Symbolic Calculus for Boundary Value Problems on Manifolds with Edges

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Abstract. Boundary value problems for (pseudo-) differential operators on a manifold with edges can be characterised by a hierarchy of symbols. The symbol structure is responsible for ellipticity and for the nature of parametrices within an algebra of "edge-degenerate" pseudo-differential operators. The edge symbol component of that hierarchy takes values in boundary value problems on an infinite model cone, with edge variables and covariables as parameters. Edge symbols play a crucial role in this theory, in particular, the contribution with holomorphic operator-valued Mellin symbols. We establish a calculus in a framework of "twisted homogeneity" that refers to strongly continuous groups of isomorphisms on weighted cone Sobolev spaces. We then derive an equivalent representation with a particularly transparent composition behaviour.

Key words: pseudo-differential boundary value problems, operators on manifolds with singularities

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Introduction

Parametrices of elliptic boundary value problems for differential operators on a manifold with edges can be characterised as elements of suitable pseudo-differential algebras with a certain principal symbol hierarchy. Algebras of that kind have been constructed by the authors in [14] in connection with applications to elliptic boundary value problems of crack theory. They have a rich structure, and there are interesting subalgebras that deserve a deeper investigation. In the present paper we develop a new machinery for the subalgebra of operators with holomorphic Mellin and flat Green symbols, here, for the case of boundary value problems with the transmission property at the smooth part of the boundary. We will focus our attention to one of the essential aspects of the edge calculus, namely to operator-valued edge symbols. These are families of (pseudo-differential) boundary value problems on an infinite cone, parametrised by edge covariables $\eta \in \mathbb{R}^q$. The case without boundary (i.e., when the base of the cone is closed and compact) has been studied by Gil, Schulze and Seiler [7]. In the present paper we shall obtain similar results for boundary value problems, especially, a new representation of operator-valued edge amplitude functions. Parameter-dependent operators on manifolds with conical singularities are useful also for other applications. In particular, they are essential as ingredients of operator-valued symbols on manifolds with corners and higher singularities, see, [26], [25]. Moreover, they belong to the structures in heat trace expansions for operators on manifolds with conical singularities, cf. Gil [8], as well as in long-time asymptotics of solutions to parabolic boundary value problems for spatial regions with geometric singularities, cf. Krainer and Schulze [16] and Krainer [17].

Our paper is organised as follows: In Chapter 1 we present material on pseudo-differential boundary value problems with parameters, cf., analogously, Boutet de Monvel [2] for the standard case. Our approach is based on operator-valued symbols, acting in spaces that are equipped with strongly continuous groups of isomorphisms. For boundary value problems on cones we apply Mellin operator conventions and kernel cut-off arguments as they have been employed in analogous form in [24], [27] for the case without boundary. We then introduce Green edge symbols without asymptotics as well as with infinite flatness in the axial variable $r \in \mathbb{R}_+$ on the cone. Chapter 2 studies Mellin pseudo-differential operators with operator-valued amplitude functions acting in Fréchet spaces. Here, we adopt definitions and arguments of Seiler [28], see also [7]. Holomorphic Mellin symbols and associated operators are of particular relevance for the rest of the paper. Recall that holomorphic operator functions play the role of conormal symbols of elliptic operators on a cone. These are Fredholm families in Sobolev spaces on the base of the cone. Parametrix constructions require the inverse of such families; these are meromorphic families, where poles and Laurent coefficients are responsible for asymptotics of solutions to elliptic equations, cf. Kondratyev [15], or Rempel and Schulze [20]. The general functional analytic background on Fredholm functions may be found in the paper of Gohberg and Sigal [9]; concerning further general results on holomorphic and meromorphic Fredholm functions, see Gramsch [11], [10]. A useful factorisation for meromorphic Mellin symbols in the specific cone situation (when the base is closed) has been constructed by Witt [30]. In Chapter 3 we develop a calculus of parameter-dependent boundary value problems on an infinite (stretched) cone $X^\wedge := \mathbb{R}_+ \times X$ where the base X is a compact C^∞ manifold with boundary. The families are operator-valued symbols of a specific structure; they represent the "non-smoothing" part of the edge symbol calculus, similarly to the case of manifolds with edges without boundary, see [27]. We then establish another approach, similarly to [7] for the case without boundary, that makes the elements of the calculus (compositions, etc.) very transparent. Chapter 4 completes the material by a new proof of the Mellin quantisation for edge-degenerate families of boundary value problems and gives characterisations of Green remainders in the edge symbol algebra that are necessary in connection with the aspect that $X^\wedge$ is a particular manifold with exit to infinity.

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1 Boundary value problems on a cone

1.1 Manifolds with conical singularities

Let us first give a definition of a manifold with boundary and conical singularities. Given any topological space X there is an associated cone $X^\triangle := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base X (in the quotient space $\{0\} \times X$ is identified with a point, is interpreted as the tip v of the cone). Setting $X^\wedge := \mathbb{R}_+ \times X$, called the open stretched cone with base X, we have a homeomorphism $X^\triangle \backslash \{v\} \cong X^\wedge$, and every such homeomorphism defines a splitting of variables (r, x) on $X^\triangle \backslash \{v\}$. In our case, X will be a compact C^∞ manifold with boundary.

Two splittings of variables (r, x) and $(\tilde{r}, \tilde{x})$ on $X^\triangle \backslash \{v\}$ are said to define the same cone structure, if $(r, x) \to (\tilde{r}, \tilde{x})$, $X^\wedge \to X^\wedge$, is the restriction of a diffeomorphism (in the sense of C^∞ manifolds with boundary) $\mathbb{R} \times X \to \mathbb{R} \times X$ to $X^\wedge$. In the following we fix any cone structure and exclude in this way cases that are, for instance, cuspidal relative to the given one. Now a manifold with boundary and conical singularities is a topological space D with a finite set $S \subset D$ of conical singularities, where $D \backslash S$ is a C^∞ manifold with boundary, and every $v \in S$ has a neighbourhood V that is homeomorphic to a cone $X^\triangle$, where $X = X(v)$ is a compact C^∞ manifold with boundary, and $X^\wedge$ is equipped with a cone structure.

A similar definition makes sense for the case when the base is a closed compact C^∞ manifold. In particular, if D is a manifold with boundary and conical singularities, $B := \partial(D \backslash S) \cup S$ is a closed manifold with conical singularities. Moreover, to every D we can form the double $2D$ which is closed and has conical singularities, where the base to a conical point $v \in S$ is the double $2X(v)$ of $X(v)$. We mainly refer to associated stretched manifolds. To recall the definition we first consider a closed manifold B with conical singularities. Base manifolds Y to conical points $v \in S$ are then closed and compact. In this case the stretched manifold $\mathbb{B}$ appears by attaching the sets $\{0\} \times Y(v)$ to $B \backslash \{v\}$, $v \in S$; then $\mathbb{B}$ is a C^∞ manifold with boundary $\partial \mathbb{B} \cong \cup_{v \in S} Y(v)$ (this is an invariant construction). Now if D is a manifold with boundary and conical singularites S, we first form the stretched manifold $2\mathbb{D}$ for $2D$ and then pass to $\mathbb{D}$ itself by taking the subset of $2\mathbb{D}$ consisting of $D \backslash S$ with the attached sets $\{0\} \times X(v)$ at the points $v \in S$. We then set $\mathbb{D}_{\text{sing}} := \bigcup_{v \in S} (\{0\} \times X(v))$ and $\mathbb{D}_{\text{reg}} := \mathbb{D} \backslash \mathbb{D}_{\text{sing}}$. Similar notation is used for the closed case.

1.2 Basics on operator-valued symbols

In our calculus we shall systematically employ a certain particular class of operator-valued symbols that turns out to be very useful for a concise description of the algebra of pseudo-differential boundary value problems.

A Hilbert space E is said to be endowed with a group action, if there is given a group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \to E$, strongly continuous in $\lambda \in \mathbb{R}_+$, where $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$ for all $\lambda, \rho \in \mathbb{R}_+$. If $\{E, \{\kappa_\lambda\}\}$, $\{\tilde{E}, \{\tilde{\kappa}_\lambda\}\}$ are Hilbert spaces with group actions, $S^\mu(U \times \mathbb{R}^p; E, \tilde{E})$ for $U \subseteq \mathbb{R}^m$ open, $\mu \in \mathbb{R}$, denotes the subspace of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^p, \mathcal{L}(E, \tilde{E}))$ such that

$$||\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \le c\langle\eta\rangle^{\mu-|\beta|}$$

for all $\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^p$ and all $y \in K$ for arbitrary $K \Subset U$, $\eta \in \mathbb{R}^p$, with constants $c = c(\alpha, \beta, K) > 0$, $\langle\eta\rangle := (1 + |\eta|^2)^{\frac{1}{2}}$.

Moreover, let $S^{(\nu)}(U \times (\mathbb{R}^p \setminus 0); E, \tilde{E})$ denote the set of all $f(y,\eta) \in C^\infty(U \times (\mathbb{R}^p \setminus 0), \mathcal{L}(E, \tilde{E}))$ such that $f(y, \lambda\eta) = \lambda^\nu \tilde{\kappa}_\lambda f(y,\eta)\kappa^{-1}_\lambda$ for all $(y,\eta) \in U \times (\mathbb{R}^p \setminus 0), \lambda \in \mathbb{R}_+$. Then $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^p; E, \tilde{E})$ is defined to be the set of all $a(y,\eta) \in S^\mu(U \times \mathbb{R}^p; E, \tilde{E})$, such that there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^p \setminus 0); E, \tilde{E})$ where

$$a(y,\eta) - \chi(\eta)\sum_{j=0}^N a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^p; E, \tilde{E})$$

for all $N \in \mathbb{N}$.

Let us set $\sigma^\mu_\wedge(a)(y,\eta) := a_{(\mu)}(y,\eta)$, called the homogeneous principal symbol of order μ in $\eta \in \mathbb{R}^p \setminus 0$. If μ is clear from the context we simply set $\sigma_\wedge := \sigma^\mu_\wedge(a)$.

Further background on operator-valued symbols of that type may be found in [24], [27], in particular, adequate Fréchet topologies in the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E})$ (here, subscript "(cl)" is used when we talk about classical or non-classical symbols).

1.3 Calculus on a manifold with boundary

We now prepare elements on (classical) pseudo-differential boundary value problems with the transmission property. Consider the half-space $\Omega \times \overline{\mathbb{R}}_+ \ni x = (x', t)$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, and let $\xi := (\xi', \tau) \in \mathbb{R}^n$, $\eta \in \mathbb{R}^q$. Define $S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+q})_{\mathrm{tr}}$ for $\mu \in \mathbb{Z}$ to be the subspace of all $a(x,\xi,\eta) \in S^\mu_{\mathrm{cl}}(\Omega_x \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+q}_{\xi,\eta})$ such that

$$D^k_t D^\alpha_{\xi',\eta}\{a_{(\mu-j)}(x', t, \xi', \tau, \eta) - (-1)^{\mu-j}a_{(\mu-j)}(x', t, -\xi', -\tau, -\eta)\} = 0 \tag{1}$$

on the set $\{(x,\xi,\eta) \in \Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+q} : x' \in \Omega, t = 0, (\xi', \eta) = 0, \tau \in \mathbb{R}\setminus 0\}$, for all $k \in \mathbb{N}, \alpha \in \mathbb{N}^{n-1+q}$ and all $j \in \mathbb{N}$. Here, $a_{(\mu-j)}$ denotes the homogeneous component of a of order $\mu - j$ in $(\xi, \eta) \ne 0$. Below we shall employ symbols with the transmission property for $k \times m$-matrices and employ invariance under symbol push-forwards, belonging to coordinate diffeomorphisms and trivialisations of bundles on manifolds of fibre dimensions m and k, respectively. With $a(x,\xi,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+q})_{\mathrm{tr}}$ we associate a family of pseudo-differential operators in $\Omega \times \mathbb{R}_+$ by setting

$$\mathrm{Op}^+(a)u(x) := \mathrm{r}^+\mathrm{Op}(\tilde{a})\mathrm{e}^+u(x), \tag{2}$$

where e^+ is the operator of extension by zero from $\Omega \times \mathbb{R}_+$ to $\Omega \times \mathbb{R}$ and r^+ the operator of restriction from $\Omega \times \mathbb{R}$ to $\Omega \times \mathbb{R}_+$; moreover, $\tilde{a} \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^{n+q}_{\xi,\eta})_{\mathrm{tr}}$ is a symbol such that $a = \tilde{a}|_{t>0}$ (clearly, (2) does not depend on the choice of $\tilde{a}$). Similarly, $a(x', t, \xi', \tau, \eta)$ gives rise to an operator family

$$\mathrm{op}^+(a)(x', \xi', \eta) = \mathrm{r}^+\mathrm{op}(\tilde{a})(x', \xi', \eta)\mathrm{e}^+$$

on $\mathbb{R}_+$, where $\operatorname{op}(a)(x',\xi',\eta)u(t) = \iint e^{i(t-t')\tau}a(x',t,\xi',\tau,\eta)u(t')dt'đ\tau$, $(x',\xi',\eta) \in \Omega \times \mathbb{R}^{n-1+q}$.

Boundary value problems will locally be generated as pseudo-differential operators with operator-valued symbols. We shall apply the formalism of "abstract" edge symbols in the sense of Section 1.2.

In our calculus we have, for instance, $E = H^s(\mathbb{R}_+)$, $\tilde{E} = H^{s-\mu}(\mathbb{R}_+)$, with κ_λ and $\tilde{\kappa}_\lambda$ being given by $u(t) \to \lambda^{\frac{1}{2}}u(\lambda t)$, $\lambda \in \mathbb{R}_+$. Then, if $a(x',t,\xi',\tau,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+q})_{\mathrm{tr}}$ is a symbol that is independent of t for large t, we have

$$\operatorname{op}^+(a)(x',\xi',\eta) \in S^\mu(\Omega \times \mathbb{R}^{n-1+q}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+))$$

for all real $s > -\frac{1}{2}$. In addition, if a is independent of t, the operator family $\operatorname{op}^+(a)(x',\xi',\eta)$ is a classical operator-valued symbol.

In boundary value problems there is another important class of operator-valued symbols, called Green symbols. First, the concept of operator-valued symbols easily extends to the case of Fréchet spaces of the following kind. If a Fréchet space E can be written as a projective limit $E = \operatorname{proj\,lim}\{E^j : j \in \mathbb{N}\}$ of Hilbert spaces E^j with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j, where E^0 is endowed with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ that restricts to a group action on E^j for every j, we say that E itself is endowed with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$.

Let us now assume that $\tilde{E} = \operatorname{proj\,lim}\{\tilde{E}^k : k \in \mathbb{N}\}$ is endowed with a group action $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$; moreover, let E be a Hilbert space with $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. We then set $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E}) := \operatorname{proj\,lim}\{S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E}^k) : k \in \mathbb{N}\}$. Finally, let both E and $\tilde{E}$ be Fréchet spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Then to every map $r : \mathbb{N} \to \mathbb{N}$ we first define the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E})_r := \operatorname{proj\,lim}\{S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E^{r(k)}, \tilde{E}^k) : k \in \mathbb{N}\}$ and then set

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E}) := \bigcup_r S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E})_r.$$

The group actions in the spaces E and $\tilde{E}$ are fixed and known by the context; therefore they are usually not indicated in the notation of symbol spaces. However, it happens that we modify the choice of actions; then we indicate them as subscripts, and then, if necessary, we write $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^p; E, \tilde{E})_{\kappa,\tilde{\kappa}}$. In particular, for the spaces

$$E := L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}, \quad \tilde{E} := \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_+}$$

where $\mathcal{S}(\overline{\mathbb{R}}_+) = \operatorname{proj\,lim}\{\langle t\rangle^{-l}H^l(\mathbb{R}_+) : l \in \mathbb{N}\}$, we employ

$$\{\operatorname{diag}(\kappa_\lambda, \lambda^{\frac{1}{2}})\}_{\lambda\in\mathbb{R}_+} =: \{\boldsymbol{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+} =: \boldsymbol{\kappa}, \tag{3}$$

$\kappa_\lambda(u)(t) := \lambda^{\frac{1}{2}}u(\lambda t)$. Another possible choice is

$$\{\operatorname{diag}(\kappa_\lambda, \lambda^{-\frac{1}{2}})\}_{\lambda\in\mathbb{R}_+} =: \{\boldsymbol{\chi}_\lambda\}_{\lambda\in\mathbb{R}_+} =: \boldsymbol{\chi}. \tag{4}$$

An element $g(x',\xi',\eta) \in S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^{n-1+q}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_+})$, $\mu \in \mathbb{R}$ is said to be a (parameter-dependent) Green symbol of type 0, if

$$g(x',\xi',\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^{n-1+q}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_+})_{\boldsymbol{\kappa},\boldsymbol{\kappa}}$$

and

$$g^*(x',\xi',\eta) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{n-1+q}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_+}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_-})_{\boldsymbol{\chi},\boldsymbol{\chi}},$$

where $*$ denotes the pointwise adjoint with respect to the standard scalar products (in particular, $(u,v)_{L^2(\mathbb{R}_+)} = \int_0^\infty u(t)\overline{v(t)}\mathrm{d}t$). Moreover, operator families of the form

$$g(x',\xi',\eta) = g_0(x',\xi',\eta) + \sum_{j=1}^{d} g_j(x',\xi',\eta) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{5}$$

for arbitrary Green symbols $g_j(x',\xi',\eta)$ of order $\mu - j$ and type 0 are called Green symbols of type d; the space of all those g will be denoted $\mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^{n-1+q}; j_-, j_+)$.

There is an obvious generalisation to $k \times m$-block matrix valued Green symbols in the upper left corners; the corresponding symbol space will be denoted by $\mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^{n-1+q}; \boldsymbol{w})$, where $\boldsymbol{w} := (m, j_-; k, j_+)$ is a tuple of given dimensions. With symbols $g(x',\xi',\eta) \in \mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^{n-1+q}; \boldsymbol{w})$ we now associate η-dependent pseudo-differential operators $\mathrm{Op}_{x'}(g)(\eta)u(x') = (2\pi)^{-\frac{n-1}{2}} \iint e^{i(x'-\tilde{x}')\xi'} g(x',\xi',\eta)u(\tilde{x}')\mathrm{d}\tilde{x}'\mathrm{d}\xi'$. Then we get η-wise continuous operators

$$\mathrm{Op}_{x'}(g)(\eta) : \begin{array}{c} H^s_{\mathrm{comp}}(\Omega \times \mathbb{R}_+, \mathbb{C}^m) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\Omega, \mathbb{C}^{j_-}) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(\Omega \times \mathbb{R}_+, \mathbb{C}^k) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}_{\mathrm{loc}}(\Omega, \mathbb{C}^{j_+}) \end{array} \tag{6}$$

for all $s > d - \frac{1}{2}$ ("comp" and "loc" in the first components only refer to x'-variables). We now define families of Green operators on a C^∞ manifold X with boundary (not necesserily compact) acting between distributional sections in bundles $E, F \in \mathrm{Vect}(X)$ and $J_-, J_+ \in \mathrm{Vect}(\partial X)$, respectively. Here $\mathrm{Vect}(.)$ means the set of all smooth complex vector bundles on the manifold in the brackets, and $H^s_{\mathrm{comp}}(X, E)$ denotes the Sobolev space of compactly supported sections in E, while $H^s_{\mathrm{loc}}(X, E)$ is the space of sections that are locally of Sobolev smoothness $s \in \mathbb{R}$.

An η-dependent family $\mathcal{G}(\eta)$ of continuous operators

$$\begin{array}{c} H^s_{\mathrm{comp}}(X, E) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial X, J_-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(X, F) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}_{\mathrm{loc}}(\partial X, J_+) \end{array} \tag{7}$$

$s > d - \frac{1}{2}$, is said to be a Green operator family of order μ and type $d \in \mathbb{N}$, if $\mathcal{G}(\eta) = \mathcal{G}_0(\eta) + \mathcal{C}(\eta)$, where $\mathcal{G}_0(\eta)$ is locally near ∂X (in local coordinates $x = (x', t)$ and with respect to trivialisations of the bundles E, F, J_-, and J_+ of fibre dimensions m, k, j_- and j_+, respectively) of the form (6). Moreover, $\mathcal{C}(\eta) = (\mathcal{C}_{ij}(\eta))_{i,j=1,2}$ has the form $\mathcal{C}^0(\eta) + \sum_{l=1}^{d} \mathcal{C}^l(\eta)\mathrm{diag}(T^l, 0)$ where T^l is an arbitrary differential operator on X of order l (with smooth coefficients up to ∂X), while $\mathcal{C}^l(\eta)$, $l = 0, \ldots, d$, is a Schwartz function in $\eta \in \mathbb{R}^q$ with values in the space of 2×2-block matrices of operators with C^∞-kernels, smooth up to ∂X in the variables referring to X.

Let $\tilde{X} := 2X$ be the double of X, written as $X_- \cup_g X_+$, where $\cup_g$ means glueing together two copies $X_\pm$ of X along ∂X; let us identify X_+ with X. Given $\tilde{E}, \tilde{F} \in \mathrm{Vect}(\tilde{X})$ we have the well-known space $L^\mu_{\mathrm{cl}}(\tilde{X}; \tilde{E}, \tilde{F}; \mathbb{R}^q)$ of classical parameter-dependent operators of order μ, acting between distributional sections in $\tilde{E}$ and $\tilde{F}$, respectively. The space $L^\mu_{\mathrm{cl}}(\tilde{X}; \tilde{E}, \tilde{F}; \mathbb{R}^q)$ in Fréchet in a natural way. In particular, $L^{-\infty}(\tilde{X}; \tilde{E}, \tilde{F}; \mathbb{R}^q)$, the spaces of all parameter-dependent smoothing operators, coinsides with $\mathcal{S}(\mathbb{R}^q, L^{-\infty}(\tilde{X}; \tilde{E}, \tilde{F}))$; here, $\mathcal{S}(\mathbb{R}^q, G)$ for some Fréchet space G is the space of all $G-valued$ Schwartz functions on $\mathbb{R}^q$. Let $L^\mu_{\mathrm{cl}}(\tilde{X};\tilde{E},\tilde{F};\mathbb{R}^q)_{\mathrm{tr}}$, $\mu\in\mathbb{Z}$, denote the subspace of all elements where the local symbols have the transmssion property at ∂X; this is a closed subspace. Then, setting $E:=\tilde{E}|_{X_+}$, $F:=\tilde{F}|_{X_+}$, we get the space $L^\mu_{\mathrm{cl}}(X;E,F;\mathbb{R}^q)_{\mathrm{tr}}$ of all operators of the form $\mathrm{r}^+\tilde{A}(\eta)\mathrm{e}^+$, $\tilde{A}(\eta)\in L^\mu_{\mathrm{cl}}(\tilde{X};\tilde{E},\tilde{F};\mathbb{R}^q)_{\mathrm{tr}}$, where e^+ denotes the extension of functions on $\mathrm{int}X_+$ by zero to $\tilde{X}$ and r^+ the restriction of distributions on $\tilde{X}$ to $\mathrm{int}X_+$.

Also $L^\mu_{\mathrm{cl}}(X;E,F;\mathbb{R}^q)_{\mathrm{tr}}$ is a Fréchet space. Notice that $L^{-\infty}_{\mathrm{cl}}(X;E,F;\mathbb{R}^q)$ (the intersection over all μ) just coincides with the space of upper left corners $\mathcal{G}_{11}(\eta)$ of families in $\mathcal{B}^{-\infty,0}(X;\boldsymbol{v};\mathbb{R}^q)$, where, $\mathcal{B}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q):=\bigcap_\mu\mathcal{B}^{\mu,d}_G(X;\boldsymbol{v};\mathbb{R}^q), d\in\mathbb{N}$.

Definition 1.1 *Given $\boldsymbol{v}:=(E,J_-;F,J_+)$ and $\mu\in\mathbb{Z}$ the space $\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)$ of parameter-dependent pseudo-differential boundary value problems on X of order $\mu\in\mathbb{Z}$ and typed $d\in\mathbb{N}$ is defined to be the set of all operator families of the form*

$$\mathcal{A}(\eta):=\begin{pmatrix}\mathrm{r}^+\tilde{A}(\eta)\mathrm{e}^+ & 0\\ 0 & 0\end{pmatrix}+\mathcal{G}(\eta)$$

for arbitrary $\tilde{A}(\eta)\in L^\mu_{\mathrm{cl}}(X;E,F;\mathbb{R}^q)_{\mathrm{tr}}$ and $\mathcal{G}(\eta)\in\mathcal{B}^{\mu,d}_G(X;\boldsymbol{v};\mathbb{R}^q)$.

Let us fix a notation for the parameter-dependent principal symbol structure for elements $\mathcal{A}(\eta)\in\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)$, namely $\sigma(\mathcal{A}):=(\sigma^\mu_\psi(\mathcal{A}),\sigma^\mu_\partial(\mathcal{A}))$ where

$$\sigma^\mu_\psi(\mathcal{A})(x,\xi,\eta):\pi^*_XE\to\pi^*_XF \tag{8}$$

is the homogeneous principal symbol of order μ in $(\xi,\eta)\neq0$ of the upper left corner of $\mathcal{A}$, $\pi_X:(T^*X\times\mathbb{R}^q)\setminus0\to X$, and

$$\sigma^\mu_\partial(\mathcal{A})(x',\xi',\eta):\pi^*_{\partial X}\begin{pmatrix}E'\otimes H^s(\mathbb{R}_+)\\ \oplus\\ J_{-,x'}\end{pmatrix}\to\pi^*_{\partial X}\begin{pmatrix}F'\otimes H^{s-\mu}(\mathbb{R}_+)\\ \oplus\\ J_{+,x'}\end{pmatrix} \tag{9}$$

$s>d-\frac{1}{2}$ is the homogeneous principal boundary symbol of order μ in $(\xi',\eta)\neq0$, $\pi_{\partial X}:(T^*\partial X\times\mathbb{R}^q)\setminus0\to\partial X$, $E':=E|_{\partial X}$, $F':=F|_{\partial X}$ here, by order we understand twisted homogeneity in the sense

$$\sigma^\mu_\partial(\mathcal{A})(x',\lambda\xi',\lambda\eta)=\lambda^\mu\boldsymbol{\kappa}_\lambda\sigma^\mu_\partial(\mathcal{A})(x',\xi',\eta)\boldsymbol{\kappa}^{-1}_\lambda$$

for all $\lambda\in\mathbb{R}_+$, cf. (3), (4).

1.4 Operators near the tip of the cone

To formulate parameter-dependent operators on the (stretched) cone $X^\wedge$ near the tip we recall the definition of holomorphic Mellin symbols with parameters.

Let $E,F\in\mathrm{Vect}(X)$, $J_-,J_+\in\mathrm{Vect}(\partial X)$ and set $\boldsymbol{v}=(E,J_-;F,J_+)$. Then $\mathcal{M}^{\mu,d}_{\mathcal{O}}(X;\boldsymbol{v};\mathbb{R}^q)$ denotes the space of all

$$h(z,\eta)\in\mathcal{A}(\mathbb{C},\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q_\eta))$$

such that $h(\beta+i\varrho,\eta)\in\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}_{\tau,\eta})$ for every $\beta\in\mathbb{R}$, uniformly in $c\leq\beta\leq c'$ for arbitrary $c\leq c'$. Moreover, let $\mathcal{M}^{\mu,d}_{\mathcal{O},G}(X;\boldsymbol{v};\mathbb{R}^q)$ be the subspace of all elements of $\mathcal{M}^{\mu,d}_{\mathcal{O}}(X;\boldsymbol{v};\mathbb{R}^q)$ that take values in the space $\mathcal{B}^{\mu,d}_G(X;\boldsymbol{v};\mathbb{R}^q)$.

The space $\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)$ is Fréchet in a canonical way, and $\mathcal{M}_{\mathcal{O},G}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)$ is a closed subspace; so we can talk about

$$C^\infty(\mathbb{R}_+\times\mathbb{R}_+,\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)) \quad\text{and}\quad C^\infty(\mathbb{R}_+\times\mathbb{R}_+,\mathcal{M}_{\mathcal{O},G}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)), \tag{10}$$

respectively.

Moreover, let $C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ defined to be the set of all $h(r,z,\eta)\in C^\infty(\mathbb{R}_+,\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ such that

$$h(r,z,\eta)=\tilde{h}(r,z,r\eta)$$

for some $\tilde{h}(r,z,\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+,\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q_{\tilde{\eta}}))$. Let $C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))_{\mathrm{const}}$ be the set of all $h(r,z,\eta)\in C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ where $\tilde{h}(z,r\eta)$ is independent of r.

Let us start from parameter-dependent boundary value problems $\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q})$ on X with parameters $(\varrho,\eta)\in\mathbb{R}^{1+q}$. Given an element $\tilde{p}(\tilde{\varrho},\tilde{\eta})\in\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q})$ we form

$$p(r,\varrho,\eta):=\tilde{p}(r\varrho,r\eta) \tag{11}$$

and pass to the space $\tilde{\mathcal{B}}^{\mu,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q_\eta)$ of all operator families of the form

$$a(\eta):=\mathrm{op}_r(p)(\eta),$$

where p runs over all $p(r,\varrho,\eta)$ for arbitrary $\tilde{p}(\tilde{\varrho},\tilde{\eta})\in\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q})$.

We now form pseudo-differential operators with respect to the Mellin transform on $\mathbb{R}_+$ with operator-valued symbols.

The Mellin transform $Mu(z)=\int_0^\infty r^{z-1}u(r)\mathrm{d}r$, first given on $C_0^\infty(\mathbb{R}_+)$ with $Mu(z)$ being an entire function in z, will be extended to larger spaces of distributions, also vector-valued ones. To introduce notation we simply consider the case of scalar functions on $\mathbb{R}_+$. Set $\Gamma_\beta=\{z\in\mathbb{C}:\mathrm{Re}\,z=\beta\}$ for any $\beta\in\mathbb{R}$. Function spaces on $\Gamma_\beta\ni z$ will be regarded as usual ones with respect to $\mathrm{Im}\,z\in\mathbb{R}$; e.g., we write $L^2(\Gamma_\beta)$, $\mathcal{S}(\Gamma_\beta)$, etc. It is well-known that the map $M:u\to(Mu)|_{\Gamma_{\frac{1}{2}-\gamma}}$ extends from $C_0^\infty(\mathbb{R}_+)$ to an isomorphism $M_\gamma: t^\gamma L^2(\mathbb{R}_+)\to L^2(\Gamma_{\frac{1}{2}-\gamma})$, with the inverse $(M_\gamma^{-1}g)(z)=(2\pi i)^{-1}\int_{\Gamma_{\frac{1}{2}-\gamma}}r^{-z}g(z)\mathrm{d}z$. We then define weighted pseudo-differential operators

$$\mathrm{op}_M^\gamma(f)u(t):=M_{\gamma,z\to r}^{-1}\{M_{\gamma,r'\to z}f(r,r',z)u(r')\} \tag{12}$$

for symbols $f(r,r',z)\in S^\mu(\mathbb{R}_+\times\mathbb{R}_+\times\Gamma_{\frac{1}{2}-\gamma})$ (in the scalar case). In the operator-valued case we take symbols $f(r,r',z)\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+,\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\Gamma_{\frac{1}{2}-\gamma}))$, or symbols $h(r,r',z,\eta)\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+,\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ depending on additional parameters $\eta\in\mathbb{R}^q$.

Let $C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{B}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$ be the set of all $p(r,\varrho,\eta)\in C^\infty(\mathbb{R}_+,\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$ such that $p(r,\varrho,\eta)=\tilde{p}(r,r\varrho,r\eta)$ for some $\tilde{p}(r,\tilde{\varrho},\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+,\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$. Moreover, let $C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{B}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))_{\mathrm{const}}$ be the subspace of all $p(r,\varrho,\eta)$ of the form (11), i.e., where $\tilde{p}(\tilde{\varrho},\tilde{\eta})$ is independent of r. The following theorem will be proved in Section 4.1 below.

Theorem 1.2 *Let $p(r,\varrho,\eta)\in C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{B}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$, and let $\varphi\in C_0^\infty(\mathbb{R}_+)$ be a function such that $\varphi\equiv 1$ in a neighbourhood of 1. Then there exists an $h(r,z,\eta)\in C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ such that*

$$\mathrm{op}_r(p)(\eta)-\mathrm{op}_M^\beta(h)(\eta)=\mathrm{op}_r\left(1-\varphi\left(r'/r\right)p\right)(\eta)\in\mathcal{B}^{-\infty,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q)$$

for every $\beta\in\mathbb{R}$.

To have a convenient notation we call $h(r,z,\eta)$ the Mellin quantisation of $p(r,\varrho,\eta)$.

Mellin pseudo-differential operators (12) will act as continuous operators in adequate weighted Sobolev spaces. Let G be a closed compact C^∞ manifold of dimension n, and set $G^\wedge := \mathbb{R}_+ \times G$, called the (open) stretched cone with base G. Then the space $\mathcal{H}^{s,\gamma}(G^\wedge)$, $s,\gamma \in \mathbb{R}$, is defined to be the completion of $C_0^\infty(G^\wedge)$ with respect to the norm

$$\Big\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^s(|\mathrm{Im}\, z|)(Mu)||^2_{L^2(G)}\mathrm{d}z\Big\}^{\frac{1}{2}}.$$

Here, $R^s(\lambda) \in L^s_{\mathrm{cl}}(G;\mathbb{R}_\lambda)$ is a parameter-dependent elliptic classical pseudo-differential operator on G that induces isomorphisms $H^t(G) \to H^{t-s}(G)$ for all $t,s \in \mathbb{R}, \lambda \in \mathbb{R}$; $L^2(G)$ refers to a fixed Riemannian metric on G. Now if X is a compact C^∞ manifold with boundary, we first form $2X$, the double of X, and then set

$$\mathcal{H}^{s,\gamma}(X^\wedge) := \{u|_{X^\wedge} : u \in \mathcal{H}^{s,\gamma}((2X)^\wedge)\}, \tag{13}$$

where X is identified with the plus-side of $2X$. Similarly, if we are given a (smooth complex) vector bundle V on G (or X) we form the spaces $\mathcal{H}^{s,\gamma}(G^\wedge,V)$ (or $\mathcal{H}^{s,\gamma}(X^\wedge,V)$) in a canonical way (bundles in the spaces on respective stretched cones are interpreted as pull-backs of corresponding bundles on the base of the cone).

In this paper a cut-off function $\omega(r)$ means any $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega(r) = 1$ in a neighbourhood of $r = 0$. Given two cut-off functions ω_1, ω_2, we write $\omega_1 \prec \omega_2$ if ω_2 equals 1 on $\mathrm{supp}\,\omega_1$.

We are mainly interested in a modification of these spaces, denoted by $\mathcal{K}^{s,\gamma}(G^\wedge,V)$, defined by the property $u \in \mathcal{K}^{s,\gamma}(G^\wedge,V) \Leftrightarrow \omega u \in \mathcal{H}^{s,\gamma}(G^\wedge,V)$ and $(1-\omega)u \in H^s_{\mathrm{cone}}(\mathbb{R}_+ \times G,V)$ for any cut-off function ω, where subscript "cone" means the standard Sobolev space definition from $\mathbb{R}^{n+1}$ near infinity, cf. [27, Section 2.1.4]. Moreover, similarly to (13) we set $\mathcal{K}^{s,\gamma}(X^\wedge) := \{u|_{X^\wedge} : u \in \mathcal{K}^{s,\gamma}((2X)^\wedge)\}$ and we use analogous notation for spaces $\mathcal{K}^{s,\gamma}(.,V)$ of distributional sections in bundles V on the cone $G^\wedge$ and $X^\wedge$, repectively. Finally, let us define spaces with weight $\beta \in \mathbb{R}$ at infinity, namely $\mathcal{K}^{s,\gamma;\beta}(.,V) := \langle r\rangle^{-\beta}\mathcal{K}^{s,\gamma}(.,V)$. Let $\boldsymbol{m} := (E,J)$ for $E \in \mathrm{Vect}(X)$, $J \in \mathrm{Vect}(\partial X)$, and set

$$\mathcal{H}^{s,\gamma}(X^\wedge;\boldsymbol{m}) := \mathcal{H}^{s,\gamma}(X^\wedge,E) \oplus \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge,J),$$
$$\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m}) := \mathcal{K}^{s,\gamma;\beta}(X^\wedge,E) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2};\beta}((\partial X)^\wedge,J),$$

where we also drop β for $\beta = 0$. The latter spaces are endowed with the group action $\mathrm{diag}(\{\kappa_\lambda^{(n)}\}_{\lambda\in\mathbb{R}_+},\{\kappa_\lambda^{(n-1)}\}_{\lambda\in\mathbb{R}_+}) =: \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. In addition, we define

$$\mathcal{S}_{\mathcal{O}}(X^\wedge,E) := \mathrm{proj\,lim}\{\mathcal{K}^{k,k;k}(X^\wedge,E) : k \in \mathbb{N}\}$$

and, analogously, $\mathcal{S}_{\mathcal{O}}((\partial X)^\wedge,J)$, and set

$$\mathcal{S}_{\mathcal{O}}(X^\wedge;\boldsymbol{m}) := \mathcal{S}_{\mathcal{O}}(X^\wedge,E) \oplus \mathcal{S}_{\mathcal{O}}((\partial X)^\wedge,J). \tag{14}$$

$\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ induces a group action on the space (14). For purposes below we set

$$\mathcal{K}^{s^*,\gamma^*;\beta}(X^\wedge;\boldsymbol{m}) := \mathcal{K}^{s,\gamma;\beta}(X^\wedge;E) \oplus \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2};\beta}((\partial X)^\wedge;J). \tag{15}$$

Moreover, define $\mathcal{S}^\gamma(X^\wedge,E) := \mathrm{proj\,lim}\{\mathcal{K}^{k,\gamma;k}(X^\wedge,E) : k \in \mathbb{N}\}$, and, analogously, $\mathcal{S}^\gamma((\partial X)^\wedge,J)$, and set

$$\mathcal{S}^\gamma(X^\wedge;\boldsymbol{m}) := \mathcal{S}^\gamma(X^\wedge,E) \oplus \mathcal{S}^{\gamma-\frac{1}{2}}((\partial X)^\wedge,J),$$

$$\mathcal{S}^(\gamma^*)(X^\wedge;\boldsymbol{m}) := \mathcal{S}^\gamma(X^\wedge,E) \oplus \mathcal{S}^{\gamma+\frac{1}{2}}((\partial X)^\wedge,J).$$

For cut-off functions $\omega(r)$, $\tilde{\omega}(r)$ we consider operator families of the form

$$a(\eta) := \omega(r[\eta])[\eta]^\nu r^{\tilde{\nu}} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}(r[\eta])$$

where $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))_{\mathrm{const}}$, $\boldsymbol{v} = (E,J_-;F,J_+)$. For every η we get continuous maps

$$a(\eta) : \mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m}) \to \mathcal{K}^{s-\mu,\gamma+\tilde{\nu};\beta'}(X^\wedge;\boldsymbol{n})$$

$s \in \mathbb{R},\ s > d - \frac{1}{2}$, $\boldsymbol{m} := (E,J_-)$, $\boldsymbol{n} := (F,J_+)$. Moreover, we have

$$a(\eta) \in S_{\mathrm{cl}}^{\nu-\tilde{\nu}}(\mathbb{R}^q;\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m}),\mathcal{K}^{s-\mu,\gamma+\tilde{\nu};\beta'}(X^\wedge;\boldsymbol{n}))$$

for all $s \in \mathbb{R},\ s > d - \frac{1}{2}$, where $\sigma_\wedge^{\nu-\tilde{\nu}}(a)(\eta) = \omega(r|\eta|)|\eta|^\nu r^{\tilde{\nu}} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}(r|\eta|)$.

1.5 Green symbols

Analogously to Green symbols for boundary value problems we now consider Green symbols with respect to the edge-covariables η. As before, the adequate context are symbols $g(\eta) = (g_{ij}(\eta))_{i,j=1,2}$ with a scheme of Douglis-Nirenberg orders

$$\begin{pmatrix} \mu & \mu-\frac{1}{2} \\ \mu+\frac{1}{2} & \mu \end{pmatrix} \tag{16}$$

where

$$g_{11}(\eta) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge,E),\mathcal{S}_{\mathcal{O}}(X^\wedge,F)),$$
$$g_{12}(\eta) \in S_{\mathrm{cl}}^{\mu-\frac{1}{2}}(\mathbb{R}^q;\mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge,J_-),\mathcal{S}_{\mathcal{O}}(X^\wedge,F)),$$
$$g_{21}(\eta) \in S_{\mathrm{cl}}^{\mu+\frac{1}{2}}(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge,E),\mathcal{S}_{\mathcal{O}}(X^\wedge,J_+)),$$
$$g_{22}(\eta) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q;\mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge,J_-),\mathcal{S}_{\mathcal{O}}(X^\wedge,J_+)),$$

for all $s \in \mathbb{R}$. The symbols refer to $\{\kappa_\lambda^{(n)}\}_{\lambda\in\mathbb{R}_+}$ for the spaces on $X^\wedge$ and to $\{\kappa_\lambda^{(n-1)}\}_{\lambda\in\mathbb{R}_+}$ for the spaces on $(\partial X)^\wedge$.

Consider the space $\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m})$ with the group action $\boldsymbol{\kappa} := \{\boldsymbol{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ where

$$\boldsymbol{\kappa}_\lambda := \mathrm{diag}\{\kappa_\lambda^{(n)}, \lambda^{\frac{1}{2}}\kappa_\lambda^{(n-1)}\}$$

and $\mathcal{K}^{s^*,\gamma^*;\beta}(X^\wedge;\boldsymbol{m})$ with the group action $\boldsymbol{\chi} := \{\boldsymbol{\chi}_\lambda\}_{\lambda\in\mathbb{R}_+}$ where

$$\boldsymbol{\chi}_\lambda := \mathrm{diag}\{\kappa_\lambda^{(n)}, \lambda^{-\frac{1}{2}}\kappa_\lambda^{(n-1)}\}.$$

We then have the symbol spaces

$$S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q;\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m}),\mathcal{K}^{s',\gamma';\beta'}(X^\wedge;\boldsymbol{n}))_{\boldsymbol{\kappa},\boldsymbol{\kappa}}$$

and

$$S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q;\mathcal{K}^{s^*,\gamma^*;\beta}(X^\wedge;\boldsymbol{m}),\mathcal{K}^{s'^*,\gamma'^*;\beta'}(X^\wedge;\boldsymbol{n}))_{\boldsymbol{\chi},\boldsymbol{\chi}};$$

recall that subscripts just refer to the chosen groups in the corresponding parameter spaces.

In the following definition we introduce so-called Green symbols, where, in contrast to those in Section 1.3, the half-axis is replaced by $X^\wedge$ and the spaces are modified. We will employ similar notation for such spaces of Green symbols that should not cause confusion.

Definition 1.3 *Let* $\boldsymbol{v} := (E, J_-; F, J_+)$, $\boldsymbol{m} := (E, J_-)$, $\boldsymbol{n} := (F, J_+)$ *and* $\mu \in \mathbb{R}$. *Then* $\mathcal{R}_G^{\mu,0}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$ *denotes the set of all operator-valued symbols* $g(\eta)$ *such that*

$$g(\eta) \in S_{\text{cl}}^{\mu}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge; \boldsymbol{m}), \mathcal{S}_{\mathcal{O}}(X^\wedge; \boldsymbol{n}))_{\boldsymbol{\kappa},\boldsymbol{\kappa}}, \tag{17}$$

for all $s, \gamma, \beta \in \mathbb{R}$, $s > -\frac{1}{2}$, *and*

$$g^*(\eta) \in S_{\text{cl}}^{\mu}(\mathbb{R}^q; \mathcal{K}^{s^*,\gamma^*;\beta}(X^\wedge; \boldsymbol{n}), \mathcal{S}_{\mathcal{O}}(X^\wedge; \boldsymbol{m}))_{\boldsymbol{\chi},\boldsymbol{\chi}} \tag{18}$$

for all $s, \gamma, \beta \in \mathbb{R}$, $s > -\frac{1}{2}$, *where* $*$ *means the* η*-wise formal adjoint with respect to the reference scalar product of the spaces* $\mathcal{K}^{0,0;0}(X^\wedge, E) \oplus \mathcal{K}^{0,0;0}((\partial X)^\wedge, J_-)$ *and* $\mathcal{K}^{0,0;0}(X^\wedge, F) \oplus \mathcal{K}^{0,0;0}((\partial X)^\wedge, J_+)$, *respectively. Moreover,* $\mathcal{R}_G^{\mu,d}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$ *for* $d \in \mathbb{N}$ *is defined to be the set of all operator families of the form*

$$g(\eta) = g_0(\eta) + \sum_{j=1}^{d} g_j(\eta) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{19}$$

where $g_j(\eta) \in \mathcal{R}_G^{\mu-j,0}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$, $j = 0, \ldots, d$.

Clearly, we have

$$\mathcal{R}_G^{\nu,d}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}} \subset S^{\nu}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge; \boldsymbol{m}); \mathcal{S}_{\mathcal{O}}(X^\wedge; \boldsymbol{n}))_{\boldsymbol{\kappa},\boldsymbol{\kappa}}$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$, and all $\beta \in \mathbb{R}$.

Remark 1.4 *Note that we have* $g(\eta) \in \mathcal{R}_G^{\mu,0}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$ *if and only if* (17) *and* (18) *hold for any fixed* $s \in \mathbb{R}$, $s > -\frac{1}{2}$, *and all* $\gamma, \beta \in \mathbb{R}$. *To get an equivalent definition of* $\mathcal{R}_G^{\mu,0}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$*in* (17) *and* (18) *we may even impose arbitrary fixed* $s \in \mathbb{R}, s > -\frac{1}{2}$ *in the spaces on* X *and* $s' \in \mathbb{R}$ *in the spaces on* ∂X. *The reason is that the respective cone algebras on* $X^\wedge$ *and* $(\partial X)^\wedge$ *contain order reducing elements for any required shift of smoothness and that there are kernel characterisations of mappings* g *when, e.g., those* s *are taken to be fixed. The operators with such kernels then have the mapping properties in the sense of relations* (17) *and* (18) *for all* $s > -\frac{1}{2}$; *then this follows also for the original operators.*

Remark 1.5 *Note that the particular choice of groups* $\boldsymbol{\kappa}$ *and* $\boldsymbol{\chi}$ *gives us operator-valued symbols with the scheme of Douglis-Nirenberg orders* (16) *for the entries of* $\boldsymbol{g} = (g_{ij})_{i,j=1,2}$.

Remark 1.6 *The space* $\mathcal{R}_G^{\mu,0}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$ *can also be defined by*

$$g(\eta) \in \bigcap S_{\text{cl}}^{\mu}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge; \boldsymbol{m}), \mathcal{K}^{s',\gamma';\beta'}, (X^\wedge; \boldsymbol{n}))_{\boldsymbol{\kappa},\boldsymbol{\kappa}}$$

where the intersection is taken over all $s, \gamma, \beta, s', \gamma', \beta' \in \mathbb{R}$, $s > -\frac{1}{2}$, *and a similar condition for* $g^*(\eta)$.

We will also employ more general Green symbols without prescribing asymptotic properties.

Definition 1.7 *Let* $\mathcal{R}_G^{\mu,0}(\mathbb{R}^q, (\gamma, \gamma'); \boldsymbol{v})$ *for* $\mu \in \mathbb{R}$ *and fixed* $\gamma, \gamma' \in \mathbb{R}$ *denote the set of all*

$$g(\eta) \in \bigcap S_{\text{cl}}^{\mu}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge; \boldsymbol{m}), \mathcal{S}^{\gamma'}(X^\wedge; \boldsymbol{n}))_{\boldsymbol{\kappa},\boldsymbol{\kappa}}$$

such that

$$g^*(\eta) \in \bigcap S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{K}^{s^*,(-\gamma')^*;\beta}(X^\wedge; \boldsymbol{n}), \mathcal{S}^{(-\gamma)^*}(X^\wedge; \boldsymbol{m}))_{\boldsymbol{\chi},\boldsymbol{\chi}}.$$

The intersections here are taken over $s, \beta \in \mathbb{R}$, $s > -\frac{1}{2}$. Moreover, $\mathcal{R}^{\mu,d}_G(\mathbb{R}^q; (\gamma,\gamma'); \boldsymbol{v})$ for $d \in \mathbb{N}$ is defined to be the set of all operator families of the form (19) *for arbitrary $g_j \in \mathcal{R}^{\mu,0}_G(\mathbb{R}^q; (\gamma,\gamma'); \boldsymbol{v})$, $j = 0, \ldots, d$.*

Example 1.8 *The operator family*

$$g(\eta) := \varphi(r[\eta])[\eta]^{\nu} r^{\tilde{\nu}} \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(\eta)\psi(r[\eta])$$

for $f(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q))_{\mathrm{const}}$ and arbitray $\varphi, \psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ where $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$ belongs to $\mathcal{R}^{\nu-\tilde{\nu},d}_G(\mathbb{R}^q, (\gamma, \gamma+\tilde{\nu}); \boldsymbol{v})$. For $f(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}^{\mu,d}_{\mathcal{O}}(X; \boldsymbol{v}; \mathbb{R}^q))_{\mathrm{const}}$ we have $g(\eta) \in \mathcal{R}^{\nu-\tilde{\nu},d}_G(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$.

Example 1.9 *Let $P(\eta) \in \mathcal{B}^{-\infty,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$ and let $\varphi, \tilde{\varphi} \in C_0^\infty(\mathbb{R}_+)$. Then we have $\varphi P(\eta)\tilde{\varphi} \in \mathcal{R}^{-\infty,d}_G(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$.*

The following relations are a simple consequence of the definition.

Proposition 1.10 *Let $g(\eta) \in \mathcal{R}^{\mu,d}_G(\mathbb{R}^q, (\gamma,\gamma'); \boldsymbol{v})$ and σ, $\tilde{\sigma} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions. Then we have the following properties:*

(i) $\sigma g,\ g\sigma \in \mathcal{R}^{\mu,d}_G(\mathbb{R}^q, (\gamma,\gamma'); \boldsymbol{v})$,

(ii) $(1-\sigma)g,\ g(1-\sigma) \in \mathcal{R}^{-\infty,d}_G(\mathbb{R}^q, (\gamma,\gamma'); \boldsymbol{v})$,

(iii) $\sigma g \tilde{\sigma} - g \in \mathcal{R}^{-\infty,d}_G(\mathbb{R}^q, (\gamma,\gamma'); \boldsymbol{v})$,

(iv) $r^k g r^l \in \mathcal{R}^{\mu-(k+l),d}_G(\mathbb{R}^q, (\gamma-l, \gamma'+k); \boldsymbol{v})$ *for arbitrary $k, l \in \mathbb{R}$.*

Analogous relations hold for $g(\eta) \in \mathcal{R}^{\mu,d}_G(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$.

1.6 Operators on a manifold with conical exits to infinity

The space $X^\wedge$ is a C^∞ manifold with boundary and conical exit to infinity, see the basics for the analogous case without boundary, cf. Parenti [19], Cordes [3] or Nierenberg and Walker [18]. We then have the operator spaces $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(X^\wedge; \boldsymbol{v})$ from [12], [13]. In contrast to [12], [13] we shall employ here the order conventions from Section 1.3.

To be more precise, we first have $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M; \boldsymbol{v})$ for any manifold M with boundary and conical exit to infinity, especially for the case $M := X^{\asymp}$, where $X^{\asymp}$ is defined to be $\mathbb{R} \times X$ with the variables (r,x), equipped with a Riemannian metric that is a cone metric $dr^2 + r^2 g_X$ for $|r| > R$ with some $R > 0$, where g_X is a Riemannian metric on X. Then $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(X^\wedge; \boldsymbol{v})$ is defined to be the subspace of all $f \in \mathcal{B}^{\mu,d}(X^\wedge; \boldsymbol{v})$ such that $\chi f \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(X^{\asymp}; \boldsymbol{v})$ for each $\chi \in C^\infty(\mathbb{R})$ where $\chi(r) = 0$ for $r < \varepsilon_0$, $\chi = 1$ for $r > \varepsilon_1$ for certain $0 < \varepsilon_0 < \varepsilon_1$ (here, $X^\wedge$ is identified with $\mathbb{R}_+ \times X \subset \mathbb{R} \times X$).

Recall from [13] that $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(X^\wedge; \boldsymbol{v})$ has a hierarchy of principal symbols, namely

$$\sigma(a) = (\sigma_\psi(a), \sigma_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a); \sigma_\partial(a), \sigma_{\mathrm{e}'}(a), \sigma_{\partial,\mathrm{e}'}(a)) \tag{20}$$

where $(\sigma_\psi(a), \sigma_\partial(a))$ for an $a \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(X^\wedge; \boldsymbol{v})$ are the standard interior and boundary symbols of a in the sense of $\mathcal{B}^{\mu,d}(X^\wedge; \boldsymbol{v})$, cf. formulas (8), (9) while the components with subscript e or e$'$ are the corresponding exit symbols.

Remark 1.11 $\mathrm{op}_r(p)(\eta) \in \tilde{\mathcal{B}}^{\mu,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q_\eta)$ *implies*

$$r^{-\mu}\mathrm{op}_r(p)(\eta) \in \mathcal{B}^{\mu,d;0}_{\mathrm{cl}}(X^\wedge; \boldsymbol{v})$$

for every fixed $\eta \in \mathbb{R}^q \setminus 0$.

2 Mellin pseudo-differential operators

2.1 Tools on oscillatory integrals in Fréchet spaces

Let E be a Fréchet space, and let $\mathcal{T}(\mathbb{R}_+ \times \mathbb{R}, E)$ denote the space of all elements $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}, E)$ such that

$$(Su)(r,\tau) := u(e^{-t}, \tau) \in \mathcal{S}(\mathbb{R}_t \times \mathbb{R}_\tau, E).$$

From the isomorphism $S : \mathcal{T}(\mathbb{R}_+ \times \mathbb{R}, E) \to \mathcal{S}(\mathbb{R} \times \mathbb{R}, E)$ we get a Fréchet topology on the space $\mathcal{T}(\mathbb{R}_+ \times \mathbb{R}, E)$. In the special case $E = \mathbb{C}$ we also write $\mathcal{T}(\mathbb{R}_+ \times \mathbb{R})$ for the corresponding space. Below we also employ the notation $\mathcal{T}(\mathbb{R}_+ \times \Gamma_0)$ where $\tau \in \mathbb{R}$ is replaced by $\Gamma_0 = \{i\tau : \tau \in \mathbb{R}\}$. We now introduce a space of E-valued amplitude functions for Mellin oscillatory integrals.

Definition 2.1 *(i)* $\mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$ *denotes the space of all* $h(r,z) \in C^\infty(\mathbb{R}_+ \times \Gamma_0, E)$ *such that for each continuous semi-norm* p *on* E *there exist reals* $\mu = \mu_p$ *and* $\delta = \delta_p$ *such that*

$$\sup\{p((\partial_z^k (r\partial_r)^l h)(r, i\tau))\langle \log r\rangle^{-\delta}\langle\tau\rangle^{-\mu} : (r,\tau) \in \mathbb{R}_+ \times \mathbb{R}\} < \infty \qquad (21)$$

for every $k, l \in \mathbb{N}$,

(ii) $\mathcal{Q}(\mathbb{R}_+ \times \mathbb{C}, E)$ *denotes the space of all* $h(r,z) \in C^\infty(\mathbb{R}_+, \mathcal{A}(\mathbb{C}, E))$ *such that for each continuous semi-norm* p *on* E *there exist reals* $\mu = \mu_p$ *and* $\delta = \delta_p$ *such that*

$$\sup\{p((\partial_z^k (r\partial_r)^l h)(r, \beta + i\tau))e^{-\delta\langle \log r\rangle}\langle\tau\rangle^{-\mu} : (r,\tau) \in \mathbb{R}_+ \times \mathbb{R},\ |\beta| \le j\} < \infty$$

for all $j, k, l \in \mathbb{N}$.

The properties of the following proposition are practically evident:

Proposition 2.2 *(i)* $h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$ *implies* $\partial_z^k (r\partial_r)^l h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$.

(ii) *If* $T : E \to \tilde{E}$ *is a continuous operator, then* $h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$ *implies* $Th \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, \tilde{E})$.

(iii) *Let* E *be the projective limit of Fréchet spaces* E_j *with respect to linear maps* $T_j : E \to E_j$, $j \in \mathbb{N}$. *Then* $h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$ *is equivalent to* $T_j h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E_j)$ *for all* $j \in \mathbb{N}$.

(iv) *Given two Fréchet spaces* E_0 *and* E_1 *and a continuous bilinear map* $\langle .,. \rangle$: $E_0 \times E_1 \to E$, *then* $h_j \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E_j)$, $j = 1, 2$, *implies* $\langle h_0, h_1\rangle \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$.

(v) *Let* V *be a closed subspace of* E *and* $h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$, *then we have* $[h] \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E/V)$.

Analogous relations hold for amplitude functions in the sense of Definition 2.1 *(ii) (that are holomorphic in* z*).*

Example 2.3 *Let X be a closed compact C^∞ manifold. Then*

$$C^\infty(\mathbb{R}_+, L^\mu_{\mathrm{cl}}(X;\Gamma_0)) \subset \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, L^\mu_{\mathrm{cl}}(X)), C^\infty(\mathbb{R}_+, \mathcal{M}^\mu_{\mathcal{O}}(X)) \subset \mathcal{Q}(\mathbb{R}_+ \times \mathbb{C}, L^\mu_{\mathrm{cl}}(X)).$$

Definition 2.4 *A function $\chi_\varepsilon(r,z) : (0,1] \times \mathbb{R}_+ \times \Gamma_0 \to \mathbb{C}$ is called regularising, if*

(i) $\chi_\varepsilon \in \mathcal{T}(\mathbb{R}_+ \times \Gamma_0)$ *for each* $\varepsilon \in (0,1]$,

(ii) $\sup\{|(\partial_z^k (r\partial_r)^l \chi_\varepsilon)(r,z)| \,:\, \varepsilon \in (0,1],\, (r,z) \in \mathbb{R}_+ \times \Gamma_0\} < \infty$ *for all* $k,l \in \mathbb{N}$,

(iii) $\partial_z^k (r\partial_r)^l \chi_\varepsilon)(r,z) \to \begin{cases} 1 \text{ for } k+l=0 \\ 0 \text{ for } k+l>0 \end{cases}$ *pointwise on* $\mathbb{R}_+ \times \Gamma_0$, *as* ε *tends to* 0.

Example 2.5 *Choose an element $\chi \in \mathcal{T}(\mathbb{R}_+ \times \mathbb{R})$ where $\chi(1,0) = 1$, and set $\chi_\varepsilon(r,i\tau) := \chi(r^\varepsilon, \varepsilon\tau)$. Then χ_ε is a regularising function in the sense of Definition* 2.4.

Definition 2.6 *A function $\chi_\varepsilon(r,z) : (0,1] \times \mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}$ is said to be holomorphically regularising if*

(i) $(\varepsilon, r, i\tau) \to \chi_\varepsilon(r, \beta + i\tau)$ *is regularising in the sense of Definition* 2.4 *for every* $\beta \in \mathbb{R}$,

(ii) $\chi_\varepsilon(r,z)$ *is an entire function in* $z \in \mathbb{C}$, *and* $\tau \to \chi_\varepsilon(r, \beta + i\tau) \in \mathcal{S}(\mathbb{R}_\tau)$ *holds uniformly in compact* β*-intervals,*

(iii) for every $\varepsilon \in (0,1]$ *there is a compact set* $K_\varepsilon \subset \mathbb{R}_+$ *such that* $\chi_\varepsilon(r,z) = 0$ *whenever* $r \notin K_\varepsilon$.

Example 2.7 *If $M : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$ is the Mellin transform and $\varphi \in C_0^\infty(\mathbb{R}_+)$ any function such that $\varphi(1) = (M\varphi)(0) = 1$, then $\chi_\varepsilon(r,z) := \varphi(r^\varepsilon)(M\varphi)(\varepsilon z)$ is holomorphically regularising in the sense of Definition* 2.6.

Theorem 2.8 *Let $h(r,z) \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$ and $\chi_\varepsilon(r,z)$ be a regularizing function. Then*

$$\iint_0^\infty r^{i\tau} h(r,i\tau) \frac{\mathrm{d}r}{r} đ\tau := \lim_{\varepsilon \to 0} \iint_0^\infty r^{i\tau} \chi_\varepsilon(r,i\tau) h(r,i\tau) \frac{\mathrm{d}r}{r} đ\tau \tag{22}$$

(*called an oscillatory integral*) *exists in E and is independent of the choice of χ_ε. An analogous result holds for $h(r,z) \in \mathcal{Q}(\mathbb{R}_+ \times \mathbb{C}, E)$ where χ_ε is holomorphically regularising. In particular, for $h \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E) \cap \mathcal{Q}(\mathbb{R}_+ \times \mathbb{C}, E)$ both definitions of oscillatory integrals coincide.*

The proof of this theorem can be given in a similar manner as that in the context of scalar amplitude functions and of the Fourier transform by using integration by parts and applying Lebegue's theorem on dominated convergence.

Remark 2.9 *Let us fix a countable semi-norm system $\{p_\iota\}_{\iota \in \mathbb{N}}$ for the Fréchet topology of E. Then from Definition* 2.1 *for every $h(r,z) \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$ there are sequences $\mu := (\mu_\iota)_{\iota \in \mathbb{N}}$ and $\delta := (\delta_\iota)_{\iota \in \mathbb{N}}$ such that the estimates* (21) *for p_ι are satisfied with the exponents $-\delta_\iota$ and $-\mu_\iota$. Let $\mathcal{Q}^{\mu;\delta}(\mathbb{R}_+ \times \Gamma_0, E)$ denote the subspace of all $h(r,z) \in \mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, E)$, where the estimates* (21) *hold for given sequences μ and δ. We then get a semi-norm system that turns $\mathcal{Q}^{\mu;\delta}(\mathbb{R}_+ \times \Gamma_0, E)$ to a Fréchet space. Then* (22) *gives rise to continuous operators $\mathcal{Q}^{\mu;\delta}(\mathbb{R}_+ \times \Gamma_0, E) \to E$ and $\mathcal{Q}^{\mu;\delta}(\mathbb{R}_+ \times \mathbb{C}, E) \to E$, respectively, for every μ, δ.*

2.2 Mellin operators

Our next objective is to formulate some elements of a global (on $\mathbb{R}_+$) calculus of Mellin pseudo-differential operators with amplitude functions taking values in boundary value problems on X. Similarly to the notation in the beginning of Section 2.1 we consider a class of weighted Schwartz spaces on $\mathbb{R}_+$ with values in a Fréchet space V. In our case V will be one of the spaces

$$V_n := C^\infty(X, E) \quad \text{or} \quad V_{n-1} := C^\infty(\partial X, J), \tag{23}$$

where X is a compact C^∞ manifold with boundary and $E \in \mathrm{Vect}(X)$, $J \in \mathrm{Vect}(\partial X)$. Because of the weight conventions in Mellin pseudo-differential operators there is always given a dimension information $m \in \mathbb{N}$ connected with V (associated with dim X in our application). In general, if V is given together with m, we set

$$\mathcal{T}^\gamma(\mathbb{R}_+, V) := \{u \in C^\infty(\mathbb{R}_+, V) : e^{(\gamma - \frac{m+1}{2})t} u(e^{-t}) \in \mathcal{S}(\mathbb{R}_t, V)\}.$$

Then the isomorphism $\mathcal{T}^\gamma(\mathbb{R}_+, V) \to \mathcal{S}(\mathbb{R}, V)$, $u(r) \to e^{(\gamma - \frac{m+1}{2})t} u(e^{-t})$, gives us a Fréchet topology in the space $\mathcal{T}^\gamma(\mathbb{R}_+, V)$. In particular, for the spaces in (23) we set $m = n$ or $m = n-1$, and we then have the spaces

$$\mathcal{T}^\gamma(\mathbb{R}_+, C^\infty(X, E)) \quad \text{and} \quad \mathcal{T}^\gamma(\mathbb{R}_+, C^\infty(\partial X, E)),$$

respectively. For abbreviation we also set

$$\mathcal{T}^\gamma(\mathbb{R}_+ \times X; \boldsymbol{m}) := \mathcal{T}^\gamma(\mathbb{R}_+, C^\infty(X, E)) \oplus \mathcal{T}^{\gamma - \frac{1}{2}}(\mathbb{R}_+, C^\infty(\partial X, E))$$

for $\boldsymbol{m} := (E, J)$.

Using the Fréchet topology of the space $\mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma})$, $\boldsymbol{v} = (E, J_-; F, J_+)$, with a corresponding semi-norm system $(p_\iota)_{\iota \in \mathbb{N}}$, we can define the space $C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma}))$ of all

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma})$$

such that $\sup\{p_\iota((r\partial r)^k (r'\partial r')^l) f) : r, r' \in \mathbb{R}_+\} < \infty$ for every $\iota, k, l \in \mathbb{N}$. In a similar manner we define $C_{\mathrm{b}}^\infty(\mathbb{R}_+, \ldots)$ for the case of only one $\mathbb{R}_+$-variable. Set (as usual)

$$\mathrm{op}_M^\gamma(f) u(r) = \iint_0^\infty \left(\frac{r}{r'}\right)^{-(\frac{1}{2} - \gamma + i\tau)} f(r, r', \frac{1}{2} - \gamma + i\tau) u(r') \frac{dr'}{r'} đ\tau \tag{24}$$

for $f(r, r', z) \in C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma}))$. We then get a continuous operator

$$\mathrm{op}_M^\gamma(f) : \mathcal{T}^{\gamma + \frac{n}{2}}(\mathbb{R}_+ \times X, \boldsymbol{m}) \to \mathcal{T}^{\gamma + \frac{n}{2}}(\mathbb{R}_+ \times X, \boldsymbol{n})$$

where $\boldsymbol{m} := (E, J_-)$ and $\boldsymbol{n} := (F, J_+)$. In this case the integrand in (24) is regarded as an amplitude function of the class $\mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, C^\infty(X, F) \oplus C^\infty(\partial X, J_+))$ for every fixed $r > 0$.

Remark 2.10 *Let $f(r, r', z) \in C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma}))$. Then (as it will be verified later on in an analogous situation) the function*

$$a(s, i\tau) := ((r, z) \to f(r, sr, z + i\tau))$$

belongs to $\mathcal{Q}(\mathbb{R}_+ \times \Gamma_0, V)$ *for* $V = C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma}))$*. Thus the oscillatory integral*

$$f_L(r,z) := \iint_0^\infty s^{i\tau} f(r, sr, z + i\tau) \frac{\mathrm{d}s}{s} đ\tau \tag{25}$$

converges in $C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma}))$*. Formula (25) is just an expression for the left symbol of the Mellin pseudo-differential calculus, i.e., we have* $\mathrm{op}_M^\gamma(f) = \mathrm{op}_M^\gamma(f_L)$*. In an analogous manner we can find a corresponding right symbol* f_R *associated with* f*.*

Proposition 2.11 *The operator* $\mathrm{op}_M^\gamma(f)$ *for* $f \in C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma}))$ *induces a continuous map*

$$\mathrm{op}_M^\gamma(f) : \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge; \boldsymbol{m}) \to \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge; \boldsymbol{n})$$

for every $s \in \mathbb{R}$*. Moreover,* $f \to \mathrm{op}_M^\gamma(f)$ *represents a continuous operator*

$$C_{\mathrm{b}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \Gamma_{\frac{1}{2}-\gamma})) \to \mathcal{L}(\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge; \boldsymbol{m}), \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge; \boldsymbol{n}))$$

for every $s \in \mathbb{R}$*.*

These properties follow from generalities on pseudo-differential operators with operator-valued symbols in a similar way as corresponding assertions for the case without boundary, cf. [7].

Let $C_{\mathrm{b,F}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ denote the subspace of all $u(r,r') \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ such that

$$\sup_{r,r' \in \mathbb{R}_+} |(r\partial_r)^k (r'\partial_{r'})^{k'} u(r,r')| < \infty$$

for all $k, k' \in \mathbb{N}$. Let $C_{\mathrm{b,F}}^\infty(\mathbb{R}_+)$ denote the subspace of all elements $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ that are independent of r'. Moreover, if E is a Fréchet space, we have a canonical definition of spaces $C_{\mathrm{b,F}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+; E)$ and $C_{\mathrm{b,F}}^\infty(\mathbb{R}_+; E)$.

Lemma 2.12 *For* $h(r,r',z) \in C_{\mathrm{b,F}}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(X; \boldsymbol{v}))$ *and arbitrary reals* γ, δ, σ *we have the following identities:*

- *(i)* $\kappa_\lambda^{-1} \mathrm{op}_M^\gamma(h) \kappa_\lambda = \mathrm{op}_M^\gamma(h_\lambda)$ *where* $h_\lambda(r,r',z) = h(\lambda^{-1}r, \lambda^{-1}r', z)$*, here,* κ_λ *is regarded as a diagonal block matrix* $\mathrm{diag}\{\kappa_\lambda^{(n)}, \kappa_\lambda^{(n-1)}\}_{\lambda \in \mathbb{R}_+}$ *composed with corresponding identity operators.*
- *(ii)* $\mathrm{op}_M^\gamma(h) r^{-\sigma} = r^{-\sigma} \mathrm{op}_M^{\gamma+\sigma}(T^\sigma h)$ *where* $(T^\sigma h)(r,r',z) = h(r,r',z+\sigma)$*,*
- *(iii)* $\mathrm{op}_M^\gamma(h) = \mathrm{op}_M^\delta(h)$ *as operators on functions with compact support with respect to* $r \in \mathbb{R}_+$*.*

Relations of that type and other technicalities on Mellin pseudo-differential operators may be found in [24].

Proposition 2.13 *Let* $\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^q))$ *be independent of* r *for large* r*. Then we have*

$$a(s,w) := ((r,z,\eta) \to \tilde{h}(sr, w+z, s\eta)) \in \mathcal{Q}(\mathbb{R}_+ \times \mathbb{C}, C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^q))).$$

Proof. By definition, the space $\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q)$ is a subspace of $\mathcal{A}(\mathbb{C},\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$, where $\operatorname{Im} z$ plays the role of an extra parameter, i.e., parameters vary on $\Gamma_\beta \times \mathbb{R}^q$ for each $\beta \in \mathbb{R}$, with a uniformity condition in finite intervals $c \le \beta \le c'$. We can dissolve this information by looking at the entries of corresponding 2×2-matrices separately and going back to the definition of $\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\Gamma_\beta \times \mathbb{R}^q)$. If our parameter-dependent operators belong to $C^\infty(\overline{\mathbb{R}}_+,\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$, we can proceed in a similar way; we then write $\tilde{h}(r,z,\eta) = (\tilde{h}_{ij}(r,z,\eta))_{i,j=1,2}$. Assume for simplicity that the bundles E, F as well J_-, J_+ in $\boldsymbol{v}$ are trivial and of fibre dimension 1. In the upper left corner we have an operator family, of the form

$$\mathrm{r}^+ A(r,z,\eta)\mathrm{e}^+ + \tilde{G}(r,z,\eta),$$

where $\tilde{G}(r,z,\eta)$ is of Green type that will be discussed afterwards. $\tilde{A}(r,z,\eta)$ is an element of $C^\infty(\overline{\mathbb{R}}_+,M_{\mathcal{O}}^{\mu}(2X;\mathbb{R}^q))$, in the sense of notation from the analogous context on a closed compact C^∞ manifold, cf. [7]. Here, $2X$ denotes the double of X and we identify X with X_+. Let e^+ denote the operator of extension by zero from $\operatorname{int}X_+$ to $2X$ and r^+ the operator of restriction from $2X$ to $\operatorname{int}X_+$. Then, using an analogue of Proposition 2.13 for the case of a closed compact manifold that corresponds to [7, Lemma 4.1], here, applied to $\tilde{A}(r,z,\eta)$ on $2X$, we immediately get the desired result also for $\mathrm{r}^+\tilde{A}(r,z,\eta)\mathrm{e}^+$. It remains to consider the Green operator-valued case, i.e., $\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{M}_{\mathcal{O},G}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^q))$, cf. formula (10). By virtue of the representation of $\tilde{h}(r,z,\eta)$ as a sum

$$\tilde{h}(r,z,\eta) = \tilde{h}_0(r,z,\eta) + \sum_{j=1}^{d} \tilde{h}_j(r,z,\eta) \begin{pmatrix} T^j & 0 \\ 0 & 0 \end{pmatrix}$$

where $\tilde{h}_j \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{M}_{\mathcal{O},G}^{\mu-j,0}(X;\boldsymbol{v};\mathbb{R}^q))$, $0 \le j \le d$, with T being a differential operator on X of first order that correspond to a vector field that is transversal to ∂X, it suffices to consider the operator functions $\tilde{h}_j$ separately. In other words, without loss of generality we assume $d = 0$. The case $\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{M}_{\mathcal{O}}^{-\infty,0}(X;\boldsymbol{v};\mathbb{R}^q))$ is easy because the entries can be represented by (r,z,η)-dependent families of C^∞-kernels that are smooth up to the boundary. The straightforward calculation is left to the reader. Thus it remains to consider the case that $\tilde{h}(r,z,\eta)$ is a family of Green operators, localised in a coordinate neighbourhood of the boundary and written in local coordinates in the form

$$\tilde{h}(r,z,\eta) = \mathrm{Op}_{x'}(\tilde{g})(r,z,\eta)$$

where $\tilde{g}(r,z,\eta;x',\xi') \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{R}_G^{\mu,0}(U_{x'} \times \mathbb{R}^{n-1+q}_{\xi',\eta} \times \mathbb{C}_z))$, see the definition below. If we now show the assertion for the upper left corners $\tilde{h}_{11}(r,z,\eta)$ of a 2×2-operator function $\tilde{h}(r,z,\eta) = (\tilde{h}_{ij}(r,z,\eta))_{i,j=1,2}$, we shall see how to argue for the other entries that are of simpler structure. In other words, the proof is complete when we study $\tilde{h}_{11}(r,z,\eta)$.

To continue the proof we need a lemma that refers to operator-valued symbols with holomorphic dependence on a covariable.

Let E and $\tilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Consider spaces of symbols $S^\mu_{(\mathrm{cl})}(\mathbb{R}^{p+q} \times \Gamma_\beta;E,\tilde{E})$ for some $p,q \in \mathbb{N}$ and $\beta \in \mathbb{R}$ with constant coefficients and covariables $(\xi,\eta) \in \mathbb{R}^{p+q}$, $\operatorname{Im} z \in \Gamma_\beta$. Then $S^\mu_{(\mathrm{cl})}(\mathbb{R}^{p+q} \times \mathbb{C}_z;E,\tilde{E})$ denotes the space of all $a(\xi,\eta,z) \in \mathcal{A}(\mathbb{C}_z,S^\mu_{(\mathrm{cl})}(\mathbb{R}^{p+q};E,\tilde{E}))$

such that $a(\xi,\eta,\beta+i\tau)\in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{p+q+1}_{\xi,\eta,\tau};E,\tilde{E})$ for every $\beta\in\mathbb{R}$, uniformly in $c\le\beta\le c'$ for every $c\le c'$. We consider $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{p+q}\times\mathbb{C};E,\tilde{E})$ in its canonical Fréchet topology and form the space

$$C^{\infty}(\overline{\mathbb{R}}_{+}\times U,S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{p+q};E,\tilde{E})) \tag{26}$$

for any open $U\subseteq\mathbb{R}^{m}$. Taking instead of a Hilbert space a Fréchet space $\tilde{E}=\operatorname{proj\,lim}\{\tilde{E}_j:j\in\mathbb{N}\}$ with group action we get a corresponding space (26) as a projective limit over j of spaces with respect to $(E,\tilde{E}_j)$. In particular, we can form the spaces

$$C^{\infty}(\overline{\mathbb{R}}_{+}\times U,S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{p+q};L^{2}(\mathbb{R}_{+}),\mathcal{S}(\overline{\mathbb{R}}_{+}))). \tag{27}$$

For the case $U=U_{x'}\subseteq\mathbb{R}^{n-1}$, and ξ' instead of ξ we get a definition of the space $C^{\infty}(\overline{\mathbb{R}}_{+},\mathcal{R}^{\mu,0}_{G}(U_{x'}\times\mathbb{R}^{n-1+q}_{\xi',\eta}\times\mathbb{C}_z))$ as the set of all elements $g(r,z,\eta;x',\xi')$ in the space (27) such that the pointwise formal adjoint $g^{*}(r,z,\eta;x',\xi')$ also belongs to (27). The remaining part of the proof of Proposition 2.13 now reduces to the following general result:

Lemma 2.14 *Let $(E,\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+})$ be a Hilbert space, $(\tilde{E},\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+})$ a Fréchet space, both endowed with corresponding group actions.*

Let $\tilde{h}(r,z,\eta;x,\xi)\in C^{\infty}(\overline{\mathbb{R}}_{+}\times U_x,S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{p+q}_{\xi,\eta}\times\mathbb{C}_z;E,\tilde{E}))$ be independent of r for large r. Then

$$a(s,w):=((r,z,\eta;x,\xi)\to\tilde{h}(sr,w+z,s\eta;x,\xi))$$

is an element of $\mathcal{Q}(\mathbb{R}_{+}\times\mathbb{C}_w,C^{\infty}(\overline{\mathbb{R}}_{+}\times U,S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{p+q}_{\xi,\eta}\times\mathbb{C}_z;E,\tilde{E})))$.

This assertion for scalar symbols (i.e., $E=\tilde{E}=\mathbb{C}$ and the identity as group actions) is just the content of the proof of [7, Lemma 4.1]. The arguments for Hilbert spaces E and $\tilde{E}$ with group actions are practically the same, since $||\kappa_{\langle\cdot\rangle}||_{\mathcal{L}(E)}$ and $||\tilde{\kappa}_{\langle\cdot\rangle}||_{\mathcal{L}(\tilde{E})}$ are of polynomial growth in the covariables. Finally, for the case of a Fréchet space $\tilde{E}$ we only have to check the proof for evey $\tilde{E}_j$ as in the step before. □

To complete the proof of Proposition 2.13 we apply Lemma 2.14 to the case of Green symbols and then repeat the arguments for pointwise adjoints that have the same structure. □

Proposition 2.15 *Let $\tilde{h}(r,z,\eta)\in C^{\infty}(\overline{\mathbb{R}}_{+},\mathcal{M}^{\mu,d}_{\mathcal{O}}(X;\boldsymbol{v};\mathbb{R}^{q}))$ be independent of r for large r, and set $h(r,z,\eta):=\tilde{h}(r,z,r\eta)$. Then*

$$\tilde{h}_R(r',z,\eta):=\iint s^{-i\tau}\tilde{h}(sr',z+i\tau,s\eta)\frac{ds}{s}đ\tau$$

converges in $C^{\infty}(\overline{\mathbb{R}}_{+},\mathcal{M}^{\mu,d}_{\mathcal{O}}(X;\boldsymbol{v};\mathbb{R}^{q}))$, and for $h_R(r',z,\eta):=\tilde{h}_R(r',z,r'\eta)$ we have

$$\mathrm{op}^{\gamma}_{M}(h_R)(\eta)=\mathrm{op}^{\gamma}_{M}(h)(\eta)$$

for every real γ.

This can be proved in a similar manner as a corresponding result for Mellin pseudodifeerential operators with “abstract” operator-valued symbols.

The same is true for the following result: Set $h_j(r,z,\eta):=\tilde{h}_j(r,z,r\eta)$, $j=0,1$, and $h(r,z,\eta):=\tilde{h}(r,z,r\eta)$.

Theorem 2.16 *Let $h_j(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\mu_j,d_j}(X;\boldsymbol{v}_j;\mathbb{R}^q))$ $j=0,1$, be independent of r for large r, where $\boldsymbol{v}_0 := (E_0,J_0;F,J_+)$, $\boldsymbol{v}_1 := (E,J_-;E_0,J_0)$. Define $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\mu_0+\mu_1,e}(X;\boldsymbol{v}_0 \circ \boldsymbol{v}_1;\mathbb{R}^q))$ for $e=\max(\mu_0+d_0,d_1)$ by*

$$\tilde{h}(r,z,\eta) := \iint s^{i\tau}\tilde{h}_0(r,z+i\tau,\eta)\tilde{h}_1(sr,z,s\eta)\frac{ds}{s}đ\tau$$

which is covergent in $C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu_0+\mu_1,e}(X;\boldsymbol{v}_0 \circ \boldsymbol{v}_1;\mathbb{R}^q))$. Then we have

$$\mathrm{op}_M^\gamma(h_0)(\eta)\mathrm{op}_M^\gamma(h_1)(\eta) = \mathrm{op}_M^\gamma(h)(\eta)$$

for each real γ.

Theorem 2.17 *Let $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{0,0}(X;\boldsymbol{v};\mathbb{R}^q))$ be independent of r for large r, and set $h^{(*)}(r,z,\eta) := \tilde{h}^{(*)}(r,z,r\eta) \in C^\infty(\mathbb{R}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{0,0}(X;\boldsymbol{v};\mathbb{R}^q))$, where*

$$\tilde{h}^{(*)}(r,z,\eta) := \iint s^{i\tau}\tilde{h}(sr,n+1-\bar{z}+i\tau,s\eta)^*\frac{ds}{s}đ\tau$$

(with $*$ denoting the pointwise formal adjoint of boundary value problems on X, cf. [27, Theorem 4.3.33]) converges in $C^\infty(\mathbb{R}_+, \mathcal{M}_{\mathcal{O}}^{0,0}(X;\boldsymbol{v}^*;\mathbb{R}^q))$, $\boldsymbol{v}^* := (F,J_+;E,J_-)$. Then

$$\mathrm{op}_M^\gamma(h)(\eta)^* = \mathrm{op}_M^{-\gamma-n}(h^{(*)})(\eta) \tag{28}$$

for every real γ, where the operator on the left hand side is the (η-wise) formal adjoint of $\mathrm{op}_M^\gamma(h)(\eta)$, i.e.,

$$(\mathrm{op}_M^\gamma(h)(\eta)u,v)_{\mathcal{H}^{0,0}(X^\wedge,F)\oplus\mathcal{H}^{0,0}((\partial X)^\wedge,J_+)} = (u,\mathrm{op}_M^\gamma(h)(\eta)^*v)_{\mathcal{H}^{0,0}(X^\wedge,E)\oplus\mathcal{H}^{0,0}((\partial X)^\wedge,J_-)},$$

for all $u \in C_0^\infty(X^\wedge,E)\oplus C_0^\infty((\partial X)^\wedge,J_-)$, $v \in C_0^\infty(X^\wedge,F)\oplus C_0^\infty((\partial X)^\wedge,J_+)$.

3 The edge symbolic calculus

3.1 Edge-degenerate families

The material on operator-valued amplitude functions will be applied to the following class of edge symbols with values in boundary value problems on $X^\wedge$. In the sequel, $\sigma_i(r)$, $\omega_i(r)$, $i=0,1,2,\ldots$, will denote arbitrary cut-off functions.

Definition 3.1 *Let $\gamma,\mu,\nu \in \mathbb{R}$, $\mu-\nu \in \mathbb{N}$, and let $\mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu)$ denote the set of all operator families*

$$a(\eta) = \sigma_1(r)\{a_M(\eta)+a_F(\eta)\}\sigma_0(r) + (1-\sigma_1(r))a_{\mathrm{int}}(\eta)(1-\sigma_2(r)) + g(\eta), \tag{29}$$

where

$$a_M(\eta) := \omega_1(r[\eta])r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\omega_0(r[\eta]), \tag{30}$$

$$a_F(\eta) := (1-\omega_1(r[\eta]))r^{-\nu}\mathrm{op}_r(p)(\eta)(1-\omega_2(r[\eta])), \tag{31}$$

with $p(r,\varrho,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$ and $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ connected with eachother via Mellin quantisation (cf. Theorem 1.2), an element $a_{\mathrm{int}}(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q)$ such that $a_{\mathrm{int}}(\eta) = \varphi_1 a_{\mathrm{int}}(\eta)\varphi_2$ for certain φ_1, $\varphi_2 \in C_0^\infty(\mathbb{R}_+)$, and $g(\eta) \in \mathcal{R}_{\mathrm{G}}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$. The cut-off functions in (29), (30), (31) are assumed to satisfy the relations $\omega_2 \prec \omega_1 \prec \omega_0$ and $\sigma_2 \prec \sigma_1 \prec \sigma_0$.

In addition, we define $\mathcal{R}^{\nu,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$ to be the set of all operator families (29), where (30) and (31) as well as $a_{\mathrm{int}}(\eta)$ are defined as before, while now we require $g(\eta) \in \mathcal{R}_{\mathrm{G}}^{\nu;d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$.

Note that we have $\mathcal{R}^{\nu,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}} \subset \mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$, $\boldsymbol{g} = (\gamma, \gamma-\mu)$, with arbitrary $\gamma,\mu,\nu \in \mathbb{R}$, $\mu-\nu \in \mathbb{N}$.

Let $a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$, and set

$$\sigma_\wedge^\nu(a)(\eta) = \sigma_\wedge^\nu(a_M)(\eta) + \sigma_\wedge^\nu(a_F)(\eta) + \sigma_\wedge^\nu(g)(\eta), \quad \eta \in \mathbb{R}^q\setminus 0,$$

with

$$\sigma_\wedge^\nu(a_M)(\eta) := \omega_1(r|\eta|)r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_\wedge)(\eta)\omega_0(r|\eta|),$$
$$\sigma_\wedge^\nu(a_F)(\eta) := (1-\omega_1(r|\eta|))r^{-\nu}\mathrm{op}_r(p_\wedge)(\eta)(1-\omega_0(r|\eta|)),$$

where $h_\wedge(r,z,\eta) = \tilde{h}(0,z,r\eta)$ and $p_\wedge(r,\varrho,\eta) = \tilde{p}(0,\varrho,r\eta)$.

Remark 3.2 *We have for $\eta \neq 0$*

$$\sigma_\wedge^\nu(a)(\eta) = \lim_{\lambda\to\infty} \lambda^{-\nu}\boldsymbol{\kappa}_\lambda^{-1}a(\lambda\eta)\boldsymbol{\kappa}_\lambda, \tag{32}$$

regarded as a family of operators $\sigma_\wedge^\nu(a)(\eta) : \mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m}) \to \mathcal{K}^{s-\nu,\gamma-\mu;\beta}(X^\wedge;\boldsymbol{n})$, where (32) *holds in the operator norm for every $s > d-\frac{1}{2}$ and $\beta \in \mathbb{R}$. In addition,*

$$\sigma_\wedge^\nu(a)(\lambda\eta) = \lambda^\nu\boldsymbol{\kappa}_\lambda\sigma_\wedge^\nu(a)(\eta)\boldsymbol{\kappa}_\lambda^{-1}$$

for all $\eta \neq 0$, $\lambda \in \mathbb{R}_+$.

Remark 3.3 *We have* (*in the notation of Definition* 3.1)

$$\mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v}) \subset S^\nu(\mathbb{R}^q;\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\boldsymbol{m});\mathcal{K}^{s-\nu,\gamma-\mu;\beta}(X^\wedge;\boldsymbol{n}))_{\boldsymbol{\kappa},\boldsymbol{\kappa}},$$

for all $s \in \mathbb{R}$, $s > d-\frac{1}{2}$, and all $\beta \in \mathbb{R}$.

3.2 An alternative representation of complete edge symbols

As announced in the beginning we now represent complete edge symbols from Definition 3.1 in a new form, where we avoid the η-dependent cut-off functions in (30), (31), but only employ the Mellin term near zero. By this we extend a corresponding result from [7] to the case of boundary value problems. In the following we use the same notation as in Definition 3.1.

Theorem 3.4 *The following conditions are equivalent:*

(*i*) $a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$.

(*ii*) *There are elements $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{M}}_{\mathcal{O}}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^q))$, $g(\eta) \in \mathcal{R}_G^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$, and $a_{\mathrm{int}}(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q)$ where $\varphi_1 a_{\mathrm{int}}(\eta)\varphi_2 = a_{\mathrm{int}}(\eta)$ for some $\varphi_1,\varphi_2 \in C_0^\infty(\mathbb{R}_+)$, such that*

$$a(\eta) = \sigma_1(r)r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0(r) + (1-\sigma_1(r))a_{\mathrm{int}}(\eta)(1-\sigma_2(r)) + g(\eta) \tag{33}$$

with cut-off functions σ_j, $j = 0,1,2$, satisfying $\sigma_2 \prec \sigma_1 \prec \sigma_0$.

An analogous result holds for $a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$ where $g(\eta) \in \mathcal{R}_G^{\nu,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$. For the principal symbol we have in both cases

$$\sigma_\wedge^\nu(a)(\eta) = r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_\wedge)(\eta) + \sigma_\wedge^\nu(g)(\eta)$$

where $h_\wedge(r,z,\eta) = \tilde{h}(0,z,r\eta)$, and $\sigma_\wedge^\nu(g)$ is the homogeneous edge symbol of order ν of the new Green symbol $g(\eta)$.

Proof. Let us start from $a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ in the notation of Definition 3.1. Then we have

$$\begin{aligned} & a_M(\eta) + a_F(\eta) \\ &= \omega_1(r[\eta])r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\omega_0(r[\eta]) + (1-\omega_1(r[\eta]))r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)(1-\omega_2(r[\eta])) \\ &\quad + (1-\omega_1(r[\eta]))r^{-\nu}\{\mathrm{op}_r(p)(\eta) - \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\}(1-\omega_2(r[\eta])) \\ &= r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) + g_1(\eta) + g_2(\eta), \end{aligned}$$

where

$$g_1(\eta) := (1-\omega_1(r[\eta]))r^{-\nu}\{\mathrm{op}_r(p)(\eta) - \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\}(1-\omega_2(r[\eta]))$$

and

$$g_2(\eta) := (\omega_1(r[\eta])-1)r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\omega_2(r[\eta]) + \omega_1(r[\eta])r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)(\omega_0(r[\eta])-1).$$

Since p and h are related via Mellin quantisation, cf. Theorem 1.2, we have $g_1(\eta) \in \mathcal{R}_G^{\nu,d}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$ due to Proposition 4.5. Moreover, Propositions 4.9 and 1.10 yield $g_2(\eta) \in \mathcal{R}_G^{\nu,d}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$. Finally, by analogous calculations for the principal edge symbol we obtain

$$\begin{aligned} \sigma^\nu_{\wedge,old}(a) &= \sigma^\nu_{\wedge,old}(a_M) + \sigma^\nu_{\wedge,old}(a_F) + \sigma^\nu_{\wedge,old}(g) \\ &= r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_\wedge) + \sigma^\nu_\wedge(g_1+g_2) = \sigma^\nu_{\wedge,new}(a) \end{aligned}$$

with obvious meaning of notation. □

3.3 Some properties of edge symbols

Let us now investigate the properties of edge symbols from Definition 3.1 under the aspect of the alternative representation of Section 3.2.

Lemma 3.5 *Let $a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu)$ and $\boldsymbol{v} = (E, J_-; F, J_+)$. Then we have $a(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$.*

Proof. Write $a(\eta)$ in the form (33). We have $g(\eta) \in \mathcal{B}^{-\infty,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$ which is easy to verify and $(1-\sigma_1)a_{\mathrm{int}}(\eta)(1-\sigma_2) \in \mathcal{B}^{\nu,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$ by the notation in (33). Moreover, the Mellin pseudo-differential operator $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)$ can also be expressed in terms of the Fourier transform on $\mathbb{R}_+ \ni r$ with an amplitude function $q(r, \varrho, \eta) \in C^\infty(\mathbb{R}_+, \mathcal{B}^{\nu,d}(X; \boldsymbol{v}; \mathbb{R}^{q+1}_{\varrho,\eta}))$, i.e.,

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) = \mathrm{op}_r(q)(\eta) \quad \mathrm{mod} \quad \mathcal{B}^{-\infty,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q), \tag{34}$$

where $\mathrm{op}_r(q)(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$. □

Lemma 3.6 *For every $\varphi,\ \psi \in C_0^\infty(\mathbb{R}_+)$ and $p(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$ we have $\varphi p(\eta)\psi \in \mathcal{R}^{\nu,d}(\mathbb{R}^q; \boldsymbol{g}; \boldsymbol{v})$ where $\sigma_\wedge(\varphi p \psi) \equiv 0$. Moreover, $\varphi p(\eta)\psi$ can be represented in the form* (33) *with h and a_{int} being compatible, i.e., there are constants $c' > c > 0$ such that*

$$\tilde{\varphi}(r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) - a_{\mathrm{int}}(\eta))\tilde{\tilde{\varphi}} \in \mathcal{B}^{-\infty,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q) \tag{35}$$

for all $\tilde{\varphi},\ \tilde{\tilde{\varphi}} \in C_0^\infty(\mathbb{R}_+)$ supported in $[c, c']$ (if h and a_{int} satisfy the latter condition, we also talk about compatibility of h and a_{int} in the interval $[c, c']$).

Proof. To write $\varphi p(\eta)\psi$ in a form as in Definition 3.1 we may set $a_{\rm int}(\eta) = \varphi p(\eta)\psi$. Choosing cut-off functions $\sigma_2 \prec \sigma_1 \prec \sigma_0$ we can write

$$a_{\rm int}(\eta) = \sigma_1 a_{\rm int}(\eta)\sigma_0 + \sigma_1 a_{\rm int}(\eta)(1-\sigma_0)$$

$$+(1-\sigma_1)a_{\rm int}(\eta)\sigma_2 + (1-\sigma_1)a_{\rm int}(\eta)(1-\sigma_2).$$

Then $\sigma_1 a_{\rm int}(\eta)(1-\sigma_0)$ and $(1-\sigma_1)a_{\rm int}(\eta)\sigma_2$ are smoothing, and we have, in fact,

$$\sigma_1 a_{\rm int}(\eta)(1-\sigma_0), \quad (1-\sigma_1)a_{\rm int}(\eta)\sigma_2 \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}},$$

cf. Proposition 1.10. In other words,

$$a_{\rm int}(\eta) = \sigma_1 a_{\rm int}(\eta)\sigma_0 + (1-\sigma_1)a_{\rm int}(\eta)(1-\sigma_2) \mod \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}.$$

By Mellin quantisation we will write $\sigma_1 a_{\rm int}(\eta)\sigma_0$ as a sum $\sigma_1 r^{-\nu}{\rm op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + g(\eta)$ with suitable h and g. Because of the presence of the functions φ and ψ there is a symbol $q(r,\varrho,\eta) \in C_0^\infty(\mathbb{R}_+,\mathcal{B}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^q))$, compactly supported in $\mathbb{R}_+$, such that $a_{\rm int}(\eta) = {\rm op}_r(q)(\eta)$ and

$$r^\nu q(r,r^{-1}\varrho,r^{-1}\eta) \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{B}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}_{\varrho,\eta}))$$

so that $p(r,\varrho,\eta) := r^\nu q(r,\varrho,\eta) \in C^\infty(\overline{\mathbb{R}}_+,\tilde{\mathcal{B}}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$. It is clear that the symbol p is compactly supported in $\mathbb{R}_+$. Due to Theorem 1.2 there exists an $h(r,z,\eta) \in C^\infty(\mathbb{R}_+,\tilde{\mathcal{M}}_{\mathcal{O}}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^q))$ such that $d(\eta) := {\rm op}_r(p)(\eta) - {\rm op}_M^{\gamma-\frac{n}{2}}(h)(\eta) \in \mathcal{B}^{-\infty}(X^\wedge;\boldsymbol{v};\mathbb{R}^q)$. Note that h is compactly supported in $\mathbb{R}_+$ as p, and $d(\eta)$ is supported away from $r=0$. Using Example 1.9 we obtain

$$\sigma_1 a_{\rm int}(\eta)\sigma_0 = \sigma_1 r^{-\nu}{\rm op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + \sigma_1 r^{-\nu}d(\eta)\sigma_0$$

$$= \sigma_1 r^{-\nu}{\rm op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 \mod \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$$

which yields the desired representation

$$a_{\rm int}(\eta) = \sigma_1 r^{-\nu}{\rm op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + (1-\sigma_1)a_{\rm int}(\eta)(1-\sigma_2) + g(\eta)$$

with $g(\eta) \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$, and the compatibility relation is satisfied for suitable constants $c' > c > 0$. □

Lemma 3.7 *Every $a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ (given of the form (29)) can be written as in (33), where h and $a_{\rm int}$ are compatible in the sense of Lemma 3.6 (i.e., with φ, $\tilde{\varphi} \in C_0^\infty(\mathbb{R}_+)$ being 1 in a suitable interval $[c,c']$).*

Proof. First, by Theorem 3.4 every $a(\eta)$ can be written in the form (33). By means of inverse Mellin quantisation we find an element $\tilde{a}_{\rm int}(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q)$ where $\tilde{\sigma}\tilde{a}_{\rm int}(\eta)\tilde{\tilde{\sigma}} = \tilde{a}_{\rm int}(\eta)$ for suitable cut-off functions $\tilde{\sigma},\tilde{\tilde{\sigma}}$. Let us set

$$\tilde{a}(\eta) := \sigma_1 r^{-\nu}{\rm op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + (1-\sigma_1)\tilde{a}_{\rm int}(\eta)(1-\sigma_2) + g(\eta).$$

We then have

$$a(\eta) - \tilde{a}(\eta) = (1-\sigma_1)(a_{\rm int} - \tilde{a}_{\rm int})(\eta)(1-\sigma_2)$$

which can be written by Lemma 3.6 in terms of a Mellin symbol with the desired compatibility condition. Then $a = \tilde{a} + (a-\tilde{a})$ has the form (33) where the asserted compatibility holds. □

Proposition 3.8 *Let*

$$h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\nu,d}(X;\boldsymbol{v};\mathbb{R}^q)), a_{\text{int}}(\eta) \in \mathcal{B}^{\nu,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q),$$

and assume h and a_{int} to be compatible in $[c,c']$. Further, let $\sigma_2 \prec \sigma_1 \prec \sigma_0$ and $\tilde{\sigma}_2 \prec \tilde{\sigma}_1 \prec \tilde{\sigma}_0$ be two sets of cut-off functions supported in $[0,c']$ where $\sigma_2 = \tilde{\sigma}_2 = 1$ in an open neighbourhood of $[0,c]$. Then the operator functions

$$a(\eta) := \sigma_1 r^{-\nu} \text{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + (1-\sigma_1)a_{\text{int}}(\eta)(1-\sigma_2),$$

$$\tilde{a}(\eta) := \tilde{\sigma}_1 r^{-\nu} \text{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\sigma}_0 + (1-\tilde{\sigma}_1)a_{\text{int}}(\eta)(1-\tilde{\sigma}_2).$$

coincide mod $\in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$. *In other words, the class $\mathcal{R}^{\nu,d}(\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ is independent of the choice of the cut-off functions whenever the above-mentioned support conditions with respect to the compatibility interval are fulfilled. The same is true if in* (33) *we interchange simultaneously $\sigma_1 \leftrightarrow \sigma_2$ and $(1-\sigma_1) \leftrightarrow (1-\sigma_2)$.*

Proof. Let $\omega_5 \prec \omega_4 \prec \omega_3$ be cut-off functions such that $\omega_3 \prec \sigma_2$, $\omega_3 \prec \tilde{\sigma}_2$ and $\omega_5 = 1$ on $[0,c]$. Set

$$a := \omega_4 a \omega_3 + (1-\omega_4)a(1-\omega_5) + \omega_4 a(1-\omega_3) + (1-\omega_4)a\omega_5$$

and $b_M(\eta) := \sigma_1 r^{-\nu}\text{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0$. Then Proposition 4.14 yields

$$a - \omega_4 b_M \omega_3 + (1-\omega_4)a(1-\omega_5) \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}. \tag{36}$$

In a similar manner, if $\omega_2 \prec \omega_1 \prec \omega_0$ where $\sigma_0 \prec \omega_2$ and $\tilde{\sigma}_0 \prec \omega_2$, we get

$$a - \omega_1 a \omega_0 + (1-\omega_1)a_{\text{int}}(1-\omega_2) \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}. \tag{37}$$

Inserting (37) into (36) yields

$$a - \{(1-\omega_4)\omega_1 a \omega_0(1-\omega_5) + \omega_4 b_M \omega_3 + (1-\omega_1)a_{\text{int}}(1-\omega_2)\} \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}.$$

Applying analogous constructions for $\tilde{a}$ we obtain

$$a - \tilde{a} - \varphi_1(a-\tilde{a})\varphi_0 \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$$

where $\varphi_1 := (1-\omega_4)\omega_1$, $\varphi_0 := \omega_0(1-\omega_5)$. As in the proof of Lemma 3.6 we see that both

$$\varphi_1(1-\sigma_1)a_{\text{int}}(1-\sigma_2)\varphi_0 - \varphi_1(a_{\text{int}} - \sigma_1 a_{\text{int}}\sigma_0)\varphi_0$$

and

$$\varphi_1(1-\tilde{\sigma}_1)a_{\text{int}}(1-\tilde{\sigma}_2)\varphi_0 - \varphi_1(a_{\text{int}} - \tilde{\sigma}_1 a_{\text{int}}\tilde{\sigma}_0)\varphi_0$$

belong to $\mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$. Thus

$$\varphi_1(a-\tilde{a})\varphi_0 - \{\varphi_1\sigma_1(b_M - a_{\text{int}})\sigma_0\varphi_0 - \varphi_1\tilde{\sigma}_1(b_M - a_{\text{int}})\tilde{\sigma_0}\varphi_0\}$$

also belongs to that space as well as the summand in $\{\ldots\}$, due to the compatibility relation (35) and Example 1.9. $\square$

As an immediate consequence of Theorem 3.4 we get the following two propositions:

Proposition 3.9 *$a(\eta) \in \mathcal{R}^{\nu,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$ implies*

$$D_\eta^\alpha a(\eta) \in \mathcal{R}^{\nu-|\alpha|,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$$

for every $\alpha \in \mathbb{N}^q$, where

$$\sigma_\wedge^{\nu-|\alpha|}(D_\eta^\alpha a)(\eta) = D_\eta^\alpha \sigma_\wedge^\nu(a)(\eta).$$

Proposition 3.10 *$a(\eta) \in \mathcal{R}^{0,0}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, J_-; F, J_+)$, implies $a^*(\eta) \in \mathcal{R}^{0,0}(\mathbb{R}^q, \boldsymbol{g}^*; \boldsymbol{v}^*)$ for $\boldsymbol{g}^* = (-\gamma + \mu, -\gamma)$, $\boldsymbol{v}^* = (F, J_+; E, J_-)$, and*

$$\sigma_\wedge^0(a^*)(\eta) = \sigma_\wedge^0(a)^*(\eta).$$

3.4 Compositions

Theorem 3.11 *Let $a_j(\eta) \in \mathcal{R}^{\nu_j,d_j}(\mathbb{R}^q, \boldsymbol{g}_j; \boldsymbol{v}_j)$, $j = 0, 1$, where $\boldsymbol{g}_j = (\gamma_j, \gamma_j - \mu_j)$, $\gamma_1 = \gamma_0 - \mu_0$, and $\boldsymbol{v}_1 = (E_0, J_0; F, J_+)$, $\boldsymbol{v}_0 = (E, J_-; E_0, J_0)$. Then we have*

$$a_1(\eta)a_0(\eta) \in \mathcal{R}^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$$

for $d = \max(\nu_1 + d_1, d_0)$, and $\boldsymbol{g} = (\gamma_0, \gamma_0 - \mu_0, -\mu_1)$, $\boldsymbol{v} = \boldsymbol{v}_1 \circ \boldsymbol{v}_0$, where

$$\sigma_\wedge^{\nu_0+\nu_1}(a_1a_0)(\eta) = \sigma_\wedge^{\nu_1}(a_1)(\eta)\sigma_\wedge^{\nu_0}(a_0)(\eta) \quad \text{for} \quad \eta \in \mathbb{R}^q \setminus 0$$

and $\sigma_\psi^{\nu_0+\nu_1}(a_1a_0) = \sigma_\psi^{\nu_1}(a_1)\sigma_\psi^{\nu_0}(a_0)$, $\sigma_\partial^{\nu_0+\nu_1}(a_1a_0) = \sigma_\partial^{\nu_1}(a_1)\sigma_\partial^{\nu_0}(a_0)$ (cf. the notation in (8) and (9) respectively).

Proof. Let $a_j := b_{j,M} + p_{j,M} + g_j$, $j = 0, 1$, where $b_{j,M}(\eta) = \sigma_1 r^{-\nu_j} \operatorname{op}_M^{\gamma_j - \frac{n}{2}}(h_j)(\eta)\sigma_0$, and $p_{j,\text{int}}(\eta) = (1 - \sigma_1)a_{j,\text{int}}(\eta)(1 - \sigma_2)$. First we consider the term

$$a_1 g_0 = b_{1,M} g_0 + p_{1,\text{int}} g_0 + g_1 g_0.$$

Clearly, we have $g_1 g_0 \in \mathcal{R}_G^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ with

$$\sigma_\wedge^{\nu_0+\nu_1}(g_1 g_0) = \sigma_\wedge^{\nu_1}(g_1)\sigma_\wedge^{\nu_0}(g_0).$$

Moreover, $p_{1,\text{int}} g_0 \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ due to the presence of $(1 - \sigma_1)$ as a factor, cf. Proposition 1.10 (*ii*). If $\omega_2 \prec \omega_1 \prec \omega_0$ are cut-off functions, Proposition 4.9 yields

$$b_{1,M}(\eta)g_0(\eta) = \omega_1(r[\eta])b_{1,M}\omega_0(r[\eta])g_0(\eta) + (1 - \omega_1(r[\eta]))b_{1,M}(1 - \omega_2(r[\eta]))g_0(\eta) \quad (38)$$

mod $\mathcal{R}_G^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$. By virtue of the basic mapping properties of Mellin operators the first summand on the right hand side of the latter equations also belongs to $\mathcal{R}_G^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$. By an appropriate choice of ω_1 the second summand can be written in the form

$$\sigma_1\{r^{-N}(1 - \omega_1(r[\eta]))\}\{\chi(\eta)r^{-\nu_1} \operatorname{op}_M^{\gamma - \frac{n}{2}}(T^N h_1)(\eta)\}\{r^N(1 - \omega_2(r[\eta]))\sigma_0 g_0(\eta)\},$$

where $\chi(\eta)$ is a certain excision function and $N \in \mathbb{N}$ arbitrary. Choosing N sufficiently large and using the mapping properties of each of the three factors, we see that also the second term on the right hand side of (38) belongs to $\mathcal{R}_G^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$. By freezing the coefficients of $\tilde{h}_1$ at $r = 0$ we can easily verify that $\sigma_\wedge^{\nu_0+\nu_1}(b_{1,M} g_0) =$

$\sigma_\wedge^{\nu_1}(b_{1,M})\sigma_\wedge^{\nu_0}(g_0)$. In a similar manner we can show $g_1 b_{0,M}+g_1 p_{0,\mathrm{int}} \in \mathcal{R}_G^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$. This gives us

$$a_1 a_0 = b_{1,M} b_{0,M} + p_{1,\mathrm{int}} p_{0,\mathrm{int}} + b_{1,M} p_{0,\mathrm{int}} + p_{1,\mathrm{int}} b_{0,M}$$

mod $\mathcal{R}_G^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$. Choosing a cut-off function $\tilde\sigma$ such that $\tilde\sigma \prec \sigma_1$ we can write

$$b_{1,M}(\eta) p_{0,\mathrm{int}}(\eta) = \{\tilde\sigma b_{1,M}(\eta)(1-\sigma_1)\} a_{0,\mathrm{int}}(\eta)(1-\sigma_2) + (1-\tilde\sigma) b_{1,M}(\eta) p_{0,\mathrm{int}}(\eta).$$

In view of Proposition 4.14 the first term on the right belongs to $\mathcal{R}_G^{-\infty,d}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}$. The second term belongs to $\mathcal{R}^{\nu_0+\nu_1,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ due to Lemma 3.6. The composition $p_{1,\mathrm{int}} b_{0,M}$ can be treated in an analogous manner. In other words, there is an element $a_{\mathrm{int}}(\eta) \in \mathcal{B}^{\nu_0+\nu_1,d}(X^\wedge; \boldsymbol{v}; \mathbb{R}^q)$ such that $\varphi a_{\mathrm{int}} \tilde\varphi = a_{\mathrm{int}}$ for suitable $\varphi, \tilde\varphi \in C_0^\infty(\mathbb{R}_+)$ such that

$$p_{1,\mathrm{int}} p_{0,\mathrm{int}} + b_{1,M} p_{0,\mathrm{int}} + p_{1,\mathrm{int}} b_{0,M} = (1-\sigma_1) a_{\mathrm{int}}(\eta)(1-\sigma_2)$$

mod $\mathcal{R}_G^{-\infty,d}(\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$. As an immediate consequence we see that this term has a vanishing principal edge symbol. Now, setting $h_0'(r,z,\eta) := \sigma_1(r) h_0(r,z,\eta)$ and applying Lemma 2.12 we obtain

$$b_{1,M}(\eta) b_{0,M}(\eta) = \sigma_1 r^{-\nu_0-\nu_1} \mathrm{op}_M^{\gamma_1+\gamma_0-\frac{n}{2}}(T^{\nu_0} h_1)(\eta) \mathrm{op}_M^{\gamma_1+\nu_0-\frac{n}{2}}(h_0')(\eta)\sigma_0 =$$

$$= \sigma_1 r^{-\nu_0-\nu_1} \mathrm{op}_M^{\gamma_0-\frac{n}{2}}(h)(\eta)\sigma_0,$$

$h \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\nu_0+\nu_1,d}(X; \boldsymbol{v}; \mathbb{R}^q))$, where the actions are defined on compactly supported functions with respect to $r \in \mathbb{R}_+$, cf. Theorem 2.16. More precisely, we have $h(r,z,\eta) = \tilde h(r,z,r\eta)$, where

$$\tilde h(r,z,\eta) = \iint s^{i\tau} \tilde h_1(r, z+\nu_0+i\tau, \eta)\sigma_1(sr)\tilde h_0(sr,z,s\eta)\frac{ds}{s} đ\tau.$$

The asserted relation for the principal edge symbols is obviously satisfied as well as the relation for interior and boundary symbols. □

Corollary 3.12 *Assume in Theorem* 3.11 *that* $a_j(\eta) \in \mathcal{R}^{\nu_j,d_j}(\mathbb{R}^q; \boldsymbol{v}_j)_{\mathcal{O}}$, $j = 0,1$*; then we have*

$$a_0(\eta) a_1(\eta) \in \mathcal{R}^{\nu_0+\nu_1,d}(\mathbb{R}^q; \boldsymbol{v})_{\mathcal{O}}.$$

4 Parameter-dependent cone calculus

4.1 Mellin quantisation

Operator families in the following theorem are interpreted in the sense

$$C_0^\infty(X^\wedge, E) \oplus C_0^\infty((\partial X)^\wedge, J_-) \to C^\infty(X^\wedge, F) \oplus C^\infty((\partial X)^\wedge, J_+)$$

for every fixed $\eta \in \mathbb{R}^q$. Let $\boldsymbol{v} = (E, J_-; F, J_+)$.

Theorem 1.2 is a consequence of the following more explicit result:

Theorem 4.1 *Let* $p(r,\varrho,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^{1+q}))$*, and let* $\varphi(r) \in C_0^\infty(\mathbb{R}_+)$ *be any function such that* $\varphi \equiv 1$ *in a neighbourhood of* 1*. Then there exists an* $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\mu;d}(X; \boldsymbol{v}; \mathbb{R}^q))$ *such that*

$$\mathrm{op}_r(\varphi(r'/r)\, p)(\eta) = \mathrm{op}_M^\beta(h)(\eta)$$

for every $\beta \in \mathbb{R}$.

The proof of Theorem 4.1 will be given in several steps. First we consider the upper left corner

$$p_{11}(r,\varrho,\eta) \quad \text{of} \quad p(r,\varrho,\eta) = (p_{ij}(r,\varrho,\eta))_{i,j=1,2}.$$

For abbreviation, in this part of the argumentation we write $p(r,\varrho,\eta)$ in place of $p_{11}(r,\varrho,\eta)$. By definition we have a representation $p(r,\varrho,\eta) = \tilde{p}(r, r\varrho, r\eta)$ for $\tilde{p}(r,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{\mu,d}(X;E,F;\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ that can be written

$$\tilde{p}(r,\tilde{\varrho},\tilde{\eta}) = \tilde{p}_0(r,\tilde{\varrho},\tilde{\eta}) + \tilde{p}_{\mathrm{int}}(r,\tilde{\varrho},\tilde{\eta}) + \tilde{c}(r,\tilde{\varrho},\tilde{\eta}),$$

where $\tilde{p}_0(r,\tilde{\varrho},\tilde{\eta})$ is supported near the boundary in the sense that $\varphi\tilde{p}_0(r,\tilde{\varrho},\tilde{\eta})\psi = 0$ for all $\varphi,\psi \in C_0^\infty(\mathrm{int}X)$ that are supported outside some collar neighbourhood of the boundary, $p_{\mathrm{int}}(r,\tilde{\varrho},\tilde{\eta})$ is supported in $\mathrm{int}X$ in the sense that there is a compact set $K \subset \mathrm{int}\,X$ such that $\varphi p_{\mathrm{int}}(r,\tilde{\varrho},\tilde{\eta})\psi = p_{\mathrm{int}}(r,\tilde{\varrho},\tilde{\eta})$ for all $\varphi,\psi \in C_0^\infty(\mathrm{int}X)$ such that $\varphi = \psi = 1$ on K, while $\tilde{c}(r,\tilde{\varrho},\tilde{\eta})$ is a smoothing family, i.e.,

$$\tilde{c}(r,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{-\infty,d}(X;E,F;\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}})).$$

This term will be treated by Proposition 4.4 below. Concerning $p_{\mathrm{int}}(r,\tilde{\varrho},\tilde{\eta})$ we can forget about the boundary, and we are in the situation of [7, Theorem 3.2]. In other words, it then remains to consider the part $\tilde{p}_0(r,\tilde{\varrho},\tilde{\eta})$ near the boundary. In view of the definition of the spaces $C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;E,F;\mathbb{R}^{1+q}))$ the summand $\tilde{p}_0(r,\tilde{\varrho},\tilde{\eta})$ can be written (up to operator push-forwards from trivialisations of respective bundles in local coordinates) as a finite sum of expressions of the form

$$\mathrm{op}_{x'}\mathrm{op}^+(\tilde{a})(r,\tilde{\varrho},\tilde{\eta}) + \mathrm{op}_{x'}(\tilde{g})(r,\tilde{\varrho},\tilde{\eta}) \tag{39}$$

for a symbol $\tilde{a}(x',y',t,\xi;r,\tilde{\varrho},\tilde{\eta})$ with the transmission property, analogously defined as in Section 1.3. Here, x',y' are variables Ω, and we may assume that a vanishes for $x',y' \notin K'$ for some set $K' \Subset \Omega$, while r and $(\tilde{\varrho},\tilde{\eta})$ play the role of extra tangential variables and covariables. Moreover, $\tilde{g}(x',y',\xi';r,\tilde{\varrho},\tilde{\eta})$ is a Green amplitude function that vanishes for $x',y' \notin K'$ for a certain $K' \Subset \Omega$ (clearly, $\mathrm{op}_{x'}(.)$ means the application of the pseudo-differential action with respect to variables x' and covariables ξ', for instance, $\mathrm{op}_{x'}(\tilde{g})(r,\tilde{\varrho},\tilde{\eta})u(x') = \iint e^{i(x'-y')\xi'}\tilde{g}(x',y',\xi';r,\tilde{\varrho},\tilde{\eta})u(y')dy'đ\xi'$.

To study our problem we have to pass to $a(x',y',t,\xi;r,\varrho,\eta) := \tilde{a}(x',y',t,\xi;r,r\varrho,r\eta)$ and $g(x',y',t,\xi;r,\varrho,\eta) := \tilde{g}(x',y',t,\xi;r,r\varrho,r\eta)$. By virtue of

$$\mathrm{op}_r\left(\varphi\left(r'/r\right)\mathrm{op}_{x'}\mathrm{op}^+(a)\right)(\eta) = \mathrm{op}^+\mathrm{op}_r\left(\varphi\left(r'/r\right)\mathrm{op}_{x'}(a)\right)(\eta) =$$

$$\mathrm{r}^+\mathrm{op}_{t,r}\left(\varphi\left(r'/r\right)\mathrm{op}_{x'}(a)\right)(\eta)\mathrm{e}^+$$

the calculation reduces again to the case without boundary, because in that case the result from [7] gives us a holomorphic Mellin symbol $h(x',y',t,\xi;r,\varrho,z,\eta) = \tilde{h}(x',y',t,\xi;r,\varrho,z,r\eta)$ with the required properties, namely

$$\mathrm{op}_{t,r}\left(\varphi\left(r'/r\right)\mathrm{op}_{x'}(a)\right) = \mathrm{op}_{t,r}\mathrm{op}_{x'}(h)(\eta)$$

which implies $\mathrm{r}^+\mathrm{op}_{t,r}\left(\varphi\left(r'/r\right)\mathrm{op}_{x'}(a)\right)(\eta)\mathrm{e}^+ = \mathrm{r}^+\mathrm{op}_{t,r}\mathrm{op}_{x'}(h)(\eta)\mathrm{e}^+$. Using information from [6, Theorem 2.3] we have

$$\tilde{h}(x',y',t,\xi;r,\varrho,z,\tilde{\eta}) = v_z(r)\mathrm{op}_M^{\frac{1}{2}}(\varphi\left(r'/r\right)\tilde{f})(x',y',t,\xi,\tilde{\eta})v_{-z}(r) \tag{40}$$

where $v_z(r) := r^z$ and

$$\tilde{f}(x',y',t,\xi;r,r',i\varrho,\tilde{\eta}) := M(r,r')r'\tilde{h}(x,x',t,\xi;r,-M(r,r')r\varrho,\tilde{\eta}) \tag{41}$$

for $M(r,r') := (\log r - \log r')(r-r')^{-1},\ r,r' \in \mathbb{R}_+$. We now have to observe analogous relations for summands of the form $\mathrm{op}_r\,(\varphi\,(r'/r)\,\mathrm{op}_{x'}(g))\,(\eta)$. In other words, to complete the proof of Theorem 4.1 it remains to show Proposition 4.4 and Theorem 4.3 below.

Definition 4.2 *Let $U \subseteq \mathbb{R}^p_{x'}$ be open, set $\boldsymbol{w} = (e, j_-; f, j_+)$, for e, f and j_-, j_+ in $\mathbb{N}$, and consider the space of Green symbols*

$$\mathcal{R}^{\mu,d}_G(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}^{1+q}_{\varrho,\eta}; \boldsymbol{w})$$

in the version of smooth dependence on $(r,x') \in \overline{\mathbb{R}}_+ \times U$ up to $r=0$ in its Fréchet topology. Then

$$\mathcal{R}^{\mu,d}_G(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{C}_z \times \mathbb{R}^q_\eta; \boldsymbol{w})$$

denotes the space of all $h(r,x',\xi',z,\eta) \in \mathcal{A}(\mathbb{C}_z, \mathcal{R}^{\mu,d}_G(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{n-1} \times \mathbb{R}^q; \boldsymbol{w}))$ such that $h(r,x',\xi',\beta+i\tau,\eta) \in \mathcal{R}^{\mu,d}_G(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{n-1} \times \mathbb{R}^{1+q}_{\tau,\eta}; \boldsymbol{w})$ for every $\beta \in \mathbb{R}$ and uniformly in $c \le \beta \le c'$ for every $c \le c'$.

We shall apply this kind of spaces for the case $U = \Omega \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^{n-1}$ also write (x',y') for the variable in $\Omega \times \Omega$.

Theorem 4.3 *Let $\tilde{g}(r,x',y',\xi',\varrho,\eta) \in \mathcal{R}^{\mu,d}_G(\overline{\mathbb{R}}_+ \times \Omega \times \Omega \times \mathbb{R}^{n-1} \times \mathbb{R}^{1+q}; \boldsymbol{w})$ and set*

$$g(r,x',y',\xi',\varrho,\eta) := \tilde{g}(r,x',y',\xi',r\varrho,r\eta).$$

Further, let $\varphi(r) \in C_0^\infty(\mathbb{R}_+)$ be a function such that $\varphi \equiv 1$ in a neighbourhood of $r=1$. Then there exists an $\tilde{h}(r,x',y',\xi',z,\tilde{\eta}) \in \mathcal{R}^{\mu,d}_G(\overline{\mathbb{R}}_+ \times \Omega \times \Omega \times \mathbb{R}^{n-1} \times \mathbb{C} \times \mathbb{R}^q; \boldsymbol{w})$ such that for

$$h(r,x',y',\xi',z,\eta) := \tilde{h}(r,x',y',\xi',z,r\eta)$$

we have $\mathrm{op}_r\,(\varphi\,(r'/r)\,\mathrm{op}_{x'}(g))\,(\eta) = \mathrm{op}^\beta_M \mathrm{op}_{x'}(h)(\eta)$ for every $\beta \in \mathbb{R}$.

Proof. The proof of Theorem 4.3 is formally analogous to that of [6, Theorem 2.3] which has stated an analogous fact for scalar symbols, see also the consideration in connection with formula (41). The difference to our case is that we apply the method to operator-valued amplitude functions instead of scalar ones, namely

$$g_j(r,x',y',\xi',\varrho,\eta) = \tilde{g}_j(r,x',y',\xi',r\varrho,r\eta),$$

where $\tilde{g}_j(r,x',y',\xi',\tilde{\varrho},\tilde{\eta}) \in \mathcal{R}^{\mu-j,0}_G(\overline{\mathbb{R}}_+ \times \Omega \times \Omega \times \mathbb{R}^{n-1} \times \mathbb{R}^{1+q})$ and

$$\tilde{g}(r,x',y',\xi',\tilde{\varrho},\tilde{\eta}) = \tilde{g}_0(r,x',y',\xi',\tilde{\varrho},\tilde{\eta}) + \sum_{j=1}^d \tilde{g}_j(r,x',y',\xi',\tilde{\varrho},\tilde{\eta}) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix},$$

cf. formula (19) in a corresponding simpler version. Similarly to (40) it is then admitted to set

$$\tilde{h}(r,x',y',\xi',z,\tilde{\eta}) = \tilde{h}_0(r,x',y',\xi',z,\tilde{\eta}) + \sum_{j=1}^d \tilde{h}_j(r,x',y',\xi',z,\tilde{\eta}) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\tilde{h}_j(r,x',y',\xi',z,\tilde{\eta}) = v_z(r)\mathrm{op}_M^{\frac{1}{2}}(\varphi\,(r'/r)\,\tilde{h}_j)(x',y',\xi',\tilde{\eta})v_{-z}(r)$$

for

$$\tilde{h}_j(r,x',y',\xi',i\varrho,\tilde{\eta}) := M(r,r')r'\tilde{g}_j(r,x',y',\xi',-M(r,r')r\varrho,\tilde{\eta}),$$

$j = 0,\ldots,d.$ $\square$

Proposition 4.4 *Let* $c(r,\varrho,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}_{\varrho,\eta}))$*, and let* $\varphi \in C_0^\infty(\mathbb{R}_+)$ *be given as in Theorem* 4.1*. Then there exists an* $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q_\eta))$ *such that* $\mathrm{op}_r(\varphi(r'/r)\,c)(\eta) = \mathrm{op}_M^\beta(h)(\eta)$ *for every* $\beta \in \mathbb{R}$.

Proof. Assume, for simplicity, that the bundles E, F and J_-, J_+ are trivial and of fibre dimension 1; these bundles are then denoted simply by $\mathbb{C}$ (we then get kernels of operators with respect to chosen Riemannian metrics on X and ∂X with measures $\mathrm{d}x$ and $\mathrm{d}x'$, respectively; otherwise the kernels refer to Hermitian metrics in the respective bundles; this is formally more complicated but does not contribute any substantial difficulty). Then, elements G in $\mathcal{B}^{-\infty,d}(X)$ (which is the corresponding space of smoothing operators for $\boldsymbol{v} = (\mathbb{C},\mathbb{C};\mathbb{C},\mathbb{C})$) have a unique representation as sums

$$G = G_0 + \sum_{l=1}^{d-1} \begin{pmatrix} K_l\gamma^l & 0 \\ Q_l\gamma^l & 0 \end{pmatrix}$$

where $G_0 \in \mathcal{B}^{-\infty,0}(X)$, γ^l acts like $\gamma^l u := (\frac{\partial}{\partial t})^l u|_{\partial X}$ for $u \in H^s(X)$, $s > d - \frac{1}{2}$, where t is the chosen (global) normal variable to ∂X, K_l is a smoothing potential operator described by a kernel M_l in $C^\infty(X \times \partial X)$, and Q_l a smoothing operator on ∂X, i.e., Q_l has a kernel N_l in $C^\infty(\partial X \times \partial X)$. The operator $G_0 = (G_{0,ij})_{i,j=1,2}$ is a matrix of operators with kernels $(C_{ij})_{i,j=1,2}$ where $C_{11} \in C^\infty(X \times X)$, $C_{12} \in C^\infty(X \times \partial X)$, $C_{21} \in C^\infty(\partial X \times X)$, $C_{22} \in C^\infty(\partial X \times \partial X)$. Then the Fréchet topology of $\mathcal{B}^{-\infty,d}(X)$ is determined by the bijection $G \to (C_{ij})_{i,j=1,2} \times (M_l, N_l)_{l=1,\dots,d-1}$, i.e., $\mathcal{B}^{-\infty,d}(X) \cong C^\infty(X\times X) \oplus C^\infty(\partial X \times X) \oplus \oplus_{l=0}^{d-1} C^\infty(X \times \partial X) \oplus \oplus_{l=0}^{d-1} C^\infty(\partial X \times \partial X)$. Similarly, for the element $c(r,\varrho,\eta) = \tilde{c}(r, r\varrho, r\eta)$ in consideration we have a unique representation of the form

$$\tilde{c}(r,\tilde{\varrho},\tilde{\eta}) = \tilde{c}_0(r,\tilde{\varrho},\tilde{\eta}) + \sum_{l=1}^{d-1} \begin{pmatrix} \tilde{k}_l(r,\tilde{\varrho},\tilde{\eta})\gamma^l & 0 \\ \tilde{q}_l(r,\tilde{\varrho},\tilde{\eta})\gamma^l & 0 \end{pmatrix} \tag{42}$$

where, for instance, when we write $\tilde{c}_0 = (\tilde{c}_{0,ij})_{ij=1,2}$, we have

$$\tilde{c}_{0,11}(r,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{S}(\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}, C^\infty(X \times X))),$$

$$\tilde{c}_{0,12}(r,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{S}(\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}, C^\infty(X \times \partial X))),$$

$$\tilde{c}_{0,21}(r,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{S}(\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}, C^\infty(\partial X \times X))),$$

etc., and, analogously, for $\tilde{k}_l$ and $\tilde{q}_l$. To find $h(r,z,\eta) = \tilde{h}(r,z,r\eta)$ for $\tilde{h}(r,z,\tilde{\eta}) \in C^\infty(\mathbb{R}_+, \mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\mathbb{R}^q_{\tilde{\eta}}))$ (where $\boldsymbol{v}$ in the notation is dropped under our assumption on the bundles) it sufficies to treat the summands in (42) and entries in the block matrices separately. For $\tilde{c}_{0,11}(r,\tilde{\varrho},\tilde{\eta})$ the arguments are practically the same as in the proof of [7, Theoerm 3.2]. Concerning the other entries of $\tilde{c}_0$ as well as the integral kernels belonging to $\tilde{k}_l$ and $\tilde{q}_l$ the only change is to replace $C^\infty(X\times X)$ by $C^\infty(X\times\partial X)$, $C^\infty(\partial X \times X)$ and $C^\infty(\partial X \times \partial X)$, respectively. Summing up, we get

$$\tilde{h}(r,z,\tilde{\eta}) := v_z(r)\mathrm{op}_M^{\frac{1}{2}}(\tilde{g})(\tilde{\eta})v_{-z}(r)$$

for $\tilde{g}(r,r',i\varrho,\tilde{\eta}) := \varphi(r'/r)\,M(r,r')r'\tilde{c}(r,-M(r,r')r\varrho,\tilde{\eta})$. $\square$

4.2 Green remainders in the Mellin quantisation

Let $H^{s;\delta}(\mathbb{R}^{1+n}_{\tilde{x}}) := \langle\tilde{x}\rangle^{-\delta}H^s(\mathbb{R}^{1+n})$ for $s,\delta \in \mathbb{R}$ be the standard weighted Sobolev spaces in $\mathbb{R}^{1+n}$ with respect to the variable $\tilde{x}$, and set

$$H^{s;\delta}(\mathbb{R}^{1+n}_+) := \{u|_{\mathbb{R}^{1+n}_+} : u \in H^{s;\delta}(\mathbb{R}^{1+n})\}.$$

Proposition 4.5 *Let $p(r,\varrho,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$ and $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}^{\mu,d}_{\mathcal{O}}(X;\boldsymbol{v};\mathbb{R}^q))$ be related via Mellin quantisation (cf. Theorem 4.1). Then, if $\omega(r)$, $\tilde{\omega}(r)$ and $\sigma(r)$, $\tilde{\sigma}(r)$ are arbitrary cut-off functions,*

$$g(\eta) := \sigma(r)(1-\omega(r[\eta]))\{\mathrm{op}_r(p)(\eta) - \mathrm{op}_M^{\frac{1}{2}}(h)(\eta)\}(1-\tilde{\omega}(r[\eta]))\tilde{\sigma}(r)$$

is an element of $\mathcal{R}^{0,d}_G(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$.

Proof. First observe that $p(r,\varrho,\eta)$ is a sum of families in $C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{B}}^{\mu-j,0}(X;\boldsymbol{v};\mathbb{R}^{1+q}))$ composed with differentiations of order j in direction transversal to the boundary, $j = 0,\dots,d$, cf. (5). Thus, without loss of generality we may assume $d = 0$, because the summands can be treated separately, and the transversal derivatives remain untouched in the constructions.

From Theorems 1.2 and 4.1 we get

$$\mathrm{op}_r(p)(\eta) - \mathrm{op}_M^{\frac{1}{2}}(h)(\eta) = \mathrm{op}_r(q)(\eta) \quad \text{for all} \quad \eta \in \mathbb{R}^q,$$

where $q(r,r',\varrho,\eta) = (1-\varphi(r/r'))p(r,\varrho,\eta)$, with $\varphi \in C_0^\infty(\mathbb{R}_+)$ being as before, where $\mathrm{op}_r(q)(\eta) \in \mathcal{B}^{-\infty,0}(X^\wedge;\boldsymbol{v};\mathbb{R}^q)$. We then have

$$g(\eta) = \sigma(r)(1-\omega(r[\eta]))\mathrm{op}_r(q)(\eta)(1-\tilde{\omega}(r[\eta]))\tilde{\sigma}(r).$$

We now choose an open covering $\{U_1,\dots,U_L,U_{L+1},\dots,U_N\}$ of X by coordinate neighbourhoods where $U_j \cap \partial X \neq \emptyset$ for $0 \le j \le L$, $U_j \cap \partial X = \emptyset$ for $L+1 \le j \le N$, and a subordinate partition of unity $\{\varphi_j\}_{j=1,\dots,N}$. Then the functions $\varphi'_j := \varphi_j|_{\partial X}$, $j = 1,\dots,L$, form a partition of unity on ∂X subordinate to $\{U'_1\dots,U'_L\}$, $U'_j := U_j \cap \partial X$. As is well-known, the covering can be chosen in such a way that also the sets $U_j \cup U_k$ are coordinate neighbourhoods on X for arbitrary $j,k = 1,\dots,N$. Writing $g(\eta)$ in the form

$$g(\eta) = \sum_{j,k=1}^N \Phi_j(x)g(\eta)\Phi_k(x')$$

for $\Phi_j = \mathrm{diag}(\varphi_j,\varphi'_j)$ (composed with corresponding identity maps) the construction reduces to the case when $q(r,r',\varrho,\eta)$ is replaced by $\Phi_j(x)q(r,r',\varrho,\eta)\Phi_k(x')$ for arbitrary j,k. Writing $q(r,r',\varrho,\eta) = (q_{lm}(r,r',\varrho,\eta))_{l,m=1,2}$, we get

$$\Phi_j q \Phi_k = \begin{pmatrix} \varphi_j q_{11}\varphi_k & \varphi_j q_{12}\varphi'_k \\ \varphi'_j q_{21}\varphi_k & \varphi'_j q_{22}\varphi'_k \end{pmatrix}.$$

Now the indices (j,k) for $L+1 \le j,k \le N$ belong to interior neighbourhoods. Corresponding summands $\varphi_j q_{11}\varphi_k$ have the same nature as expressions from the case of operators on a manifold without boundary. This has been treated in [6, Proposition A.4]. Also $\varphi'_j q_{22}\varphi'_k$ for $1 \le j,k \le L$ corresponds to the case on a closed manifold which is the boundary. Thus this case is clear as well. There remain the following cases: the entries of $g(\eta)$ are given in terms of $\mathrm{op}_r(\varphi_j q_{11}\varphi_k)(\eta)$ for $1 \le j \le L$ or $1 \le k \le L$ or $\mathrm{op}_r(\varphi'_j q_{21}\varphi_k)(\eta)$ for $1 \le j \le L$, $1 \le k \le N$ or $\mathrm{op}_r(\varphi_j q_{12}\varphi'_k)(\eta)$ for $1 \le j \le N$, $1 \le k \le L$. We shall show the assertion by verifying the conditions of Remark 1.6 for the respective entries separately. This will be done by looking at kernel representations with integrals over $(\mathbb{R}_+ \times U_j) \times (\mathbb{R}_+ \times U_k)$ for g_{11}, over $(\mathbb{R}_+ \times U'_j) \times (\mathbb{R}_+ \times U_k)$ for g_{21}, and over $(\mathbb{R}_+ \times U_j) \times (\mathbb{R}_+ \times U'_k)$ for g_{12}. We will study $g(\eta) := g_{11}(\eta)$. The arguments for $g_{21}(\eta)$ and $g_{12}(\eta)$ are completely analogous and left to the reader. Without loss of generality we consider scalar operators, i.e., where the involved vector bundles are trivial and of fibre dimension 1. Our operator family now has the form

$$g(\eta) = \sigma(r)(1-\omega(r[\eta]))\mathrm{op}_r(q)(\eta)(1-\tilde{\omega}(r[\eta]))\tilde{\sigma}(r)$$

where $q(r, r', \varrho, \eta) = (1 - \varphi(r'/r))\, p(r, \varrho, \eta)$, $p(r, \varrho, \eta) = \tilde{p}(r, r\varrho, r\eta)$ for a $\tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{\mu,0}(\overline{\mathbb{R}}^n_+; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$, such that $\tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) = \psi(x)\tilde{p}(r, \tilde{\varrho}, \tilde{\eta})\psi(x')$ for a suitable element $\psi \in C_0^\infty(\overline{\mathbb{R}}^n_+)$ (referred below to as a localising function in x and x'). We now observe that integration by parts gives us

$$\mathrm{op}_r\{(1-\varphi(r'/r))\, p\}(\eta) = \mathrm{op}_r\{(1-\varphi(r'/r))\,(r'/r-1)^{-N}\, p_N\}(\eta)$$

for every N, where $p_N(r, \varrho, \eta) = (D_\varrho^N \tilde{p})(r, r\varrho, r\eta)$. In other words we may look at the representation

$$g(\eta) = \sigma(r)(1-\omega(r[\eta]))\mathrm{op}_r(q_N)(\eta)(1-\tilde{\omega}(r[\eta]))\tilde{\sigma}(r), \tag{43}$$

for $q_N(r, r', \varrho, \eta) = (1-\varphi(r'/r))\, p_N(r, \varrho, \eta)$. Observe that $p_N(r, \varrho, \eta) = \tilde{p}_N(r, r\varrho, r\eta)$ for $\tilde{p}_N(r, \tilde{\varrho}, \tilde{\eta}) := D_\varrho^N \tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{\mu-N,0}(\overline{\mathbb{R}}^n_+; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$. The variable $x \in \overline{\mathbb{R}}^n_+$ is to be interpreted as an angular variable of polar coordinates (r, x) in $\mathbb{R}^{n+1}_{\tilde{x}} \setminus 0$, $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_{n+1})$. In other words, $x \in \mathbb{R}^n_+$ plays the role of local coordinates under a chart from an open set on $S^n_+ := \{S^n : \tilde{x}_{n+1} \ge 0\}$ to $\overline{\mathbb{R}}^n_+$. To show our assertion we have to characterise the operator families

$$\kappa^{-1}_{[\eta]}\{D^\alpha_\eta g(\eta)\}\kappa_{[\eta]} =: f_\alpha(\eta) \tag{44}$$

for every multi-index $\alpha \in \mathbb{N}^q$ and to show that (44) for arbitrary $s, \gamma, \beta, s', \gamma', \beta'$, $s > -\frac{1}{2}$, defines a family of continuous operators

$$f_\alpha(\eta) : \mathcal{K}^{s,\gamma;\beta}((S^n_+)^\wedge) \to \mathcal{K}^{s',\gamma';\beta'}((S^n_+)^\wedge) \tag{45}$$

with estimates for the operator norms

$$||f_\alpha(\eta)|| \le c\langle\eta\rangle^{-|\alpha|} \quad \text{for all} \quad \eta \in \mathbb{R}^q \tag{46}$$

with constants $c = c(\alpha; s, \gamma, \beta, s', \gamma', \beta') > 0$. The same has to be done for $f^*_\alpha(\eta)$. This yields the symbol property of $g(\eta)$. In addition, we have to show that $g(\eta)$ is classical in η. We shall investigate the case $\alpha = 0$ in detail. The structure of expressions for arbitrary α is completely analogous; differentiation in η generates a factor $r^{|\alpha|}$, and we may use the relation $\kappa^{-1}_{[\eta]} r^{|\alpha|} \kappa_{[\eta]} = r^{|\alpha|}[\eta]^{-|\alpha|}$, which directly yields (46).

To analyse (45) we now employ the following fact: For every $M \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $\tilde{p}_N(r, \tilde{\varrho}, \tilde{\eta})$ acts with respect to x-variables of the form

$$\tilde{p}_N(r, \tilde{\varrho}, \tilde{\eta})v(x) = \int_{\mathbb{R}^n_+} K(x, x'; r, \tilde{\varrho}, \tilde{\eta})v(x')\mathrm{d}x',$$

where the integral kernel has the following properties:

(*i*) $K(x,x';r,\tilde{\varrho},\tilde{\eta}) \in C_0^M(\overline{\mathbb{R}}^n_{+,x} \times \overline{\mathbb{R}}^n_{+,x'})$ for every fixed $(r,\tilde{\varrho},\tilde{\eta})$,

(*ii*) $(\tilde{\varrho},\tilde{\eta}) \to K(x,x';r,\tilde{\varrho},\tilde{\eta})$ defines a continuous operator $\mathbb{R}^{1+q} \to C^\infty(\overline{\mathbb{R}}_+, \langle\tilde{\varrho},\tilde{\eta}\rangle^{-M} C_{\mathrm{b}}^M(\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}, C_0^M(\overline{\mathbb{R}}^n_+ \times \overline{\mathbb{R}}^n_+)))$.

Let us express the action of $f(\eta)$ first on functions $u(r,x) \in C_0^\infty(\mathbb{R}_+ \times \overline{\mathbb{R}}^n_+)$. We have

$$f(\eta)u(r,x) = \int_{\mathbb{R}_\varrho}\int_{\mathbb{R}_+}\int_{\overline{\mathbb{R}}^n_+} e^{i(r-r')\varrho}\sigma(r[\eta]^{-1})(1-\omega(r))\,(1-\varphi\,(r'/r))$$
$$K(x,x';r,r\varrho,r\eta[\eta]^{-1})(1-\tilde{\omega}(r'))\tilde{\sigma}(r'[\eta]^{-1})u(r',x')\mathrm{d}x'\mathrm{d}r'\text{đ}\varrho. \tag{47}$$

To show the estimates (46) for $\alpha = 0$ we observe that the integral kernel in (47) vanishes identically for $r, r' < \varepsilon$ where $\varepsilon > 0$ is determined by $1-\omega(r) = 0$ and $1-\tilde{\omega}(r') = 0$. This allows us to completely forget about the weights γ and γ', because we achieve at once mappings to corresponding spaces with infinite weights at zero, starting from functions with arbitrary weights. By construction our kernel is supported in a set $\Gamma \subset \overline{\mathbb{R}}^n_+$ with respect to the variable (r,x) as well as to (r',x'), where $(r,x) \in \Gamma \Rightarrow (\lambda r, x) \in \Gamma$ for every $\lambda \geq 1, r \geq c$ for some $c > 0$, and, as explained before, $0 \notin \Gamma$. Therefore, it suffices to replace the spaces $\mathcal{K}^{s,\gamma;\beta}((S^n_+)^\wedge)$ by $H^{s;\beta}(\mathbb{R}^{n+1}_+)$. Because the kernel is expressed in polar coordinates (r,x) for $\tilde{x} \in \mathbb{R}^{1+n}$, it is convenient to pass to the "cylindrical" weighted Sobolev spaces

$$H^{s;\beta}(\mathbb{R}_+ \times S^n_+) := \{u|_{\mathbb{R}_+ \times S^n_+} : u \in H^{s;\beta}(\mathbb{R} \times S^n_+)\}$$

where $H^{s;\beta}(\mathbb{R} \times S^n_+) := \langle r\rangle^{-\beta} H^s(\mathbb{R} \times S^n_+)$ with the space $H^s(\mathbb{R} \times S^n_+) := H^s(\mathbb{R} \times S^n)|_{\mathbb{R}\times S^n_+}$ based on $\text{d}r\text{d}x$. By virtue of an evident half-space variant of [21, Lemma 4.2.2] we may employ the spaces $H^{s;\beta}(\mathbb{R} \times S^n_+)$ instead of $H^{s;\beta}(\mathbb{R}^{n+1}_+)$; finite shifts of s and β are obviously compensated in such a change, because smoothness and weight are arbitrary in the required operator norm.

We then use the fact that the mapping property to be checked in Green symbols as well as in adjoints is only necessary for an arbitrary fixed smoothness and any fixed weight in the argument function. For instance, we check the case $u(r,x) \in L^2(\mathbb{R}_+ \times \mathbb{R}^n_+)$, where we may assume $u(r,x) = 0$ for $x \notin A$ for some $A \Subset \overline{\mathbb{R}}^n_+$ or for $r < \delta$ for some $\delta > 0$ (we may take $A = \operatorname{supp}\psi$ with the above-mentioned localising function ψ). To simplify the notation we write formula (47) in the form

$$f(\eta)u(r,x) = \int_{\mathbb{R}_\varrho}\int_{\mathbb{R}_+}\int_{\overline{\mathbb{R}}^n_+} e^{i(r-r')\varrho}\sigma(r[\eta]^{-1})\langle r\varrho, r\eta[\eta]^{-1}\rangle^{-M} K_M(r,x,r',x',r\varrho,r\eta[\eta]^{-1})$$
$$\tilde{\sigma}(r'[\eta]^{-1})u(r',x')\mathrm{d}x'\mathrm{d}r'\text{đ}\varrho, \tag{48}$$

where

$$K_M(r,x,r',x',r\varrho,r\eta[\eta]^{-1})$$
$$= \langle r\varrho, r\eta[\eta]^{-1}\rangle^M (1-\omega(r))\,(1-\varphi\,(r'/r))\,K(x,x';r,r\varrho,r\eta[\eta]^{-1})(1-\tilde{\omega}(r')).$$

To show that $||f(\eta)u||_{H^{s;\beta}(\mathbb{R}_+\times\mathbb{R}^{n+1}_+)}$ is bounded for every $s,\beta \in \mathbb{R}$, uniformly in $\eta \in \mathbb{R}^q$, it suffices to check the case $s \in \mathbb{N}$ and to show that

$$||\langle r\rangle^{-\beta} D^\alpha_{r,x} f(\eta)u||_{L^2(\mathbb{R}_+\times\mathbb{R}^{n+1}_+)} < \infty \tag{49}$$

for every $\eta \in \mathbb{R}^q$, $|\alpha| \leq s$. The differentiations with respect to x make no problem; thus, for simplicity, we consider D^k_r for $k \leq s$. We may suppose M to be sufficiently large in order to make the following considerations possible. In particular, differentiations under the integral sign are admitted. The structure of the expressions will be preserved under differentiations with respect to r, except for extra powers of the variable ϱ under the inegral, where the exponents are $\le s$. We now set for a moment $\zeta = r\eta[\eta]^{-1}$ and look at the behaviour of $\varrho^k\langle r\varrho,\zeta\rangle^{-M}$, $k \le s$. We have

$$|\frac{\varrho^k}{\langle r\varrho,\zeta\rangle^M}| = r^{-k}|\frac{r\varrho^k}{\langle r\varrho,\zeta\rangle^M}| \le cr^{-k}|\frac{\langle r\varrho,\zeta\rangle^k}{\langle r\varrho,\zeta\rangle^M}| = cr^{-k}\langle r\varrho,\zeta\rangle^{-M+s}$$
$$= cr^{-k}\langle r\varrho,\zeta\rangle^{\frac{-M+s}{2}}\langle r\varrho,\zeta\rangle^{\frac{-M+s}{2}} \le cr^{-k}\langle r\varrho\rangle^{\frac{-M+s}{2}}\langle\zeta\rangle^{\frac{-M+s}{2}}.$$

Then $|D_r^k f(\eta)u|$ can be estimated by an integral $\int_{\mathbb{R}_+}\int_{\overline{\mathbb{R}}^n_+} I\mathrm{d}r'\mathrm{d}x'$ for

$$I := r^{-k}\langle r\eta[\eta]^{-1}\rangle^{\frac{-M+s}{4}} g(r,x;\eta),$$

with

$$g(r,x;\eta) := \int_{\mathbb{R}_\varrho} |e^{i(r-r')\varrho}\sigma(r[\eta]^{-1})\langle r\varrho\rangle^{\frac{-M+s}{2}}\langle r\eta[\eta]^{-1}\rangle^{\frac{-M+s}{4}}$$
$$K_M(r,x,r',x',r\varrho,r\eta[\eta]^{-1})\tilde\sigma(r'[\eta]^{-1})u(r',x')|\mathrm{d}\varrho.$$

Clearly, for M sufficiently large, the integral over ϱ converges, and it is trivial that $g(r,x;\eta) \in L^2(\mathbb{R}_+\times\mathbb{R}^{n+1}_+)$, i.e., for every choice of β we can choose M so large that (49) holds uniformly in η, for all $|\alpha| \le s$.

Summing up we have proved that $g(\eta)$ satisfies the required symbol estimates. It remains to verify that $g(\eta)$ is classical. To this end we look once again at expression (43) and write it in the form $g(\eta) = (1-\omega(r[\eta]))\mathrm{op}_r(a_N)(\eta)(1-\tilde\omega(r[\eta]))$ where

$$a_N(r,r',\varrho,\eta) := \tilde a_N(r,r',r\varrho,r\eta),\quad \tilde a_N(r,r',\tilde\varrho,\tilde\eta) = \sigma(r)\,(1-\varphi\,(r'/r))\,\tilde\sigma(r')\tilde p_N(r,\tilde\varrho,\tilde\eta).$$

Let us simply write $\tilde a_N(\boldsymbol{r},\tilde\varrho,\tilde\eta)$ for $\boldsymbol{r} := (r,r')$. Taylor expansion in $\boldsymbol{r}$ at $(0,0)$ gives us for every N

$$\tilde a_N(\boldsymbol{r},\tilde\varrho,\tilde\eta) = \sum_{|\alpha|\le N}\boldsymbol{r}^\alpha\tilde a_{N,\alpha}(\tilde\varrho,\tilde\eta) + \sum_{|\alpha|=N+1}\boldsymbol{r}^\alpha\tilde{\boldsymbol{a}}_N(\boldsymbol{r},\tilde\varrho,\tilde\eta) \tag{50}$$

for $(1-\omega(r[\eta]))\mathrm{op}_r(\boldsymbol{a}_N)(\eta)(1-\tilde\omega(r[\eta]))$, where $\boldsymbol{a}_N(r,r',\varrho,\eta) := \tilde{\boldsymbol{a}}_N(r,r',r\varrho,r\eta)$ behaves like a symbol of order $-(N+1)$ (because $\boldsymbol{a}_N(r,r',\varrho,\eta)$ gives rise to analogous expressions as in the explicit calculations in the first part of the proof, where the order comes from the factor $\boldsymbol{r}^\alpha$ for $|\alpha| = N+1$). Moreover, we get the contributions from the first sum on the right of (50), i.e., expressions like

$$g_\alpha(\eta) := (1-\omega(r[\eta]))\mathrm{op}_r((r,r')^\alpha\boldsymbol{a}_{N,\alpha})(\eta)(1-\tilde\omega(r[\eta]))$$

for $\boldsymbol{a}_{N,\alpha}(r,\varrho,\eta) := \tilde{\boldsymbol{a}}_{N,\alpha}(r\varrho,r\eta)$ where $g_\alpha(\lambda\eta) = \lambda^{-|\alpha|}\kappa_\lambda g_\alpha(\eta)\kappa_\lambda^{-1}$ for all $\lambda \ge 1$, $|\eta| \ge c$ for some $c > 0$, i.e., these symbols are classical. Because N is arbitrary, we see altogether that $g(\eta)$ is a classical symbol.

To complete the proof we have to do the same things for the formal adjoint $g^*(\eta)$. The only relevant point is that we have to exchange the role of r and r'. This is harmless in all terms, except (perhaps) for the analogue of p_N that is now of the form $p_N(r',\varrho,\eta) = \tilde p_N(r',r'\varrho,r'\eta)$ with a corresponing $\tilde p_N(r',\tilde\varrho,\tilde\eta)$ of analogous structure as before. It is now a standard procedure to pass from "right" symbols, i.e., with (r',ϱ)-dependence to "left" symbols with dependence on (r,ϱ) with respect to variables r and covariables ϱ. In this change the η-dependence will preserve its character, i.e., we get an alternative representation of our operator function where $\tilde p_N(r',r'\varrho,r'\eta)$ is replaced by $\tilde p_N^\vee(r,r\varrho,r\eta)$ for some $\tilde p_N^\vee(r,\tilde\varrho,\tilde\eta)$ of analogous structure as $\tilde p_N$. The remaining part of the proof is as before. $\square$

4.3 Flat elements of the algebra of edge symbols

Our next objective is to investigate Green and Mellin edge symbols (with constant coefficients) as they appear in compositions. Starting from holomorphic Mellin symbols the only remainders will consist of flat Green symbols. We shall extend here the results of [7, Section A.3] from operators on a closed manifold to pseudo-differential boundary value problems. In this case the values of symbols are 2×2-block matrices of operator functions, containing upper left corners from the interior as well as the trace and potential entries, and lower right corners operating on the boundary. Let us consider in our proof the upper left corners. The other entries can be treated in an analogous manner; lower right corners correspond to the case studied in [7, Section A.3].

Let X be a compact C^∞ manifold with boundary. As before, vector bundles $E \in \mathrm{Vect}(X)$ give rise to bundles on $\mathrm{Vect}(X^\wedge)$ (by pull-back with respect to $X^\wedge \to X$) we denote them again by E. Operators of multiplication by functions in spaces of distributional sections will be simply denoted by the functions themselves where identity maps in corresponding bundles will be omitted.

Lemma 4.6 *Let $s, \gamma \in \mathbb{R}$, $E \in \mathrm{Vect}(X)$, let $\omega(r)$ be a cut-off function, and set $\chi(r) := 1 - \omega(r)$.*

(*i*) *For arbitrary $L \in \mathbb{R}$ and $\gamma' \in \mathbb{R}$ we have*

$$\chi(r[\eta])r^{-L} \in S^L_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{H}^{s,\gamma}(X^\wedge, E), \mathcal{K}^{s,\gamma';\beta}(X^\wedge, E))$$

for an appropriate $\beta = \beta(s, \gamma, L)$.

(*ii*) *For every $\gamma', \beta \in \mathbb{R}$ there exists an $L = L(s, \gamma, \gamma', \beta) \geq 0$ such that*

$$\chi(r[\eta])r^{-L} \in S^L_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge, E), \mathcal{H}^{s,\gamma'}(X^\wedge, E)).$$

Lemma 4.7 *Let $\omega(r)$ be any cut-off function and $E \in \mathrm{Vect}(X)$. For arbitrary $s, \gamma, \beta, \gamma', \beta' \in \mathbb{R}$ we have the following relations:*

(*i*)

$$\omega(r[\eta])r^L \in S^{-L}_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{H}^{s,\gamma'}(X^\wedge, E), \mathcal{K}^{s,\gamma'+L;\beta}(X^\wedge, E))$$

for every $L \in \mathbb{R}$.

(*ii*)

$$\omega(r[\eta])r^L \in S^{-L}_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge, E), \mathcal{H}^{s,\gamma}(X^\wedge, E))$$

for every $L \geq 0$.

Lemmas 4.6 and 4.7 can be obtained in an analogous manner as the corresponding assertions in [7, Section A.3]. The following Lemma 4.8 corresponds to [7, Lemma A.7].

Lemma 4.8 *Let $\omega_2 \prec \omega_1$ be cut-off functions and $N \in \mathbb{N}$, and set $\chi_1(r) = 1 - \omega_1(r)$,*

$$f(r, r', \eta) := \omega_2(r[\eta])(\log r/r')^{-N}\chi_1(r'[\eta])$$

for $r, r' \in \mathbb{R}_+$ and $\eta \in \mathbb{R}^q$. We then have the following relations:

(*i*) *$f(\lambda^{-1}r, \lambda^{-1}r', \lambda\eta) = f(r, r', \eta)$ for all $\lambda \geq 1$, $r, r' \in \mathbb{R}_+$, and all $|\eta| \geq$const.*

(*ii*) *For every* $k, k' \in \mathbb{N}$, $\alpha \in \mathbb{N}^q$ *we have*

$$\sup_{r,r',\eta} \{|(r\partial_r)^k (r'\partial_{r'})^{k'} \partial_\eta^\alpha f(r,r',\eta)|[\eta]^{|\alpha|}\} < \infty$$

In particular, $(r,r') \to [\eta]^\alpha \partial_\eta^\alpha f(r,r',\eta)$ *as an* η*-dependent family of elements in* $C^\infty_{\mathrm{b,F}}(\mathbb{R}_+ \times \mathbb{R}_+)$ *is bounded in* $\eta \in \mathbb{R}^q$.

Proposition 4.9 *Let* $\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^{\mu;d}(X;\boldsymbol{v};\mathbb{R}^q))$ *be independent of* r *for large* r, *and set* $h(r,z,\eta) = \tilde{h}(r,z,r\eta)$. *Then for arbitrary cut-off functions* $\omega_2 \prec \omega_1$ *both*

$$g_0(\eta) = \omega_2(r[\eta])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)(1-\omega_1(r[\eta]))$$

and

$$g_1(\eta) = (1-\omega_1(r[\eta]))\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\omega_2(r[\eta])$$

belong to $\mathcal{R}_G^{0,d}(\mathbb{R}^q;\boldsymbol{v})_\mathcal{O}$.

Proof. As announced in the beginning we content ourselves with upper left corners, i.e., we assume $\boldsymbol{v} = (E,F)$ for $E,F \in \mathrm{Vect}(X)$. Since the method does not depend on the bundle aspect we simply consider trivial bundles of fibre dimension 1 and then omit $\boldsymbol{v}$. By definition, $\tilde{h}(r,z,\tilde{\eta})$ has the form

$$\tilde{h}(r,z,\tilde{\eta}) = \sum_{j=0}^{d} \tilde{h}_j(r,z,\tilde{\eta})T^j \tag{51}$$

where T is any first order differential operator on X operating between sections in E, where $T^j|_V = \partial^j/\partial t^j \cdot \mathrm{id}_E$ in a collar neighbourhood of ∂X, with t being the normal variable to the boundary, and $\tilde{h}_j(r,z,\tilde{\eta})$ belonging to $C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^{\mu-j,0}(X;\mathbb{R}^q))$ for $j = 0,\dots,d$. It suffices then to show the assertion for $h(r,z,\eta) := \tilde{h}_j(r,z,r\eta)$. In other words, without loss of generality we assume

$$\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^{\mu,0}(X;\mathbb{R}^q)).$$

The remaining part of the proof is analogous to that of [7, Proposition A.8] and is left to the reader. Only note that the spaces $M_\mathcal{O}^{\mu-N}(X;\mathbb{R}^q)$, $\mathcal{H}^{s,\gamma}(X^\wedge), \dots$ should be replaced by $\mathcal{M}_\mathcal{O}^{\mu-N,0}(X;\mathbb{R}^q)$, $\mathcal{H}^{s,\gamma}(X^\wedge)$, etc., though here for a manifold X with boundary. Lemmas 4.6, 4.7 and 4.8 in the case of boundary value problems play the same role as the analogous of results from [7, Lemmas A.5, A.6, and A.7]. The arguments from [7] in terms of adjoints can be applied in the present situation, since they refer to Mellin symbols in $\mathcal{M}_\mathcal{O}^{\mu-N,0}$ for sufficiently large N. For $N \geq \mu$ we reach the case of non-positive orders, where adjoints of zero-type can be formulated as in the boundaryless case. □

Definition 4.10 *Let* E *be a Fréchet space. We define* $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, E)$ *to be the subspace of all functions* $k \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, E)$ *that satisfy*

$$\sup\{p(\partial_r^l \partial_{r'}^{l'} k(r,r'))\langle r\rangle^N \langle r'\rangle^N : r,r' \geq \varepsilon\} < \infty$$

for each $\varepsilon > 0$, $l, l', N \in \mathbb{N}$ *and each continuous semi-norm* p *on* E. *These expressions define a semi-norm system that induces a Fréchet topology on the space* $S(\mathbb{R}_+ \times \mathbb{R}_+, E)$.

Lemma 4.11 *Let* $\tilde{h}(r,r',z,\eta) \in C^\infty_{\mathrm{b,F}}(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q))$, *and set*

$$h_0(r,r',z,\eta) := \tilde{h}(r,r',z,r\eta), \quad h_1(r,r',z,\eta) := \tilde{h}(r,r',z,r'\eta).$$

We have the identity

$$\mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q) = \mathcal{M}_{\mathcal{O}}^{-\infty}(\mathbb{R}^q)\hat{\otimes}_\pi \mathcal{B}^{-\infty,d}(X;\boldsymbol{v}).$$

With $h_j(r,r',z,\eta)$, $j=0,1$, *we can associate an operator-valued kernel*

$$k_j(r,r',\eta) := \int \left(\frac{r}{r'}\right)^{-i\varrho} h_j(r,r',i\varrho,\eta)\mathrm{d}\varrho,$$

where

$$k_j(r,r',\eta) \in C^\infty(\mathbb{R}^q, C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})))$$

and

$$k_j(r,r',\eta)|_{\mathbb{R}_+\times\mathbb{R}_+\times(\mathbb{R}^q\setminus 0)} \in C^\infty(\mathbb{R}^q\setminus 0, \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}^{-\infty,d}(X;\boldsymbol{v}))).$$

The mappings $\tilde{h} \to k_j$ *induced by this construction are continuous,* $j=0,1$.

Proof. First we concentrate on upper left corners, the other entries behave analogously, and consider again the case of trivial bundles of fibre dimension 1. We reduce the assertion to the case $d=0$ by applying a decomposition of $\tilde{h}(r,r',z,\eta)$ like (51) and treating the factors at T^j separately. Then we may replace $\mathcal{B}^{-\infty,0}(X)$ by $C^\infty(X\times X)$ that turns the assertion into a form that is analogous to [7, Lemma A.10]. From that point on there is no essential difference between the arguments for the case of X without or with boundary. In other words, the proof can be completed in a similar way as [7, Lemma A.10]. □

Proposition 4.12 *Let* $\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q))$, *and set*

$$h_0(r,z,\eta) := \tilde{h}(r,z,r\eta), \quad h_1(r,z,\eta) := \tilde{h}(r',z,r'\eta).$$

If $\psi(\eta)$ *is an excision function (i.e.,* $\psi \in C^\infty(\mathbb{R}^q)$, $\psi \equiv 0$ *near* $\eta = 0$, $\psi \equiv 1$ *outside some neighbourhood of* $\eta = 0$*), we have*

$$g_j(\eta) := \psi(\eta)\chi_1(r[\eta])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_j)(\eta)\chi_2(r[\eta]) \in \mathcal{R}_G^{0,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$$

for $j=0,1$, *where* $\chi_1(r) = 1-\omega_1(r)$, $\chi_2(r) = 1-\omega_2(r)$ *for cut-off cunctions* $\omega_1(r)$, $\omega_2(r)$. *Moreover, for cut-off functions* $\sigma(r)$, $\tilde{\sigma}(r)$ *we have*

$$c_j(\eta) = \sigma(r)\chi_1(r[\eta])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_j)(\eta)\chi_2(r[\eta])\tilde{\sigma}(r) \in \mathcal{R}_G^{0,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}},$$

$j=0,1$.

Proof. Similarly to the proof of Lemma 4.11 we simply take $d=0$ and omit $\boldsymbol{v}$. Then the arguments are practically the same as in [7, Proposition A.11] for the case without boundary; the role of kernels is the same as in [7]. □

Remark 4.13 *Let* $h(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{\mu;d}(X;\boldsymbol{v};\mathbb{R}^q))$, *and let* $\varphi,\psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ *be functions with disjoint support. Then there is a* $c(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q))$ *such that*

$$\varphi\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\psi = \varphi\mathrm{op}_M^{\gamma-\frac{n}{2}}(c)(\eta)\psi$$

for arbitrary $\gamma \in \mathbb{R}$.

Proposition 4.14 *Let $\tilde{h}(r,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu;d}(X;\boldsymbol{v};\mathbb{R}^q))$, and let $\sigma(r)$, $\tilde{\sigma}(r)$, $\sigma_1(r)$, $\sigma_2(r)$ be cut-off functions where $\sigma_2(r) \prec \sigma_1(r)$, and set $b_M(\eta) := \sigma \mathrm{op}_M^{\gamma-\frac{n}{2}}(\eta)\tilde{\sigma}$ for $h(r,z,\eta) := \tilde{h}(r,z,r\eta)$. Then*

$$\sigma_2(r)b_M(\eta)(1-\sigma_1(r)) \quad \text{and} \quad (1-\sigma_1(r))b_M(\eta)\sigma_2(r)$$

belong to $\mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$.

Proof. Choose cut-off functions $\omega_2 \prec \omega_1$ such that $\omega_2 \prec \sigma_2$ and $\omega_1 \prec \sigma_1$. Then

$$\sigma_2(r)b_M(\eta)(1-\sigma_1(r)) = \sigma_2(r)(g_1(\eta)+g_2(\eta))(1-\sigma_1(r)) =: g(\eta)$$

with $g_1(\eta) = \omega_2(r[\eta])b_M(\eta)(1-\omega_1(r[\eta]))$ and $g_2(\eta) = (1-\omega_2(r[\eta]))b_M(\eta)(1-\omega_1(r[\eta]))$. Proposition 4.9 yields $g_1 \in \mathcal{R}_G^{0,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$. Using Remark 4.13 we may assume that $h \in C^\infty(\overline{\mathbb{R}}_+, \tilde{\mathcal{M}}_{\mathcal{O}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^q))$ so that $g_2 \in \mathcal{R}_G^{0,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$ because of Proposition 4.12. Finally, Remark 1.10 gives us $g \in \mathcal{R}_G^{-\infty,d}(\mathbb{R}^q;\boldsymbol{v})_{\mathcal{O}}$. The family $(1-\sigma_1(r))b_M(\eta)\sigma_2(r)$ can be treated in an analogous manner. □

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