right) \otimes 1_{\pi_! E|_{\partial M}}. \end{aligned}$$

The latter isomorphism is induced by a vector bundle isomorphism

$$\pi_! \, E|_{\partial M} \overset{\sigma'}{\simeq} \pi_! \, F|_{\partial M} .$$

The symbol $(\pi_!\sigma \oplus \pi_!\sigma^{-1}) \otimes 1_{\pi_! \, E|_{\partial M}}$ is trivially homotopic to the identity

$$\begin{pmatrix} \pi_!\sigma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi_!\sigma^{-1} \end{pmatrix} \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}, \quad \tau \in [0, \pi/2] .$$

Thus, we have extended the symbol (34) up to a nonlocal elliptic symbol on M. Now the desired product is defined as the difference construction of this symbol, which is denoted by

$$[\sigma] \times [E, F, \sigma'] \in K_0 \left(\mathcal{A}_{T^*M,\pi} \right) .$$

The product naturally defines the pairing

$$\langle , \rangle : K^0 \left(\overline{T^*M}^{\pi} \right) \times K_0 \left(\mathcal{A}_{M,\pi} \right) \longrightarrow K_0 \left(\mathcal{A}_{T^*M,\pi} \right) \xrightarrow{\text{ind}_t} \mathbb{Z}$$

as a composition with the topological index mapping of Section 3. This pairing generalizes the Poincaré duality on the manifold $\overline{T^*M}^{\pi}$ with singularities (this is teated in detail in Section 9).

3. Covering on the boundary defines a K-theory element. Let us first consider the classifying space BS_n. The universal vector bundle is denoted by γ. Starting from this bundle, we construct an element

$$[\widetilde{\gamma}] \in K_0 \left(\mathcal{A}_{\mathcal{M}_\infty}, \mathbb{Q}/n\mathbb{Z} \right)$$

of the K-group with coefficients.

To this end, we take finite-dimensional approximations $\mathcal{M}_N$ of the universal space $\mathcal{M}_\infty$ and obtain, similar to (22), an expression of the K-group in topological terms

$$K_* \left(\mathcal{A}_{\mathcal{M}_N,\pi'} \right) \simeq K^{*+1} \left((BS_n)_N \right) .$$

Lemma 4 *The coboundary mapping ∂ of the exact sequence*

$$\to K^1 \left((BS_n)_N \right) \otimes \mathbb{Q} \longrightarrow K^1 \left((BS_n)_N , \mathbb{Q}/n\mathbb{Z} \right) \xrightarrow{\partial} K^0 \left((BS_n)_N \right) \xrightarrow{\times n} K^0 \left((BS_n)_N \right) \otimes \mathbb{Q},$$

corresponding to the monomorphism of coefficient groups $n\mathbb{Z} \subset \mathbb{Q}$, induces an isomorphism

$$\widetilde{K}^0 \left(BS_n \right) \simeq K^1 \left(BS_n, \mathbb{Q}/n\mathbb{Z} \right),$$

where the K-groups of the infinite-dimensional space BS_n are defined as an inverse limit of K-groups of the subspaces $(BS_n)_N$ as $N \to \infty$.

The proof is given in the Appendix.

The universal bundle $\gamma \in \mathrm{Vect}\,(BS_n)$ is an element of the inverse limit

$$[\gamma] - n \in \lim_{\longleftarrow} \widetilde{K}^0\left((BS_n)_N\right).$$

Using the lemma, it defines the desired element

$$[\widetilde{\gamma}] \in K^1\left(BS_n, \mathbb{Q}/n\mathbb{Z}\right) \simeq K_0\left(\mathcal{A}_{\mathcal{M}_\infty}, \mathbb{Q}/n\mathbb{Z}\right).$$

Let us now assume that we are given a pair (M, π), consisting of a manifold with a covering $\pi : \partial M \to X$ on its boundary. Consider the classifying mapping

$$\widetilde{f} : M \longrightarrow \mathcal{M}_\infty.$$

We obtain a well-defined element

$$\widetilde{f}^*\,[\widetilde{\gamma}] \in K_0\left(\mathcal{A}_{M,\pi}, \mathbb{Q}/n\mathbb{Z}\right). \tag{35}$$

Let us denote it for brevity by $\left[\widetilde{\pi_! 1}\right]$. We now give a geometric realization of this element.

Firstly, note that the group $\mathbb{Q}/n\mathbb{Z}$ is a direct limit of finite groups $\mathbb{Z}_{nN}$ with regard for the embeddings

$$\begin{array}{ccc} \mathbb{Z}_{nN} & \subset & \mathbb{Q}/n\mathbb{Z}, \\ x & \mapsto & x/N. \end{array}$$

Thus, the K-theory with coefficients $\mathbb{Q}/n\mathbb{Z}$ is defined as a direct limit

$$K_0\left(\mathcal{A}_{M,\pi}, \mathbb{Q}/n\mathbb{Z}\right) = \lim_{\longrightarrow} K_0\left(\mathcal{A}_{M,\pi}, \mathbb{Z}_{nN}\right)$$

of groups with finite coefficients.

Elements of K-groups with finite coefficients can be defined by means of the following proposition (cf. [21]).

Proposition 4 *A triple* (E, G, σ), *where*

$$E \in \mathrm{Vect}\,(M),\; G \in \mathrm{Vect}\,(X), \qquad \pi_!\left(E|_{\partial M}\right) \stackrel{\sigma}{\simeq} kG, \tag{36}$$

and σ *is a bundle isomorphism, defines an element in* $K_0\left(\mathcal{A}_{M,\pi}, \mathbb{Z}_k\right)$.

Proof. Similarly to the topological case (e.g., see [20]), the groups with coefficients in $\mathbb{Z}_k$ are defined in terms of the *Moore space* M_k of this group by the following formula

$$K_0\left(\mathcal{A}_{M,\pi}, \mathbb{Z}_k\right) = K_0\left(\widetilde{C}_0\left(M_k, \mathcal{A}_{M,\pi}\right)\right), \tag{37}$$

where $C_0\left(M_k, \mathcal{A}_{M,\pi}\right)$ is the algebra of $\mathcal{A}_{M,\pi}$-valued functions on the Moore space that vanish at the marked point.

One can easily generalize Lemma 3 to the case of families. More precisely, the same method shows that the group $K_0\left(\widetilde{C}_0\left(M_k,\mathcal{A}_{M,\pi}\right)\right)$ is isomorphic to the group of stable homotopy classes of triples (E',F',σ') where $E',F'\in \mathrm{Vect}\,(M\times M_k)$ and an isomorphism

$$\sigma' : \pi_!\, E'|_{\partial M} \longrightarrow \pi_!\, F'|_{\partial M}$$

over $X\times M_k$. Let us fix a one-dimensional bundle ε over the Moore space such that it defines the generator

$$[\varepsilon]-1\in \widetilde{K}\,(M_k)\simeq \mathbb{Z}_k$$

(more details on Moore spaces can be found in [20, 21]). Let us also fix a trivialization

$$\rho : k\varepsilon \to \mathbb{C}^k.$$

A triple (E,G,σ) defines an element

$$[E\otimes\varepsilon, E, \sigma'] \in K_0\left(\widetilde{C}_0\left(M_k,\mathcal{A}_{M,\pi}\right)\right),$$

where the isomorphism σ' is defined as a composition (see [21])

$$\pi_!\, E|_{\partial M}\otimes\varepsilon \stackrel{\sigma\otimes 1}{\to} kG\otimes\varepsilon \simeq G\otimes k\varepsilon \stackrel{1\otimes\rho}{\to} G\otimes\mathbb{C}^k \simeq kG \stackrel{\sigma^{-1}\otimes 1}{\to} \pi_!\, E|_{\partial M}\,. \tag{38}$$

□

In terms of this realization, the element $\left[\widetilde{\pi_! 1}\right]$ is defined for N large enough by a triple of the form $(N,1,\alpha)$, where N and 1 stand for the trivial bundles of the corresponding dimensions

$$N\in \mathrm{Vect}\,(M)\,,\qquad 1\in \mathrm{Vect}\,(X)\,,$$

and a trivialization

$$\pi_! N \stackrel{\alpha}{\simeq} \mathbb{C}^{nN}$$

is chosen as follows.

Let the classifying mapping

$$f : M \longrightarrow \mathcal{M}_\infty$$

to the universal space take M to a finite-dimensional subspace $\mathcal{M}_{N'}$. Then, for N' there exists a number L' such that the property (66) is valid. We now choose N such that the restriction of the universal bundle γ to the skeleton $(BS_n)_{N'+L'}$ in N copies becomes trivial with some trivialization

$$N\gamma \stackrel{\alpha'}{\simeq} \mathbb{C}^{nN}.$$

Hence, over M we have an induced trivialization

$$\alpha = f^*\alpha'.$$

A diagram similar to (71) from the Appendix shows that the element

$$[N,1,\alpha]\in K_0\left(\mathcal{A}_{M,\pi},\mathbb{Q}/n\mathbb{Z}\right)$$

is independent of the choice of α' and coincides with $\left[\widetilde{\pi_! 1}\right]$, defined previously in (35).

4. Index defect theorem.

Theorem 7 *For a manifold M with a covering $\pi:\partial M\to X$ on its boundary there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Ell}\left(\overline{M}^{\pi}\right) & \xrightarrow{\chi} & K\left(\overline{T^*M}^{\pi}\right) \\ \downarrow \widetilde{\operatorname{ind}} & \swarrow \langle\cdot,[\widetilde{\pi_!1}]\rangle & \\ \mathbb{R}/n\mathbb{Z}, & & \end{array}$$

where $\langle,\rangle$ denotes the Poincaré pairing with coefficients

$$\langle,\rangle : K\left(\overline{T^*M}^{\pi}\right)\times K_0\left(\mathcal{A}_{M,\pi},\mathbb{Q}/n\mathbb{Z}\right)\longrightarrow K_0\left(\mathcal{A}_{T^*M,\pi},\mathbb{Q}/n\mathbb{Z}\right)\overset{\operatorname{ind}_t}{\to}\mathbb{Q}/n\mathbb{Z}. \tag{39}$$

Proof. Let φ be the mapping defining the product with $\left[\widetilde{\pi_!1}\right]$

$$\varphi : K\left(\overline{T^*M}^{\pi}\right)\longrightarrow K_0\left(\mathcal{A}_{T^*M,\pi},\mathbb{Q}/n\mathbb{Z}\right).$$

1. We construct a similar mapping Φ in elliptic theory such that the following diagram commutes

$$\begin{array}{ccc} \operatorname{Ell}\left(\overline{M}^{\pi}\right) & \xrightarrow{\Phi} & \operatorname{Ell}\left(M,\pi,\mathbb{Q}/n\mathbb{Z}\right) \\ \chi\downarrow & & \downarrow\chi' \\ K\left(\overline{T^*M}^{\pi}\right) & \xrightarrow{\varphi} & K\left(\mathcal{A}_{T^*M,\pi},\mathbb{Q}/n\mathbb{Z}\right). \end{array} \tag{40}$$

To this end, we first define elliptic theory $\operatorname{Ell}(M,\pi,\mathbb{Q}/n\mathbb{Z})$ with coefficients in $\mathbb{Q}/n\mathbb{Z}$. The definition can be given in terms of the direct limit

$$\operatorname{Ell}\left(M,\pi,\mathbb{Q}/n\mathbb{Z}\right)=\varinjlim \operatorname{Ell}\left(M,\pi,\mathbb{Z}_{nN}\right),\quad \mathbb{Z}_{nN}\subset\mathbb{Z}_{nNM}\subset\mathbb{Q}/n\mathbb{Z},$$

of groups with finite coefficients. Moreover, the groups with coefficients in $\mathbb{Z}_k$ are defined as

$$\operatorname{Ell}\left(M,\pi,\mathbb{Z}_k\right)=\operatorname{Ell}_{M_k}\left(M,\pi\right)$$

in terms of families of nonlocal elliptic first-order operators over the Moore space M_k for $\mathbb{Z}_k$ (more details on elliptic theory modulo k can be found in [21]).

Let us consider the following family of elliptic first-order operators on M

$$D^*\oplus\left(D\otimes 1_\varepsilon\right): C^\infty\left(M,F\oplus E\otimes\varepsilon\right)\longrightarrow C^\infty\left(M,E\oplus F\otimes\varepsilon\right),$$

parametrized by the Moore space M_{nN} (the number N will be chosen later on). Here D^* is the adjoint operator, while the family $D\otimes 1_\varepsilon$ stands for D with coefficients in the bundle ε. We obviously have

$$\operatorname{ind}\left(D^*\oplus\left(D\otimes 1_\varepsilon\right),\Pi_-\oplus\left(\Pi_+\otimes 1_\varepsilon\right)\right)=\operatorname{ind}\left(D,\Pi_+\right)\left([\varepsilon]-1\right)\in\widetilde{K}\left(M_{nN}\right),$$

for the corresponding family of Atiyah–Patodi–Singer boundary value problems. On the other hand, consider the direct sum of N copies of this family. It turns out that this sum for N large enough has a well-posed boundary condition. Indeed, for N large the sum $N\pi_!1$ of the flat bundle is trivial. Consider a trivialization

$$N\pi_!1 \overset{\alpha}{\simeq} \mathbb{C}^{nN}. \tag{41}$$

Then, on the base of the covering we obtain a vector bundle isomorphism

$$\pi_!\left(N\ E|_{\partial M}\right) \simeq \pi_! N \otimes E_0 \overset{\alpha\otimes 1}{\longrightarrow} \mathbb{C}^{nN} \otimes E_0$$

and, similarly,

$$\pi_!\left(N\ E|_{\partial M}\right) \otimes \varepsilon \simeq \pi_! N \otimes E_0 \otimes \varepsilon \overset{\alpha\otimes 1}{\longrightarrow} \mathbb{C}^{nN} \otimes E_0 \otimes \varepsilon \simeq nN\varepsilon \otimes E_0 \overset{\rho\otimes 1}{\longrightarrow} \mathbb{C}^{nN} \otimes E_0.$$

The corresponding isomorphisms on sections are denoted by

$$B_1 = \alpha \otimes 1 : C^\infty\left(X, \pi_!\left(N\ E|_{\partial M}\right)\right) \longrightarrow C^\infty\left(X, \mathbb{C}^{nN} \otimes E_0\right),$$

$$B_2 = \rho \otimes 1\left(\alpha \otimes 1\right) : C^\infty\left(X, \pi_!\left(N\ E|_{\partial M}\right) \otimes \varepsilon\right) \longrightarrow C^\infty\left(X, \mathbb{C}^{nN} \otimes E_0\right).$$

In this notation we define a family of nonlocal elliptic boundary value problems

$$\begin{cases} ND^*u = f_1, \qquad N\left(D \otimes 1_\varepsilon\right) v = f_2, \\ B_1\beta_E\ u|_{\partial M} + B_2\beta_E\ v|_{\partial M} = g, \quad g \in C^\infty\left(X, \mathbb{C}^{nN} \otimes E_0\right). \end{cases} \tag{42}$$

Let us now fix N (see the subsection 3 of this section) such that the direct sum of the universal bundle γ in N copies over $\left(BS_n\right)_{N'}$ is also trivial

$$N\gamma \overset{\alpha'}{\simeq} \mathbb{C}^{Nn}.$$

We define the trivialization (41) to be induced by α'. Now the mapping

$$\operatorname{Ell}\left(\overline{M}^\pi\right) \overset{\Phi}{\longrightarrow} \operatorname{Ell}\left(M, \pi, \mathbb{Q}/n\mathbb{Z}\right)$$

takes an operator D to a family of nonlocal problems (42) with this special choice of trivialization α.

The mapping Φ is actually a realization of the product (39) in K-theory. More precisely, the following lemma is valid.

Lemma 5 *The diagram* (40) *is commutative.*

Proof. Substituting the definitions of $[\sigma(D)]$ and $\left[\widetilde{\pi_!1}\right]$ (according to (38) and the geometric definition of $\left[\widetilde{\pi_!1}\right]$) in the formula for the product (34), we obtain that the desired product is defined by a family of nonlocal symbols that far from the boundary are

$$\left(N\sigma(D) \otimes 1_\varepsilon\right) \oplus \left(N\sigma(D)^{-1} \otimes 1\right). \tag{43}$$

The direct image of this symbol restricted to the boundary is

$$\begin{aligned} N\pi_!\left(\sigma(D)\otimes 1_\varepsilon \oplus \sigma(D)^{-1}\otimes 1\right) &= \left(\sigma(D_0)\otimes 1_{N\varepsilon\otimes\pi_!1}\oplus\sigma(D_0)^{-1}\otimes 1_{N\pi_!1}\right)\otimes 1_{E_0}\simeq \\ &\simeq \left(\sigma(D_0)\oplus\sigma(D_0)^{-1}\right)\otimes 1_{\mathbb{C}^{nN}\otimes E_0}. \end{aligned} \tag{44}$$

In the last equivalence we use isomorphisms

$$N\pi_!1 \overset{\alpha}{\simeq} \mathbb{C}^{nN}, \quad \mathbb{C}^{nN}\varepsilon \overset{\rho}{\simeq} \mathbb{C}^{nN}.$$

In a neighborhood of the boundary, the symbol is defined by a homotopy of the direct sum $\sigma(D_0)\oplus\sigma(D_0)^{-1}$ to the identity.

It remains to prove that the difference constructions of the principal symbol for the problem (42) and the expression (43), (44) are equal.

This is obvious far from the boundary, since the only distinction is in the components $\sigma(D^*)$ and $\sigma(D)^{-1}$, which are joined by a standard homotopy

$$\sigma(D^*)\left[\sigma(D)\,\sigma(D^*)\right]^{-s}, \qquad s\in[0,1].$$

It is also easy to prove the coincidence in a neighborhood of the boundary, if one uses the formulas for order reduction of boundary value problems, see [14].

□

2. To complete the proof of the theorem, it suffices to show that Φ preserves the "index". More precisely, in elliptic theory with coefficients $\mathbb{Q}/n\mathbb{Z}$ the index mapping

$$\mathrm{ind}:\mathrm{Ell}(M,\pi,\mathbb{Q}/n\mathbb{Z})\longrightarrow\mathbb{Q}/n\mathbb{Z}$$

is defined on an element

$$[\mathcal{D}]\in\mathrm{Ell}(M,\pi,\mathbb{Z}_{nN})=\mathrm{Ell}_{M_{nN}}(M,\pi)$$

(recall that an element of $\mathrm{Ell}_{M_{nN}}(M,\pi)$ is a family $\mathcal{D}$ of elliptic operators parametrized by the Moore space M_{nN}) by the formula

$$\mathrm{ind}\,[\mathcal{D}]=\mathrm{ind}\mathcal{D}\in\widetilde{K}(M_{nN})\simeq\mathbb{Z}_{nN}\subset\mathbb{Q}/n\mathbb{Z}.$$

Here $\mathrm{ind}\mathcal{D}\in\widetilde{K}(M_{nN})$ is the reduced index of a family.

Lemma 6 *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Ell}\left(\overline{M}^{\pi}\right) & \xrightarrow{\quad\Phi\quad} & \mathrm{Ell}(M,\pi,\mathbb{Q}/n\mathbb{Z}) \\ & {\scriptstyle\widetilde{\mathrm{ind}}}\searrow \quad \swarrow{\scriptstyle\mathrm{ind}} & \\ & \mathbb{R}/n\mathbb{Z}. & \end{array}$$

Proof of Lemma. 1) The boundary value problem (42) is linearly homotopic to

$$\begin{cases} ND^*u = f_1, \qquad N\left(D\otimes 1_\varepsilon\right)v = f_2, \\ B_1\beta_E\Pi_-\, u|_{\partial M} + B_2\beta_E\Pi_+\, v|_{\partial M} = g, \quad g\in C^\infty\left(X,\mathbb{C}^{nN}\otimes E_0\right). \end{cases}$$

This implies that the index of the family $\Phi\left[D\right]$ is a sum of the indices of the spectral boundary value problems for ND^* and $N\left(D\otimes 1_\varepsilon\right)$, and the index of a family

$$N\mathrm{Im}\,\Pi_-\left(A\right)\oplus N\mathrm{Im}\,\Pi_+\left(A\right)\otimes\varepsilon \overset{B_1+B_2}{\longrightarrow} C^\infty\left(X,\mathbb{C}^{nN}\otimes E_0\right) \tag{45}$$

on the boundary. Here and in what follows in the notation we specify self-adjoint operators defining the corresponding spectral projections. Let us compute the index of this family.

2) Consider a decomposition

$$C^\infty\left(X,\mathbb{C}^{nN}\otimes E_0\right)\simeq nN\mathrm{Im}\,\Pi_-\left(A_0\right)\oplus nN\varepsilon\otimes\mathrm{Im}\,\Pi_+\left(A_0\right)$$

of the right-hand sides space in (45). Here the isomorphism is defined as

$$nN\mathrm{Im}\,\Pi_-\left(A_0\right)\oplus nN\varepsilon\otimes\mathrm{Im}\,\Pi_+\left(A_0\right)\overset{1+(\rho\otimes 1)}{\longrightarrow} C^\infty\left(X,\mathbb{C}^{nN}\otimes E_0\right).$$

Using this isomorphism, we obtain for the index of the family (45)

$$= \mathrm{ind}\left(N\mathrm{Im}\,\Pi_+\left(A\right)\overset{\Pi_+(A_0)\beta_E}{\longrightarrow} nN\mathrm{Im}\,\Pi_+\left(A_0\right)\right)\left(\left[\varepsilon\right]-1\right)\in\widetilde{K}\left(M_{nN}\right).$$

Finally, pushing forward $\mathrm{Im}\,\Pi_+\left(A\right)$ to the base of the covering we have

$$= \mathrm{ind}\left(N\mathrm{Im}\,\Pi_+\left(\pi_! A\right)\overset{\Pi_+(A_0)}{\longrightarrow} nN\mathrm{Im}\,\Pi_+\left(A_0\right)\right)\left(\left[\varepsilon\right]-1\right).$$

The index of the operator (not a family !) in the last formula is expressed in terms of the η-invariants by the Atiyah–Patodi–Singer formula [19]

$$\mathrm{ind}\left(N\mathrm{Im}\,\Pi_+\left(\pi_! A\right)\overset{\Pi_+(A_0)}{\longrightarrow} nN\mathrm{Im}\,\Pi_+\left(A_0\right)\right) = N\eta\left(A\right)-nN\eta\left(A_0\right)+\left\langle\left[\sigma\left(A_0\right)\right],\left[\pi_! 1\right]\right\rangle, \tag{46}$$

where the angular brackets denote the evaluation of the pairing

$$\langle,\rangle : K^1\left(T^*X\right)\times K^1\left(X,\mathbb{Q}\right)\longrightarrow\mathbb{Q} \tag{47}$$

on the difference construction of an elliptic self-adjoint operator

$$\left[\sigma\left(A_0\right)\right]\in K^1\left(T^*X\right)$$

and the element $\left[\pi_! 1\right]\in K^1\left(X,\mathbb{Q}\right)$, which is defined by the trivialized flat bundle $N\pi_! 1$ (more details concerning this formula can be found also in the book [22]).

3) It turns out that the last term in (46) is equal to zero for our choice of trivialization (41) .

Indeed, consider the classifying mapping $f : X \to (BS_n)_{N'}$. Then, the pairing (47) can be computed on the classifying space:

$$\langle [\sigma (A_0)], [\pi_! 1] \rangle = \langle f_! [\sigma (A_0)], [\gamma] \rangle, \quad [\pi_! 1] = f^* [\gamma] \in K^1 (X, \mathbb{Q}), \tag{48}$$

where $[\gamma] \in K^1((BS_n)_{N'}) \otimes \mathbb{Q}$ is an element defined by the trivialized flat bundle $N\gamma$. The embedding $(BS_n)_{N'} \subset (BS_n)_{N'+L'}$ induces a commutative diagram

$$\begin{array}{ccccc} K^1 (T^* (BS_n)_{N'}) & \times & K^1 ((BS_n)_{N'}) \otimes \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \uparrow & & \| \\ K^1 \left(T^* (BS_n)_{N'+L'}\right) & \times & K^1 \left((BS_n)_{N'+L'}\right) \otimes \mathbb{Q} & \longrightarrow & \mathbb{Q}. \end{array}$$

This diagram together with property (66) give the desired triviality of the pairing (48):

$$\langle f_! [\sigma (A_0)] = 0, [\gamma] \rangle .$$

Thus, the expression for $\operatorname{ind}\Phi [D]$ reduces to the desired equality

$$\operatorname{ind}\Phi [D] = \widetilde{\operatorname{ind}} [D] .$$

This proves Lemma 6.

□

Now the theorem follows from Lemmas 5 and 6. Theorem is proved.

□

Remark 6 The evaluation of the Poincaré duality pairing can be done on the universal space $\mathcal{M}_N$. Indeed, for an embedding $f : (M, \pi) \to (M', \pi')$, there is a direct image mapping in K-theory (cf. [6])

$$f_! : K \left(\overline{T^* M}^{\pi}\right) \longrightarrow K \left(\overline{T^* M'}^{\pi}\right).$$

Hence, for an embedding $f : M \subset \mathcal{M}_N$, one has a commutative diagram

$$\begin{array}{ccc} K \left(\overline{T^* M}^{\pi}\right) & \xrightarrow{\times [\widetilde{\pi_! 1}]} & K \left(\mathcal{A}_{T^* M, \pi}, \mathbb{Q}/n\mathbb{Z}\right) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ K \left(\overline{T^* \mathcal{M}_N}^{\pi}\right) & \xrightarrow{\times [\widetilde{\gamma}]} & K \left(\mathcal{A}_{T^* \mathcal{M}_N, \pi'}, \mathbb{Q}/n\mathbb{Z}\right). \end{array}$$

6 Applications

1. Theorem 7 on the index defect enables one to express the fractional part of the η-invariant in the following situation.

Let M be an even-dimensional spin manifold with boundary represented as the total space of a covering such that the spin structure on the boundary is a pull-back of a spin structure on the base. Let us also fix $E \in \operatorname{Vect}(M)$ that near the boundary is also pulled back $E|_{\partial M} = E_0$ from the base. We choose a metric on M that near the boundary is a product metric induced by a metric on the base. Finally, we choose a similar connection in E.

Proposition 5 *The Dirac operator D_M on M with coefficients in E satisfies the assumptions of Theorem 7 and the fractional part of the η-invariant is equal to*

$$\{\eta(D_X)\} = \frac{1}{n}\left(\int_M \widehat{A}(M)\,\mathrm{ch}E - \left\langle [\sigma(D_M)], \left[\widetilde{\pi_! 1}\right]\right\rangle\right) \in \mathbb{R}/\mathbb{Z},$$

where D_X denotes the self-adjoint Dirac operator with coefficients in E_0 on X.

Proof. The formula follows from Theorem 7 if we decompose the index of the spectral problem using the Atiyah–Patodi–Singer formula

$$\mathrm{ind}\,(D_M, \Pi_+) = \int_M \widehat{A}(M)\,\mathrm{ch}E - \eta(D_{\partial M}).$$

□

2. The invariant $\widetilde{\mathrm{ind}}$ can be effectively computed via Lefschetz theory. Suppose that π is regular, i.e. the boundary is a principal G-bundle for a finite group G with a natural projection on the quotient space

$$\pi : \partial M \longrightarrow \partial M/G.$$

In addition, suppose that the action extends to an action on the entire bounding manifold M (the action is not free in general).

Let D be a G-invariant elliptic operator on M. For $g \in G$ let $L(D, g) \in \mathbb{C}$ be the usual contribution to the Lefschetz formula (see [23]) for D of the fixed point set of the diffeomorphism $g : M \to M$.

The invariant $\widetilde{\mathrm{ind}}D$ is expressed in terms of the contributions for the nontrivial elements of the group.

Proposition 6 *One has*

$$\widetilde{\mathrm{ind}}D \equiv -\sum_{g\neq e} L(D, g) \pmod{n}.$$

Proof. Consider the equivariant index $\mathrm{ind}_g(D, \Pi_+)$ of the Atiyah–Patodi–Singer problem and the equivariant η-function (see [24])

$$\eta(A, g) = \frac{1}{2}\left(\sum_{\lambda\in \mathrm{Spec}A} \mathrm{sgn}\lambda\, |\lambda|^{-s}\, \mathrm{Tr}\,(g : E_\lambda \to E_\lambda) + \mathrm{Tr}\,(g : E_0 \to E_0)\right)\Bigg|_{s=0}$$

of the tangential operator A on the boundary, where E_λ stands for the eigenspace of A with eigenvalue λ.

Denote by $(D,\Pi_+)^G$ and A^G the restrictions of the corresponding operators to the subspaces of G-invariant sections. Clearly, A^G is isomorphic to A_0 on X. On the other hand, one can express the usual objects in terms of their equivariant counterparts

$$\operatorname{ind}(D,\Pi_+)^G = \frac{1}{|G|}\sum_{g\in G}\operatorname{ind}_g(D,\Pi_+) \qquad \eta\left(A^G\right) = \frac{1}{|G|}\sum_{g\in G}\eta(A,g)$$

(these are obtained by an elementary character theory). Using these expressions, we rewrite $\widetilde{\operatorname{ind}}D$ as

$$\widetilde{\operatorname{ind}}D = \operatorname{ind}_e(D,\Pi_+) + \eta(A,e) - |G|\,\eta\left(A^G\right)$$

and rearrange it as follows

$$\widetilde{\operatorname{ind}}D = \operatorname{ind}_e(D,\Pi_+) - \sum_{g\neq e}\eta(A,g).$$

Let us substitute into this formula the expression for the η-invariant given by the equivariant Atiyah–Patodi–Singer formula (see [24])

$$-\eta(A,g) = \operatorname{ind}_g(D,\Pi_+) - L(D,g).$$

The result is

$$\widetilde{\operatorname{ind}}D = |G|\operatorname{ind}(D,\Pi_+)^G - \sum_{g\neq e}L(D,g).$$

This gives the desired equality

$$\widetilde{\operatorname{ind}}D \equiv -\sum_{g\neq e}L(D,g) \bmod |G|.$$

$\square$

Remark 7 The problem of finding a bounding manifold M with a free action of G is considered in the theory of free G-bordisms (e.g., see [25]).

7 Exact sequence in elliptic theory

The aim of this section is to construct an exact sequence in elliptic theory similar to the K-theory sequence (17).

1. Let us first construct an exact sequence

$$\begin{array}{ccccc} & \operatorname{Ell}(\partial M) & \longrightarrow & \operatorname{Ell}(M) & \\ \nearrow & & & & \searrow \\ \operatorname{Ell}_1(M,\partial M) & & & & \operatorname{Ell}(M,\partial M) \\ \nwarrow & & & & \swarrow \\ & \operatorname{Ell}_1(M) & \longleftarrow & \operatorname{Ell}_1(\partial M) & \end{array} \tag{49}$$

on a manifold with boundary (without the structure of a covering). Each of the groups here is defined as a group of stable homotopy classes in a suitable set of elliptic operators modulo a subset of trivial operators.

Definition of the groups. First of all, the well-known group $\mathrm{Ell}(\partial M)$ corresponds to all elliptic operators on a closed manifold ∂M, while trivial operators are induced by vector bundle isomorphisms (see [1]). The group $\mathrm{Ell}_1(\partial M)$ is generated by self-adjoint elliptic operators on the boundary, modulo operators induced by vector bundle isomorphisms (see [19]).

Now the group $\mathrm{Ell}(M, \partial M)$ is defined in terms of first-order elliptic operators near the boundary having the form

$$\Gamma\left(\frac{\partial}{\partial t} + A\right) \tag{50}$$

for a bundle isomorphism Γ and some elliptic self-adjoint pseudodifferential operator A of first order on the boundary. In this case, the class of trivial operators consists of operators $D_\pm$ (cf. Example 2, for the case of the identity covering $\pi = Id$). The group $\mathrm{Ell}(M)$ corresponds to the classical boundary value problems for operators of the above described type, while trivial problems are the boundary value problems $\mathcal{D}_\pm$.

Further, the group $\mathrm{Ell}_1(M, \partial M)$ is generated by symmetric elliptic operators that near the boundary have the form (50), where Γ satisfies relations

$$\Gamma^2 = -1, \quad \Gamma^* = -\Gamma, \quad \Gamma A + A\Gamma = 0 \tag{51}$$

(the last two guarantee that (50) is symmetric), and the tangential operator A is self-adjoint. In this case the trivial operators

$$D'_\pm : C^\infty(M, E_+ \oplus E_-) \longrightarrow C^\infty(M, E_+ \oplus E_-). \tag{52}$$

are defined as follows. They act on the sections of a direct sum of bundles $E_\pm \in \mathrm{Vect}(M)$ equipped with an isomorphism

$$E_+|_U \simeq E_-|_U \tag{53}$$

in a neighborhood of the boundary. To define this operator, consider a composition of a linear homotopy

$$(1-\tau)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\left[\frac{\partial}{\partial t} + \begin{pmatrix} 0 & \sqrt{\Delta_{\partial M}} \\ \sqrt{\Delta_{\partial M}} & 0 \end{pmatrix}\right] + \tau\begin{pmatrix} 0 & -i\sqrt{\Delta_M} \\ i\sqrt{\Delta_M} & 0 \end{pmatrix}, \quad \tau \in [0,1]$$

and a homotopy

$$\begin{pmatrix} \sin\varphi\sqrt{\Delta_M} & -i\sqrt{\Delta_M}\cos\varphi \\ i\sqrt{\Delta_M}\cos\varphi & -\sin\varphi\sqrt{\Delta_M} \end{pmatrix}, \qquad \varphi \in [0, \pi/2]$$

of elliptic self-adjoint expressions. Let $D(\tau)$, $\tau \in [0,2]$ be the smoothed composition of these homotopies. Then the desired operator $D'_\pm$ in a collar neighborhood of the boundary is

$$
D'_{\pm}\big|_U = \frac{1}{2}\left[D(t) + (D(t))^*\right] : \begin{array}{c} C^\infty(U, E_+|_U) \\ \oplus \\ C^\infty(U, E_-|_U) \end{array} \longrightarrow \begin{array}{c} C^\infty(U, E_+|_U) \\ \oplus \\ C^\infty(U, E_-|_U), \end{array} \qquad (54)
$$

on the remaining part of M it has a diagonal form

$$
D'_{\pm}\big|_{M\backslash U} = \begin{pmatrix} \sqrt{\Delta_M} & 0 \\ 0 & -\sqrt{\Delta_M} \end{pmatrix} : \begin{array}{c} C^\infty(M\backslash U, E_+) \\ \oplus \\ C^\infty(M\backslash U, E_-) \end{array} \longrightarrow \begin{array}{c} C^\infty(M\backslash U, E_+) \\ \oplus \\ C^\infty(M\backslash U, E_-). \end{array}
$$

The two expressions define an operator $D'_{\pm}$ with smooth coefficients on the entire manifold by virtue of the isomorphism (53). Let us note that near the boundary the operator is

$$
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left[\frac{\partial}{\partial t} + \begin{pmatrix} 0 & \sqrt{\Delta_{\partial M}} \\ \sqrt{\Delta_{\partial M}} & 0 \end{pmatrix}\right]. \qquad (55)
$$

Finally, the group $\mathrm{Ell}_1(M)$ corresponds to homotopy classes of boundary value problems for elliptic self-adjoint operators coinciding with (55) near the boundary, while the boundary value problems for $D'_{\pm}$ define the class of trivial elements. Here we assume that the expressions like (55) are equipped with homogeneous boundary conditions

$$
u_+|_{\partial M} - u_-|_{\partial M} = 0, \qquad u_{\pm} \in C^\infty(M, E_{\pm}), \qquad (56)
$$

defining a symmetric boundary value problem.

Remark 8 It is convenient to consider the groups Ell as periodically graded. In other words, we define

$$
\mathrm{Ell}_{2k}(\cdot) = \mathrm{Ell}(\cdot), \qquad \mathrm{Ell}_{2k+1}(\cdot) = \mathrm{Ell}_1(\cdot).
$$

Definition of the mappings in diagram (49).

First of all, the mappings

$$
\mathrm{Ell}_*(M) \longrightarrow \mathrm{Ell}_*(M, \partial M)
$$

are induced by forgetting the boundary conditions.

Secondly, we define the boundary mappings

$$
\mathrm{Ell}(M, \partial M) \longrightarrow \mathrm{Ell}_1(\partial M), \qquad \mathrm{Ell}_1(M, \partial M) \longrightarrow \mathrm{Ell}(\partial M).
$$

The first takes an operator on M to its tangential operator A (see (50)), and the second takes a self-adjoint expression (50) to an elliptic operator on the boundary

$$
A : C^\infty\left(\partial M, \Lambda^+\right) \longrightarrow C^\infty\left(\partial M, \Lambda^-\right),
$$

where the vector bundles $\Lambda^{\pm} \in \mathrm{Vect}(\partial M)$ are the $\pm i$-eigenspaces of the automorphism Γ.

Thirdly, the mapping $\operatorname{Ell}(\partial M) \longrightarrow \operatorname{Ell}(M)$ was defined previously (consider in (19) the identity covering). Recall that it takes an elliptic operator B on ∂M to a boundary value problem $(D_\pm, B^{-1})$ with B entering as the boundary condition.

Finally, the mapping $\operatorname{Ell}_1(\partial M) \longrightarrow \operatorname{Ell}_1(M)$ can be defined as follows. It takes an elliptic self-adjoint first-order operator A to a self-adjoint operator that in the collar neighborhood $[-\pi, \pi] \times \partial M$ of the boundary is equal to

$$\begin{pmatrix} i\frac{\partial}{\partial t} & -i\cos t\sqrt{\Delta_X} + A\sin t \\ i\cos t\sqrt{\Delta_X} + A\sin t & -i\frac{\partial}{\partial t} \end{pmatrix}$$

(in this notation, the boundary of M corresponds to $t = -\pi$, it is also supposed that A acts in a trivial bundle). At the endpoints $t = \pm\pi$ this operator coincides with (55). This makes it possible to extend the operator inside the manifold and pose the boundary condition (56) at $t = -\pi$. Note that for A trivial (i.e., when the principal symbol on the cosphere bundle lifts from ∂M) this expression is homotopic to a trivial problem for a suitable operator $D'_\pm$.

To prove that the sequence (49) defines a complex and to prove its exactness, we shall prove that this sequence is isomorphic to the exact sequence in K-theory.

Difference constructions. Let us define the difference construction mappings, which take elliptic operators to the K-theory elements.

For even groups $\operatorname{Ell}(\partial M)$, $\operatorname{Ell}(M)$, $\operatorname{Ell}(M, \partial M)$, the difference construction

$$\begin{array}{ccc} \operatorname{Ell}(\partial M) & \longrightarrow & K(T^*\partial M), \\ \operatorname{Ell}(M, \partial M) & \longrightarrow & K(T^*M), \\ \operatorname{Ell}(M) & \longrightarrow & K(T^*(M \backslash \partial M)), \\ [D] & \mapsto & [\sigma(D)] \end{array} \tag{57}$$

is well known (in the first two cases this is just the usual difference construction of the principal symbol of an elliptic operator, see [1], in the third case this is the Atiyah–Bott construction for classical boundary value problems, see [26], and also [14]).

The difference construction for groups Ell_1 generated by elliptic self-adjoint operators

$$\begin{array}{ccc} \operatorname{Ell}_1(\partial M) & \longrightarrow & K^1(T^*\partial M), \\ \operatorname{Ell}_1(M, \partial M) & \longrightarrow & K^1(T^*M), \\ \operatorname{Ell}_1(M) & \longrightarrow & K^1(T^*(M \backslash \partial M)), \end{array} \tag{58}$$

is defined in the following way (see [19]). In the first two cases the difference construction of an operator A is an element

$$[i\tau + \sigma(A)] \in K(\mathbb{R} \times T^*M') \equiv K^{-1}(T^*M') \simeq K^1(T^*M'), \qquad \tau \in \mathbb{R},$$

where M' denotes either ∂M or M. Here we suppose that the principal symbol $\sigma(A)$ is homogeneous of order one. Under these assumptions, bundle homomorphism $i\tau + \sigma(A)(x, \xi)$ defines an isomorphism for (τ, x, ξ) outside a compact set in $\mathbb{R} \times T^*M'$.

In the last case, an element of $K^1(T^*(M\backslash\partial M))$ is obtained from $i\tau+\sigma(A)$ for operator (55) if we modify the expression near the boundary so that it becomes isomorphism over $\mathbb{R}\times T^*M|_{\partial M}$. To this end, consider the expression

$$i\tau+\sigma(A)(x,\xi)+\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}(1-|\xi|)\chi(x), \qquad \text{for } |\xi|\le 1, \tag{59}$$

where χ denotes a cut-off function equal to one near the boundary. This bundle homomorphism is an isomorphism outside a compact set in $\mathbb{R}\times T^*(M\backslash\partial M)$, i.e. it defines the desired element in $K^1(T^*(M\backslash\partial M))\simeq K(\mathbb{R}\times T^*(M\backslash\partial M))$.

We define the *difference construction* of a self-adjoint boundary value problem as the element of $K(\mathbb{R}\times T^*(M\backslash\partial M))$ determined by the extension (59).

Theorem 8

1. *The difference constructions* (57), (58) *are isomorphisms;*
2. *The sequences* (49) *and* (17) *are isomorphic. In other words, there is a commutative diagram*

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{Ell}(\partial M) & \longrightarrow & \mathrm{Ell}(M) & \longrightarrow & \mathrm{Ell}(M,\partial M) & \stackrel{\partial}{\longrightarrow} & \mathrm{Ell}_1(\partial M) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K(T^*\partial M) & \longrightarrow & K(T^*(M\backslash\partial M)) & \longrightarrow & K(T^*M) & \longrightarrow & K^1(T^*\partial M) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccccc}
\stackrel{\partial}{\longrightarrow} & \mathrm{Ell}_1(\partial M) & \longrightarrow & \mathrm{Ell}_1(M) & \longrightarrow & \mathrm{Ell}_1(M,\partial M) & \longrightarrow & \mathrm{Ell}(\partial M) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K^1(T^*\partial M) & \longrightarrow & K^1(T^*(M\backslash\partial M)) & \longrightarrow & K^1(T^*M) & \longrightarrow & K(T^*\partial M) & \longrightarrow
\end{array}$$

Corollary 2 *The sequence* (49) *is exact.*

Proof of Theorem. 1) The isomorphism for the groups $\mathrm{Ell}(\partial M)$, $\mathrm{Ell}_1(\partial M)$, corresponding to closed manifolds, was proved in [1, 19]. The isomorphism for the classical boundary value problems, i.e. the group $\mathrm{Ell}(M)$, was obtained in [14].

Let us prove the isomorphism for $\mathrm{Ell}_1(M)$. Similarly to [19], we prove that the group $K^1(T^*(M\backslash\partial M))$ is isomorphic to the group of stable homotopy classes of subbundles

$$L\subset\pi^*E, \qquad \pi: S^*M\to M, \quad E\in\mathrm{Vect}(M)$$

that near the boundary ∂M are pull-backs from the base. While the trivial elements are those subbundles that are pulled back globally on M. The isomorphism is defined via the coboundary mapping

$$K(S^*M,\, S^*M|_{\partial M})\longrightarrow K^1(T^*(M\backslash\partial M))$$

of the exact sequence of the pair $S^*M/(S^*M|_{\partial M})\subset B^*M/(B^*M|_{\partial M})$.

Let us construct the inverse mapping

$$K^1\left(T^*\left(M\backslash\partial M\right)\right)\longrightarrow \mathrm{Ell}_1\left(M\right). \tag{60}$$

Indeed, an arbitrary element of $K^1\left(T^*\left(M\backslash\partial M\right)\right)$ is realized by means of a subbundle $L\subset\pi^*E$. Adding trivial elements (which can be constructed be means of a nonsingular vector field on M), i.e. vector bundles on M, we can suppose that E is a trivial even-dimensional bundle, while in a neighborhood of the boundary the subbundle L coincides with a constant subspace

$$L|_U=\mathbb{C}^N\subset\mathbb{C}^{2N}.$$

Now the mapping (60) takes the bundle L to an elliptic self-adjoint operator

$$D_L: C^\infty\left(M,\mathbb{C}^{2N}\right)\longrightarrow C^\infty\left(M,\mathbb{C}^{2N}\right),$$

which is defined far from the boundary by the formula

$$\sqrt[4]{\Delta_M}\left(2P_L-1\right)\sqrt[4]{\Delta_M},$$

here P_L denotes a self-adjoint pseudodifferential projection with principal symbol projecting on L. Near the boundary, D_L is a diagonal operator $\sqrt{\Delta_M}\oplus-\sqrt{\Delta_M}$ in the sections of a direct sum $\mathbb{C}^N\oplus\mathbb{C}^N$, similarly to the standard operator $D'_\pm$. Hence, this expression can be extended up to the boundary by the same expression as for $D'_\pm$ (see (54)). It is easy to see that the correspondence $L\mapsto D_L$ defines the desired inverse mapping (60) to the difference construction.

Let us prove that the difference construction is an isomorphism for $\mathrm{Ell}\left(M,\partial M\right)$. We construct the inverse mapping

$$K\left(T^*M\right)\longrightarrow \mathrm{Ell}\left(M,\partial M\right).$$

Every element of $K\left(T^*M\right)$ is a difference construction for some elliptic symbol $\sigma\left(D\right)$. However, the symbol may not have such a simple form as the symbol of (50). Let us show that $\sigma\left(D\right)$ can be simplified. Indeed, the restriction of $\sigma\left(D\right)$ to the boundary is stably homotopic to a symbol of the form $i\tau+\sigma\left(A\right)$ for an elliptic self-adjoint operator A. This follows from the periodicity isomorphism

$$K^1\left(T^*M\right)\xrightarrow{\simeq}K^{-1}\left(T^*M|_{\partial M}\right)=K\left(T^*M|_{\partial M}\right)$$

and an explicit formula $\sigma\left(A\right)\mapsto i\tau+\sigma\left(A\right)$ for it. One can also verify that the resulting element of $\mathrm{Ell}\left(M,\partial M\right)$ is independent of the homotopy. Similarly, using the Bott periodicity, on proves that the isomorphism is valid also for the odd group $\mathrm{Ell}_1\left(M,\partial M\right)$.

This completes the proof of the first part of the theorem.

2) The commutativity of the second, third and the fifth squares of the diagram follows from the definitions. The commutativity of the remaining squares can be obtained using the formula for the Bott periodicity isomorphism

$$\begin{array}{ccc} K\left(X\right) & \xrightarrow{\approx} & K\left(X\times\mathbb{R}^2\right), \\ \left[\sigma\right] & \mapsto & \left[\sigma\#\left(\xi+i\eta\right)\right], \end{array}$$

here σ is a vector bundle isomorphism outside a compact set, and $\#$ denotes the crossed product with the Bott element $[\xi + i\eta] \in K(\mathbb{R}^2)$.

$\square$

2. Let us now extend the exact sequence to the theory of nonlocal boundary value problems.

The group $\mathrm{Ell}\,(M,\pi)$ was already defined in Section 2, and the group $\mathrm{Ell}_1\,(M,\pi)$ is also easy to define. To this end, we should consider only nonlocal elliptic self-adjoint operators D with the direct image near the boundary satisfying (50), i.e. operator D is such that

$$(\pi \times 1)_!\,(D|_U) = \Gamma\left(\frac{\partial}{\partial t} + A\right),$$

where Γ satisfies (51).

We are now in a position to define the sequence

$$\begin{array}{ccccc} & \mathrm{Ell}\,(X) & \longrightarrow & \mathrm{Ell}\,(M,\pi) & \\ \nearrow & & & & \searrow \\ \mathrm{Ell}_1\,(M,\partial M) & & & & \mathrm{Ell}\,(M,\partial M)\,. \\ \nwarrow & & & & \swarrow \\ & \mathrm{Ell}_1\,(M,\pi) & \longleftarrow & \mathrm{Ell}_1\,(X) & \end{array} \tag{61}$$

Here as before there are forgetful mappings

$$\mathrm{Ell}_*\,(M,\pi) \longrightarrow \mathrm{Ell}_*\,(M,\partial M)\,.$$

The boundary mapping

$$\mathrm{Ell}_*\,(M,\partial M) \to \mathrm{Ell}_{*-1}\,(\partial M) \xrightarrow{\pi_!} \mathrm{Ell}_{*-1}\,(X)$$

is a composition of the boundary mapping for M and the direct image $\pi_!$.

As a corollary of the previous theorem, we prove the exactness of this sequence.

Theorem 9 *The sequence* (61) *is exact.*

Proof. Consider a diagram

$$\begin{array}{cccccccccc} \to & \mathrm{Ell}\,(X) & \longrightarrow & \mathrm{Ell}\,(M,\pi) & \longrightarrow & \mathrm{Ell}\,(M,\partial M) & \longrightarrow & \mathrm{Ell}_1\,(X) & \to \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to & K\,(T^*X) & \longrightarrow & K\,(\mathcal{A}_{T^*M,\pi}) & \longrightarrow & K\,(T^*M) & \longrightarrow & K^1\,(T^*X) & \to \\ & & \longrightarrow & \mathrm{Ell}_1\,(M,\pi) & \longrightarrow & \mathrm{Ell}_1\,(M,\partial M) & \longrightarrow & \mathrm{Ell}\,(X) & \to \\ & & & \downarrow & & \downarrow & & \downarrow & \\ & & \longrightarrow & K^1\,(\mathcal{A}_{T^*M,\pi}) & \longrightarrow & K^1\,(T^*M) & \longrightarrow & K\,(T^*X) & \to, \end{array}$$

where the bottom row was defined in Section 2. All the vertical mappings are isomorphisms defined by the corresponding difference constructions. One can also show the commutativity of the diagram. This implies the exactness of the upper row.

The theorem is proved.

$\square$

3. Elliptic theory of Sections 4 and 5 for operators corresponding to the covering also has an exact sequence, which we now describe. Let $\mathrm{Ell}\left(\overline{M}^{\pi}\right)$ be the group of stable homotopy classes of elliptic operators on M that near the boundary have the form

$$\Gamma\left(\frac{\partial}{\partial t}+A\right)$$

and satisfy the Assumptions 1 and 2 of Section 4. Denote by $\mathrm{Ell}_1\left(\overline{M}^{\pi}\right)$ a similar group, corresponding to elliptic operators satisfying (50), (51). Then we define a sequence

$$\begin{array}{ccccc} & \mathrm{Ell}\,(X) & \longrightarrow & \mathrm{Ell}\,(M) & \\ \nearrow & & & & \searrow \\ \mathrm{Ell}_1\left(\overline{M}^{\pi}\right) & & & & \mathrm{Ell}\left(\overline{M}^{\pi}\right). \\ \nwarrow & & & & \swarrow \\ & \mathrm{Ell}_1\,(M) & \longleftarrow & \mathrm{Ell}_1\,(X) & \end{array} \qquad (62)$$

Here the mapping

$$\mathrm{Ell}_*\,(X) \longrightarrow \mathrm{Ell}_*\,(M)$$

is a composition of the pull-back $\mathrm{Ell}_*\,(X) \xrightarrow{\pi^*} \mathrm{Ell}_*\,(\partial M)$ and the mapping $\mathrm{Ell}_*\,(\partial M) \to \mathrm{Ell}_*\,(M)$ defined at the beginning of this section (the pull-back of a pseudodifferential operator from the base of the covering is defined here in terms of the pull-back of the complete symbol of the operator). The boundary mapping is a composition as well

$$\mathrm{Ell}_*\left(\overline{M}^{\pi}\right) \longrightarrow \mathrm{Ell}_*\,(X\times[0,1)) \to \mathrm{Ell}_{*-1}\,(X)\,,$$

where we first take the operator on $X\times[0,1)$ that induces the original operator, and then apply the boundary mapping. For example, in the notation of Section 4, the boundary mapping takes D to the tangential operator A_0 on the base of the covering.

It turns out that this sequence corresponds to the sequence of the pair $\mathbb{R}\times T^*X \subset \overline{T^*M}^{\pi}$ in K-theory:

$$\begin{array}{ccccc} & K\,(T^*X) & \longrightarrow & K\,(T^*\,(M\backslash\partial M)) & \\ \nearrow & & & & \searrow \\ K^1\left(\overline{T^*M}^{\pi}\right) & & & & K\left(\overline{T^*M}^{\pi}\right). \\ \nwarrow & & & & \swarrow \\ & K^1\,(T^*\,(M\backslash\partial M)) & \longleftarrow & K^1\,(T^*X) & \end{array} \qquad (63)$$

Theorem 10 *The sequences* (62) *and* (63) *are isomorphic. The isomorphism is given by the difference constructions.*

Proof. One readily checks the exactness of (62) in the term $\mathrm{Ell}_*\left(\overline{M}^{\pi}\right)$, and also the commutativity of the diagram connecting the two sequences by means of the difference constructions. Hence, using the 5-lemma, we obtain the desired isomorphisms

$$\mathrm{Ell}_*\left(\overline{M}^{\pi}\right) \simeq K^*\left(\overline{T^*M}^{\pi}\right).$$

□

Remark 9 This approach to the periodic exact sequences in elliptic theory is remarkable, since it makes all the mappings in the sequence explicitly defined. The groups Ell can be defined in different terms. For instance, one can consider only zero-order operators. We have chosen first-order operators, since those are sufficient for most of the applications of index theory.

8 Poincaré isomorphism for some manifolds with singularities

It turns out that the exact sequences of the previous Section 7 are also isomorphic to sequences in the K-homological theory of Atiyah–Kasparov–Baum–Douglas–Fillmore [27, 28, 29]. As before, we start with the manifolds with boundary (without the covering). This case is well known (e.g., see [30, 31]). To fix the notation, we will briefly outline the corresponding results.

1. Elliptic operators of order zero define elements in K-theory

$$[\sigma(D)] \in K^*\left(T^*\left(M \backslash \partial M\right)\right), \quad [D] \in K^*\left(C\left(M\right)\right) \equiv K_*\left(M\right),$$

where $C(M)$ is the algebra of continuous functions. The latter group is the analytic K-homology group, where the grading is odd for self-adjoint operators and even otherwise. The first element is the difference construction of the operator. To define the second element, we recall that an elliptic operator D of order zero is a Fredholm operator

$$D : L^2\left(M, E\right) \longrightarrow L^2\left(M, F\right),$$

where both L^2-spaces are modules over $C\left(M\right)$ (the module structure is given by the product of functions). In addition, D commutes with the module structure up to compact operators. Thus, for a self-adjoint D (of course in this case the bundles coincide) the pair $\left(L^2\left(M, E\right), D\right)$ is an element

$$[D] \in K^1\left(C\left(M\right)\right).$$

For a nonself-adjoint D, we consider a self-adjoint matrix operator

$$T = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

in a naturally $\mathbb{Z}_2$-graded $C(M)$-module $L^2(M,E)\oplus L^2(M,F)$. Operator T is odd with respect to the grading. Hence, it defines a K-theory element

$$[D]\in K^0(C(M)).$$

On the other hand, elliptic operators of first order define similar elements

$$[\sigma(D)]\in K^i(T^*M),\quad [D]\in K_i(M\backslash\partial M).$$

The former is the Atiyah–Singer difference construction, while the latter is defined as follows. Consider an embedding

$$M\subset\widetilde{M}$$

of M in some closed manifold $\widetilde{M}$ of the same dimension. Denote by $\widetilde{D}$ an arbitrary extension of D to $\widetilde{M}$. On $\widetilde{M}$ consider a zero-order operator

$$\widetilde{F}=\left(1+\widetilde{D}^*\widetilde{D}\right)^{-1/2}\widetilde{D}.$$

We define a restriction of this operator to M as a following bounded operator

$$F=i^*\widetilde{F}i_*:L^2(M,E)\longrightarrow L^2(M,F),\tag{64}$$

where $i_*:L^2(M)\to L^2\left(\widetilde{M}\right)$ stands for the extension as zero, and $i^*:L^2\left(\widetilde{M}\right)\to L^2(M)$ is the restriction operator.

For a symmetric D, we obtain that F is also self-adjoint and the following relations are valid

$$F-F^*\in\mathcal{K},\quad f\left(F^2-1\right)\in\mathcal{K},\quad [F,f]\in\mathcal{K},$$

for functions $f\in C_0(M\backslash\partial M)$ vanishing on the boundary. Here $\mathcal{K}$ denotes the ideal of compact operators. These relations show that F defines an element of $K^1(C_0(M\backslash\partial M))$ (see [32]).

If D is nonself-adjoint then instead of F we consider a matrix operator T, as in the above. It defines an element of $K^0(C_0(M\backslash\partial M))$.

Proposition 7 *The following Poincaré isomorphisms are valid*

$$\begin{array}{ccc} K^*(T^*(M\backslash\partial M)) & \longrightarrow & K_*(M), \\ K^*(T^*M) & \longrightarrow & K_*(M\backslash\partial M), \\ [\sigma(D)] & \mapsto & [D]. \end{array}$$

Proof see in [33].

□

2. Elliptic nonlocal zero-order operators define elements

$$[\sigma(D)]\in K_*(\mathcal{A}_{T^*M,\pi}),[D]\in K^*\left(C\left(\overline{M}^\pi\right)\right)\simeq K_*\left(\overline{M}^\pi\right).$$

The first is the difference construction of Section 2. To define the second element, we note that a nonlocal elliptic operator D of order zero does not almost commute with the entire algebra $C(M)$, but only with the functions that are pulled back from the quotient space $\overline{M}^{\pi}$. This implies that we have an element of $K^*\left(C\left(\overline{M}^{\pi}\right)\right)$.

On the other hand, the operators of Sections 4, 5 define similar elements

$$[\sigma(D)] \in K^*\left(\overline{T^*M}^{\pi}\right), [D] \in K_*\left(\mathcal{A}_{M,\pi}\right).$$

However, in this case the corresponding operators (64), on the contrary, almost commute with functions $C_0(M\backslash\partial M)$, as well as with the elements of the algebra $\mathcal{A}_{M,\pi}$.

Theorem 11 *The following Poincaré isomorphisms are valid*

$$\begin{array}{ccc} K_*\left(\mathcal{A}_{T^*M,\pi}\right) & \longrightarrow & K_*\left(\overline{M}^{\pi}\right), \\ K^*\left(\overline{T^*M}^{\pi}\right) & \longrightarrow & K^*\left(\mathcal{A}_{M,\pi}\right), \\ [\sigma(D)] & \mapsto & [D]. \end{array}$$

Proof. 1) Consider a commutative diagram

$$\begin{array}{ccccccccc} \to & K^0(T^*X) & \to & K_0\left(\mathcal{A}_{T^*M,\pi}\right) & \to & K^0(T^*M) & \to & K^1(T^*X) & \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to & K_0(X) & \to & K_0\left(\overline{M}^{\pi}\right) & \to & K_0(M,\partial M) & \to & K_1(X) & \ldots \end{array}$$

Here the lower sequence is the exact sequence of the pair $X \subset \overline{M}^{\pi}$ in K-homology. The vertical mappings of the diagram (except the second one) are isomorphisms (see [29, 33]). Thus, using the 5-lemma, we obtain that the mapping in the middle

$$K_*\left(\mathcal{A}_{T^*M,\pi}\right) \longrightarrow K_*\left(\overline{M}^{\pi}\right)$$

is isomorphism as well. This gives the desired Poincaré isomorphism.

2) In the second case, the proof follows the same scheme, but one uses the exact sequence

$$\begin{array}{ccccccccc} \leftarrow & K^1(T^*X) & \leftarrow & K^0\left(\overline{T^*M}^{\pi}\right) & \leftarrow & K^0(T^*(M\backslash\partial M)) & \leftarrow & K^0(T^*X) & \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \leftarrow & K_1(X) & \leftarrow & K^0\left(\mathcal{A}_{M,\pi}\right) & \leftarrow & K_0(M) & \leftarrow & K_0(X) & \ldots \end{array}$$

(the upper row corresponds to the pair $\mathbb{R} \times T^*X \subset \overline{T^*M}^{\pi}$).

The theorem is proved.

$\square$

9 Poincaré duality

Remark 10 An analog of the pairing for the groups $K^0\left(\overline{T^*M}^{\pi}\right)$ and $K_0\left(\mathcal{A}_{M,\pi}\right)$ of Section 5 is also valid for the odd groups. The definition is left to the reader.

Theorem 12 *On a manifold M with a covering π on the boundary, the pairings*

$$K^i\left(\overline{T^*M}^{\pi}\right)\times K_i\left(\mathcal{A}_{M,\pi}\right)\longrightarrow \mathbb{Z} \tag{65}$$

are nondegenerate on the free parts of the groups.

Proof. Fixing the first argument of the pairing, we obtain a mapping

$$K^i\left(\overline{T^*M}^{\pi}\right)\otimes\mathbb{Q}\longrightarrow K_i'\left(\mathcal{A}_{M,\pi}\right),$$

where we put for brevity $G' = \mathrm{Hom}\,(G,\mathbb{Q})$. This mapping is part of a commutative diagram

$$\begin{array}{ccccccc}
K^1\left(T^*X\right)\otimes\mathbb{Q} & \leftarrow & K^0\left(\overline{T^*M}^{\pi}\right)\otimes\mathbb{Q} & \leftarrow & K^0\left(T^*\left(M\backslash\partial M\right)\right)\otimes\mathbb{Q} & \leftarrow & K^0\left(T^*X\right)\otimes\mathbb{Q}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
K^{1\prime}\left(X\right) & \leftarrow & K_0'\left(\mathcal{A}_{M,\pi}\right) & \leftarrow & K^{0\prime}\left(M\right) & \leftarrow & K^{0\prime}\left(X\right).
\end{array}$$

Here the vertical mappings, except the second one, are isomorphisms (by virtue of the Poincaré duality on a closed manifold or on a manifold with boundary). Thus, the 5-lemma implies that the second mapping is also isomorphism. Thus, the pairing (65) is nondegenerate with respect to the second argument.

The nondegeneracy with respect to the first argument is proved similarly.

□

As an example consider M with a $spin^c$-structure that on the boundary is induced by a $spin^c$-structure on the base X of the covering π. Then the group $K^*\left(\overline{T^*M}^{\pi}\right)$ is a free $K^{*+n}\left(\overline{M}^{\pi}\right)$-module with one generator (where $n=\dim M$), as a generator one can take the difference construction

$$\left[\sigma\left(D\right)\right]\in K^n\left(\overline{T^*M}^{\pi}\right)$$

of the principal symbol of the Dirac operator on M (this is proved similarly to the usual case of closed manifolds, e.g., see [34]). Consequently, one can define the Poincaré duality pairing

$$K^{*+n}\left(\overline{M}^{\pi}\right)\times K_*\left(\mathcal{A}_{M,\pi}\right)\longrightarrow\mathbb{Z}$$

by means of a composition with $K^{*+n}\left(\overline{M}^{\pi}\right)\to K\left(\overline{T^*M}^{\pi}\right)$. The above theorem shows that this pairing is nondegenerate.

Remark 11 One can also define the linking pairing for the torsion subgroups and the Pontryagin duality pairing (e.g., see [35]). Then, it is possible to prove other dualities as the following isomorphisms

$$\begin{aligned}
K^*\left(\overline{T^*M}^{\pi}\right) &\simeq \operatorname{Hom}\left(K_*\left(\mathcal{A}_{M,\pi},\mathbb{R}/\mathbb{Z}\right),\mathbb{R}/\mathbb{Z}\right),\\
K_*\left(\mathcal{A}_{M,\pi},\mathbb{R}/\mathbb{Z}\right) &\simeq \operatorname{Hom}\left(K^*\left(\overline{T^*M}^{\pi}\right),\mathbb{R}/\mathbb{Z}\right),\\
K^*\left(\overline{T^*M}^{\pi}\right)/\mathrm{Tor} &\simeq \operatorname{Hom}\left(K_*\left(\mathcal{A}_{M,\pi}\right),\mathbb{Z}\right),\\
K_*\left(\mathcal{A}_{M,\pi}\right)/\mathrm{Tor} &\simeq \operatorname{Hom}\left(K^*\left(\overline{T^*M}^{\pi}\right),\mathbb{Z}\right),\\
\mathrm{Tor}K^{*+1}\left(\overline{T^*M}^{\pi}\right) &\simeq \operatorname{Hom}\left(\mathrm{Tor}K_*\left(\mathcal{A}_{M,\pi}\right),\mathbb{Q}/\mathbb{Z}\right),\\
\mathrm{Tor}K_*\left(\mathcal{A}_{M,\pi}\right) &\simeq \operatorname{Hom}\left(\mathrm{Tor}K^{*+1}\left(\overline{T^*M}^{\pi}\right),\mathbb{Q}/\mathbb{Z}\right),
\end{aligned}$$

induced by fixing the arguments in the corresponding pairings. The first two generalize the Pontryagin duality, while the following define the Poincaré duality for the free parts and for the torsion subgroups, respectively.

Appendix

Proof of Lemma 2. Using isomorphism (22), the direct limit of K-groups for algebras can be expressed in topological terms

$$\varinjlim K_0\left(\mathcal{A}_{T^*\mathcal{M}_N}\right) = \varinjlim K\left(T^*\left(BS_n\right)_N\right).$$

Let us compute the last limit. We shall use the Poincaré duality on $T^*\left(BS_n\right)_N$. Denote[1] for brevity

$$K_0(X) = K^0\left(T^*X\right)$$

for a smooth closed manifold X.

Consider the exact sequence

$$0 \to \mathrm{Tor}K_0\left(\left(BS_n\right)_N\right) \to \widetilde{K}_0\left(\left(BS_n\right)_N\right) \to \widetilde{K}_0\left(\left(BS_n\right)_N\right)/\mathrm{Tor} \to 0,$$

where $\widetilde{K}$ are the reduced K-groups, and Tor is the torsion subgroup. The sequence $\widetilde{K}_0\left(\left(BS_n\right)_N\right)$ in the middle of the formula satisfies the following condition (e.g., see [20, 37]). For an arbitrary N there exists a positive number L such that the range of the mapping

$$\widetilde{K}_*\left(\left(BS_n\right)_N\right) \longrightarrow \widetilde{K}_*\left(\left(BS_n\right)_{N+L}\right) \tag{66}$$

is in the torsion subgroup (this can be obtained passing via the Chern character to cohomology and considering the skeletons of BS_n). This gives an isomorphism of the direct limits

$$\varinjlim \mathrm{Tor}K_0\left(\left(BS_n\right)_N\right) \simeq \varinjlim \widetilde{K}_0\left(\left(BS_n\right)_N\right).$$

[1] If we use the homological K-functor, then the following equality is actually the Poincaré duality (e.g., see [36, 33]).

Let us prove that $\varinjlim \mathrm{Tor} K_0\left((BS_n)_N\right)$ is trivial. An arbitrary element is realized as a sequence $x_N \in \mathrm{Tor} K_0\left((BS_n)_N\right)$. By virtue of the Poincaré duality for torsion groups (e.g., see [35])

$$\mathrm{Tor} K_0\left((BS_n)_N\right) \times \mathrm{Tor} K^1\left((BS_n)_N\right) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

the sequence x_N defines a sequence of functionals

$$x'_N \in \mathrm{Hom}\left(\mathrm{Tor} K^1\left((BS_n)_N\right), \mathbb{Q}/\mathbb{Z}\right)$$

and this correspondence is one-to-one. The desired triviality relies essentially on the following result.

M. Atiyah in [37] computed the K-groups of classifying spaces of finite groups. In particular, it turned out that the odd groups are trivial

$$K^1\left(BS_n\right) = 0, \tag{67}$$

where the K-group of the infinite-dimensional space BS_n is defined as an inverse limit

$$K^1\left(BS_n\right) \stackrel{def}{=} \varprojlim K^1\left((BS_n)_N\right). \tag{68}$$

Let us now apply (67). The classifying space BS_n satisfies the so-called Mittag–Leffler condition (see [37]), i.e. for an arbitrary N there exists $L \geq 0$ such that

$$\mathrm{Im}\left(K^1\left((BS_n)_{L+M}\right) \to K^1\left((BS_n)_N\right)\right) = \mathrm{Im}\left(K^1\left((BS_n)_{N+M+L}\right) \to K^1\left((BS_n)_N\right)\right) \tag{69}$$

for all nonnegative M. For completeness of the presentation we note that this property is obtained in two steps. First, by duality, we obtain for K-groups an analog of (66)

$$\mathrm{Im}\left(\widetilde{K}_0\left((BS_n)_N\right) \longrightarrow \widetilde{K}_0\left((BS_n)_{N+L}\right)\right) \qquad \text{— torsion group.}$$

Then a diagram search argument gives the Mittag–Leffler property.

Let us define an infinite sequence

$$\ldots \to \mathrm{Tor} K^1\left((BS_n)_{N+L_2}\right) \xrightarrow{i_2} \mathrm{Tor} K^1\left((BS_n)_{N+L}\right) \xrightarrow{i_1} \mathrm{Tor} K^1\left((BS_n)_N\right),$$

where the numbers $L, L_i, i = 2, 3, \ldots \quad 0 < L < L_2 < \ldots$ are consecutively chosen according to condition (69).

Let us now show that the functional x'_{N+L} is equal to zero (by Poincaré duality this would give the desired equality $x_N = 0$ of the original elements).

Consider an arbitrary $y \in \mathrm{Tor} K^1\left((BS_n)_{N+L}\right)$. Starting from y, we inductively define an element of the inverse limit of groups. By definition, put

$$y_N = i_1 y \in \mathrm{Tor} K^1\left((BS_n)_N\right).$$

It follows from the Mittag–Leffler property that $y_N = i_1 i_2 y'$ for some element $y' \in \mathrm{Tor} K^1\left((BS_n)_{N+L_2}\right)$. Let us now take

$$y_{N+L} = i_2 y'.$$

The following steps are similar. As a result, we obtain an infinite sequence

$$\ldots \mapsto y_{N+L_2} \mapsto y_{N+L} \mapsto y_N.$$

By construction, we have

$$x'_{N+L}(y) = x'_{N+L}(y_{N+L}).$$

On the other hand, this infinite sequence is an element of the inverse limit (68). The inverse limit is trivial. This gives the desired equality

$$x'_{N+L} = 0.$$

The lemma is proved.

□

Proof of Lemma 4. Let us rewrite the sequence

$$\to K^1((BS_n)_N) \otimes \mathbb{Q} \longrightarrow K^1((BS_n)_N, \mathbb{Q}/n\mathbb{Z}) \longrightarrow K^0((BS_n)_N) \xrightarrow{\times n} K^0((BS_n)_N) \otimes \mathbb{Q}$$

as a short exact sequence

$$0 \to K^1((BS_n)_N) \otimes \mathbb{Q}/n\mathbb{Z} \longrightarrow K^1((BS_n)_N, \mathbb{Q}/n\mathbb{Z}) \longrightarrow \mathrm{Tor} K^0((BS_n)_N) \to 0.$$

We get (possibly not exact) sequence of inverse limits as $N \to \infty$

$$0 \to \varprojlim K^1((BS_n)_N) \otimes \mathbb{Q}/n\mathbb{Z} \longrightarrow K^1(BS_n, \mathbb{Q}/n\mathbb{Z}) \longrightarrow \varprojlim \mathrm{Tor} K^0((BS_n)_N) \to 0. \quad (70)$$

From the property (66) of the classifying space, one can obtain the following equalities for the limits of groups

$$\varprojlim K^1((BS_n)_N) \otimes \mathbb{Q}/n\mathbb{Z} = 0 \text{ and } \varprojlim \mathrm{Tor} K^0((BS_n)_N) = \varprojlim \widetilde{K}^0((BS_n)_N) = \widetilde{K}^0(BS_n).$$

Combining this with (66), one proves the exactness of (70). This is done by considering commutative diagrams like

$$\begin{array}{ccccc}
0 \to K^1((BS_n)_N) \otimes \mathbb{Q}/n\mathbb{Z} & \longrightarrow & K^1((BS_n)_N, \mathbb{Q}/n\mathbb{Z}) & \longrightarrow & \mathrm{Tor} K^0((BS_n)_N) \to 0 \\
0 \uparrow & & \uparrow & & \uparrow \\
0 \to K^1\big((BS_n)_{N+L}\big) \otimes \mathbb{Q}/n\mathbb{Z} & \longrightarrow & K^1\big((BS_n)_{N+L}, \mathbb{Q}/n\mathbb{Z}\big) & \longrightarrow & \mathrm{Tor} K^0\big((BS_n)_{N+L}\big) \to 0,
\end{array} \quad (71)$$

where L is chosen from condition (66).

Hence, we obtain the desired isomorphism

$$\varprojlim K^1((BS_n)_N, \mathbb{Q}/n\mathbb{Z}) \simeq \varprojlim \widetilde{K}^0((BS_n)_N).$$

□

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