Weyl calculus for a class of subelliptic operators

L.Maniccia* and M.Mughetti
maniccia@dm.unibo.it mughetti@dm.unibo.it
Department of Mathematics
University of Bologna
Piazza di Porta S.Donato 5
40127 Bologna, ITALY

Abstract

Weyl-Hörmander calculus is used to get a parametrix in $\mathrm{OPS}^{1-m}_{\frac{1}{2},\frac{1}{2}}(\Omega)$ for a class of subelliptic pseudodifferential operators in $\mathrm{OPS}^{m}_{1,0}(\Omega)$ with real non-negative principal symbol.

1 Introduction

In this paper we consider a classical pseudodifferential operator $P = p(x,D) \in \mathrm{OPS}^m(\Omega)$ (where Ω is an open set of $\mathbb{R}^n$) with asymptotic expansion $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$ such that $p_m(x,\xi)$ is real non-negative on $T^*\Omega \setminus 0$ (see [7] Vol.III)[1]. We denote by $Tr^+ F_{x,\xi}$ the positive trace of the fundamental matrix $F_{x,\xi}$ associated with P and defined as

$$\sigma(v, F_{x,\xi}\, w) = \frac{1}{2} < Hess\, p_m(x,\xi)\, v, w > \qquad v,w \in T_{(x,\xi)}T^*\Omega\,.$$

$Hess\, p_m(x,\xi)$ denotes the Hessian of p_m in $(x,\xi) \in T^*\Omega \setminus 0$ and $\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$ the canonical symplectic form on $T^*\Omega$. In explicit form, $Tr^+ F_{x,\xi} = \sum_{\mu>0} \mu$ where $i\mu$ is in the spectrum of $F_{x,\xi}$. We show that, under suitable conditions on the positive trace $Tr^+ F_{x,\xi}$ and on the subprincipal

*supported by the EU Research and Training Network “Geometric Analysis”

[1]Unexplained notations used troughout are standard and can be found in [7]

symbol

$$p_{m-1}^s(x,\xi) = p_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=1}^{n} \partial_{x_j}\partial_{\xi_j} p_m(x,\xi)\,,$$

P admits a pseudodifferential parametrix $Q = q(x,D) \in \mathrm{OPS}^{1-m}_{\frac{1}{2},\frac{1}{2}}(\Omega)$. More precisely, we prove the following

Theorem 1.1 *Let $P \in \mathrm{OPS}^m(\Omega)$ be a properly supported operator as described above and $\Sigma = \{(x,\xi) \in T^*\Omega \setminus 0 \,|\, p_m(x,\xi) = 0\}$ be its characteristic set. Suppose that*

$$(1.1) \qquad (x,\xi) \in \Sigma \Longrightarrow p_{m-1}^s(x,\xi) + Tr^+\, F_{x,\xi} \notin \mathbb{R}_- = \{z \in \mathbb{C} |\, \mathrm{Im}\, z = 0,\, \mathrm{Re}\, z \le 0\,\}.$$

Then there exists a properly supported pseudodifferential operator $Q = q(x,D) \in \mathrm{OPS}^{1-m}_{\frac{1}{2},\frac{1}{2}}(\Omega)$ such that $I - PQ$ and $I - QP$ are smoothing operators (i.e., integral operators with C^∞ kernel). Furthermore, better estimates are satisfied by the total symbol $q(x,\xi)$ when $m = 2$. Precisely, for any compact $K \subseteq \Omega$, for any multi-index $\alpha, \beta \in \mathbb{Z}^n_+$ there exists a constant $C = C(K,\alpha,\beta)$ such that, for every $(x,\xi) \in K \times (\mathbb{R}^n \setminus 0)$

$$(1.2) \qquad |\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \le C \langle\xi\rangle^{|\alpha|} M(x,\xi)^{-2-|\alpha|-|\beta|}\,,$$

where $M(x,\xi) = \big(p_2(x,\xi)^2 + \langle\xi\rangle^2\big)^{\frac{1}{4}}$.

Here are some remarks.

(1) For any point $(x_0,\xi_0) \notin \Sigma$, there exists a conic neighborhood $\Gamma \subseteq T^*\Omega \setminus 0$ of (x_0,ξ_0) such that $M(x,\xi) \ge c\langle\xi\rangle \quad \forall (x,\xi) \in \Gamma$; hence it follows that $q_{|\Gamma} \in S^{-2}(\Gamma)$.

(2) It is enough to prove Theorem 1.1 for $m = 2$; in fact, denote by Λ_s ($s \in \mathbb{R}$) a properly supported self-adjoint pseudodifferential operator with principal symbol $|\xi|^s$, and, when $m > 2$, consider $\widetilde{P} = \Lambda_{2-m} P \in \mathrm{OPS}^2(\Omega)$. Observe that the principal and the subprincipal symbol of $\widetilde{P}$ are, respectively, $|\xi|^{2-m} p_m(x,\xi)$ and $|\xi|^{2-m} p_{m-1}^s(x,\xi)$ whenever $(x,\xi) \in \Sigma$; moreover, denoting by $Tr^+\, \widetilde{F}_{x,\xi}$ the positive trace of the fundamental matrix associated with $\widetilde{P}$, we have $Tr^+\, \widetilde{F}_{x,\xi} = |\xi|^{2-m} Tr^+\, F_{x,\xi}$ for any $(x,\xi) \in \Sigma$. Therefore $\widetilde{P}$ satisfies hypotheses of Theorem 1.1, then $\widetilde{P}$ has a parametrix $\widetilde{Q} \in \mathrm{OPS}^{-1}_{\frac{1}{2},\frac{1}{2}}(\Omega)$, whence $\widetilde{Q}\Lambda_{2-m} \in \mathrm{OPS}^{1-m}_{\frac{1}{2},\frac{1}{2}}(\Omega)$ is a left parametrix for P. In a similar way we can obtain a right parametrix for P in $\mathrm{OPS}^{1-m}_{\frac{1}{2},\frac{1}{2}}(\Omega)$.

(3) Let $P \in \mathrm{OPS}^2(\Omega)$ be a classical operator with real non-negative principal symbol $p_2(x,\xi)$ and Σ be its smooth characteristic manifold. Then P is in the class $\mathrm{OP}N^{2,2}(\Omega,\Sigma)$, introduced by Sjöstrand [12], since $p_2(x,\xi)$ must vanish to 2nd-order on Σ (see Example 1.4 [1]). However, P is not in general transversally elliptic with respect to Σ (i.e., $p_2(x,\xi)$ does not vanish exactly to 2nd-order on Σ). Thus techniques developed by Boutet de Monvel in [1], and by Boutet

de Monvel, Grigis, Helffer in [2] do not apply. We want to point out that in Theorem 1.1 no assumption on the geometry of the characteristic set Σ and on trasversal ellipticity of P is made (see examples 1.1 and 1.2 below). When P is transversally elliptic with respect to Σ, Boutet de Monvel (see [1] Theorem 8.6) gives a better condition for the existence of a parametrix in $\mathrm{OPS}^{1-m}_{\frac{1}{2},\frac{1}{2}}(\Omega)$: it suffices that $p^s_{m-1}(x,\xi) + Tr^+ F_{x,\xi}$ avoid a discrete set of values in $\mathbb{R}_-$; more precisely, for any $(x,\xi) \in \Sigma$

$$p^s_{m-1}(x,\xi) + < Hess\, p_m(x,\xi)\, v, \overline{v} > + \sum_{j=1}^{n} (2\alpha_j + 1)\lambda_j \neq 0$$

for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \geq 0$, and for every $v \in Ker(F_{x,\xi})^{2n}$.

(4) From continuity properties in Sobolev spaces, it follows that an operator P satisfying hypotheses of Theorem 1.1 is C^∞-hypoelliptic with minimal loss of 1-derivative. This is a well-known fact (see [7] Theorem 22.3.4), but here it is obtained in a constructive way, i.e., by exhibting a parametrix in a suitable class of Hörmander.

Theorem 1.1 above represents a generalization of results given by Cancelier, Chemin and Xu in [5], concerning second order pseudodifferential operators in "sum of squares" form; more precisely, the authors deal with generalized Laplacians Δ_H defined as

$$\Delta_H = \sum_{j=1}^{m} B_j^* B_j \,,$$

where $m \in \mathbb{N}$, $B_j = b_j(x,D)$ is a classical operator in $\mathrm{OPS}^1(\Omega)$ with real symbol $b_j(x,\xi) \sim \sum_{l\geq 0} b_{j,1-l}(x,\xi)$ $(j = 1, ..., m)$. They assume that the "step 2" condition of Hörmander holds for Δ_H, namely for any compact $K \subseteq \Omega$, there exist some positive constants c, C such that

$$\sum_{j=1}^{m} b_{j,1}(x,\xi)^2 + \sum_{i,j=1}^{m} \{b_{i,1}, b_{j,1}\}^2(x,\xi) \geq c|\xi|^2 \,, \tag{1.3}$$

whenever $(x,\xi) \in K \times \mathbb{R}^n \,:\, |\xi| > C$.

In this setting they prove the existence of a parametrix $Q = q(x,D) \in \mathrm{OPS}^{-1}_{\frac{1}{2},\frac{1}{2}}(\Omega)$ for Δ_H such that $q(x,\xi)$ satisfies estimates (1.2)(see [5] for more details). Note that when the "step 2" condition (1.3) holds, then given any compact set $K \subseteq \Omega$, there exists a positive constant C_K such that

$$\|u\|_1^2 \leq C_K\big(\|\Delta_H u\|_0^2 + \|u\|_0^2\big) \quad \forall u \in C_0^\infty(K)\,, \tag{1.4}$$

where $\|u\|_s^2 = (2\pi)^{-n} \int (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi$ denotes the usual Sobolev norm of u.
If Λ_s ($s \in \mathbb{R}$) is as above, the subelliptic estimate (1.4) yields $C_K(\Delta_H + I)^2 \geq \Lambda_2$, as self-adjoint operators. From Functional Analysis, one has that $A^2 \geq B^2$, $A = A^*, B = B^* \geq 0$, implies $A \geq B$. Hence we get, for some positive constants $c_1 = c_1(K)$, $c_2 = c_2(K)$

$$\big(\Delta_H u, u\big) \geq c_1 \|u\|_{\frac{1}{2}}^2 - c_2 \|u\|_0^2 \quad \forall u \in C_0^\infty(K)\,,$$

which is the so-called Melin inequality for Δ_H (see [10]). Whence, Δ_H satisfies condition (1.1) (see [7] Chapter XXII). So it is clear that Theorem 1.1 does not improve the precision of the results given in [5], but rather extends the class of operators they can be applied to.
We now consider the following examples:

Example 1.1 In $\mathbb{R}^3$

$$P = D_{x_1}^2 + \mu_1 D_{x_2}^2 + \mu_2 x_2^2 D_{x_3}^2 + \phi(x,D)^4 + \alpha(x) D_{x_1} + \beta(x) D_{x_2} + \gamma(x) D_{x_3} + c(x)\ ,$$

where

- $D_j = \dfrac{1}{i}\dfrac{\partial}{\partial x_j} \quad (i = \sqrt{-1})$;
- $\mu_1, \mu_2 \in \mathbb{R} : \mu_1 > 0,\ \mu_2 > 0$;
- $\phi(x,\xi) \in C^\infty(T^*\mathbb{R}^3 \setminus 0; \mathbb{R})$ is positively homogeneuos of degree $\frac{1}{2}$ such that $\big(\partial_{x_1}, \partial_{x_3}, \partial_{\xi_3}\big)\phi \neq (0,0,0)$ on $\Sigma = \{(x,\xi) \in T^*\mathbb{R}^3 \setminus 0 \mid \xi_1 = \xi_2 = x_2 = \phi(x,\xi) = 0\ \}$;
- $\alpha, \beta, \gamma, c \in C^\infty(\mathbb{R}^3; \mathbb{R}) \ : \ |\gamma(x)| < \sqrt{\mu_1 \mu_2}$ when $(x,\xi) \in \Sigma$.

It is easy to show that P is not a generalized laplacian Δ_H if for some $(\overline{x}, \overline{\xi}) \in \Sigma$ we have $\gamma(\overline{x}) \neq 0$. Furthermore, the results in [2] cannot be applied to the operator P; in fact, it is not transversally elliptic with respect to Σ (due to the term $\phi(x,\xi)^4$ in its principal symbol). However, since for any $(x,\xi) \in \Sigma$

$$p_1^s(x,\xi) + Tr^+ F_{x,\xi} = \gamma(x)\ \xi_3 + \sqrt{\mu_1 \mu_2}|\xi_3| > 0\,,$$

Theorem 1.1 applies.

Example 1.2 In $\mathbb{R}^2$ (for a fixed positive integer h)

$$P = D_{x_1}^2 + f(x)^{2h} D_{x_2}^2 + \alpha(x) D_{x_1} + \beta(x) D_{x_2} + c(x)\ ,$$

where

- $f \in C^\infty(\mathbb{R}^2;\mathbb{R})$, $\partial_{x_1} f \neq 0$ on $\Sigma = \{(x,\xi) \in T^*\mathbb{R}^2 \setminus 0 \mid \xi_1 = f(x) = 0\}$;
- $\alpha,\ \beta,\ c \in C^\infty(\mathbb{R}^2;\mathbb{C})$, such that, for any $(x,\xi) \in \Sigma$,
  $$\operatorname{Im}\beta(x) \neq 0 \ \text{ if } \begin{cases} h = 1 \text{ and } |\operatorname{Re}\beta(x)| \geq |\partial_{x_1} f(x)| \\ h > 1 \end{cases} .$$

Once more, P is not a generalized laplacian and it is not transversally elliptic with respect to Σ (when $h > 1$); however, Theorem 1.1 holds for such a P.

The proof of Theorem 1.1 is fully based on the Weyl-Hörmander calculus developed in [5] and can be obtained by largely using the method shown there in a microlocal version. For the sake of completeness in Section 2 we have collected definitions and results we need. The main difference with [5] concerns the way we obtain the following lower bound

$$\|(\mathcal{A}_\Gamma + cI)u\|_0 \geq C_\Gamma \|u\|_1 \qquad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Here c is a suitable costant and $\mathcal{A}_\Gamma$ is an appropriated microlocal extension of P outside of a conic set $\Gamma \subseteq T^*\Omega$ (see Section 4).

Acknowledgements. We wish to thank A. Parmeggiani for suggesting the problem, introducing us to the Weyl-Hörmander calculus, and for the many interesting discussions on this subject.

2 Symbol classes

In this section we recall some definitions and results about Weyl-Hörmander calculus. For missing details we refer the reader to [5].

Let, for $X = (x,\xi) \in \mathbb{R}^{2n}$, g_X be a Riemannian metric on $\mathbb{R}^{2n}$, i.e. $g : \mathbb{R}^{2n} \to Sym^+(2n,\mathbb{R})$, $X \longmapsto g_X$ ($Sym^+(2n,\mathbb{R})$ is the set of real symmetric positive definite $(2n)\times(2n)$ matrices). By g_X^σ we denote the dual metric of g_X with respect to the symplectic form σ on $\mathbb{R}^{2n}$ ($\sigma(X,Y) = y\xi - x\eta$, $Y = (y,\eta)$) defined, for every $Z = (z,\zeta) \in \mathbb{R}^{2n}$, as

$$\sup_{Z\neq 0} \frac{\sigma(Y,Z)^2}{g_X(Z)} .$$

The metric g_X is called a Hörmander metric if it satisfies the following properties :

- g_X is **slowly varying** i.e. there exists a constant $C \geq 1$ such that, given $X, Y, Z \in \mathbb{R}^{2n}$

$$g_X(Y - X) \leq \frac{1}{C} \Longrightarrow C^{-1} g_Y(Z) \leq g_X(Z) \leq g_Y(Z) . \tag{2.1}$$

- g_X is σ **-temperate** i.e. there exists a positive constant C' and a positive integer $N_0 \in \mathbb{Z}_+$ such that , for every $X, Y, Z \in \mathbb{R}^{2n}$

$$g_X(Y) \leq g_X^\sigma(Y) \quad \text{and} \quad g_X(Z) \leq C' g_Y(Z)(1 + g_X^\sigma(Y - X))^{N_0} \ . \tag{2.2}$$

A positive function $M : \mathbb{R}^{2n} \to (0, +\infty)$ is called a **g-admissible weight** if there exists some positive constants C, C' and a positive integer $N_1 \in \mathbb{Z}_+$ such that, for every $X, Y, Z \in \mathbb{R}^{2n}$,

$$g_X(X - Y) \leq C^{-1} \Longrightarrow \frac{1}{C'} M(X) \leq M(Y) \leq C' M(X) \tag{2.3}$$

and

$$\frac{M(X)}{M(Y)} \leq C'(1 + g_X^\sigma(Y - X))^{N_1} \ . \tag{2.4}$$

If $q \in C^\infty(\mathbb{R}^{2n})$ we say that $q \in S(M, g)$ if, for any integer $k \geq 0$, we have

$$||q||_{k;S(M,g)} = \sup_{l \leq k, X \in \mathbb{R}^{2n}} \frac{|q|_l^g(X)}{M(X)} < +\infty \tag{2.5}$$

where, denoting by $q^{(l)}$ the l-th differential of q,

$$|q|_l^g(X) = \sup_{Z_1, \dots, Z_l \in \mathbb{R}^{2n}} \frac{|q^{(l)}(X; Z_1, \dots, Z_l)|}{\Pi_{j=1}^l g_X(Z_j)^{1/2}} \ .$$

The space $S(M, g)$ is a Fréchet space with topology given by the seminorms in (2.5). Once we have $q \in S(M, g)$ (with g_X satisfying (2.1) and (2.2) and M satisfying (2.3) and (2.4)), $q^W(x, D)$ is a bounded operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ which can be extended to $\mathcal{S}'(\mathbb{R}^n)$. Here we denote by $q^W(x, D)$ the usual Weyl quantization, namely

$$q^W(x, D)u(x) = \iint e^{i(x-y,\xi)} q\Big(\frac{x + y}{2}, \xi\Big) u(y) dy đ\xi \qquad u \in \mathcal{S}(\mathbb{R}^n) \ .$$

In what follows, $q(x, \xi) \in \mathrm{S}^m(\mathbb{R}^n)$ is called a global symbol iff for any multi-index $\alpha, \beta \in \mathbb{Z}_+^n$, there exists a positive constant $C = C(\alpha, \beta)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C \langle \xi \rangle^{m - |\beta|} \quad \text{for every } (x, \xi) \in T^* \mathbb{R}^n \setminus 0.$$

Accordingly, we say that $q(x, D) \in \mathrm{OPS}^m(\mathbb{R}^n)$ is global if $q(x, \xi)$ is.
We now define the metric and the weight we will consider throughout the paper.

Definition 2.1 *Let $a(x,\xi)$ be a classical global symbol in $S^2(\mathbb{R}^n)$ with $a(x,\xi) \sim \sum_{j\geq 0} a_{2-j}(x,\xi)$, $a_{2-j}(x,\xi) \in C^\infty(T^*\mathbb{R}^n \setminus 0)$ positively homogeneous of order $2-j$ and $a_2(x,\xi) \geq 0$. Set*

$$M(X) = (a_2^2(x,\xi) + \langle\xi\rangle^2)^{1/4} \quad X = (x,\xi) \in \mathbb{R}^{2n} \tag{2.6}$$

$$g_X(Y) = M^{-2}(X)(\langle\xi\rangle^2|y|^2 + |\eta|^2) \quad X = (x,\xi), Y = (y,\eta) \in \mathbb{R}^{2n} \ . \tag{2.7}$$

The metric and the weight defined above have all the good properties required by the calculus as stated in the following Lemma.

Lemma 2.2 *Let $a(x,\xi)$ be as in Definition 2.1 and let M, g_X be as in (2.6) and (2.7) respectively. We have*

(1) *g_X is a Hörmander metric and M^2 is an admissible weight with respect to g_X;*

(2) *$a \in S(M^2, g)$.*

Proof.

(1) For this part we refer to [5], Lemma 1.2.1.

(2) We need to prove the following estimate

$$|\partial_x^\alpha \partial_\xi^\beta a(X)| \leq CM(X)^{2-|\alpha|-|\beta|}\langle\xi\rangle^{|\alpha|} \qquad (\alpha,\beta) \in \mathbb{Z}_+^{2n}\ , X = (x,\xi) \in \mathbb{R}^{2n} \ .$$

When $|\alpha| + |\beta| \geq 2$ the estimate obviously follows from the fact that $a \in S^2(\mathbb{R}^n)$, the only case to check is then $|\alpha| + |\beta| \leq 1$.

We write $a = a_2 + (a - a_2)$ with $a - a_2 \in S^1(\mathbb{R}^n)$. Noting that $M(X) \geq \langle\xi\rangle^{\frac{1}{2}} \geq 1$, we have

$$|a_2(X)| \leq M^2(X), \qquad |(a - a_2)(X)| \leq C\langle\xi\rangle \leq CM^2(X),$$

and

$$|\partial_{x_j}(a - a_2)(X)| \leq C\langle\xi\rangle \leq C\langle\xi\rangle M(X)\ , \qquad |\partial_{\xi_j}(a - a_2)(X)| \leq C \leq CM(X)\ .$$

To treat derivatives of the term a_2, recall that for any non-negative function $f \in C^2(\mathbb{R})$ we have

$$(f'(t))^2 \leq 2||f||_\infty \cdot f(t). \tag{2.8}$$

This gives

$$|(\partial_{x_i} a_2)(X)\langle\xi\rangle^{-1}M^{-1}(X)| \leq C|a_2(X)|^{\frac{1}{2}} \cdot M^{-1}(X) \leq C\ ,$$

$$|(\partial_{\xi_i} a_2)(X)M^{-1}(X)| \leq C|a_2(X)|^{\frac{1}{2}} \cdot M^{-1}(X) \leq C\ .$$

The parametrix construction we develop is based on the following theorem which gives a sufficient condition for the existence of an inverse of $a^W(x,D)$ in the calculus. We do not repeat here the proof that can be found in [5] but only say that it uses a general inversion result due to Bony and Chemin ([3] Corollary 7.7).

Theorem 2.3 *Let $a(x,\xi)$ be a classical global symbol in $S^2(\mathbb{R}^n)$ with $a(x,\xi) \sim \sum_{j\geq 0} a_{2-j}(x,\xi)$ and $a_2(x,\xi) \geq 0$. If a^W is invertible and $(a^W)^{-1}$ is a continuous operator from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$, then there exists a symbol $b \in S(M^{-2}, g)$ (with M and g defined as in (2.6) and (2.7) respectively) such that*

$$a\sharp b = b\sharp a = 1 \ .$$

Here the composition between symbols is defined by means of $(a\sharp b)^W = a^W b^W$.

3 A preliminary inversion result

We now show how estimates on $\big(Au,u\big)$, $A \in$OPS$^2(\mathbb{R}^n)$, can be used to get a parametrix of A via Theorem 2.3. In what follows, when nothing else is specified, by C, C', or C'' we mean any real positive constant arising in the estimates. We start with a preliminary result.

Lemma 3.1 *Let $A = a(x,D)$ be a classical global operator in $\mathrm{OPS}^2(\mathbb{R}^n)$ with principal symbol $a_2(x,\xi) \geq 0$. Then for any $\epsilon_1 > 0$, $\epsilon_2 > 0$ there exists a positive constant $C = C(\epsilon_1, \epsilon_2)$ such that, for $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$||Au||_0^2 \geq -(\epsilon_1+\epsilon_2)||u||_1^2 + ||A'u||_0^2 + (1-\epsilon_2)||A''u||_0^2 - C||u||_0^2 \ ,$$

where the self-adjoint operators

$$A' = \frac{A-A^*}{2i} \quad \text{and} \quad A'' = \frac{A+A^*}{2}$$

represent the immaginary and the real part of A, i.e. $\mathrm{Im}\big(Au,u\big) = \big(A'u,u\big)$ and $\mathrm{Re}\big(Au,u\big) = \big(A''u,u\big)$.

Proof. Since $A = A'' + iA'$, one has, for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$||Au||_0^2 = \big((A''+iA')u,(A''+iA')u\big) = ||A'u||_0^2 + ||A''u||_0^2 - 2\,\mathrm{Im}\big(A'u, A''u\big) \ .$$

Note that $\mathrm{Im}\big(A'u,A''u\big) = -\frac{i}{2}\big([A'',A']u,u\big)$, we can then write

$$||Au||_0^2 = ||A'u||_0^2 + ||A''u||_0^2 + i\big([A'',A']u,u\big) \ . \tag{3.1}$$

If we denote by $a'_1 \in \mathrm{S}^1(\mathbb{R}^n)$ and $a''_2 \in \mathrm{S}^2(\mathbb{R}^n)$ the principal symbols of A' and A'' respectively, $\{a''_2, a'_1\}$ is the principal symbol of $i[A'', A']$.
Applying (2.8) to $a''_2(x,\xi)$ and $a''_2(x, \frac{\xi}{|\xi|})$, we obtain

$$|\partial_{\xi_j} a''_2(x,\xi)| \leq C(a''_2(x,\xi))^{\frac{1}{2}} \quad \text{and} \quad |\partial_{x_j} a''_2(x,\xi)| \leq C|\xi|(a''_2(x,\xi))^{\frac{1}{2}} \ .$$

Thus, for every $\delta_1 > 0$,

$$\begin{aligned} |\{a''_2, a'_1\}| &\leq C\sum_{j=1}^{n}\Big(|\partial_{x_j} a'_1(x,\xi)|(a''_2(x,\xi))^{\frac{1}{2}} + |\partial_{\xi_j} a'_1(x,\xi)||\xi|(a''_2(x,\xi))^{\frac{1}{2}}\Big) \\ &\leq \delta_1 \sum_{j=1}^{n}\Big((\partial_{x_j} a'_1(x,\xi))^2 + (\partial_{\xi_j} a'_1(x,\xi))^2|\xi|^2\Big) + \frac{1}{\delta_1}a''_2(x,\xi) \ . \end{aligned}$$

Set now $q_2(x,\xi) = \sum_{j=1}^{n}\Big((\partial_{x_j} a'_1(x,\xi))^2 + (\partial_{\xi_j} a'_1(x,\xi))^2|\xi|^2\Big)$ and consider a classical global self-adjoint operator $Q = q(x,D) \in \mathrm{OPS}^2(\mathbb{R}^n)$ with $\sigma_{princ}(Q) = q_2(x,\xi)$; then the principal symbol of $i[A'', A'] + \delta_1 Q + \frac{1}{\delta_1}A'' \in \mathrm{OPS}^2(\mathbb{R}^n)$ is real non-negative. Sharp Gårding inequality (see [7] Theorem 18.1.14) yields, for any $\delta_2 > 0$,

$$\big((i[A'', A'] + \delta_1 Q + \frac{1}{\delta_1}A'')u, u\big) \geq -\delta_2||u||_1^2 - C_{\delta_2}||u||_0^2 \ .$$

Note that, for every $\delta_3 > 0$, $|\big(A''u, u\big)| \leq \delta_3||A''u||_0^2 + \frac{1}{\delta_3}||u||_0^2$ and that $\big(Qu, u\big) \leq ||Qu||_{-1}||u||_1 \leq C||u||_1^2$. Hence

$$i\big([A'', A']u, u\big) \geq -\frac{\delta_3}{\delta_1}||A''u||_0^2 - (\delta_1 C + \delta_2)||u||_1^2 - (C_{\delta_2} + \frac{1}{\delta_1\delta_3})||u||_0^2 \ .$$

Therefore from (3.1) one has

$$||Au||_0^2 \geq ||A'u||_0^2 + (1 - \frac{\delta_3}{\delta_1})||A''u||_0^2 - (\delta_1 C + \delta_2)||u||_1^2 - (C_{\delta_2} + \frac{1}{\delta_1\delta_3})||u||_0^2,$$

and the proof immediatelly follows. ■

Proposition 3.2 *Let $A = a(x,D) \in \mathrm{OPS}^2(\mathbb{R}^n)$ is as in Lemma 3.1 and suppose that one of the following inequalities holds, for any $u \in \mathcal{S}(\mathbb{R}^n)$:*

(3.2) $$\mathrm{Re}\ (Au, u) \geq C||u||_{\frac{1}{2}}^2 - C'||u||_0^2 \ ,$$

(3.3) $$\text{Im}\ (Au,u) \geq C||u||^2_{\frac{1}{2}} - C'||u||^2_0\ ,$$

(3.4) $$-\text{Im}\ (Au,u) \geq C||u||^2_{\frac{1}{2}} - C'||u||^2_0\ .$$

If M and g are defined as in (2.6) and (2.7) respectively, then there exists $B = b(x,D)$, with $b \in S(M^{-2}, g)$ such that the operators $I - AB$ and $I - BA$ are regularizing, i.e. they map continuously $\mathcal{E}'(\mathbb{R}^n)$ in $C^\infty(\mathbb{R}^n)$.

Proof. We start proving

(3.5) $$(||Au||^2_0 + ||u||^2_0) \geq C||u||^2_1 \qquad \text{for every } u \in \mathcal{S}(\mathbb{R}^n)\ .$$

This allows us to apply Theorem 2.3 to $A + \lambda I$ for a suitable constant λ.
Suppose that (3.2) holds, and let Λ_s ($s \in \mathbb{R}$) be the self-adjoint pseudodifferential operator defined in remark (2) of Section 1, for $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\text{Re}\ (Au, \Lambda_1 u) = \text{Re}\ (A\Lambda_{\frac{1}{2}}u, \Lambda_{\frac{1}{2}}u) + \text{Re}\ (\Lambda_{\frac{1}{2}}[\Lambda_{\frac{1}{2}}, A]u, u)\ .$$

Note that $(\Lambda_{\frac{1}{2}}[\Lambda_{\frac{1}{2}}, A]u, u) = T_1 u + T_2 u$ with $T_1 \in$OPS$^2(\mathbb{R}^n)$ skew-symmetric and $T_2 \in$OPS$^1(\mathbb{R}^n)$, so one has, for any $\delta_1 > 0$,

$$|\text{Re}\ (\Lambda_{\frac{1}{2}}[\Lambda_{\frac{1}{2}}, A]u, u)| = |(T_2 u, u)| \leq C||u||_1||u||_0 \leq \delta_1||u||^2_1 + \frac{1}{\delta_1}||u||^2_0\ ,$$

and then, by means of (3.2)

$$\text{Re}\ (Au, \Lambda_1 u) \geq C||u||^2_1 - C'||u||^2_{\frac{1}{2}} - \delta_1||u||^2_1 - \frac{1}{\delta_1}||u||^2_0\ .$$

Since, for every $\epsilon > 0$, there exists some positive constant C_ϵ such that

(3.6) $$||u||^2_{\frac{1}{2}} \leq \epsilon\|u\|^2_1 + C_\epsilon\|u\|^2_0$$

one easily has

$$\text{Re}\ (Au, \Lambda_1 u) \geq C||u||^2_1 - C'||u||^2_0\ .$$

For any $\delta_2 > 0$

$$\text{Re}\ (Au, \Lambda_1 u) \leq \delta_2||u||^2_1 + \frac{1}{\delta_2}||Au||^2_0,$$

so that (3.5) finally follows.

Let us now assume that (3.3) holds and consider $A' = \dfrac{A - A^*}{2i}$. For any $\delta_1 > 0$ we have

$$(A'u, \Lambda_1 u) \le \frac{1}{\delta_1} ||A'u||_0^2 + \delta_1 ||u||_1^2 \ . \tag{3.7}$$

Furthermore

$$(A'u, \Lambda_1 u) = (A'\Lambda_{\frac{1}{2}} u, \Lambda_{\frac{1}{2}} u) + (\Lambda_{\frac{1}{2}}[\Lambda_{\frac{1}{2}}, A']u, u) \ .$$

Since $\Lambda_{\frac{1}{2}}[\Lambda_{\frac{1}{2}}, A'] \in$OPS$^1(\mathbb{R}^n)$, for any $\delta_2 > 0$,

$$|(\Lambda_{\frac{1}{2}}[\Lambda_{\frac{1}{2}}, A']u, u)| \le \delta_2 ||u||_1^2 + \frac{1}{\delta_2} ||u||_0^2 \ .$$

In view of (3.3) and (3.6) one has, for any $\delta_3 > 0$, $(A'\Lambda_{\frac{1}{2}} u, \Lambda_{\frac{1}{2}} u) \ge (C - \delta_3)||u||_1^2 - C_{\delta_3}||u||_0^2$. Hence, from (3.7),

$$\frac{1}{\delta_1} ||A'u||_0^2 \ge (C - \delta_1 - \delta_3 - \delta_2)||u||_1^2 - (\frac{1}{\delta_2} + C_{\delta_3})||u||_0^2 \ .$$

Now (3.5) follows from Lemma 3.1. Same arguments show that (3.5) is a consequence of (3.4).

Let us observe that to apply Theorem 2.3 to $A + \lambda I$, with a suitable constant λ, we need to prove the estimate

$$||(A + \lambda I)u||_0 \ge C||u||_1 \qquad \text{for any } u \in \mathcal{S}(\mathbb{R}^n) \ . \tag{3.8}$$

Now, if one of inequalities (3.2), (3.3) or (3.4) holds we have, for a suitable constant $\lambda \in \mathbb{R}$, $|((A + \lambda I)u, u)| \ge C||u||_0^2$, whence

$$||(A + \lambda I)u||_0 \ge C||u||_0 \qquad \text{for any } u \in \mathcal{S}(\mathbb{R}^n) \ .$$

Thus (3.5) yields estimate (3.8).

Theorem 2.3 gives existence of $s \in S(M^{-2}, g)$ such that

$$(A + \lambda I) \circ s^W = s^W \circ (A + \lambda I) = I \ .$$

Observe that

$$u = s^W Au + \lambda s^W = s^W Au + \lambda (s^W)^2 Au + \lambda^2 (s^W)^2 u = \ldots$$

If, for $j \in \mathbb{N}$, we define $b'_j = \lambda^{j-1} s^{\sharp j}$ (where $s^{\sharp j}$ denotes the product $s\sharp ... \sharp s$ j-times), from Lemma 3.1 of [5], there exists $b' \in S(M^{-2}, g)$ such that

$$b' - \sum_{j<N} b'_j \in S(M^{-2N}, g) \ .$$

Let $B = (b')^W$, then $B - \sum_{j<N} (b'_j)^W \in \mathrm{OPS}^{-N}(\mathbb{R}^n)$ for any $N > 0$. Hence $I - AB$ and $I - BA$ are continous operators from $\mathcal{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. We can finally take $b \in S(M^{-2}, g)$ such that $B = b(x, D)$ in fact the metric g is symmetric with respect to the x and to the ξ variable (see [7], [8] and [11]). ■

Remark 3.3 *The operator B constructed above turns out to be in* $\mathrm{OPS}^{-1}{}_{\frac{1}{2},\frac{1}{2}}(\mathbb{R}^n)$ *since* $M(x,\xi) \geq \langle \xi \rangle^{\frac{1}{2}}$.

4 Parametrix construction

In this section we prove Theorem 1.1; our strategy is based on the following steps:

(1) suitable "modification" of the operator P outside of a conic set Γ of $T^*\Omega \setminus 0$ in order to obtain a global operator $\mathcal{A}_\Gamma$ having the properties required in Proposition 3.2;

(2) application of Theorem 2.3 to $\mathcal{A}_\Gamma + \lambda I$ for some large constant λ;

(3) construction of a parametrix for P by means of a microlocal partition of unity.

As observed in the Introduction (remark (2)), we can consider only the case $m = 2$. From now on, $P = p(x, D) \in \mathrm{OPS}^2(\Omega)$ is a classical second order operator satisfying hypotheses of Theorem 1.1. First of all, from (1.1) we have that, for every $\varrho = (x_0, \xi_0) \in T^*\Omega \setminus 0$, there exists an open conic neighborhood $\widetilde{\Gamma}_\varrho$ of ϱ in $T^*\Omega \setminus 0$ such that one of the following cases holds:

(I) $p_2(x,\xi) > 0\,, \ \forall (x,\xi) \in \widetilde{\Gamma}_\varrho$;

(II) $\mathrm{Re}\, p_1^s(x,\xi) + Tr^+ F_{x,\xi} > 0, \ \forall (x,\xi) \in \widetilde{\Gamma}_\varrho$;

(III) $\mathrm{Im}\, p_1^s(x,\xi) \neq 0, \ \forall (x,\xi) \in \widetilde{\Gamma}_\varrho$.

Choose now another open conic neighborhood $\Gamma_\varrho \subset\subset \widetilde{\Gamma}_\varrho$ of ϱ in $T^*\Omega \setminus 0$ and fix a positively homogeneous real function $\psi(x,\xi) \in \mathrm{S}^0(\mathbb{R}^n)$ supported in $\widetilde{\Gamma}_\varrho$ such that $\psi \equiv 1$ on Γ_ϱ. We define

$$\mathcal{A}_{\Gamma_\varrho} = \Psi^* P \Psi + (I - \Psi^*) L_\varrho (I - \Psi) \tag{4.1}$$

where $\Psi = \psi(x, D) \in \mathrm{OPS}^0(\mathbb{R}^n)$ and L_ϱ is a global pseudodifferential operator in $\mathrm{OPS}^2(\mathbb{R}^n)$ suitably chosen. It is immediately checked that $\mathcal{A}_{\Gamma_\varrho}$ is a global operator in $\mathrm{OPS}^2(\mathbb{R}^n)$.
In case (I) we take L_ϱ to be the positive laplacian operator $\Delta = \sum_{j=1}^n D_j^2$ in $\mathbb{R}^n$ so that the

principal symbol of $\mathcal{A}_{\Gamma_\varrho}$ is positive in $T^*\mathbb{R}^n \setminus 0$; therefore, from Gårding inequality it follows that there exist some positive constants c, c' for which

$$\text{Re}\,\big(\mathcal{A}_{\Gamma_\varrho} u, u\big) \geq c\|u\|^2_{\frac{1}{2}} - c'\|u\|^2_0 \quad \text{for any } u \in \mathcal{S}(\mathbb{R}^n)\ . \tag{4.2}$$

Again, in case (II) we choose $L_\varrho = \Delta$, but now the principal symbol is non-negative and for every (x,ξ) in the characteristic set of $\mathcal{A}_{\Gamma_\varrho}$, we have

$$\text{Re}\, sub(\mathcal{A}_{\Gamma_\varrho})(x,\xi) + Tr^+ F_{x,\xi}(\mathcal{A}_{\Gamma_\varrho}) = \text{Re}\, p^1_s(x,\xi) + Tr^+ F_{x,\xi} > 0\ ,$$

where $F_{x,\xi}(\mathcal{A}_{\Gamma_\varrho})$ and $sub(\mathcal{A}_{\Gamma_\varrho})(x,\xi)$ denote the fundamental matrix and the subprincipal symbol of $\mathcal{A}_{\Gamma_\varrho}$, respectively. Hence, by Melin inequality applied to $\dfrac{\mathcal{A}_{\Gamma_\varrho} + \mathcal{A}^*_{\Gamma_\varrho}}{2}$ we get again (4.2).
Finally, in case (III) we start supposing $\text{Im}\, p^s_1 > 0$ on $\widetilde{\Gamma}_\varrho$ and we take L_ϱ to be $i\sqrt{\Delta}$ where $\sqrt{\Delta}$ represents the pseudodifferential operator with symbol $|\xi|$ in $\text{S}^1(\mathbb{R}^n)$. Using Gårding inequality for $\dfrac{\mathcal{A}_{\Gamma_\varrho} - \mathcal{A}^*_{\Gamma_\varrho}}{2i}$, we get for some positive constants c, c'

$$\text{Im}\,\big(\mathcal{A}_{\Gamma_\varrho} u, u\big) \geq c\|u\|^2_{\frac{1}{2}} - c'\|u\|^2_0 \quad \text{for any } u \in \mathcal{S}(\mathbb{R}^n)\ .$$

If $\text{Im}\, p^s_1 < 0$ on $\widetilde{\Gamma}_\varrho$, we choose $L_\varrho = -i\sqrt{\Delta}$ so we have

$$-\text{Im}\,\big(\mathcal{A}_{\Gamma_\varrho} u, u\big) \geq c\|u\|^2_{\frac{1}{2}} - c'\|u\|^2_0 \quad \text{for any } u \in \mathcal{S}(\mathbb{R}^n)\ .$$

We can apply Proposition 3.2 to $\mathcal{A}_{\Gamma_\varrho}$ for any $\varrho \in T^*\Omega \setminus 0$ and get a parametrix $B_\varrho = b_\varrho(x,D)$ for $\mathcal{A}_{\Gamma_\varrho}$ with symbol $b_\varrho(x,\xi) \in S(M^{-2}, g)$. We now show how to obtain a local parametrix of P by suitably glueing operators B_ϱ.
If we fix a compact set $K \subset \Omega$, there exists a finite family of cones $\Gamma_{\varrho_j} \subset\subset \widetilde{\Gamma}_{\varrho_j}$ $\quad j = 1,\dots,N$ in $T^*\Omega \setminus 0$ with the properties described above, for which, denoting by $\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ the canonical projection, we have $K \subset\subset \pi\big(\bigcup_{j=1}^N \Gamma_{\varrho_j}\big)$. For every $j = 1,\dots,N$ denote by $\mathcal{A}_j$ the operator

$$\mathcal{A}_{\Gamma_{\varrho_j}} = \Psi^*_j P \Psi_j + (I - \Psi^*_j) L_{\varrho_j} (I - \Psi_j) \tag{4.3}$$

defined in (4.1) and related to cones Γ_{ϱ_j} , $\widetilde{\Gamma}_{\varrho_j}$; furthermore, denote by B_j its parametrix $B_{\Gamma_{\varrho_j}}$ so that

$$\mathcal{A}_j\, B_j \equiv B_j\, \mathcal{A}_j \equiv I\ , \tag{4.4}$$

where $A \equiv B$ means $A - B \in \mathrm{OPS}^{-\infty}$. Take a family $\phi_j(x,\xi) \in \mathrm{S}^0(\mathbb{R}^n)$, $j = 1, ..., N$ of positively homogeneous real function, such that $\phi_j \equiv 1$ on some conic neighborhood of ϱ_j, supp $\phi_j \subset \Gamma_{\varrho_j}$ and

$$\sum_{j=1}^{N} \phi_j(x,\xi) \equiv 1 \quad \text{on} \quad \pi^{-1}(U)\,, \tag{4.5}$$

for some open set $U \supset K$ in Ω. Finally, for every $j = 1, ..., N$ choose a positively homogeneous real function $\theta_j(x,\xi) \in \mathrm{S}^0(\mathbb{R}^n)$ supported in Γ_{ϱ_j}, such that $\theta_j \equiv 1$ on some neighborhood of supp ϕ_j. In the following Θ_j and Φ_j denote usual pseudodifferential operators with symbol $\theta_j(x,\xi)$, $\phi_j(x,\xi)$ respectively.
Pseudolocality of pseudodifferential operators and (4.4) yields

$$\Phi_j \;\equiv\; \mathcal{A}_j\, B_j\, \Phi_j \;\equiv\; \mathcal{A}_j\, \Theta_j\, B_j \Phi_j, \tag{4.6}$$

and from (4.3) it follows

$$\Phi_j \;\equiv\; \Psi_j^* P \Psi_j\, \Theta_j\, B_j \Phi_j \equiv\; \Psi_j^* P\, \Theta_j\, B_j \Phi_j \;\equiv\; P\, \Theta_j\, B_j \Phi_j\,.$$

Thus

$$P\big(\sum_{j=1}^{N} \Theta_j\, B_j \Phi_j\big) \equiv I + \big(\sum_{j=1}^{N} \Phi_j - I\big)\,.$$

In view of (4.5), $\sum_{j=1}^{N} \Phi_j - I$ is a linear continuous map from $\mathcal{E}'(K)$ to $C_0^\infty(\Omega)$; whence, denoting by Q_K the operator $\sum_{j=1}^{N} \Theta_j\, B_j \Phi_j$, one gets

$$PQ_K - I : \mathcal{E}'(K) \longrightarrow C_0^\infty(\Omega) \;\; \text{continuously.}$$

Finally, defining $Q'_K = \sum_{j=1}^{N} \Phi_j\, B_j \Theta_j$ and arguing in the same way, we show that $Q'_K P - I$: $\mathcal{E}'(K) \longrightarrow C_0^\infty(\Omega)$ continuously. This concludes the construction of a local parametrix for P on the compact set K. Standard arguments easily give the parametrix $Q \in \mathrm{OPS}^{-1}_{\frac{1}{2},\frac{1}{2}}(\Omega)$ (see [5] proof of Theorem A). ■

References

[1] L.Boutet de Monvel: *Hypoelliptic operators with double characteristics and related pseudodifferential operators.* Comm. Pure Appl. Math. **27**, (1974), 585-639.

[2] L.Boutet de Monvel, A.Grigis and B.Helffer: *Paramétrixes d'opérateurs pseudo-différentiels à caractéristiques multiples.* Astérisque, **34-35**, (1976), 93-121 .

[3] J.-M.Bony, J.-Y.Chemin: *Espaces fonctionelles associés au calcul de Weyl-Hörmander.* Bull. Soc. Math. France **122**, 1 (1994).

[4] J.-M.Bony, N.Lerner: *Quantification Asymptotique et Microlocalisations d'Ordre Supérieur I.* Ann. Scient. Éc, Norm. Sup., 4^e serie, t.22 (1989).

[5] C.E.Cancellier, J.-Y.Chemin and C.J.Xu: *Calcul de Weyl et operateurs sous-elliptiques.* Ann. Inst. Fourier, Grenoble **43**, 4 (1997), 1157-1178.

[6] N.Denker: *The Weyl Calculus with Locally Temperate Metrics and Weights.* Arkiv für Mat. **24**, 1 (1986), 1157-1178.

[7] L.Hörmander: *The Analysis of Linear Partial Differential Operators, Vol.I-III.* Springer-Verlag (1983/85).

[8] L.Hörmander: *The Weyl Calculus of Pseudodifferential Operators.* Comm. on Pure and Appl. Math., **XXXII**, (1979).

[9] N.Lerner, J.Nourrigat: *Lower Bounds for Pseudodifferential Operators.* Ann. Inst. Fourier, Grenoble **40**, 1 (1990), 1157-1178.

[10] A.Melin: *Lower Bounds for Pseudo-Differential Operators.* Arkiv för Matematik, **9** (1971), 117-140.

[11] A.Parmeggiani: *A Crash Introduction to Pseudodifferential Calculus.* Seminari di Geometria del Dipartimento di Matematica di Bologna (2000).

[12] J.Sjöstrand: *Parametrices for pseudodifferential operators with multiple characteristics.* Arkiv för Matematik, **12**, (1974), 85-130.