The edge algebra structure of boundary value problems

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Abstract

Boundary value problems for pseudodifferential operators (with or without the transmission property) are characterised as a substructure of the edge pseudodifferential calculus with constant discrete asymptotics. The boundary in this case is the edge and the inner normal the model cone of local wedges. Elliptic boundary value problems for non-integer powers of the Laplace symbol belong to the examples as well as problems for the identity in the interior with a prescribed number of trace and potential conditions. Transmission operators are characterised as smoothing Mellin and Green operators with meromorphic symbols.

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Introduction

Boundary value problems for differential (or pseudodifferential) operators on a smooth manifold with boundary have much in common with problems for operators on a manifold with edges. This relies on the fact that the "half-space" $\overline{\mathbb{R}}_+ \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^q$, $q = \dim Y$, can be regarded as a wedge with model cone $\overline{\mathbb{R}}_+$ and edge Ω. Let us illustrate this for a differential operator $A = \sum_{|\alpha| \leq \mu} a_\alpha(x) D_x^\alpha$ in $\mathbb{R}_+ \times \Omega$ with smooth coefficients $a_\alpha(x) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$. Inserting $x = (t, y)$ for $t \in \mathbb{R}$, $y \in \Omega$, the operator A takes the form

$$A = t^{-\mu} \sum_{j+|\beta| \leq \mu} a_{j\beta}(t, y) \left(-t \tfrac{\partial}{\partial t}\right)^j (t D_y)^\beta \tag{0.0.1}$$

with coefficients $a_{j\beta}(t, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$ for all j, β. In general, a differential operator A in $\mathbb{R}_+ \times \Omega$ of the form (0.0.1) will be called edge-degenerate. The class of such operators is much larger than that induced by operators with smooth coefficients, though there are far reaching similarities between boundary value problems and edge-degenerate operators in the framework of the edge algebra, see, for instance, [23] or [27]. Operators without the transmission property at the boundary are of particular relevance in this connection (even for $q = 0$). A result of Eskin [5] says that zero-order pseudodifferential operators on the half-axis can be reformulated as Mellin pseudodifferential operators with meromorphic symbols, modulo Hilbert-Schmidt operators. Similar ideas have been used by Rempel and Schulze [18] in relation to the calculus of Vishik and Eskin [35] for higher dimensions, though an edge algebra characterisation was not given in [18]. Also Plamenevskij [16] has derived certain Mellin reformulations of pseudo-differential operators. An essential point in such constructions is a suitable control of corresponding symbols and remainders near 0. Eskin's result for the half-axis has been refined later on in [19] to a representation with a complete sequence of meromorphic Mellin symbols of lower conormal order, modulo so-called Green operators, defined by their mapping properties to spaces with asymptotics, see also the monograph [27]. This calculus became a model for a specific cone algebra (for cones with non-trivial base) with a similar Mellin and Green operator structure, cf. [20] and [24], [27], or the papers [21], [22]. The cone algebra completes the class of differential operators on a manifold with conical singularities of Kondrat'ev [13] to a corresponding pseudodifferential algebra containing the parametrices of elliptic elements. Also the edge, corner, and higher singular pseudodifferential algebras of [25], [26], [30] belong to this development. The operators in these algebras are degenerate in a typical way (in fact, edge-degenerate, corner-degenerate, etc.). There are ideals of smoothing operators with asymptotics, generated by certain operator-valued symbols that are contributed by the geometric singularities of the configuration. In particular, they describe the nature of additional trace and potential conditions along the lower-dimensional strata.

There is then the following natural question. Consider, for instance, a bounded domain G in $\mathbb{R}^n$, say, with a piecewise smooth boundary (e.g., with conical, edge, etc. singularities). Let $a(x, \xi) \in S^0_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol of order zero in Hörmander's symbol space (subscript "cl" stands for the space of classical symbols), and consider the pseudodifferential operator

$$A = \mathrm{r}^+ \operatorname{Op}(a) \mathrm{e}^+ : L^2(G) \to L^2(G), \tag{0.0.2}$$

$\operatorname{Op}(a)u(x) = \iint e^{i(x-x')\xi} a(x, \xi) u(x')\, dx' đ\xi$, $đ\xi = (2\pi)^{-n}\, d\xi$, with $\mathrm{e}^+ : L^2(G) \to L^2(\mathbb{R}^n)$ being the operator of extension from G to $\mathbb{R}^n$ by 0 in $\mathbb{R}^n \setminus G$ and $\mathrm{r}^+ : L^2(\mathbb{R}^n) \to L^2(G)$ the restriction to G. The problem is then to what extent A can be characterised as an element in the above-mentioned "corner operator algebra" on G that is intrinsically defined, without any reference to a neighbouring space of G . For higher order operators we can ask the same; then the space $L^2(G)$ has to be replaced by weighted Sobolev spaces that are typical for geometric singularities.

In the present paper we shall solve this problem for a domain (or a manifold) with smooth boundary and symbols that have not necessarily the transmission property at the boundary. For our characterisation we employ the edge algebra from [23] with constant discrete asymptotics, (see also [28], or Egorov and Schulze [4], Gil, Schulze, and Seiler [6], [7]). As a byproduct we get a new characterisation of Boutet de Monvel's algebra [3]. To have a convenient method for reductions of orders in our calculus we establish a corresponding parameter-dependent variant, cf., analogously, Agranovich and Vishik [1] for a simpler situation.

Operators of the form (0.0.2) occur in a number of interesting applications, see, for instance, Widom [37], or Kapanadze and Schulze [11]. For the authors the problem became relevant in connection with boundary value problems, where the Atiyah-Bott obstruction for the existence of Shapiro-Lopatinskij elliptic conditions does not vanish, cf. Atiyah and Bott [2], see also [29] and the joint paper [31] of the authors. In [31] we will give a transparent construction of the set of all global projection boundary conditions (analogues of APS boundary conditions for operators without the transmission property) in terms of the edge symbol machinery. Our results also contribute to the description of a pseudo-differential algebra with asymptotics, where mixed elliptic problems have their parametrices. There are so-called transmission operators in this algebra, and we show that they have the form of smoothing Mellin and Green operators in the edge calculus. This extends a similar relation of Eskin [5] for zero order operators on the half-axis, see also [27] for arbitrary orders, with lower order conormal symbols; moreover, the characterisation specialises to an observation of Grubb [9] for the case of symbols with the transmission property (in arbitrary dimensions), see also the article of Myshkis [14].

Moreover, we construct elliptic elements of special interest, in particular, elliptic boundary value problems for non-integer powers of the symbol of the Laplacian, or boundary value problems for the identity operator (plus a smoothing Mellin operator) with a prescribed number of elliptic boundary and potential conditions. Furthermore, as a refinement of a corresponding result in [18], we construct homotopies through elliptic elements to operators with interior symbols that have the transmission property at the boundary.

Finally, we apply our characterisation of transmission operators to the construction of a large general class of operators (with or without the transmission property) that are elliptic (and thus Fredholm) without additional boundary and potential conditions.

1 Edge symbols on the half-space

1.1 Cone Sobolev spaces and Green symbols

The aim of this subsection is to fix some terminology for the pseudodifferential analysis on manifolds with conical and edge singularities.

For $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, we let $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ be the Hilbert space of all distributions $u \in \mathcal{D}'(\mathbb{R}_+)$ such that

$$t^{-\gamma}(t\partial_t)^k u(t) \in L^2(\mathbb{R}_+, dt) \qquad \text{for all } k \leq s.$$

This definition extends in a natural way to arbitrary real s. Throughout this paper a cut-off function is an arbitrary element $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega \equiv 1$ in a neighbourhood of $t = 0$.

Definition 1.1.1 *Let $s, \gamma \in \mathbb{R}$. Then*

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) := \{u \in \mathcal{D}'(\mathbb{R}_+) : \ \omega u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \text{ and } (1-\omega)u \in H^s(\mathbb{R})\},$$

(with $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ being an arbitrary cut-off function and $H^s(\mathbb{R})$ the standard Sobolev space of smoothness s) is a Hilbert space with the norm

$$\|u\|_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} + \|(1-\omega)u\|_{H^s(\mathbb{R})}.$$

The scalar product in $L^2(\mathbb{R}_+) = \mathcal{K}^{0,0}(\mathbb{R}_+)$ induces a pairing

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) \times \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+) \to \mathbb{C}, \qquad (u,v) \mapsto \langle u, v\rangle_{L^2(\mathbb{R}_+)},$$

that admits an identification of the dual space $(\mathcal{K}^{s,\gamma}(\mathbb{R}_+))'$ with $\mathcal{K}^{-s,-\gamma}(\mathbb{R}_+)$.

The cone Sobolev spaces contain subspaces of distributions with a certain asymptotic behaviour at 0 that is typical for solutions of elliptic equations of Fuchs type.

Definition 1.1.2 *Let $\gamma, \theta \in \mathbb{R}$ and $\theta > 0$. An asymptotic type $Q \in \mathrm{As}(\gamma,\theta)$ is a finite set*

$$Q = \{(q,l) \in \mathbb{C} \times \mathbb{N}_0 : \ \tfrac{1}{2} - \gamma - \theta < \operatorname{Re} q < \tfrac{1}{2} - \gamma\}.$$

The complex conjugate type of Q is defined as $\overline{Q} := \{(\overline{q}, l) : \ (q,l) \in Q\}$.

With such a $Q \in \mathrm{As}(\gamma,\theta)$ we associate the function space

$$\mathcal{E}_Q(\mathbb{R}_+) := \Big\{ t \mapsto \omega(t) \sum_{(q,l)\in Q} \sum_{k=0}^{l} c_{qk} t^{-q} \log^k t : \ c_{qk} \in \mathbb{C} \Big\}, \tag{1.1.1}$$

which is a finite-dimensional subspace of $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. Here, ω is some fixed cut-off function.

Definition 1.1.3 *Let $s, \gamma \in \mathbb{R}$ and $Q \in \mathrm{As}(\gamma,\theta)$. Then*

$$\begin{aligned} \mathcal{K}_Q^{s,\gamma}(\mathbb{R}_+) &:= \mathcal{E}_Q(\mathbb{R}_+) \oplus \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma+\theta-\varepsilon}(\mathbb{R}_+), \\ \mathcal{S}_Q^{\gamma}(\mathbb{R}_+) &:= \{u \in \mathcal{K}_Q^{\infty,\gamma}(\mathbb{R}_+) : \ (1-\omega)u \in \mathcal{S}(\mathbb{R})\} \end{aligned}$$

are Fréchet subspaces of $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. If $Q = \emptyset$ is the empty set, we agree to write $\mathcal{K}_\theta^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{S}_\theta^{\gamma}(\mathbb{R}_+)$.

These spaces can be written as a projective limit $E = \varprojlim_{k\in\mathbb{N}} E_k$ of Hilbert spaces E_k, namely

$$\begin{aligned} E_k &= \mathcal{E}_Q(\mathbb{R}_+) + \mathcal{K}^{s,\gamma+\theta-\frac{1}{k}}(\mathbb{R}_+) \qquad &&\text{if } E = \mathcal{K}_Q^{s,\gamma}(\mathbb{R}_+), \\ E_k &= \mathcal{E}_Q(\mathbb{R}_+) + \langle t\rangle^k \mathcal{K}^{k,\gamma+\theta-\frac{1}{k}}(\mathbb{R}_+) \qquad &&\text{if } E = \mathcal{S}_Q^{\gamma}(\mathbb{R}_+). \end{aligned} \tag{1.1.2}$$

So far we have only treated $Q \in \mathrm{As}(\gamma,\theta)$ for finite θ. The extension to $\theta = \infty$ is straightforward by saying that $Q \in \mathrm{As}(\gamma,\infty)$ if

$$Q(k) := \{(q,l) \in Q : \ \tfrac{1}{2} - \gamma - k < \operatorname{Re} q < \tfrac{1}{2} - \gamma\} \in \mathrm{As}(\gamma, k) \qquad \text{for all } k \in \mathbb{N}$$

and then defining

$$\mathcal{K}_Q^{s,\gamma}(\mathbb{R}_+) = \varprojlim_k \mathcal{K}_{Q(k)}^{s,\gamma}(\mathbb{R}_+), \qquad \mathcal{S}_Q^{\gamma}(\mathbb{R}_+) = \varprojlim_k \mathcal{S}_{Q(k)}^{\gamma}(\mathbb{R}_+).$$

Example 1.1.4 *If $T := \{(-j,0) : \ j \in \mathbb{N}_0\} \in \mathrm{As}(0,\infty)$, then $\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\mathbb{R}_+} = \mathcal{S}_T^0(\mathbb{R}_+)$. This simply means that smoothness up to zero corresponds to Taylor asymptotics at $t = 0$.*

The following class of smoothing operators – the so-called Green operators – will be of particular interest in the sequel.

Definition 1.1.5 *Let $\gamma, \gamma' \in \mathbb{R}$, $\theta, \theta' \in \mathbb{R}_+ \cup \{\infty\}$, and asymptotic types $Q \in \mathrm{As}(-\gamma, \theta)$, $Q' \in \mathrm{As}(\gamma', \theta')$ be given. If $\boldsymbol{g} = (\gamma, \theta; \gamma', \theta')$, then $C_G(\mathbb{R}_+, \boldsymbol{g})_{Q,Q'}$ consists of all operators $G \in \mathcal{L}(\mathcal{K}^{0,\gamma}(\mathbb{R}_+), \mathcal{K}^{0,\gamma'}(\mathbb{R}_+))$ such that for all $s \in \mathbb{R}$*

$$G : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+), \qquad G^* : \mathcal{K}^{s,-\gamma'}(\mathbb{R}_+) \to \mathcal{S}^{-\gamma}_{Q}(\mathbb{R}_+).$$

Here, $*$ denotes the adjoint with respect to the $L^2(\mathbb{R}_+)$-scalar product. The union over all such types Q, Q' is denoted by $C_G(\mathbb{R}_+, \boldsymbol{g})$. If $\theta = \theta'$ we agree to write $\boldsymbol{g} = (\gamma, \gamma', \theta)$.

Let E be a Hilbert space and κ a strongly continuous group of isomorphisms on E, i.e., $\kappa : \mathbb{R}_+ \to \mathcal{L}(E)$ where $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho}$ for all $\lambda, \varrho > 0$, and $\lambda \mapsto \kappa_\lambda e : \mathbb{R}_+ \to E$ is continuous for any $e \in E$.

Example 1.1.6 *On $E = \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ the "standard" group action is defined by*

$$(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}} u(\lambda t), \qquad u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+). \tag{1.1.3}$$

The factor $\lambda^{\frac{1}{2}}$ appears in order to ensure that $\kappa_\lambda \in \mathcal{L}(L^2(\mathbb{R}_+))$ are unitary operators.

If E, $\widetilde{E}$ are Hilbert spaces with corresponding group actions κ, $\widetilde{\kappa}$, and $\Omega \subset \mathbb{R}^m$ is an open set, we say that a is an operator-valued symbol of order μ, written $a \in S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$, if $a \in C^\infty(\Omega_y \times \mathbb{R}^q_\eta, \mathcal{L}(E, \widetilde{E}))$ and

$$\sup_{y \in K, \eta \in \mathbb{R}^q} \|\widetilde{\kappa}^{-1}_{\langle\eta\rangle} \{\partial^\alpha_\eta \partial^\beta_y a(y,\eta)\} \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\widetilde{E})} \langle\eta\rangle^{|\alpha|-\mu} < \infty$$

for all multi-indices $\alpha \in \mathbb{N}^q_0$, $\beta \in \mathbb{N}^m_0$ and any compact subset K of Ω. Note that for $E = \widetilde{E} = \mathbb{C}$ and $\kappa = \widetilde{\kappa} \equiv 1$ we recover the standard symbol classes $S^\mu(\Omega \times \mathbb{R}^q)$.

If $\widetilde{E}$ is a Fréchet space that can be written as a projective limit $\widetilde{E} = \varprojlim_{k \in \mathbb{N}} \widetilde{E}_k$ of Hilbert spaces $\ldots \hookrightarrow \widetilde{E}_{k+1} \hookrightarrow \widetilde{E}_k \hookrightarrow \ldots \hookrightarrow \widetilde{E}_1$ and $\widetilde{\kappa}$ is a group action on $\widetilde{E}_1$ that restricts for each k to a group action on $\widetilde{E}_k$, then we define $S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E}) := \bigcap_{k \in \mathbb{N}} S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E}_k)$.

Example 1.1.7 *If $\mathcal{S}^\gamma_Q(\mathbb{R}_+) = \varprojlim_{k \in \mathbb{N}} E_k$ as in* (1.1.2)*, the standard group action from Example* 1.1.6 *induces a group action on any E_k.*

Similarly to the scalar case, the subspace $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ of classical symbols consists of all a that admit an zero expansion $a \sim \sum_{k=0}^\infty \chi a_{(\mu-k)}$ with an excision function $\chi(\eta)$ and "κ-homogeneous" functions $a_{(\mu-k)}$, i.e., $a_{(\mu-k)} \in C^\infty(\Omega \times (\mathbb{R}^q \setminus 0), \mathcal{L}(E, \widetilde{E}))$ and

$$a_{(\mu-k)}(y, \lambda\eta) = \lambda^{\mu-k} \widetilde{\kappa}_\lambda a_{(\mu-k)}(y,\eta) \kappa^{-1}_\lambda \qquad \text{for all } \lambda > 0.$$

Green symbols are now particular operator-valued symbols that are pointwise Green operators as described above:

Definition 1.1.8 *Let $\gamma, \gamma' \in \mathbb{R}$, $\theta, \theta' \in \mathbb{R}_+ \cup \{\infty\}$, and asymptotic types $Q \in \mathrm{As}(-\gamma, \theta)$, $Q' \in \mathrm{As}(\gamma', \theta')$ be given. Then $R^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g})_{Q,Q'}$, $\boldsymbol{g} = (\gamma, \theta; \gamma', \theta')$ denotes the space of all symbols that satisfy*

$$g \in \bigcap_{s \in \mathbb{R}} S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)), \qquad g^* \in \bigcap_{s \in \mathbb{R}} S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\gamma'}(\mathbb{R}_+), \mathcal{S}^{-\gamma}_{Q}(\mathbb{R}_+)),$$

where $*$ refers to the pointwise $L^2(\mathbb{R}_+)$-adjoint. If $\theta = \theta'$ we agree to write $\boldsymbol{g} = (\gamma, \gamma', \theta)$.

1.2 Green symbols and parameter-dependent integral kernels

Our aim is to show that Green symbols are families of integral operators (with respect to the $L^2(\mathbb{R}_+)$-scalar product) with smooth kernels of a specific structure. To this end we first investigate Green operators on $\mathbb{R}_+$.

Let $G \in \mathcal{L}(\mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s',\gamma'}(\mathbb{R}_+))$ for some fixed $s, s', \gamma, \gamma' \in \mathbb{R}$ such that

$$G : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+), \qquad G^* : \mathcal{K}^{-s',-\gamma'}(\mathbb{R}_+) \to \mathcal{S}^{-\gamma}_{Q}(\mathbb{R}_+)$$

for some asymptotic types $Q \in \mathrm{As}(\gamma, \theta)$ and $Q' \in \mathrm{As}(\gamma', \theta')$ for $0 < \theta, \theta' \leq \infty$. Again, $*$ refers to the scalar product of $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$. Then general facts on operators in Hilbert spaces tell us that G has an integral kernel

$$k_G \in \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+) \widehat{\otimes}_\pi \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+) \cap \mathcal{K}^{s',\gamma'}(\mathbb{R}_+) \widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+), \tag{1.2.1}$$

i.e., $Gu(t) = \langle k_G(t,\cdot), \overline{u} \rangle_{0,0} = \int_0^\infty k_G(t,t')u(t')\, dt'$ for any function $u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$.

We are now going to refine the statement from (1.2.1). Therefore, for arbitrary $\gamma \in \mathbb{R}$ we set

$$\mathcal{S}^{\gamma}_{0}(\mathbb{R}_+) := \{u \in \mathcal{K}^{\infty,\gamma}(\mathbb{R}_+) : \ (1-\omega)u \in \mathcal{S}(\mathbb{R}), \\ (\log^k t)\omega(t)u(t) \in \mathcal{K}^{\infty,\gamma}(\mathbb{R}_+) \quad \text{for all } k \in \mathbb{N}_0\}, \tag{1.2.2}$$

where ω is an arbitrary cut-off function.

Lemma 1.2.1 *If $0 \leq \theta \leq \infty$, then*

$$L^2(\mathbb{R}_+) \widehat{\otimes}_\pi \mathcal{S}^0_\theta(\mathbb{R}_+) \cap \mathcal{S}^0_0(\mathbb{R}_+) \widehat{\otimes}_\pi L^2(\mathbb{R}_+) = \mathcal{S}^0_0(\mathbb{R}_+) \widehat{\otimes}_\pi \mathcal{S}^0_\theta(\mathbb{R}_+). \tag{1.2.3}$$

Proof. The right-hand side of (1.2.3) is clearly contained in the left one. Writing an element $g = g(s,t)$ from the left-hand side as $g = \omega(s)\omega(t)g + \omega(s)(1-\omega)(t)g + (1-\omega)(s)\omega(t)g + (1-\omega)(s)(1-\omega)(t)g$ and using the push-forward $S_{\frac{1}{2}}$ defined by $(S_{\frac{1}{2}}f)(x) = e^{-\frac{1}{2}x}f(e^{-x})$ reduces the statement to the proof of

$$L^2(\mathbb{R}) \widehat{\otimes}_\pi \mathcal{S}_\theta(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}) \widehat{\otimes}_\pi L^2(\mathbb{R}) = \mathcal{S}(\mathbb{R}) \widehat{\otimes}_\pi \mathcal{S}_\theta(\mathbb{R}) \tag{1.2.4}$$

where $\mathcal{S}_\theta(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ for $\theta = 0$ and otherwise $\mathcal{S}_\theta(\mathbb{R}) = \{u \in \mathcal{S}(\mathbb{R}) : e^{\widetilde{\theta}t}u(t) \in \mathcal{S}(\mathbb{R}) \text{ for all } 0 \leq \widetilde{\theta} < \theta\}$. The result is known to be true for $\theta = 0$. The proof for $\theta = 0$ is actually a simpler version of that for $\theta > 0$, which we shall give below.

It suffices to show for some $g = g(x,y)$ belonging to the left-hand side of (1.2.4) that

$$g_{\widetilde{\theta}}(x,y) := e^{\widetilde{\theta}y}g \in \mathcal{S}(\mathbb{R}) \widehat{\otimes}_\pi \mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R} \times \mathbb{R})$$

for any fixed $\widetilde{\theta}$ with $0 < \widetilde{\theta} < \theta$. Thus we have to show

$$\langle x \rangle^{k'} \langle y \rangle^{l'} \langle D_x \rangle^k \langle D_y \rangle^l e^{\widetilde{\theta}y} g(x,y) \in L^2(\mathbb{R} \times \mathbb{R}) \qquad \text{forall } k, k', l, l' \in \mathbb{N}_0.$$

Here, we write $\langle D_x \rangle^\mu = \mathrm{op}(\langle \xi \rangle^\mu)$ for any $\mu \in \mathbb{R}$. In the following $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R} \times \mathbb{R})$. Using repeatedly the inequality $\alpha\beta \leq \alpha^2 + \beta^2$ and Plancherel's formula, it is straightforward to see that

$$\|\langle x \rangle^{k'} \langle y \rangle^{l'} \langle D_x \rangle^k \langle D_y \rangle^l g_{\widetilde{\theta}}\| \leq \|\langle x \rangle^{4k'} \langle D_x \rangle^k g_{\widetilde{\theta}}\| + \|\langle D_x \rangle^k \langle D_y \rangle^{2l} g_{\widetilde{\theta}}\| + \\ + \|\langle D_x \rangle^{2k} \langle D_y \rangle^l g_{\widetilde{\theta}}\| + \|\langle y \rangle^{4l'} \langle D_y \rangle^l g_{\widetilde{\theta}}\|. \tag{1.2.5}$$

The fourth term is finite, since $g_{\widetilde{\theta}} \in L^2(\mathbb{R})\widehat{\otimes}_\pi \mathcal{S}(\mathbb{R})$. To treat the other terms choose $p > 1$ such that $\widetilde{\theta} < p\widetilde{\theta} < p^2\widetilde{\theta} < \theta$ and denote by p' its dual coefficient, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\alpha\beta \leq \alpha^p + \beta^{p'}$ for all $\alpha, \beta \geq 0$. By passing to the Fourier image, we first obtain

$$\|\langle D_x\rangle^k \langle D_y\rangle^l g_{\widetilde{\theta}}\| \leq \|\langle D_x\rangle^{2k} g_{\widetilde{\theta}}\| + \|\langle D_y\rangle^{2l} g_{\widetilde{\theta}}\|.$$

The second summand is finite. For the first, let $\hat{g}(\xi, y) = \mathcal{F}_{x\to\xi} g(\xi, y)$. Then

$$\begin{aligned}\|\langle D_x\rangle^{2k} g_{\widetilde{\theta}}\| &= \|\langle\xi\rangle^{2k} e^{\widetilde{\theta}y}\hat{g}(\xi, y)\| \leq \|\langle\xi\rangle^{2p'k}\hat{g}(\xi, y)\| + \|e^{p\widetilde{\theta}y}\hat{g}(\xi, y)\| \\ &= \|\langle D_x\rangle^{2p'k} g\| + \|e^{p\widetilde{\theta}y} g\| < \infty.\end{aligned}$$

Therefore, the second and third term on the right-hand side of (1.2.5) are finite. Finally,

$$\begin{aligned}\|\langle x\rangle^{k'} \langle D_x\rangle^k g_{\widetilde{\theta}}\| &\leq \|\langle x\rangle^{p'k'} \langle D_x\rangle^k g\| + \|\langle D_x\rangle^k e^{p\widetilde{\theta}y} g\| \\ &\leq \|\langle x\rangle^{p'k'} \langle D_x\rangle^k g\| + \|\langle D_x\rangle^{p'k} g\| + \|e^{p^2\widetilde{\theta}y} g\| < \infty\end{aligned}$$

shows that the right-hand side of (1.2.5) is finite. This yields (1.2.4). $\square$

Proposition 1.2.2 *Let $0 < \theta, \theta' \leq \infty$ and $Q \in \mathrm{As}(-\gamma, \theta)$, $Q' \in \mathrm{As}(\gamma', \theta')$ be given asymptotic types. Let $G \in \mathcal{L}(\mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s',\gamma'}(\mathbb{R}_+))$ with the property that*

$$G : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+), \qquad G^* : \mathcal{K}^{-s',-\gamma'}(\mathbb{R}_+) \to \mathcal{S}^{-\gamma}_{Q}(\mathbb{R}_+).$$

Then G has an integral kernel k_G (with respect to the $L^2(\mathbb{R}_+)$-scalar product) satisfying:

a) $k_G \in \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+) := \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_0(\mathbb{R}_+) \cap \mathcal{S}^{\gamma'}_0(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+)$.

b) *If $Q_p = \{(q,l) \in Q :\ q > \frac{1}{2} + \gamma - \frac{\theta}{p}\} \in \mathrm{As}(-\gamma, \frac{\theta}{p})$, and similarly $Q'_{p'} \in \mathrm{As}(\gamma', \frac{\theta'}{p'})$ for $\frac{1}{p} + \frac{1}{p'} = 1$, then $k_G \in \mathcal{S}^{\gamma'}_{Q'_{p'}}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{\overline{Q}_p}(\mathbb{R}_+)$.*
In particular, for $\theta = \theta' = \infty$ we have $k_G \in \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+)$.

Proof. a) First let θ and θ' be finite. Using reductions of order, we may asume that $s = s' = \gamma = \gamma' = 0$. Define $A = A(Q) = \mathrm{op}^0_M(h)$ with $h(z) = \prod_{(q,l)\in Q}(z-q)^l(z+\gamma)^{-l}$. Here, without loss of generality, Q is written as a sequence of pairs (q,l) where $(q,k) \notin Q$ for all $k < l$. Then $A \in \mathcal{L}(L^2(\mathbb{R}_+))$ is an isomorphism that restricts to isomorphisms $\mathcal{S}^0_Q(\mathbb{R}_+) \to \mathcal{S}^0_\theta(\mathbb{R}_+)$ and $\mathcal{S}^0_0(\mathbb{R}_+) \to \mathcal{S}^0_0(\mathbb{R}_+)$. Similarly, we construct A' corresponding to Q'. If we set $\widetilde{G} = A'GA^*$, then $\widetilde{G} : L^2(\mathbb{R}_+) \to \mathcal{S}^0_\theta(\mathbb{R}_+)$ and $\widetilde{G}^* : L^2(\mathbb{R}_+) \to \mathcal{S}^0_{\theta'}(\mathbb{R}_+)$. By (1.2.1) and Lemma 1.2.1, $\widetilde{G}$ has a kernel $\widetilde{k} \in \mathcal{S}^0_{\theta'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^0_\theta(\mathbb{R}_+)$. Then the result follows by expressing $G = (A^{-1}(A^{-1}\widetilde{G})^*)^*$ on the level of kernels. The corresponding result for $\theta = \theta' = \infty$ is obtained by passing to the limit $\theta_k = \theta'_k = k$ for $k \to \infty$.

b) The proof is similar to that of a), by using the fact (cf. [34]) that

$$\mathcal{S}^0_{\theta'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^0_\theta(\mathbb{R}_+) = \bigcap_{1\leq p\leq\infty} \mathcal{S}^0_{\frac{\theta'}{p'}}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^0_{\frac{\theta}{p}}(\mathbb{R}_+). \qquad \square$$

Let us now turn to the description of Green symbols. In the formulation of the corresponding result, we use the following notation: If E is a Fréchet space, then $S^\mu(\Omega \times \mathbb{R}^q, E)$ denotes the space of all symbols $a \in C^\infty(\Omega_y \times \mathbb{R}^q_\eta, E)$ satisfying

$$\sup_{y\in K, \eta\in\mathbb{R}^q} \{|\!|\!|\partial^\alpha_\eta \partial^\beta_y a(y,\eta)|\!|\!| \langle\eta\rangle^{|\alpha|-\mu}\} < \infty \qquad \text{for all } \alpha \in \mathbb{N}^q_0,\ \beta \in \mathbb{N}^m_0,$$

for any continuous seminorm $|\!|\!| \cdot |\!|\!|$ of E and any compact subset K of Ω. Classical symbols again are those having asymptotic expansions $a \sim \sum_{k=0}^{\infty} \chi a_{(\mu-k)}$ with an excision function χ and $a_{(\mu-k)}$ satisfying $a_{(\mu-k)}(y, \lambda\eta) = \lambda^{\mu-k} a_{(\mu-k)}(y,\eta)$ for positive λ.

$S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta)$ is a nuclear Fréchet space (a proof may be found in [38]), and we have

$$S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, E) = S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta)\widehat{\otimes}_\pi E. \tag{1.2.6}$$

Moreover, we have $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q) = S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q, C^\infty(\Omega))$.

Proposition 1.2.3 *Let $Q \in \mathrm{As}(-\gamma, \theta)$ and $Q' \in \mathrm{As}(\gamma', \theta')$ with $0 \le \theta, \theta' \le \infty$. Then $g \in R^{\mu}_{G}(\Omega \times \mathbb{R}^q, (\gamma, \theta; \gamma', \theta'))_{Q,Q'}$ if and only if there exists a kernel-function*

$$k(y, \eta, t, t') \in S^{\mu}_{\mathrm{cl}}(\Omega_y \times \mathbb{R}^q_\eta, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+))$$

such that $g(y,\eta)$ is the integral operator with kernel $k_g(y, \eta, t, t') = [\eta]k(y, \eta, t[\eta], t'[\eta])$.

Proof. The y-variable is here unessential and thus will be dropped for the proof. If the kernel k_g is of the required form, it is straightforward to show that g is a Green symbol. The homogeneous components $g_{(\mu-j)}$ are then given by the kernels

$$k_{g_{(\mu-j)}}(\eta, t, t') = |\eta| k_{(\mu-j)}(\eta, t|\eta|, t'|\eta|), \qquad \eta \neq 0.$$

Vice versa, if g is a Green symbol, each homogeneous component $g_{(\mu-j)}$ possesses a kernel $k_{g_{(\mu-j)}} \in C^\infty(\mathbb{R}^q_\eta \setminus 0, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+))$ due to Proposition 1.2.2. By the κ-homogeneity of $g_{(\mu-j)}$ we obtain $k_{g_{(\mu-j)}}(\lambda\eta, t, t') = \lambda^{\mu-j+1} k_{g_{\mu-j)}}(\eta, \lambda t, \lambda t')$ forall $\lambda > 0$. This shows that $k_{(\mu-j)}(\eta, t, t') := |\eta|^{-1} k_{g_{(\mu-j)}}(\eta, t|\eta|^{-1}, t'|\eta|^{-1})$ is homogeneous of order $\mu - j$ in η and belongs to $C^\infty(\mathbb{R}^q \setminus 0, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+))$. If we now choose some kernel $\widetilde{k}$ with $\widetilde{k} \sim \sum_{j=0}^{\infty} k_{(\mu-j)}$, then $\widetilde{k} \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+))$, and if $\widetilde{g}(\eta)$ is defined via the kernel $[\eta]\widetilde{k}(\eta, t[\eta], t'[\eta])$. This yields $\widetilde{g} \in R^{\mu}_{G}(\mathbb{R}^q_\eta, (\gamma, \theta; \gamma', \theta'))_{Q,Q'}$ by the first part of the proof, and $g - \widetilde{g} \in R^{-\infty}_{G}(\mathbb{R}^q_\eta, (\gamma, \theta; \gamma', \theta'))_{Q,Q'}$ by construction of $\widetilde{g}$ and $\widetilde{k}$, respectively. Therefore,

$$\widetilde{\widetilde{k}}(\eta, t, t') := k_g(\eta, t, t') - [\eta]\widetilde{k}(\eta, t[\eta], t'[\eta]) \in \mathcal{S}(\mathbb{R}^q_\eta, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+)).$$

However, the assertion then holds if we set $k(\eta, t, t') := \widetilde{k}(\eta, t, t') + [\eta]^{-1}\widetilde{\widetilde{k}}(\eta, t[\eta]^{-1}, t'[\eta]^{-1})$, since the second term on the right-hand side again belongs to $\mathcal{S}(\mathbb{R}^q_\eta, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\Gamma \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+))$. □

Corollary 1.2.4 *Let $g \in R^{\mu}_{G}(\Omega \times \mathbb{R}^q, (\gamma, \theta; \gamma', \theta'))_{Q,Q'}$, and let k be its kernel function as in Proposition* 1.2.3, *then*

$$k \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q, \mathcal{S}^{\gamma'}_{Q'_{p'}}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{\overline{Q}_p}(\mathbb{R}_+)),$$

where Q_p, $Q'_{p'}$ are as in Proposition 1.2.2 b). *In particular, if $\theta = \theta' = \infty$,*

$$k \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{Q}(\mathbb{R}_+)).$$

Example 1.2.5 *With $a = a(t, \tau, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ we associate the operator family*

$$\mathrm{op}^+(a)(\eta) = \mathrm{r}^+ \,\mathrm{op}(a)(\eta)\mathrm{e}^+ : C^\infty_0(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+),$$

where e^+ denotes the extension by zero and r^+ the restriction to $\mathbb{R}_+$.

a) *If* $\sigma_0, \sigma_1 \in C_0^\infty(\overline{\mathbb{R}}_+)$ *are cut-off functions such that* σ_0 *and* $1-\sigma_1$ *have disjoint support, then we have* $g(\eta) := \sigma_0 \operatorname{op}^+(a)(\eta)(1-\sigma_1) \in R_G^{-\infty}(\mathbb{R}^q_\eta, (0,0,\infty))_{T,T}$ *for the Taylor asymptotics* $T = \{(-j,0) : j \in \mathbb{N}_0\}$. *Indeed, it is straightforward to show that* $g(\eta)$ *posesses a kernel* $k_g(\eta,t,t') \in \mathcal{S}(\mathbb{R}^q_\eta, \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+))$. *It remains to observe that* $\mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+) = \mathcal{S}(\overline{\mathbb{R}}_+)\widehat{\otimes}_\pi \mathcal{S}(\overline{\mathbb{R}}_+)$ *and* $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}^0_T(\mathbb{R}_+)$, *cf. Example* 1.1.4.

b) *If* $\omega_0, \omega_1 \in C_0^\infty(\overline{\mathbb{R}}_+)$ *are cut-off functions such that* ω_0 *and* $1-\omega_1$ *have disjoint support, then* $g(\eta) := \omega_0(t[\eta]) \operatorname{op}^+(a)(\eta)(1-\omega_1)(t[\eta]) \in R_G^\mu(\mathbb{R}^q_\eta, (0,0,\infty))_{T,T}$, *where* T *denotes Taylor asymptotics as above. Indeed, if* $\kappa(\eta) = \kappa_{[\eta]}$ *with the standard group action from Example* 1.1.6, *then* $\widetilde{g}(\eta) := \kappa^{-1}(\eta)g(\eta)\kappa(\eta) = \omega_0(t)\operatorname{op}^+(\widetilde{a})(\eta)(1-\omega_1)(t)$ *with* $\widetilde{a}(t,\tau,\eta) = a(t[\eta]^{-1}, \tau[\eta], \eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$ *by Proposition* 2.2.1 *below. Therefore,* $\widetilde{g}(\eta)$ *has a kernel-function* $\widetilde{k}(\eta,t,t') \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+))$. *Thus the assertion holds in view of Proposition* 1.2.3, *since* $g(\eta)$ *has the kernel* $k(\eta,t,t') = [\eta]\widetilde{k}(\eta, t[\eta], t'[\eta])$.

If we require that $a \in S^{\mu,0}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$, *we can interchange the role of* σ_0 *and* $(1-\sigma_1)$ *in* a) *and of* ω_0 *and* $(1-\omega_1)$ *in* b), *and still obtain the same statement on* $g(\eta)$. *Moreover, corresponding results hold for* $a = a(y,t,\tau,\eta) \in S^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\tau,\eta})$.

1.3 Edge-degenerate symbols and holomorphic Mellin symbols

A symbol $p \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega_y \times \mathbb{R}^{1+q}_{\tau,\eta})$ is said to be edge-degenerate, if there exists a symbol $\widetilde{p}(t,y,\tau,\eta) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\tau,\eta})$ such that

$$p(t,y,\tau,\eta) = \widetilde{p}(t,y,t\tau,t\eta). \tag{1.3.1}$$

If $a \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\tau,\eta})$ we call p an edge-degenerate symbol associated with a, if

$$a(t,y,\tau,\eta) - t^{-\mu}p(t,y,\tau,\eta) \in S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}).$$

Lemma 1.3.1 *To any* $a \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q})$ *there exists an edge-degenerate symbol* p *associated with* a. *If* $\widetilde{p}$ *is connected with* p *as in* (1.3.1), $\widetilde{p}$ *is uniquely determined modulo* $S^{-\infty}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q})$ *by the asymptotic expansion*

$$\widetilde{p} \sim \sum_{j=0}^\infty \widetilde{p}_{(\mu-j)}, \qquad \widetilde{p}_{(\mu-j)}(t,y,\tau,\eta) = t^j a_{(\mu-j)}(t,y,\tau,\eta),$$

where $a_{(\mu-j)}$ *are the homogeneous components of* a. (*Vice versa, if* p *is as in* (1.3.1) *we can find an* $a \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q})$ *such that* $a - t^{-\mu}p \in S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q})$ *if and only if the homogeneous components* $\widetilde{p}_{(\mu-j)}$ *of* $\widetilde{p}$ *have a zero of order* j *in* $t=0$).

Proof. If $\widetilde{p}$ is of the described form, then, in $S^\mu(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q})$,

$$t^{-\mu}p(t,y,\tau,\eta) \sim \sum_{j=0}^\infty t^{-\mu}t^j a_{(\mu-j)}(t,y,t\tau,t\eta) = \sum_{j=0}^\infty a_{(\mu-j)}(t,y,\tau,\eta) \sim a(t,y,\tau,\eta).$$

If $\widetilde{p}_0 \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q})$ also satisfies $t^{-\mu}p_0(t,y,\tau,\eta) = t^{-\mu}\widetilde{p}_0(t,y,t\tau,t\eta) \equiv a(t,y,\tau,\eta)$ modulo $S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q})$, then $p - p_0 \in S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q})$ and thus

$$\widetilde{p} - \widetilde{p}_0 \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}) \cap S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}) = S^{-\infty}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}).$$

□

Let $M^\mu_{\mathcal{O}}(\mathbb{R}^q_\eta)$ denote the space of all smooth functions $h : \mathbb{C} \times \mathbb{R}^q_\eta \to \mathbb{C}$ holomorphic in the first variable such that $h_\beta(\tau, \eta) := h(\beta + i\tau, \eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{1+q}_{\tau,\eta})$ for any real β, and that $\beta \mapsto h_\beta$ is locally bounded in $\beta \in \mathbb{R}$. The corresponding space of (t, y)-dependent symbols is denoted by $M^\mu_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times \Omega_y \times \mathbb{R}^q_\eta)$ $(= C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(\mathbb{R}^q))$ with $M^\mu_{\mathcal{O}}(\mathbb{R}^q)$ being equipped with its natural Fréchet space structure).

With such symbols (and also meromorphic ones introduced below) we associate operator families

$$[\mathrm{op}^\gamma_M(h)(y, \eta)u](t) = \int_{\Gamma_{1/2-\gamma}} t^{-z} h(t, y, z, \eta)(\mathcal{M}u)(z)\, đz, \qquad u \in C^\infty_0(\mathbb{R}_+),$$

where $\gamma \in \mathbb{R}$ and $(\mathcal{M}u)(z) = \int_0^\infty t^z u(t) \frac{dt}{t}$ is the Mellin transform of u.

Lemma 1.3.2 *To any edge-degenerate symbol $p \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q})$ as in* (1.3.1) *there exists an $\widetilde{h} \in M^\mu_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q_\eta)$ such that for $h(t, y, z, \eta) := \widetilde{h}(t, y, z, t\eta)$ the following relation holds*

$$\mathrm{op}^\gamma_M(h)(y, \eta) - \mathrm{op}(p)(y, \eta) \in C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+; \mathbb{R}^q_\eta)). \tag{1.3.2}$$

If $p_0(y, \tau, \eta) = \widetilde{p}(0, y, t\tau, t\eta)$ and $h_0(y, z, \eta) = \widetilde{h}(0, y, z, t\eta)$, then also

$$\mathrm{op}^\gamma_M(h_0)(y, \eta) - \mathrm{op}(p_0)(y, \eta) \in C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+; \mathbb{R}^q_\eta)). \tag{1.3.3}$$

Moreover, if $\widetilde{p}$ is independent of y, also $\widetilde{h}$ may be chosen to be independent of y.

A proof of Lemma 1.3.2 can be found, for instance, in [27, Section 2.1.3]. The symbol $\widetilde{h}$ associated with p is uniquely determined modulo $M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q_\eta)$, and h is called a Mellin quantization of p.

1.4 Complete edge symbols

We are going to study a class of operator–valued symbols, that shall serve later on as local symbols for (parameter–dependent) pseudodifferential operators on manifolds with edges (in particular on manifolds with boundary). For more details and elements of the calculus we refer to [25], [4], and [32].

Definition 1.4.1 a) *A set $P \subset \mathbb{C} \times \mathbb{N}_0$ is called a Mellin asymptotic type if the projection $\pi_{\mathbb{C}} P$ of P to the complex plane contains for any $\beta > 0$ only finitely many points lying in the vertical strip $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq \beta\}$.*

b) *A meromorphic, smoothing Mellin symbol $h = h(z)$ associated with an asymptotic type P as in* a) *is a meromorphic function on $\mathbb{C}$ with poles at most in any $p \in \pi_{\mathbb{C}} P$ of order at most $n + 1$ if $(p, n) \in P$. In addition, if χ is a $\pi_{\mathbb{C}} P$–excision function, then*

$$\tau \to (\chi h)(\beta + i\tau) \in S^{-\infty}(\mathbb{R}_\tau)$$

for any real β, locally bounded in $\beta \in \mathbb{R}$. We denote by $M^{-\infty}_P$ the space of all such functions; the corresponding y–dependent functions by $M^{-\infty}_P(\Omega)$ $(= C^\infty(\Omega, M^{-\infty}_P))$.

For given data $\boldsymbol{g} = (\gamma, \gamma - \mu, k)$ with $k \in \mathbb{N}$, we define $R^\mu_{M+G}(\Omega_y \times \mathbb{R}^q_\eta, \boldsymbol{g})$ to be the space of all operator–functions of the form $m(y, \eta) + g(y, \eta)$ with arbitrary $g \in R^\mu_G(\Omega \times \mathbb{R}^q_\eta, \boldsymbol{g})$, cf. Definition 1.1.8, and

$$m(y, \eta) = \omega(t[\eta]) \sum_{j=0}^{k-1} t^{-\mu+j} \Big(\sum_{|\alpha| \leq j} \mathrm{op}^{\gamma_{j\alpha}}_M(h_{j\alpha})(y)\eta^\alpha \Big) \widetilde{\omega}(t[\eta]). \tag{1.4.1}$$

Here, $h_{j\alpha}(y) \in M^{-\infty}_{P_{j\alpha}}(\Omega)$ for certain Mellin asymptotic types $P_{j\alpha}$ and weights $\gamma_{j\alpha} \in \mathbb{R}$ such that $\Gamma_{\frac{1}{2}-\gamma_{j\alpha}} \cap \pi_{\mathbb{C}} P_{j\alpha} = \emptyset$ and $\gamma - j \le \gamma_{j\alpha} \le \gamma$ for all j, α. Further, ω and $\widetilde{\omega}$ are arbitrary cut–off functions.

Definition 1.4.2 *The space $R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, k)$ and $k \in \mathbb{N}$ is defined to be the set of all symbols of the form*

$$\begin{aligned} a(y,\eta) = \sigma\big\{&\omega(t[\eta])t^{-\mu}\operatorname{op}^\gamma_M(h)(y,\eta)\omega_0(t[\eta]) + (1-\omega)(t[\eta])t^{-\mu}\operatorname{op}(p)(y,\eta)(1-\omega_1)(t[\eta])\big\}\sigma_0 \\ &+ (1-\sigma)\operatorname{op}(q)(y,\eta)(1-\sigma_1) + m(y,\eta) + g(y,\eta) \end{aligned}$$

where $m + g \in R^\mu_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, h is a Mellin quantization of an edge–degenerate symbol $p(t,y,\tau,\eta) = \widetilde{p}(t,y,t\tau,t\eta)$ with $\widetilde{p} \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\tau,\eta})$, and $q \in S^{\mu,0}(\mathbb{R} \times \Omega \times \mathbb{R}^{1+q}_{\tau,\eta})$. Moreover, σ, σ_j and ω, ω_j are arbitrary cut–off functions satisfying $\omega\omega_0 = \omega$, $\omega_1\omega = \omega_1$ and $\sigma\sigma_0 = \sigma$, $\sigma_1\sigma = \sigma$.

Recall, e.g. from [28], that any $a \in R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ is an operator–valued symbol of the classes $S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+))$ and $S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}_Q(\mathbb{R}_+), \mathcal{K}^{s-\mu,\gamma-\mu}_R(\mathbb{R}_+))$ for any asymptotic type $Q \in \mathrm{As}(\gamma,k)$ with a resulting type $R = R(Q,a) \in \mathrm{As}(\gamma-\mu,k)$.

If $a \in R^\mu_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ the analogous statement is true for $\mathcal{K}^{s-\mu,\gamma-\mu}_{(R)}(\mathbb{R}_+)$ replaced by $\mathcal{K}^{\infty,\gamma-\mu}_{(R)}(\mathbb{R}_+)$.

As a slight generalization of Definition 1.4.2 we introduce matrices of complete edge symbols,

$$R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; l, m) = R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}) \otimes \mathbb{C}^m \otimes \mathbb{C}^l, \tag{1.4.2}$$

where we identify $\mathbb{C}^m \otimes \mathbb{C}^l$ with the space of $(m \times l)$–matrices. Equivalently, we could require all components h, p, q, g from Definition 1.4.2 and all $h_{j\alpha}$ from (1.4.1) to be matrix–valued. Analogously, we have the subclasses $R^\mu_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; l, m)$ and $R^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; l, m)$. Elements from (1.4.2) act pointwise as operators $\mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+, \mathbb{C}^l) \to \mathcal{K}^{s-\mu,\gamma-\mu}_{(R)}(\mathbb{R}_+, \mathbb{C}^m)$, where

$$\mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+, \mathbb{C}^l) = \mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+) \otimes \mathbb{C}^l = \bigoplus_{j=1}^{l} \mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+) \tag{1.4.3}$$

In fact, if we endow $\mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+, \mathbb{C}^l)$ again with the standard group action $(\kappa_\lambda u)(t) = \lambda^{1/2} u(\lambda t)$, then $R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; l, m) \subset S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+, \mathbb{C}^l), \mathcal{K}^{s-\mu,\gamma-\mu}_{(R)}(\mathbb{R}_+, \mathbb{C}^m))$.

2 The edge symbol structure of boundary value problems

For a given symbol $a = a(t,y,\tau,\eta) \in S^{\mu,0}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\tau,\eta})$ set $\operatorname{op}^+(a)(y,\eta) = \mathrm{r}^+ \operatorname{op}(a)(y,\eta)\mathrm{e}^+$, cf. the notation in Example 1.2.5 ($\operatorname{op}^+(a)(y,\eta)$ is first regarded as a family of maps $C^\infty_0(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+)$). The main purpose of this section is to show that the op^+-action on $\mathbb{R}_+$ leads to an edge symbol in the sense of Subsection 1.3. More precisely, for any $\gamma \in \mathbb{R}$ and $N \in \mathbb{N}$

$$\operatorname{op}^+(a)(y,\eta) - g(y,\eta) \in R^\mu(\Omega \times \mathbb{R}^q, (\gamma, \gamma-\mu, N)), \tag{2.0.4}$$

where $g := g_{\gamma,N}$ is an appropriate Green symbol. Since the y-variable never enters explicitly in the proofs, we shall drop it in the following. Now let $\sigma, \sigma_0, \sigma_1 \in C^\infty_0(\overline{\mathbb{R}}_+)$ be cut-off functions such that σ and $(1-\sigma_0)$, respectively σ_1 and $(1-\sigma)$ have disjoint support. Then we write

$$\operatorname{op}^+(a)(\eta) = \sigma \operatorname{op}^+(a)(\eta)\sigma_0 + (1-\sigma)\operatorname{op}(a)(\eta)(1-\sigma_1) + g_1(\eta)$$

with the remainder $g_1(\eta) = \sigma \operatorname{op}^+(a)(\eta)(1-\sigma_0) + (1-\sigma)\operatorname{op}^+(a)(\eta)\sigma_1$. Due to Example 1.2.5 a), g_1 belongs to $R_G^{-\infty}(\mathbb{R}^q,(0,0,\infty))_{T,T}$ if $a \in S^{\mu,0}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q})$.

If $\omega, \omega_0, \omega_1 \in C_0^\infty(\overline{\mathbb{R}}_+)$ are cut-off functions satisfying the same conditions as the σ's above,

$$\begin{aligned}\sigma \operatorname{op}^+(a)(\eta)\sigma_0 = \sigma\{&\omega(t[\eta])\operatorname{op}^+(a)(\eta)\omega_0(t[\eta]) + \\ &+ (1-\omega)(t[\eta])\operatorname{op}(a)(\eta)(1-\omega_1)(t[\eta]) + g_2(\eta)\}\sigma_1\end{aligned}$$

with a remainder $g_2(\eta) \in R_G^\mu(\mathbb{R}^q;(0,0,\infty))_{T,T}$, cf. Example 1.2.5 b). The term $(1-\omega)(t[\eta])\operatorname{op}(a)(\eta)(1-\omega_1)(t[\eta])$ will be treated in Subsection 2.1 (see Proposition 2.1.5 below). The crucial step towards a verification of (2.0.4) is the analysis of the operator family

$$\omega(t[\eta])\operatorname{op}^+(a)(\eta)\omega_0(t[\eta]).$$

Since $\kappa_\lambda^{-1}\operatorname{op}^+(a)(\eta)\kappa_\lambda = \operatorname{op}^+(a_\lambda)(\eta)$ for $a_\lambda(t,\tau,\eta) = a(t\lambda^{-1},\tau\lambda,\eta)$, the analysis of the above operator-family is – by conjugation with $\kappa_{[\eta]}$ – essentially equivalent to the investigation of $\omega_0 \operatorname{op}^+(\widetilde{a})(\eta)\omega_1$ with

$$\widetilde{a}(t,\tau,\eta) := a_{[\eta]}(t,\tau,\eta) = a(t[\eta]^{-1},\tau[\eta],\eta). \tag{2.0.5}$$

The symbol $\widetilde{a}$ shall be investigated in the following Subsection 2.2. Then a combination of results from the theory of cone pseudodifferential operators from [27] (i.e., the analysis of $\operatorname{op}^+(p)$ for $p \in S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau)$) together with those from Subsections 1.2 and 2.1 shall lead to (2.0.4).

2.1 Symbol structures far from the boundary

Lemma 2.1.1 *Let $p(t,\tau,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times (\mathbb{R}^{1+q}_{\tau,\eta} \setminus 0))$ be positively homogeneous (of arbitrary degree) in (τ,η) and $\chi(\tau,\eta) \in C^\infty(\mathbb{R}^{1+q})$ be a 0-excision function. Then*

$$r(t,\tau,\eta) = (\chi(\tau,\eta) - \chi(t\tau,t\eta))p(t,\tau,\eta) \in S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\tau,\eta}).$$

Proof. Without loss of generality let us assume that $\chi(\tau,\eta) = 0$ for $|(\tau,\eta)| \le 1$ and $\chi(\tau,\eta) = 1$ for $|(\tau,\eta)| \ge 2$. If we define $\widetilde{r}(t,\tau,\eta) = \chi(\tau,\eta) - \chi(t\tau,t\eta)$, then $\widetilde{r}$ is smooth, and if $t \in [a,b] \subset \mathbb{R}_+$ then $\widetilde{r}(t,z,\eta) = 0$ for $|(\tau,\eta)| \le \min\{1,b^{-1}\}$ and for $|(\tau,\eta)| \ge \max\{2,2a^{-1}\}$. Thus $\widetilde{r} \in S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^{1+q})$ and, since $\widetilde{r}$ excises 0 for any $t > 0$, this is also true for r. □

Lemma 2.1.2 *Let $\omega, \omega_0, \sigma, \sigma_0 \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions and define r as in Lemma 2.1.1. Then χ can be chosen in such a way that, for any $\gamma \in \mathbb{R}$,*

$$g(\eta) = \sigma(1-\omega)(t[\eta])\operatorname{op}(r)(\eta)(1-\omega_0)(t[\eta])\sigma_0 \in R_G^{-\infty}(\mathbb{R}^q,(\gamma,\gamma,\infty))_{\mathcal{O},\mathcal{O}}.$$

Proof. By Lemma 2.1.1 we know that $g(\eta)$ has an integral kernel $\widetilde{k}(\eta,t,t') \in C^\infty(\mathbb{R}^q \times \mathbb{R}_+ \times \mathbb{R}_+)$. By Proposition 1.2.3 it suffices to show that

$$k(\eta,t,t') = [\eta]^{-1}\widetilde{k}(\eta,t[\eta]^{-1},t'[\eta]^{-1}) \in \mathcal{S}(\mathbb{R}^q_\eta, \mathcal{S}^\gamma_{\mathcal{O}}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^\gamma_{\mathcal{O}}(\mathbb{R}_+)). \tag{2.1.1}$$

By definition of g we have

$$\begin{aligned}k(\eta,t,t') &= [\eta]^{-1}(1-\omega)(t)\sigma(t[\eta]^{-1})\int e^{i(t-t')[\eta]^{-1}\tau}\{\chi(\tau,\eta) - \chi(t[\eta]^{-1}(\tau,\eta))\}p(t[\eta]^{-1},\tau,\eta)\,đ\tau \\ &\quad\cdot(1-\widetilde{\omega})(t')\widetilde{\sigma}(t'[\eta]^{-1}) \\ &= (1-\omega)(t)\sigma(t[\eta]^{-1})\int e^{i(t-t')\tau}\{\chi(\tau[\eta],\eta) - \chi(t(\tau,\eta/[\eta]))\}p(t[\eta]^{-1},\tau[\eta],\eta)\,đ\tau \\ &\quad\cdot(1-\widetilde{\omega})(t')\widetilde{\sigma}(t'[\eta]^{-1}).\end{aligned}$$

Due to the factors $(1-\omega)$, $(1-\widetilde{\omega})$ and σ, $\widetilde{\sigma}$ we clearly have $k \in C^\infty(\mathbb{R}^q_\eta, \mathcal{S}^\gamma_{\mathcal{O}}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^\gamma_{\mathcal{O}}(\mathbb{R}_+))$. For $|\eta|$ so large that $|\eta| = [\eta]$, we have

$$|(\tau[\eta], \eta)| = [\eta]\langle\tau\rangle \geq [\eta] \qquad \text{and} \qquad |(t(\tau, \eta/[\eta]))| = t\langle\tau\rangle \geq t. \tag{2.1.2}$$

There exists a $c > 0$ such that $(1-\omega)(t) = 0$ for all $t \leq c$. Now we choose a χ such that $\chi(\tau, \eta) = 1$ for all $|(\tau, \eta)| \geq c$. Then $k(\eta, t, t') = 0$ whenever $t \leq c$, and by (2.1.2), $k(\eta, t, t') = 0$ whenever $t \geq c$ and $[\eta] \geq c$. This shows that $k(\eta, t, t') \neq 0$ only for η in a bounded subset of $\mathbb{R}^q$, and therefore $k \in C_0^\infty(\mathbb{R}^q \times \mathbb{R}_+ \times \mathbb{R}_+)$, which obviously implies (2.1.1). □

Lemma 2.1.3 *Let $p(t, \tau, \eta) = \widetilde{p}(t, t\tau, t\eta)$ be an edge-degenerate symbol with $\widetilde{p}(t, \tau, \eta) \in S^{-L}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ and set $g_L(\eta) := \sigma(1-\omega)(t[\eta])t^{-\mu}\operatorname{op}(p)(\eta)(1-\omega_0)(t[\eta])\sigma_0$. Then to any given $k \in \mathbb{N}$, the number L can be chosen so large that*

$$g_L, g_L^* \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta; H^{s,\delta}(\mathbb{R}), H^{s+k,\delta+k}(\mathbb{R})) \qquad \text{for all } s, \delta \in \mathbb{R}.$$

Here, $H^{s,\delta}(\mathbb{R}) = \langle\cdot\rangle^{-\delta}H^s(\mathbb{R})$ are weighted Sobolev spaces.

Proof. See the proof of Lemma 3.17 in [6]. □

Lemma 2.1.4 *Let $a(t, \tau, \eta) \in S^{-L}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ and set $g_L(\eta) := \sigma(1-\omega)(t[\eta])\operatorname{op}(a)(\eta)(1-\omega_0)(t[\eta])\sigma_0$. Then to any given $k \in \mathbb{N}$, the number L can be chosen so large that*

$$g_L, g_L^* \in S^{-k}(\mathbb{R}^q_\eta; H^{s,\delta}(\mathbb{R}), H^{s+k,\delta+k}(\mathbb{R})) \qquad \text{for all } s, \delta \in \mathbb{R}.$$

Here, $H^{s,\delta}(\mathbb{R}) = \langle\cdot\rangle^{-\delta}H^s(\mathbb{R})$ are weighted Sobolev spaces.

Proof. It suffices to verify the statement for g_L, since it then automatically is also true for g_L^* by duality of the Sobolev spaces. By [33, Lemma 5.3] for any $s, \delta \in \mathbb{R}$ we have $\sigma(1-\omega)(t[\eta]) \in S^k(\mathbb{R}^q_\eta; H^{s,\delta}(\mathbb{R}), H^{s,\delta+k}(\mathbb{R}))$ for all $k \geq 0$. Hence we only have to show that we can choose L such that

$$\operatorname{op}(p)(\eta) \in S^{-2k}(\mathbb{R}^q_\eta; H^{s,\delta}(\mathbb{R}), H^{s+k,\delta}(\mathbb{R})) \tag{2.1.3}$$

for any s and δ. Due to the factor σ we may assume that a has exit order 0 in t. Since $a(t, \tau, \eta) \in C^\infty(\mathbb{R}^q_\eta, S^{-L,0}(\mathbb{R}\times\mathbb{R}_\tau))$, it is clear that $\operatorname{op}(a)(\eta) \in C^\infty(\mathbb{R}^q_\eta, \mathcal{L}(H^{s,\delta}(\mathbb{R}), H^{s+k,\delta}(\mathbb{R})))$ for $L \geq k$. Moreover, we have $\widetilde{g}(\eta) := \kappa^{-1}(\eta)g(\eta)\kappa(\eta) = \operatorname{op}(\widetilde{a})(\eta)$ with $\widetilde{a}$ being given by (2.0.5). Since $\widetilde{a} \in S^{-L}(\mathbb{R}^q_\eta, S^{-L,0}(\mathbb{R} \times \mathbb{R}_\tau))$ (which is a relation of similar type as (2.2.1) below), this means that $\widetilde{g}(\eta) : H^{s,\delta}(\mathbb{R}) \to H^{s+k,\delta}(\mathbb{R})$ for $L \geq k$ and $\|\widetilde{g}(\eta)\|_{\mathcal{L}(H^{s,\delta}(\mathbb{R}), H^{s+k,\delta}(\mathbb{R}))} \leq c\langle\eta\rangle^{-L}$.

The derivatives of $\operatorname{op}(a)(\eta)$ can be treated in the same way, since $D^\alpha_\eta \operatorname{op}(a)(\eta) = \operatorname{op}(D^\alpha_\eta a)(\eta)$ and $D^\alpha_\eta a \in S^{-L-|\alpha|}(\mathbb{R} \times \mathbb{R}^{1+q})$. Hence, (2.1.3) holds for $L \geq 2k$. □

Proposition 2.1.5 *Let $a \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ and $p \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ be an edge-degenerate symbol associated with a. Then, for any $\gamma \in \mathbb{R}$,*

$$g(\eta) := \sigma(1-\omega)(t[\eta])\{\operatorname{op}^+(a)(\eta) - t^{-\mu}\operatorname{op}(p)(\eta)\}(1-\omega_0)(t[\eta])\sigma_0 \in R^\mu_G(\mathbb{R}^q_\eta, (\gamma, \gamma, \infty))_{\mathcal{O},\mathcal{O}}.$$

Proof. Let $p(t,\tau,\eta)=\widetilde{p}(t,t\tau,t\eta)$ with $\widetilde{p}$ as in (1.3.1). If $\chi(\tau,\eta)$ is a 0-excision function (chosen as in Lemma 2.1.2), we have $a-\sum_{j=0}^{L-1}\chi a_{(\mu-j)}\in S^{M-L}(\overline{\mathbb{R}}_+\times\mathbb{R}^{1+q})$ for any $L\in\mathbb{N}$, and

$$t^{-\mu}p(t,\tau,\eta)-\sum_{j=0}^{L-1}\chi(t\tau,t\eta)a_{(\mu-j)}=t^{-\mu}p_L(t,\tau,\eta)=t^{-\mu}\widetilde{p}_L(t,t\tau,t\eta)$$

where $\widetilde{p}_L\in S^{M-L}(\overline{\mathbb{R}}_+\times\mathbb{R}^{1+q}_{\tau,\eta})$. Hence, for $r_L(t,\tau,\eta):=\sum_{j=0}^{L-1}\{\chi(\tau,\eta)-\chi(t\tau,t\eta)\}a_{(\mu-j)}(t,\tau,\eta)$ we obtain $g(\eta)=\sigma(1-\omega)(t[\eta])\{\mathrm{op}(a_L)(\eta)-t^{-\mu}\,\mathrm{op}(p_L)(\eta)+\mathrm{op}(r_L)(\eta)\}(1-\widetilde{\omega})(t[\eta])\widetilde{\sigma}$. Since L is arbitrary, the result then follows from Lemmas 2.1.2, 2.1.3, and 2.1.4. □

2.2 Vector-valued interpretation of symbols on the half-space

Throughout this subsection we let $a\in S^{\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}^{1+q}_{\tau,\eta})$ and $\widetilde{a}$ as in (2.0.5). A simple observation, using the chain rule and the relation $\langle\tau[\eta],\eta\rangle\sim\langle\tau\langle\eta\rangle,\eta\rangle=\langle\tau\rangle\langle\eta\rangle$, is that

$$a\mapsto\widetilde{a}:S^{\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}^{1+q})\to S^{\mu}(\mathbb{R}^q,S^{\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}))\tag{2.2.1}$$

is a continuous map. Since the unit sphere in $\mathbb{R}$ consists of the two points 1, -1, the homogeneous components of a are of the form

$$a_{(\mu-k)}(t,\tau)=a_k^+(t)\theta^+(\tau)\tau^{\mu-k}+a_k^-(t)\theta^-(\tau)\tau^{\mu-k},$$

where $\theta^{\pm}$ is the characteristic function of $\mathbb{R}_{\pm}$, and $a_k^{\pm}\in C^{\infty}(\overline{\mathbb{R}}_+)$. For abbreviation we then write

$$a\sim\sum_{k=0}^{\infty}a_k^{\pm}(t)\tau^{\mu-k}\qquad\text{for }\tau\to\pm\infty.$$

Proposition 2.2.1 *If $a\in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}^{1+q}_{\tau,\eta})$, then $\widetilde{a}\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_{\eta},S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tau}))$ and the mapping $a\mapsto\widetilde{a}$ is continuous. In particular, if χ is an arbitrary zero excision function, then*

$$\widetilde{a}(t,\tau,\eta)-\chi(\tau)\sum_{k=0}^{N}a_k^{\pm}(t[\eta]^{-1},\eta)[\eta]^{\mu-k}(i\tau)^{\mu-k}\in S^{\mu}(\mathbb{R}^q_{\eta},S^{\mu-N-1}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tau}))\tag{2.2.2}$$

for $\tau\to\pm\infty$, where the coefficients of $a_k^{\pm}$ are given by

$$a_k^{\pm}(t,\eta)=\sum_{j+|\alpha|=k}\frac{1}{\alpha!}a_{j\alpha}^{\pm}(t)\eta^{\alpha},\qquad a_{j\alpha}^{\pm}(t)=(-i)^{\mu-j-|\alpha|}(\partial_{\eta}^{\alpha}a_{(\mu-j)})(t,\pm1,0).$$

Note that the symbol in (2.2.2) then automatically belongs to $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_{\eta},S^{\mu-N}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tau}))$.

Proof. Since the functions $a_k^{\pm}(t,\eta)$ are polynomials in η of degree at most k, it is clear that $a_k^{\pm}(t[\eta]^{-1},\eta)[\eta]^{\mu-k}\in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}^q_{\eta})$. Thus to show – as the first step – that $\widetilde{a}\in S^{\mu}(\mathbb{R}^q_{\eta},S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tau}))$, we have to verify (2.2.2). To this end let $c>0$ such that $\chi(\tau)=0$ for $|\tau|\le c$, hence $\chi(\tau/[\eta])=0$ for $|\tau|\le c[\eta]$. Thus we can choose a zero excision function $\widetilde{\chi}(\tau,\eta)$ such that $\widetilde{\chi}(\tau,\eta)\chi(\frac{\tau}{[\eta]})=\chi(\frac{\tau}{[\eta]})$ for all τ,η, or, equivalently, $\widetilde{\chi}(\tau[\eta],\eta)\chi(\tau)=\chi(\tau)$ forall $(\tau,\eta)\in\mathbb{R}\times\mathbb{R}^q$. By (2.2.1) we have $(1-\chi)(\tau)\widetilde{a}(t,\tau,\eta)\in S^{\mu}(\mathbb{R}^q_{\eta},S^{-\infty}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tau}))$. Moreover,

$$\widetilde{a}(t,\tau,\eta)-\chi(\tau)\sum_{k=0}^{N}a_{(\mu-k)}(t[\eta]^{-1},\tau[\eta],\eta)=\widetilde{a}_N(t,\tau,\eta)+(1-\chi)(\tau)\widetilde{a}(t,\tau,\eta),$$

with $\widetilde{a}_N(t,\tau,\eta) = a_N(t[\eta]^{-1},\tau[\eta],\eta)$, where

$$a_N(t,\tau,\eta) = \chi(\tau[\eta]^{-1})\Big\{a(t,\tau,\eta) - \widetilde{\chi}(\tau,\eta)\sum_{k=0}^{N} a_{(\mu-k)}(t,\tau,\eta)\Big\}.$$

Again by (2.2.1), we see that $\widetilde{a}_N \in S^{\mu-N-1}(\mathbb{R}^q_\eta, S^{\mu-N-1}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$. Thus to show (2.2.2) it suffices to investigate for $\tau \to \pm\infty$

$$\chi(\tau)\sum_{k=0}^{N}\Big\{a_{(\mu-k)}(t[\eta]^{-1},\tau[\eta],\eta) - a_k^{\pm}(t[\eta]^{-1},\eta)[\eta]^{\mu-k}(i\tau)^{\mu-k}\Big\}. \tag{2.2.3}$$

For $\tau \neq 0$ a Taylor expansion yields

$$a_{(\mu-k)}(t,\tau,\eta) = \sum_{|\alpha|\le N}\frac{1}{\alpha!}a^{\pm}_{k\alpha}(t)\eta^\alpha(i\tau)^{\mu-k-|\alpha|} + \sum_{|\sigma|=N+1}\frac{\eta^\sigma}{(N+1)!}\int_0^1 (\partial^\sigma_\eta a_{(\mu-k)})(t,\tau,\theta\eta)\,d\theta.$$

Inserting $(t[\eta]^{-1},\tau[\eta],\eta)$, summing from $k=0$ to $k=N$, and rearranging terms gives us

$$\sum_{k=0}^{2N}\Big(\sum_{\substack{j+|\alpha|=k\\ j,|\alpha|\le N}} a^{\pm}_{j\alpha}(t[\eta]^{-1})\eta^\alpha\Big)(i[\eta]\tau)^{\mu-k} + \\ + \sum_{k=0}^{N}\sum_{|\sigma|=N+1}\frac{\eta^\sigma}{(N+1)!}\int_0^1 (\partial^\sigma_\eta a_{(\mu-k)})(t[\eta]^{-1},\tau[\eta],\theta\eta)\,d\theta.$$

Hence, by definition of the $a_k^{\pm}$'s, the function in (2.2.3) equals

$$\sum_{k=0}^{2N}\Big(\sum_{\substack{j+|\alpha|=k\\ j,|\alpha|\le N}} a^{\pm}_{j\alpha}(t[\eta]^{-1})\eta^\alpha\Big)(i[\eta]\tau)^{\mu-k}\chi(\tau)(i\tau)^{\mu-k} + \\ + \sum_{k=0}^{N}\sum_{|\sigma|=N+1}\frac{\eta^\sigma}{(N+1)!}\chi(\tau)\int_0^1 (\partial^\sigma_\eta a_{(\mu-k)})(t[\eta]^{-1},\tau[\eta],\theta\eta)\,d\theta. \tag{2.2.4}$$

The first term in (2.2.4) is easily seen to be an element of $S^\mu(\mathbb{R}^q_\eta, C^\infty(\overline{\mathbb{R}}_+))\otimes S^{\mu-N-1}(\mathbb{R}_\tau) \subset S^\mu(\mathbb{R}^q_\eta, S^{\mu-N-1}(\overline{\mathbb{R}}_+\times\mathbb{R}_\tau))$. For the second term in (2.2.4) observe that (uniformly for t in compact subsets of $\overline{\mathbb{R}}_+$ and $(\tau,\eta)\in\mathbb{R}\times\mathbb{R}^q$)

$$\begin{aligned}\chi(\tau)|\partial^\sigma_\eta a_{(\mu-k)}(t[\eta]^{-1},\tau[\eta],\theta\eta)| &\le C\chi(\tau)\langle\tau[\eta],\theta\eta\rangle^{\mu-k-|\sigma|}\\ &= C\chi(\tau)[\eta]^{\mu-k-|\sigma|}(\tau^2+\langle\theta\eta\rangle^2[\eta]^{-2})^{\frac12(\mu-k-|\sigma|)}.\end{aligned}$$

If $\mu-k-|\sigma|\ge 0$, this can be estimated from above by $C[\eta]^{\mu-k-|\sigma|}\langle\tau\rangle^{\mu-k-|\sigma|}$, since $\langle\theta\eta\rangle^2[\eta]^{-2}$ is uniformly bounded in $\eta\in\mathbb{R}^q$ and $0\le\theta\le 1$. If $\mu-k-|\sigma|<0$, we estimate by

$$C[\eta]^{\mu-k-|\sigma|}\chi(\tau)\tau^{\mu-k-|\sigma|} = C[\eta]^{\mu-k-|\sigma|}\langle\tau\rangle^{\mu-k-|\sigma|}\chi(\tau)\frac{\tau^{\mu-k-|\sigma|}}{\langle\tau\rangle^{\mu-k-|\sigma|}} \le C[\eta]^{\mu-k-|\sigma|}\langle\tau\rangle^{\mu-k-|\sigma|}.$$

Derivatives with respect to t,τ,η can be similarly treated by chain rule, to obtain that the second term in (2.2.4) belongs to $S^{\mu-k}(\mathbb{R}^q_\eta, S^{\mu-N-1}(\overline{\mathbb{R}}_+\times\mathbb{R}_\tau))$. Thus (2.2.2) is proven, and this shows that $\widetilde{a}\in S^\mu(\mathbb{R}^q_\eta, S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}_\tau))$.

We now show that $\widetilde{a}$ is classical in η. First of all, using the expansion of a into homogeneous terms in (τ,η), we can write $\widetilde{a}(t,\tau,\eta) - \sum_{k=0}^{N-1} \widetilde{\chi}(\tau[\eta],\eta) a_{(\mu-k)}(t[\eta]^{-1},\tau[\eta],\eta) = \widetilde{a}_N(t,\tau,\eta)$ where $a_N \in S_{\mathrm{cl}}^{\mu-N}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{(\tau,\eta)})$. Therefore, as we just have shown, $\widetilde{a}_N \in S^{\mu-N}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{\mu-N}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$. Thus we are done if we can verify that

$$\widetilde{\chi}(\tau[\eta],\eta) a_{(\mu-k)}(t[\eta]^{-1},\tau[\eta],\eta) \in S_{\mathrm{cl}}^{\mu-k}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{\mu-k}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau)). \tag{2.2.5}$$

We have $\widetilde{\chi}(\tau[\eta],\eta) a_{(\mu-k)}(t[\eta]^{-1},\tau[\eta],\eta) = \sum_{l=0}^{N-1} \frac{1}{l!}[\eta]^{-l} r_{kl}(t,\tau,\eta) + [\eta]^{-N} r_N(t,\tau,\eta)$ by Taylor expansion, where $r_{kl}(t,\tau,\eta) = t^l \widetilde{\chi}(\tau[\eta],\eta)(\partial_t^l a_{(\mu-k)})(0,\tau[\eta],\eta) \in S^{\mu-k}(\mathbb{R}^q, S_{\mathrm{cl}}^{\mu-k}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$,

$$r_N(t,\tau,\eta) = t^N \widetilde{\chi}(\tau[\eta],\eta) \int_0^1 (\partial_t^N \partial_\eta^l a_{(\mu-k)})(\theta t[\eta]^{-1},\tau[\eta],\eta)\, d\theta \in S^{\mu-k}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{\mu-k}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau)).$$

For sufficiently large $|\eta|$, we have $\widetilde{\chi}(\tau[\eta],\eta) \equiv 1$ and $[\eta] = |\eta|$, thus

$$[\eta]^{-l} r_{kl}(t,\tau,\eta) = |\eta|^{\mu-k-l} t^l (\partial_t^l a_{(\mu-k)})(0,\tau,\eta|\eta|^{-1}) \qquad \text{for all } |\eta| \geq C,$$

for some constant $C > 0$. Therefore, $[\lambda\eta]^{-l} r_{kl}(t,\tau,\lambda\eta) = \lambda^{\mu-k-l}[\eta]^{-l} r_{kl}(t,\tau,\eta)$ for all $|\eta| \geq C$, $\lambda \geq 1$ shows that $[\eta]^{-l} r_{kl}(t,\tau,\eta) \in S^{\mu-k-l}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{\mu-k}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$ is homogeneous of degree $\mu-k-l$ for large $|\eta|$. This yields (2.2.5). Finally, by (2.2.1) and the closed graph theorem, the map $a \mapsto \widetilde{a}$ between the spaces of classical symbols is also continuous. □

Let us finish this subsection by proving a result on asymptotic summation in a certain symbol class, which will be relevant later on.

Lemma 2.2.2 *Let $(\mu_j)_{j\in\mathbb{N}}$ be a decreasing sequence tending to $-\infty$ and such that $\mu_0 - \mu_j \in \mathbb{N}_0$ for all $j \in \mathbb{N}_0$. Further let symbols $h_j \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{\mu_j}(\overline{\mathbb{R}}_+))$ be given. Then there exists an $h \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{\mu_0}(\overline{\mathbb{R}}_+))$ such that for any $N \in \mathbb{N}_0$*

$$h(t,z,\eta) - \sum_{j=0}^{N-1} t^j h_j(t;z,\eta) \in t^N S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{\mu_N}(\overline{\mathbb{R}}_+)) + S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{-\infty}(\overline{\mathbb{R}}_+)).$$

Proof. Set $a_j^k(t,i\tau,\eta) := t^{j-k} h_j(t,i\tau,\eta) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{\mu_j}(\overline{\mathbb{R}}_+ \times \Gamma_0)) = S_{\mathrm{cl}}^{\mu_j}(\Gamma_0, S_{\mathrm{cl}}^{\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_\eta))$ for $j \geq k$. The latter identification is justified by virtue of (1.2.6). For fixed k we can construct a sequence $0 \leq d_j \to +\infty$ such that $a_k := \sum_{j\geq k} \chi(\frac{\tau}{d_j}) a_j^k$ converges (absolutely) in $S^{\mu_k}(\Gamma_0, S_{\mathrm{cl}}^{\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_\eta))$. Here, χ is a zero excision function. By a standard diagonal sequence-argument, we can assume that this convergence holds simultaneously for all $k \in \mathbb{N}_0$. We then have for any $N \in \mathbb{N}_0$

$$a_0(t,i\tau,\eta) - \sum_{j=0}^{N-1} t^j a_j^j(t,i\tau,\eta) = t^N a_N(t,i\tau,\eta) + \sum_{j=0}^{N-1} t^j (1-\chi)\left(\frac{\tau}{d_j}\right) a_j^j(t,i\tau,\eta).$$

The second term on the right-hand side clearly belongs to $S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_0))$. To obtain the corresponding statement for the h_j's, let us recall the existence of the so-called kernel cut-off operator. This is a $C^\infty(\overline{\mathbb{R}}_+)$-linear, continuous map $K : \bigcup_{\nu\in\mathbb{R}} S_{\mathrm{cl}}^{\nu}(\overline{\mathbb{R}}_+ \times \Gamma_0) \to \bigcup_{\nu\in\mathbb{R}} M_{\mathcal{O}}^{\nu}(\overline{\mathbb{R}}_+)$ that restricts to maps $K : S_{\mathrm{cl}}^{\nu}(\overline{\mathbb{R}}_+ \times \Gamma_0) \to M_{\mathcal{O}}^{\nu}(\overline{\mathbb{R}}_+)$ for any ν and has the property that $a - K(a) \in S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_0)$ and $a - K(a) \in M_{\mathcal{O}}^{-\infty}(\overline{\mathbb{R}}_+)$ if $a \in M_{\mathcal{O}}^{\nu}(\overline{\mathbb{R}}_+)$. This K clearly induces mappings $S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, S_{\mathrm{cl}}^{\nu}(\overline{\mathbb{R}}_+ \times \Gamma_0)) \to S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{\nu}(\overline{\mathbb{R}}_+))$, which we again denote by K. Then the assertion of the lemma follows if we define $h := K(a_0) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{\mu}(\overline{\mathbb{R}}_+))$, since $h_j - K(a_j^j) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{-\infty}(\overline{\mathbb{R}}_+))$. □

For purposes below we consider the particular case of Lemma 2.2.2 when $h_j(t,z,\eta) = a_j(t,\eta)\widetilde{h}_j(z)$ with $a_j \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_\eta)$ and $\widetilde{h}_j \in M^{\mu_j}_{\mathcal{O}}$.

With $h^0_j(i\tau) := \widetilde{h}_j(i\tau)$ we form $h(t,z,\eta) = \sum_{j=0}^{\infty} t^j a_j(t,\eta) K(\chi_j h^0_j)(z)$ for $\chi_j(i\tau) = \chi(\frac{\tau}{d_j})$ for an appropriate sequence $d_j \to +\infty$; then, modulo $t^N S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{\mu_N}_{\mathcal{O}}(\overline{\mathbb{R}}_+))$,

$$h(t,z,\eta) - \sum_{j=0}^{N-1} t^j h_j(t,z,\eta) \equiv \sum_{j=0}^{N-1} t^j a_j(t,\eta)[K(h^0_j)(z) - \widetilde{h}_j(z) + K((1-\chi_j)h^0_j)(z)].$$

2.3 The case of weight and order zero

In this subsection we shall investigate the operator family $\omega_1(t[\eta])\,\mathrm{op}^+(a)(\eta)\omega_2(t[\eta])$ for a symbol $a \in S^0(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q})$ of order 0. We consider this case separately, since it contains all basic ideas relevant for the case of arbitrary order and weight, but is a little more transparent and concrete. We shall show, in particular, that

$$\omega_1(t[\eta])\,\mathrm{op}^+(a)(\eta)\omega_2(t[\eta]) \in R^0(\mathbb{R}^q_\eta, (0,0,\infty)),$$

i.e., in (2.0.4) we need no correction term g, at least for $\gamma = 0$. To begin with, let us define the following Mellin symbols

$$\begin{gathered} g^{\pm}(z) = (1 - e^{\mp 2\pi i z})^{-1} \in M^0_{\{(j,0):\ j \in \mathbb{Z}\}}, \\ f_0 \equiv 1, \qquad f_k(z) = \prod_{j=1}^{k} (j-z)^{-k} \in M^{-k}_{\{(j,0):\ 1 \le j \le k\}} \qquad \text{for } k \in \mathbb{N}. \end{gathered} \tag{2.3.1}$$

We choose some splittings $g^{\pm} = g^{\pm}_0 + \widetilde{g}^{\pm}$, $f_k = f_{k,0} + \widetilde{f}_k$ with holomorphic $g^{\pm}_0 \in M^0_{\mathcal{O}}$, $f_{k,0} \in M^{-k}_{\mathcal{O}}$, and meromorphic smoothing Mellin symbols $\widetilde{g}^{\pm}$ and $\widetilde{f}_k$.

For $\widetilde{a} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$ we define

$$\sigma^{-k}_M(\widetilde{a})(t,z,\eta) = \{\widetilde{a}^+_k(t,\eta)g^+(z) + \widetilde{a}^-_k(t,\eta)g^-(z)\}f_k(z),$$

where the coefficients $\widetilde{a}^{\pm}_k$ are defined by the expression $\widetilde{a} \sim \sum_{k=0}^{\infty} \widetilde{a}^{\pm}_k(t,\eta)(i\tau)^{-k}$ as $\tau \to \pm\infty$, i.e., $\widetilde{a}^{\pm}_k \in S^0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_\eta)$ and for any $N \in \mathbb{N}$

$$\widetilde{a}(t,\tau,\eta) - \chi(\tau)\sum_{k=0}^{N-1}(\widetilde{a}^+_k(t,\eta)\theta^+(\tau) + \widetilde{a}^-_k(t,\eta)\theta^-(\tau))(i\tau)^{-k} \in S^0(\mathbb{R}^q_\eta, S^{-N}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau)),$$

for any 0-excision function $\chi(\tau)$. According to the above splittings of $g^{\pm}$ and f_k we obtain a splitting of $\sigma^{-k}_M(\widetilde{a})$ into

$$\begin{aligned} \sigma^{-k}_{M,0}(\widetilde{a})(t,z,\eta) &= (\widetilde{a}^+_k(t,\eta)g^+_0(z) + \widetilde{a}^-_k(t,\eta)g^-_0(z))f_{k,0}(z) \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{-k}_{\mathcal{O}}(\overline{\mathbb{R}}_+)), \\ \widetilde{\sigma}^{-k}_M(\widetilde{a})(t,z,\eta) &= (\sigma^{-k}_M(\widetilde{a}) - \sigma^{-k}_{M,0}(\widetilde{a}))(t,z,\eta) \\ &= \widetilde{a}^+_k(t,\eta)l^+_k(z) + \widetilde{a}^-_k(t,\eta)l^-_k(z) \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{-\infty}_{R_k}(\overline{\mathbb{R}}_+)), \end{aligned}$$

with meromorphic Mellin symbols,

$$l^{\pm}_k(z) = (\widetilde{g}^{\pm} f_{k,0} + g^{\pm}\widetilde{f}_k)(z) \in M^{-\infty}_{R_k} \tag{2.3.2}$$

and asymptotic types $R_k = \{(j,0):\ j \in \mathbb{Z} \setminus \{1,\dots,k\}\} \cup \{(j,1):\ j \in \mathbb{Z}, 1 \le j \le k\}$. Moreover, let $G^{(m)}_w(\mathbb{R}_+)$, $m \in \mathbb{N}$, be the space of all "weak Green operators" $R \in \mathcal{L}(L^2(\mathbb{R}_+))$, which by definition satisfy

$$R : L^2(\mathbb{R}_+) \to \widetilde{\mathcal{S}}^m(\overline{\mathbb{R}}_+), \qquad R^* : L^2(\mathbb{R}_+) \to \widetilde{\mathcal{S}}^m(\overline{\mathbb{R}}_+) + \mathcal{S}^0_{\{(-j,1):\ j \in \mathbb{N}_0\}}(\mathbb{R}_+), \tag{2.3.3}$$

where $\widetilde{\mathcal{S}}^m(\overline{\mathbb{R}}_+)$ consists of all functions $u \in C^m(\overline{\mathbb{R}}_+)$ such that $|d_t^k u(t)|\langle t\rangle^m < \infty$ uniformly in $t \in \overline{\mathbb{R}}_+$. $G_w^{(m)}(\mathbb{R}_+)$ is a Fréchet space in an obvious way.

Note that $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}_T^0(\mathbb{R}_+) = \varprojlim_{m\in\mathbb{N}} \widetilde{\mathcal{S}}^m(\overline{\mathbb{R}}_+)$, $C_G(\mathbb{R}_+, (0,0,\infty))_{P,T} = \varprojlim_{m\in\mathbb{N}} G_w^{(m)}(\overline{\mathbb{R}}_+)$ for $P = \{(-j,1) : j \in \mathbb{N}_0\} \in \mathrm{As}(0,\infty)$, cf. Definition 1.1.5.

Proposition 2.3.1 *Let $\widetilde{a} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$. Then there exist $\widetilde{d}_k(\widetilde{a}) \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{-\infty}_{R_k})$ such that for any $N \in \mathbb{N}_0$*

$$\omega_1 \operatorname{op}^+(\widetilde{a})(\eta)\omega_2 = \omega_1 \sum_{k=0}^{N} t^k \operatorname{op}_M^0(\sigma_{M,0}^{-k}(\widetilde{a}))(\eta)\omega_2 + \omega_1 \sum_{k=0}^{N} t^k \operatorname{op}_M^0(\widetilde{d}_k(\widetilde{a}))(\eta)\omega_2 + \widetilde{R}_N(\widetilde{a})(\eta)$$

with a remainder $\widetilde{R}_N \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, G_w^m(\mathbb{R}_+))$ and $m = m(N) \to \infty$ as $N \to \infty$.

Proof. First assume that $\widetilde{a} \in S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+, \mathbb{R}_\tau)$ is independent of η. Then the result is true by (the proof of) Theorem 2.1.25 in [27], if we set $\widetilde{d}_k(\widetilde{a}) = \sum_{p+j=k} d_{pj}(\widetilde{a})$, where the $d_{kj}(\widetilde{a})(z)$ are the Taylor coefficients of $\widetilde{\sigma}_M^{-k}(\widetilde{a})(t,z) \sim \sum_{j=0}^\infty t^j d_{kj}(\widetilde{a})(z)$. Certainly, the map $\widetilde{a} \mapsto \widetilde{R}_N(\widetilde{a}) : S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau) \to \mathcal{L}(L^2(\mathbb{R}_+))$ is linear and continuous. From the closed graph theorem we see that $\widetilde{a} \mapsto \widetilde{R}_N(\widetilde{a})$ defines a continuous map $S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau) \to G_w^{(m)}(\mathbb{R}_+)$. Hence the result follows, if we set $\widetilde{d}_k(\widetilde{a})(z,\eta) = \widetilde{d}_k(\widetilde{a}(\eta))(z)$. $\square$

Theorem 2.3.2 *Let $\widetilde{a} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$. Then there exists an $\widetilde{h} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+))$ and $d_k \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{-\infty}_{R_k})$ such that*

$$\omega_1 \operatorname{op}^+(\widetilde{a})(\eta)\omega_2 = \omega_1 \operatorname{op}_M^0(\widetilde{h})(\eta)\omega_2 + \omega_1 \sum_{k=0}^{N} t^k \operatorname{op}_M^0(d_k)(\eta)\omega_2 + R_N(\eta)$$

with R_N having a kernel in $S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{S}^0_{T(N)}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^0_0(\mathbb{R}_+) \cap \mathcal{S}^0_0(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^0_{P(N)}(\mathbb{R}_+))$, with asymptotic types $P(N), T(N) \in \mathrm{As}(0, N+1)$ given by

$$P(N) = \{(-j,1) : 0 \le j \le N\}, \qquad T(N) = \{(-j,0) : 0 \le j \le N\} \in \mathrm{As}(0, N+1).$$

Proof. We apply the above Lemma 2.2.2 to the symbols $h_k = \sigma_{M,0}^{-k}(\widetilde{a})$ to obtain a symbol $\widetilde{h}$ such that $\widetilde{h} - \sum_{j=0}^N t^j h_k = t^{N+1} r_{N+1} + r$, where $r_{N+1} \in S^0(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{-N-1}(\overline{\mathbb{R}}_+))$ and $r \in S^0(\mathbb{R}^q_\eta, M_{\mathcal{O}}^{-\infty}(\overline{\mathbb{R}}_+))$. Using the expansion $r(t,z,\eta) = \sum_{j=0}^N t^j r_{(j)}(z,\eta) + t^{N+1}\widetilde{r}_{N+1}(t,z,\eta)$, we can set $d_k = \widetilde{d}_k(\widetilde{a}) + r_{(k)}$, $\quad R_N = \widetilde{R}_N(\widetilde{a}) + \omega_1 t^{N+1} \operatorname{op}_M^0(r_{N+1} + \widetilde{r}_{N+1})$, where $\widetilde{d}_k$ and $\widetilde{R}_N$ are those from the previous Proposition 2.3.1. We then have $R_N \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, G_w^{(m)}(\mathbb{R}_+))$. It remains to show the kernel structure of R_N. Write

$$R_N(\eta) = \omega_1 \sum_{k=N+1}^{M} t^k \operatorname{op}_M^0(d_k)(\eta)\omega_2 + R_M(\eta) =: R_{N,M}(\eta) + R_M(\eta)$$

for any $M > N$. However, by the mapping properties of smoothing Mellin symbols, we know that $R_{N,M} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{L}(L^2(\mathbb{R}_+), \mathcal{S}^0_{N+1}(\mathbb{R}_+)))$ and $R^*_{N,M} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{L}(L^2(\mathbb{R}_+), \mathcal{S}^0_{P(N)}(\mathbb{R}_+)))$. Since we can choose M arbitrarily large, it follows that $R_N \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{L}(L^2(\mathbb{R}_+), \mathcal{S}_{T(N)}(\mathbb{R}_+)))$ and $R^*_N \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{L}(L^2(\mathbb{R}_+), \mathcal{S}_{P(N)}(\mathbb{R}_+)))$. Hence the result holds in view of the above kernel characterization, Proposition 1.2.3. $\square$

As inspiration of the latter proof shows that

$$\widetilde{h}(t,z,\eta) = \sum_{j=0}^{\infty} t^j \widetilde{a}_j^{+}(t,\eta) h_j^{+}(z) + \sum_{j=0}^{\infty} t^j \widetilde{a}_j^{-}(t,\eta) h_j^{-}(z) \tag{2.3.4}$$

with $h_j^{\pm} \in M_{\mathcal{O}}^{-j}$, and for $l_j^{\pm}$ from (2.3.2)

$$d_k(z,\eta) = \sum_{p+j=k} \frac{1}{j!} (\partial_t^p \widetilde{a}_j^{+})(0,\eta) l_j^{+}(z) + \sum_{p+j=k} \frac{1}{j!} (\partial_t^p \widetilde{a}_j^{-})(0,\eta) l_j^{-}(z), \tag{2.3.5}$$

provided $\widetilde{a} \sim \sum_{k=0}^{\infty} \widetilde{a}_k^{\pm}(t,\eta)(i\tau)^{-k}$ for $\tau \to \pm\infty$.

Let us now look at the particular case, when $\widetilde{a}(t,\tau,\eta) = a(t[\eta]^{-1}, \tau[\eta], \eta)$ for some $a \in S_{\mathrm{cl}}^0(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{(\tau,\eta)})$. By Proposition 2.2.1 we have $\widetilde{a} \sim \sum_{k=0}^{\infty} \widetilde{a}_k^{\pm}(t,\eta)(i\tau)^{-k}$ with $\widetilde{a}_k^{\pm}(t,\eta) = a_k^{\pm}(t[\eta]^{-1},\eta)[\eta]^{-k}$ and $a_k^{\pm}(t,\eta) = \sum_{p+|\alpha|=k} \frac{1}{\alpha!} a_{p\alpha}^{\pm}(t)\eta^{\alpha}$.

It follows that $\partial_t^p \widetilde{a}_j^{\pm}(0,\eta) = [\eta]^{-k} \sum_{n+|\alpha|=j} \frac{1}{\alpha!} (\partial_t^p a_{n\alpha}^{\pm})(0)\eta^{\alpha}$, $j+p=k$. Inserting this into (2.3.5) we obtain $d_k(z,\eta) = [\eta]^{-k} \sum_{|\alpha|\le k} c_{\alpha k}(z)\eta^{\alpha}$ where

$$c_{\alpha k}(z) = \frac{1}{\alpha!} \sum_{j=|\alpha|}^{k} \frac{1}{j!} \{ (\partial_t^{k-j} a_{j-|\alpha|,\alpha}^{+})(0) l_j^{+}(z) + (\partial_t^{k-j} a_{j-|\alpha|,\alpha}^{-})(0) l_j^{-}(z) \}.$$

Hence, by Theorem 2.3.2,

$$\begin{aligned} &\omega_1(t[\eta]) \operatorname{op}^{+}(a)(\eta)\omega_2(t[\eta]) = \kappa(\eta)\{\omega_1 \operatorname{op}^{+}(\widetilde{a})(\eta)\omega_2\}\kappa^{-1}(\eta) = \\ &\quad \omega_1(t[\eta]) \operatorname{op}_M^0(h)\omega_2(t[\eta]) + \omega_1(t[\eta]) \sum_{k=0}^{N} t^k \Big(\sum_{|\alpha|\le k} \operatorname{op}_M^0(c_{\alpha k})\eta^{\alpha} \Big) \omega_2(t[\eta]) + g_N(\eta), \end{aligned} \tag{2.3.6}$$

where

$$h(t,z,\eta) = \widetilde{h}(t[\eta],z,\eta) \in C^{\infty}(\mathbb{R}^q_{\eta}, M_{\mathcal{O}}^0(\overline{\mathbb{R}}_+)),$$

and $g_N(\eta) = \kappa(\eta) R_N(\eta)\kappa^{-1}(\eta) \in R_G^0(\mathbb{R}^q, (0,0,N+1))_{P(N),T(N)}$ which is a Green symbol by Proposition 1.2.3. Identity (2.3.4) yields

$$\begin{aligned} h(t,z,\eta) &= \sum_{j=0}^{\infty} t^j \Big(\sum_{|\alpha|\le j} b_{j\alpha}(t,z)\eta^{\alpha} \Big) = \sum_{j=0}^{\infty} \big(\sum_{|\alpha|\le j} t^{j-|\alpha|} b_{j\alpha}(t,z)(t\eta)^{\alpha} \big), \\ b_{j\alpha}(t,z) &= \frac{1}{\alpha!} (a_{j-|\alpha|,\alpha}^{+}(t) h_j^{+}(z) + a_{j-|\alpha|,\alpha}^{-}(t) h_j^{-}(z)). \end{aligned}$$

In particular, we see that $h(t,z,\eta) = H(t,z,t\eta)$ with $H(t,z,\eta) \in C^{\infty}(\mathbb{R}^q_{\eta}, M_{\mathcal{O}}^0(\overline{\mathbb{R}}_+))$. (For this one has to note that $h_j^{\pm}(z) = K(\chi_j \widetilde{f}_j^{\pm})(z)$ for $\chi_j = \chi(\tau/d_j)$ with $d_j \to +\infty$ and $\widetilde{f}_j^{\pm} \in S_{\mathrm{cl}}^{-j}(\Gamma_0)$ by construction of $\widetilde{h}$. By eventually enlarging the d_j's, without loss of generality we may assume that the series $\sum_j (\sum_{|\alpha|\le j} t^{j-|\alpha|} b_{j\alpha}(t,z)\eta^{\alpha})$ converges absolutely in $C^{\infty}(\mathbb{R}^q_{\eta}, M_{\mathcal{O}}^0(\overline{\mathbb{R}}_+))$.)

On the other hand, by Mellin quantization, there exists an $F(t,\tau,\eta) \in M_{\mathcal{O}}^0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q)$ such that $\operatorname{op}_M^0(f)(\eta) = \operatorname{op}^{+}(a)(\eta) \in L^{-\infty}(\mathbb{R}_+; \mathbb{R}^q_{\eta})$ for $f(t,z,\eta) = F(t,z,t\eta)$. Note that F is parameter-dependent on η, while H is not. Therefore,

$$\omega_1(t[\eta])\{\operatorname{op}_M^0(f)(\eta) - \operatorname{op}_M^0(h)(\eta)\}\omega_2(t[\eta]) \in L^{-\infty}(\mathbb{R}_+; \mathbb{R}^q_{\eta}).$$

Since this is true for any choice of $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbb{R}}_+)$, we obtain

$$\mathrm{op}_M^0(f)(\eta) - \mathrm{op}_M^0(h)(\eta) \in C^\infty(\mathbb{R}^q_\eta, L^{-\infty}(\mathbb{R}_+)).$$

This yields $f-h \in C^\infty(\mathbb{R}^q_\eta, M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+)) \cap C^\infty(\mathbb{R}^q_\eta, S^{-\infty}(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}})) = C^\infty(\mathbb{R}^q_\eta, M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+))$. The latter identity holds, since $S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}}) \cap S^{-\infty}(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}}) = S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}})$ and $M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+) \cap S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}}) = M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$. Then

$$(F-H)(t,z,\eta) \in C^\infty(\mathbb{R}^q_\eta, M^{-\infty}_{\mathcal{O}}(\mathbb{R}_+)) \cap C^\infty(\mathbb{R}^q_\eta, M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+)) = C^\infty(\mathbb{R}^q_\eta, M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+)).$$

Due to the following Lemma 2.3.3, a replacement of the symbol h in (2.3.6) thus generates only remainders of "allowed" type.

Lemma 2.3.3 *Let $\widetilde{p}(t,z,\eta) \in C^\infty(\mathbb{R}^q_\eta, M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+))$ and $p(t,z,\eta) := \widetilde{p}(t,z,t\eta)$. Then*

$$\omega_1(t[\eta])\,\mathrm{op}_M^0(p)(\eta)\omega_2(t[\eta]) = \omega_1(t[\eta]) \sum_{k=0}^{N-1} t^k \Big(\sum_{|\alpha|=0}^{k} \mathrm{op}_M^0(p_{\alpha k})\eta^\alpha \Big) \omega_2(t[\eta]) + r_N(\eta)$$

with $p_{\alpha k}(z) = \frac{1}{k!}(\partial_t^{k-|\alpha|}\partial_\eta^\alpha \widetilde{p})(0,z,0) \in M^{-\infty}_{\mathcal{O}}$ and $r_N \in R^0_G(\mathbb{R}^q_\eta, (0,0,N))_{\mathcal{O},\mathcal{O}}$.

Proof. By a Taylor expansion of p in t at $t=0$, it is immediately seen that $r_N(\eta) = \omega_1(t[\eta]) \sum_{|\alpha|=0}^N t^N \eta^\alpha\, \mathrm{op}_M^0(p_\alpha^{(N)})(\eta)\omega_2(t[\eta])$ with $p_\alpha^{(N)}(t,z,\eta) = \widetilde{p}_\alpha^{(N)}(t,z,t\eta)$ and

$$\widetilde{p}_\alpha^{(N)}(t,z,\eta) = \int_0^1 (\partial_t^{N-|\alpha|}\partial_\eta^\alpha p)(\theta t, z, \theta\eta)\, d\theta \in C^\infty(\mathbb{R}^q_\eta, M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+)).$$

It is now routine to check that r_N is indeed a Green symbol. The presence of the factor t^N shows that r_N and r_N^* generate "flatness" $\mathcal{O} \in \mathrm{As}(0,N)$, a Taylor expansion of $\widetilde{p}_{\alpha,(N)}$ in t at $t=0$ shows that r_N, r_N^* are classical symbols. We shall not go into further details here. □

Summing up, we now have proved the following theorem:

Theorem 2.3.4 *Let $a \in S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ and $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbb{R}}_+)$ be arbitrary cut-off functions. If $\widetilde{h} \in M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_\eta)$ is a Mellin quantization of a and $h(t,z,\eta) = \widetilde{h}(t,z,t\eta)$, then for any $N \in \mathbb{N}_0$ we have*

$$\omega_1(t[\eta])\,\mathrm{op}^+(a)(\eta)\omega_2(t[\eta]) = \omega_1(t[\eta])\,\mathrm{op}_M^0(h)(\eta)\omega_2(t[\eta]) + $$
$$+ \omega_1(t[\eta]) \sum_{k=0}^{N} t^k \Big(\sum_{|\alpha|\le k} \mathrm{op}_M^0(h_{\alpha k})\eta^\alpha \Big)\omega_2(t[\eta]) + g_N(\eta)$$

for certain meromorphic Mellin symbols

$$h_{\alpha k} \in M^{-\infty}_{R_k}, \qquad R_k = \{(j,0):\ j \in \mathbb{Z} \setminus \{1,\dots,k\}\} \cup \{(j,1):\ j = 1,\dots,k\},$$

and an element $g_N \in R^0_G(R^q_\eta, (0,0,N+1))_{P(N),T(N)}$ with asymptotic types $P(N), T(N) \in$ $\mathrm{As}(0,N+1)$ *given by $P(N) = \{(-j,1):\ j=0,\dots,N\}$, $T(N) = \{(-j,0):\ j=0,\dots,N\}$.*

If $a \sim \sum_{k=0}^\infty a_{(-k)}$ in $S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q})$, the conormal symbols are

$$\sigma_M^{-j}(\mathrm{op}^+(a))(z,\eta) = \sum_{p+k=j} (a^+_{pk}(\eta)g^+(z) + a^-_{pk}(\eta)g^-(z)) f_p(z)$$

with polynomials $a^\pm_{pk}(\eta) = \frac{(-i)^p}{k!} \sum_{l+|\alpha|=p} \frac{1}{\alpha!}(\partial_\eta^\alpha \partial_t^k a_{(-l)})(0,\pm1,0)\eta^\alpha$ and $g^\pm, f_p$ as in (2.3.1)*. For the principal conormal symbol this reduces to*

$$\sigma_M^0(\mathrm{op}^+(a))(z) = a_{(0)}(0,1,0)g^+(z) + a_{(0)}(0,-1,0)g^-(z).$$

2.4 The case of large weight and arbitrary order

We now turn to the verification of (2.0.4) for the case of large γ compared with the order μ. To be more precise, we shall assume

$$\gamma \geq \mu_+, \qquad \tfrac{1}{2} + \mu - \gamma \notin \mathbb{Z}. \tag{2.4.1}$$

Here, $\mu_+ := \max(0, \mu)$ and $\mu_- := \min(0, \mu)$. For $\widetilde{a} \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$ with $\widetilde{a} \sim \sum_{k=0}^\infty \widetilde{a}^\pm_k(t,\eta)(i\tau)^{\mu-k}$ for $\tau \to \pm\infty$ we set

$$\sigma^{\mu-k}_M(\widetilde{a})(t,\eta,z) = \{\widetilde{a}^+_k(t,\eta)g^+(z+\mu) + \widetilde{a}^-_k(t,\eta)g^-(z+\mu)\}f_{k-\mu}(z)$$

where $g^\pm$ is as in (2.3.1) and $f_p(z) = \frac{\Gamma(1-z)}{\Gamma(1-z+p)}$ with Euler's Γ-function. Note that if $\mu \notin \mathbb{Z}$, then $f_{k-\mu}$ has simple poles in $1, \dots, k-\mu$ provided $k-\mu \geq 1$, otherwise $f_{k-\mu}$ is holomorphic. This yields $\sigma^{\mu-k}_M(\widetilde{a}) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{\mu-k}_{R_k}(\overline{\mathbb{R}}_+))$ with an asymptotic type

$$R_k = \begin{cases} \{(j,0): \ j \in \mathbb{Z}\} & \text{if } \mu \in \mathbb{Z} \text{ and } k \leq \mu, \\ \{(j,0),(l,1): \ l \in \{1,\dots,k-\mu\}, \ j \in \mathbb{Z} \setminus \{1,\dots,k-\mu\} & \text{if } \mu \in \mathbb{Z} \text{ and } k > \mu, \\ \{(j,0),(l-\mu,0): \ j \in \mathbb{N}, \ l \in \mathbb{Z}\} & \text{if } \mu \notin \mathbb{Z}. \end{cases} \tag{2.4.2}$$

Therefore, $\pi_{\mathbb{C}} R_k = \mathbb{Z}$ if $\mu \in \mathbb{Z}$ and $\pi_{\mathbb{C}} R_k = \mathbb{N} \cup (\mathbb{Z} - \mu)$ if $\mu \notin \mathbb{Z}$. In any case, condition (2.4.1) ensures that $\Gamma_{\frac{1}{2}-\gamma} \cap \pi_{\mathbb{C}} R_k = \emptyset$.

As we did in the zero order case we split $\sigma^{\mu-k}_M(\widetilde{a})$ into a holomorphic part of order μ and a smoothing meromorphic part,

$$\sigma^{\mu-k}_M(\widetilde{a})(t,\eta,z) = \sigma^{\mu-k}_{M,0}(\widetilde{a})(t,\eta,z) + \{\widetilde{a}^+_k(t,\eta)l^+_{k-\mu}(z) + \widetilde{a}^-_k(t,\eta)l^-_{k-\mu}(z)\},$$

where

$$l^\pm_{k-\mu} = \widetilde{g}^\pm(z+\mu)f_{k-\mu}(z) + g^\pm(z+\mu)\widetilde{f}_{k-\mu}(z) \in M^{-\infty}_{R_k}. \tag{2.4.3}$$

We introduce asymptotic types $Q = Q(\gamma) \in \mathrm{As}(-\gamma,\infty)$, $P = P(\gamma) \in \mathrm{As}(0,\infty)$ by

$$Q = \{(k,1): \ k \in \mathbb{Z}, \ k < \tfrac{1}{2} + \gamma\}, \qquad P = \{(j,0),(k,1): \ j,k \in \mathbb{Z}, \ k < \tfrac{1}{2} + \mu - \gamma < j < \tfrac{1}{2}\} \tag{2.4.4}$$

if $\mu \in \mathbb{Z}$, and if $\mu \notin \mathbb{Z}$,

$$\begin{aligned} Q &= \{(k,0),(j+\mu,0): \ j,k \in \mathbb{Z}, \ k, j+\mu < \tfrac{1}{2} + \gamma\}, \\ P &= \{(k,0),(j+\mu,0): \ j,k \in \mathbb{Z}, \ k < \tfrac{1}{2}, \ j < \tfrac{1}{2} - \gamma\}. \end{aligned} \tag{2.4.5}$$

By $G^{(m)}_w(\mathbb{R}_+,\gamma)$ we denote the Fréchet space of all operators $R \in \mathcal{L}(\mathcal{K}^{0,\gamma}(\mathbb{R}_+), L^2(\mathbb{R}_+))$ with the property $R: \mathcal{K}^{0,\gamma}(\mathbb{R}_+) \to \widetilde{S}^m(\overline{\mathbb{R}}_+)$, $R^*: L^2(\mathbb{R}_+) \to \widetilde{S}^m(\overline{\mathbb{R}}_+) + \mathcal{S}^{-\gamma}_Q(\mathbb{R}_+)$. Using [27, Theorems 2.1.44 and 2.1.40] we have the following proposition:

Proposition 2.4.1 *Let $\widetilde{a} \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$ and ω_1, ω_2 be cut-off functions. Then, as an operator-family $C^\infty_0(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+)$, and for any $N \in \mathbb{N}_0$, we have*

$$\omega_1 \operatorname{op}^+(\widetilde{a})(\eta)\omega_2 = \omega_1 t^{-\mu} \sum_{k=0}^N t^k \operatorname{op}^\gamma_M(\sigma^{\mu-k}_M(\widetilde{a}))(\eta)\omega_2 + \widetilde{R}_{N,\gamma}(\eta)$$

with a remainder $\widetilde{R}_{N,\gamma} = \widetilde{R}_{N,\gamma}(\widetilde{a}) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta, G^{(m)}_w(\mathbb{R}_+,\gamma))$ and $m = m(N) \to \infty$ for $N \to \infty$.

Proceeding as in the proofs of Proposition 2.3.1 and Theorem 2.3.2 we get:

Proposition 2.4.2 *Let* $\widetilde{a} \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_\tau))$. *Then there exists an* $\widetilde{h} \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q, M^{\mu}_{\mathcal{O}}(\overline{\mathbb{R}}_+))$ *and* $d_k \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, M^{-\infty}_{R_k})$ *such that for any* $N \in \mathbb{N}_0$

$$\omega_1 \operatorname{op}^+(\widetilde{a})(\eta)\omega_2 = \omega_1 t^{-\mu} \operatorname{op}^{\gamma}_M(\widetilde{h})(\eta)\omega_2 + \omega_1 \sum_{k=0}^{N} t^{-\mu+k} \operatorname{op}^{\gamma}_M(d_k)(\eta)\omega_2 + R_{N,\gamma}(\eta)$$

with $R_{N,\gamma}$ *having a kernel in* $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{S}^0_{P(N)}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_0(\mathbb{R}_+) \cap \mathcal{S}^0_0(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{Q(N)}(\mathbb{R}_+))$, *where, if* Q *and* P *are as introduced above in* (2.4.4), (2.4.5),

$$\begin{aligned} P(N) &= \{(k,m) \in P :\ \tfrac{1}{2} - N - 1 < k < \tfrac{1}{2}\} \in \mathrm{As}(0, N+1), \\ Q(N) &= \{(k,m) \in Q :\ \tfrac{1}{2} + \gamma - N - 1 < k < \tfrac{1}{2} + \gamma\} \in \mathrm{As}(-\gamma, N+1). \end{aligned} \tag{2.4.6}$$

Again, we obtain formulas both for $\widetilde{h}$ and d_k, namely

$$\begin{aligned} \widetilde{h}(t,z,\eta) &= \sum_{j=0}^{\infty} t^j \widetilde{a}^+_j(t,\eta) h^+_{\mu-j}(z) + \sum_{j=0}^{\infty} t^j \widetilde{a}^-_j(t,\eta) h^-_{\mu-j}(z), \\ d_k(z,\eta) &= \sum_{p+j=k} \frac{1}{j!}\{(\partial^p_t \widetilde{a}^+_j)(0,\eta) l^+_{\mu-j}(z) + (\partial^p_t \widetilde{a}^-_j)(0,\eta) l^-_{\mu-j}(z)\} \end{aligned}$$

for certain $h^{\pm}_{\mu-j} \in M^{\mu-j}_{\mathcal{O}}$ and $l^{\pm}_{\mu-j}$ from (2.4.3). Similarly to the zero order case we finally obtain the following theorem.

Theorem 2.4.3 *Let* $\gamma, \mu \in \mathbb{R}$ *with* $\gamma \geq \mu_+$ *and* $\frac{1}{2} + \mu - \gamma \notin \mathbb{Z}$. *Let* $a \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tau,\eta})$ *and* ω_1, ω_2 *be arbitrary cut-off functions. If* $\widetilde{h} \in M^{\mu}_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_\eta)$ *is a Mellin quantization of* a *and* $h(t,z,\eta) = \widetilde{h}(t,z,t\eta)$, *then for any* $N \in \mathbb{N}_0$

$$\begin{aligned} &\omega_1(t[\eta]) \operatorname{op}^+(a)(\eta)\omega_2(t[\eta]) = \\ &\omega_1(t[\eta]) t^{-\mu} \operatorname{op}^{\gamma}_M(h)(\eta)\omega_2(t[\eta]) + \omega_1(t[\eta]) \sum_{k=0}^{N} t^{-\mu+k} \Big(\sum_{|\alpha|\leq k} \operatorname{op}^{\gamma}_M(h_{\alpha k})\eta^\alpha\Big)\omega_2(t[\eta]) + g_N(\eta) \end{aligned}$$

for certain meromorphic Mellin symbols $h_{\alpha k} \in M^{-\infty}_{R_k}$, R_k *as in* (2.4.2), *and a Green symbol* $g_N(\eta) = g_N(\eta)(\gamma) \in R^{\mu}_G(\mathbb{R}^q_\eta, (\gamma, 0, N+1))_{Q(N),P(N)}$ *with* $Q(N)$, $P(N)$ *from* (2.4.6).

2.5 The general case

Lemma 2.5.1 *Let* $g \in R^{\mu}_G(\mathbb{R}^q, (\gamma, \theta, \gamma', \theta'))$ *have a kernel* $k \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q, \mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^{-\gamma}_{\overline{Q}}(\mathbb{R}_+))$. *Then there exists a* $g_0 \in R^{\mu}_G(\mathbb{R}^q, (\gamma, \theta - \varepsilon; \gamma', \theta' - \varepsilon'))$ *such that*

$$g - g_0 \in R^{\mu}_G(\mathbb{R}^q, (\gamma - \varepsilon, \theta - \varepsilon; \gamma' + \varepsilon', \theta' - \varepsilon')),$$

and $g_0(\eta)$ *is a finite rank operator for any* $\eta \in \mathbb{R}^q$.

Proof. If we define $Q'(\varepsilon') := \{(q', l') \in Q :\ \operatorname{Re} q' < \frac{1}{2} - \gamma' - \varepsilon'\} \in \mathrm{As}(\gamma' + \varepsilon', \theta' - \varepsilon')$, $Q'_0(\varepsilon') := Q' \setminus Q'(\varepsilon') \in \mathrm{As}(\gamma', \infty)$, we have a direct decomposition $\mathcal{S}^{\gamma'}_{Q'}(\mathbb{R}_+) = \mathcal{S}^{\gamma'+\varepsilon'}_{Q'(\varepsilon')}(\mathbb{R}_+) \oplus \mathcal{E}_{Q'_0(\varepsilon')}(\mathbb{R}_+)$ with the finite-dimensional space $\mathcal{E}_{Q'_0(\varepsilon')}(\mathbb{R}_+) \subset \mathcal{S}^{\gamma'}_{Q'_0(\varepsilon')}(\mathbb{R}_+)$ as defined in (1.1.1).

Analogously, we obtain $\mathcal{S}_{\overline{Q}}^{-\gamma}(\mathbb{R}_+) = \mathcal{S}_{\overline{Q}(\varepsilon)}^{-(\gamma-\varepsilon)}(\mathbb{R}_+) \oplus \mathcal{E}_{\overline{Q}_0'(\varepsilon)}(\mathbb{R}_+)$. Since $S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q, E) = S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q)\widehat{\otimes}_\pi E$ for any Fréchet space E, cf. (1.2.6), we thus can write $k = k_0 + \widetilde{k}$ with

$$\begin{aligned}
\widetilde{k} &\in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q, \mathcal{S}_{Q'(\varepsilon')}^{\gamma'+\varepsilon'}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}_{\overline{Q}(\varepsilon)}^{-(\gamma-\varepsilon)}(\mathbb{R}_+)), \\
k_0 &\in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^q, \mathcal{S}_{Q'(\varepsilon')}^{\gamma'+\varepsilon'}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{E}_{\overline{Q}_0'(\varepsilon)}(\mathbb{R}_+) + \mathcal{E}_{Q_0'(\varepsilon')}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}_{\overline{Q}(\varepsilon)}^{-(\gamma-\varepsilon)}(\mathbb{R}_+) + \\
&\qquad + \mathcal{E}_{Q_0'(\varepsilon')}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{E}_{\overline{Q}_0'(\varepsilon)}(\mathbb{R}_+)).
\end{aligned}$$

We now define g_0 via the kernel function k_0. Then $g_0 \in R_G^{\mu}(\mathbb{R}^q, (\gamma, \infty; \gamma' + \varepsilon', \theta' - \varepsilon')) + R_G^{\mu}(\mathbb{R}^q, (\gamma - \varepsilon, \theta - \varepsilon; \gamma', \infty)) + R_G^{\mu}(\mathbb{R}^q, (\gamma, \infty; \gamma', \infty))$, by Proposition 1.2.3, and g_0 is pointwise of finite rank, since the $\mathcal{E}$-spaces are finite dimensional. □

Corollary 2.5.2 *Let $\theta > 0$ be given and $g \in R_G^{\mu}(\mathbb{R}^q, (\gamma, \gamma', \widetilde{\theta}))$ with $\widetilde{\theta} > 2\theta$. Then there exists a Green symbol $g_0 \in R_G^{\mu}(\mathbb{R}^q, (\gamma, \gamma', \theta))$ which is pointwise of finite rank, and*

$$g - g_0 \in R_G^{\mu}(\mathbb{R}^q, (\gamma - \theta, \gamma' + \theta, \theta)).$$

This is an immediate consequence of Corollary 1.2.4 and Lemma 2.5.1. Now we are in the position to state (2.0.4) in a precise way.

Theorem 2.5.3 *Let $a \in S_{\mathrm{cl}}^{\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q})$, $N \in \mathbb{N}$, and $\gamma \in \mathbb{R}$ with $\frac{1}{2} - \gamma \notin \mathbb{N} \cup (\mathbb{Z} - \mu)$ (i.e., $\frac{1}{2} - \gamma \notin \mathbb{Z}$ if $\mu \in \mathbb{Z}$). Then to any $\gamma' \in \mathbb{R}$ with $\gamma' \geq \max(\mu_+, \gamma)$ and $\frac{1}{2} - \gamma' \notin \mathbb{Z} - \mu$ there exists a Green symbol $g(\eta) = g(\eta)(\gamma, \gamma', N) \in R_G^{\mu}(\mathbb{R}^q, (\gamma', (\gamma - \mu)_-, N))$ which is pointwise of finite rank, and such that*

$$\begin{aligned}
\omega_1(t[\eta])\,\mathrm{op}^+(a)(\eta)\omega_2(t[\eta]) - g(\eta) &\equiv \omega_1(t[\eta])t^{-\mu}\,\mathrm{op}_M^{\gamma}(h)(\eta)\omega_2(t[\eta]) + \\
&\quad + \omega_1(t[\eta])t^{-\mu}\sum_{k=0}^{N-1} t^k\Big(\sum_{|\alpha\leq k} \mathrm{op}_M^{\gamma}(h_{\alpha k})\eta^{\alpha}\Big)\omega_2(t[\eta])
\end{aligned} \tag{2.5.1}$$

$\operatorname{mod} R_G^{\mu}(\mathbb{R}_\eta^q, (\gamma, \gamma - \mu, N))$ where h and $h_{\alpha k}$ are as in Theorem 2.4.3.

Proof. Write H_γ for the first term on the right-hand side of (2.5.1), and m_γ^N for the second term. By Theorem 2.4.3 we then have a representation

$$\omega_1(t[\eta])\,\mathrm{op}^+(a)(\eta)\omega_2(t[\eta]) = H_{\gamma'}(\eta) + m_{\gamma'}^{2N'+1}(\eta) + g_{\gamma'}^{2N'+1}(\eta)$$

with a Green symbol $g_{\gamma'}^{2N'+1} \in R_G^{\mu}(\mathbb{R}^q, (\gamma', 0, 2N' + 1))$ for any $N' \in \mathbb{N}$. By Corollary 2.5.2 there is a Green symbol $g_0 \in R_G^{\mu}(\mathbb{R}^q, (\gamma', 0, N'))$, pointwise of finite rank, such that $g_{\gamma'}^{2N'+1} - g_0 \in R_G^{\mu}(\mathbb{R}^q, (\gamma' - N', N', N'))$. Thus, if we choose N' such that $N' \geq \max(\gamma' - \gamma, \gamma - \mu, N)$, then $g_{\gamma'}^{2N'+1} - g_0 \in R_G^{\mu}(\mathbb{R}^q, (\gamma, \gamma - \mu, N))$. Thus we can write

$$\omega_1(t[\eta])\,\mathrm{op}^+(a)(\eta)\omega_2(t[\eta]) = H_\gamma(\eta) + m_\gamma^N(\eta) + g_1(\eta) + g(\eta)$$

with $g_1 = (g_{\gamma'}^{2N'+1} - g_0) + (m_\gamma^{2N'+1} - m_\gamma^N)$, $g = (m_{\gamma'}^{2N'+1} - m_\gamma^{2N'+1}) + g_0$.

Since $t^{\mu-k}\omega_1(t[\eta])\,\mathrm{op}_M^{\gamma}(h_{\alpha k})\eta^{\alpha}\omega_2(t[\eta]) \in R_G^{\mu}(\mathbb{R}_\eta^q, (\gamma, \gamma - \mu, N))$ whenever $k \geq N$, the Green symbol g_1 belongs to $R_G^{\mu}(\mathbb{R}_\eta^q, (\gamma, \gamma - \mu, N))$. Moreover, g is as required, since

$$t^{\mu-k}\omega_1(t[\eta])\{\mathrm{op}_M^{\gamma'}(h_{\alpha k}) - \mathrm{op}_M^{\gamma}(h_{\alpha k})\}\eta^{\alpha}\omega_2(t[\eta])$$

is an element of $R_G^{\mu}(\mathbb{R}_\eta^q, (\gamma', \gamma - \mu, \infty)) \subset R_G^{\mu}(\mathbb{R}^q, (\gamma, \gamma - \mu, N))$, which is pointwise of finite rank, cf. [27]. □

In Theorem 2.5.3 one, of course, tries to choose the remainder $g(\gamma,\gamma',N)$ as "small" as possible, i.e., having a kernel which is as regular as possible. In case $\gamma \geq \mu_+$, one can take $g \equiv 0$, cf. Theorem 2.4.3. If $\gamma < \mu_+$ this is not true, instead one has to choose γ' as small as possible. The best case would be $\gamma' = \mu_+$; however this is only possible if $\frac{1}{2} - \mu_- \notin \mathbb{Z}$. But if this condition is not satisfied, the construction of g shows that g is independent of γ' for $\gamma' \to \mu_+$. In any case, we can find a unique Green symbol $\widetilde{g} = \widetilde{g}(\gamma, N)$ such that Theorem 2.5.3 holds with $\widetilde{g}$ instead of g.

2.6 Transmission operators

In this section we are going to investigate the operators (as mapping $C_0^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+)$)

$$A(y,\eta) = \varepsilon^* \mathrm{r}^- \operatorname{op}(a)(y,\eta)\mathrm{e}^+, \qquad A'(y,\eta) = \mathrm{r}^+ \operatorname{op}(a)(y,\eta)\mathrm{e}^- \varepsilon^*$$

for a symbol $a(t,y,\tau,\eta) \in S^{\mu,0}(\mathbb{R}_t \times \Omega_y \times \mathbb{R}^{1+q}_{\tau,\eta})$, where ε^* denotes simultaneously both the pull-back $C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}_-)$ and $C^\infty(\mathbb{R}_-) \to C^\infty(\mathbb{R}_+)$ under the reflection $t \to -t$. Roughly speaking, the operator $A(y,\eta)$ measures what $\operatorname{op}(a)(y,\eta)$ brings from "the right of zero" (i.e., from $\mathbb{R}_+$) to the left (i.e., to $\mathbb{R}_-$), and $A'(y,\eta)$ vice versa. By symmetry, it suffices to analyze $A(\eta)$. To be precise, if $b(t,y,\tau,\eta) \in S^{\mu,0}(\mathbb{R} \times \Omega \times \mathbb{R}^{1+q})$ is the (unique) symbol such that $\operatorname{op}(b)(y,\eta) = \operatorname{op}(\check{a})(y,\eta)$, where $\check{a}(t',y,\tau,\eta) = a(-t',y,\tau,\eta)$, then

$$A'(y,\eta) = \varepsilon^* \mathrm{r}^- \operatorname{op}(b)(y,\eta)\mathrm{e}^+.$$

We shall show now that for any $\gamma \in \mathbb{R}$ and $N \in \mathbb{N}$

$$A(y,\eta) - g(y,\eta) \in R^\mu_{M+G}(\Omega \times \mathbb{R}^q, (\gamma, \gamma-\mu, N)), \tag{2.6.1}$$

where $g = g_{\gamma,N}$ is an appropriate Green symbol. Again the variable $y \in \Omega$ is unessential, and we shall drop it in the following.

An easy calculation shows, that the kernel of $A(\eta)$ is given by

$$(t,t') \to -\int e^{i(t+t')\tau} a(-t,-\tau,\eta)\, đ\tau.$$

Therefore, if $\sigma, \sigma_0 \in C_0^\infty(\overline{\mathbb{R}}_+)$ are arbitrary cut-off functions, then

$$\sigma A(\eta)(1-\sigma_0), (1-\sigma)A(\eta)\sigma_0, (1-\sigma)A(\eta)(1-\sigma_0) \in R_G^{-\infty}(\mathbb{R}^q, (0,0,\infty)_{T,T},$$

$T = \{(-j,0) : \ j \in \mathbb{N}\}$, since all these operator families have a kernel in $\mathcal{S}(\mathbb{R}^q_\eta, \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+))$. Since

$$\kappa^{-1}(\eta)A(\eta)\kappa(\eta) = \varepsilon^* \mathrm{r}^- \operatorname{op}(\widetilde{a})(\eta)\mathrm{e}^+, \qquad \widetilde{a}(t,\tau,\eta) = a(t[\eta]^{-1}, \tau[\eta], \eta) \in S^\mu(\mathbb{R}^q_\eta, S^{\mu,0}(\mathbb{R}_t \times \mathbb{R}_\tau)),$$

it follows that for cut-off functions $\omega, \omega_0 \in C_0^\infty(\overline{\mathbb{R}}_+)$

$$\omega(t[\eta])A(\eta)(1-\omega_0)(t[\eta]), \quad (1-\omega)(t[\eta])A(\eta)\omega_0(t[\eta]), \quad (1-\omega)(t[\eta])A(\eta)(1-\omega_0)(t[\eta])$$

belong to $R^\mu_G(\mathbb{R}^q_\eta, (0,0,\infty))_{T,T}$, cf. the argumentation in Example 1.2.5 b). Hence we have

$$A(\eta) = \sigma\{\omega(t[\eta])A(\eta)\omega_0(t[\eta]) + g(\eta)\}\sigma_0 \bmod R_G^{-\infty}(\mathbb{R}^q_\eta, (0,0,\infty))_{T,T} \tag{2.6.2}$$

with a Green symbol $g \in R^\mu_G(\mathbb{R}^q_\eta, (0,0,\infty))_{T,T}$. Thus the remaining term to consider is $\omega(t[\eta])A(\eta)\omega_0(t[\eta])$. First we shall do this in the case of order 0 and then for arbitrary order.

Let $f_0 = 1$ and $f_k(z) = \prod_{j=1}^k (j-z)^{-k}$ as in (2.3.1) and

$$g(z) := e^{-i\pi z}(1-e^{2\pi i z})^{-1} \in M^{-\infty}_{\{(j,0):\ j \in \mathbb{Z}\}} \tag{2.6.3}$$

Proposition 2.6.1 *Let $\widetilde{a} \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^0_{\mathrm{cl}}(\mathbb{R}_t \times \mathbb{R}_\tau))$ have the asymptotic expansion*

$$\widetilde{a} \sim \sum_{k=0}^{\infty} a_k^{\pm}(t,\eta)\tau^{-k} \qquad \text{for } \tau \to \pm\infty.$$

If we set

$$\sigma_M^{-k}(\widetilde{a})(z,\eta) = g(z) \sum_{p+j=k} \frac{(-1)^p}{j!}(\partial_t^j a_p^+(0,\eta) - \partial_t^j a_p^-(0,\eta)) f_p(z),$$

then for any $N \in \mathbb{N}_0$

$$\omega(t[\eta])\varepsilon^* \mathrm{r}^- \operatorname{op}(\widetilde{a})(\eta)\mathrm{e}^+\omega_0(t[\eta]) = \omega(t[\eta]) \sum_{k=0}^{N} t^k \operatorname{op}_M^0(\sigma_M^{-k}(\widetilde{a}))(\eta)\omega_0(t[\eta]) + R_N(\eta)$$

with R_N having a kernel in $S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, \mathcal{S}^0_{T(N)}(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^0_0(\mathbb{R}_+) \cap \mathcal{S}^0_0(\mathbb{R}_+)\widehat{\otimes}_\pi \mathcal{S}^0_{P(N)}(\mathbb{R}_+))$ with asymptotic types $P(N), T(N) \in \mathrm{As}(0, N+1)$ given by

$$P(N) = \{(-j,0) :\ 0 \le j \le N\}, \qquad T(N) = \{(-j,0) :\ 0 \le j \le N\}.$$

This statement follows by the method used in the proof of Theorem 2.3.2, now applied to Theorem 2.1.191, 2.1.192 of [27] for $\widetilde{a}$ independent of η.

Applying this result together with Proposion 2.2.1 to

$$\kappa^{-1}(\eta)\{\omega(t[\eta])\varepsilon^* \mathrm{r}^- \operatorname{op}(a)(\eta)\mathrm{e}^+\omega_0(t[\eta])\}\kappa(\eta) = \omega\varepsilon^* \mathrm{r}^- \operatorname{op}(\widetilde{a})(\eta)\mathrm{e}^+\omega_0,$$

with $\widetilde{a}(t,\tau,\eta) = a(t[\eta]^{-1}, \tau[\eta], \eta)$ then yields the following theorem.

Theorem 2.6.2 *Let $a \in S^0(\mathbb{R} \times \mathbb{R}^{1+q})$, $A(\eta) = \varepsilon^* \mathrm{r}^- \operatorname{op}(a)(\eta)\mathrm{e}^+$, and $\omega, \omega_0 \in C_0^\infty(\overline{\mathbb{R}}_+)$ be arbitrary cut-off functions. Then there exist Mellin symbols*

$$h_{\alpha k} \in M^{-\infty}_{R_k}, \qquad R_k = \{(j,0) :\ j \in \mathbb{Z} \setminus \{1,\dots,k\}\} \cup \{(j,1) :\ j = 1,\dots,k\}$$

such that for any $N \in \mathbb{N}_0$

$$\omega(t[\eta])\Big\{A(\eta) - \sum_{k=0}^{N} t^k \Big(\sum_{|\alpha| \le k} \operatorname{op}_M^0(h_{\alpha k})\eta^\alpha\Big)\Big\}\omega_0(t[\eta]) \in R_G^0(\mathbb{R}^q_\eta, (0,0,N+1))_{P(N),T(N)},$$

where the asymptotic types $P(N), T(N) \in \mathrm{As}(0, N+1)$ are given by

$$P(N) = \{(-j,1) :\ 0 \le j \le N\}, \qquad T(N) = \{(-j,0) :\ 0 \le j \le N\}.$$

In particular, (2.6.1) *is true for $\gamma = \mu = 0$ and $g \equiv 0$.*

If $a \sim \sum_{k=0}^{\infty} a_{(-k)}$ in $S^0(\mathbb{R} \times \mathbb{R}^{1+q})$, the conormal symbols of $A(\eta)$ are

$$\sigma_M^{-j}(A)(z,\eta) = g(z) \sum_{p+k=j} (a_{pk}^+(\eta) - a_{pk}^-(\eta)) f_p(z),$$

$$a_{pk}^{\pm}(\eta) = \frac{(-i)^p}{k!} \sum_{l+|\alpha|=p} \frac{1}{\alpha!}(\partial_\eta^\alpha \partial_t^k a_{(-l)})(0,\pm 1, 0)\eta^\alpha.$$

Proceeding analogously to Sections 2.4 and 2.5 using Theorem 2.1.210 and Remark 2.1.211 of [27], we get the following result.

Theorem 2.6.3 *Let $a \in S^{\mu}_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R}^{1+q})$ and $A(\eta) = \varepsilon^* \mathrm{r}^- \operatorname{op}(a)(\eta)\mathrm{e}^+$. Moreover, let $\gamma \in \mathbb{R}$ satisfy $\frac{1}{2} - \gamma \notin \mathbb{N} \cup (\mathbb{Z} - \mu)$ (i.e., $\frac{1}{2} - \gamma \notin \mathbb{Z}$ if $\mu \in \mathbb{Z}$) and $N \in \mathbb{N}$. Then to any $\gamma' \geq \max(\mu_+, \gamma)$ and $\frac{1}{2} - \gamma' \notin \mathbb{Z} - \mu$ there exists a Green symbol $r = r(\gamma, \gamma', N) \in R^{\mu}_G(\mathbb{R}^q, (\gamma', (\gamma - \mu)_-, N))$ which is pointwise of finite rank, and such that*

$$\omega(t[\eta])\Big\{A(\eta) - t^{-\mu} \sum_{k=0}^{N-1} t^k \Big(\sum_{|\alpha| \leq k} \operatorname{op}^{\gamma}_M(h_{\alpha k})\eta^{\alpha}\Big)\Big\}\omega_0(t[\eta] - r(\eta) \in R^{\mu}_G(\mathbb{R}^q_{\eta}, (\gamma, \gamma - \mu, N)),$$

for certain $h_{\alpha k} \in M^{-\infty}_{R_k}$ and R_k as in (2.4.2). In particular, $r \equiv 0$ in case $\gamma \geq \mu_+$. The conormal symbols of $A(\eta) - r(\eta)$ are

$$\sigma^{\mu - j}_M(A - r)(z, \eta) = g(z + \mu) \sum_{p+k=j} (a^+_{pk}(\eta) - a^-_{pk}(\eta)) f_{p-\mu}(z),$$

where g is as in (2.6.3), $f_{\varrho}(z) = \Gamma(1 - z)/\Gamma(1 - z + \varrho)$, and

$$a^{\pm}_{pk}(\eta) = \frac{(-i)^p}{k!} \sum_{l+|\alpha|=p} \frac{1}{\alpha!}(\partial^{\alpha}_{\eta}\partial^k_t a_{(\mu-l)})(0, \pm 1, 0)\eta^{\alpha}.$$

In the same sense as we explained after Theorem 2.5.3, there exists a unique, best possible remainder $\widetilde{r} = \widetilde{r}(\gamma, N)$ such that Theorem 2.6.3 holds true for $\widetilde{r}$ instead of r.

3 Global calculus

In this chapter we formulate a calculus of pseudodifferential boundary value problems on a smooth manifold X with boundary Y. For the used notation, see the Appendix.

Given a (classical) pseudodifferential operator $A \in L^{\mu}_{\mathrm{cl}}(2X)$, $\mu \in \mathbb{R}$, we cannot expect that

$$\mathrm{r}^+ A \mathrm{e}^+ : C^{\infty}_0(\operatorname{int} X) \to C^{\infty}(\operatorname{int} X) \tag{3.0.4}$$

has an extension to a continuous map $\mathrm{r}^+ A \mathrm{e}^+ : H^s(X) \to H^{s-\mu}(X)$ (at least for $s > -\frac{1}{2}$) unless A has the transmission property at the boundary Y (if A has the transmission property, such a continuity holds, cf. [3] or [17]). For this reason Vishik and Eskin [35] have studied operators of the form $\mathrm{r}^+ A : H^s_0(X) \to H^{s-\mu}(X)$ that are continuous for all $s \in \mathbb{R}$, cf. also Section 4.5 below. However, there is no algebra property in this formulation. Therefore, we shall change the scales of spaces and pass to weighted Sobolev spaces as they are employed in the theory of operators on a manifold with edges; here, the edge is $Y = \partial X$ and the inner normal $\overline{\mathbb{R}}_+$ the model cone of the "wedge" $\overline{\mathbb{R}}_+ \times Y$. We then have the space $Y^{\mu}(X, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \gamma - \mu, k)$, of edge pseudodifferential operators on X, whose elements act continuously in weighted Sobolev spaces and can be composed in the same class. From the calculus of [28] (using Mellin quantization) we know that to any operator $\mathrm{r}^+ A \mathrm{e}^+$ as in (3.0.4) there exists an $A_{\gamma} \in Y^{\mu}(X, \boldsymbol{g})$ such that

$$C_{\gamma} := \mathrm{r}^+ A \mathrm{e}^+ - A_{\gamma} \in L^{-\infty}(\operatorname{int} X). \tag{3.0.5}$$

An essential point of our investigations in this chapter is to give a much more precise description of the remainder C_{γ} than in (3.0.5), i.e., to describe the singular structure of the kernel of C_{γ} near the boundary.

3.1 Weighted Sobolev spaces on a manifold

Given a Hilbert space E, endowed with a strongly continuous group of isomorphisms $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}_+$, we define the space $\mathcal{W}^s(\mathbb{R}^q, E)$ to be the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\left(\int \langle \eta \rangle^{2s} \| \kappa^{-1}_{\langle \eta \rangle} \hat{u}(\eta) \|^2_E \, d\eta \right)^{\frac{1}{2}}.$$

If a Fréchet space E is written as a projective limit $E = \varprojlim_{k \in \mathbb{N}} E_k$ of Hilbert spaces E_k, $k \in \mathbb{N}$, such that there are continuous embeddings $E_{k+1} \hookrightarrow E_k \hookrightarrow \ldots \hookrightarrow E_1$ for all k and E_1 is endowed with a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ that restricts to strongly continuous groups of isomorphisms $\kappa_\lambda|_{E_k} : E_k \to E_k$ for all k, we say that E is endowed with a group action. In that case we have the spaces $\mathcal{W}^s(\mathbb{R}^q, E_k)$ and continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E_{k+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E_k)$ for all k, and we set $\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{k \in \mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E_k)$.

Example 3.1.1 *For $E = \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ with the standard group action $(\kappa_\lambda u)(t) = \lambda^{1/2} u(\lambda t)$ we get the space*

$$\mathcal{W}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1}) := \mathcal{W}^s(\mathbb{R}^{n-1}, \mathcal{K}^{s,\gamma}(\mathbb{R}_+)).$$

If $Q \in \mathrm{As}(\gamma, \theta)$ and $\mathcal{K}^{s,\gamma}_Q(\mathbb{R}_+) = \varprojlim_{k \in \mathbb{N}} E_k$ in the sense of (1.1.2)*, we obtain*

$$\mathcal{W}^{s,\gamma}_Q(\mathbb{R}_+ \times \mathbb{R}^{n-1}) := \mathcal{W}^s(\mathbb{R}^{n-1}, \mathcal{K}^{s,\gamma}_Q(\mathbb{R}_+)),$$

which are subspaces of $\mathcal{W}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1})$.

Definition 3.1.2 *Let $\{U_1, \ldots, U_N\}$ be a covering of the manifold X as described in the beginning of this chapter and $\{\varphi_1, \ldots, \varphi_N\}$ a subordinate partition of unity. Then $\mathcal{W}^{s,\gamma}(X)$ is defined to be the completion of $C_0^\infty(\operatorname{int} X)$ with respect to the norm*

$$\left(\sum_{j=1}^{M} \| (\chi_j^{-1})^* (\varphi_j u) \|^2_{\mathcal{W}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1})} + \sum_{j=M+1}^{N} \| (\chi_j^{-1})^* (\varphi_j u) \|^2_{H^s(\mathbb{R}^n)} \right)^{\frac{1}{2}}. \tag{3.1.1}$$

Similarly (i.e., replace in (3.1.1) *$\| \cdot \|_{\mathcal{W}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1})}$ by $\| \cdot \|_{E_k}$ with E_k as in* (1.1.2) *and pass to the projective limit) we get the Fréchet space $\mathcal{W}^{s,\gamma}_Q(X) \subset \mathcal{W}^{s,\gamma}(X)$ for asymptotic types $Q \in \mathrm{As}(\gamma, \theta)$.*

Throughout this exposition we fix a Riemannian metric on X that induces the product metric of $[0,1] \times Y$ on a collar neighbourhood of Y. We then have a natural identification

$$L^2(X) = \mathcal{W}^{0,0}(X)$$

and, via the $L^2(X)$-scalar product, a non-degenerate sesquilinear pairing

$$\mathcal{W}^{s,\gamma}(X) \times \mathcal{W}^{-s,-\gamma}(X) \to \mathbb{C}.$$

Analogous definitions make sense for the case of distributional sections in vector bundles. Let $\mathrm{Vect}(\cdot)$ denote the set of all smooth, complex vector bundles on the space in the brackets. For every $E \in \mathrm{Vect}(X)$ we have an analogue $\mathcal{W}^{s,\gamma}(X, E)$ of the above space of scalar functions, locally modelled by $\mathcal{W}^{s,\gamma}(\mathbb{R}^{n-1}, \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^l))$, where $l \in \mathbb{N}_0$ corresponds to the fibre dimension of E, cf. [28, Section 3.5.2].

For each E we fix a Hermitian metric. We then have a corresponding space $L^2(X, E)$ that equals $\mathcal{W}^{0,0}(X, E)$. Moreover, there is a straightforward generalisation of spaces with constant discrete asymptotics of type Q associated with weight data (γ, θ), namely $\mathcal{W}^{s,\gamma}_Q(X, E)$.

3.2 Pseudodifferential boundary value problems

We are going to recall some details on the class $Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ of parameter-dependent edge pseudodifferential operators on X. Here, $\Lambda = \mathbb{R}^l$ for some $l \in \mathbb{N}_0$ and $E, F \in \mathrm{Vect}(X)$.

Definition 3.2.1 *If $Q \in \mathrm{As}(\delta, \theta)$ and $R \in \mathrm{As}(-\gamma, \theta)$ are given asymptotic types and $\boldsymbol{g} = (\gamma, \delta, \theta)$, then $Y^{-\infty}(X, \boldsymbol{g}; E, F)_{Q,R}$ denotes the subspace of all continuous operators*

$$G \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}^{s,\gamma}(X,E), \mathcal{W}^{\infty,\delta}(X,F)_Q) \quad \text{where} \quad G^* \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}^{s,-\delta}(X,F), \mathcal{W}^{\infty,-\gamma}(X,E)_R)$$

with G^* being the formal adjoint of G with respect to the corresponding L^2-scalar products. This is a Fréchet space, and we introduce the parameter-dependent class as $Y^{-\infty}(X \times \Lambda, \boldsymbol{g}; E, F)_{Q,R} := \mathcal{S}(\Lambda, Y^{-\infty}(X, \boldsymbol{g}; E, F)_{Q,R})$ and then set

$$Y^{-\infty}(X \times \Lambda, \boldsymbol{g}; E, F) = \bigcup_{Q,R} Y^{-\infty}(X \times \Lambda, \boldsymbol{g}; E, F)_{Q,R}. \tag{3.2.1}$$

Let $\{U_1, \dots, U_N\}$ be a covering of X as described in the Appendix and assume that E and F are trivial over U_j, $E|_{U_j} \cong U_j \times \mathbb{C}^l$, $F|_{U_j} \cong U_j \times \mathbb{C}^m$. Using the charts $\chi_j : U_j \to \Omega_j \times \overline{\mathbb{R}}_+$ for $j = 1, \dots, M$, we can associate with any operator $A : C_0^\infty(\Omega_j \times \mathbb{R}_+) \to C^\infty(\Omega_j \times \mathbb{R}_+)$ the pulled-back operator

$$A_j : C_0^\infty(E|_{\mathrm{int}\, U_j}) \to C^\infty(F|_{\mathrm{int}\, U_j}).$$

Specializing this to operator families $A(\lambda) = \mathrm{op}_y(a)(\lambda)$, $a \in R^\mu(\Omega_j \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; l, m)$, we denote the resulting space of parameter-dependent operators $A_j(\lambda)$ by

$$Y^\mu(U_j \times \Lambda, \boldsymbol{g}; E|_{U_j}, F|_{U_j}). \tag{3.2.2}$$

For the following definition let $\{\varphi_1, \dots, \varphi_N\}$ be a partition of unity subordinate to $\{U_1, \dots, U_N\}$ and $\psi_j \in C_0^\infty(U_j)$ such that $\varphi_j\psi_j = \varphi_j$. Moreover, set $\Phi = \sum_{j=M+1}^N \varphi_j$ and choose a function $\Psi \in C_0^\infty(U_{M+1} \cup \ldots \cup U_N)$ such that $\Phi\Psi = \Phi$.

Definition 3.2.2 *The space $Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, k)$, $k \in \mathbb{N}$, is defined to be the set of all operators $A(\lambda) = \sum_{j=1}^M \varphi_j A_j(\lambda)\psi_j + \Phi A_{\mathrm{int}}(\lambda)\Psi + C(\lambda)$ such that*

(i) $A_j(\lambda) \in Y^\mu(U_j \times \Lambda, \boldsymbol{g}; E|_{U_j}, F|_{U_j})$ *for all* $j = 1, \dots, M$,

(ii) $A_{\mathrm{int}}(\lambda) \in L^\mu_{\mathrm{cl}}(\mathrm{int}\, X \times \Lambda; E, F)$,

(iii) $C(\lambda) \in Y^{-\infty}(X \times \Lambda, \boldsymbol{g}; E, F)$.

Furthermore, $Y^\mu_{M+G}(X \times \Lambda, \boldsymbol{g}; E, F)$ (resp. $Y^\mu_G(X \times \Lambda, \boldsymbol{g}; E, F)$) will denote the subspaces of all $A(\lambda) \in Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ such that $A_{\mathrm{int}} \in L^{-\infty}(\mathrm{int}\, X \times \Lambda; E, F)$ and the local operator-valued symbols for $A_j(\lambda)$ all belong to $R^\mu_{M+G}(\Omega_j \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; l, m)$ (resp. $R^\mu_G(\Omega_j \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; l, m)$).

Given $A(\lambda) \in Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$, then pointwise $A(\lambda) : C_0^\infty(\mathrm{int}\, X, E) \to C^\infty(\mathrm{int}\, X, F)$ has continuous extensions to

$$A(\lambda) : \mathcal{W}^{s,\gamma}_{(Q)}(X, E) \to \mathcal{W}^{s-\mu,\gamma-\mu}_{(R)}(X, F) \tag{3.2.3}$$

(where $R \in \mathrm{As}(\gamma - \mu, k)$ is a resulting asymptotic type to $Q \in \mathrm{As}(\gamma, k)$). It is also worth mentioning that

$$Y^\mu_{M+G}(X \times \Lambda, \boldsymbol{g}; E, F) = Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F) \cap L^{-\infty}(\mathrm{int}\, X \times \Lambda; E, F). \tag{3.2.4}$$

Let us finish this subsection by giving some remarks on the principal symbols (which determine the ellipticity) associated with such operators. Since by construction

$$Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F) \subset L^\mu_{\mathrm{cl}}(\operatorname{int} X \times \Lambda; E, F)$$

we can associate with any $A(\lambda) \in Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ its homogeneous principal symbol,

$$\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F, \qquad \pi_X : (T^*X \times \Lambda) \setminus 0 \to X \tag{3.2.5}$$

with π_X being the canonical projection. (3.2.5) is a bundle homomorphism that is smooth up to the boundary. In fact, the local amplitude functions near the boundary are given by $t^{-\mu} p(t, y, \tau, \eta, \lambda) = t^{-\mu} \widetilde{p}(t, y, t\tau, t\eta, t\lambda)$ for certain $\widetilde{p}(t, y, \tau, \eta, \lambda)$ (modulo symbols of order $-\infty$ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n \times \Lambda$), and the homogeneous principal part of order μ equals

$$t^{-\mu} p_{(\mu)}(t, y, \tau, \eta, \lambda) = t^{-\mu} \widetilde{p}_{(\mu)}(t, y, t\tau, t\eta, t\lambda) = \widetilde{p}_{(\mu)}(t, y, \tau, \eta, \lambda),$$

where $\widetilde{p}_{(\mu)}(t, y, \tau, \eta, \lambda)$ is the homogeneous principal part of $\widetilde{p}(t, y, \tau, \eta, \lambda)$ in $(\tau, \eta, \lambda) \neq 0$ of order μ that is smooth up to $t = 0$. Moreover, we have the operator-valued principal edge symbol

$$\sigma_\partial(A) : \pi_Y^*(\mathcal{K}^{s,\gamma}(\mathbb{R}_+) \otimes E') \to \pi_Y^*(\mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \otimes F') \tag{3.2.6}$$

where $E' = E|_Y$, $F' = F|_Y$, and $\pi_Y : (T^*Y \times \Lambda) \setminus 0 \to Y$ is the canonical projection. $\sigma_\partial(A)$ is invariantly defined by corresponding local expressions, using Definition 1.4.2, namely

$$\begin{aligned} \sigma_\partial(a)(y, \eta, \lambda) = {} & t^{-\mu} \omega_1(t|\eta, \lambda|) \operatorname{op}_M^\gamma(h_0)(y, \eta, \lambda) \omega_2(t|\eta, \lambda|) \\ & + t^{-\mu}(1 - \omega_1(t|\eta, \lambda|)) \operatorname{op}_t(p_0)(y, \eta, \lambda)(1 - \omega_3(t|\eta, \lambda|)) \\ & + \sigma_\partial(m)(y, \eta, \lambda) + \sigma_\partial(g)(y, \eta, \lambda), \end{aligned} \tag{3.2.7}$$

where for a smoothing Mellin symbol m as in (1.4.1) we set

$$\sigma_\partial(m)(y, \eta, \lambda) := t^{-\mu} \omega(t|\eta, \lambda|) \sum_{j=0}^{k} t^j \sum_{|\alpha|=j} \operatorname{op}_M^{\gamma_j}(h_{j\alpha})(y)(q, \lambda)^\alpha \widetilde{\omega}(t|\eta, \lambda|).$$

Remark 3.2.3 *Operator families from* (3.2.7) *are pointwise cone operators on* $\mathbb{R}_+$ *with discrete asymptotics. As such they have a principal conormal symbol that is independent of the (co)variables* (η, λ)*, namely*

$$\sigma_M(a)(y, z) := \sigma_M \sigma_\partial(a)(y, z) = h_0(y, z) + h_{00}(y, z). \tag{3.2.8}$$

Let us recall that any $A(\lambda) \in Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ particularly is a parameter-dependent pseudodifferential operator in $L^\mu_{\mathrm{cl}}(\operatorname{int} X \times \Lambda; E, F)$. Therefore, passing to local trivializations of the bundles near the boundary, with $A(\lambda)$ we can associate local symbols belonging to $S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^n \times \Lambda; \mathbb{C}^l, \mathbb{C}^m)$, where l and m are the fibre dimensions of E and F, respectively.

Definition 3.2.4 *By* $L^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ *we denote the subspace of* $Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ *consisting of all operator-families* $A(\lambda)$*, whose local symbols near the boundary all belong to* $S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^n \times \Lambda; \mathbb{C}^l, \mathbb{C}^m)$ *modulo* $S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^n \times \Lambda; \mathbb{C}^l, \mathbb{C}^m)$.

Equivalently, we could require that $A(\lambda)$ belongs to $L^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ if and only if $A(\lambda) \in Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$, and all homogeneous components of the local symbols near the boundary are smooth up to $t=0$. Or, to write it globally, as operators $C_0^\infty(\operatorname{int} X, E) \to C^\infty(\operatorname{int} X, F)$,

$$\begin{aligned} &L^\mu(X \times \Lambda, \boldsymbol{g}; E, F) \\ &\qquad = Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F) \cap \{\mathrm{r}^+ L^\mu_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)\mathrm{e}^+ + L^{-\infty}(\operatorname{int} X \times \Lambda; E, F)\} \\ &\qquad = \{Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F) \cap \mathrm{r}^+ L^\mu_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)\mathrm{e}^+\} + Y^\mu_{M+G}(X \times \Lambda, \boldsymbol{g}; E, F). \end{aligned}$$

Remark 3.2.5 *If $A(\lambda) \in L^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ its principal conormal symbol $\sigma_M(A)$ only depends on the homogeneous principal interior symbol of $A(\lambda_0)$ of order μ (for any $\lambda_0 \in \Lambda$, evaluated in the sense without parameters), frozen at the boundary.*

3.3 Connection to the results of Sections 1 and 2

We now formulate the relation of the local results from Sections 1 and 2 to the present global calculus on the manifold X. For any parameter-dependent pseudodifferential operator $A(\lambda) \in L^\mu_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)$ we can form $\mathrm{r}^+ A(\lambda)\mathrm{e}^+ : C_0^\infty(\operatorname{int} X, E) \to C^\infty(\operatorname{int} X, F)$ and, for the pull-backs ε^* under $\varepsilon : X_+ \to X_-$ or $\varepsilon : X_- \to X_+$,

$$\varepsilon^* \mathrm{r}^- A(\lambda)\mathrm{e}^+ : C_0^\infty(\operatorname{int} X, E) \to C^\infty(\operatorname{int} X, F), \quad \mathrm{r}^+ A(\lambda)\mathrm{e}^- \varepsilon^* : C_0^\infty(\operatorname{int} X, E) \to C^\infty(\operatorname{int} X, F).$$

We shall now relate these operators to the classes of edge pseudodifferential operators we introduced above. We will consider separately the case of order zero and the general case.

Theorem 3.3.1 *For every $A(\lambda) \in L^0_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)$ we have*

$$\mathrm{r}^+ A(\lambda)\mathrm{e}^+ \in L^0(X \times \Lambda, \boldsymbol{g}; E, F),$$

where $\boldsymbol{g} = (0, 0, \infty)$. In particular, for any $k \in \mathbb{N} \cup \{+\infty\}$, we obtain

$$L^0(X \times \Lambda, (0,0,k); E, F) = \mathrm{r}^+ L^0_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)\mathrm{e}^+ + Y^0_{M+G}(X \times \Lambda, (0,0,k); E, F).$$

Proof. Let us decompose $A(\lambda)$ as

$$A(\lambda) = A^+_{\mathrm{int}}(\lambda) + A^-_{\mathrm{int}}(\lambda) + \sum_{j=1}^N A_j(\lambda) + C(\lambda), \tag{3.3.1}$$

where $A^\pm_{\mathrm{int}}(\lambda)$ are zero order pseudodifferential operators supported in $X_\pm$ (away from the boundary), $C(\lambda) \in L^{-\infty}(2X \times \Lambda; 2E, 2F)$ a smoothing remainder, and all A_j located in charts covering $Y = X_+ \cap X_-$ and corresponding to local operators

$$\mathrm{op}_{(t,y)}(a_j)(\lambda) = \mathrm{op}_y(\mathrm{op}_t(a_j))(\lambda), \qquad a_j \in S^0_{\mathrm{cl}}(\mathbb{R} \times \Omega_y \times \mathbb{R}^{1+q}_{\tau,\eta} \times \Lambda; \mathbb{C}^l, \mathbb{C}^m)$$

where l and m denote the fibre dimensions of E and F, respectively. Then

$$\mathrm{r}^+ A(\lambda)\mathrm{e}^+ = \sum_{j=1}^N r^+ A_j(\lambda)\mathrm{e}^+ + A^+_{\mathrm{int}}(\lambda) + \mathrm{r}^+ C(\lambda)\mathrm{e}^+.$$

Clearly, $\mathrm{r}^+ C(\lambda)\mathrm{e}^+$ belongs to $Y^{-\infty}(X \times \Lambda, \boldsymbol{g}; E, F)_{T,T}$ for Taylor asymptotics $T = \{(-j, 0) : j \in \mathbb{N}_0\} \in \mathrm{As}(0, \infty)$. By Theorem 2.4.3 we have $\mathrm{op}^+(a_j)(y, \eta, \lambda) \in R^0(\Omega \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; l, m)$. Hence $\mathrm{r}^+ A(\lambda)\mathrm{e}^+ \in Y^0(X \times \Lambda, \boldsymbol{g}; E, F)$ by Definition 3.2.2. Obviously, the local symbols near the boundary of $\mathrm{r}^+ A(\lambda)\mathrm{e}^+$ are smooth up to the boundary. This yields the result in view of Definition 3.2.4. $\square$

Let us mention that starting from given X and $E, F \in \mathrm{Vect}(X)$, it is not essential for the structure of $\mathrm{r}^+ A \mathrm{e}^+$ to have A and extensions $2E$, $2F$ of the bundles on the double $2X$; it suffices to know these data in a neighbourhood of X in a larger open manifold $\widetilde{X}$ (say, X glued together with $2V \setminus \mathrm{int}\, X$ along Y, where V is a collar neighbourhood of Y in X). Nevertheless, for convenience we mainly talk about $2X$, and $2E, 2F \in \mathrm{Vect}(2X)$, etc.

Theorem 3.3.2 *If $A(\lambda) \in L^0_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)$ and $\boldsymbol{g} = (0, 0, \infty)$, then*

$$\varepsilon^* \mathrm{r}^- A(\lambda) \mathrm{e}^+, \mathrm{r}^+ A(\lambda) \mathrm{e}^- \varepsilon^* \in Y^0_{M+G}(X \times \Lambda, \boldsymbol{g}; E, F).$$

Proof. If we write $A(\lambda)$ as in the beginning of the proof of the previous theorem, we obtain

$$\varepsilon^* \mathrm{r}^- A(\lambda) \mathrm{e}^+ = \sum_{j=1}^{N} \varepsilon^* \mathrm{r}^- A_j(\lambda) \mathrm{e}^+ + \varepsilon^* \mathrm{r}^- C(\lambda) \mathrm{e}^+.$$

Then the second summand belongs to $Y_G^{-\infty}(X \times \Lambda, \boldsymbol{g}; E, F)_{T,T}$ for Taylor asymptotics T, and the (local) terms $\varepsilon^* \mathrm{r}^- A_j(\lambda) \mathrm{e}^+$ can be treated with Theorem 2.6.2. The argumentation for $\mathrm{r}^+ A(\lambda) \mathrm{e}^- \varepsilon^*$ is the same. □

In an analogous way, using the local results from Theorem 2.5.3 and Theorem 2.6.3, these statements can be generalized to the case of arbitrary order $\mu \in \mathbb{R}$. They read as follows:

Theorem 3.3.3 *Let $A(\lambda) \in L^\mu_{\mathrm{cl}}(2X \times \Lambda; 2E, 2F)$, $N \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ with $\frac{1}{2} - \gamma \notin \mathbb{N} \cup (\mathbb{Z} - \mu)$. then to any $\gamma' \in \mathbb{R}$ with $\gamma' \geq \max(\mu_+, \gamma)$ and $\frac{1}{2} - \gamma' \notin \mathbb{Z} - \mu$, there exist Green operators*

$$G(\lambda) = G(\lambda)(\gamma, \gamma', N), R(\lambda) = R(\lambda)(\gamma, \gamma', N) \in Y_G^\mu(X \times \Lambda, (\gamma', (\gamma - \mu)_-, N)),$$

such that

$$\mathrm{r}^+ A(\lambda) \mathrm{e}^+ - G(\lambda) \in L^\mu(X \times \Lambda, (\gamma, \gamma - \mu, N); E, F),$$
$$\varepsilon^* \mathrm{r}^- A(\lambda) \mathrm{e}^+ - R(\lambda) \in Y^\mu_{M+G}(X \times \Lambda, (\gamma, \gamma - \mu, N); E, F)$$

The analogue of the last statement also holds for $\mathrm{r}^+ A(\lambda) \mathrm{e}^- \varepsilon^*$. In case $\gamma \geq \mu_+$, it is possible to take $G(\lambda) = R(\lambda) = 0$.

As described immediately after Theorem 2.6.3, there exist best possible choices $\widetilde{G} = \widetilde{G}(\gamma, N)$ and $\widetilde{R} = \widetilde{R}(\gamma, N)$ for which the previous theorem remains valid.

4 Boundary value problems

4.1 Boundary value problems on a manifold

We let X be a compact manifold with smooth boundary Y as described in the Appendix.

Boundary value problems for operators $A(\lambda) \in Y^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ will be studied in terms of 2×2-block matrices of continuous operators

$$\mathcal{A}(\lambda) : \begin{array}{c} \mathcal{W}^{s,\gamma}(X, E) \\ \oplus \\ H^s(Y, J_-) \end{array} \rightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{array} \tag{4.1.1}$$

for additional vector bundles $J_-, J_+ \in \mathrm{Vect}(Y)$ over the boundary, and the upper left corner (u. l. c.) of $\mathcal{A}$ equals A. More precisely, $\mathcal{Y}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J_-, J_+)$ consists of all operators $\mathcal{A}$ of the form

$$\mathcal{A}(\lambda) = \begin{pmatrix} A(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}(\lambda) \tag{4.1.2}$$

for some $A(\lambda) \in \mathcal{Y}^\mu(X\times\Lambda, \boldsymbol{g}; E, F)$ and some Green operator $\mathcal{G} \in \mathcal{Y}^\mu_G(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})$, the latter space being defiend below. The elements $\mathcal{A}(\lambda) \in \mathcal{Y}^\mu(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})$ are continuous as operators (4.1.1) and $\mathcal{A}(\lambda) : \mathcal{W}^{s,\gamma}_Q(X, E) \oplus H^s(Y, J_-) \to \mathcal{W}^{s-\mu,\gamma-\mu}_R(X, F) \oplus H^{s-\mu}(Y, J_+)$ for all $s \in \mathbb{R}$ and every asymptotic type Q with some resulting R.

Moreover, $\mathcal{L}^\mu(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})$ will denote the subspace of all elements of $\mathcal{Y}^\mu(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})$ where the upper left corners belong to $L^\mu(X\times\Lambda, \boldsymbol{g}; E, F)$.

Remark 4.1.1 *The space $\mathcal{Y}^\mu(X, \boldsymbol{g}; \boldsymbol{v})$ is a particular case of the general space of edge problems for edge-degenerate pseudodifferential operators on a manifold W with edges. In the present case W is simply X, and the edge is the boundary Y. Here, we have singled out the space with constant (in $y \in Y$) discrete asymptotics, though parametrices of elliptic operators require a generalisation of the notion of asymptotics to continuous (or variable in $y \in Y$ and pointwise discrete) ones; however, this is not the main aspect of the present paper.*

Definition 4.1.2 *Let $\boldsymbol{g} = (\gamma, \delta, \theta)$ and, as above, $\boldsymbol{v} = (E, F; J_-, J_+)$. For asymptotic types $Q \in \mathrm{As}(\delta, \theta)$ and $R \in \mathrm{As}(-\gamma, \theta)$ we let $\mathcal{Y}^{-\infty}_G(X, \boldsymbol{g}; \boldsymbol{v})_{Q,R}$ denote the space of all operators G satisfying $\mathcal{G} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}^{s,\gamma}(X, E) \oplus H^s(Y, J_-), \mathcal{W}^{\infty,\delta}_Q(X, F) \oplus C^\infty(Y, J_+))$ such that $\mathcal{G}^* \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}^{s,-\delta}(X, E) \oplus H^s(Y, J_+), \mathcal{W}^{\infty,-\gamma}_R(X, E) \oplus C^\infty(Y, J_-))$ with $\mathcal{G}^*$ being the formal adjoint with respect to the corresponding L^2-scalar products.*

This is a Fréchet space, and we introduce the parameter-dependent analogue as

$$\mathcal{Y}^{-\infty}_G(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})_{Q,R} := \mathcal{S}(\Lambda, \mathcal{Y}^{-\infty}_G(X, \boldsymbol{g}; \boldsymbol{v})_{Q,R}).$$

Moreover, we set $\mathcal{Y}^{-\infty}_G(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v}) := \bigcup_{Q,R} \mathcal{Y}^{-\infty}_G(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})_{Q,R}$.

Let us now employ the spaces $\mathcal{K}^{s,\gamma}_{(Q)}(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^k$, $\mathcal{S}^\gamma_Q(\mathbb{R}_+, \mathbb{C}^e) \oplus \mathbb{C}^k$ for $Q \in \mathrm{As}(\gamma, \theta)$, $k, l \in \mathbb{N}_0$ with the group action $\kappa_\lambda \oplus 1$, where κ_λ denotes the standard group action (1.1.3). If $\boldsymbol{g} = (\gamma, \delta, \theta)$ and $l, m, j_-, j_+ \in \mathbb{N}_0$ we define

$$\mathcal{R}^\mu_G(\Omega\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v}), \qquad \boldsymbol{v} = (l, m, j_-, j_+),$$

in analogy to Definition 1.1.8 to be the space of all symbols g satisfying

$$\begin{aligned} g(y,\eta) &\in \bigcap_{s\in\mathbb{R}} S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-}, \mathcal{S}^\delta_Q(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_+}), \\ g^*(y,\eta) &\in \bigcap_{s\in\mathbb{R}} S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, \mathcal{K}^{s,-\delta}(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_+}, \mathcal{S}^{-\gamma}_Q(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-}) \end{aligned} \tag{4.1.3}$$

for some asymptotic types $Q \in \mathrm{As}(\delta, \theta)$, $R \in \mathrm{As}(-\gamma, \theta)$. If we want to point out the specific asymptotic types involved, we write $\mathcal{R}^\mu_G(\Omega\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_{R,Q}$.

Let $\{U_1, \dots, U_N\}$ be a covering of X as described in the Appendix. Let $U'_j := U_j \cap Y \neq \emptyset$ for $j = 1, \dots, M$, and assume all bundles E, F and J_-, J_+ to be trivial over U_j and U'_j, respectively, $E|_{U_j} \cong U_j \times \mathbb{C}^l$, $F|_{U_j} \cong U_j \times \mathbb{C}^m$, $J_\pm|_{U_j} \cong U'_j \times \mathbb{C}^{j_\pm}$. If we use charts $\chi_j : U_j \to \Omega_j \times \overline{\mathbb{R}}_+$, then any operator

$$g : \begin{array}{c} C^\infty_0(\mathbb{R}_+\times\Omega_j, \mathbb{C}^l) \\ \oplus \\ C^\infty_0(\Omega_j, \mathbb{C}^{j_-}) \end{array} \to \begin{array}{c} C^\infty_0(\mathbb{R}_+\times\Omega_j, \mathbb{C}^m) \\ \oplus \\ C^\infty_0(\Omega_j, \mathbb{C}^{j_+}) \end{array}$$

induces via its pull-back under χ_j an operator

$$\mathcal{G} : \begin{array}{c} C^\infty_0(E|_{U_j}) \\ \oplus \\ C^\infty_0(J_-|_{U'_j}) \end{array} \to \begin{array}{c} C^\infty(F|_{U_j}) \\ \oplus \\ C^\infty(J_+|_{U'_j}) \end{array}.$$

Specializing this to operator-families $g(\lambda) \in \mathcal{R}_G^{\mu}(\Omega_j \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{v} = (l, m; j_-, j_+)$, we obtain the space $\mathcal{Y}_G^{\mu}(U_j \times \Lambda, \boldsymbol{g}; E|_{U_j}, F|_{U_j}; J_-|_{U_j'}, J_+|_{U_j'})$. For the following definition let $\{\varphi_1, \dots, \varphi_N\}$ be a partition of unity subordinate to the covering $\{U_1, \dots, U_N\}$, and $\psi_j \in C_0^\infty(U_j)$ wth $\psi_j\varphi_j = \varphi_j$. Set $\varphi_j' = \varphi_j|_{U_j'}$ and $\psi_j' = \psi_j|_{U_j'}$ for $1 \le j \le M$.

Definition 4.1.3 *The space $\mathcal{Y}_G^{\mu}(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_-, J_+)$, consists of all operator families*

$$\mathcal{G}(\lambda) = \sum_{j=1}^{M} \begin{pmatrix} \varphi_j & 0 \\ 0 & \varphi_j' \end{pmatrix} \mathcal{G}_j(\lambda) \begin{pmatrix} \psi_j & 0 \\ 0 & \psi_j' \end{pmatrix} + \mathcal{C}(\lambda) \tag{4.1.4}$$

with $\mathcal{G}_j(\lambda) \in \mathcal{Y}_G^{\mu}(U_j \times \Lambda, \boldsymbol{g}; E|_{U_j}, F|_{U_j}, J_-|_{U_j'}, J_+|_{U_j'})$ and $\mathcal{C}(\lambda) \in \mathcal{Y}_G^{-\infty}(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$.

Remark 4.1.4 *Any $\mathcal{A} \in \mathcal{Y}^{\mu}(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ has, in the sense of* (4.1.1)*, a block-matrix representation $\mathcal{A} = \left(\begin{smallmatrix} A & K \\ T & Q \end{smallmatrix}\right)$ with $A \in Y^{\mu}(X \times \Lambda, \boldsymbol{g}; E, F)$ and $\left(\begin{smallmatrix} 0 & K \\ T & Q \end{smallmatrix}\right) \in \mathcal{Y}_G^{\mu}(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$.*

The ellipticity in the class $\mathcal{Y}^{\mu}(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ is based on a pair of principal symbols $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$. If $\mathcal{A} = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) + \mathcal{G}$ in the above sense, then

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(A) : \pi_X^* E \to \pi_X^* F, \qquad \pi_X : (T^*X \times \Lambda) \setminus 0 \to X, \tag{4.1.5}$$

is the homogeneous principal symbol of the upper left corner of $\mathcal{A}$, as it was described in (3.2.5). Moreover, the principal boundary symbol of $\mathcal{A}$ is defined as

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}, \tag{4.1.6}$$

$\pi_Y : T^*Y \times \Lambda \setminus 0 \to Y$, where $E' = E|_Y$, $F' = F|_Y$, and $s \in \mathbb{R}$ is arbitrary (the choice of s is unessential). In (4.1.6), $\sigma_\partial(A)$ is as described in (3.2.6), (3.2.7), and $\sigma_\partial(\mathcal{G})$ is obtained from the homogeneous principal symbols of the Green symbols involved in the construction of $\mathcal{G}$, cf. (4.1.3) and Definition 4.1.3.

4.2 Ellipticity and reductions of orders

The calculus of edge problems, cf. [28], specialised to a manifold with boundary, gives us parametrices of elliptic elements in $\mathcal{Y}^{\mu}(X, \boldsymbol{g}; \boldsymbol{v})$ in a larger space with continuous asymptotics. For our purposes it suffices to establish parametrices in operator spaces, where Green operators induce only some weight improvements relative to a given reference weight γ. To this end, we introduce

$$\mathcal{Y}_G^{\mu}(X \times \Lambda, (\gamma, \delta); \boldsymbol{v})_{\mathcal{O}} = \bigcup_{0<\varepsilon<1} \mathcal{Y}_G^{\mu}(X \times \Lambda, \boldsymbol{g}_\varepsilon; \boldsymbol{v})_{\mathcal{O},\mathcal{O}} \tag{4.2.1}$$

where $\gamma, \delta \in \mathbb{R}$, $\boldsymbol{g}_\varepsilon = (\gamma, \delta, \varepsilon)$. The subscript $\mathcal{O}, \mathcal{O}$ on the right of (4.2.1) indicates that in (4.1.4) the asymptotic types involved in $\mathcal{C}(\lambda)$ as well as in the local amplitude functions associated with $\mathcal{G}_j(\lambda)$ are the empty in $\mathrm{As}(\gamma, \varepsilon)$ and $\mathrm{As}(\delta, \varepsilon)$, respectively.

Then in the sense of (4.1.1) and (4.1.2), we let $\mathcal{Y}^{\mu}(X \times \Lambda, (\gamma, \gamma - \mu); \boldsymbol{v})_{\mathcal{O}}$, be the space of all block-matrices

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}, \qquad A \in Y^{\mu}(X \times \Lambda, (\gamma, \gamma - \mu); \boldsymbol{v})_{\mathcal{O}}, \qquad \mathcal{G} \in \mathcal{Y}_G^{\mu}(X \times \Lambda, (\gamma, \gamma - \mu); \boldsymbol{v})_{\mathcal{O}},$$

where $Y^{\mu}(X \times \Lambda, (\gamma, \gamma - \mu); \boldsymbol{v})_{\mathcal{O}}$ is defined as in Definition 3.2.2 with the following modifications: The local amplitude functions of the operator-families $A_j(\lambda)$ are defined as in

Definition 1.4.2 (extended to the matrix case) but there the Green symbol g is required to belong to $\bigcup_{0<\varepsilon<1} R^{\mu}_{G}(\Omega \times \mathbb{R}^{n-1} \times \Lambda, (\gamma, \gamma-\mu, \varepsilon))_{\mathcal{O},\mathcal{O}}$ and the Mellin symbol m is replaced by the expression (1.4.1) but now $h_{j\alpha} \equiv 0$ for $j + |\alpha| > 0$, and

$$h_{00} \in M^{-\infty}_{\gamma} := \bigcup_{0<\varepsilon<1} M^{-\infty}_{\gamma,\varepsilon}, \tag{4.2.2}$$

where $M^{-\infty}_{\gamma,\varepsilon}$ denotes the space of all functions $h(z)$, which are holomorphic on $S_{\gamma,\varepsilon} := \{z \in \mathbb{C} : \ |\operatorname{Re} z + \gamma - \frac{1}{2}| < \varepsilon\}$ and such that $h(\varrho + i\tau) \in \mathcal{S}(\mathbb{R}_\tau)$ locally uniformly in $\{\varrho \in \mathbb{R} : \ |\varrho + \gamma - \frac{1}{2}| < \varepsilon\}$. The spaces $M^{-\infty}_{\gamma,\varepsilon}$ are Fréchet in a canonical way. Moreover, the form $C(\lambda)$ from Definition 3.2.2 is required to be of such a form that $\left(\begin{smallmatrix} C(\lambda) & 0 \\ 0 & 0 \end{smallmatrix}\right)$ belongs to $\mathcal{Y}^{\mu}_{G}(X \times \Lambda, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$.

Remark 4.2.1 *Every $A \in \mathcal{Y}^{\mu}(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \theta)$, belongs to $\mathcal{Y}^{\mu}(X \times \Lambda, (\gamma, \gamma - \mu); \boldsymbol{v})_{\mathcal{O}}$.*

Definition 4.2.2 *An operator $\mathcal{A} \in \mathcal{Y}^{\mu}(X \times \Lambda, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$ for $\boldsymbol{v} = (E, F; J_-, J_+)$, is called parameter-dependent elliptic, if* (4.1.5) *and* (4.1.6) *are isomorphisms. For $p = 0$ we simply talk about ellipticity (without parameters).*

The ellipticity condition for $\sigma_\partial(\mathcal{A})$ is required for some $s_0 \in \mathbb{R}$. It is then automatically satisfied for all $s \in \mathbb{R}$. Clearly, if $\mathcal{A}(\lambda) \in \mathcal{Y}^{\mu}(X \times \Lambda, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$ is parameter-dependent elliptic, $\mathcal{A}(\lambda_0)$ is elliptic in the sense without parameters, for every fixed $\lambda_0 \in \Lambda$.

Remark 4.2.3 *For convenience, elements $\mathcal{A} \in \mathcal{Y}^{\mu}(X \times \Lambda, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$ will also be written in the form*

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & K \\ T & Q \end{pmatrix} \tag{4.2.3}$$

where $A \in Y^{\mu}(X \times \Lambda, (\gamma, \gamma-\mu); E, F)_{\mathcal{O}}$ and $\mathcal{G} = \left(\begin{smallmatrix} 0 & K \\ T & Q \end{smallmatrix}\right) \in \mathcal{Y}^{\mu}(X, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$. Observe that the ellipticity of $\mathcal{A}$ implies that $A = \mathrm{u.l.c.}\, \mathcal{A} \in Y^{\mu}(X, (\gamma, \gamma-\mu); E, F)_{\mathcal{O}}$ is elliptic in the sense of the second part of Definition 4.2.2, *and $\sigma_\partial(A) = \mathrm{u.l.c.}\, \sigma_\partial(\mathcal{A})$,*

$$\sigma_\partial(A) : \pi_Y^* \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \otimes E' \to \pi_Y^* \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \otimes F', \tag{4.2.4}$$

$\pi_Y : (T^*Y \times \Lambda) \setminus 0 \to Y$, is a family of Fredholm operators between the fibres. Set $S^* = \{(y, \eta, \lambda) \in (T^*Y \times \Lambda) \setminus 0 : \ |\eta|^2 + |\lambda|^2 = 1\}$, and let $\pi_{Y,1} : S^* \to Y$ be the canonical projection. Then $\operatorname{ind}_{S^*} \sigma_\partial(A) = [\pi^*_{Y,1} J_+] - [\pi^*_{Y,1} J_-] \in \pi^*_{Y,1} K(Y)$. In general, the ellipticity of an element $A \in Y^{\mu}(X \times \Lambda, \boldsymbol{g}; E, F)_{\mathcal{O}}$ only implies that (4.2.4) *is a Fredholm family where*

$$\operatorname{ind}_{S^*} \sigma_\partial(A) \in K(S^*). \tag{4.2.5}$$

The following two theorems are special cases of the general calculus of [28].

Theorem 4.2.4 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(X \times \Lambda, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$ be parameter-dependent elliptic, $\boldsymbol{v} = (E, F; J_-, J_+)$. Then there is a parametrix $\mathcal{P} \in \mathcal{Y}^{-\mu}(X \times \Lambda, (\gamma-\mu, \gamma); \boldsymbol{v}^{-1})_{\mathcal{O}}$, $\boldsymbol{v}^{-1} = (F, E; J_+, J_-)$, in the sense*

$$\mathcal{I} - \mathcal{P}\mathcal{A} \in \mathcal{Y}^{-\infty}(X \times \Lambda, (\gamma, \gamma); \boldsymbol{v}_l)_{\mathcal{O}}, \qquad \mathcal{I} - \mathcal{A}\mathcal{P} \in \mathcal{Y}^{-\infty}(X \times \Lambda, (\gamma-\mu, \gamma-\mu); \boldsymbol{v}_r)_{\mathcal{O}},$$

with $\boldsymbol{v}_l = (E, E; J_-, J_-)$ and $\boldsymbol{v}_r = (F, F; J_+, J_+)$.

Theorem 4.2.5 *Let $\mathcal{A} \in \mathcal{Y}^\mu(X \times \Lambda, (\gamma, \gamma-\mu); \boldsymbol{v})_{\mathcal{O}}$ be elliptic. Then*

$$\mathcal{A}(\lambda) : \begin{array}{c} \mathcal{W}^{s,\gamma}(X,E) \\ \oplus \\ H^s(Y,J_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y,J_+) \end{array} \tag{4.2.6}$$

is a Fredholm operator for each $s \in \mathbb{R}$, $\lambda \in \Lambda$, and the index $\operatorname{ind} \mathcal{A}(\lambda)$ is independent of s and λ. For non-trivial Λ we have $\operatorname{ind} \mathcal{A}(\lambda) = 0$ for all s and λ, and there is a $c > 0$ such that (4.2.6) is an isomorphism for $|\lambda| > c$, $s \in \mathbb{R}$.

The main aspect of the present section is the construction of order (and weight) reducing elements in our algebra of boundary value problems with a prescribed number of trace and potential conditions. In this connection we want to construct some particular Mellin symbols.

Proposition 4.2.6 *For every $k \in \mathbb{Z}$ there exists an element $f_k(z) \in M_{\mathcal{O}}^{-\infty}$ such that $1 + f_k(z) \neq 0$ for all $z \in \mathbb{C}$, and $1 + \omega \operatorname{op}_M^\gamma(f_k)\widetilde{\omega} : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ is a Fredholm operator of index k for every $s, \gamma \in \mathbb{R}$; here $\omega(t)$ and $\widetilde{\omega}(t)$ are arbitrary cut-off functions. In addition, keeping ω, $\widetilde{\omega}$ and γ fixed, there exists an operator g_k with kernel in $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ such that*

$$1 + \omega \operatorname{op}_M^\gamma(f_k)\widetilde{\omega} + g_k : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$$

is surjective for $k \geq 0$ and injective for $k \leq 0$, $s \in \mathbb{R}$.

Proof. As is known, cf. [5] and [27], the index of a Fredholm operator of the form $1 + \omega \operatorname{op}_M^\gamma(f)\widetilde{\omega} : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ for $f \in M_R^{-\infty}$, $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$, equals

$$\frac{1}{2\pi} \Delta \arg(1 + f(z))|_{\Gamma_{\frac{1}{2}-\gamma}}\Big|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty},$$

where Δ denotes the change of the arguments of $1 + f(z)$ when z runs from $\operatorname{Im} z = -\infty$ to $\operatorname{Im} z = \infty$ on the line $\Gamma_{\frac{1}{2}-\gamma}$. To construct $f_k(z)$ we first consider the case $k = 1$, $\gamma = 0$, and start from an arbitrary function $m(z) \in C^\infty(\Gamma_{\frac{1}{2}})$ such that $m(z) = 0$ for $\operatorname{Im} z \leq 0$ and $m(z) = 2\pi i$ for $\operatorname{Im} z \geq 1$, then $\Delta \arg m(z)|_{\Gamma_{\frac{1}{2}}}\big|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty} = 2\pi$. Moreover, we have $m(z) \in S_{\mathrm{cl}}^0(\Gamma_{\frac{1}{2}})$. By kernel cut-off we find an $h(t) \in M_{\mathcal{O}}^0$ such that $m(z) - h(z)|_{\Gamma_{\frac{1}{2}}} \in S^{-\infty}(\Gamma_{\frac{1}{2}})$, where $\Delta \arg h(z)|_{\Gamma_{\frac{1}{2}}}\big|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty} = 2\pi$, cf. [28, Remark 1.1.51]. Let us set $l(z) := e^{h(z)}$; we then have $l(z) \in M_{\mathcal{O}}^0$ and $l^{-1}(z) \in M_{\mathcal{O}}^0$, and $\Delta \arg l(z)|_{\Gamma_\beta}\big|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty} = 2\pi$ for every $\beta \in \mathbb{R}$. Now we may set $f_1(z) = l(z) - 1$, and we have $f_1(z) \in M_{\mathcal{O}}^{-\infty}$ because $f_1(z)|_{\Gamma_{\frac{1}{2}}} \in \mathcal{S}(\Gamma_{\frac{1}{2}})$ which implies $f_1(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_{\frac{1}{2}})$ for each β. Similarly, we can set $f_k(z) = l^k(z) - 1$. Let us now consider the case $k \geq 0$. Set for abbreviation $a = 1 + \omega \operatorname{op}_M^\gamma(f_k)\widetilde{\omega}$. Without loss of generality we may consider the case $s = \gamma = 0$. In fact, the choice of s is unessential anyway, because $\ker a$ and $\operatorname{coker} a$ are independent of s. Moreover, we may pass to the operator $a_0 := k^{-\gamma} a k^\gamma$ where $k^\gamma(t) \in C^\infty(\mathbb{R}_+)$ is any strictly positive function where $k^\gamma(t) = t^\gamma$ for $0 < t < c_0$, $k^\gamma(t) = 1$ for $t > c_1$ for certain $0 < c_0 < c_1$. Let c_0 be so large that ω, $\widetilde{\omega}$ vanish for $t > c_0$. Then we get $a_0 = 1 + \omega t^{-\gamma} \operatorname{op}_M^\gamma(f_k) t^{-\gamma}\widetilde{\omega} = 1 + \omega \operatorname{op}_M(T^{-\gamma} f_k)\widetilde{\omega} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$. Set $\mathcal{N}_- := (\operatorname{im} a_0)^\perp$ (the orthogonal complement in $L^2(\mathbb{R}_+)$) and set $N_- := \dim \mathcal{N}_-$. Since $\operatorname{ind} a_0 = k > 0$ we have $\dim \ker a_0 = N_- + k$. Choose a subspace $\widetilde{\mathcal{N}}_- \subset \ker a_0$ of dimension N_-, let $P : L^2(\mathbb{R}_+) \to \widetilde{\mathcal{N}}_-$ be the orthogonal projection to $\widetilde{\mathcal{N}}_-$, and choose an isomorphism $G : \widetilde{\mathcal{N}}_- \to \mathcal{N}_-$. Then $a_0 + GP : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is surjective; the operator GP has an integral kernel in $\mathcal{N}_- \otimes \widetilde{\mathcal{N}}_-$ that can be approximated by a kernel b in $C_0^\infty(\mathbb{R}_+) \otimes C_0^\infty(\mathbb{R}_+)$ in such a way that $a_0 + g : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ remains surjective, when g is the operator with kernel b. It suffices then to set $g_k = t^{-\gamma} g t^{\gamma}$. For $k < 0$ we can again first reduce the construction to $s = \gamma = 0$, then carry out the construction for the adjoint operator that has non-negative index and then take the adjoint of the resulting operator. $\square$

Lemma 4.2.7 *For every fixed $\gamma \in \mathbb{R}$ and $p \in \Gamma_{\frac{1}{2}-\gamma}$ there exists an element $g(z) \in M_R^{-\infty}$ for a suitable R such that $g(z)$ has exactly one pole at p of first order, and $(1+g(z))^{-1} \neq 0$ for all $z \neq p$, $|\operatorname{Re} z - (\frac{1}{2} - \gamma)| < \varepsilon$ for some $\varepsilon = \varepsilon(g) > 0$.*

Proof. Without loss of generality we assume $\gamma = \frac{1}{2}$ and $p = 0$; the general case can be reached by a translation in the complex plane. We set $g(z) = \int_0^\infty t^{z-1}\omega(t)\,dt$ for a suitable choice of a cut-off function $\omega(t)$. First it is clear that such a $g(z)$ has a pole of first order at $z = 0$ and $g(z) \in M_R^{-\infty}$ for a discrete asymptotic type R with $\pi_{\mathbb{C}} R = \{0\}$. Choose $\omega(t)$ in a way that $\omega(t) \equiv 1$ for $0 \leq t \leq 1$ and $\omega(t) \geq 0$ for $1 \leq t \leq 1+\sigma$, $\omega(t) = 0$ for $t \geq 1+\sigma$ for a $\sigma > 0$, to be defined in a suitable way below. Write $g(z) = \int_0^1 t^{z-1}\,dt + r(z)$, for $r(z) = \int_1^{1+\sigma} t^{z-1}\omega(t)\,dt$. We then have $1 + g(z) = z^{-1}(1 + z + zr(z))$. Thus, $1 + g(z) = 0$, $z \neq 0$, if and only if $1 + z + zr(z) = 0$, i.e., $z(1 + r(z)) = -1$. For every $\delta > 0$ there is a $\sigma > 0$ such that $|r(z)| < \delta$ for $|\operatorname{Re} z| < \frac{1}{2}$. Now we have $z \neq -1$ for all $|\operatorname{Re} z| < \frac{1}{2}$, thus $z(1 + r(z)) \neq -1$ for all $|\operatorname{Re} z| < \frac{1}{2}$ when $\sigma > 0$ is chosen sufficiently small. $\square$

Theorem 4.2.8 *For every $\gamma, \mu \in \mathbb{R}$ and $E \in \operatorname{Vect}(X)$ there exists an elliptic operator $D^\mu \in L^\mu(X, \boldsymbol{g}; E, E)$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \infty)$ such that*

(i) $\sigma_\psi(D^\mu)(\xi) = |\xi|^\mu \operatorname{id}_E$,

(ii) *D^μ induces isomorphisms $D^\mu : \mathcal{W}^{s,\gamma}(X, E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(X, E)$ for all $s \in \mathbb{R}$,*

(iii) *$(D^\mu)^{-1} \in L^{-\mu}(X, \boldsymbol{g}^{-1}; E, E)$ where $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \infty)$.*

Theorem 4.2.9 *For every $J \in \operatorname{Vect}(Y)$ and suitable $N \in \mathbb{N}$ (such that J is a subbundle of $Y \times \mathbb{C}^N$), and for every $\gamma \in \mathbb{R}$ there exist elliptic operators $\mathcal{A}_+ = \begin{pmatrix} A_+ \\ T \end{pmatrix}$, $\mathcal{A}_- = \begin{pmatrix} A_- & K \end{pmatrix}$ in $\mathcal{Y}^0(X, \boldsymbol{g}; \boldsymbol{v}_\pm)$ for $\boldsymbol{g} = (\gamma, \gamma, \infty)$, $\boldsymbol{v}_+ = (N, N; 0, J)$, $\boldsymbol{v}_- = (N, N; J, 0)$ (where N stands for the trivial bundle $X \times \mathbb{C}^N$) such that*

(i) $A_\pm = 1 \bmod Y^0_{M+G}(X, \boldsymbol{g}; N, N)$,

(ii) *the operators*

$$\mathcal{A}_+ : \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \to \begin{matrix} \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \\ \oplus \\ H^s(Y, J) \end{matrix}, \qquad \mathcal{A}_- : \begin{matrix} \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \\ \oplus \\ H^s(Y, J) \end{matrix} \to \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \tag{4.2.7}$$

are isomorphisms for all $s \in \mathbb{R}$,

(iii) $\mathcal{A}_\pm^{-1} \in \mathcal{Y}^0(X, \boldsymbol{g}, \boldsymbol{v}_\mp)$.

Proof of Theorem 4.2.8. We will show a parameter-dependent variant of the assertion and then get the result by freezing the parameter at a point of sufficiently large absolute value. For simplicity, we consider the case of the trivial bundle $E = X \times \mathbb{C}$. The general case is analogous, we only have to multiply symbols by the identity map in the fibres of E. First assume that $|\xi|$ refers to a Riemannian metric on X that is the product metric from $[0, 1) \times Y$ in a collar neighbourhood of the boundary with a given Riemannian metric on Y. Then we have $|\xi|^2 = |\tau|^2 + |\eta|^2$ near the boundary. Starting from local parameter-dependent amplitude functions $(1 + |\xi|^2 + |\lambda|^2)^{\mu/2}$, $\lambda \in \Lambda$, and applying an operator convention based on the Fourier transform, then a summation, using a finite atlas and a subordinate partition of unity, we get an element $D_0^\mu(\lambda) \in L_{\mathrm{cl}}^\mu(\operatorname{int} X \times \Lambda)$ with $|\xi, \lambda|^\mu$ as the parameter-dependent principal symbol of order μ with respect to the covariables ξ including parameters λ. Let us show that for every $\gamma \in \mathbb{R}$ there is an element $D^\mu(\lambda) \in L^\mu(X \times \Lambda, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \infty)$, such that $D^\mu(\lambda) - D_0(\lambda) \in L^{-\infty}(\operatorname{int} X \times \Lambda)$. To this end we choose functions $\sigma, \sigma_0, \sigma_1 \in C^\infty(X)$ supported in a collar neighbourhood of the boundary Y, such that $\sigma \equiv 1$, $\sigma_0 \equiv 1$ and $\sigma_1 \equiv 1$ near Y, and $\sigma\sigma_0 = \sigma$, $\sigma\sigma_1 = \sigma_1$ and set

$$D^\mu(\lambda) = H^\mu(\lambda) + (1-\sigma) D_0^\mu(\lambda)(1-\sigma_1). \tag{4.2.8}$$

Locally near Y in the variables $(t, y) \in \overline{\mathbb{R}}_+ \times \Omega$ we set $H^\mu(\lambda) = \operatorname{Op}(a)(\lambda)$ for an operator-valued symbol $a(y, \eta, \lambda) := \sigma(t)\{b(y, \eta, \lambda) + m(\eta, \lambda) + g(\eta, \lambda)\}\sigma_0(t)$, where

$$\begin{aligned} b(y, \eta, \lambda) = t^{-\mu}\omega_1(t[\eta, \lambda]) \operatorname{op}_M^\gamma(h)(y, \eta, \lambda)\omega_2(t[\eta, \lambda]) \\ + (1 - \omega_1(t[\eta, \lambda])) \operatorname{op}_t(p)(y, \eta, \lambda)(1 - \omega_3(t[\eta, \lambda])) \end{aligned} \tag{4.2.9}$$

for a symbol $p \in S_{\mathrm{cl}}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}_\xi^n \times \Lambda)$ with $|\xi, \lambda|^\mu = (|\xi|^2 + |\lambda|^2)^{\mu/2}$ as the parameter-dependent homogeneous principal part, and

$$(m + g)(\eta, \lambda) \in R_{M+G}^\mu(\mathbb{R}_\eta^{n-1} \times \Lambda, \boldsymbol{g}). \tag{4.2.10}$$

Here, according to relation (1.3.3), the Mellin symbol h is chosen in the form $h(t, y, z, \eta, \lambda) = \widetilde{h}(t, y, z, t\eta, t\lambda)$, where $\widetilde{h}(t, y, z, \widetilde{\eta}, \widetilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^\mu(\mathbb{R}_{\widetilde{\eta}}^{n-1} \times \Lambda))$, such that

$$t^{-\mu} \operatorname{op}_M^\gamma(h)(y, \eta, \lambda) = \operatorname{op}_t(p)(y, \eta, \lambda) \bmod C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+; \mathbb{R}_\eta^{n-1} \times \Lambda)).$$

The homogeneous principal boundary symbol of $b(y, \eta, \lambda)$ equals

$$\begin{aligned} \sigma_\partial(b)(\eta, \lambda) = t^{-\mu}\omega_1(t|\eta, \lambda|) \operatorname{op}_M^\gamma(h_0)(\eta, \lambda)\omega_2(t|\eta, \lambda|) \\ + (1 - \omega_1(t|\eta, \lambda|)) \operatorname{op}_t(p_{(\mu)})(\eta, \lambda)(1 - \omega_3(t|\eta, \lambda|)). \end{aligned} \tag{4.2.11}$$

It is invariant as an operator function parametrised by points $(y, \eta, \lambda) \in T^*Y \times \Lambda \setminus 0$. By virtue of the special form of $p_{(\mu)}$ we can choose h in such a way that (4.2.11) only depends on $|\eta, \lambda|$.

We now apply the result of Lemma 4.2.11 below, namely the existence of an element (4.2.10) such that

$$\sigma_\partial(a)(\eta, \lambda) = \sigma_\partial(b)(\eta, \lambda) + \sigma_\partial(m + g)(\eta, \lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{4.2.12}$$

is a family of isomorphisms for all $(y, \eta, \lambda) \in T^*Y \times \Lambda \setminus 0$, $s \in \mathbb{R}$, also, where (4.2.12) depends on $|\eta, \lambda|$. This shows that the operator $D^\mu(\lambda)$ is parameter-dependent elliptic in the sense of Definition 4.2.2, where the bundles J_-, J_+ do not occur (they are of fibre dimension 0). From Theorem 4.2.5 we conclude that $D^\mu(\lambda) : \mathcal{W}^{s,\gamma}(X) \to \mathcal{W}^{s-\mu,\gamma-\mu}(X)$ is a family of isomorphisms for all $|\lambda| > c$ for some $c > 0$. Thus it suffices to set $D^\mu := D^\mu(\lambda_0)$ for a $\lambda_0 \in \mathbb{R}$, $|\lambda_0| > c$. Let us now consider the case when the Riemannian metric on X is not necessarily the product metric from $[0, 1) \times Y$. In this case it suffices to observe that in the bijectivity condition for (4.2.12) only the restiction of $(|\xi|^2 + |\lambda|^2)^{\mu/2}$ to the boundary is involved. However, this has again the behaviour of $(|\tau|^2 + |\eta|^2 + |\lambda|^2)^{\mu/2}$ i.e., we can argue as before. To complete the proof we have to verify property (iii). What we always know is the invertibility of D^μ in $\mathcal{Y}^{-\mu}(X, (\gamma - \mu, \gamma); E, E)_{\mathcal{O}}$ for some $\varepsilon > 0$. But we have, in fact, $(D^\mu)^{-1} \in L^{-\mu}(X, \boldsymbol{g}^{-1}; E, E)$, because the interior symbol of the $(D^\mu)^{-1}$ is as required, and the occurring smoothing Mellin and Green operators have asymptotic types that are independent of $y \in Y$, cf. formula (4.2.11). $\square$

Remark 4.2.10 *The proof of Theorem* 4.2.8 *shows that there is a parameter-dependent analogue of the assertion, i.e., there exists an element* $D^\mu(\lambda) \in L^\mu(X \times \Lambda, \boldsymbol{g}; E, E)$ *for* $\boldsymbol{g} = (\gamma, \gamma - \mu, \infty)$ *with* $|\xi, \lambda|^\mu \cdot \mathrm{id}_E$ *as the parameter-dependent principal interior symbol such that* $D^\mu(\lambda) : \mathcal{W}^{s,\gamma}(X, E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(X, E)$ *are isomorphisms for all* $\lambda \in \mathbb{R}^p$. *It suffices to carry out the construction first with the parameter set* $\Lambda \times \mathbb{R} \ni (\lambda, \delta)$ *and then to insert* $\delta = \delta_0 \in \mathbb{R}$ *sufficiently large.*

Lemma 4.2.11 *Let* $p \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^n_\xi \times \Lambda)$ *be a symbol with* $(|\xi|^2 + |\lambda|^2)^{\mu/2}$ *as the parameter-dependent homogeneous principal part. Then for every* $\gamma \in \mathbb{R}$ *there is an element* (4.2.10) *such that* (4.2.12) *is an isomorphism for all* $(y, \eta, \lambda) \in \Omega \times (\mathbb{R}^{n-1} \times \Lambda \setminus \{0\})$, $s \in \mathbb{R}$.

Proof. Since the homogeneous principal part $p_{(\mu)}$ of p is independent of $y \in \Omega$, also h_0 does not depend on y, and hence it suffices to construct (4.2.10) in y-independent form.

$\sigma_\partial(b)(\eta, \lambda)$ is a family of operators in the cone algebra on $\mathbb{R}_+$, and its principal conormal symbol $u(z)$ is independent of η, λ. We have (with Euler's Γ-function)

$$u(z) = \{a^+ g^+(z + \mu) + a^- g^-(z + \mu)\}\Gamma(1 - z)/\Gamma(1 - z - \mu) \tag{4.2.13}$$

modulo a meromorphic Mellin symbol of order $-\infty$, where $a^\pm$ are non-vanishing constants. The latter relation is a consequence of the results in Chapter 2.

The function $u(z)$ is holomorphic in $\mathbb{C}$, and we have $u|_{\Gamma_\beta} \in S^\mu_{\mathrm{cl}}(\Gamma_\beta)$ for each β, uniformly in compact β-intervals. The symbols $u|_{\Gamma_\beta}$ are elliptic in $\operatorname{Im} z$ for each β. On the given weight line $\Gamma_{\frac{1}{2}-\gamma}$ the function $u(z)$ has at most a finite number of zeros of finite multiplicity. Let p_j denote these points and l_j the corresponding multiplicities, $j = 1, \ldots, M$. Then, if $g(z)$ is the function of Lemma 4.2.7, we have $\prod_{j=1}^m (1 + g(z - p_j))^{l_j} = 1 + f(z)$ for an element $f(z) \in M_R^{-\infty}$ for some asymptotic type R for Mellin symbols, and $u(z)(1 + f(z))$ has no zeros in a strip $|\operatorname{Re} z - (\frac{1}{2} - \gamma)| < \varepsilon$ for some $\varepsilon > 0$.

Setting $\varphi(\eta, \lambda) := 1 + \omega(t[\eta, \lambda]) \operatorname{op}^\gamma_M(f)\tilde{\omega}(t[\eta, \lambda])$ for some choice of cut-off functions ω, $\tilde{\omega}$, the composition

$$\sigma_\partial(b)(\eta, \lambda)\sigma_\partial(\varphi)(\eta, \lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{4.2.14}$$

is a family of Fredholm operators for $(\eta, \lambda) \neq 0$, $s \in \mathbb{R}$. Let k be the index of (4.2.14) (it is always independent of s) and let $f_{-k}(z)$ be a Mellin symbol in the sense of Proposition 4.2.6. Set $\psi(\eta, \lambda) := 1 + \omega(t[\eta, \lambda]) \operatorname{op}^\gamma_M(f_{-k})\tilde{\omega}(t[\eta, \lambda])$. Then

$$c_\partial(\eta, \lambda) := \sigma_\partial(b)(\eta, \lambda)\sigma_\partial(\varphi)(\eta, \lambda)\sigma_\partial(\psi)(\eta, \lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{4.2.15}$$

is a family of Fredholm operators. Applying Lemma 4.2.12 below we find a family $r_\partial(\eta, \lambda)$ of operators $\mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$, smooth in $(\eta, \lambda) \neq 0$, with $r_\partial(\delta\eta, \delta\lambda) = \delta^\mu \kappa_\delta r_\partial(\eta, \lambda)\kappa_\delta^{-1}$ for all $\delta \in \mathbb{R}_+$, such that $r_\partial(\eta, \lambda)$ has a kernel in $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$, and $c_\partial(\eta, \lambda) + r_\partial(\eta, \lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$ is a family of isomorphisms for all $s \in \mathbb{R}$. Set $r(\eta, \lambda) := \chi(\eta, \lambda) r_\partial(\eta, \lambda)$ for an excision function χ in (η, λ). Then $a(y, \eta, \lambda) := b(y, \eta, \lambda)\varphi(\eta, \lambda)\psi(\eta, \lambda) + r(\eta, \lambda)$ has the form $b(y, \eta, \lambda) + m(y, \eta, \lambda) + g(y, \eta, \lambda)$ where $(m + g)(y, \eta, \lambda) \in R^\mu_{M+G}(\Omega \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \infty)$ and $\sigma_\partial(a)(\eta, \lambda) = c_\partial(\eta, \lambda) + r_\partial(\eta, \lambda)$ is an isomorphism for all $(\eta, \lambda) \neq 0$, $s \in \mathbb{R}$. $\square$

Lemma 4.2.12 *The operators* (4.2.15) *are of index zero for all* $s \in \mathbb{R}$, *and there is an element* $r_\partial(\eta, \lambda) \in \sigma_\partial R^\mu_G(\mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g})$ *for* $\boldsymbol{g} = (\gamma, \gamma - \mu, \infty)$ *that has a kernel in* $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ *for each* $(\eta, \lambda) \neq 0$, *where* $(c_\partial + r_\partial)(\eta, \lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$ *is a family of isomorphisms for all* $s \in \mathbb{R}$, $(\eta, \lambda) \neq 0$.

Proof. The operators (4.2.15) are (η,λ)-wise elliptic in the cone algebra on $\mathbb{R}_+$, and it is known, cf. [28], that dimensions of kernels and cokernels are independent of s. In this proof the dimension of covariables does not play any role; therefore, we simply use $\eta \in \mathbb{R}^q \setminus \{0\}$ for some q instead of $(\eta,\lambda) \in \mathbb{R}^{n-1} \times \Lambda \setminus \{0\}$. Moreover, for abbreviation we write c and r in place of c_∂ and r_∂, respectively. To construct $r(\eta)$ it suffices to assume $|\eta| = 1$ because our operator families are uniquely determined by their values on the unit sphere when their order is given. Fix s and set $H_1 := \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, $H_2 := \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$. Then $c(\eta) : H_1 \to H_2$ is a family of Fredholm operators of index zero, $\eta \in S^{q-1}$. A standard construction gives us the existence of some $M \in \mathbb{N}$ and of an isomorphism $k : \mathbb{C}^M \to H_2$ such that the row matrix $\begin{pmatrix} c(\eta) & k \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^M \end{matrix} \to H_2$ is surjective for all η and that in addition there is an isomorphism of the form

$$\begin{pmatrix} c(\eta) & k \\ b(\eta) & d(\eta) \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^M \end{matrix} \to \begin{matrix} H_2 \\ \oplus \\ \mathbb{C}^M \end{matrix}, \tag{4.2.16}$$

where $\begin{pmatrix} b(\eta) & d(\eta) \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^M \end{matrix} \xrightarrow{\cong} \mathbb{C}^M$ can be represented by choosing an orthonormal base in $\ker \begin{pmatrix} c(\eta) & k \end{pmatrix}$ and defining the map as a projection first to $\ker \begin{pmatrix} c(\eta) & k \end{pmatrix}$ composed with an isomorphism $\ker \begin{pmatrix} c(\eta) & k \end{pmatrix} \to \mathbb{C}^M$. This can be done in terms of scalar products with elements in $H_1 \oplus \mathbb{C}^M$. Since $C_0^\infty(\mathbb{R}_+)$ is dense both in H_1 and H_2, a simple approximation argument allows us to take k in the form $k : w \to \sum_{j=1}^M w_j \varphi_j$ for $w = (w_1, \dots, w_M)$, with functions $\varphi_j \in C_0^\infty(\mathbb{R}_+)$, and $\begin{pmatrix} b(\eta) & d(\eta) \end{pmatrix} \binom{u}{v} = (\int_0^\infty u(t)\psi(t) + \sum_{l=1}^M d_{jl} v_l)_{j=1,\dots,M}$ for $u \in H_1$, $v = (v_1, \dots, v_M) \in \mathbb{C}^M$ with functions $\psi_j \in C_0^\infty(\mathbb{R}_+)$. In general, ψ_j and d_{jl} depend on η. However, in the present situation they may be chosen to be independent of η, since our operator functions only depend on $|\eta|$ which equals 1. Without loss of generality we may assume that the $M \times M$-matrix d is invertible. Otherwise, if this is not the case for the original choice we can perturb d to an invertible matrix without violating the isomorphy of (4.2.16). Now a simple algebraic argument shows that (4.2.16) is invertible if and only if $c - kd^{-1}b$ is invertible. This allows us to set $r = -kd^{-1}b$. □

Proof of Theorem 4.2.9. By assumption, $J_1 := J$ is a subbundle of $Y \times \mathbb{C}^N$. Let J_0 be a complementary bundle such that $J_0 \oplus J_1 = Y \times \mathbb{C}^N$. Let $\omega(t)$, $\tilde{\omega}(t)$ be fixed cut-off functions and choose elements $f_k(z) \in M_{\mathcal{O}}^{-\infty}$ and $g_k(t,t') \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ according to Proposition 4.2.6 for $k = 0, 1$. Set $l_k(\eta,\lambda)u(t) := |\eta,\lambda| \int_0^\infty g_k(t|\eta,\lambda|, t'|\eta,\lambda|)u(t')\,dt'$ and

$$b_k(y,\eta,\lambda) := \{\omega(t|\eta,\lambda|)\,\mathrm{op}_M^\gamma(f_k)\tilde{\omega}(t|\eta,\lambda|) + l_k(\eta,\lambda)\} \otimes \mathrm{id}_{\pi_Y^* J_k},$$

where $\pi_Y : T^*Y \times \Lambda \setminus 0 \to Y$ is the canonical projection. If 1 denotes the identity operator in $\mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^N)$, then $\sigma_\partial(A_+)(y,\eta,\lambda) := 1 + \mathrm{diag}(b_0, b_1)(y,\eta,\lambda)$ locally in $y \in \Omega$ is a boundary symbol belonging to $1 + \sigma_\partial R^0_{M+G}(\Omega \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; N, N)$. The invariance allows us to interpret (y,η,λ) as points in $T^*Y \times \Lambda \setminus 0$. There is then an operator family $A(\lambda) := 1 + (M+G)(\lambda)$ for $(M+G)(\lambda) \in Y^0_{M+G}(X \times \Lambda, \boldsymbol{g}; N, N)$ that has $\sigma_\partial(A_+)$ as parameter-dependent principal boundary symbol. By construction

$$\sigma_\partial(A_+)(y,\eta,\lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^N) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^N)$$

is surjective for all $(y,\eta,\lambda) \in T^*Y \times \Lambda \setminus 0$, and we have $\ker \sigma_\partial(A_+)(y,\eta,\lambda) = J_{1,y}$ (which is the fibre of J_1 over $y \in Y$). Choose a family of maps

$$\sigma_\partial(T)(y,\eta,\lambda) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^N) \to \pi_Y^* J_1$$

such that $\sigma_\partial(T)(y,\delta\eta,\delta\lambda) = \sigma_\partial(T)(y,\eta,\lambda)\kappa_\delta^{-1}$ for all $\delta \in \mathbb{R}_+$ and that $\sigma_\partial(T)(y,\eta,\lambda)$ induces an isomorphism

$$\sigma_\partial(T) : \ker \sigma_\partial(A_+) \to \pi_Y^* J_1 .$$

Since $C_0^\infty(\mathbb{R}_+, \mathbb{C}^N)$ is dense in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^N)$, we find $\sigma_\partial(T)$ in such a way that it acts for each y as a vector of scalar products $u(t) \to \left(\int_0^\infty (u(t'), \varphi_j(y,\eta,\lambda,t'))_{\mathbb{C}^N}\, dt' \right)_{j=1,\dots,d}$ (where d is the fibre dimension of J_1) for suitable $\varphi_j(y,\eta,\lambda) \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^N)$, smoothly dependent on (y,η,λ). From $\sigma_\partial(t)$ we can pass to a family of operators $T(\lambda) : \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \to H^s(Y, J_1)$ belonging to $\mathcal{Y}^0(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v}_+)$ with $\sigma_\partial(T)$ as parameter-dependent principal symbol.

Since $\sigma_\partial(\mathcal{A}_+) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^N) \to \begin{matrix} \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^N) \\ \oplus \\ \pi_Y^* J \end{matrix}$ is an isomorphism (recall that we set $J_1 = J$), the operator family

$$\mathcal{A}_+(\lambda) := \begin{pmatrix} A_+(\lambda) \\ T(\lambda) \end{pmatrix} : \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \to \begin{matrix} \mathcal{W}^{s,\gamma}(X, \mathbb{C}^N) \\ \oplus \\ H^s(Y, J) \end{matrix}$$

is parameter-dependent elliptic and hence induces isomorphisms for all sufficiently large $|\lambda|$. Thus, we may set $\mathcal{A}_+ := \mathcal{A}_+(\tilde{\lambda})$ for any $\tilde{\lambda} \in \Lambda$ of sufficiently large absolute value. The property (iii) is a consequence of the fact that $\sigma_M(A_+)(z) : \mathbb{C}^N \to \mathbb{C}^N$ is a family of isomorphisms for all $z \in \mathbb{C}$, i.e., Mellin and Green operators belonging to $\mathcal{A}_+^{-1}$ have trivial asymptotic types. The construction of $\mathcal{A}_-$ is completely analogous; details will be omitted. □

Similarly to Remark 4.2.10 there is a parameter-dependent analogue of Theorem 4.2.9.

Remark 4.2.13 *Let $J := Y \times \mathbb{C}^N$ be a trivial bundle (where we also write N instead of J); then there exist elliptic operators $\mathcal{A}_+ = \begin{pmatrix} A_+ \\ T \end{pmatrix}$, $\mathcal{A}_- = \begin{pmatrix} A_- & K \end{pmatrix}$ in $\mathcal{Y}^0(X, \boldsymbol{g}; \boldsymbol{v}_\pm)$ for $\boldsymbol{g} = (\gamma, \gamma, \infty)$, $\boldsymbol{v}_+ = (1,1;0,N)$, $\boldsymbol{v}_- = (1,1;N,0)$, such that*

(i) $A_\pm = 1 \bmod Y^0_{M+G}(X, \boldsymbol{g}; 1, 1)$,

(ii) *the operators*

$$\mathcal{A}_+ : \mathcal{W}^{s,\gamma}(X) \to \begin{matrix} \mathcal{W}^{s,\gamma}(X) \\ \oplus \\ H^s(Y, \mathbb{C}^N) \end{matrix}, \qquad \mathcal{A}_- : \begin{matrix} \mathcal{W}^{s,\gamma}(X) \\ \oplus \\ H^s(Y, \mathbb{C}^N) \end{matrix} \to \mathcal{W}^{s,\gamma}(X) \tag{4.2.17}$$

are isomorphisms for all $s \in \mathbb{R}$ where $\mathcal{A}_\pm^{-1} \in \mathcal{Y}^0(X, \boldsymbol{g}; \boldsymbol{v}_\mp)$.

In fact, using the arguments in the proof of Theorem 4.2.9 we first find isomorphisms (4.2.17) for the case $N = 1$. Let $\mathcal{A}_\pm^1$ denote these operators. To construct (4.2.17) for arbitrary N, say, for the plus-case, it suffices to successively form compositions $\mathcal{A}_+^N := \begin{pmatrix} \mathcal{A}_+^1 & 0 \\ 0 & \mathrm{id}_{H^s(Y)} \end{pmatrix} \mathcal{A}_+^{N-1}$ for arbitrary $N \geq 2$. Similarly, we can proceed in the minus-case.

4.3 Homotopies and operators with the transmission property

To analyse the nature of elliptic pseudodifferential boundary value problems it is often useful to apply homotopies through elliptic elements. In the present section we shall show that elliptic operators in $\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \theta)$, $\boldsymbol{v} = (E, F; J_-, J_+)$, can be connected by a homotopy with elliptic boundary value problems with the transmission property (up to a stabilisation).

Let us briefly summarise what we have constructed above in connection with the algebra of boundary value problems $\mathcal{A}$ of the class $\mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$, $\mu \in \mathbb{R}$. Every $\mathcal{A}$ has a (parameter-dependent) principal symbol $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})) =: (p_{(\mu)}, a_{(\mu)})$ consisting of interior and boundary components. These are homomorphisms

$$p_{(\mu)} : \pi_X^* E \to \pi_X^* F, \tag{4.3.1}$$

$\pi_X : T^*X \times \Lambda \setminus 0 \to X$, and

$$a_{(\mu)} : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}, \tag{4.3.2}$$

$\pi_Y : T^*Y \times \Lambda \setminus 0 \to Y$, where $p_{(\mu)}(x, \xi, \lambda)$ is smooth up to the boundary and homogeneous of order μ in $(\xi, \lambda) \neq 0$ in the usual sense, while $a_{(\mu)}(y, \eta, \lambda)$ is homogeneous in the sense

$$a_{(\mu)}(y, \delta\eta, \delta\lambda) = \delta^\mu \operatorname{diag}(\kappa_\lambda, \mathrm{id}_{J_y^+}) a_{(\mu)}(y, \eta, \lambda) \operatorname{diag}(\kappa_\lambda, \mathrm{id}_{J_y^-})$$

for all $(y, \eta, \lambda) \in T^*Y \times \Lambda \setminus 0$, $\delta \in \mathbb{R}_+$. In addition, we have a subordinate principal conormal symbol, cf. Remark 3.2.3, that is a family of homomorphisms

$$\sigma_M(a_{(\mu)})(y, z) : E'_y \to F'_y \tag{4.3.3}$$

$y \in Y$, $z \in \Gamma_{\frac{1}{2}-\gamma}$. Let $\nu^\pm(y) : E'_y \to F'_y$ denote the limits of $|\operatorname{Im} z|^{-\mu} \sigma_M(a_{(\mu)})(y, z)$ for $\operatorname{Im} z \to \mp\infty$.

There is a compatibility between (4.3.1) and (4.3.2). In fact, the restriction $p_0(y, \xi, \lambda)$ of $p_{(\mu)}(x, \xi, \lambda)$ to $T^*X|_Y \times \Lambda \setminus 0$ is uniquely determined by the upper left corner of (4.3.2). In addition, there is a compatibility condition between (4.3.1) and (4.3.3).

Consider $p_{(\mu)}(0, y, \tau, \eta, \lambda)$, the restriction of $p_{(\mu)}(x, \xi, \lambda)$ to the boundary. Then, for $\tau \neq 0$ we have $p_{(\mu)}(0, y, \tau, \eta, \lambda) = |\tau|^\mu p_{(\mu)}(0, y, \tau/|\tau|, \eta/|\tau|, \lambda/|\tau|)$, and the compatibility is

$$\nu^\pm(y) = p_{(\mu)}(0, y, \pm 1, 0, 0). \tag{4.3.4}$$

According to the constructions of Section 3.2 the upper left corner $a_{11,(\mu)}$ of $a_{(\mu)}$ is given by an expression of the form (3.2.7), where, in particular $g =: g_{11,(\mu)}$ is the upper left corner of a homogeneous Green symbol $g_{(\mu)} = (g_{ij,(\mu)})_{i,j=1,2}$, that is locally of the class $\sigma_\partial \mathcal{R}_G^\mu(\Omega \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}; \boldsymbol{v})_{P,Q}$ for some asymptotic types P, Q. More precisely, we have

$$a_{(\mu)}(y, \eta, \lambda) = \begin{pmatrix} (b_{(\mu)} + m_{(\mu)})(y, \eta, \lambda) & 0 \\ 0 & 0 \end{pmatrix} + g_{(\mu)}(y, \eta, \lambda), \tag{4.3.5}$$

where

$$\begin{aligned} b_{(\mu)}(y, \eta, \lambda) = {} & t^{-\mu} \omega_1(t|\eta, \lambda|) \operatorname{op}_M^\gamma(h_0)(y, \eta, \lambda) \omega_2(t|\eta, \lambda|) \qquad (4.3.6) \\ & + t^{-\mu}(1 - \omega_1(t|\eta, \lambda|)) \operatorname{op}_t(p_0)(y, \eta, \lambda)(1 - \omega_3(t|\eta, \lambda|)), \\ m_{(\mu)}(y, \eta, \lambda) = {} & t^{-\mu} \omega(t|\eta, \lambda|) \operatorname{op}_M^\gamma(h_{00})(y) \widetilde{\omega}(t|\eta, \lambda|) + n_{(\mu)}(y, \eta, \lambda) \qquad (4.3.7) \end{aligned}$$

for $n_{(\mu)}(y, \eta, \lambda) = t^{-\mu} \omega(t|\eta, \lambda|) \sum_{j=1}^k t^j \sum_{|\alpha|=j} \operatorname{op}_M^{\gamma_j}(h_{j\alpha})(y)(\eta, \lambda)^\alpha \widetilde{\omega}(t|\eta, \lambda|)$, which is an element of $\sigma_\partial R_G^\mu(\Omega \times \mathbb{R}^{n-1} \times \Lambda, (\gamma, \gamma - \mu); l, m)_{\mathcal{O}}$.

Remark 4.3.1 *To each pair* $(p_{(\mu)}, a_{(\mu)})$ *of this structure there is an* $\mathcal{A} \in \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ *such that* $\sigma_\psi(\mathcal{A}) = p_{(\mu)}$ *and* $\sigma_\partial(\mathcal{A}) = a_{(\mu)}$.

In other words, $\operatorname{symb} \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v}) := \{(\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})) : \mathcal{A} \in \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})\}$ *consists of the space of all pairs* $(p_{(\mu)}, a_{(\mu)})$ *of the described kind. Let* op *denote the choice of a map* $(p_{(\mu)}, a_{(\mu)}) \to \mathcal{A}$ *such that* $\sigma(\mathcal{A}) = (p_{(\mu)}, a_{(\mu)})$.

An element $\sigma = (p_{(\mu)}, a_{(\mu)}) \in \operatorname{symb} \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ is called elliptic, if the components define isomorphisms (4.3.1) and (4.3.2), respectively.

Recall that for every $a_{(\mu)}$ we have a subordinate principal conormal symbol $\sigma_M(a_{(\mu)})(y, z)$, cf. Remark 3.2.3, that only depends on the upper left corner of $a_{(\mu)}$. If $(p_{(\mu)}, a_{(\mu)})$ is elliptic,

$$\sigma_M(a_{(\mu)})(\cdot, z) : E' \to F' \tag{4.3.8}$$

is a family of isomorphisms for all $z \in \Gamma_{\frac{1}{2}-\gamma}$. A pair $(p_{(\mu)}, h)$, where $p_{(\mu)}$ is of the form (4.3.1) and $h(y, z)$ of the form of a principal conormal symbol $\sigma_M(\text{u.l.c.}\, a_{(\mu)})$, i.e., $h(y, z) = \widetilde{h}(0, y, z, 0, 0) + h_{00}(y, z)$ associated with $p_{(\mu)}$ is said to be elliptic (with respect to the weight γ), if both $p_{(\mu)} : \pi_X^* E \to \pi_X^* F$ and $h(\cdot, z) : E' \to F'$ are isomorphisms, the latter ones for all $z \in \Gamma_{\frac{1}{2}-\gamma}$.

Definition 4.3.2 *A family* $(p^r_{(\mu)}, a^r_{(\mu)})_{0 \le r \le 1}$ *of elements in* $\operatorname{symb} \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ *is said to be a homotopy, if* $(p^r_{(\mu)})_{0 \le r \le 1}$ *is a homotopy of homogeneous symbols* $\pi_X^* E \to \pi_X^* F$ *as usual, while the other ingredients of* $a^r_{(\mu)}$ *are continuous in* $r \in [0, 1]$. *More precisely, writing (in local coordinates)* $a^r_{(\mu)}(y, \eta, \lambda)$ *analogously to* (4.3.5), *with upper subscript* r *in the involved summands and* h_0, p_0 *and* $h_{j\alpha}$ *in* (4.3.6), (4.3.7) *being replaced by* h^r_0, p^r_0 *and* $h^r_{j\alpha}$, *respectively, (where* γ^j *in* (4.3.7) *is independent of* r*) we require* $\widetilde{h}^r(0, y, z, \widetilde{\eta}, \widetilde{\lambda}) \in C([0, 1], C^\infty(\Omega, M^\mu_{\mathcal{O}}(\mathbb{R}^{n-1}_{\widetilde{\eta}, \widetilde{\lambda}} \times \Lambda))) \otimes \mathbb{C}^m \otimes \mathbb{C}^l$ *for* $h^r_0(t, y, z, \eta, \lambda) = \widetilde{h}^r(0, y, z, \widetilde{\eta}, \widetilde{\lambda})|_{\widetilde{\eta} = t\eta, \widetilde{\lambda} = t\lambda}$ *further* $h^r_{00}(y, z) \in C([0, 1], C^\infty(\Omega, M^{-\infty}_{\gamma, \varepsilon})) \otimes \mathbb{C}^m \otimes \mathbb{C}^l$, *and* $g^r_{(\mu)}(y, \eta, \lambda) \in C([0, 1], \sigma_\partial \mathcal{R}^\mu_G(\Omega \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}_\varepsilon; \boldsymbol{v})), n^r_{(\mu)}(y, \eta, \lambda) \in C([0, 1], \sigma_\partial R^\mu_G(\Omega \times \mathbb{R}^{n-1} \times \Lambda, \boldsymbol{g}_\varepsilon; l, m))$ *for some* $0 < \varepsilon < 1$, $\boldsymbol{g}_\varepsilon = (\gamma, \gamma - \mu, \varepsilon)$.

Finally, a family of pairs $(p^r_{(\mu)}, h^r)_{0 \le r \le 1}$ *is said to be a* (σ_ψ, σ_M)*-homotopy, if* $p^r_{(\mu)} : \pi_X^* E \to \pi_X^* F$ *is as before, and* $h^r(y, z)$, *supposed to be of the form* $h^r(y, z) = \widetilde{h}^r(0, y, z, 0, 0) + h^r_{00}(y, z)$ *with* $\widetilde{h}^r$ *and* h^r_{00} *as mentioned above, is required to be continuous in the sense that* $\widetilde{h}^r(0, y, z, 0, 0) \in C([0, 1], C^\infty(\Omega, M^\mu_{\mathcal{O}})) \otimes \mathbb{C}^m \otimes \mathbb{C}^l$ *and* $h^r_{00}(y, z) \in C([0, 1], C^\infty(\Omega, M^{-\infty}_{\gamma, \varepsilon})) \otimes \mathbb{C}^m \otimes \mathbb{C}^l$ *for some* $0 < \varepsilon < 1$, *where the compatibility condition* (4.3.4) *is fulfilled for all* $r \in [0, 1]$ *with* $\nu^{\pm, r}(y) := \lim\{|\operatorname{Im} z|^{-\mu} h^r(y, z) : \operatorname{Im} z \to \mp\infty\}$.

Lemma 4.3.3 *Let* $\sigma^0 := (p^0_{(\mu)}, a^0_{(\mu)}) \in \operatorname{symb} \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{v} = (E, F; J_-, J_+)$ *be elliptic, and let* $(p^r_{(\mu)}, h^r)_{0 \le r \le 1}$ *be a* (σ_ψ, σ_M)*-homotopy, where both components are elliptic for all* $r \in [0, 1]$, *i.e.,* $p^r_{(\mu)} : \pi_X^* E \to \pi_X^* F$, *as well as* $h^r(\cdot, z) : E' \to F'$ *are isomorphisms (the second one for all* $z \in \Gamma_{\frac{1}{2}-\gamma}$*), and assume* $\sigma_M(a^0_{(\mu)}) = h^0$. *Then there exists an* $L \in \operatorname{Vect}(Y)$ *and a homotopy through elliptic elements* $\widetilde{\sigma}^r := (p^r_{(\mu)}, \widetilde{a}^r_{(\mu)}) \in \operatorname{symb} \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \widetilde{\boldsymbol{v}})$, $r \in [0, 1]$, $\widetilde{\boldsymbol{v}} = (E, F; J_- \oplus L, J_+ \oplus L)$, *such that* $\sigma_M(\widetilde{a}^r_{(\mu)}) = h^r$ *for all* $r \in [0, 1]$ *and* $\widetilde{\sigma}^0 = (p^0_{(\mu)}, \operatorname{diag}(a^0_{(\mu)}, r_{(\mu)} \otimes \mathrm{id}_{\pi_Y^* L}))$, *where* $r_{(\mu)}(\eta, \lambda) := |\eta, \lambda|^\mu$.

Proof. Let us define a homotopy $(a^r_{11,(\mu)})_{0\le r\le 1}$ of operator families

$$\begin{aligned}a^r_{11,(\mu)}(y,\eta,\lambda) = t^{-\mu}\omega_1(t|\eta,\lambda|)\operatorname{op}^\gamma_M(h^r_0)(y,\eta,\lambda)\omega_2(t|\eta,\lambda|)\\ + t^{-\mu}(1-\omega_1(t|\eta,\lambda|))\operatorname{op}^+(p^r_{(\mu)})(y,\eta,\lambda)(1-\omega_3(t|\eta,\lambda|))\\ + m^r_{(\mu)}(y,\eta,\lambda) + g^r_{11,(\mu)}(y,\eta,\lambda), \quad (4.3.9)\end{aligned}$$

where $m^0_{(\mu)}$ and $g^0_{11,(\mu)}$ are the smoothing Mellin and Green summands for $r=0$ that occur in (3.2.7) in the notation $\sigma_\partial(m)$ and $\sigma_\partial(g)$, respectively (here, taken with parameters). In other words, $a^r_{11,(\mu)}|_{r=0}$ is the upper left corner of $a^0_{(\mu)}$. To get (4.3.9) it suffices to construct $h^r_0(t,y,z,\eta,\lambda)$ and $m^r_{(\mu)}(y,\eta,\lambda)$. For the Green summands that are families of compact operators we simply set $g^r_{11,(\mu)} = (1-r)g^0_{11,(\mu)}$. Moreover, we may choose $m^r_{(\mu)}(y,\eta,\lambda)$ to be of the form $m^r_{(\mu)}(y,\eta,\lambda) = t^{-\mu}\omega(t|\eta,\lambda|)\operatorname{op}^\gamma_M(h^r_{00})(y)\widetilde{\omega}(t|\eta,\lambda|) + (1-r)n^0_{(\mu)}(y,\eta,\lambda)$, cf., similarly, formula (4.3.7), where $n^0_{(\mu)}$ is the compact remainder belonging to $m^0_{(\mu)}$. In other words, we have to define $h^r_{00}(y,z)$. Let $\chi(\tau,\eta,\lambda)$ be an excision function and set $p^r(y,\tau,\eta,\lambda) := \chi(\tau,\eta,\lambda)p^r_{(\mu)}(0,y,\tau,\eta,\lambda)$. There is an $\widetilde{h}^r(y,z,\widetilde{\eta},\widetilde{\lambda}) \in C^\infty(\Omega, M^\mu_{\mathcal{O}}(\mathbb{R}^{n-1}_{\widetilde{\eta}}\times\Lambda_{\widetilde{\lambda}}))$ such that, when we set $h^r_0(t,y,z,\eta,\lambda) := \widetilde{h}^r(y,z,t\eta,t\lambda)$, we have

$$\operatorname{op}^\gamma_M(h^r_0)(y,\eta,\lambda) = t^{-\mu}\operatorname{op}_t(p^r)(y,\eta,\lambda)$$

mod $C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+;\mathbb{R}^{n-1}_\eta\times\Lambda_\lambda))$. Here, we first argue in local terms with respect to $y\in\Omega$, $\Omega\subseteq\mathbb{R}^{n-1}$ open, but then, when we include $h^r_{00}(y,z)$ to be defined yet, our final operator functions will be invariant as families on $(y,\eta,\lambda)\in(T^*Y\times\Lambda)\setminus 0$.

We now have $h^r_{00}(y,z) := h^r(y,z) - \widetilde{h}^r(y,z,0,0) \in M^{-\infty}_R$ for every y, where $R=R(y)$ is a certain asymptotic type for Mellin symbols that is inherited from the meromorphic summand of h^r of order $-\infty$ (since $\widetilde{h}^r_0(y,z,0,0)$ itself is holomorphic in z). Those summands of order $-\infty$ are elements of $M^{-\infty}_{\gamma,\varepsilon}$ for some $\varepsilon>0$ for all $r\in[0,1]$. Thus, $h^r_{00}(y,z)$ is as desired.

Then,

$$\begin{pmatrix} a^r_{11,(\mu)} & a^0_{12,(\mu)}\end{pmatrix} : \pi^*_Y\begin{pmatrix}\mathcal{K}^{s,\gamma}(\mathbb{R}_+)\otimes E'\\ \oplus\\ J_-\end{pmatrix} \to \pi^*_Y\mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)\otimes F'$$

is a family of Fredholm operators, surjective for $r=0$. It is then clear that there exists an $L\in\operatorname{Vect}(Y)$ and a potential symbol $k:\pi^*_Y L\to\pi^*_Y\mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)\otimes F'$ such that

$$\begin{pmatrix} a^r_{(\mu)} & rk\end{pmatrix} : \pi^*_Y\begin{pmatrix}\mathcal{K}^{s,\gamma}(\mathbb{R}_+)\otimes E'\\ \oplus\\ J_-\\ \oplus\\ L\end{pmatrix} \to \pi^*_Y\mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)\otimes F' \quad (4.3.10)$$

is surjective for all $0\le r\le 1$ and all $(y,\eta,\lambda)\in T^*Y\times\Lambda\setminus 0$. The operator family

$$\begin{pmatrix} a^0_{21,(\mu)} & a^0_{22,(\mu)} & 0\\ 0 & 0 & r_{(\mu)}\otimes\operatorname{id}_{\pi^*_Y L}\end{pmatrix} \quad (4.3.11)$$

maps the kernel of (4.3.10) for $r=0$ isomorphically to $\pi^*_Y(J_+\oplus L)$. The contractibility of the interval $[0,1]$ then implies the existence of an extension of (4.3.11) to a family of isomorphisms

$$\begin{pmatrix} a^r_{21,(\mu)} & a^r_{22,(\mu)} & rk \\ a^r_{31,(\mu)} & a^r_{32,(\mu)} & a^r_{33,(\mu)} \end{pmatrix} : \ker \begin{pmatrix} a^r_{(\mu)} & rk \end{pmatrix} \to \pi^*_Y \begin{pmatrix} J_+ \\ \oplus \\ L \end{pmatrix} \tag{4.3.12}$$

continuous in $r \in [0,1]$ where $(a^0_{3j,(\mu)})_{1\le j\le 3} = (0, 0, \mathrm{id}_{\pi^*_Y L})$, and the entries of (4.3.12) belong to the class $\sigma_\partial \mathcal{R}^\mu_G(\dots)$ for an evident choice of weight and bundle data. □

A family $(\mathcal{A}^r)_{0\le r\le 1}$ of operators in $\mathcal{L}^\mu(X\times\Lambda, \boldsymbol{g}; \boldsymbol{v})$ is said to be a homotopy if it is continuous in $r \in [0,1]$ with respect to $\mathcal{L}(\mathcal{W}^{s,\gamma}(X,E)\oplus H^s(Y,J_-), \mathcal{W}^{s-\mu,\gamma-\mu}(X,F)\oplus H^{s-\mu}(Y,J_+))$ for every $s \in \mathbb{R}$. We talk about a homotopy through elliptic operators if $\mathcal{A}^r$ is elliptic for each $r \in [0,1]$.

Theorem 4.3.4 *Let $\mathcal{A}^0 \in \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ be an elliptic operator, $\boldsymbol{g} = (\gamma, \gamma-\mu, \theta)$, $\boldsymbol{v} = (E, F; J_-, J_+)$. Moreover, let $(p^r_{(\mu)}, h^r)_{0\le r\le 1}$ be a (σ_ψ, σ_M)-homotopy through elliptic elements in the sense of Lemma 4.3.3, where $\sigma^\mu_\psi(\mathcal{A}^0) = p^0_{(\mu)}$ and $\sigma_M\sigma_\partial(\mathcal{A}^0) = h^0$. Then there exists an $L \in \mathrm{Vect}(Y)$ and a family $(\widetilde{\mathcal{A}}^r)_{0\le r\le 1}$ of elements in $\mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \widetilde{\boldsymbol{v}})$ for $\widetilde{\boldsymbol{v}} = (E, F; J_- \oplus L, J_+ \oplus L)$ such that*

$$\widetilde{\mathcal{A}}^r(\lambda) : \begin{matrix} \mathcal{W}^{s,\gamma}(X,E) \\ \oplus \\ H^s(Y, J_- \oplus L) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y, J_+ \oplus L) \end{matrix} \tag{4.3.13}$$

is a continuous family of Fredholm operators for each $s \in \mathbb{R}$, and $\widetilde{\mathcal{A}}^0(\lambda) = \mathcal{A}^0(\lambda) \oplus R^\mu_L(\lambda)$, where $R^\mu_L(\lambda) : H^s(Y,L) \to H^{s-\mu}(Y,L)$ is an (elliptic) element of $L^\mu_{\mathrm{cl}}(Y \times \Lambda; L, L)$.

Proof. Starting from $\mathcal{A}^0$ and setting $\sigma^0 := (p^0_{(\mu)}, a^0_{(\mu)}) = (\sigma_\psi(\mathcal{A}^0), \sigma_\partial(\mathcal{A}^0))$ we have the situation of Lemma 4.3.3. We then find an $L \in \mathrm{Vect}(Y)$ and a homotopy $\widetilde{\sigma}^r := (p^r_{(\mu)}, \widetilde{a}^r_{(\mu)}) \in \mathrm{symb}\,\mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \widetilde{\boldsymbol{v}})$ through elliptic elements that connects $\widetilde{\sigma}^0 = (p^0_{(\mu)}, \mathrm{diag}(a^0_{(\mu)}, r_{(\mu)} \otimes \mathrm{id}_{\pi^*_Y L}))$ with $\widetilde{\sigma}^1 = (p^1_{(\mu)}, \widetilde{a}^1_{(\mu)})$. Then $\mathcal{A}^r(\lambda) = \mathrm{op}(\widetilde{\sigma}^r)$ (cf. the notation of Remark 4.3.1) is a Fredholm operator (4.3.13) for every $r \in [0,1]$, $\lambda \in \Lambda$, $s \in \mathbb{R}$, and it is clear that we have continuity in r with respect to the operator norm. □

We want to give a relation between operators in $\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})$ and boundary value problems with the transmission property at the boundary. Homogeneous principal symbols (4.3.1) may assumed to be induced by (classical) local symbols of integer order with the transmission property in the sense of [3], see also [17]. Such symbols (here considered with parameters $\lambda \in \Lambda = \mathbb{R}^l$), give rise to operators in $L^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \theta)$, when we apply the operator convention of Remark 4.3.1. Let $B^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$ for $\mu \in \mathbb{Z}$ denote the subspace of all $L^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$, where the local (parameter-dependent) symbol $p(t, y, \tau, \eta, \lambda)$ (in the variables $(t, y, \tau, \eta, \lambda)$ in a collar neighbourhood of the boundary) has the transmission property at the boundary ($t = 0$), and such that the conormal symbol to the order μ equals $c\Gamma(1-z)/\Gamma(1-z-\mu)$, where $c : E' \to F'$ is the homomorphism induced by the restriction of $p_{(\mu)}$ to the points $(r, y, \tau, \eta, \lambda) = (0, y, 1, 0, 0)$, cf. also formula (4.2.13). Moreover, let $\mathcal{B}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ denote the subspace of all $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2} \in \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ such that $\mathcal{A}_{11} \in B^\mu(X \times \Lambda, \boldsymbol{g}; E, F)$.

Remark 4.3.5 *By definition we have $\mathcal{B}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v}) \subset \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v}) \subset \mathcal{Y}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ (where the first inclusion refers to $\mu \in \mathbb{Z}$). The subspaces $\mathcal{B}^{-\infty} \subset \mathcal{L}^{-\infty} \subset \mathcal{Y}^{-\infty}$ (unions over all μ) remain preserved under compositions (if weight and bundle data fit together) and the* *components of the principal symbols behave multiplicative. $\mathcal{B}^{-\infty}$ is closed under parametrix construction of elliptic elements. For the other classes this is not the case when we insist on constant discrete asymptotic types in the Mellin and Green operators. However, if we enlarge the classes by operators with continuous asymptotics, cf.* [28], *we get parametrices within the respective classes .*

Lemma 4.3.6 *Let $p_{(\mu)}$ be an elliptic symbol of order $\mu \in \mathbb{Z}$, cf.* (4.3.1), *and let h be of the form of a principal conormal symbol associated with $p_{(\mu)}$, assumed to be a family of isomorphisms $h(\cdot, z) : E' \to F'$ parametrised by $z \in \Gamma_{\frac{1}{2}-\mu}$ (such that the pair $(p_{(\mu)}, h)$ is elliptic in the above mentioned sense with respect to the weight μ). Then there is a (σ_ψ, σ_M)-homotopy $(p^r_{(\mu)}, h^r)_{0\le r\le 1}$ through elliptic pairs, such that $(p^0_{(\mu)}, h^0) = (p_{(\mu)}, h)$, and $p^1_{(\mu)}$ has the transmission property at the boundary and $h^1(z) = c\Gamma(1-z)/\Gamma(1-z-\mu)$ for an isomorphism $c : E' \to F'$ that is induced by the restriction of $p^1_{(\mu)}$ to the points $(r, y, \tau, \eta, \lambda) = (0, y, 1, 0, 0)$.*

Proof. First observe that when we have two (σ_ψ, σ_M)-homotopies of elliptic pairs, the componentwise composition (defined, if the bundle from the image of the first factor fits to the one of the domain of the second factor) is again a (σ_ψ, σ_M)-homotopy. To be more precise, we set $(p^r_{(\mu)}, h^r)_{0\le r\le 1} \cdot (\widetilde{p}^r_{(\nu)}, \widetilde{h}^r)_{0\le r\le 1} = (p^r_{(\mu)}\widetilde{p}^r_{(\nu)}, (T^{-\nu}h^r)\widetilde{h}^r)_{0\le r\le 1}$ where $(T^{-\nu}h^r)(z) = h^r(z-\nu)$. Now, as is well-known, to each order $\mu \in \mathbb{Z}$ there exists a homogeneous elliptic symbol with the transmission property. Near the boundary we may set

$$b_{(-\mu)}(t, y, \tau, \eta, \lambda) := \left(\chi\left(\frac{\tau}{\alpha|\eta, \lambda|}\right)|\eta, \lambda| - i\tau\right)^{-\omega(t)\mu} |\tau, \eta, \lambda|^{-(1-\omega(t))\mu} \cdot \mathrm{id}_{\pi^* E}$$

where $\chi(\tau) \in \mathcal{S}(\mathbb{R})$ is a function such that $\mathrm{supp}(F^{-1}_{\tau\to t}\chi) \subset \mathbb{R}_-$, $\chi(0) = 1$, and $\alpha \gg \sup\{|(\partial_\tau\chi)(\tau)| : \tau \in \mathbb{R}\}$ a constant, cf. [9], or [21, Section 5.3], and ω is a cut-off function and $b_{(-\mu)}(x, \xi, \lambda) := |\xi, \lambda|^{-\mu} \cdot \mathrm{id}_{\pi^* E}$ for all x where $1-\omega$ vanishes. To $b_{(-\mu)}$ we choose an $\widetilde{h}$ such that $(b_{(-\mu)}, \widetilde{h})$ is an elliptic pair with respect to the weight $\gamma = 0$, namely $\widetilde{h}(z) := b\Gamma(1-z)/\Gamma(1-z+\mu)$, where b is the restriction of $b_{(\mu)}$ to $(0, y, 1, 0, 0)$. Assume for a moment that $(p^r_{(\mu)}, h^r)_{0\le r\le 1}$ is already found. Then

$$(p^r_{(\mu)}, h^r)_{0\le r\le 1}(b_{(-\mu)}, \widetilde{h}) = (p^r_{(\mu)}b_{(-\mu)}, (T^\mu h^r)\widetilde{h})_{0\le r\le 1} \tag{4.3.14}$$

is a (σ_ψ, σ_M)-homotopy through pairs that are elliptic of order 0 and with respect to the weight 0. Formula (4.3.14) reduces the construction of $p^r_{(\mu)}$ and $h^r(z)$ to the case of order zero and weight zero. In other words, without loss of generality we may assume $\mu = 0$ and construct $p^r_{(0)}$ and $h^r(z)$ on $\Gamma_{\frac{1}{2}}$. For simplicity, we consider the case of trivial vector bundles E and F with fibre $\mathbb{C}$. Set $S^*Y := \{(y, \tau, \eta, \lambda) \in T^*X|_Y \times \mathbb{R}^p : |\tau, \eta, \lambda| = 1\}$ and $N := \{(y, \tau) : y \in Y, -1 \le \tau \le 1\}$, where τ is the covector to the normal variable t. Let us fix a diffeomorphism $\nu : Y \times (-1, 1) \to Y \times \Gamma_{\frac{1}{2}}$, such that $\nu(y, \tau) := (y, z(\tau))$, where $z(\tau) \in \Gamma_{\frac{1}{2}}$ and $\mathrm{Im}\, z(\tau) \to \pm\infty$ when $\tau \to \mp 1$. Then the ellipticity of the given pair $(p_{(0)}, h)$ means that $f^0 := (p_{(0)}, \nu^* h)$ represents a non-vanishing function on $S^*Y \cup N$. Setting $p^\pm(y) := p_{(0)}(0, y, \pm 1, 0, 0)$, we have $h(y, z) = p^+(y)g^+(z) + p^-(y)g^-(z) + l(y, z)$ where $l(y, z) \in C^\infty(Y, M^{-\infty}_{0,\varepsilon})$ for some $\varepsilon > 0$. A trivial geometric consideration shows that there is a homotopy $(f^r)_{0\le r\le 1}$ through non-vanishing functions on $S^*Y \cup N$ such that f^1 is represented by a pair of the form $(p^1_{(0)}, 1)$; here, we write $f^r = (p^r_{(0)}, g^r)$, i.e., $p^0_{(0)} = p_{(0)}$, $\nu^* h = g^0$. It is also evident that the functions g^r can be chosen in such a way that $(\nu^*)^{-1} g^r =: \widetilde{h}^r$ have the form $\widetilde{h}^r(y, z) = p^{r,+}(y)g^+(z) + p^{r,-}(y)g^-(z) + \widetilde{l}^r(y, z)$ where $p^{r,\pm}(y) := p^r_{(0)}(0, y, \pm 1, 0, 0)$,

$\widetilde{l}^r(y,z) \in C^\infty(Y, \mathcal{S}(\Gamma_{\frac{1}{2}}))$, and $\widetilde{l}^1(y,z) = 0$. Applying now a kernel cut-off argument to $\widetilde{l}^r(y,z)$ we can pass to a family $l^r(y,z) \in C^\infty(Y, M_{0,\varepsilon}^{-\infty})$ such that

$$h^r(y,z) := p^{r,+}(y)g^+(z) + p^{r,-}(y)g^-(z) + l^r(y,z)$$

is as desired, in particular, $l^1(y,z) = 0$ due to $\widetilde{l}^r(y,z) = 0$. □

Theorem 4.3.7 *Let $\mathcal{A}^0 \in \mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \boldsymbol{v})$ be an elliptic operator, $\mu \in \mathbb{Z}$, $\boldsymbol{g} = (\mu, 0, \theta)$, $\boldsymbol{v} = (E, F; J_-, J_+)$. Then there exists an $L \in \mathrm{Vect}(Y)$ and a homotopy $(\widetilde{\mathcal{A}}^r)_{0\le r\le 1}$ through elliptic elements in $\mathcal{L}^\mu(X \times \Lambda, \boldsymbol{g}; \widetilde{\boldsymbol{v}})$ for $\widetilde{\boldsymbol{v}} = (E, F; J_- \oplus L, J_+ \oplus L)$ such that $\widetilde{\mathcal{A}}^0 = \mathcal{A}^0 \oplus R_L^\mu$ (cf. the notation in Theorem 4.3.4) and $\widetilde{\mathcal{A}}^1 \in \mathcal{B}^\mu(X \times \Lambda, \boldsymbol{g}; \widetilde{\boldsymbol{v}})$.*

Proof. Let $(p^0_{(\mu)}, h^0)$ denote the pair consisting of the principal interior symbol and the principal conormal symbol of $\mathcal{A}^0$. Then Lemma 4.3.6 gives us a (σ_ψ, σ_M)-homotopy $(p^r_{(\mu)}, h^r)_{0\le r\le 1}$ through elliptic pairs such that $(p^1_{(\mu)}, h^1)$ is a pair where $p^1_{(\mu)}$ has the transmission property and h^1 is as in Lemma 4.3.6. To complete the proof it suffices to apply Theorem 3.3.2. □

4.4 Elliptic operators without additional conditions

In Theorem 4.2.8 above we have constructed elements D^μ in $L^\mu(X, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \theta)$, $\boldsymbol{v} = (E, E)$, that are elliptic without additional boundary conditions. In general, if we start from an arbitrary homogeneous principal symbol

$$p_{(\mu)} : \pi_X^* E \to \pi_X^* F \tag{4.4.1}$$

on X, it may happen that there is no choice of bundles J_-, J_+ on Y such that there is an elliptic operator $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_-, J_+)$, where $\sigma_\psi(\mathcal{A}) = p_{(\mu)}$. This phenomenon is well known for differential operators, cf. Atiyah and Bott [2], or, more generally, for pseudodifferential boundary value problems with the transmission property, cf. [3], [29]. In a forthcoming paper [31] we shall investigate the case of arbitrary symbols (4.4.1) (i.e., when the transmission property is not necessarily satisfied).

In this section we want to show other general cases where ellipticity always holds, again without boundary conditions.

Let X be a compact C^∞ manifold with boundary Y and $2X$ the double of X. We employ notation from the Appendix, in particular, the reflection map $\varepsilon : 2X \to 2X$ and the notation $2E$ for bundles E on X. Write $2X = X_- \cup X_+$ where $Y = X_- \cap X_+$, and $E_\pm = 2E|_{X_\pm}$.

Starting from an element $A \in L^0_{\mathrm{cl}}(2X; 2E, 2E)$ we can form the continuous operator

$$A : L^2(2X, 2E) \to L^2(2X, E). \tag{4.4.2}$$

We now apply the decomposition $L^2(2X, 2E) = L^2(X_+, E_+) \oplus L^2(X_-, E_-)$ and the operators $\mathrm{e}^\pm : L^2(X_\pm, E_\pm) \to L^2(2X, 2E)$, $\mathrm{r}^\pm : L^2(2X, 2E) \to L^2(X_\pm, E_\pm)$. This transforms (4.4.2) to the operator

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ A \mathrm{e}^+ & \mathrm{r}^+ A \mathrm{e}^- \\ \mathrm{r}^- A \mathrm{e}^+ & \mathrm{r}^- A \mathrm{e}^- \end{pmatrix} : \begin{matrix} L^2(X_+, E_+) \\ \oplus \\ L^2(X_-, E_-) \end{matrix} \to \begin{matrix} L^2(X_+, E_+) \\ \oplus \\ L^2(X_-, E_-) \end{matrix}.$$

By substituting the reflection map $\varepsilon : X_- \to X_+$ we get

$$\mathcal{B} := \begin{pmatrix} \mathrm{r}^+ A \mathrm{e}^+ & \mathrm{r}^+ A \mathrm{e}^- \varepsilon^* \\ \varepsilon^* \mathrm{r}^- A \mathrm{e}^+ & \varepsilon^* \mathrm{r}^- A \mathrm{e}^- \varepsilon^* \end{pmatrix} : \begin{matrix} L^2(X, E) \\ \oplus \\ L^2(X, E) \end{matrix} \to \begin{matrix} L^2(X, E) \\ \oplus \\ L^2(X, E) \end{matrix}. \tag{4.4.3}$$

Theorem 4.4.1 *$A \in L^0_{\mathrm{cl}}(2X; 2E, 2E)$ implies $\mathcal{B} \in L^0(X; \boldsymbol{g}; E \oplus E, E \oplus E)$ for $\boldsymbol{g} = (0, 0, \infty)$. Moreover, the ellipticity of A is equivalent to the ellipticity of $\mathcal{B}$.*

Proof. Let $A^\vee \in L^0_{\mathrm{cl}}(2X; 2E, 2E)$ denote the push-forward of A under $\varepsilon : 2X \to 2X$. Then $\varepsilon^* \mathrm{r}^- A \mathrm{e}^- \varepsilon^* = \mathrm{r}^+ A^\vee \mathrm{e}^+$, and Theorem 3.3.2 yields $\mathrm{r}^+ A \mathrm{e}^- \varepsilon^*, \varepsilon^* \mathrm{r}^- A \mathrm{e}^+ \in Y^0_{M+G}(X, \boldsymbol{g}; E, E)$. Thus the first assertion is proved. To show the second one we first note that the ellipticity of A is equivalent to the Fredholm property of (4.4.2) which is equivalent to the Fredholm property of (4.4.3). Finally, $\mathcal{B}$ is Fredholm, if and only if $\mathcal{B}$ is elliptic with respect to $(\sigma_\psi(\mathcal{B}), \sigma_\partial(\mathcal{B}))$. Such a result is proved in [18] for boundary value problems of order zero, where operators are also expressed by using r^+ and e^+, modulo compact operators. □

Remark 4.4.2 *The constructions for Theorem 4.4.1 can also be carried out in analogous form for parameter-dependent operators with parameter $\lambda \in \Lambda$. Then the resulting $\mathcal{B}(\lambda)$ induces isomorphisms for sufficiently large $|\lambda|$.*

Given an elliptic operator (4.4.3) we can pass to an elliptic operator

$$\mathcal{B}^\mu := \mathcal{B} \,\mathrm{diag}(D^\mu, D^\mu) \in L^\mu(X, \boldsymbol{g}; E \oplus E, E \oplus E)$$

where D^μ is the operator from Theorem 4.2.8 for weight data $\boldsymbol{g}_\mu := (\mu, 0, \infty)$. This yields a Fredholm operator $\mathcal{B}^\mu : \mathcal{W}^{\mu,\mu}(X, E \oplus E) \to L^2(X, E \oplus E)$ in $L^\mu(X, \boldsymbol{g}_\mu; E \oplus E, E \oplus E)$ without additional conditions. Returning to the transmission situation we get a Fredholm operator

$$\mathcal{A}^\mu := \begin{pmatrix} B^\mu_{11} & B^\mu_{12}\varepsilon^* \\ \varepsilon^* B^\mu_{21} & \varepsilon^* B^\mu_{22}\varepsilon^* \end{pmatrix} : \begin{matrix} \mathcal{W}^{\mu,\mu}(X_+, E_+) \\ \oplus \\ \mathcal{W}^{\mu,\mu}(X_-, E_-) \end{matrix} \to \begin{matrix} L^2(X_+, E_+) \\ \oplus \\ L^2(X_-, E_-) \end{matrix}, \tag{4.4.4}$$

$\mathcal{B}^\mu := (B^\mu_{ij})_{i,j=1,2}$.

Remark 4.4.3 *The way to construct $\mathcal{A}^\mu$ allows us to start from an arbitrary elliptic principal symbol $\widetilde{p}_{(\mu)} : \pi_{2X}(2E) \to \pi^*_{2X}(2E)$, $\pi_{2X} : T^*(2X) \setminus 0 \to 2X$, to form a zero order symbol $\widetilde{p}_{(0)}(x, \xi) := \widetilde{p}_{(\mu)}(x, \xi)|\xi|^{-\mu}\,\mathrm{id}_{2E}$ and an associated operator $A^0 \in L^0_{\mathrm{cl}}(2X; 2E, 2E)$ and to pass to an operator $\mathcal{B}^0$ via Theorem 4.4.1. Then (4.4.4) is a Fredholm operator where*

$$\sigma_\psi(B^\mu_{11}) = \widetilde{p}_{(\mu)}|_{T^*X_+ \setminus 0}, \qquad \sigma_\psi(\varepsilon^* B^\mu_{22}\varepsilon^*) = \widetilde{p}_{(\mu)}|_{T^*X_- \setminus 0}. \tag{4.4.5}$$

Here, the left hand sides of (4.4.5) mean the homogeneous principal symbols of corresponding operators in $L^\mu(X_\pm, \boldsymbol{g}_\mu; E_\pm, E_\pm)$.

4.5 A relation to the operator convention in Vishik-Eskin's theory

As noted in the introduction there is a theory of pseudodifferential boundary value problems of Vishik and Eskin, see, [36] or the book [5]. The transmission property of interior symbols is not required in this framework. The operator convention, say, for operators $\mathrm{Op}(a)$ in the half-space with symbols $a(\xi) \in S^\mu(\mathbb{R}^n_\xi)$ with constant coefficients, is

$$\mathrm{r}^+ \mathrm{Op}(a) : H^s_0(\overline{\mathbb{R}}^n_+) \to H^{s-\mu}(\mathbb{R}^n_+). \tag{4.5.1}$$

Here $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$, and $H^s_0(\overline{\mathbb{R}}^n_+)$ is the subspace of all $u \in H^s(\mathbb{R}^n)$ such that $\mathrm{supp}\, u \subseteq \overline{\mathbb{R}}^n_+$. Clearly, (4.5.1) is continuous for all $s \in \mathbb{R}$.

Our operator convention employs weighted spaces $\mathcal{W}^{s,\gamma}(\mathbb{R}^n_+) = \mathcal{W}^s(\mathbb{R}^{n-1}, \mathcal{K}^{s,\gamma}(\mathbb{R}_+))$, and we have continuous operators

$$\mathrm{r}^+ \mathrm{Op}(a)\mathrm{e}^+ + C_\gamma : \mathcal{W}^{s,\gamma}(\mathbb{R}^n_+) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{R}^n_+) \tag{4.5.2}$$

for all $s \in \mathbb{R}$. Here, $\gamma \in \mathbb{R}$ is fixed, and C_γ is a suitably chosen operator with a kernel in $C^\infty(\mathbb{R}^n_+ \times \mathbb{R}^n_+)$.

Let us now assume $s = \gamma > -\frac{1}{2}$. Then there is a canonical identification $H^s_0(\overline{\mathbb{R}}^n_+) = \mathcal{W}^{s,s}(\mathbb{R}^n_+)$. Thus, for $s-\mu > -\frac{1}{2}$ we have $H^{s-\mu}_0(\overline{\mathbb{R}}^n_+) = \mathcal{W}^{s-\mu,s-\mu}(\mathbb{R}^n_+)$, and (4.5.2) takes the form

$$\mathrm{r}^+ \operatorname{Op}(a)\mathrm{e}^+ + C_s : H^s_0(\overline{\mathbb{R}}^n_+) \to H^{s-\mu}_0(\overline{\mathbb{R}}^n_+). \tag{4.5.3}$$

Let us give a construction for C_s when $r := s-\mu \notin \frac{1}{2}\mathbb{N}$. Form the η-dependent family of maps $d(\eta) : \mathbb{C}^{[r]+1} \to \mathcal{S}(\overline{\mathbb{R}}_+)$, $d(\eta) : c \to \sum_{j=0}^{[r]} c_j (t[\eta])^j [\eta]^{\frac{1}{2}} \omega(t[\eta])$, $c = (c_0, \dots, c_{[r]})$, where ω is a cut-off function. Moreover, consider $b(\eta) : H^r(\mathbb{R}_+) \to \mathbb{C}^{[r]+1}$, where $(b(\eta)v)_j := \frac{1}{j!}[\eta]^{-j-\frac{1}{2}} \partial_t^j v|_{t=0}$, $j = 0, \dots, [r]$. We then have $d(\eta)b(\eta) \in S^0_{\mathrm{cl}}(\mathbb{R}^{n-1}_\eta; H^r(\mathbb{R}_+), H^r(\mathbb{R}_+))$ and $1 - d(\eta)b(\eta) \in S^0_{\mathrm{cl}}(\mathbb{R}^{n-1}_\eta; H^r(\mathbb{R}_+), H^r_0(\overline{\mathbb{R}}_+))$. Note that $d(\eta)b(\eta)$ is a Green symbol in Boutet de Monvel's theorey of type $[r]+1$. It can easily be verified that $G_r := \operatorname{Op}_y(db)$ and $1 - G_r = \operatorname{Op}_y(1-db)$ are complementary projections in the space $\mathcal{W}^r(\mathbb{R}^{n-1}, H^r(\mathbb{R}_+)) = H^r(\mathbb{R}^n_+)$, where $\operatorname{Op}_y(1-db) : H^s(\mathbb{R}^n_+) \to H^r_0(\overline{\mathbb{R}}^n_+)$. Now composing $\mathrm{r}^+ \operatorname{Op}(a)$ in (4.5.1) from the left with $1 - G_{s-\mu}$ gives us

$$\mathrm{r}^+ \operatorname{Op}(a) - G_{s-\mu}\mathrm{r}^+ \operatorname{Op}(a) : H^s_0(\overline{R}^n_+) \to H^{s-\mu}_0(\overline{\mathbb{R}}^n_+)$$

which agrees with (4.5.3) when we set $C_s = G_{s-\mu}\mathrm{r}^+ \operatorname{Op}(a)$.

Appendix: Notation and miscellaneous

If $\Omega \subset \mathbb{R}^n$ is an open set we let $S^\mu(\Omega \times \mathbb{R}^q) = S^\mu(\Omega_y \times \mathbb{R}^q_\eta)$ denote the space of all symbols $a \in C^\infty(\Omega \times \mathbb{R}^q)$ satisfying

$$\sup_{\substack{y \in K \Subset \Omega \\ \eta \in \mathbb{R}^q}} |\partial_\eta^\alpha \partial_y^\beta a(y,\eta)| \langle \eta \rangle^{|\alpha|-\mu} < \infty \qquad \text{for all } \alpha \in \mathbb{N}_0^q \text{ and } \beta \in \mathbb{N}_0^n$$

for any compact subset K of Ω. The subspace of classical symbols $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q)$ consists of all a that admit an asymptotic expansion into homogeneous components, i.e., there exist functions $a_{(\mu-j)} \in C^\infty(\Omega \times (\mathbb{R}^q \setminus 0))$ such that $a_{(\mu-j)}(y, r\eta) = r^{\mu-j} a_{(\mu-j)}(y,\eta)$ for any positive r and all (y,η), and for any $N \in \mathbb{N}$

$$a(y,\eta) - \sum_{j=0}^{N-1} \chi(\eta) a_{(\mu-j)}(y,\eta) \ \in \ S^{\mu-N}(\Omega \times \mathbb{R}^q),$$

where $\chi \in C^\infty(\mathbb{R}^q)$ is a 0-excision function, i.e., $\chi \equiv 0$ near $\eta = 0$ and $\chi \equiv 1$ outside a compact set.

In an analogous way we define the spaces $S^\mu_{(\mathrm{cl})}(\mathbb{R} \times \Omega \times \mathbb{R}^q_\eta)$ and set

$$S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q_\eta) := S^\mu_{(\mathrm{cl})}(\mathbb{R} \times \Omega \times \mathbb{R}^q_\eta)|_{\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q_\eta}.$$

Occasionally we will require that symbols from $S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q_\eta)$ have a certain exit-behaviour for $t \to \infty$, namely

$$\sup_{t \in \mathbb{R}} \sup_{\substack{y \in K \Subset \Omega \\ \eta \in \mathbb{R}^q}} |\partial_\eta^\alpha \partial_y^\beta \partial_t^k a(t,y,\eta)| \langle t \rangle^{-k} \langle \eta \rangle^{|\alpha|-\mu} < \infty$$

for any compact set K and all indices α, β, k. That symbol space will be referred to as $S^{\mu,0}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q_\eta)$ (in case of classical symbols we also require the homogeneous components to have a corresponding exit-behaviour in t).

In the above definitions, the dimensions of $\Omega \subset \mathbb{R}^n$ and $\mathbb{R}^q$ or $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}^q$, are admitted to be different. In applications $\mathbb{R}^q_\eta$ is replaced by $\mathbb{R}^n_\eta \times \Lambda$ or $\mathbb{R}^{1+n}_{\tau,\eta} \times \Lambda$, where η is considered as the covariable to $y \subset \Omega$, τ as the covariable to $t \in \mathbb{R}_+$, and $\Lambda = \mathbb{R}^l$ for some $l \in \mathbb{N}$, is an additional parameter space. Then we shall use notations as $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n \times \Lambda)$, $S^\mu_{(\mathrm{cl})}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+\mu}_{\tau,\eta} \times \Lambda)$, and so on. This leads to the definition of parameter-dependent pseudodifferential operators $\mathrm{op}(a)(\lambda)$, which are given by

$$[\mathrm{op}(a)(\lambda)u](y) = \int e^{-iy\eta} a(y,\eta,\lambda)\hat{u}(\eta)\, đ\eta, \qquad u \in C_0^\infty(\Omega),$$

in case of $a \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n \times \Lambda)$, and similarly for a from other symbol spaces. Then

$$L^\mu_{(\mathrm{cl})}(\Omega \times \Lambda) := \{\mathrm{op}(a)(\lambda) + c(\lambda) : \ a \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n \times \Lambda), c \in L^{-\infty}(\Omega \times \Lambda)\}$$

where $L^{-\infty}(\Omega \times \Lambda) \cong \mathcal{S}(\Lambda, C^\infty(\Omega \times \Omega))$ consists of all integral operators having a smooth kernel, depending rapidly descreasing on Λ.

In case of M being a smooth manifold (without boundary), we let $L^\mu_{(\mathrm{cl})}(M \times \Lambda)$ consist of all operators which, modulo $L^{-\infty}(M \times \Lambda)$, locally equal $\mathrm{op}(a)(\lambda)$ with $a \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n \times \Lambda)$ and Ω corresponding to local coordinates of M. Again, $L^{-\infty}(M \times \Lambda) \cong \mathcal{S}(\Lambda, C^\infty(M \times M))$ consists of all Schwartz functions with values in integral operators with smooth kernel.

Throughout the exposition we let X denote a smooth compact manifold with smooth boundary Y. We let $\{U_1, \ldots U_N\}$ be an open covering of X by coordinate neighbourhoods, such that $U_j \cap Y \neq \emptyset$ for $j = 1, \ldots, M < N$ and $U_j \cap Y = \emptyset$ for $j = M+1, \ldots, N$.

The corresponding charts are denoted by

$$\chi_j : U_j \to \Omega_j \times \overline{\mathbb{R}}_+, \quad j = 1, \ldots, M, \qquad \chi_j : U_j \to \Sigma_j, \quad j = M+1, \ldots, N, \quad \text{for } \Sigma_j \subseteq \mathbb{R}^n.$$

By the collar theorem, without loss of generality we can assume that, for $j, k = 1, \ldots, M$, the transition diffeomorphisms

$$\kappa_{kj} := \chi_k \chi_j^{-1}|_{U_j \cap U_k} : \mathbb{R}_+ \times (\Omega_j \cap \Omega_k) \to \mathbb{R}_+ \times (\Omega_j \cap \Omega_k)$$

have the form $\kappa_{kj}(t,y) = (t, \kappa'_{kj}(y))$ for diffeomorphisms $\kappa'_{kj} : \Omega_j \cap \Omega_k \to \Omega_j \cap \Omega_k$.

We let $\mathrm{Vect}(X)$ denote the set of all smooth (complex) vector bundles on X, where the transition maps near the boundary are assumed to be independent of the normal direction t.

With X we associate the double $2X$. This is a smooth closed manifold which is obtained by glueing together two copies X_+, X_- of X along their common boundary. We then let

$$\varepsilon : 2X \to 2X$$

be the canonical reflection diffeomorphism which induces diffeomorphisms $\varepsilon : X_+ \to X_-$ and $\varepsilon : X_- \to X_+$ (where $Y = X_+ \cap X_-$ remains fixed). Accordingly, to each bundle $E \in \mathrm{Vect}(X)$ we have its double $2E$ consisting of two copies E_+, E_- of E. The reflection ε induces bundle isomorphisms $\varepsilon^* : E_+ \to E_-$ and $\varepsilon^* : E_- \to E_+$.

On $2X$ we have the standard distribution spaces, in particular, the Sobolev spaces $H^s(2X)$ of smoothness $s \in \mathbb{R}$. Let r^+ denote the operator of restriction $u \mapsto u|_{\mathrm{int}\, X}$ of a distribution u on $2X$ to $\mathrm{int}\, X$, where X is identified with X_+, and set $H^s(X) = \{\mathrm{r}^+ u : \ u \in H^s(2X)\}$. Further, let e^+ be the operator of extension by zero from X_+ to $2X$ that is defiend as a map

$e^+ : H^s(X) \to \mathcal{D}'(2X)$ for all $s > -\frac{1}{2}$. Moreover, define $H_0^s(X)$, $s \in \mathbb{R}$, to be the subspace of all $u \in H^s(2X)$ such that $\operatorname{supp} u \subset X = X_+$.

Analogously, for $E \in \mathrm{Vect}(X)$, we have $H^s(2X, 2E)$, $H^s(X, E)$, and $H_0^s(X, E)$.

Let $\mathbb{Z}$ denote the integer numbers, $\mathbb{N}$ the positive integers, $\mathbb{N}_0$ the non-negative ones. $\mathbb{R}$ are the real numbers, $\mathbb{R}_+$ the positive reals, $\overline{\mathbb{R}}_+$ the non-negative reals. The complex numbers are denoted by $\mathbb{C}$.

For $\gamma \in \mathbb{R}$ we let $\Gamma_\gamma = \{z \in \mathbb{C} : \operatorname{Re} z = \gamma\}$ be a vertical line in the complex plane, which occasionally is identified with $\mathbb{R}$ via $\gamma + i\tau \mapsto \tau$.

If E, F are Fréchet spaces, the set of all linear continuous operators $E \to F$ is written as $\mathcal{L}(E, F)$. By $E \widehat{\otimes}_\pi F$ we denote the completed projective tensor product of E and F, which itself is a Fréchet space.

A smooth function ω on $\overline{\mathbb{R}}_+$ is called a cut-off function if $0 \leq \omega \leq 1$, $\omega \equiv 1$ in a neighbourhood of 0, and ω has compact support.

A function $\chi \in C^\infty(\mathbb{R}^q)$ is called a zero excision function of $0 \leq \chi \leq 1$, $\chi \equiv 0$ in a neighbourhood of 0, and $1 - \chi$ has compact support.

We let $[\cdot]$ denote a smoothed norm function, i.e., a smooth, positive function on $\mathbb{R}^q$ (for some q) such that $[\eta] = |\eta|$ for $|\eta|$ sufficiently large.

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