Local solvability for semilinear Fuchsian equations

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1 Introduction

In literature there are many known results concerning local solvability for nonlinear partial differential equations defined in an open set Ω of $\mathbb{R}^n$.
Consider a general linear partial differential operator:

$$P = P(x, D) = \sum_{|\alpha| \le m} c_\alpha(x) D^\alpha. \tag{1.1}$$

In (1.1) we use standard notations, in particular $x = (x_1, ..., x_n)$ is the variable in an open subset Ω of $\mathbb{R}^n$, $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, with $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D_{x_j} = -i\partial_{x_j}$.
As usual in the general theory, we assume the coefficients $c_\alpha(x)$ to be smooth, let us say C^∞, functions of $x \in \Omega$, allowing complex values.
The most primitive question one can ask concerning the equation $Pu = g$, for a given right-hand side g, is if there exists a solution u, at least locally, and not subjected to any additional condition. More precisely, according to the standard terminology, we say that P in (1.1) is solvable in Ω if for every $g \in C_0^\infty(\Omega)$ there exists a distribution u in Ω such that $Pu = g$; the operator P is locally solvable at $x^0 \in \Omega$ if it is solvable in some neighborhood of x^0 contained in Ω.

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During these last years the attention in the literature has been mainly addressed to the semi-linear case:

$$P(x,D)u + f(x, D^{\alpha}u)_{|\alpha|\leq m-1} = g(x), \tag{1.2}$$

where the local solvability of $P(x,D)$ is assumed to be already known, see Gramchev, Popivanov[6] and Dehman[3] where, exploiting the fact that the nonlinear part of the equation (1.2) involves derivatives of order $\leq m-1$, one is reduced to applications of the classical contraction principle and Brower's fixed point theorem, provided the linear part is subelliptic. The general case of $P(x,D)$ satisfying the famous $(\mathcal{P})$ condition of Nirenberg and Trèves (cf. Nirenberg-Trèves [11]) has been settled in Hounie, Santiago[7], by combining the contraction principle with compactness arguments. Let us also mention Messina-Rodino [9], treating the case when $P(x,D) = P(D)$ is an operator with constant coefficients.
We shall focus here on the case of a manifold B with a conical singularity. As far as we know there are no precise references about the problem of local solvability of semilinear partial differtial equations defined on this type of manifold. We recall that B, Hausdorff space, is a smooth manifold outside the singular point, while, close to this point it has a structure of a cone with smooth, closed cross-section X. Blowing up B near its tip, we obtain a manifold $\mathbb{B}$ with boundary $\partial\mathbb{B} = X$.
Near the boundary, we fix a splitting of coordinates $(r,x) \in [0,1[\times X$. Rather than on B the analysis will be performed on $\mathbb{B}$ (respectively on the interior of $\mathbb{B}$). We consider so-called cone or Fuchs-type differential operators, i.e., operators which close to the boundary are of the form

$$A = r^{-\nu}\sum_{j=0}^{\nu} a_j(r)(-r\partial_r)^j. \tag{1.3}$$

where each $a_j \in \mathcal{C}^{\infty}(\overline{\mathbb{R}_+}, Diff^{\nu-j}(X))$ is a smooth family of differential operators on the cross-section. Such an A acts as an unbounded operator $A : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-\nu,\gamma-\nu}(\mathbb{B})$ where the space $\mathcal{H}^{s,\gamma}(\mathbb{B})$ away from the boundary coincides with $H^s(\mathbb{B})$ and near the boundary $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$ is such that

$$r^{\frac{n+1}{2}-\gamma}(r\partial_r)^k D_l u \in L_2\left([0,1[\times X, \frac{dr}{r}dx\right), \quad \forall k+l \leq s, \quad n = \dim X.$$

The general theory about the weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{B})$ and the pseudodifferential operators defined on manifold with conical singularity has been

introduced by Schulze and it has been developed in many books and papers, for example Egorov-Schulze [5], Schulze [12], Schulze [13], Seiler [14].
In the present paper we want to discuss the case of a semi-linear equation, with the linear part given by the general operator of Fuchs type A elliptic in a suitable sense with respect a fixed weight γ_0 (see Egorov-Schulze [5], Seiler [14], Witt [16]) in (1.3):

$$Au + f(x, Q_1 u, ..., Q_M u) = g. \tag{1.4}$$

where Q_i are differential operators of Fuchs type for every $1 \leq i \leq M$, $f(x, v)$, with $x \in \mathbb{B}$ and $v \in \mathbb{C}^M$, is entire analytic with respect to v with the coefficients belonging to $\bigcup_{s\in\mathbb{R}} \mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})$, $f(x, 0) = 0$ for every $x \in \mathbb{B}$ and g belonging to $\mathcal{H}^{s,\gamma_0}(\mathbb{B})$.
Under suitable assumptions on the smoothness order s and the weight γ the weighted Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{B})$ forms an algebra and we may give meaning to the nonlinear term, cf. Liu-Witt [10], Witt [16]. The local solvability Theorem is stated in Section 4 and proved in section 6 by the Schauder Fixed Point Theorem. We assume the nonlinearity involves only Fuchs operators Q_i of strictly lower order than the linear term A. The first step is to extend a well known property of the standard Sobolev spaces on $\mathbb{R}^n$ to the case of the weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{B})$, namely, let $s > s'$,$s' \geq 0$,$\gamma > \gamma'$,$\gamma' \geq \frac{n}{2}$, then

$$||u||_{\mathcal{H}^{s',\gamma'}(\mathbb{B})} \leq C(\epsilon)||u||_{\mathcal{H}^{s,\gamma}(\mathbb{B})}$$

for every $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$ with $\operatorname{supp} u \subset [0, \epsilon) \times X$ and $C(\epsilon) \to 0$ for $\epsilon \to 0$.
Let us also note that the hypotheses on s and γ under which $\mathcal{H}^{s,\gamma_0}(\mathbb{B})$ is an algebra assure that a function u_0 solution of (1.4) in the interior satisfies (1.4) up to the boundary. We finally emphasize that our solutions are local near the singular point; the more difficult problem of the global existence of solutions in $\mathbb{B}$ to (1.4) is outside the aims of the present paper.

2 Differential operators on manifolds with conical singularities

In this section we consider spaces, called manifolds with conical singularities, common to many applications in mathematical physics, mechanics, engineering, and also in branches of pure mathematics such as geometry and topology.

Then we will consider a generalization of the standard Sobolev spaces defined on them and the typical differential operators on those manifolds. We limit ourselves here to definitions and statement of results, addressing for details and proofs to Egorov-Schulze [5], Schulze [12],[13], Seiler [14], Witt [16].
First we introduce same notations.

- $\mathbb{R}_+ = (0, +\infty)$, $\overline{\mathbb{R}_+} = [0, +\infty)$
- X is a smooth compact manifold without boundary;
- $X^\wedge := \mathbb{R}_+ \times X$, interpreted as the open stretched cone (or as the open cilinder) with base X, where $\mathbb{R}_+ = (0, \infty)$
- $X^\triangle := (\overline{\mathbb{R}_+} \times X)/(\{0\} \times X)$, interpreted as the cone.

Definition 2.1. *Let B be an Hausdorff topological compact space and $b \in B$ such that $B \setminus b$ is a smooth manifold (without boundary) of dimension $n+1$. To say that B is a manifold with conical singularity b means to require that the following properties hold:*

i) the existence of a neighbourhood U of b and of a diffeomorphism

$$\Theta : U \setminus b \to X^\wedge \tag{2.1}$$

for same closed compact smooth manifold X (without boundey) of dimension n,

ii) Θ is extendible to a homeomorphism $\bar{\Theta} : U \to X^\triangle$.

If

$$\Phi : U \setminus b \to X^\wedge \tag{2.2}$$

is another diffeomorphism, analogously extendible to U, we say that (2.1) and (2.2) are equivalent if $\Theta\Phi^{-1} : X^\wedge \to X^\wedge$ is the restriction of same diffeomorphism $\overline{\mathbb{R}_+} \times X \to \overline{\mathbb{R}_+} \times X$ to $\mathbb{R}_+ \times X$.
The equivalence classes of the map (2.1) are regarded as part of the "geometry" on $B \setminus b$ in the neighbourhood of the conical singularity, which is fixed once and for all.
If we keep the map (2.1) fixed, then

$$\partial_\lambda u := \phi^{-1}(\lambda r, x) \quad \text{for} \quad \phi(u) = (r, x),\ \lambda \in \mathbb{R}_+, u \in U \setminus \{b\}, \tag{2.3}$$

induces an $\mathbb{R}_+$ action on $U \setminus \{b\}$.
The manifold X is called the base of the cone, and $\mathbb{R}_+$ the cone axis.
It follow easly from the assumption that there is a smooth manifold $\mathbb{B}$ with compact $\mathcal{C}^\infty$ boundary $\partial\mathbb{B} \cong X$ such that there is a diffeomorphism

$$B \setminus \{b\} \cong \mathbb{B} \setminus \partial\mathbb{B}$$

the restriction of which to $U_1 \setminus \{b\}$ is diffeomorphism $U_1 \setminus \{b\} \cong V_1 \setminus \partial\mathbb{B}$ for an open neighbourhood $U_1 \subset B$ of b and a collar neighboorhood $V_1 \subset \mathbb{B}$ of $\partial\mathbb{B}$, i.e. $V_1 \cong \{[0,1] \times X\}$.
$\mathbb{B}$ is also called the stretched manifold with "conical singularity" associated with B. Via this identification the analysis of differential operators on B is carried out on $\mathbb{B}$.
The typical differential operators that we will considered on manifolds with conical singularity are defined as in the following.
An operator $A \in Diff^m(X^\wedge)$, espressed in $(r,x) \in \mathbb{R}_+ \times X = X^\wedge$, is said to be of Fuchs type if A is in a neighbourhood of $r = 0$ of the form

$$A = r^{-m} \sum_{k=0}^{m} a_k(r) \left(-r\frac{\partial}{\partial r}\right)^k \tag{2.4}$$

with coefficients

$$a_k(r) \in \mathcal{C}^\infty(\overline{\mathbb{R}_+}, Diff^{m-k}(X)), \tag{2.5}$$

r^{-m} is also called a weight factor, with the weight $-m$.
The expression (2.4) is equivalent to the following

$$A = r^{-m} \sum_{k+|\alpha|=0}^{m} a_{j,k}(r,x) \left(-r\frac{\partial}{\partial r}\right)^k \partial_x^\alpha, \tag{2.6}$$

where $(r,x) \in \overline{\mathbb{R}_+} \times \mathbb{R}^n$ refers to the local coordinates near the sigular point b and $a_{jk} \in \mathcal{C}^\infty(\overline{\mathbb{R}_+} \times \mathbb{R}^n)$.

Definition 2.2. *Let B be a manifold with a conical singularity b. Then $A \in Diff^m(B \setminus \{b\})$ is said to be of Fuchs type if b has a neighbourhood such that A in local coordinates (r,x) is of Fuchs type in the sense of* (2.4).
Similarly we say that $A \in Diff^m(int\mathbb{B})$ is of Fuchs type if it is in a neighbourhood of $\partial\mathbb{B}$ of the form (2.4).

It is possible to prove, see for example Egorov-Schulze [5], that the previous definition is corret in the sense that it only depends on the equivalence class of (2.1). It means that if A is of Fuchs type in the coordinates associated to the diffeomorphism Θ defined as in (2.1) it remains of Fuchs type in the coordinates associated to an equivalent diffeomorphism Φ.
Let us denote by

$$Diff^m(\mathbb{B})_{Fuchs}$$

the space of all differential operators of order m of fuchs type on $\mathbb{B}$.

Corollary 2.1. *The operators of Fuchs type over a manifold B with a conical singularity form an algebra.*

The term $-r\frac{\partial}{\partial r}$ has an important role also the properties of the Mellin Transform

$$\mathcal{M}u(z) := \int_0^{\infty} r^{z-1}u(r)dr. \tag{2.7}$$

The basic properties of the Mellin Transform (which is a classical integral transform) may be found in Schulze [12] . Note that

$$\mathcal{M}(r^{\beta}u)(z) = (\mathcal{M}u)(z+\beta). \tag{2.8}$$

It is easy to prove that

$$\mathcal{M}(-r\partial_r u)(z) = z\mathcal{M}(u)(z). \tag{2.9}$$

Another usefull property is

$$\mathcal{M}v(z) = \mathcal{M}u(-z) \tag{2.10}$$

where $v(r) = u\left(\frac{1}{r}\right)$.
We will consider the weight Mellin Transform

$$\mathcal{M}_{\delta}u(Imz) := \int_0^{\infty} r^{z-1}u(r)dr\Big|_{Rez=\frac{1}{2}-\delta}$$

where Imz is the immaginary part of z, Rez is the real part of z, $\Gamma_{\delta} := \{z \in \mathbb{C}/Rez = \delta\}$, $\delta \in \mathbb{R}$ (a line in the complex space $\mathbb{C}$).

There exists a relation between the Mellin Transform and the Fourier Transform. Consider the following isomorphism

$$S_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\mathbb{R}), \tag{2.11}$$

with $r^\gamma L^2(\mathbb{R}_+) = \{u/r^{-\gamma}u \in L^2(\mathbb{R}_+)\}$, defined as

$$S_\gamma u(x) := e^{\left(\gamma - \frac{1}{2}\right)x} u(e^{-x}). \tag{2.12}$$

Hence

$$\mathcal{F}S_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \tag{2.13}$$

where $\mathcal{F}$ is the Fourier Transform, is also an isomorphirsm.
Then we get the following relation.

Proposition 2.1. *Let* $\gamma \in \mathbb{R}$

$$\mathcal{M}_\gamma u\left(\frac{1}{2} - \gamma + \tau\right) = \mathcal{F}(S_\gamma u)(\tau). \tag{2.14}$$

As a consequence we get

$$\mathcal{M}_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2\left(\Gamma_{\frac{1}{2}-\gamma}\right) \tag{2.15}$$

is also an isomorphism.
The expression of the inverse of the Mellin Transform follows easily from (2.14).

Proposition 2.2. *Let* $\gamma \in \mathbb{R}$

$$((\mathcal{M}_\gamma)^{-1}h)(r) = \int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} h(z) đz \tag{2.16}$$

where $đz = \frac{dz}{2\pi i}$.

For a smooth compact manifold X, we denote by $L^\mu(X;\Lambda)$ the set of all parameter-dependent classical pseudodifferential operators of order $\mu \in \mathbb{R}$ on X, i.e.

i) the local symbols satisfy the estimates

$$\sup_{(x,\xi,\lambda)\in\mathbb{R}^n\times\mathbb{R}^n\times\Lambda} \mid \langle\xi,\lambda\rangle\partial_\xi^\alpha\partial_\lambda^k\partial_x^\beta a(x,\xi,\lambda)\mid < +\infty$$

for all multi-index α,β and for all k. Here, n is the dimension of X and, as usual, $\langle\xi,\lambda\rangle = (1+\mid\xi\mid^2+\mid\lambda\mid^2)^{\frac{1}{2}}$

ii) the local symbols are classical, i.e. we require that these symbols to allow asymptotic expansions $a\sim\sum a_{(\mu-j)}$ with $a_{(\mu-j)}$ positively homogeneous of degree $\mu-j$ in (ξ,λ). Thus, the parameter λ is treaded ad an additional covariable.

Definition 2.3. *Let* $\omega\in\mathcal{C}_0^\infty(X^\wedge)$*, we say that it is a cut-off function if depends only on* $r\in\mathbb{R}$*,* $\operatorname{supp}\omega\subset[0,1)\times X$*,* $0\le\omega\le1$ *and* $\omega\equiv1$ *close to* $r=0$*.*

Definition 2.4. $\mathcal{H}^{s,\gamma}(X^\wedge)$ *for* $s,\gamma\in\mathbb{R}$ *is the space of all* $u\in H^s_{loc}(X^\wedge)$ *such that for every* $\phi\in\mathcal{C}_0^\infty(X^\wedge)$ *supported close to* $r=0$ *we have*

$$\left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}}\int_{\mathbb{R}^n}(1+\mid z\mid^2+\mid\xi\mid^2)^s\mid\mathcal{M}_{\gamma-\frac{n}{2},r\to z}\mathcal{F}_{x\to\xi}(\phi u)(z,\xi)\mid^2 d\xi dz\right\}^{\frac{1}{2}} < +\infty \tag{2.17}$$

and $v(r,x)=r^{-n-1}u(r^{-1},x)$ *satisfies the analogues conditions with* $-\gamma$ *instead of* γ*.*

From (2.9) and (2.10) one can deduce an equivalent definition for $s\in\mathbb{N}$.

Definition 2.5. *We say that* $u\in\mathcal{H}^{s,\gamma}(X^\wedge)$*,* $s\in\mathbb{N},\gamma\in\mathbb{R}$ *if*

$$r^{\frac{n+1}{2}-\gamma}(r\partial_r)^k D_l u\in L_2\left(\mathbb{R}_+\times X,\frac{dr}{r}dx\right)\quad\forall k+l\le s.$$

Here $\frac{dr}{r}$ *is the measure on* $\mathbb{R}$*,* dx *is the measure on* X *and* D_l *are arbitrary differential operators on* X *of order* l*.*

Proposition 2.3. *For every* $m\in\mathbb{R}$ *there exists* $B^m\in L^m(X,\mathbb{R})$ *such that*

$$B^m(\lambda): H^s(X)\longrightarrow H^{s-m}(X)$$

is an isomorphism for some $s\in\mathbb{R}$*. Its local symbol have to be of the form*

$$b^m(\lambda)=(c+\lambda^2+\mid\xi\mid^2)^{\frac{m}{2}}$$

with c sufficiently large.

Reminding the definition of pseudodifferential operators on a smooth compact manifold and the Fourier Transform properties we get an equivalent definition of $\mathcal{H}^{s,\gamma}(X^\wedge)$ that gives us a Banach structure of this spaces.

Definition 2.6. *$\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ is the closure of $\mathcal{C}_0^\infty(X^\wedge)$ with respect to the norm*

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|B^s(Imz)\big(\mathcal{M}_{\gamma-\frac{n}{2}}u)(z,\cdot)\|^2_{L_2(X)} dz \right\}^{\frac{1}{2}}. \quad (2.18)$$

If $\widetilde{B^s}(z)$ is another reducing family of the above kind then the analogous norm in terms of $\widetilde{B^s}(z)$ is equivalent to (2.18)*.*

Let us observe that there exists another rappresentation of $\mathcal{H}^{s,\gamma}(X^\wedge)$ but before giving it we have to recall the definition of $H^s(\mathbb{R}\times X)$.

Definition 2.7. *$H^s(\mathbb{R}\times X)$ is the subspace of all $u \in \mathcal{D}'(\mathbb{R}\times X)$ with $f^*(\varphi u) \in H^s(\mathbb{R}^{n+1})$ for every chart $f : V \to \mathbb{R}^n$ on X and an arbitrary $\varphi \in \mathcal{C}_0^\infty(V)$.*
It suffices to take a finite open covering $\mathcal{V} = \{V_1, .., V_N\}$ by coordinate neighbourhoods and fixed charts $f_j : V_j \to \mathbb{R}^n$, $1 \le j \le N$. Then, if $\{\varphi_j\}_{1\le j\le N}$ is a subordinated partition of unity,

$$\|u\|_{H^s(\mathbb{R}\times X)} = \left\{ \sum_{j=1}^{N} \|f_{j,*}(\varphi_j u)\|^2_{H^s(\mathbb{R}^{n+1})} \right\}^{1/2}$$

is a norm on $H^s(\mathbb{R}\times X)$ under which it becames a Banach space. Any other choise of $\mathcal{V}, f_j, \varphi_j$ gives rise to an equivalent norm.
Note that $\mathcal{C}_0^\infty(\mathbb{R}\times X)$ is dense in $H^s(\mathbb{R}\times X)$.

Proposition 2.4. *The trasformation*

$$\begin{aligned} S_{\gamma-\frac{n}{2}} : \mathcal{C}_0^\infty(\mathbb{R}_+ \times X) &\to \mathcal{C}_0^\infty(\mathbb{R}\times X) \\ u(r,x) &\to (S_{\gamma-\frac{n}{2}}u)(r,x), \end{aligned}$$

cf. (2.12)*, has an extension to an isomorphism*

$$S_{\gamma-\frac{n}{2}} : \mathcal{H}^{s,\gamma}(X^\wedge) \to H^s(\mathbb{R}\times X) \quad (2.19)$$

for every $s, \gamma \in \mathbb{R}$, and in particular

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} \sim \|S_{\gamma-\frac{n}{2}}u\|_{H^s(\mathbb{R}\times X)}$$

i.e. there exist $C_1, C_2 > 0$ such that

$$C_1\|S_{\gamma-\frac{n}{2}}u\|_{H^s(\mathbb{R}\times X)} \leq \|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} \leq C_2\|S_{\gamma-\frac{n}{2}}u\|_{H^s(\mathbb{R}\times X)}. \tag{2.20}$$

Let us observe that an immediate consequence of the previous proposition is

$$\mathcal{H}^{s,\gamma+\delta}(X^\wedge) = r^\delta\mathcal{H}^{s,\gamma}(X^\wedge)$$

for all $s, \gamma, \delta \in \mathbb{R}$, then

$$\mathcal{H}^{0,\frac{n}{2}}(X^\wedge) = L^2(X^\wedge). \tag{2.21}$$

From the results on the norm growth of parameter-dependent families of pseudodifferential operators on X and the formula (2.18), we obtain continuos embeddings

$$\mathcal{H}^{s',\gamma}(X^\wedge) \hookrightarrow \mathcal{H}^{s,\gamma}(X^\wedge) \tag{2.22}$$

for all $s, s' \in \mathbb{R}$ with $s' \geq s$ and for all $\gamma \in \mathbb{R}$. They are not compact for $s' > s$.

Proposition 2.5. *We have $\mathcal{H}^{s,\gamma}(X^\wedge) \subset H^s_{loc}(X^\wedge)$ for all $s, \gamma \in \mathbb{R}$.*

Proof. Suppose for semplicity $s \in \mathbb{N}$. Note that in the definition of $H^s_{loc}(X^\wedge)$ compares the term $r\partial_r$, otherwise in the definition of $\mathcal{H}^{s,\gamma}(X^\wedge)$ compares also the term $r^{\frac{n+1}{2}-\gamma}$. But we are far from the boundary and so we can estimates these terms on compact sets. □

Definition 2.8. *Let $s, \gamma \in \mathbb{R}$, $\mathcal{H}^{s,\gamma}(\mathbb{B})$ is the space of all u such that for ω some cut-off function, $\omega u \in \mathcal{H}^{s,\gamma}(X^\wedge)$ and $(1-w)u \in H^s(2\mathbb{B})$.*
We denote by $2\mathbb{B}$ the double of $\mathbb{B}$ or some smooth closed compact manifold in which $\mathbb{B}$ is embedded.
We can also define a norm under which $\mathcal{H}^{s,\gamma}(\mathbb{B})$ is a Sobolev space

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \|(1-\omega)u\|_{H^s(2\mathbb{B})}. \tag{2.23}$$

Proposition 2.6. *The norm defined by (2.23) is independent of the cut-off function ω, i.e. that if $\widetilde{\omega}$ is another cut-off function it gives rise an equivalent norm.*

Remark : It follows immediately from the definition of the weighted Sobolev spaces defined on $\mathbb{B}$ that if ω is a cut-off function, then

$$||\omega u||_{\mathcal{H}^{s},\gamma(X^\wedge)} \leq C(\omega)||u||_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \tag{2.24}$$

and

$$||(1-\omega)(u)||_{H^{s}(2\mathbb{B})} \leq C(\omega)||u||_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \tag{2.25}$$

for every $s, \gamma \in \mathbb{R}$ and for every $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$.
An obvious consequence is that ,let $\widetilde{\omega}$ a cut-off function, then

$$||\widetilde{\omega} u||_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \leq C(\widetilde{\omega})||u||_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \tag{2.26}$$

for every $s, \gamma \in \mathbb{R}$ and for every $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$.

Theorem 2.1. *There are canonical continuos embeddings*

$$\mathcal{H}^{s',\gamma'}(\mathbb{B}) \hookrightarrow \mathcal{H}^{s,\gamma}(\mathbb{B}) \tag{2.27}$$

for all $s, s' \in \mathbb{R}$ with $s' \geq s$ and for all $\gamma, \gamma' \in \mathbb{R}$ with $\gamma' \geq \gamma$ which are compact for $s' > s$, $\gamma' > \gamma$.

Now we can study how the Fuchsian operators act on these weighted Sobolev spaces already defined.

Theorem 2.2. *If $A \in Diff^{m}(int\mathbb{B})$ is of Fuchs type, then A induces continuos operators*

$$\mathcal{H}^{s,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}^{s-m,\gamma-m}(\mathbb{B})$$

for all $s, \gamma \in \mathbb{R}$.

Proof. We start to show that, let $\omega, \omega_0, \omega_1$ be cut-off functions such that $\omega\omega_0 = \omega$ and $\omega\omega_1 = \omega_1$, then there exist A_0 that in local coordinates has the form (2.4) and A_1 an usual linear differential operator defined on $2\mathbb{B}$ such that

$$A = \omega A_0 \omega_0 + (1-\omega)A_1(1-\omega_1).$$

Indeed $\omega(1-\omega_0) \equiv 0$ and $(1-\omega)\omega_1 \equiv 0$ and the differential operators do not encrease the support, then

$$\begin{aligned} Au &= \omega Au + (1-\omega)Au \\ &= \omega A(\omega_0 u + (1-\omega_0)u) + (1-\omega)A(\omega_1 u + (1-\omega_1)u) \\ &= \omega A\omega_0 u + (1-\omega)A(1-\omega_1)u, \end{aligned}$$

and so the first term acts near the boundary and the second term acts far from the boundary.
It's easy to deduce the following continuos mappings

$$\begin{aligned} &\omega A_0 : \mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{H}^{s-m,\gamma-m}(X^\wedge) \\ &(1-\omega)A_1 : H^s(2\mathbb{B}) \hookrightarrow H^{s-m}(2\mathbb{B}), \end{aligned}$$

from which we get

$$\|Au\|_{\mathcal{H}^{s-m,\gamma-m}(\mathbb{B})} \leq C\|\omega_0 u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \|(1-\omega_1)u\|_{H^s(2\mathbb{B})}.$$

Therefore from the estimates (2.24) and (2.25) we obtain the thesis. □

The next problem is to define an adequate notion of ellipticity. This is done in terms of the symbolic structure.
We first have the map

$$\sigma_\psi^m(A) : Diff^m(\Omega) \longrightarrow S^{(m)}(T^*\Omega \setminus 0)$$

with $S^{(m)}(T^*\Omega \setminus 0)$ being the space of all $\mathcal{C}^\infty$ functions on $T^*\Omega \setminus 0$ which are (positevely) homogeneous of order m with respect to the canonical $\mathbb{R}_+$ actions on the fibres of $T^*\Omega$. This can be applied, in particular, to $\Omega = int\mathbb{B}$. We will call $\sigma_\psi^m(A)$ the homogeneous principal symbol of A.

Definition 2.9. *$A \in Diff^m(\mathbb{B})_{Fuchs}$ is called elliptic (of order m, with respect to a weight $\gamma \in \mathbb{R}$) if*

i) $\sigma_\psi^m(A)$ is invertible on $T^*\mathbb{B} \setminus 0$ and, in coordinates $(r,x) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^n$ near the boundary, $r^m\sigma_\psi^m(A)(r,x,r^{-1}\tau,\xi)$ is invertible up to $r=0$ for $(\tau,\xi) \neq 0$,

ii) the Mellin symbol $\sigma_M^m(A)(z) = \sum_{|\alpha|\leq m} a_\alpha(0)z^\alpha$ that, when I fix $z \in \mathbb{C}$, is a linear continuos map $H^s(X) \to H^{s-m}(X)$ for every $s \in \mathbb{R}$, is a family of isomorphisms for some $s \in \mathbb{R}$ and for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

Now we can introduce the definition of parametrix of a differential operator of Fuchs type.

Definition 2.10. *An operator*

$$P \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{H}^{s+m,\gamma}(\mathbb{B})) \tag{2.28}$$

is called a parametrix of $A \in Diff^m_{Fuchs}(\mathbb{B})$ with respect to a fixed weight $\gamma \in \mathbb{B}$, if there exists an $\epsilon > 0$ such that

$$\begin{aligned} AP - I &\in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma-m}(\mathbb{B}), \mathcal{H}^{\infty,\gamma-m+\epsilon}(\mathbb{B})) \\ PA - I &\in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{H}^{\infty,\gamma+\epsilon}(\mathbb{B})). \end{aligned} \tag{2.29}$$

We will use the following important result.

Theorem 2.3. *If $A \in Diff^m(\mathbb{B})$ is elliptic with respect to γ, there exists a parametrix P with respect to the fixed weight γ of A.*

3 When $\mathcal{H}^{s,\gamma}(\mathbb{B})$ is an algebra

We introduce some conditions regarding the smoothness order $s \in \mathbb{R}$ and the weight $\gamma \in \mathbb{R}$ under which the corresponding weighted Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{B})$ has the property of algebra. This result was already in Liu-Witt [10], Witt [16]; since there details were missing, we provide a complete proof in the following.
We will prove the following theorem

Theorem 3.1. *Let $s > \frac{n+1}{2}$, $\gamma, \delta \in \mathbb{R}$, then the pointwise multiplication induces a bilinear continuos map*

$$\mathcal{H}^{s,\gamma}(\mathbb{B}) \times \mathcal{H}^{s,\delta}(\mathbb{B}) \to \mathcal{H}^{s,\gamma+\delta-\frac{(n+1)}{2}}(\mathbb{B}),$$

i.e. there exists $D > 0$ such that

$$\|uv\|_{\mathcal{H}^{s,\gamma+\delta-\frac{(n+1)}{2}}(\mathbb{B})} \leq D\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\|v\|_{\mathcal{H}^{s,\delta}(\mathbb{B})}. \tag{3.1}$$

An obvious consequence is

Corollary 3.1. *Let $s > \frac{n+1}{2}$ and $\gamma \geq \frac{n+1}{2}$, then $\mathcal{H}^{s,\gamma}(\mathbb{B})$ is an algebra.*

To be definite, to prove theorem 3.1 first we verify when the standard Sobolev spaces defined on a smooth closed compact manifold have the property of algebra.

Theorem 3.2. *Let M be a smooth closed compact manifold of dimension m and $s > \frac{m}{2}$, then $H^s(M)$ has the property of algebra.*

Proof. Let $\mathcal{V} = \{V_1, .., V_N\}$ be a finite open covering by coordinate neighbourhoods and fix charts $f_j : V_j \to \mathbb{R}^n$, $1 \leq j \leq N$. Then, if $\{\varphi_j\}_{1\leq j\leq N}$ is a subordinated partition of unity, we have

$$\|uv\|_{H^s(M)} = \left\{ \sum_{j=1}^{N} \|\varphi_j(f_j^{-1}(x))u(f_j^{-1}(x))v(f_j^{-1}(x))\|^2_{H^s(\mathbb{R}^m)} \right\}^{1/2}.$$

Consider $\widetilde{\varphi}_j \in \mathcal{C}_0^\infty(V_j)$ such that $\varphi_j \widetilde{\varphi}_j = \varphi_j$ for every $j = 1, ..N$, then

$$\|uv\|_{H^s(M)} = \left\{ \sum_{j=1}^{N} \|\varphi_j(f_j^{-1}(x))u(f_j^{-1}(x))\widetilde{\varphi}_j(f_j^{-1}(x))v(f_j^{-1}(x))\|^2_{H^s(\mathbb{R}^m)} \right\}^{1/2};$$

by observing that the hypothesis $s > m/2$ is the condition under which the standard Sobolev spaces on $\mathbb{R}^m$ has the property of algebra we get

$$\|uv\|^2_{H^s(M)} \leq \sum_{j=1}^{N} C\|\varphi_j(f_j^{-1})u(f_j^{-1})\|^2_{H^s(\mathbb{R}^m)}\|\widetilde{\varphi}_j(f_j^{-1})v(f_j^{-1})\|^2_{H^s(\mathbb{R}^m)}. \quad (3.2)$$

Now we have to observe that, let $u \in H^s(M)$, $U_0 \subseteq M$, $f_0 : U_0 \subseteq M \to \mathbb{R}^m$ be a chart, $K \subseteq U_0$ be compact, $\operatorname{supp} u \subseteq K$, then

$$\|u\|_{H^s(M)} \simeq \|u \circ f_0^{-1}\|_{H^s(\mathbb{R}^m)}. \quad (3.3)$$

Indeed, let $\psi_0 \in \mathcal{C}_0^\infty(U_0)$ with $\psi_0 \equiv 1$ on K, $0 \leq \psi_0 \leq 1$ and $\psi_0, .., \psi_N$ be a finite partition of unity where ψ_0 is the first term, we get

$$\begin{aligned}\|u\|_{H^s(M)} &\simeq \left(\sum_{j=1}^{N} \|(\psi_j u) \circ f_j^{-1}\|^2_{H^s(\mathbb{R}^m)} \right)^{\frac{1}{2}} \\ &= \|(\psi_0 u) \circ f_0^{-1}\|_{H^s(\mathbb{R}^m)} = \|u \circ f_0^{-1}\|_{H^s(\mathbb{R}^m)}.\end{aligned}$$

A consequence is that, let $\psi \in \mathcal{C}_0^\infty(U_0)$, there exists a positive constant C such that

$$\|(\psi u) \circ f_0^{-1}\|_{H^s(\mathbb{R}^m)} \le C\|u\|_{H^s(M)}. \tag{3.4}$$

Indeed, applying (3.3) to ψu, we obtain

$$\begin{aligned} \|(\psi u) \circ f_0^{-1}\|_{H^s(\mathbb{R}^m)} \simeq \|\psi u\|_{H^s(M)} &= \left(\sum_{j=1}^N \|(\psi_j \psi u) \circ f_j^{-1}\|^2_{H^s(\mathbb{R}^m)}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{j=1}^N C_j \|(\psi_j u) \circ f_j^{-1}\|^2_{H^s(\mathbb{R}^m)}\right)^{\frac{1}{2}} \le C\|u\|_{H^s(M)}. \end{aligned} \tag{3.5}$$

From (3.2) and (3.4) we can deduce

$$\|uv\|^2_{H^s(M)} \le \sum_{j=1}^N CC_j \|u\|^2_{H^s(M)} \widetilde{C}_j \|v\|^2_{H^s(M)} \le \widetilde{C} \|u\|^2_{H^s(M)} \|v\|^2_{H^s(M)}$$

and so the theorem is proved. □

In analogous way it is possible to prove the following theorem regarding the condition under which the space $H^s(\mathbb{R} \times X)$ has the propery of algebra.

Theorem 3.3. *Let X be a smooth closed compact manifold of dimension m and $s > \frac{m+1}{2}$, then $H^s(\mathbb{R} \times X)$ is an algebra.*

We need also to know when the wieghted Sobolev spaces defined on $X^\wedge$ have the property of algebra.

Theorem 3.4. *Let $s > \frac{n+1}{2}$, with $n = dim X$, then $\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)$ is an algebra.*

Proof. Remind the relation between $\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)$ and $H^s(\mathbb{R} \times X)$, cf. proposition 2.4, in particular from the right part of (2.20) we get

$$\begin{aligned} \|uv\|_{\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)} &\le C\|S_{\frac{1}{2}}(uv)\|_{H^s(\mathbb{R}\times X)} \\ &= \|u(e^{-r}, x)v(e^{-r}, x)\|_{H^s(\mathbb{R}\times X)}. \end{aligned} \tag{3.6}$$

But, from theorem 3.3 and from the left part of (2.20), we obtain

$$\begin{aligned} \|u(e^{-r}, x)v(e^{-r}, x)\|_{H^s(\mathbb{R}\times X)} &\le C_1 \|S_{\frac{1}{2}} u\|_{H^s(\mathbb{R}\times X)} \|S_{\frac{1}{2}} v\|_{H^s(\mathbb{R}\times X)} \\ &\le C_2 \|u\|_{\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)} \|v\|_{\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)}. \end{aligned} \tag{3.7}$$

□

Now we have all the arguments to introduce the condition on the smoothness order $s \in \mathbb{R}$ under which $\mathcal{H}^{s,\frac{n+1}{2}}(2\mathbb{B})$ has the property of algebra.

Theorem 3.5. *Let $s > \frac{n+1}{2}$ then $\mathcal{H}^{s,\frac{n+1}{2}}(2\mathbb{B})$ is an algebra.*

Proof. Let ω be an arbitrary cut-off function, we have to estimate

$$\|uv\|_{\mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})} = \|\omega uv\|_{\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)} + \|(1-\omega)u\|_{H^s(2\mathbb{B})}. \tag{3.8}$$

Let ω_0 and ω_1 two cut-off functions such that $\omega\omega_0 = \omega$ and $\omega\omega_1 = \omega_1$, then, using the same arguments developed to prove Theorem 2.2

$$\|uv\|_{\mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})} = \|\omega u\omega_0 v\|_{\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)} + \|(1-\omega)u(1-\omega_1)v\|_{H^s(2\mathbb{B})} \tag{3.9}$$

and applying theorem 3.2 and 3.4 the thesis follows immediately. □

Let us observe

Proposition 3.1. *Let $k^\delta \in \mathcal{C}^\infty(\mathbb{B})$ such that*

$$k^\delta(r,t) = \begin{cases} r^\delta & \text{for } r \text{ near } 0 \\ 1 & \text{otherwise} \end{cases}$$

then $k^\delta \mathcal{H}^{s,\gamma}(\mathbb{B}) = \mathcal{H}^{s,\gamma+\delta}(\mathbb{B})$ and in particular the map

$$\mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s,\gamma+\delta}(\mathbb{B})$$
$$u \to k^\delta u$$

is continuos.

Proof. We have to estimate

$$\|k^\delta u\|_{\mathcal{H}^{s,\gamma+\delta}(\mathbb{B})} = \|\omega k^\delta u\|_{\mathcal{H}^{s,\gamma+\delta}(X^\wedge)} + \|(1-\omega)k^\delta u\|_{H^s(\mathbb{B})}.$$

where ω is an arbitrary cut-off function.
Let us choose ω such that $\omega k^\delta u = \omega r^\delta u$, then

$$\|k^\delta u\|_{\mathcal{H}^{s,\gamma+\delta}(\mathbb{B})} = \|\omega r^\delta u\|_{\mathcal{H}^{s,\gamma+\delta}(X^\wedge)} + \|(1-\omega)k^\delta u\|_{H^s(\mathbb{B})}.$$

As usual we separately analyse the two terms,

$$\|\omega r^\delta u\|_{\mathcal{H}^{s,\gamma+\delta}(X^\wedge)} = \left(\frac{1}{2\pi}\int_{\mathbb{R}} \|B^s(\lambda)\big(\mathcal{M}_{\gamma+\delta-\frac{n}{2}}(\omega r^\delta u)(\lambda,\cdot)\big)\|^2_{L^2(X)} d\lambda\right)^{\frac{1}{2}} \tag{3.10}$$

and

$$\begin{aligned}\mathcal{M}_{\gamma+\delta-\frac{n}{2}}(\omega r^\delta u) &= \mathcal{F}\big(S_{\gamma+\delta-\frac{n}{2}}(\omega r^\delta u)\big) = \mathcal{F}\big(e^{(\gamma+\delta-\frac{n}{2}+1)r}\omega(e^{-r})e^{-\delta r}u(e^{-r})\big)\\ &= \mathcal{F}\big(e^{(\gamma-\frac{n}{2})r}\omega(e^{-r})u(e^{-r})\big) = \mathcal{F}S_{\gamma-\frac{n}{2}+1}(\omega u)\\ &= \mathcal{M}_{\gamma-\frac{n}{2}}(\omega u).\end{aligned} \tag{3.11}$$

Replacing (3.11) in (3.10) we get

$$\begin{aligned}\|\omega r^\delta u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} &= \left(\frac{1}{2\pi}\int_{\mathbb{R}} \|B^s(\lambda)\big(\mathcal{M}_{\gamma-\frac{n}{2}}(\omega u)(\lambda,\cdot)\big)\|^2_{L^2(X)}d\lambda\right)^{\frac{1}{2}}\\ &= \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)}.\end{aligned}$$

Moreover, from the boundness of the function k^δ and of all its derivatives, we have

$$\|(1-\omega)k^\delta u\|_{H^s(2\mathbb{B})} \leq C(\delta)\|(1-\omega)u\|_{H^s(2\mathbb{B})}.$$

□

Now we have all the struments to prove theorem 3.1.

Proof of Theorem 3.1. Let us set $u_\gamma = k^{\frac{n+1}{2}-\gamma}u$ and $v_\delta = k^{\frac{n+1}{2}-\delta}v$ with $k^{\frac{n+1}{2}-\gamma}$ and $k^{\frac{n+1}{2}-\delta}$ defined as in the previous proposition, we have to estimate

$$\|uv\|_{\mathcal{H}^{s,\gamma\delta-\frac{n+1}{2}}} = \|k^{\gamma+\delta-(n+1)}u_\gamma v_\delta\|_{\mathcal{H}^{s,\gamma+\delta-\frac{n+1}{2}}}. \tag{3.12}$$

Applying proposition 3.1

$$\|uv\|_{\mathcal{H}^{s,\gamma\delta-\frac{n+1}{2}}} \leq C_{\gamma\delta}\|u_\gamma v_\delta\|_{\mathcal{H}^{s,\frac{n+1}{2}}} \tag{3.13}$$

and from theorem 3.5

$$\|uv\|_{\mathcal{H}^{s,\gamma+\delta-\frac{n+1}{2}}} \leq \widetilde{C}_{\gamma\delta}\|u_\gamma\|_{\mathcal{H}^{s,\frac{n+1}{2}}}\|v_\delta\|_{\mathcal{H}^{s,\frac{n+1}{2}}}. \tag{3.14}$$

The thesis follows immediately using proposition 3.1 again. □

It's possible to generalize this result to the invariance of the weighted Sobolev spaces on $\mathbb{B}$ under the composition with analytic functions.

Theorem 3.6. *Let us consider a non-linear function $f(x,v)$ with $x \in \mathbb{B}$ and $v \in \mathbb{C}^M$ such that*

$$f(x,0) = 0 \tag{3.15}$$

and

$$f(x,v) = \sum_{|\alpha|\geq 1} c_\alpha(x) v^\alpha \tag{3.16}$$

i.e. that f is entire with respect the variable v, and the coefficients

$$c_\alpha \in \bigcup_s \mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B}), \tag{3.17}$$

and, for every $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}$, there exist two positive constants $c_s, \lambda_\alpha > 0$ such that

$$\|c_\alpha\|_{\mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})} \leq c_s \lambda_\alpha \qquad \text{with} \quad \sum \lambda_\alpha v^\alpha < +\infty \quad \forall v \in \mathbb{C}^M. \tag{3.18}$$

Moreover let $s' > \frac{n+1}{2}$, $\delta \geq \frac{n+1}{2}$ and $u_i \in \mathcal{H}^{s',\delta}(\mathbb{B})$, for every $1 \leq i \leq M$, then

$$f(x, u_1, .., u_M) \in \mathcal{H}^{s',\delta}(\mathbb{B}). \tag{3.19}$$

Proof. Set $u := (u_1, ..u_M)$ and $\|u\|_{\mathcal{H}^{s',\delta}(\mathbb{B})} := (\|u_1\|_{\mathcal{H}^{s',\delta}(\mathbb{B})}, .., \|u_M\|_{\mathcal{H}^{s',\delta}(\mathbb{B})})$; under these hypotheses on s' and δ we can apply corollary 3.1 and then $u^\alpha \in \mathcal{H}^{s,\delta}(\mathbb{B})$ for every multiindex α, in particular

$$\|u^\alpha\|_{\mathcal{H}^{s',\delta}(\mathbb{B})} \leq \frac{1}{C}\big(C\|u\|_{\mathcal{H}^{s',\delta}(\mathbb{B})}\big)^\alpha. \tag{3.20}$$

But, using theorem 3.1,

$$\begin{aligned} \|c_\alpha u^\alpha\|_{\mathcal{H}^{s',\delta}(\mathbb{B})} &\leq \|c_\alpha\|_{\mathcal{H}^{s',\frac{n+1}{2}}(\mathbb{B})} \|u^\alpha\|_{\mathcal{H}^{s',\delta}(\mathbb{B})} \\ &\leq \|c_\alpha\|_{\mathcal{H}^{s',\frac{n+1}{2}}(\mathbb{B})} \frac{1}{C}\big(C\|u\|_{\mathcal{H}^{s',\delta}(\mathbb{B})}\big)^\alpha. \end{aligned} \tag{3.21}$$

From the hypothesis (3.18) on the coefficients

$$\begin{aligned} \|f(x,u_1,..,u_M)\|_{\mathcal{H}^{s',\delta}(\mathbb{B})} &\leq \frac{1}{C} \sum_{|\alpha|=1}^{\infty} \|c_\alpha\|_{\mathcal{H}^{s',\frac{n+1}{2}}(\mathbb{B})} \big(C\|u\|_{\mathcal{H}^{s',\delta}(\mathbb{B})}\big)^\alpha \\ &\leq \frac{1}{C} c_s \sum_{|\alpha|=1}^{\infty} \lambda_\alpha \omega^\alpha < \infty \end{aligned} \tag{3.22}$$

where $\omega := C\|u\|_{\mathcal{H}^{s',\delta}(\mathbb{B})}$. ∎

Let us observe that the conditions under which $\mathcal{H}^{s,\gamma}(\mathbb{B})$ has the property of algebra assure another important property of these spaces.

Proposition 3.2. *Let* $s > \frac{n+1}{2}$, $\gamma \geq \frac{n+1}{2}$, $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$ *then*

$$u\Big|_{\partial\mathbb{B}} \equiv 0. \tag{3.23}$$

Proof. First suppose $\gamma = \frac{n+1}{2}$, then if $u \in \mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})$ we have that

$$\|\omega(t,x)u(t,x)\|_{\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge)} \simeq \|\omega(e^{-t},x)u(e^{-t},x)\|_{H^s(\mathbb{R}\times X)} \tag{3.24}$$

with ω an arbitrary cut-off function.
Now we have to observe that $H^s(\mathbb{R}\times X)$ is included with continuity in $L^\infty(\mathbb{R}\times X)$ (see for example Adams [1]). Indeed, reminding the well known property on $H^s(\mathbb{R}^m)$ which, if $s > \frac{m+1}{2}$, are included with continuity in $L^\infty(\mathbb{R}^m)$, and denoting by

$$\omega(e^{-t},x) := \widetilde{\omega}(t,x) \quad \text{and} \quad u(e^{-t},x) := \widetilde{u}(t,x)$$

we get

$$\begin{aligned}
\|v(t,x)\|^2_{H^s(\mathbb{R}\times X)} &= \sum_{j=1}^{N} \|f_{j,*}(\varphi_j v)\|^2_{H^s(\mathbb{R}^{n+1})} \geq C \sum_{j=1}^{N} \|f_{j,*}(\varphi_j v)\|^2_{L^\infty(\mathbb{R}^{n+1})} \\
&= C \sum_{j=1}^{N} \|(\varphi_j v)\|^2_{L^\infty(\mathbb{R}\times X)} \geq C_1 \left\| \sum_{j=1}^{N} (\varphi_j v) \right\|^2_{L^\infty(\mathbb{R}\times X)} \\
&\geq C_2 \|v\|^2_{L^\infty(\mathbb{R}\times X)}.
\end{aligned} \tag{3.25}$$

According to density of $\mathcal{C}_0^\infty(\mathbb{R}\times X)$ in $H^s(\mathbb{R}\times X)$ there exists $\{u_j(t,x)\}$ belonging to $\mathcal{C}_0^\infty(\mathbb{R}\times X)$ such that $\|u_j(t,x) - \omega(e^{-t},x)u(e^{-t},x)\|_{H^s(\mathbb{R}\times X)} \to 0$ for $j \to \infty$ and for the previous consideration

$$\|u_j(t,x) - \omega(e^{-t},x)u(e^{-t},x)\|_{L^\infty(\mathbb{R}\times X)} \to 0 \quad \text{for} \quad j \to \infty.$$

Let us fix $\epsilon > 0$, for j sufficiently large,

$$\|u_j(t,x) - \omega(e^{-t},x)u(e^{-t},x)\|_{L^\infty(\mathbb{R}\times X)} < \epsilon,$$

but for t sufficiently large $u_j \equiv 0$ then $|\omega(e^{-t},x)u(e^{-t},x)| < \epsilon$ and $|u(t,x)| < \epsilon$ for t near 0.
If $u \in \mathcal{H}^{s',\gamma}(\mathbb{B})$ with $\gamma \geq \frac{n+1}{2}$, we observe that $\mathcal{H}^{s',\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}^{s',\frac{n+1}{2}}(\mathbb{B})$, and then the proof is complete. □

4 Statement of the main result

We will study the following semilinear partial differential operator

$$F(u) = Au + f(x, Q_1 u, ..., Q_M u) \tag{4.1}$$

where

(1) A is a differential operator of Fuchs type of order m_0 elliptic with respect to the weight γ_0, defined on $\mathbb{B}$;

(2) $Q_1, ..., Q_M$ are differential operators of Fuchs type defined on $\mathbb{B}$ of order strictly less than m_0;

(3) $f(x, v)$ with $x \in \mathbb{B}$ and $v \in \mathbb{C}^M$ satisfies the conditions (3.15), (3.16), (3.17) and (3.18).

We want to solve locally near the boundary $\partial\mathbb{B}$ the next equation

$$F(u) = g, \quad g \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}). \tag{4.2}$$

In view of theorem 2.2 we know that if $u \in \mathcal{H}^{s,\gamma_0}(\mathbb{B})$ then Au belongs to $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$ and, from condition (2) and theorem 2.1, it follows that $Q_i u$ belongs to $\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})$, with $l > 0$, which is included with compactness in $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$. Therefore, in view of theorem 3.6, one should require $s - m_0 > \frac{n+1}{2}$ and $\gamma \geq \frac{n+1}{2}$ in order to obtain $f(x, Q_1 u, .., Q_M u)$ belonging to $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$. Under these conditions the equation (4.2) will be well defined and we can state the main result of this paper:

Theorem 4.1. *Let $g \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$, with $s - m_0 > \frac{n+1}{2}$ and $\gamma \geq \frac{n+1}{2}$ and consider the operator F defined by* (4.1), *one can find a neighbourhood $U = [0, \epsilon_0] \times X$ of the boundary $\partial\mathbb{B}$ and $u \in \mathcal{H}^{s,\gamma_0}(\mathbb{B})$ such that*

$$Fu = g \qquad \text{in } U. \tag{4.3}$$

In the proof we will use the well known fixed point theorem of Leray-Schauder and so at first we recall its precise statement (see for example Deimling [4]).

Definition 4.1. *Consider two Banach spaces X and Y, a subset Ω of X and a map $F : \Omega \to Y$. Then F is said to be compact if it is continuos and such that $F(\Omega)$ is relatively compact.*

Note that if $F : X \to Y$ is linear and maps bounded sets into relatively compact sets then it's automatically continuos.

Schauder's Fixed Point Theorem *Let X a real Banach space, $C \subset X$ nonempty closed bounded and convex, $F : C \to C$ compact. Then F has a fixed point.*

We will apply the following obvious corollary:

Corollary 4.1. *Let X be a real Banach space, $M > 0$, $B_M(x^0) := \{x \in X / \|x - x^0\| < M\}$, $F : B_M(x^0) \to B_M(x^0)$ compact. Then F has a fixed point.*

5 Proof of the theorem

In the proof of Theorem 4.1 we will use the fixed point Theorem of Schauder. To apply this theorem we have to extend a property of the Sobolev spaces to the case of the weighted Sobolev spaces defined on the manifold with singularities $\mathbb{B}$.

Theorem 5.1. *Let $s > s'$,$s' \geq 0$,$\gamma > \gamma'$,$\gamma' \geq \frac{n}{2}$, then*

$$\|u\|_{\mathcal{H}^{s',\gamma'}(\mathbb{B})} \leq C(\epsilon)\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \tag{5.1}$$

for every $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$ with $\operatorname{supp} u \subset [0, \epsilon) \times X$ *and $C(\epsilon) \to 0$ for $\epsilon \to 0$.*

Proof. First of all note that from Theorem 2.1 it follows that the injection of $\mathcal{H}^{s,\gamma}(\mathbb{B})$ into $\mathcal{H}^{s',\gamma'}(\mathbb{B})$ is compact.
Let us denote by $\mathcal{H}^{s,\gamma}(\mathbb{B})_\epsilon := \{u \in \mathcal{H}^{s,\gamma}(\mathbb{B}) / \operatorname{supp} u \subset [0, \epsilon) \times X\}$, to prove the theorem we have to verify that

$$\sup_{u \in \mathcal{H}^{s,\gamma}(\mathbb{B})_\epsilon} \frac{\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}}{\|u\|_{\mathcal{H}^{s',\gamma'}(\mathbb{B})}} = C(\epsilon) \quad \text{where} \quad C(\epsilon) \to 0 \tag{5.2}$$

that is equivalent to prove that

$$\sup_{u \in \mathcal{H}^{s,\gamma}(\mathbb{B})_\epsilon, \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = 1} \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = C(\epsilon) \quad \text{where} \quad C(\epsilon) \to 0. \tag{5.3}$$

We suppose, ab absurdo, that $C(\epsilon)$ doesn't tend to 0 when ϵ tends to 0, then there exists a sequence $\{\epsilon_\nu\}_{\nu \in \mathbb{N}}$ such that $\epsilon_\nu \to 0$ when $\nu \to \infty$ and $C(\epsilon_\nu) \nrightarrow 0$

when $\nu \to \infty$. We can deduce that there exists a positive constant r and a subsequence $\{\epsilon_{\nu_j}\}_{j\in\mathbb{N}}$ such that $\epsilon_{\nu_j} \nrightarrow 0$ when $j \to \infty$ and $C(\epsilon_{\nu_j}) \geq r$ for every j. Then we obtain that

$$\sup_{u\in\mathcal{H}^{s,\gamma}(\mathbb{B})_\epsilon, \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}=1} \|u\|_{\mathcal{H}^{s',\gamma'}(\mathbb{B})} = C(\epsilon_{\nu_j}) \geq r \quad \text{for every } j. \tag{5.4}$$

From the definition of supremum there exists a sequence of distributions $u_{\nu_j} \in \mathcal{H}^{s,\gamma}(\mathbb{B})$ with support contained in $[0, \epsilon_{\nu_j})$ such that

$$\|u_{\nu_j}\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = 1 \quad \text{and} \quad \|u_{\nu_j}\|_{\mathcal{H}^{s',\gamma'}(\mathbb{B})} \geq r. \tag{5.5}$$

The sequence $\{u_{\nu_j}\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{H}^{s,\gamma}(\mathbb{B})$, then, according to the compactness of the injection of $\mathcal{H}^{s,\gamma}(\mathbb{B})$ into $\mathcal{H}^{s',\gamma'}(\mathbb{B})$, we may assume that there exists a subsequence, still denoted by u_{ν_j}, and a distribution $u \in \mathcal{H}^{s',\gamma'}(\mathbb{B})$ such that

$$\|u_{\nu_j} - u\|_{\mathcal{H}^{s',\gamma'}(\mathbb{B})} \to 0 \quad \text{when} \quad j \to \infty. \tag{5.6}$$

It follows

$$\|u_{\nu_j} - u\|_{H^{s'}(int\mathbb{B})} \to 0 \quad \text{when} \quad \text{j} \to \infty, \tag{5.7}$$

since u necessarily has support contained in $\partial\mathbb{B}$.
From the hypotheses on the smoothness order s' and the weight γ' it follows that $\mathcal{H}^{s',\gamma'}(X^\wedge) \subset L^2(\mathbb{R}^+ \times X)$, the boundary of $\mathbb{B}$ has measure zero in $R^+ \times X$, then $\|u\|_{L^2(\mathbb{R}^+\times X)} = 0$ and so $\|u\|_{\mathcal{H}^{s',\gamma'}(\mathbb{B})} = 0$ that is an absurd. $\square$

Let φ_ϵ be a cut-off function with support contained in $[0, \epsilon) \times X$ and P the parametrix, cf. definition 2.10 and Theorem 2.3, of the linear elliptic term A of the operator F defined in (4.1); we consider the operator

$$\tilde{F}_\epsilon : \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}) \to \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}) \tag{5.8}$$

given by the following expression

$$\tilde{F}_\epsilon[v] = g - f(x, \varphi_\epsilon Q_1(Pv), ..., \varphi_\epsilon Q_M(Pv)) - \varphi_\epsilon Rv \tag{5.9}$$

where R is the continuos map from $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$ into $\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})$, with $l > 0$, such that $AP - I = R$.
We have already noted that, from the hypotesis (2) after (4.1) it follows

that the injection of $\mathcal{H}^{s-m_0-l,\gamma_0-m_0-l}(\mathbb{B})$ into $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$ is compact. In theorem 4.1 we have also supposed that $s - m_0 > \frac{n+1}{2} > 0$ and $\gamma_0 - m_0 \geq \frac{n+1}{2} \geq \frac{n}{2}$, then the hypotheses of Theorem 5.1 are satisfied and so we obtain that there exists a function C of the variable ϵ such that $C(\epsilon) \to 0$ when $\epsilon \to 0$ and

$$\|u\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq C(\epsilon)\|u\|_{\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})} \tag{5.10}$$

for every $u \in \mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})$ with $\operatorname{supp} u \subset [0,\epsilon) \times X$.
Let $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L := \{u \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}) / \|u\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq L\}$ with $L > 2\|g\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}$, to prove Theorem 4.1 we will verify that there exists $\epsilon_0 > 0$ such that the corresponding operator defined by

$$\begin{gathered}\tilde{F}_{\epsilon_0} : \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L \to \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}) \\ \tilde{F}_{\epsilon_0}[v] = g - f(x, \varphi_{\epsilon_0}Q_1 v, ..., \varphi_{\epsilon_0}Q_M v) - \varphi_{\epsilon_0}Rv\end{gathered} \tag{5.11}$$

satisfies the hypotheses of the corollary 4.1 of the Schauder Fixed Point Theorem. Namely, it's continuous, it is defined from $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$ to itself and it maps bounded sets into relatively compact sets.
First of all we will prove that $\tilde{F}_\epsilon(\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L)$ is relatively compact for every $\epsilon > 0$ and so we will consider a sequence $\{v_j\}_{j\in\mathbb{N}}$ belonging to $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$ and we will verify that there exists a convergent subsequence belonging to $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$ of the sequence $\{\tilde{F}_\epsilon[v_j]\}_{j\in\mathbb{N}}$.
Set $\{u_j\}_{j\in\mathbb{N}} := \{Pv_j\}_{j\in\mathbb{N}}$, then, according to continuity of the operator P, $\|u_j\|_{\mathcal{H}^{s,\gamma_0}(\mathbb{B})} \leq L_1$.
From Theorem 2.2 it follows that $\|Q_i u_j\|_{\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})} \leq L_2$ for every $1 \leq i \leq M$ and $j \in \mathbb{N}$, and applying the estimate (2.25) we get that also $\{\varphi_\epsilon Q_i u_j\}_{j\in\mathbb{N}}$ are bounded in $\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})$.
Denote by

$$E_{i,\epsilon}v_j = (E_{1,\epsilon}v_j, ..., E_{M,\epsilon}v_j) := \varphi_\epsilon Q u_j = (\varphi_\epsilon Q_1 u_j, ..., \varphi_\epsilon Q_M u_j);$$

therefore, according to the compactness of the injection of $\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})$ into $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$, we may suppose that there exists a subsequence $\{v_{j_\nu}\}_{\nu\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ and a function z_1 such that

$$\|E_{1,\epsilon}v_{j_\nu} - z_1\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \to 0 \quad \text{when} \quad \nu \to \infty.$$

The sequence $\{E_{2,\epsilon}v_{j_\nu}\}_{\nu\in\mathbb{N}}$ is a subsequence of $\{E_{2,\epsilon}v_j\}_{j\in\mathbb{N}}$ and so is still bounded in $\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})$, then there exists $\{v_{j_{\nu_l}}\}_{l\in\mathbb{N}}$ and a function z_2 such that

$$\|E_{2,\epsilon}v_{j_{\nu_l}} - z_2\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \to 0 \quad \text{when} \quad l \to \infty.$$

Iterating this process for every $i = 1, .., M$ we will find a subsequence $\{v_{j_n}\}_{n\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ and M functions $z_1, .., z_M$ such that

$$\|E_{i,\epsilon}v_{j_n} - z_i\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \to 0 \quad \text{when} \quad n \to \infty.$$

According to continuity of the injection

$$i : \mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B}) \hookrightarrow \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}), \tag{5.12}$$

we may assume that the sequences $\{E_{i,\epsilon}v_{j_n}\}_{n\in\mathbb{N}}$ are bounded with respect to the norm of $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$ and so $\|E_{i,\epsilon}v_{j_n}\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \le L^*(\epsilon)$ for every $1 \le i \le M$ and $n \in \mathbb{N}$.
To complete the proof we have to verify that $\{f(x, E_{1,\epsilon}v_{j_n}, .., E_{M,\epsilon}v_{j_n})\}_{n\in\mathbb{N}}$ is convergent in $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$.
We will prove that the map

$$F : \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_{L^*} \times ... \times \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_{L^*} \to \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}) \tag{5.13}$$

defined as

$$F[w^1, .., w^M] = f(x, w^1, .., w^M) \tag{5.14}$$

is sequentially continuous, then

$$\|f(x, E_{1,\epsilon}v_{j_n}, ..., E_{M,\epsilon}v_{j_n}) - f(x, z_1, ..., z_M)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \to 0 \text{ for } n \to \infty.$$

Let $\{w_n\}_{n\in\mathbb{N}} := \{(w_n^1, ...w_n^M)\}_{n\in\mathbb{N}} \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_{L^*} \times ... \times \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_{L^*}$ and $w := (w^1, .., w^M)$ belong to $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_{L^*} \times ... \times \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_{L^*}$ such that $(w_n^1, ...w_n^M) \to (w^1, ..., w^M)$ for $n \to +\infty$, we should show that

$$F[w_n^1, ..., w_n^M] \to F[w^1, ..., w^M]. \tag{5.15}$$

Set $G_i(x,v,\tilde v) := \int_0^1 \partial_{v_i} f(x, v^1 + t(\tilde v^1 - v^1), .., v^M + t(\tilde v^M - v^M))dt$, we have to estimate

$$\begin{aligned}
&\|F[w_n^1,...w_n^M] - F[w^1,...w^M]\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} = \\
&= \|f(x,w_n^1,..,w_n^M) - f(x,w^1,..,w^M)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} = \\
&= \|\sum_{i=1}^{M}(w_n^i - w^i)G_i(x,w_n,w)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}.
\end{aligned} \tag{5.16}$$

Note that $G_i(x,v,\tilde v)$ is such that the conditions (3.16) , (3.17) , (3.18) hold for every $i = 1,..,M$. Then

$$G_i(x,v,\tilde v) = \sum_{\alpha\geq 0} a_{i,\alpha}(x)(v,\tilde v)^\alpha \quad \text{where} \quad a_{i,\alpha}(x) \in \bigcup_{s\in\mathbb{R}} \mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})$$

and there exist $c_s, \lambda_\alpha > 0$ such that

$$\|a_{i,\alpha}\|_{\mathcal{H}^{s,\frac{n+1}{2}}(\mathbb{B})} < c_s\lambda_\alpha \quad \text{with} \quad \sum_{|\alpha|\geq 0} \lambda_\alpha (v,\tilde v)^\alpha \leq \infty \quad \forall (v,\tilde v) \in \mathbb{C}^{2M}.$$

Applying Corollary 3.1, we have

$$\begin{aligned}
&\|\sum_{i=1}^{M}(w_n^i - w^i)G_i(x,w_n,w)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \\
&\leq C\sum_{i=1}^{M} \|(w_n^i - w^i)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}\|G_i(x,w_n,w)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}.
\end{aligned} \tag{5.17}$$

Reminding Theorem 3.1 it follows that

$$\begin{aligned}
&\|G_i(x,w_n,w)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq A\sum_{\alpha\geq 0}\|a_{i,\alpha}\|_{\mathcal{H}^{s-m_0,\frac{n+1}{2}}(\mathbb{B})}\|(w_n,w)^\alpha\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \\
&\leq A'\sum_{\alpha\geq 0}\|a_{i,\alpha}\|_{\mathcal{H}^{s-m_0,\frac{n+1}{2}}(\mathbb{B})}\left(C^{|\alpha|}(\|w_n\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}, \|w\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})})^\alpha\right) \\
&\leq A'\sum_{\alpha\geq 0} c_{s-m_0}\lambda_\alpha (CL^*)^{|\alpha|} \\
&= A'''\sum_{\alpha\geq 0}\lambda_\alpha (CL^*)^{|\alpha|} < D.
\end{aligned} \tag{5.18}$$

Replacing (5.18) in (5.17) we have

$$\|F[w_n^1,...w_n^M]-F[w^1,...w^M]\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}\leq\sum_{i=1}^{M}D\|w_n^i-w^i\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}$$

that tends to zero when $\|w_n^i-w^i\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}$ tends to zero for every $i=1,..M$.
Using the same arguments it is possible to prove that there exists a subsequence of $\{v_{j_n}\}_{n\in\mathbb{N}}$, still denoted with $\{v_{j_n}\}_{n\in\mathbb{N}}$, such that $\{\varphi_\epsilon Rv_{j_n}\}_{n\in\mathbb{N}}$ is convergent in $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$, then also $\{\widetilde{F}_\epsilon v_{j_n}\}_{n\in\mathbb{N}}$ is convergent with respect to the norm of $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$.
Now we will prove that there exists a constant $\epsilon_0>0$ sufficiently small such that the corresponding operator $\widetilde{F}_{\epsilon_0}$ is defined from $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$ to itself. This is equivalent to verify that for $v\in\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$ such that $\|v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}\leq L$, we get $\|\widetilde{F}_{\epsilon_0}[v]\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}\leq L$.
We have to estimate

$$\begin{aligned}\|\widetilde{F}_\epsilon[v]\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}\leq&\|f(x,\varphi_\epsilon Q_1Pv,..,\varphi_\epsilon Q_MPv)\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}+\\&+\|g\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}+\|\varphi_\epsilon Rv\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}.\end{aligned}\tag{5.19}$$

Let us observe that from the estimate (2.25) we get

$$\|\varphi_\epsilon Q_iPv\|_{\mathcal{H}^{s-m_0-l,\gamma_0-m_0-l}(\mathbb{B})}\leq A(\varphi_\epsilon)\|Q_iPv\|_{\mathcal{H}^{s-m_0-l,\gamma_0-m_0-l}(\mathbb{B})}.\tag{5.20}$$

We have to assume that the cut-off fuctions φ_ϵ have a particular form such that the constant A introduced in (5.20) does not depent on ϵ.
Let us introduce the function $\chi(y)\in C^\infty(\mathbb{R})$, $\chi(y)\in[0,1]$, defined in the following way

$$\chi(y)=\begin{cases}0 & \text{for } y<0\\ 1 & \text{for } y>1\end{cases}$$

and suppose

$$\varphi_\epsilon(e^{-y})=\chi\bigg(y-\frac{1}{\epsilon}\bigg).\tag{5.21}$$

It is trivial to verify that φ_ϵ is a cut-off function such that $\operatorname{supp}\varphi_\epsilon \in [0,\epsilon)$, then, let $\{f_j\}_{1\leq j\leq N}$ be an atlant of X, ψ_j a subordinated partition of unity, ω another cut-off function such that $\varphi_\epsilon\omega = \varphi_\epsilon$, we have to estimate

$$\begin{aligned}
&\|\varphi_\epsilon u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = \|\varphi_\epsilon \omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \|\varphi_\epsilon u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \|S_{\gamma-\frac{n}{2}}(\varphi_\epsilon\omega u)\|_{H^s(\mathbb{R}\times X)}\\
&= \|e^{(\gamma-\frac{n+1}{2})y}\varphi_\epsilon(e^{-y})u(e^{-y},x)\|_{H^s(\mathbb{R}\times X)}\\
&= \left\{\sum_{j=1}^N \|\varphi_j(f_j^{-1}(\xi))e^{(\gamma-\frac{n+1}{2})y}\varphi_\epsilon(e^{-y})u(e^{-y},f_j^{-1}(\xi))\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)}\right\}^{1/2}\\
&= \left\{\sum_{j=1}^N \left\|\varphi_j(f_j^{-1}(\xi))e^{(\gamma-\frac{n+1}{2})y}\chi\left(y-\frac{1}{\epsilon}\right)u(e^{-y},f_j^{-1}(\xi))\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)}\right\}^{1/2},
\end{aligned} \tag{5.22}$$

by replacing $y-\frac{1}{\epsilon}$ with $\tilde{y}$ and from the boundness of χ and of its derivatives we get

$$\begin{aligned}
&\left\{\sum_{j=1}^N \left\|\varphi_j(f_j^{-1}(\xi))e^{(\gamma-\frac{n+1}{2})y}\chi\left(y-\frac{1}{\epsilon}\right)u(e^{-y},f_j^{-1}(\xi))\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)}\right\}^{1/2}\\
&= \left\{\sum_{j=1}^N \|\varphi_j(f_j^{-1}(\xi))e^{(\gamma-\frac{n+1}{2})(\tilde{y}+\frac{1}{\epsilon})}\chi(\tilde{y})u(e^{-(\tilde{y}+\frac{1}{\epsilon})},f_j^{-1}(\xi))\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)}\right\}^{1/2}\\
&\leq \left\{\sum_{j=1}^N D^2\|\varphi_j(f_j^{-1}(\xi))e^{(\gamma-\frac{n+1}{2})y}u(e^{-y},f_j^{-1}(\xi))\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)}\right\}^{1/2}\\
&= \left\{D^2\sum_{j=1}^N \|\varphi_j(f_j^{-1}(\xi))e^{(\gamma-\frac{n+1}{2})y}u(e^{-y},f_j^{-1}(\xi))\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)}\right\}^{1/2}\\
&= D\|e^{(\gamma-\frac{n+1}{2})y}u(e^{-y},x)\|_{H^s(\mathbb{R}\times X)} \leq D_1\|u(t,x)\|_{\mathcal{H}^{s,\gamma}(X^\wedge)}.
\end{aligned} \tag{5.23}$$

Therefore, if $\|v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq L$, we have that $\|Pv\|_{\mathcal{H}^{s,\gamma_0}(\mathbb{B})} \leq L_1$ and $\|Q_iPv\|_{\mathcal{H}^{s,\gamma_0}(\mathbb{B})} \leq L_2$ for every $1\leq i\leq M$, and by the previous consideration $\|\varphi_\epsilon Q_iPv\|_{\mathcal{H}^{s,\gamma_0}(\mathbb{B})} \leq L_3$. Applying theorem 5.1 we obtain

$$\begin{aligned}
\|\varphi_\epsilon Q_iPv\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} &\leq C(\epsilon)\|\varphi_\epsilon Q_iv\|_{\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})}\\
&= C(\epsilon)\|E_{i,\epsilon}v\|_{\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})} \leq C(\epsilon)L_3
\end{aligned}$$

with $C(\epsilon) \to 0$ when $\epsilon \to 0$.
But from the assumptions on the nonlinear function $f(x, v)$ we have

$$\begin{aligned}
&\|f(x, \varphi_\epsilon Q_1 Pv, .., \varphi_\epsilon Q_M Pv)\|\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B}) \\
&= \|\sum_{|\alpha|>0} c_\alpha(x)(E_{1,\epsilon}, .., E_{M,\epsilon}v)^\alpha\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \\
&\leq \sum_{|\alpha|>0} C_1\|c_\alpha\|_{\mathcal{H}^{s-m_0,\frac{n+1}{2}}(\mathbb{B})}(\|E_{1,\epsilon}v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}, .., \|E_{M,\epsilon}v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})})^\alpha C^{|\alpha|} \\
&\leq \sum_{|\alpha|>0} C_2\lambda_\alpha(L_3C(\epsilon), .., L_3C(\epsilon))^\alpha C^{|\alpha|} = \sum_{|\alpha|>0} C_2\lambda_\alpha(\widetilde{C}(\epsilon))^{|\alpha|} \\
&= \sum_{l>0} C_2\mu_l(\widetilde{C}(\epsilon))^l \leq C_3\widetilde{C}(\epsilon)\sum_{l\geq 0}\mu_{l+1}(\widetilde{C}(\epsilon))^l \leq C_3\widetilde{C}(\epsilon)\sum_{l\geq 0}\mu_{l+1} = C_4\widetilde{C}(\epsilon)
\end{aligned} \tag{5.24}$$

where we have chosen ϵ such that $\widetilde{C}(\epsilon) < 1$.
Using again Theorem 5.1 it is obvious to prove that, if $\|v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq L$ then

$$\|\varphi_\epsilon Rv\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq C_5C(\epsilon) \tag{5.25}$$

Replace (5.24) and (5.25) in (5.19) and choose ϵ_0 such that $\widetilde{C}(\epsilon_0) < 1$ and $C_4\widetilde{C}(\epsilon_0) + C_5C(\epsilon_0) \leq L - \|g\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})}$, then $\|\widetilde{F}_{\epsilon_0}[v]\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq L$ for every $v \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$.
In the end continuity of the operator $\widetilde{F}_{\epsilon_0}$ defined from $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$ to itself is trivial to prove. We will prove sequentially continuity; consider the sequence $\{v_j\}_{j\in\mathbb{N}} \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$ such that there exists v belonging to $\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$

$$\|v_j - v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \to 0 \quad \text{for} \quad j \to \infty;$$

but the operators $P, Q_1, .., Q_M$ are continuos, therefore, for every $i = 1, .., M$,

$$\|Q_iPv_j - Q_iPv\|_{\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})} \to 0 \quad \text{for} \quad j \to \infty$$

and consequensly

$$\|\varphi_\epsilon Q_iPv_j - \varphi_\epsilon Q_iPv\|_{\mathcal{H}^{s-m_0+l,\gamma_0-m_0+l}(\mathbb{B})} \to 0 \quad \text{for} \quad j \to \infty$$

According to continuity of the injection (5.6) we can deduce that

$$\|\varphi_\epsilon Q_i P v_j - \varphi_\epsilon Q_i P v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \to 0 \quad \text{for} \quad j \to \infty.$$

As proved before if $\|v\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq L$ then $\|\varphi_\epsilon Q_j R v_j\|_{\mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})} \leq L^*$, therefore the conclusion follows from continuity of the operator F defined in (5.13) and (5.14) and of the operator R.
According to the corollary 4.1 we can assume that there exists a fixed point $v^0 \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})_L$ of the operator $\widetilde{F}_{\epsilon_0}$

$$v^0 = g - f(x, \varphi_{\epsilon_0} Q P v^0) - \varphi_{\epsilon_0} R v^0. \tag{5.26}$$

Let us denote by $U = [0, \epsilon_0) \times X$ the set such that $\varphi_{\epsilon_0}(U) \equiv 1$. Multiplying the left and right hand-side of (5.26) by a cut-off function ψ such that $\psi \varphi_{\epsilon_0} = \varphi_{\epsilon_0}$ and reminding the definition of the operator R we obtain that the function $u^0 := R v^0 \in \mathcal{H}^{s,\gamma_0}(\mathbb{B})$ satisfies the theorem, therefore

$$A u^0 = g - f(x, Q(D) u^0)$$

in U.

Remark: Let us observe that the local solution $u^0 \in \mathcal{H}^{s,\gamma_0}(\mathbb{B})$ of Theorem 4.1 satisfies (4.2) up to the boundary $\partial\mathbb{B}$, indeed $Au^0, Q_i u^0, g \in \mathcal{H}^{s-m_0,\gamma_0-m_0}(\mathbb{B})$, then, reminding the Proposition 3.2,

$$A u^0\big|_{\partial\mathbb{B}} = Q_i u^0\big|_{\partial\mathbb{B}} = g\big|_{\partial\mathbb{B}} \equiv 0,$$

but, from the condition (3.15), it follows that

$$f(x, Q_1 u^0, ..., Q_M u^0)\big|_{\partial\mathbb{B}} \equiv 0.$$

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