A DECOMPOSITION OF FUNCTIONS WITH VANISHING MEAN OSCILLATION

MICHAEL BRIAN KOREY

Abstract. A function has vanishing mean oscillation (VMO) on $\mathbb{R}^n$ if its mean oscillation—the local average of its pointwise deviation from its mean value—both is uniformly bounded over all cubes within $\mathbb{R}^n$ and converges to zero with the volume of the cube. The more restrictive class of functions with vanishing lower oscillation (VLO) arises when the mean value is replaced by the minimum value in this definition. It is shown here that each VMO function is the difference of two functions in VLO.

1. Introduction

To say that a locally integrable function f on $\mathbb{R}^n$ has *bounded mean oscillation*, $f \in \mathrm{BMO}$, means that

$$\sup_Q \frac{1}{|Q|} \int |f(x) - f_Q|\, dx < \infty, \tag{1}$$

where the supremum runs over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, $|Q|$ denotes the measure of Q, and f_Q is the mean value of f over Q, i.e., $f_Q = (1/|Q|) \int_Q f$. A function f is said to have *bounded lower oscillation* if the term f_Q in (1) can be replaced by $\inf_Q f$, the essential infimum[1] of f over Q; in other words, $f \in \mathrm{BLO}$ if

$$\sup_Q \frac{1}{|Q|} \int (f(x) - \inf_Q f)\, dx = \sup_Q (f_Q - \inf_Q f) < \infty. \tag{2}$$

The suprema in (1) and (2) are denoted by $\|f\|_{\mathrm{BMO}}$ and $\|f\|_{\mathrm{BLO}}$, respectively, and when only the cubes Q within a given cube Q_0 are considered, the symbols $\mathrm{BMO}(Q_0)$ and $\mathrm{BLO}(Q_0)$ will be used.

It is not difficult to see that each BLO function is in BMO; in fact, the estimate $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{\mathrm{BLO}}$ holds. Unlike BMO, however, the set BLO is not stable under multiplication by negative numbers ($\log|x|$ is in BLO, but $-\log|x|$ is not), and $\mathrm{BLO} \cap (-\mathrm{BLO}) = L^\infty$, as follows readily from the definition. On the other hand, Coifman and Rochberg [3], invoking a rather subtle argument of Carleson, showed that each BMO function can be written as the difference of two BLO functions, in short:

$$\mathrm{BMO} = \mathrm{BLO} - \mathrm{BLO}.$$

1991 *Mathematics Subject Classification.* Primary 42B25; Secondary 26A15, 46E30.

Key words and phrases. Bounded mean oscillation, bounded lower oscillation, vanishing mean oscillation, A_p condition, uniform continuity.

[1] In general, all pointwise statements in this paper should be understood to hold only almost everywhere with respect to Lebesgue measure. In particular, $\inf_Q f$ and $\sup_Q f$ denote the essential infimum and essential supremum of f on Q, and no distinction will be made between two functions agreeing on the complement of a set of measure zero.

This is likewise a consequence of the factorization of A_p weights obtained by Jones [9];[2] in this regard, see also [2], [4], and [5].

The authors of [3] concluded their paper by noting that "many results about functions in BMO have analogs for functions of vanishing mean oscillation ... [but we] do not know what the analogs of our results are in that context." The purpose of this paper is to provide such an analog in the form of the following result:

Theorem. *Each* VMO *function is the difference of two* VLO *functions. That is, if* $f \in \mathrm{VMO}$*, then there exist* VLO *functions* F *and* G *such that* $f = F - G$ *and*

$$\|F\|_{\mathrm{BLO}} + \|G\|_{\mathrm{BLO}} \leq C\|f\|_{\mathrm{BMO}}.$$

The constant C *depends only on the dimension* n.

A word of explanation about the terms in the theorem is necessary. To say that a BMO function has *vanishing mean oscillation* means not only that the supremum in (1) is bounded over all cubes, but also that it vanishes in the asymptotic limit of ever smaller scales. In other words, $f \in \mathrm{VMO}$ if both $f \in \mathrm{BMO}$ and

$$\frac{1}{|Q|}\int |f(x) - f_Q|\, dx = o(1) \qquad (\ell(Q) \to 0),$$

where $\ell(Q)$ denotes the edge-length (or "size") of the cube Q. Each bounded, uniformly continuous (BUC) function is in VMO, uniform continuity being expressable in the form

$$\sup_Q f - \inf_Q f = o(1) \qquad (\ell(Q) \to 0), \tag{3}$$

and VMO is actually the closure of BUC with respect to the BMO norm defined in (1), as was shown in [14]. The space VMO was introduced by by Sarason in [14] in the context of algebras on the unit circle and has since found manifold applications in diverse areas of analysis: [1] and [10] give two recent examples in the context of partial differential equations.

The new class VLO considered here, the set of functions with *vanishing lower oscillation*, consists of those BMO functions f for which

$$f_Q - \inf_Q f = o(1) \qquad (\ell(Q) \to 0). \tag{4}$$

VLO is a proper subset of VMO: while both $\sqrt{\log(1/|x|)}$ and $-\sqrt{\log(1/|x|)}$ are in VMO, only the former is in VLO. Moreover, the portion of VLO that is stable under multiplication by a negative number is a familiar space:

$$\mathrm{VLO} \cap (-\mathrm{VLO}) = \mathrm{BUC},$$

for when both f and $-f$ satisfy (4), then f also satisfies (3). What the theorem then says is that

$$\mathrm{VMO} = \mathrm{VLO} - \mathrm{VLO}.$$

Although the statement of the theorem is a straightforward generalization of that in [3], the proof is not. One essential reason for the difficulty is that while L^∞ functions are in BLO (and thus in BMO), they are not generally in VMO; the characteristic function of the unit interval $[0, 1]$ on the line, for example, has mean oscillation $1/2$ over each interval of the form $[-r, r]$, no matter how small r is. This presents a non-trivial obstacle to proving the theorem, because the techniques used

[2] In fact, $\mathrm{BMO} = \{\alpha \log w : \alpha \geq 0, w \in A_2\}$ and $\mathrm{BLO} = \{\alpha \log w : \alpha \geq 0, w \in A_1\}$, so that the statement $\mathrm{BMO} = \mathrm{BLO} - \mathrm{BLO}$ is the logarithm of the factorization result $A_2 = A_1/A_1$ in [9].

to characterize BMO in [3] and the Calderon-Zygmund decomposition as applied in [9] typically lead to bounded remainder terms. In the decomposition of BMO, this level of precision suffices and such remainder terms need not be broken down any further; on the other hand, if VMO functions are to be decomposed, then attention must be paid not only to the size but also to the smoothness of these remainders. (A similar difficulty is encountered when trying to extend VMO functions defined on a measurable set to all of $\mathbb{R}^n$, as in [7].)

The method of proof for the VMO theorem stated above is first to show the result in the dyadic model case, applying ideas used by Jones [9] for the factorization of Muckenhoupt A_p weights. Translations of this dyadic result will then be averaged to give the general case, along the lines of of Garnett and Jones [6]. The same combination of techniques was also used by the present author in [12]. There the focus was on showing that each Muckenhoupt weight with small A_p bound—that is, with bound near 1, the bound being *fixed and valid over all scales*—could be factored as a product of suitable powers of two A_1 weights with weight bounds also close to 1.[3] Here, by contrast, the challenge is to decompose a given VMO function—a BMO function whose oscillatory behavior *improves over ever smaller scales* in an asymptotically optimal manner—as the difference of two simpler functions that also display this asymptotically optimal improvement.[4]

2. The dyadic setting

The first goal is to prove a dyadic form of the decomposition theorem. This model version is stated for the collection $\mathcal{D}(Q_0)$ of all dyadic subcubes of an arbitrary, fixed cube Q_0 in $\mathbb{R}^n$, that is, all those cubes obtained by dividing Q_0 into 2^n congruent cubes of half its length, dividing each of these into 2^n congruent cubes, and so on; by convention, Q_0 itself belongs to $\mathcal{D}(Q_0)$. The space $\mathrm{BMO}^{\mathrm{dy}}(Q_0)$ then denotes the set of all integrable functions on Q_0 for which condition (1) holds when the supremum there is taken only over cubes Q in $\mathcal{D}(Q_0)$; dyadic versions of BLO, VMO, VLO, and BUC are defined analogously.

At first, we derive a preliminary result (Lemma 1) valid only on large dyadic scales; with the help of a suitably localized iteration procedure, we shall then extend this to the full dyadic version (Lemma 2).

Lemma 1. *Suppose $f \in \mathrm{BMO}^{\mathrm{dy}}(Q_0)$. Let δ be a dyadic scale (i.e., $\delta \in 2^{-\mathbb{N}}$). Then there exist bounded functions F_1, G_1 and R_1, all locally constant[5] on the mesh $\{Q_1 \in \mathcal{D}(Q_0) : \ell(Q_1) = \delta\ell(Q_0)\}$ of dyadic subcubes of relative size δ, satisfying both the pointwise identity*

$$f(x) - f_{Q_0} = F_1(x) - G_1(x) + R_1(x) + \sum_{\ell(Q_1)=\delta\ell(Q_0)} (f(x) - f_{Q_1})\chi_{Q_1}(x) \qquad (x \in Q_0)$$

[3]The sharp result is that a weight with A_2 bound $1+\varepsilon$ can be factored as the quotient of two A_1 weights with bounds $1 + \mathcal{O}(\sqrt{\varepsilon})$, as $\varepsilon \to 0$; see [12] for details.

[4]In the language of asymptotically optimal Muckenhoupt weights (see [11]), the theorem developed in the present paper gives the factorization $A_{2,\mathrm{as}} = A_{1,\mathrm{as}}/A_{1,\mathrm{as}}$. The statement $w \in A_{2,\mathrm{as}}$ means both that $\sup_Q w_Q(1/w)_Q < \infty$ and that $w_Q(1/w)_Q \to 0$ as $\ell(Q) \to 0$; the stronger statement $w \in A_{1,\mathrm{as}}$ arises when the product $w_Q(1/w)_Q$ is replaced by the ratio $w_Q/\inf_Q w$.

[5]Local constancy on a mesh of congruent cubes simply means constancy on each cube within the mesh.

and the estimate

$$\|F_1\|_{\mathrm{BLO^{dy}}(Q_0)} + \|G_1\|_{\mathrm{BLO^{dy}}(Q_0)} + \|R_1\|_{L^\infty(Q_0)} \le C\|f\|_{\mathrm{BMO}(Q_0)}. \tag{5}$$

The constant C depends only on the dimension n.

The proof uses an iterative Calderón-Zygmund decomposition to single out those cubes on which the mean oscillation of f is large. Note that in this first lemma only cubes of comparable size enter—here all dyadic Q with edges of length between $\delta\ell(Q_0)$ and $\ell(Q_0)$; in Lemma 2, the full range of dyadic scales will be considered, and special attention will be paid to how the oscillatory behavior of VMO functions improves as the scale decreases.

Proof. Without loss of generality, let $Q_0 = [0,1]^n$ and $\|f\|_{\mathrm{BMO^{dy}}(Q_0)} = 2^{-n}$. As is well known, the BMO condition strongly restricts how the average values of a function vary with the size of the averaging set. In particular, for any cube Q and any other cube $2Q$ containing it and with edges twice as long,

$$|f_Q - f_{2Q}| \le 2^n \frac{1}{|2Q|}\int_{2Q} |f - f_{2Q}|. \tag{6}$$

When $f \in$ BMO, the right-hand side is bounded; when $f \in$ VMO, the right-hand side vanishes as $\ell(Q) \to 0$. This simple observation drives the stopping-time argument below.

Let $\mathcal{G}_1^0 = \{Q_0\}$. Inequality (6) guarantees that $|f_Q - f_{Q_0}| \le 1$, when Q is any one of the 2^n subcubes of Q_0 obtained from it by bisecting its edges. Fix $\delta \in 2^{-\mathbb{N}}$. Of interest are the largest subcubes Q obtained by further bisection on which $|f_Q - f_{Q_0}| > 1$ (and $\ell(Q) > \delta$). In detail, define

$$\mathcal{G}_1^1 = \{Q \in \mathcal{D}(Q_0) : |f_Q - f_{Q_0}| > 1,\ \ell(Q) > \delta,\ Q\,\text{maximal}\} \tag{7}$$

and, inductively,

$$\mathcal{G}_1^{m+1} = \{Q \in \mathcal{D}(Q') : Q' \in \mathcal{G}_1^m,\ |f_Q - f_{Q'}| > 1,\ \ell(Q) > \delta,\ Q\,\text{maximal}\}. \tag{8}$$

Write $\mathcal{G}_1 = \bigcup_{m=0}^\infty \mathcal{G}_1^m$ for the full collection of all such selected cubes. Let Ω_1^m be the union of the cubes in $\mathcal{G}_1^m$; then, by construction, $\Omega_1^{m+1} \subsetneq \Omega_1^m \subsetneq \cdots \subsetneq \Omega_1^0$. For each proper dyadic subcube Q of $\mathcal{D}(Q_0)$ with $\ell(Q) \ge \delta$, let $\widetilde{Q}$ denote the minimal cube in $\mathcal{G}_1$ that strictly contains it;[6] this means, in the special case when $Q \in \mathcal{G}_1^{m+1}$, that $\widetilde{Q}$ is the unique cube in $\mathcal{G}_1^m$ containing Q.

Each dyadic subcube Q_1 of edge-length δ is then contained in a unique decreasing chain of selected cubes of the form[7]

$$Q_0 \supseteq Q_1^1 \supseteq Q_1^2 \supseteq \cdots \supseteq Q_1^m = \widetilde{Q}_1 \qquad (Q_1^i \in \mathcal{G}_1^i,\ 1 \le i \le m).$$

The difference $f(x) - f_{Q_0}$ can be expanded as a telescoping sum over pairs of adjacent cubes in this chain, extended by the additional inclusion $\widetilde{Q}_1 \supset Q_1$. What

[6] That is, $\widetilde{Q} = \bigcap\{Q' \in \mathcal{G}_1 : Q \subset Q'\}$.

[7] The length of the chain is at least 1 (in the extreme case when no predecessor of Q_1 is selected, so that $\widetilde{Q}_1 = Q_0$) and at most $1 + \log_2(1/\delta)$; the actual length varies over the mesh of dyadic subcubes Q_1 within Q_0 of size δ. Accordingly, all but finitely many of the sets $\mathcal{G}_1^m$ are empty, so that the double sum in (9) is actually a sum over only finitely many cubes.

results from this is the pointwise identity

$$f(x) - f_{Q_0} = \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{G}_1^m} (f_Q - f_{\widetilde{Q}})\chi_Q(x) + (f_{Q_1} - f_{\widetilde{Q}_1}) + f(x) - f_{Q_1} \qquad (x \in Q_1), \tag{9}$$

which holds on each dyadic $Q_1 \in \mathcal{D}(Q_0)$ of size δ.

How large are the terms in (9)? Maximality in the selection criteria (7) and (8) and the basic BMO estimate (6) give rise to the mean-value inequality

$$1 < |f_Q - f_{\widetilde{Q}}| \leq 2 \qquad (Q \in \bigcup_{m=1}^{\infty} \mathcal{G}_1^m). \tag{10}$$

They also lead to the relative density estimate

$$|Q \cap \Omega_1^{m+1}| \leq 2^{-n}|Q| \qquad (Q \in \mathcal{G}_1^m), \tag{11}$$

which is valid for each $m \geq 0$. Summing this last estimate over all the cubes Q in $\mathcal{G}_1^m$ and iterating leads to the bound

$$|\Omega_1^m| \leq 2^{-mn}|Q_0| \qquad (m \in \mathbb{N}). \tag{12}$$

Next, to obtain suitable dyadic BLO summands of f, split the double sum in (9) according to the sign of the difference $f_Q - f_{\widetilde{Q}}$. Then

$$f(x) - f_{Q_0} = F_1(x) - G_1(x) + R_1(x) + \sum_{\ell(Q_1)=\delta} \big(f(x) - f_{Q_1}\big)\chi_{Q_1}(x) \qquad (x \in Q_0),$$

where

$$F_1(x) = \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{G}_1^m} (f_Q - f_{\widetilde{Q}})^+ \chi_Q(x), \tag{13}$$

$$G_1(x) = \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{G}_1^m} (f_{\widetilde{Q}} - f_Q)^+ \chi_Q(x), \tag{14}$$

and

$$R_1(x) = \sum_{\ell(Q_1)=\delta} (f_{Q_1} - f_{\widetilde{Q}_1})\chi_{Q_1}(x).$$

It is important to note that the functions F_1 and G_1 defined in (13) and (14) are non-negative; where they are positive, their value must, by (10), exceed 1.

Proving the L^∞ estimate in (5) for R_1 is straightforward: since, by the definition of $\widetilde{Q}_1$, no dyadic cube between Q_1 and $\widetilde{Q}_1$ is selected in the stopping-time argument (in other words, $|f_Q - f_{\widetilde{Q}_1}| \leq 1$ whenever $Q_1 \subset Q \subset \widetilde{Q}_1$), then either $|f_{Q_1} - f_{\widetilde{Q}_1}| \leq 1$ or $1 < |f_{Q_1} - f_{\widetilde{Q}_1}| \leq 2$. In either case, $|R_1(x)| \leq 2$ for a.e. $x \in Q_0$, as desired.

As F_1 and G_1 are locally constant on the mesh $\{Q_1 \in \mathcal{D}(Q_0) : \ell(Q_1) = \delta\}$, it remains to estimate their (dyadic) BLO bounds on cubes larger than those in this mesh. To confirm (5), then, we must show, for each $Q \in \mathcal{D}(Q_0)$ with $\ell(Q) > \delta$, that there exists a constant C_Q (with $\sup_Q C_Q < \infty$) such that both

$$(F_1)_Q \leq C_Q + \inf_Q F_1 \tag{15}$$

and

$$(G_1)_Q \leq C_Q + \inf_Q G_1. \tag{16}$$

To prove this we now consider three cases.

Case I: The initial cube. We first verify (15) in the case when $Q = Q_0$, the original cube. In this case, $\inf_Q F_1 = 0$, for the choice of the height 1 in the stopping-time argument defining the selection procedure (7) ensures that the set $Q_0 \setminus \Omega_1^1$ has positive measure; see (12). Since $\int_Q F_1 = \int_0^\infty |E_t|\, dt$ with $E_t = \{x \in Q : F_1(x) > t\}$, estimating the dyadic VLO bound of F_1 then reduces to estimating the measure of the set E_t. But condition (10) ensures that $E_t \subseteq \Omega_1^1$, when $0 \leq t < 2$, and, in general, that $E_t \subseteq \Omega_1^k$, when $k \in \mathbb{N}$ and $2(k-1) \leq t < 2k$. Thus, by (12),

$$(F_1)_Q = \sum_{k=1}^{\infty} \int_{2(k-1)}^{2k} \frac{|E_t|}{|Q|}\, dt \leq 2 \sum_{k=1}^{\infty} \frac{|\Omega_1^k|}{|Q|} \leq 2 \sum_{k=1}^{\infty} 2^{-nk} \leq 2 + \inf_Q F_1, \tag{17}$$

which is (15) for $Q = Q_0$.

Case II: A cube with a large jump in mean value. Suppose now that $Q \in \mathcal{G}_1^m$ for some positive m and that $f_Q - f_{\widetilde{Q}} > 1$.[8] Then $(F_1)_Q - \inf_Q F_1 = |Q|^{-1} \int_0^\infty |\widetilde{E}_t|\, dt$, where $\widetilde{E}_t = \{x \in Q : F_1(x) - \inf_Q F_1 > t\}$. In analogy to the first case, we find from (10) and (11) that $\widetilde{E}_t \subset Q \cap \Omega_1^{m+k}$, when $k \in \mathbb{N}$ and $2(k-1) \leq t < 2k$. So for t in this range, $|\widetilde{E}_t| \leq 2^{-nk}|Q|$, from which the desired estimate (15) once again follows, with $C_Q = 2$.

Case III: Cubes with no large jump in the mean. In Case I, we considered Q_0; in Case II, we treated those dyadic cubes Q within Q_0 (of length exceeding δ) for which $f_Q - f_{\widetilde{Q}} > 1$. To handle the remaining case efficiently, a bit of further notation is helpful. Recall that, for each proper dyadic subcube Q of Q_0 with $\ell(Q) \geq \delta$, the symbol $\widetilde{Q}$ denotes the minimal cube in $\mathcal{G}_1 = \cup_{m=1}^\infty \mathcal{G}_1^m$ that strictly contains Q; now set

$$\begin{aligned} \mathcal{P}(Q) &= \{I \in \mathcal{D}(Q) : f_I - f_{\widetilde{Q}} > 1,\ \ell(I) > \delta,\ I \,\text{maximal}\}, \\ \mathcal{N}(Q) &= \{I \in \mathcal{D}(Q) : f_I - f_{\widetilde{Q}} < -1,\ \ell(I) > \delta,\ I \,\text{maximal}\}. \end{aligned}$$

Note that the union of $\mathcal{P}(Q)$ and $\mathcal{N}(Q)$ is exactly the set of the cubes in $\cup_{m=1}^\infty \mathcal{G}_1^m$ that lie within Q. In this notation, the remaining case now consists of proving (15) on each dyadic cube Q (of size exceeding δ) for which $Q \notin \mathcal{P}(Q)$.

Fix such a cube Q. To estimate F_1, split Q into the union of its subcubes in $\mathcal{P}(Q)$ and the complement of this union. On the one hand, if $I \in \mathcal{P}(Q)$, then $\widetilde{I} = \widetilde{Q}$; Case II then applies, so that $\int_I F_1 \leq (2 + \inf_I F_1)|I|$. Moreover,

$$1 < \inf_I F_1 - \inf_{\widetilde{Q}} F_1 = f_I - f_{\widetilde{I}} \leq 2,$$

[8] Unlike in (8), the sign of the difference is important here.

by (10). On the other hand, on the complement in Q of $\cup_{I\in\mathcal{P}(Q)} I$ the value of F_1 is exactly $\inf_{\widetilde{Q}} F_1$. In sum, then,

$$\begin{aligned}\int_Q F_1 &\le \sum_{I\in\mathcal{P}(Q)} (2+\inf_I F_1)|I| + (\inf_{\widetilde{Q}} F_1)|Q\setminus \cup_{I\in\mathcal{P}(Q)} I| \\ &\le (4+\inf_{\widetilde{Q}} F_1)\sum_{I\in\mathcal{P}(Q)} |I| + (\inf_{\widetilde{Q}} F_1)|Q\setminus \cup_{I\in\mathcal{P}(Q)} I| \\ &\le (4+\inf_{\widetilde{Q}} F)|Q|.\end{aligned}$$

Since $\inf_{\widetilde{Q}} F_1 \le \inf_Q F_1$, the bound (15) with $C_Q = 4$ thus also holds for the cubes Q in this, the last case.

The justification of the dyadic BLO bound (16) is similar, with G_1 in place of F_1, $\mathcal{N}(Q)$ in place of $\mathcal{P}(Q)$, etc. This completes the proof of Lemma 1. □

When a function has (dyadic) vanishing mean oscillation, then we can apply the previous lemma in an iterative way that takes advantage of the function's improved behavior at small scales. This is the essence of the next result.

Lemma 2. *Suppose* $f \in \mathrm{VMO}^{\mathrm{dy}}(Q_0)$*. Then there exist non-negative functions* $F, G \in \mathrm{VLO}^{\mathrm{dy}}(Q_0)$ *and a function* $R \in \mathrm{BUC}^{\mathrm{dy}}(Q_0)$ *such that*

$$f(x) - f_{Q_0} = F(x) - G(x) + R(x) \qquad (x\in Q_0). \tag{18}$$

Writing $f(x)$ as the difference $(F(x)+R(x)+f_{Q_0}) - G(x)$ thus gives the dyadic form of the decomposition theorem.

Proof. As above, let $\|f\|_{\mathrm{BMO}^{\mathrm{dy}}(Q_0)} = 2^{-n}$. Since $f\in \mathrm{VMO}^{\mathrm{dy}}(Q_0)$, we can find a strictly decreasing sequence of dyadic scales on which the mean oscillation of f vanishes exponentially fast; that is, set $\delta_0 = \ell(Q_0)$ and choose $\{\delta_j\}_{j\in\mathbb{N}} \subset 2^{-\mathbb{N}}$ such that both $\delta_{j+1} < \delta_j$ (hence $\delta_j \to 0$ as $j\to\infty$) and

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le 2^{-n-j} \qquad (Q\in\mathcal{D}(Q_0), \ell(Q)\le\delta_j). \tag{19}$$

Further, let $Q_j(x)$ denote the dyadic subcube of Q_0 that contains x and has edge-length δ_j; then $\{Q_j(x) : x\in Q_0\}$ is a family of non-overlapping, congruent subcubes whose union is Q_0, and $Q_j(x)\to\{x\}$ as $j\to\infty$ for a.e. x. (When no confusion can arise, we shall simply write Q_j for $Q_j(x)$.)

The previous lemma (with $\delta = \delta_1/\delta_0$) gives rise to the decomposition

$$f(x) - f_{Q_0} = F_1(x) - G_1(x) + R_1(x) + \sum_{\ell(Q_1)=\delta_1} \big(f(x) - f_{Q_1}\big)\chi_{Q_1}(x) \qquad (x\in Q_0)$$

as well as to a suitable estimate on the terms F_1, R_1, and G_1. Now apply the lemma to f on each cube Q_1 of size δ_1 (with $\delta = \delta_2/\delta_1$) in place of Q_0. Then

$$f(x) - f_{Q_1} = F_2(x) - G_2(x) + R_2(x) + \sum_{\ell(Q_2)=\delta_2} \big(f(x) - f_{Q_2}\big)\chi_{Q_2}(x) \qquad (x\in Q_1).$$

Repeat the procedure, applying the lemma on each cube Q_{j-1} of size δ_{j-1} (with $\delta = \delta_j/\delta_{j-1}$) to obtain[9]

$$f(x)-f_{Q_{j-1}} = F_j(x)-G_j(x)+R_j(x)+\sum_{\ell(Q_j)=\delta_j}\big(f(x)-f_{Q_j}\big)\chi_{Q_j}(x) \qquad (x\in Q_{j-1}).$$

The functions F_j, G_j, and R_j so obtained[10] are all locally constant on the mesh of dyadic subcubes of Q_0 of size δ_j. On account of (5) and (19) they also satisfy the estimate

$$\|F_j\|_{\mathrm{BLO^{dy}}(Q_{j-1})}+\|G_j\|_{\mathrm{BLO^{dy}}(Q_{j-1})}+\|R_j\|_{L^\infty(Q_{j-1})}\le C2^{-j}. \tag{20}$$

uniformly on all dyadic subcubes Q_{j-1} within Q_0 of size δ_{j-1}.

All together, the first J iterations of the procedure just described yield

$$f(x)-f_{Q_0}=\sum_{j=1}^{J}F_j(x)-\sum_{j=1}^{J}G_j(x)+\sum_{j=1}^{J}R_j(x)+\sum_{\ell(Q_J)=\delta_J}\big(f(x)-f_{Q_J}\big)\chi_{Q_J}(x),$$

which is valid for $x\in Q_0$. Since $\lim_{J\to\infty} f_{Q_J(x)}\to f(x)$ for a.e. x, the last sum vanishes in the limit. As a result,

$$\begin{aligned} f(x)-f_{Q_0} &= \sum_{j=1}^{\infty}F_j(x)-\sum_{j=1}^{\infty}G_j(x)+\sum_{j=1}^{\infty}R_j(x)\\ &= F(x)-G(x)+R(x) \qquad (x\in Q_0), \end{aligned}\tag{21}$$

which is (18). What remains is to show the appropriate estimates for the functions F, G, and R, so defined.

For this, fix an arbitrary $Q\in\mathcal{D}(Q_0)$ and find the unique number $J\in\mathbb{N}$ such that $\delta_J<\ell(Q)\le\delta_{J-1}$. On Q, the functions F_j, G_j, and R_j ($1\le j\le J-1$) are then all constant. Hence

$$F_Q-\inf_Q F=\Big(\sum_{j=J}^{\infty}F_j\Big)_Q-\inf_Q\sum_{j=J}^{\infty}F_j\le\sum_{j=J}^{\infty}\big((F_j)_Q-\inf_Q F_j\big) \tag{22}$$

and

$$\sup_Q R-\inf_Q R=\sup_Q\sum_{j=J}^{\infty}R_j-\inf_Q\sum_{j=J}^{\infty}R_j\le\sum_{j=J}^{\infty}2\sup_Q|R_j|.$$

On account of (20), the latter estimate on the (dyadic) modulus of continuity of R becomes

$$\sup_Q R-\inf_Q R\le 2C\sum_{j=J}^{\infty}(2^{-j}) \qquad (Q\in\mathcal{D}(Q_0),\ell(Q)\le\delta_{J-1}). \tag{23}$$

---

[9] For each scale index $j\ge 0$, let $\mathcal{G}_j^0=\{Q_{j-1}\in\mathcal{D}(Q_0):\ell(Q_{j-1})=\delta_{j-1}\}$ denote the full mesh of subcubes of size δ_{j-1}. Then, for $m\ge 0$, the above procedure inductively defines $\mathcal{G}_j^{m+1}$ from $\mathcal{G}_j^m$ by the rule: $\mathcal{G}_j^{m+1}=\{Q\in\mathcal{D}(Q'):Q'\in\mathcal{G}_j^m,\ |f_Q-f_{Q'}|>2^{1-j},\ \ell(Q)>\delta_j,\ Q\,\text{maximal}\}$. In addition, when $Q\in\mathcal{D}(Q_0)$ and $\delta_j<\ell(Q)\le\delta_{j-1}$, then $\tilde{Q}$ denotes the minimal cube in $\mathcal{G}_j=\cup_{m=0}^{\infty}\mathcal{G}_j^m$ that properly contains Q.

[10] Strictly speaking, the argument yields functions F_j, G_j, and R_j that are defined separately on each dyadic subcube Q_{j-1} of size δ_{j-1}; however, as the cubes $\{Q_{j-1}(x):x\in Q_0\}$ of this size are non-overlapping and together cover Q_0, we shall view these functions as being defined a.e. on all of Q_0. Hence, the statement $\|R_j\|_{L^\infty(Q_{j-1})}\le C2^{-j}$ (for all such Q_{j-1}) is equivalent to the global bound $\|R_j\|_{L^\infty(Q_0)}\le C2^{-j}$.

Since $\sum_{j=J}^{\infty} 2^{-j} \to 0$ as $J \to \infty$ (i.e., as $\delta_J \to 0$), then $R \in \mathrm{BUC}^{\mathrm{dy}}(Q_0)$.

The corresponding estimate for F is a bit more involved. The key is to determine how large each difference $(F_j)_Q - \inf_Q F_j$ in (22) can be. Recall that $\delta_J < \ell(Q) \le \delta_{J-1}$. When $j = J$, then $(F_j)_Q - \inf_Q F_j \le C2^{-j}$, by (20), since Q is contained in some dyadic cube Q_{J-1} of size δ_{J-1}.[11] Next, when $j \ge J+1$, then Q is a finite union of non-overlapping dyadic cubes Q_{j-1} of size δ_{j-1}. On each of these, the selection procedure guarantees that $\inf_{Q_{j-1}} F_j = 0$ and that $(F_j)_{Q_{j-1}} \le C2^{-j}$.[12] Over all of Q, then, $\inf_Q F_{j-1} = 0$ and (taking the average of the mean values $(F_j)_{Q_{j-1}}$ over the Q_{j-1} comprising Q) also $(F_j)_Q \le C2^{-j}$. In sum,

$$(F_j)_Q - \inf_Q F_j \le C2^{-j} \qquad \big(Q \in \mathcal{D}(Q_0)\big) \tag{24}$$

for each $j \in \mathbb{N}$. Combining (22) and (24) gives the BLO bound

$$F_Q - \inf_Q F \le C \sum_{j=J}^{\infty} (2^{-j}) \qquad (Q \in \mathcal{D}(Q_0),\, \ell(Q) \le \delta_{J-1}).$$

The estimate for the BLO bound of G runs the same way. Letting $J \to \infty$ then shows that $F, G \in \mathrm{VMO}^{\mathrm{dy}}(Q_0)$, as claimed. □

Several remarks on the construction are in order; for simplicity, these are stated under the assumption that $\|f\|_{\mathrm{BMO}^{\mathrm{dy}}(Q_0)} \le 2^{-n}$. First, observe that the functions F and G (resp. R) can be expressed as sums over all the dyadic subcubes of Q_0, not just over those where the mean oscillation of f is large (resp. not just over all dyadic subcubes having size δ_j). This means that

$$F(x) = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{G}_j^m} (f_Q - f_{\widetilde{Q}})^+ \chi_Q(x) = \sum_{Q_k \in \mathcal{D}(Q_0)} a_k \chi_{Q_k}(x), \tag{25}$$

$$G(x) = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{G}_j^m} (f_{\widetilde{Q}} - f_Q)^+ \chi_Q(x) = \sum_{Q_k \in \mathcal{D}(Q_0)} b_k \chi_{Q_k}(x), \tag{26}$$

and

$$R(x) = \sum_{j=1}^{\infty} \sum_{\ell(Q_j)=\delta_j} (f_{Q_j} - f_{\widetilde{Q}_j}) \chi_{Q_j}(x) = \sum_{Q_k \in \mathcal{D}(Q_0)} c_k \chi_{Q_k}(x), \tag{27}$$

where the latter sum in each line runs over all of $\mathcal{D}(Q_0)$. In (25), for example, whenever $Q_k \notin \cup_{j=1}^{\infty} \cup_{m=1}^{\infty} \mathcal{G}_j^m$ or whenever $Q_k \in \cup_{m=1}^{\infty} \mathcal{G}_j^m$ for some j but $f_{Q_k} - f_{\widetilde{Q}_k} < 0$, then $a_k = 0$; otherwise, $a_k = f_{Q_k} - f_{\widetilde{Q}_k} \in (2^{1-j}, 2^{2-j}]$. A similar statement applies to the coefficients b_k. In (27), $c_k = (f_{Q_k} - f_{\widetilde{Q}_k}) \in [-2^{2-j}, 2^{2-j}]$ if $\ell(Q_k) = \delta_j$ (for some $j \in \mathbb{N}$), whereas $c_k = 0$ otherwise. This reformulation will prove useful below.

The second remark concerns the rate at which the mean oscillation of a given VMO function vanishes; in the notation above, this is captured by the relative size of the sequence of scales $\{\delta_j\}$ in (19). As was noted in [15], each function in VMO is actually in BMO_φ—meaning $(1/|Q|)\int_Q |f(x) - f_Q|\,dx < \varphi(|Q|)$ for all Q—for some bounded, continuous, non-decreasing function φ on $\mathbb{R}_+$ having $\varphi(0) = 0$. Building on the work of Campanato and Meyers, Spanne [16] showed that BMO_φ functions

[11] In other words, one of the Cases I–III arising in the proof of Lemma 1 applies to F_J on Q.

[12] This corresponds to Case I for F_j on Q_{j-1}.

are essentially uniformly continuous just in case the Dini condition $\int_0^1 \varphi(t)/t\,dt < \infty$ is fulfilled; this corresponds to the condition $\sum_{j=1}^{\infty} 2^{-j} \log(\delta_{j-1}/\delta_j) < \infty$ in the notation used here. Shi and Torchinsky took this analysis one step further in [15] and gave an elegant proof of a version of the John-Nirenberg inequality for BMO_φ. Their iterative Calderón-Zygmund decomposition has much in common with that used in the proof of the dyadic version of the theorem above (Lemmas 1 and 2 taken together), with the exception that it disregards bounded remainder terms and focuses only on the size, not the sign, of the jumps arising in the stopping-time argument. Precisely these two points, fine control over the remainders and a partitioning of the jumps by sign, are key features of the argument given above: the former explains why a Calderón-Zygmund decomposition has been applied to all cubes in each mesh $\{Q_j \in \mathcal{D}(Q_0) : \ell(Q_j) = \delta_j\}$, rather than just to those having predecessors ("dyadic ancestors") that were selected within some coarser mesh, and the latter provides the basis for splitting the VMO function f into the VLO summands F and G.

3. The general setting

The proof of the theorem follows the argument in [6, pp. 361–64], except for certain technical modifications which are introduced to reflect the improved oscillatory behavior of VMO functions over small scales. For completeness, the full proof is given here. Let S_N denote the cube $\left[-2^N, 2^N\right]^n$.

Lemma 3. *Suppose f is in* VMO*. Then, for each $N \in \mathbb{N}$, there exist non-negative functions $\bar{F}_N, \bar{G}_N \in \mathrm{VLO}(S_N)$ and a function $\bar{R}_N \in \mathrm{BUC}(S_N)$ satisfying both the identity*

$$f(x) - f_{S_N} = \bar{F}_N(x) - \bar{G}_N(x) + \bar{R}_N(x) \qquad (x \in S_N) \tag{28}$$

and the estimate

$$\|\bar{F}_N\|_{\mathrm{BLO}(S_N)} + \|\bar{G}_N\|_{\mathrm{BLO}(S_N)} + \|\bar{R}_N\|_{L^\infty(S_N)} \le C\|f\|_{\mathrm{BMO}}. \tag{29}$$

The constant C depends only on the dimension n.

Moreover, for each $M \in \mathbb{N}$, the family $\{\bar{R}_N : N \ge M\}$ is equicontinuous on S_M and the families $\{\bar{F}_N : N \ge M\}$ and $\{\bar{G}_N : N \ge M\}$ have uniformly vanishing lower oscillation on S_M; this means that the quantity $C_Q = C_Q(M)$ defined by

$$C_Q = \sup_{N \ge M} \left((\bar{F}_N)_Q - \inf_Q \bar{F}_N, (\bar{G}_N)_Q - \inf_Q \bar{G}_N \right) \tag{30}$$

is finite for each $Q \subseteq S_M$ and that $C_Q \to 0$ as $\ell(Q) \to 0$.

Note that the estimates on the lower oscillation of $\bar{F}_N$ and $\bar{G}_N$ are asserted to hold over all subcubes of S_N, not just the dyadic ones; the uniform continuity of $\bar{R}_N$ on S_N is also meant in the standard sense (without respect to any dyadic mesh).

Let us first show how this last lemma implies the theorem. The identity (28) can be re-written, after subtracting off the mean value of each side on S_0, as

$$\begin{aligned} f(x) - f_{S_0} &= [\bar{F}_N(x) - (\bar{F}_N)_{S_0}] - [\bar{G}_N(x) - (\bar{G}_N)_{S_0}] + [\bar{R}_N(x) - (\bar{R}_N)_{S_0}] \\ &= \tilde{F}_N(x) - \tilde{G}_N(x) + \tilde{R}_N(x). \end{aligned} \tag{31}$$

Then (29), in conjunction with the John-Nirenberg inequality of [8], yields a uniform bound on the quadratic mean oscillation of $\tilde{F}_N$,[13] namely

$$\frac{1}{|Q|}\int_Q |\tilde{F}_N - (\tilde{F}_N)_Q|^2 \le C\|f\|^2_{\mathrm{BMO}} \qquad (Q \subseteq S_N).$$

Suppose now that $M \le N$. When $Q = S_M$, the last line becomes

$$\frac{1}{|S_M|}\int_{S_M} |\tilde{F}_N|^2 \le C\|f\|^2_{\mathrm{BMO}} + |(\tilde{F}_N)_{S_M}|^2. \tag{32}$$

To control the right-hand side here, form a telescoping sum of mean values:

$$(\tilde{F}_N)_{S_M} = (\tilde{F}_N)_{S_0} + [(\tilde{F}_N)_{S_1} - (\tilde{F}_N)_{S_0}] + \cdots + [(\tilde{F}_N)_{S_M} - (\tilde{F}_N)_{S_{M-1}}]. \tag{33}$$

Since $|S_1|/|S_0| = \cdots = |S_M|/|S_{M-1}| = 2^n$ and $(\tilde{F}_N)_{S_0} = 0$, then (33) and (6) imply that $|(F_N)_{S_M}| \le M2^n\|f\|_{\mathrm{BMO}}$. Insert this back into (32) to obtain the uniform quadratic bound

$$\frac{1}{|S_M|}\int_{S_M} |\tilde{F}_N|^2 \le C'\|f\|^2_{\mathrm{BMO}} < \infty \qquad (N = M, M+1, M+2, \dots).$$

An analogous estimate is also valid for $\{\tilde{G}_N : N \ge M\}$.

For each M, the sequences $\{\tilde{F}_N : N \ge M\}$ and $\{\tilde{G}_N : N \ge M\}$ are consequently bounded in $L^2(S_M)$. By the lemma, the sequence $\{\tilde{R}_N : N \ge M\}$ is also uniformly bounded and equicontinuous on S_M. It is therefore possible to choose a subsequence $N_k \to \infty$, so that $\tilde{F}_{N_k} \rightharpoonup F$ and $\tilde{G}_{N_k} \rightharpoonup G$ weakly in $L^2(S_M)$ and so that $\tilde{R}_{N_k} \to R$ uniformly on S_M.[14] A diagonal argument ensures (with the help of a further subsequence, if necessary) that this convergence holds simultaneously for all M. Since, by (31), $f - f_{S_0} - \tilde{R}_{N_k} = \tilde{F}_{N_k} - \tilde{G}_{N_k}$, then the sequence $\{\tilde{F}_{N_k} - \tilde{G}_{N_k}\}_{N_k}$ must also converge pointwise a.e. on S_M to the difference $F - G$.[15] In the limit, (31) thus becomes the asserted decomposition

$$f(x) - f_{S_0} = F(x) - G(x) + R(x) \qquad (x \in \mathbb{R}^n).$$

Note that by construction, R is bounded and uniformly continuous on all of $\mathbb{R}^n$.

To see that $F \in \mathrm{VLO}(\mathbb{R}^n)$, fix an arbitrary cube Q in $\mathbb{R}^n$ and choose M so large that $Q \subseteq S_M$. On this cube S_M, the weak convergence described above implies that there is a sequence $\{\varphi_K\}_{K\in\mathbb{N}}$ of finite convex combinations of the $\tilde{F}_{N_k}$, i.e.,

$$\varphi_K = \sum_{k=1}^{K} t_k \tilde{F}_{N_k} \qquad (t_k \ge 0, \sum_{k=1}^{K} t_k = 1), \tag{34}$$

that converges to F both in $L^2(S_M)$ and (taking a further subsequence, if necessary) pointwise a.e.[16] Now apply Fatou's lemma[17] to this new sequence $\{\varphi_K\}_{K\in\mathbb{N}}$ to

---

[13] The estimate on $\bar{F}_N$ carries over to $\tilde{F}_N$, since they differ only by a constant.

[14] The John-Nirenberg inequality has been invoked to move from uniform boundedness in L^1 to that in L^2; otherwise, weak compactness would have only guaranteed the existence of subsequences $\{\tilde{F}_{N_k}\}$ and $\{\tilde{G}_{N_k}\}$ converging to measures.

[15] It is not claimed—*separately*—that $\tilde{F}_{N_k}(x) \to F(x)$ and $\tilde{G}_{N_k}(x) \to G(x)$ for a.e. x.

[16] See Theorem 3.13 in [13] or Theorem V.1.2 in [17]; in the latter work, this result is attributed to S. Mazur. Note that the coefficients t_k $(1 \le k \le K)$ of φ_K in (34) may depend on K.

[17] Suppose that $\{\varphi_K\}$ is a sequence of non-negative, measurable functions that converges a.e. to φ. What is needed here is the (standard) L^1 form of Fatou's lemma, $\int \varphi \le \underline{\lim}_K \int \varphi_K$, as well as its L^∞ form: $\underline{\lim}_K(\inf \varphi_K) \le \inf \varphi$; the latter can be verified via a simple proof by contradiction. Recall that we write $\inf \varphi$ for $\operatorname{ess\,inf} \varphi$, as indicated in the introduction.

obtain from (30) the following bound on the lower oscillation of F over Q:

$$\begin{aligned}\frac{1}{|Q|}\int_Q F &\leq \underline{\lim}_K \frac{1}{|Q|}\int_Q \varphi_K = \underline{\lim}_K \sum_{k=1}^{K} \frac{t_k}{|Q|}\int_Q \tilde{F}_{N_k} \\ &\leq \underline{\lim}_K \sum_{k=1}^{K} t_k(\inf_Q \tilde{F}_{N_k} + C_Q) \leq \underline{\lim}_K \sum_{k=1}^{K}(t_k \inf_Q \tilde{F}_{N_k}) + C_Q \\ &\leq \underline{\lim}_K \inf_Q \sum_{k=1}^{K} t_k \tilde{F}_{N_k} + C_Q = \underline{\lim}_K \inf_Q \varphi_K + C_Q \leq \inf_Q F + C_Q.\end{aligned}$$

As, by assumption, $\sup_{Q\subseteq S_M} C_Q < \infty$ and $\lim_{\ell(Q)\to 0} C_Q = 0$, then $F \in \mathrm{VLO}(S_M)$ for each M. The treatment of G is identical. Modulo the proof of Lemma 3, the proof of the theorem is now complete.

Proof of Lemma 3. We use the averaging procedure of [6] to move from the dyadic version of the theorem (Lemma 2) to the general, local version (Lemma 3). As above (but now over all of $\mathbb{R}^n$), assume that $\|f\|_{\mathrm{BMO}} = 2^{-n}$. For each $\alpha \in \mathbb{R}^n$, let $T_\alpha f$ denote the translate of f by α, where $T_\alpha f(x) = f(x-\alpha)$. Choose a single, strictly decreasing sequence $\{\delta_j\}_{j\in\mathbb{N}} \subset 2^{-\mathbb{N}}$ of dyadic scales so that (19) holds uniformly for $T_\alpha f$ (in place of f) as α varies over $\mathbb{R}^n$; this is possible due to the assumption that $f \in \mathrm{VMO} = \mathrm{VMO}(\mathbb{R}^n)$.

Fix N and assume, without loss of generality, that $f_{S_N} = 0$. Set $Q_0 = S_{N+1}$ and $\delta_0 = \ell(S_{N+1})$. For each $\alpha \in S_N$, apply Lemma 2 to $T_\alpha f$ on Q_0. The result is that

$$T_\alpha f(x) - (T_\alpha f)_{S_{N+1}} = F^{(\alpha)}(x) - G^{(\alpha)}(x) + R^{(\alpha)}(x) \qquad (x \in S_{N+1}),$$

where $F^{(\alpha)}$, $G^{(\alpha)}$, and $R^{(\alpha)}$ are defined as in (21). In particular, $F^{(\alpha)} = \sum_{j=1}^{\infty} F_j^{(\alpha)}$, where $F_j^{(\alpha)}$ is locally constant on the mesh of dyadic subcubes of Q_0 of edge-length δ_j. In addition, the expansion (25) guarantees that there are non-negative coefficient functions $a_k^{(\alpha)}$, depending measurably on α, such that

$$F^{(\alpha)}(x) = \sum_{Q_k\in\mathcal{D}(S_{N+1})} a_k^{(\alpha)} \chi_{Q_k}(x).$$

Note that this sum runs over $\mathcal{D}(S_{N+1})$, a fixed, countable collection of cubes that is indexed by k and independent of α; as discussed in the remark at the end of §2, when $\delta_j < \ell(Q_k) \leq \delta_{j-1}$, each coefficient $a_k^{(\alpha)}$ is either 0 or in the interval $(2^{1-j}, 2^{2-j}]$. Equation (26) leads to a similar representation for $G^{(\alpha)}$, and (27) gives rise to the sum

$$R^{(\alpha)}(x) = \sum_{Q_k\in\mathcal{D}(S_{N+1})} c_k^{(\alpha)} \chi_{Q_k}(x); \tag{35}$$

when $\ell(Q_k) = \delta_j$ for some $j \in \mathbb{N}$, then $|c_k^{(\alpha)}| \leq 2^{2-j}$, and $c_k^{(\alpha)} = 0$ otherwise.

Since $f(x) = (1/|S_N|)\int_{S_N} f(x)\,d\alpha$, then for a.e. $x \in S_N$

$$\begin{aligned}f(x) &= \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}(T_\alpha f)(x)\,d\alpha \\ &= \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}\big(F^{(\alpha)} - G^{(\alpha)} + R^{(\alpha)} + (T_\alpha f)_{S_{N+1}}\big)(x)\,d\alpha;\end{aligned}$$

that is, $f(x) = \bar{F}_N(x) - \bar{G}_N(x) + \bar{R}_N(x)$, where

$$\bar{F}_N(x) = \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}(F^{(\alpha)})(x)\,d\alpha = \frac{1}{|S_N|}\int_{S_N} F^{(\alpha)}(x+\alpha)\,d\alpha,$$

$\bar{G}_N(x)$ is defined analogously, and

$$\begin{aligned}\bar{R}_N(x) &= \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}\big(R^{(\alpha)}\,d\alpha + (T_\alpha f)_{S_{N+1}}\big)(x)\,d\alpha \\ &= \frac{1}{|S_N|}\int_{S_N} R^{(\alpha)}(x+\alpha) + \frac{1}{|S_N|}\int_{S_N} (T_\alpha f)_{S_{N+1}}\,d\alpha. \end{aligned} \tag{36}$$

It remains to show, for the functions so defined, that $\bar{F}_N$ and $\bar{G}_N$ are in VLO on the cube S_N and that $\bar{R}_N$ is uniformly continuous there. In particular, we wish to show that

$$\frac{1}{|Q|}\int_Q \bar{F}_N - \inf_Q \bar{F}_N = o(1) \qquad \big(\ell(Q)\to 0\big). \tag{37}$$

and that

$$\sup_Q \bar{R}_N - \inf_Q \bar{R}_N = o(1) \qquad \big(\ell(Q)\to 0\big). \tag{38}$$

To reach this goal, fix an arbitrary cube Q within S_N, and find J such that $\ell(Q) \le \delta_J/(2\sqrt{n})$. Split the terms comprising $F^{(\alpha)}$ according to the size of Q by writing $F^{(\alpha)} = F^{(\alpha)}_{\text{large}} + F^{(\alpha)}_{\text{small}} = \sum_{j=1}^{J} F_j^{(\alpha)} + \sum_{j=J+1}^{\infty} F_j^{(\alpha)}$. Then

$$F^{(\alpha)}_{\text{large}}(x) = \sum_{\ell(Q_k)>\delta_J} a_k^{(\alpha)}\chi_{Q_k}(x), \qquad F^{(\alpha)}_{\text{small}}(x) = \sum_{\ell(Q_k)\le\delta_J} a_k^{(\alpha)}\chi_{Q_k}(x). \tag{39}$$

Note that only finitely many coefficients $a_k^{(\alpha)}$ enter into the first sum in (39), since this sum runs only over those dyadic subcubes $Q_k \in \mathcal{D}(S_{N+1})$ of size greater than δ_J (and this includes only cubes significantly larger than Q).

The corresponding averaged forms are given by

$$\bar{F}_{\text{large}}(x) = \frac{1}{|S_N|}\int_{S_N} F^{(\alpha)}_{\text{large}}(x+\alpha)\,d\alpha, \qquad \bar{F}_{\text{small}}(x) = \frac{1}{|S_N|}\int_{S_N} F^{(\alpha)}_{\text{small}}(x+\alpha)\,d\alpha;$$

thus, $\bar{F}_N = \bar{F}_{\text{large}} + \bar{F}_{\text{small}}$. To prove (37) we shall verify the bounds

$$\sup_Q \bar{F}_{\text{large}} \le \inf_Q \bar{F}_{\text{large}} + CJ2^{-J} \tag{40}$$

and

$$\frac{1}{|Q|}\int_Q \bar{F}_{\text{small}} \le \inf_Q \bar{F}_{\text{small}} + C2^{-J}. \tag{41}$$

Now, the first of these is a consequence of the following Lipschitz estimate[18] on the contribution to $\bar{F}_N$ of the terms associated to cubes of a similar size. For this, let

$$\hat{F}_j(x) = \frac{1}{|S_N|}\int_{S_N} \sum_{\delta_j<\ell(Q_k)\le\delta_{j-1}} a_k^{(\alpha)}\chi_{Q_k}(x+\alpha)\,d\alpha,$$

so that $\bar{F}_{\text{large}}(x) = \sum_{j=1}^{J} \hat{F}_j(x)$.

[18] This is Lemma 3.2 in [6], re-interpreted to highlight only certain scales.

Lemma 4. *If* $\max_{1\le i\le n}|x_i-y_i|\le\delta_j$, *then*

$$|\hat{F}_j(x)-\hat{F}_j(y)|\le C2^{-j}\frac{|x-y|}{\delta_j},$$

with C *dependent only on the dimension* n (*and, in particular, not on* j).

Proof. Fix $x,y\in S_N$ with $\max_{1\le i\le n}|x_i-y_i|\le\delta_j$. When $\ell(Q_k)\le\delta_{j-1}$, then the corresponding coefficient $a_k^{(\alpha)}$ is no more than 2^{2-j}, as discussed above. So,

$$\begin{aligned}|\hat{F}_j(x)-\hat{F}_j(y)| &\le \frac{1}{|S_N|}\int_{S_N}\sum_{\delta_j<\ell(Q_k)\le\delta_{j-1}}|a_k^{(\alpha)}||\chi_{Q_k}(x+\alpha)-\chi_{Q_k}(y+\alpha)|\,d\alpha\\ &\le C2^{-j}\sum_{r=0}^{r_j}\frac{1}{|S_N|}\int_{S_N}\sum_{\ell(Q_k)=2^r\delta_j}|\chi_{Q_k}(x+\alpha)-\chi_{Q_k}(y+\alpha)|\,d\alpha,\end{aligned}$$

where $r_j=\log_2(\delta_{j-1}/\delta_j)$. The last integrand is twice the characteristic function of the set of all $\alpha\in S_N$ for which $x+\alpha$ and $y+\alpha$ lie in different dyadic subcubes Q_k of S_{N+1} of size $\ell(Q_k)=2^r\delta_j$. The relative density of this set in S_N is majorized by a constant multiple of $\sum_{i=1}^n|x_i-y_i|/(2^r\delta_j)$. Thus,

$$|\hat{F}_j(x)-\hat{F}_j(y)|\le C2^{-j}\sum_{r=0}^{r_j}\frac{|x-y|}{2^r\delta_j}\le C'2^{-j}\frac{|x-y|}{\delta_j},$$

as claimed. □

Now, suppose that $x,y\in Q\subseteq S_N$ with $\ell(Q)\le\delta_J/(2\sqrt{n})$, as above. Then $|x-y|\le\sqrt{n}\ell(Q)\le\delta_J<\delta_{J-1}<\cdots<\delta_1$. The lemma then gives

$$|\bar{F}_{\mathrm{large}}(x)-\bar{F}_{\mathrm{large}}(y)|\le\sum_{j=1}^{J}|\hat{F}_j(x)-\hat{F}_j(y)|\le C\sum_{j=1}^{J}2^{-j}\frac{\delta_J}{\delta_j}.$$

Let σ_J denote this last sum. Since $\sigma_1=1/2$ and

$$\sigma_{J+1}=\frac{\delta_{J+1}}{\delta_J}\sigma_J+\frac{1}{2^{J+1}}\le\frac{1}{2}\left(\sigma_J+\frac{1}{2^J}\right),$$

then an elementary induction argument shows that $\sigma_J\le J/2^J$. This confirms (40), the bound on the modulus of continuity of $\bar{F}_{\mathrm{large}}$.

What about (41)? The right-hand side there can in fact be simplified further by noting that $\bar{F}_{\mathrm{small}}\ge 0$. As for the left-hand side, from Fubini's theorem it follows that

$$\frac{1}{|Q|}\int_Q\bar{F}_{\mathrm{small}}=\frac{1}{|S_N|}\int_{S_N}\frac{1}{|Q|}\int_{Q+\alpha}F^{(\alpha)}_{\mathrm{small}}(x)\,dx\,d\alpha.$$

For the proof of (41) and hence of (37), it thus suffices to obtain a suitable estimate on the inner integral here, i.e., to show that

$$\frac{1}{|Q|}\int_{Q+\alpha}F^{(\alpha)}_{\mathrm{small}}(x)\,dx=\sum_{j=J+1}^{\infty}\frac{1}{|Q|}\int_{Q+\alpha}F_j^{(\alpha)}(x)\,dx\le C2^{-J}\tag{42}$$

uniformly for all $\alpha\in S_N$. But the translated cube $Q+\alpha$ serving as the region of integration is contained within a union of 2^n congruent dyadic subcubes of S_{N+1}, each having edge-length less than twice that of Q (and hence not more than δ_J). Applying (24) in §2 to each of these subcubes and summing up leads to the bound

$\int_{Q+\alpha} F_j^{(\alpha)}(x)\,dx \le C2^{-j}|Q|$. A further sum in j (for $j \ge J+1$) leads to (42). The proof of estimate (37) for $\bar{G}_N$ is similar.

It remains to show that $\bar{R}_N$, as defined in (36), is uniformly continuous on S_N. The second term in (36), the average of $(T_\alpha f)_{S_{N+1}}$ over $\alpha \in S_N$, is constant and can safely be ignored. In addition, the countable sum giving $R^{(\alpha)}$ in (35) can be split according to the size of Q, so that $R^{(\alpha)} = R^{(\alpha)}_{\text{large}} + R^{(\alpha)}_{\text{small}}$, as in (39). The partition $\bar{R}_N = \bar{R}_{\text{large}} + \bar{R}_{\text{small}}$ likewise applies to the corresponding averaged forms, up to the constant just described. Since $|c_k^{(\alpha)}| \le C2^{-j}$ when $\ell(Q) \le \delta_j$ (as was the case for the coefficients $a_k^{(\alpha)}$), then the argument in Lemma 4 for $\bar{F}_{\text{large}}$ carries over to $\bar{R}_{\text{large}}$, so that estimate (40) also holds for the latter. On the other hand, since only relatively few of the coefficients $c_k^{(\alpha)}$ may be non-zero (namely those corresponding to cubes Q_k of size exactly δ_j, for some $j \in \mathbb{N}$), the BLO estimate (41) can be replaced by a stronger estimate on the modulus of continuity of $\bar{R}_{\text{small}}$. In fact,

$$\sup_Q |\bar{R}_{\text{small}}| \le C \sum_{j=J}^{\infty} 2^{-j} \le C2^{-J},$$

by (20). All together, then, $\sup_Q \bar{R}_N - \inf_Q \bar{R}_N \le CJ2^{-J}$, when $\ell(Q) \le \delta_J/(2\sqrt{n})$, giving (38). Thus, $\bar{R}_N$ can be re-defined on a set of measure zero to yield a uniformly continuous function.

This settles the last remaining step in the proof of the lemma, and the decomposition theorem is therefore complete. $\square$

References

[1] F. Chiarenza, M. Frasca, and P. Longo, *$W^{2,p}$-solvability of the Dirichlet problem for non-divergence elliptic equations with* VMO *coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853.

[2] M. Carozza and R. Mellone, *The distance to L^∞ or to* VMO *for* BMO *functions*, Ricerce Mat. **44** (1995), 439–448.

[3] R. R. Coifman, R. Rochberg, *Another characterization of* BMO, Proc. Amer. Math. Soc. **79** (1980), 249–254.

[4] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, New York, and Oxford, 1985.

[5] J. Garnett and P. Jones, *The distance in* BMO *to L^∞*, Ann. of Math. **108** (1978), 373–393.

[6] J. Garnett and P. Jones, BMO *from dyadic* BMO, Pacific J. Math. **99** (1982), 351–371.

[7] P. J. Holden, *Extension theorems for functions of vanishing mean oscillation*, Pacific J. Math. **142** (1990), 277–295.

[8] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[9] P. Jones, *Factorization of A_p weights*, Ann. of Math. **111** (1980), 511–530.

[10] C. E. Kenig and T. Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. by 150 (1999), 369–454.

[11] M. B. Korey, *Ideal weights: asymptotically optimal versions of doubling, absolute continuity, and mean oscillation*, J. Fourier Anal. Appl. **4** (1998), 491–519.

[12] M. B. Korey, *Optimal factorization of Muckenhoupt weights*, Trans. Amer. Math. Soc. **353** (2001), 839–851.

[13] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[14] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405.

[15] X. L. Shi and A. Torchinsky, *Functions of vanishing mean oscillation*, Math. Nachr. **133** (1987), 289–296.

[16] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa **19** (1965), 593–608.

[17] K. Yosida, *Functional analysis* Grundlehren Math. Wiss., vol. 123, Springer-Verlag, Berlin, Heidelberg, and New York, 1965.

Institut für Mathematik, Universität Potsdam, 14415 Potsdam, Germany
E-mail address: mike@math.uni-potsdam.de