# Asymptotic Expansions for Bounded Solutions to Semilinear Fuchsian Equations

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Abstract. It is shown that bounded solutions to semilinear elliptic Fuchsian equations obey complete asymptotic expansions in terms of powers and logarithms in the distance to the boundary. For that purpose, Schulze's notion of asymptotic type for conormal asymptotics close to a conical point is refined. This in turn allows to perform explicit calculations on asymptotic types — modulo the resolution of the spectral problem for determining the singular exponents in the asymptotic expansions.

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Contents

## 1 Introduction

In this paper, we shall study solutions $u = u(x)$ to semilinear elliptic equations of the form

$$Au = F(x, B_1u, \dots, B_Ku) \text{ on } X^\circ = X \setminus \partial X. \tag{1.1}$$

Here, $X$ is a smooth compact manifold with boundary, $\partial X$, and of dimension $n+1$, $A$, $B_1, \dots, B_K$ are Fuchsian differential operators on $X^\circ$, see Definition 2.1, with real-valued coefficients and of orders $\mu$, $\mu_1, \dots, \mu_K$, respectively, where $\mu_J < \mu$ for $1 \le J \le K$, and $F = F(x,\nu)\colon X^\circ \times \mathbb{R}^K \to \mathbb{R}$ is a smooth function subject to further conditions as $x \to \partial X$. In case $A$ is elliptic in the sense of Definition 2.2 (a) we shall prove that bounded solutions $u\colon X^\circ \to \mathbb{R}$ to Eq. (1.1) possess complete conormal asymptotic expansion of the form

$$u(t,y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} t^{-p_j} \log^k t \, c_{jk}(y) \text{ as } t \to +0. \tag{1.2}$$

Here, $(t,y) \in [0,1) \times Y$ are normal coordinates on a neighbourhood $\mathcal{U}$ of $\partial X$, $Y$ is diffeomorphic to $\partial X$, and the exponents $p_j \in \mathbb{C}$ appear in conjugated pairs, $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$, $m_j \in \mathbb{N}$, and $c_{jk}(y) \in C^\infty(Y)$. Note that such conormal asymptotic expansions are typical for solutions $u$ to linear equations of the form (1.1), i.e., in the case that $F(x) = F(x,\nu)$ is independent of $\nu \in \mathbb{R}^K$. The general form (1.2) of asymptotics was first thoroughly investigated by Kondrat'ev in his nowadays classical paper [9]. After that to assign asymptotic types to conormal asymptotic expansions of the form (1.2) has been proved to be very fruitful. In its consequence, it provides a functional-analytic set-up for treating singular problems, both linear and non-linear ones, of the kind (1.1). Function spaces with asymptotics will be discussed in Section 2.4. In the standard setting, going back to Rempel–Schulze [14] in case $n = 0$, i.e., $Y$ is a point, and Schulze [15] in the general case, an asymptotic type $P$ for conormal asymptotic expansions of the form (1.2) is given by a sequence $\{(p_j, m_j, L_j)\}_{j=0}^{\infty}$, where $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$ are as in (1.2), and $L_j$ is a finite-dimensional linear subspace of $C^\infty(Y)$ to which the coefficients $c_{jk}(y)$ for $0 \le k \le m_j$ are required to belong. (In case $n = 0$, the spaces $L_j$ equal $\mathbb{C}$ and disappear from the consideration.) A function $u(x)$ is said to have conormal asymptotics of type $P$ as $x \to \partial X$ if $u(x)$ obeys a conormal asymptotic expansion of the form (1.2), with the data given by $P$.

When treating semilinear equations we shall encounter asymptotic types belonging to bounded functions $u(x)$, i.e., asymptotic types $P$ for which

$$\begin{cases} p_0 = 0,\ m_0 = 0,\ L_0 = \operatorname{span}\{1\}, \\ \operatorname{Re} p_j < 0 \text{ for all } j \ge 1, \end{cases} \tag{1.3}$$

where $1 \in L_0$ denotes the function on $Y$ being constant equal to 1.

It turns out that the notion of asymptotic type just described resolves asymptotics not fine enough in order to be suitable for treating semilinear problems.

The difficulty with this notion is that only the aspect of the production of asymptotics — via the finite-dimensionality of the spaces $L_j$ — is emphasized, cf. Proposition 2.29 (b), but not the aspect of their annihilation. For semilinear problems, however, the latter affair becomes crucial. Therefore, in Section 2.2, we shall introduce a refined notion of asymptotic type, where additionally linear relations between the various coefficients $c_{jk}(y) \in L_j$, even for different $j$, are taken into account.

Let $\underline{\mathrm{As}}(Y)$ be the set of all these refined asymptotic types, while $\underline{\mathrm{As}}^\sharp(Y) \subset \underline{\mathrm{As}}(Y)$ denotes the set of all asymptotic types belonging to bounded functions according to (1.3). For $R \in \underline{\mathrm{As}}(Y)$, let $C_R^\infty(X)$ be the space of all $u \in C^\infty(X^\circ)$ having conormal asymptotic expansions of type $R$, and $C_R^\infty(X \times \mathbb{R}^K) = C^\infty(\mathbb{R}^K; C_R^\infty(X))$, where $C_R^\infty(X)$ is equipped with its natural Fréchet topology. In the formulation of Theorem 1.1, we will assume that $F \in C_R^\infty(X \times \mathbb{R}^K)$, where

$$\omega(t) t^{\mu - \bar{\mu} - \varepsilon} C_R^\infty(X) \subset L^\infty(X) \tag{1.4}$$

for some $\varepsilon > 0$. Here, $\bar{\mu} = \max_{1 \le J \le K} \mu_J < \mu$ and $\omega = \omega(t)$ is a cut-off function supported on $\mathcal{U}$, i.e., $\omega \in C^\infty(X)$, $\operatorname{supp} \omega \Subset \mathcal{U}$. Here and in the sequel, we will always assume that $\omega = \omega(t)$ only depends on $t$ for $0 < t < 1$ and $\omega(t) = 1$ for $0 < t \le 1/2$. Condition (1.4) means that, for the given operator $A$, and then compared to the operators $B_1, \ldots, B_K$, functions in $C_R^\infty(X)$ cannot be too singular as $t \to +0$.

There is a small difference between the set $\underline{\mathrm{As}}^b(Y)$ of all bounded asymptotic types and the set $\underline{\mathrm{As}}^\sharp(Y)$ of asymptotic types described by (1.3); $\underline{\mathrm{As}}^\sharp(Y) \subsetneq \underline{\mathrm{As}}^b(Y)$. The set $\underline{\mathrm{As}}^\sharp(Y)$ actually appears as the set of all multiplicatively closable asymptotic types, see Lemma 3.5. This proves itself in the fact that when only boundedness is presumed we have to exclude asymptotic types belonging to $\underline{\mathrm{As}}^b(Y)$, but not to $\underline{\mathrm{As}}^\sharp(Y)$ from the consideration by the following condition of non-resonance type (1.5):

Let $\mathcal{H}^{-\infty,\delta}(X) = \bigcup_{s \in \mathbb{R}} \mathcal{H}^{s,\delta}(X)$ for $\delta \in \mathbb{R}$ be the space of all distributions $u = u(x)$ on $X^\circ$ having conormal order at least $\delta$. (The weighted Sobolev spaces $\mathcal{H}^{s,\delta}(X)$, where $s \in \mathbb{R}$ is Sobolev regularity, are introduced in (2.33).) Note that $\mathcal{H}^{-\infty,\delta}(X) \subseteq \mathcal{H}^{-\infty,\delta'}(X)$ if and only if $\delta \ge \delta'$, and $\bigcup_{\delta \in \mathbb{R}} \mathcal{H}^{-\infty,\delta}(X)$ is the space of all extendable distributions on $X^\circ$ which is dual to the space $C_{\mathcal{O}}^\infty(X)$ of all smooth functions on $X$ vanishing to the infinite order on $\partial X$. (The subscript in $C_{\mathcal{O}}^\infty(X)$ anticipates the empty asymptotic type, $\mathcal{O}$.) Moreover, the conormal order $\delta$ for $\delta \to \infty$ is the parameter in which the asymptotics (1.2) are understood.

Now, fix a certain $\delta \in \mathbb{R}$ and suppose that $u \in \mathcal{H}^{-\infty,\delta}(X)$, being real-valued and satisfying $Au \in C_{\mathcal{O}}^\infty(X)$, has an asymptotic expansion of the form

$$u(x) \sim \operatorname{Re} \left( \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} t^{l+j+i\beta} \log^k t \, c_{jk}(y) \right) \quad \text{as } t \to +0,$$

where $l \in \mathbb{Z}$, $\beta \in \mathbb{R}$, $\beta \neq 0$ (and $l > \delta - 1/2$ provided that $c_{0,m_0}(y) \not\equiv 0$ due to the fact $u \in \mathcal{H}^{-\infty,\delta}(X)$). Then, for each $1 \leq J \leq K$, it is additional required that

$$B_J u = O(1) \text{ as } t \to +0 \text{ implies } B_J u = o(1) \text{ as } t \to +0. \tag{1.5}$$

Here, $O$ and $o$ are Landau's symbols. Condition (1.5) means that there is no real-valued $u \in \mathcal{H}^{-\infty,\delta}(X)$ with $Au \in C^\infty_{\mathcal{O}}(X)$ such that $B_J u$ admits an asymptotic series in the sense of Remark 2.54 starting with the term $\mathrm{Re}(t^{i\beta} d(y))$ for some $\beta \in \mathbb{R} \setminus \{0\}$, $d(y) \in C^\infty(Y)$. This condition is void if $\delta \geq 1/2 + \bar{\mu}$. Then our main theorem is stated as follows:

THEOREM 1.1. *Let $\delta \in \mathbb{R}$ and $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ be elliptic in the sense of* Definition 2.2 (a)*, $B_J \in \mathrm{Diff}^{\mu_J}_{\mathrm{Fuchs}}(X)$ for $1 \leq J \leq K$, where $\mu_J < \mu$, and $F \in C^\infty_R(X \times \mathbb{R}^k)$ for some asymptotic type $R \in \underline{\mathrm{As}}(Y)$ satisfying* (1.4)*. Further, let the non-resonance condition* (1.5) *be satisfied. Then there exists an asymptotic type $P \in \underline{\mathrm{As}}(Y)$ expressible in terms of $A$, $B_1, \dots, B_K$, $R$, and $\delta$ such that each solution $u \in \mathcal{H}^{-\infty,\delta}(X)$ to* Eq. (1.1) *satisfying $B_J u \in L^\infty(X)$ for $1 \leq J \leq K$ belongs to the space $C^\infty_P(X)$.*

Under the assumptions of Theorem 1.1, interior elliptic regularity already implies $u \in C^\infty(X^\circ)$. Thus, the statement concerns the fact that $u$ possesses a complete conormal asymptotic expansion of type $P$ close to $\partial X$. Furthermore, the asymptotic type $P$ can, in principle, be calculated once $A$, $B_1, \dots, B_K$, $R$, and $\delta$ are known.

Some remarks concerning Theorem 1.1 seem to be in order. First of all, the solution $u$ is asked to belong to the space $\mathcal{H}^{-\infty,\delta}(X)$. Thus, if the non-resonance condition (1.5) is satisfied for all $\delta \in \mathbb{R}$ — which is generically true — then the foregoing requirement can be replaced by the requirement for $u$ just being an extendable distribution. In this case, $P_\delta \preccurlyeq P_{\delta'}$ for $\delta \geq \delta'$ in the natural ordering of asymptotic types, see before Proposition 2.29, where $P_\delta$ denotes the asymptotic type associated with the conormal order $\delta$. Moreover, jumps in this relation occur only for a discrete set of values for $\delta \in \mathbb{R}$.

Secondly, for a solution $u \in C^\infty_P(X)$ to Eq. (1.1), neither $u$ nor the right-hand side $F(x, B_1 u(x), \dots, B_K u(x))$ must be bounded. Unboundedness of $u$, however, requires that asymptotics governed by the elliptic operator $A$ are cancelled jointly by the operators $B_1, \dots, B_K$, up to a certain degree, and this, in turn, is the non-generic situation. Furthermore, in applications one often has that one of the operators $B_J$, say $B_1$, is the identity belonging to $\mathrm{Diff}^0_{\mathrm{Fuchs}}(X)$, i.e., we have $B_1 u = u$ for all $u$. Then this leads to $u \in L^\infty(X)$ and explains the term "bounded solutions" in the paper's title.

*Remark* 1.2. (a) A spatial conical singularity leads via blow-up, i.e., the introduction of polar coordinates, to a manifold with boundary. Vice versa, each manifold with boundary gives rise to a space with a conical singularity via shrinking the boundary to a point. Since in both situations the analysis takes place over the interior of the underlying configuration, i.e., away from the conical singularity and the boundary, respectively, there is no essential difference

between these two situations. Stated this in a different way, the geometric singularity considered is prescribed by the type of degeneracy admitted, e.g., for differential operators. In our case, this degeneracy is of Fuchsian type.
(b) Theorem 1.1 continues to hold for sectional solutions in vector bundles over $X$. Let $E_0, E_1, E_2$ be smooth vector bundles over $X$, $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X; E_0, E_1)$ be elliptic in the sense above, $B \in \mathrm{Diff}^{\mu-1}_{\mathrm{Fuchs}}(X; E_0, E_2)$, and $F \in C^{\infty}_{R}(X, E_2; E_1)$. Then, under the same technical assumptions as above, each solution $u$ to $Au = F(x, Bu)$ in the class of extendable distributions with $Bu \in L^{\infty}(X; E_2)$ belongs to the space $C^{\infty}_{P}(X; E_0)$ for some resulting asymptotic type $P$.

Theorem 1.1 has actually been stated as one, though basic example to a more general method for deriving and then justifying conormal asymptotic expansions for solutions to semilinear elliptic Fuchsian equations. This method always works if one has boundedness assumptions as made above, but boundedness can often be successfully replaced by structural assumptions on the nonlinearity. An example is provided in Appendix 3.A. The proposed method works indeed not only for elliptic Fuchsian equations, but for other Fuchsian equations as well. What counts is the invertible of the complete sequence of conormal symbols in the algebra of all conormal symbols under the Mellin translation product, and this is equivalent to the (parameter-dependent) ellipticity of the principal conormal symbol. For elliptic Fuchsian differential operator, the latter condition is always fulfilled.
The derivation of conormal asymptotic expansions for solutions to semilinear Fuchsian equations is a purely algebraic business once the singular exponents and their multiplicities for the linear part are known. A strict justification of these conormal asymptotic expansions, in the generality supplied in this paper, requires the introduction of the refined notion of asymptotic type and corresponding function spaces with asymptotics. For this reason, from a technical point of view the main result of this paper is Theorem 2.43 which among others states the existence of a complete sequence of holomorphic conormal symbols realizing a given proper asymptotic type in the sense of exactly annihilating asymptotics of the given type. (The term "proper" is explained in Definition 2.22.) The construction of such conormal symbols relies on the factorization result of WITT [20].

The paper is organized as follows: The first part of this paper, Section 2, is devoted to the linear theory and the introduction of the refined notion of asymptotic type. Then, in the second part in Section 3, Theorem 1.1 is proved. Both parts are accompanied by appendices either explaining technical details or providing an example.
Let us conclude with a technical remark. Behind a good deal of argument, there is Schulze's cone pseudodifferential calculus. The interested reader is referred to Appendix 2.B and SCHULZE [15, 16]. We will not go to much into the details, since all the arguments below can be carried out without reference to this calculus. Indeed, the important thing to control the production and

annihilation of asymptotics is the algebra of complete conormal symbols, and it is this algebra which is discussed with great care. From time to time, however, we fall back on Schulze's cone pseudodifferential calculus to shorten proofs.

## 2 Asymptotic types

In this section, we shall introduce the notion of a discrete asymptotic type. A comparison of this notion with the formerly known notions of a weakly discrete asymptotic type and a strongly discrete asymptotic type, respectively, can be found in Figure 1. The definition of a discrete asymptotic type is modelled on part of the Gohberg-Sigal theory of the inversion of finitely meromorphic, operator-valued functions at a point. This is reviewed in Appendix 2.A. It is recommended to the reader to look up there to get a grasp of the main ideas in Sections 2.2, 2.3. Finally, in Section 2.4, function spaces with asymptotics are introduced. The definition of such function spaces relies on the existence of complete (holomorphic) conormal symbols realizing a prescribed proper asymptotic type. The existence of these complete conormal symbols is stated and proved in Theorem 2.43.

### 2.1 Fuchsian differential operators

Let $X$ be a compact $C^\infty$-manifold with boundary, $\partial X$. Throughout, we fix a collar neighbourhood $\mathcal{U}$ of $\partial X$ and a diffeomorphism $\chi\colon \mathcal{U} \to [0,1) \times Y$, with $Y$ being a closed $C^\infty$-manifold diffeomorphic to $\partial X$. Hence, we work in a fixed splitting of coordinates $(t,y)$ on $\mathcal{U}$, where $t \in [0,1)$ and $y \in Y$. Let $(\tau,\eta)$ be the covariables to $(t,y)$. The compressed covariable $t\tau$ to $t$ is denoted by $\tilde{\tau}$, i.e., $(\tilde{\tau},\eta)$ is the linear variable in the fibre of the compressed cotangent bundle $\tilde{T}^*\mathcal{U}$. Finally, let $\dim X = n+1$.

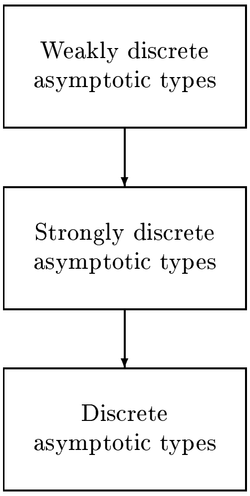
Definition 2.1. A differential operator $A$ with smooth coefficients of order $\mu$ on $X^\circ = X \setminus \partial X$ is called Fuchsian if

$$\chi_* \left(A\big|_{\mathcal{U}}\right) = t^{-\mu} \sum_{k=0}^{\mu} a_k(t) \big(-t\partial_t\big)^k, \tag{2.1}$$

where $a_k \in C^\infty([0,1); \mathrm{Diff}^{\mu-k}(Y))$ for $0 \le k \le \mu$. The class of all Fuchsian differential operators of order $\mu$ on $X^\circ$ is denoted by $\mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$.

Henceforth, we shall suppress writing the restriction $\cdot\big|_{\mathcal{U}}$ and the operator push-forward $\chi_*$ in expressions like (2.1). For $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$, we denote by $\sigma^{\mu}_{\psi}(A)$ the principal symbol of $A$, by $\tilde{\sigma}^{\mu}_{\psi}(A)$ its compressed principal symbol defined on $\tilde{T}^*\mathcal{U}$ and related to $\sigma^{\mu}_{\psi}(A)$ via

$$\sigma^{\mu}_{\psi}(A)(t,y,\tau,\eta) = t^{-\mu}\tilde{\sigma}^{\mu}_{\psi}(A)(t,y,t\tau,\eta), \quad (t,y,\tau,\eta) \in T^*\mathcal{U} \setminus 0, \tag{2.2}$$




Singular exponents with multiplicities, $(p_j, m_j)$, are prescribed, the coefficients $c_{jk}(y) \in C^\infty(Y)$ are arbitrary. The general form of asymptotics is observed, cf., e.g., KONDRAT'EV (1967), MELROSE (1993), SCHULZE (1998).

Singular exponents with multiplicities, $(p_j, m_j)$, are prescribed, $c_{jk}(y) \in L_j \subset C^\infty(Y)$, where $\dim L_j < \infty$. The *production of asymptotics* is observed, cf. REMPEL–SCHULZE (1989), SCHULZE (1991).

Linear relation between the various coefficients $c_{jk}(y) \in L_j$, even for different $j$, are additionally allowed. Thus the *production/annihilation of asymptotics* is observed, cf. this article.

Figure 1: Schematic overview of asymptotic types

and by $\sigma_M^\mu(A)(z)$ its principal conormal symbol,

$$\sigma_M^\mu(A)(z) = \sum_{k=0}^{\mu} a_k(0) z^k, \quad z \in \mathbb{C}. \tag{2.3}$$

Further, we introduce the $j$th conormal symbol $\sigma_M^{\mu-j}(A)(z)$ for $j = 1, 2, \ldots$ by

$$\sigma_M^{\mu-j}(A)(z) = \sum_{k=0}^{\mu} \frac{\partial^j a_k}{\partial t^j}(0) z^k, \quad z \in \mathbb{C}.$$

Note that $\tilde{\sigma}_\psi^\mu(A)(t, y, \tilde{\tau}, \eta)$ is smooth up to $t = 0$ and that $\sigma_M^{\mu-j}(z)$ for $j = 0, 1, 2, \ldots$ is a holomorphic function in $z$ taking values in $\mathrm{Diff}^\mu(Y)$. Moreover, if $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$, $B \in \mathrm{Diff}^\rho_{\mathrm{Fuchs}}(X)$, then $AB \in \mathrm{Diff}^{\mu+\rho}_{\mathrm{Fuchs}}(X)$,

$$\sigma_M^{\mu+\rho-l}(AB)(z) = \sum_{j+k=l} \sigma_M^{\mu-j}(A)(z + \rho - k) \sigma_M^{\rho-k}(B)(z) \tag{2.4}$$

for all $l = 0, 1, 2, \ldots$ This formula is called the Mellin translation product.

DEFINITION 2.2. (a) The operator $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is called elliptic if $A$ is an elliptic differential operator on $X^\circ$ and

$$\tilde{\sigma}_\psi^\mu(A)(t, y, \tilde{\tau}, \eta) \neq 0 \text{ for all } (t, y, \tilde{\tau}, \eta) \in \tilde{T}^*\mathcal{U} \setminus 0. \tag{2.5}$$

(b) The operator $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ is called elliptic with respect to the weight $\delta \in \mathbb{R}$ if $A$ is elliptic in the sense of (a) and, in addition,

$$\sigma^{\mu}_{M}(A)(z)\colon H^{s}(Y) \to H^{s-\mu}(Y),\quad z \in \Gamma_{(n+1)/2-\delta}, \tag{2.6}$$

is invertible for some $s \in \mathbb{R}$ (and then for all $s \in \mathbb{R}$). Here, $\Gamma_{\beta} = \{z \in \mathbb{C};\ \mathrm{Re}\, z = \beta\}$ for $\beta \in \mathbb{R}$.

Under the assumption of interior ellipticity of $A$, (2.5) can be reformulated as

$$\sum_{j=0}^{\mu} \sigma^{\mu-j}_{\psi}(a_j(0))(y,\eta)\big(-\tilde{\tau}\big)^{j} \neq 0$$

for all $(y,\tilde{\tau},\eta)$ with $(\tilde{\tau},\eta) \neq 0$. This relation implies that $\sigma^{\mu}_{M}(A)$ is parameter-dependent elliptic as an element of $L^{\mu}_{\mathrm{cl}}\big(Y;\Gamma_{(n+1)/2-\delta}\big)$, where the latter is the space of all classical pseudodifferential operators on $Y$ of order $\mu$ with parameter $z \in \Gamma_{(n+1)/2-\delta}$, for

$$\sigma^{\mu}_{\psi}(\sigma^{\mu}_{M}(A))(y,z,\eta)\big|_{z=(n+1)/2-\delta-\tilde{\tau}} = \tilde{\sigma}^{\mu}_{\psi}(A)(0,y,\tilde{\tau},\eta),$$

and $\sigma^{\mu}_{\psi}$ on the left-hand side denotes the parameter-dependent principal symbol. Thus, if (a) is fulfilled, then it follows that $\sigma^{\mu}_{M}(A)(z)$ in (2.6) is invertible for $z \in \Gamma_{(n+1)/2-\delta}$, $|z|$ large enough.

LEMMA 2.3. *If $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ is elliptic, then there exists a discrete set $\mathcal{D} \subset \mathbb{C}$ with $\mathcal{D} \cap \{z \in \mathbb{C};\ c_0 \leq \mathrm{Re}\, z \leq c_1\}$ is finite for all $-\infty < c_0 < c_1 < \infty$ such that (2.6) is invertible for all $z \in \mathbb{C} \setminus \mathcal{D}$. In particular, there is a discrete set $D \subset \mathbb{R}$ such that $A$ is elliptic with respect to the weight $\delta$ for all $\delta \in \mathbb{R} \setminus D$; $D = \mathrm{Re}\, \mathcal{D}$.*

*Proof.* Since $\sigma^{\mu}_{M}(A)(z) \in L^{\mu}(X;\Gamma_{\beta})$ is parameter-dependent elliptic for all $\beta \in \mathbb{R}$, for each $c > 0$ there is a $C > 0$ such that $\sigma^{\mu}_{M}(A)(z) \in L^{\mu}(X)$ is invertible for all $z$ with $|\,\mathrm{Re}\, z| \leq c$, $|\,\mathrm{Im}\, z| \geq C$. Then the assertion follows from results on the invertibility of holomorphic operator-valued functions. See Proposition 2.6 below or in SCHULZE [16]. ☐

Next, we introduce the class of meromorphic functions arising in point-wise inverting parameter-dependent elliptic conormal symbols $\sigma^{\mu}_{M}(A)(z)$, see again SCHULZE [16].

DEFINITION 2.4. (a) $\mathcal{M}^{\mu}_{\mathcal{O}}(Y)$ for $\mu \in \mathbb{R} \cup \{-\infty\}$ is the space of all holomorphic functions $f(z)$ on $\mathbb{C}$ with values in $L^{\mu}_{\mathrm{cl}}(Y)$ such that $f(z)\big|_{z=\beta+i\tau} \in L^{\mu}_{\mathrm{cl}}(Y;\mathbb{R}_{\tau})$ uniformly in $\beta \in [\beta_0,\beta_1]$ for all $-\infty < \beta_0 < \beta_1 < \infty$.
(b) $\mathcal{M}^{-\infty}_{\mathrm{as}}(Y)$ is the space of all meromorphic functions $f(z)$ on $\mathbb{C}$ taking values in $L^{-\infty}(Y)$ and satisfying the following conditions:
(i) The Laurent expansion around each pole $z = p$ of $f(z)$ has the form

$$f(z) = \frac{f_0}{(z-p)^{\nu}} + \frac{f_1}{(z-p)^{\nu-1}} + \cdots + \frac{f_{\nu-1}}{z-p} + \sum_{j\geq 0} f_{\nu+j}(z-p)^{j}, \tag{2.7}$$

where $f_0, f_1, \dots, f_{\nu-1} \in L^{-\infty}(Y)$ are finite-rank operators.
(ii) If the poles of $f(z)$ are numbered in a certain way, $p_1, p_2, \dots$, then $|\operatorname{Re} p_j| \to \infty$ as $j \to \infty$ if the number of poles is infinite.
(iii) For any $\bigcup_j \{p_j\}$-excision function $\chi(z) \in C^\infty(\mathbb{C})$, i.e., $\chi(z) = 0$ if $\operatorname{dist}(z, \bigcup_j \{p_j\}) \leq 1/2$ and $\chi(z) = 1$ if $\operatorname{dist}(z, \bigcup_j \{p_j\}) \geq 1$, we have $\chi(z) f(z)\big|_{z=\beta+i\tau} \in L^{-\infty}(Y; \mathbb{R}_\tau)$ uniformly in $\beta \in [\beta_0, \beta_1]$ for all $-\infty < \beta_0 < \beta_1 < \infty$.
(c) Finally, we set $\mathcal{M}^\mu_{\mathrm{as}}(Y) = \mathcal{M}^\mu_{\mathcal{O}}(Y) + \mathcal{M}^{-\infty}_{\mathrm{as}}(Y)$ for $\mu \in \mathbb{R}$ understood as a non-direct sum. (Note that $\mathcal{M}^\mu_{\mathcal{O}}(Y) \cap \mathcal{M}^{-\infty}_{\mathrm{as}}(Y) = \mathcal{M}^{-\infty}_{\mathcal{O}}(Y)$.)
Functions $f(z)$ belonging to $\mathcal{M}^\mu_{\mathrm{as}}(Y)$ are called Mellin symbols.

For $f(z) \in \mathcal{M}^\mu_{\mathrm{as}}(Y)$, $p \in \mathbb{C}$, and $N \in \mathbb{N}$, in connection with (2.7), we shall denote by $[f(z)]^N_p$ the Laurent series of $f(z)$ around $z = p$ truncated after $N + 1$ terms, i.e.,

$$[f(z)]^N_p = \frac{f_0}{(z-p)^\nu} + \cdots + \frac{f_{\nu-1}}{z-p} + f_\nu + f_{\nu+1}(z-p) + \cdots + f_N (z-p)^{N-\nu}. \quad (2.8)$$

Furthermore, $[f(z)]^*_p = [f(z)]^{\nu-1}_p$ when $\nu \geq 1$ and $[f(z)]^*_p = 0$ otherwise shall denote the principal part of $f(z)$ at $z = p$.

*Remark* 2.5. The space $\mathcal{N}^\mu_{\mathrm{as}}(Y)$ for $\mu \in \mathbb{R} \cup \{-\infty\}$ is defined in a similar way, upon replacing the spaces $L^\mu_{\mathrm{cl}}(Y)$ and $L^\mu_{\mathrm{cl}}(Y; \mathbb{R}_\tau)$, respectively, with $L^\mu(Y)$ and $L^\mu(Y; \mathbb{R}_\tau)$ (= spaces of pseudodifferential operators of type $1, 0$). Then $\bigcup_{\mu \in \mathbb{R}} \mathcal{N}^\mu_{\mathrm{as}}(Y)$ is an algebra under the pointwise composition as multiplication which is filtered by the order, i.e., $\mathcal{N}^\mu_{\mathrm{as}}(Y) \subseteq \mathcal{N}^{\mu'}_{\mathrm{as}}(Y)$ for $\mu \leq \mu'$ and $\mathcal{N}^\mu_{\mathrm{as}}(Y) \cdot \mathcal{N}^\rho_{\mathrm{as}}(Y) \subseteq \mathcal{N}^{\mu+\rho}_{\mathrm{as}}(Y)$.
We have $\mathcal{M}^\mu_{\mathrm{as}}(Y) \subsetneq \mathcal{N}^\mu_{\mathrm{as}}(Y)$ for $\mu \in \mathbb{R}$, $\mathcal{M}^\mu_{\mathrm{as}}(Y) \cdot \mathcal{M}^\rho_{\mathrm{as}}(Y) \subseteq \mathcal{M}^{\mu+\rho}_{\mathrm{as}}(Y)$, and, for $\mu \in \mathbb{R} \cup \{-\infty\}$, $\rho \in \mathbb{R}$,

$$\mathcal{M}^\mu_{\mathrm{as}}(Y) \subseteq \mathcal{M}^\rho_{\mathrm{as}}(Y) \text{ if and only if } \rho - \mu \in \mathbb{N} \cup \{\infty\}.$$

For that reason, the algebra $\bigcup_{\mu \in \mathbb{R}} \mathcal{M}^\mu_{\mathrm{as}}(Y)$ is referred to as *quasi-filtered*.

For $f(z) \in \mathcal{M}^\mu_{\mathrm{as}}(Y)$ for some $\mu \in \mathbb{R}$, $f(z) = f_0(z) + f_1(z)$, where $f_0(z) \in \mathcal{M}^\mu_{\mathcal{O}}(Y)$, $f_1(z) \in \mathcal{M}^{-\infty}_{\mathrm{as}}(Y)$, the parameter-dependent principal symbol $\sigma^\mu_\psi\big(f_0(z)\big|_{z=\beta+i\tau}\big)$ is independent of the choice of the decomposition of $f(z)$ and also independent of $\beta \in \mathbb{R}$. It is called the principal symbol of $f(z)$. The Mellin symbol $f(z) \in \mathcal{M}^\mu_{\mathrm{as}}(Y)$ is called elliptic if its principal symbol is everywhere invertible.

PROPOSITION 2.6 (SCHULZE [16]). *The Mellin symbol $f(z) \in \mathcal{M}^\mu_{\mathrm{as}}(Y)$ for $\mu \in \mathbb{R}$ is invertible in the quasi-filtered algebra $\bigcup_{\mu \in \mathbb{R}} \mathcal{M}^\mu_{\mathrm{as}}(Y)$, i.e., there is a $g(z) \in \mathcal{M}^{-\mu}_{\mathrm{as}}(Y)$ such that $f(z)g(z) = g(z)f(z) = 1$ holds on $\mathbb{C}$, if and only if $f(z)$ is elliptic.*

PROPOSITION 2.7. *Let $\mu \in \mathbb{R}$ and $\{p_j\}_{j=1,2,\dots} \subset \mathbb{C}$ be a sequence obeying the property mentioned in* Definition 2.4 (b). *Further let, for each $j = 1, 2, \dots$, the finite part*

$$\frac{f_0^j}{(z-p_j)^{\nu_j}} + \cdots + \frac{f_{\nu_j-1}^j}{z-p_j} + f_{\nu_j}^j + f_{\nu_j+1}^j(z-p_j) + \cdots + f_{N_j}^j(z-p_j)^{N_j-\nu_j}, \quad (2.9)$$

*where $\nu_j \in \mathbb{Z}$, $N_j \in \mathbb{N}$, $f_0^j, \dots, f_{\nu_j-1}^j \in L^{-\infty}(Y)$ are finite-rank operators, $f_{\nu_j}^j \in L_{\mathrm{cl}}^\mu(Y)$ is a Fredholm operator of index 0, and $f_{\nu_j+1}^j, \dots, f_{N_j}^j \in L_{\mathrm{cl}}^\mu(Y)$, be given. (The two cases $N_j < \nu_j$ and $\nu_j \le 0$ are not excluded.) Then there is an elliptic $f(z) \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$ such that, for all $j = 1, 2, \dots$, $[f(z)]_{p_j}^{N_j}$ equals the finite part given in (2.9), while $f(q) \in L_{\mathrm{cl}}^\mu(Y)$ is invertible for all $q \in \mathbb{C} \setminus \bigcup_{j=1,2,\dots}\{p_j\}$.*

*Proof.* This can be derived using the results of WITT [20]. In particular, the factorization result therein yields directly the existence of $f(z)$ in the case that the sequence $\{p_j\} \subset \mathbb{C}$ is void. Concerning the assumptions made on the coefficients $f_k^j \in L_{\mathrm{cl}}^\mu(Y)$, see also Definition 2.64, Proposition 2.65. □

For further reference, we also note:

LEMMA 2.8. *Given $\phi_0, \dots, \phi_{m-1}, \psi_0, \dots, \psi_{m-1} \in C^\infty(Y)$, there are pseudodifferential operators $A_0, A_1, \dots, A_{m-1} \in L_{\mathrm{cl}}^\mu(Y)$ such that $A_0 \in L_{\mathrm{cl}}^\mu(Y)$ is invertible and*

$$\begin{aligned} A_0\phi_0 &= \psi_0, \\ A_1\phi_0 + A_0\phi_1 &= \psi_1, \\ &\cdots\cdots\cdots \\ A_{m-2}\phi_0 + \cdots + A_0\phi_{m-2} &= \psi_{m-2}, \\ A_{m-1}\phi_0 + A_{m-2}\phi_1 + \cdots + A_0\phi_{m-1} &= \psi_{m-1}. \end{aligned} \quad (2.10)$$

Now, we are going to introduce the basic object of study — the algebra of all complete conormal symbols. This algebra will enable us to introduce the refined notion of asymptotic type and to study the behaviour of conormal asymptotics under the action of Fuchsian differential operators.

DEFINITION 2.9. (a) For $\mu \in \mathbb{R}$, the space $\mathrm{Symb}_M^\mu(Y)$ consists of all sequences $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \subset \mathcal{M}_{\mathrm{as}}^\mu(Y)$.
(b) An element $\mathfrak{S}^\mu \in \mathrm{Symb}_M^\mu(Y)$ is called holomorphic if $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \subset \mathcal{M}_{\mathcal{O}}^\mu(Y)$.
(c) $\bigcup_{\mu\in\mathbb{R}} \mathrm{Symb}_M^\mu(Y)$ is a quasi-filtered algebra under the Mellin translation product, $\sharp_M$. Namely, for $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \in \mathrm{Symb}_M^\mu(Y)$, $\mathfrak{T}^\rho = \{\mathfrak{t}^{\rho-k}(z);\, k \in \mathbb{N}\} \in \mathrm{Symb}_M^\rho(Y)$, we define $\mathfrak{U}^{\mu+\rho} = \mathfrak{S}^\mu \sharp_M \mathfrak{T}^\rho \in \mathrm{Symb}_M^{\mu+\rho}(Y)$, where $\mathfrak{U}^{\mu+\rho} = \{\mathfrak{u}^{\mu+\rho-l}(z);\, l \in \mathbb{N}\}$, by

$$\mathfrak{u}^{\mu+\rho-l}(z) = \sum_{j+k=l} \mathfrak{s}^{\mu-j}(z+\rho-k)\mathfrak{t}^{\rho-k}(z) \quad (2.11)$$

for $l = 0, 1, 2, \dots$ See (2.55).

From Proposition 2.6 we immediately get:

LEMMA 2.10. *$\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \in \operatorname{Symb}_M^\mu(Y)$ is invertible in the quasi-filtered algebra $\bigcup_{\mu\in\mathbb{R}} \operatorname{Symb}_M^\mu(Y)$ if and only if $\mathfrak{s}^\mu(z) \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$ is elliptic.*

In the case of the preceding lemma, $\mathfrak{S}^\mu \in \operatorname{Symb}_M^\mu(Y)$ is called elliptic. It is called elliptic with respect to the weight $\delta \in \mathbb{R}$ if the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $\mathfrak{s}^\mu(z)$. Notice that an elliptic $\mathfrak{S}^\mu \in \operatorname{Symb}_M^\mu(Y)$ is elliptic for all, but a discrete set of $\delta$. The inverse to $\mathfrak{S}^\mu$ with respect to the Mellin translation product is denoted by $(\mathfrak{S}^\mu)^{-1}$. The set of all elliptic elements of $\operatorname{Symb}_M^\mu(Y)$ is then denoted by $\operatorname{Ell}\operatorname{Symb}_M^\mu(Y)$.
Furthermore, there is a homomorphism of filtered algebras,

$$\bigcup_{\mu\in\mathbb{N}} \operatorname{Diff}_{\mathrm{Fuchs}}^\mu(X) \to \bigcup_{\mu\in\mathbb{Z}} \operatorname{Symb}_M^\mu(Y),\ \ A \mapsto \left\{\sigma_M^{\mu-j}(A)(z);\, j \in \mathbb{N}\right\}. \tag{2.12}$$

By the remark preceding Lemma 2.3, $\left\{\sigma_M^{\mu-j}(A)(z);\, j \in \mathbb{N}\right\} \in \operatorname{Symb}_M^\mu(Y)$ is elliptic if $A \in \operatorname{Diff}_{\mathrm{Fuchs}}(X)$ is elliptic in the sense of Definition 2.2 (a).

## 2.2 DEFINITION OF ASYMPTOTIC TYPES

We now start introducing discrete asymptotic types.

### 2.2.1 THE SPACES $\mathcal{E}_V^\delta(Y)$

First, we construct a space $\mathcal{E}^\delta(Y) = \bigcup_{V\in\mathcal{C}^\delta} \mathcal{E}_V^\delta(Y)$ allowing a non-canonical isomorphism

$$C_{\mathrm{as}}^{\infty,\delta}(X)/C_{\mathcal{O}}^\infty(X) \xrightarrow{\cong} \mathcal{E}^\delta(Y), \tag{2.13}$$

where $C_{\mathrm{as}}^{\infty,\delta}(X)$ is the space of all smooth functions on $X^\circ$ obeying conormal asymptotic expansions of conormal order at least $\delta$, i.e., $\operatorname{Re} p_j < (n+1)/2-\delta$ for all $j$, of the form (1.2) (with the condition that the singular exponents $p_j$ appear in conjugated pairs skipped) and $C_{\mathcal{O}}^\infty(X)$ is the subspace of all smooth functions on $X^\circ$ vanishing to the infinite order on $\partial X$. "Non-canonical" in (2.13) means that the isomorphism depends explicitly on the chosen splitting of coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, close to $\partial X$.

DEFINITION 2.11. A carrier $V$ of asymptotics for distributions of conormal order $\delta$ is a discrete subset of $\mathbb{C}$ contained in the half-space $\{z \in \mathbb{C};\, \operatorname{Re} z < (n+1)/2-\delta\}$ such that, for all $\beta_0,\, \beta_1 \in \mathbb{R}$, $\beta_0 < \beta_1$, the intersection $V \cap \{z \in \mathbb{C};\, \beta_0 < \operatorname{Re} z < \beta_1\}$ is finite. The set of all these carriers is denoted by $\mathcal{C}^\delta$.

In particular, $V_p = p - \mathbb{N}$ for $p \in \mathbb{C}$ is a carrier of asymptotics. Note that $V_p \in \mathcal{C}^\delta$ if and only if $\operatorname{Re} p < (n+1)/2-\delta$. We set $T^{-\varrho}V = -\varrho + V \in \mathcal{C}^{-\operatorname{Re}\varrho+\delta}$ for $\varrho \in \mathbb{C}$ and $V \in \mathcal{C}^\delta$. We further set $\mathcal{C} = \bigcup_{\delta\in\mathbb{R}} \mathcal{C}^\delta$.
Let $[C^\infty(Y)]^\infty$ be the space of all finite sequences in $C^\infty(Y)$, as in (2.41), with $E$ replaced with $C^\infty(Y)$. For $V \in \mathcal{C}^\delta$, we set $\mathcal{E}_V^\delta(Y) = \prod_{p\in V} [C^\infty(Y)]_p^\infty$, where

| Local asymptotics | Discrete asymptotics |
|---|---|
| $E$ | $C^\infty(Y)$ |
| $\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ | $\bigcup_{\mu\in\mathbb{R}} \text{Symb}_M^\mu(Y)$ |
| $\mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$ | $\bigcup_{\mu\in\mathbb{R}} \text{Ell}\,\text{Symb}_M^\mu(Y)$ |
| $E^\infty$ | $\bigcup_{\delta\in\mathbb{R}} \mathcal{E}^\delta(Y)$, $\mathcal{E}^\delta(Y)$ |
| $L_F$ | $L_{\mathfrak{S}^\mu}$, $L^\delta_{\mathfrak{S}^\mu}$ |
| $\mathcal{J}(E)$ | $\underline{\text{As}}(Y)$, $\underline{\text{As}}^\delta(Y)$ |
| $J$, $K$ | asymptotic types $P$, $Q$ |
| $J^F$ | $\mathcal{Q}^{\delta-\mu}(P;A)$ |
| $K^{F^{-1}}$, $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$ | $\mathcal{P}^\delta(Q;A)$, $A$ elliptic |
| $\mathcal{M}_p(E,J)$ | $C_P^\infty(X)$, $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ |

Figure 2: Local asymptotics versus discrete asymptotics

$[C^\infty(Y)]_p^\infty$ is an isomorphic copy of $[C^\infty(Y)]^\infty$, and define $\mathcal{E}^\delta(Y)$ to be the space of all families $\Phi \in \mathcal{E}_V^\delta(Y)$ for some $V \in \mathcal{C}^\delta$ depending on $\Phi$. Thereby, $\Phi \in \mathcal{E}_V^\delta(Y)$, $\Phi' \in \mathcal{E}_{V'}^\delta(Y)$ for possibly different $V$, $V' \in \mathcal{C}^\delta$ are identified if $\Phi(p) = \Phi'(p)$ for all $p \in V \cap V'$, and $\Phi(p) = 0$ for $p \in V \setminus V'$, $\Phi'(p) = 0$ for $p \in V' \setminus V$. Under this identification,

$$\mathcal{E}^\delta(Y) = \bigcup_{V\in\mathcal{C}^\delta} \mathcal{E}_V^\delta(Y) \tag{2.14}$$

and $\mathcal{E}_V^\delta(Y) \cap \mathcal{E}_{V'}^\delta(Y) = \mathcal{E}_{V\cap V'}^\delta(Y)$. The right shift operator $T$, see (2.42), acts on $\mathcal{E}^\delta(Y)$ component-wise, i.e., $(T\Phi)(p) = T(\Phi(p))$ for $\Phi \in \mathcal{E}_V^\delta(Y)$ and all $p \in V$.

*Remark* 2.12. (a) In Section 3, we shall write $\mathcal{E}_V(Y)$ instead of $\mathcal{E}_V^\delta(Y)$ for $V \in \mathcal{C}^\delta$ without ambiguity.
(b) To designate different shift operators with the same symbol $T$, once $T^{-\varrho}$ for $\varrho \in \mathbb{C}$ for carriers of asymptotics, once $T$, $T^2$ etc. for vectors in $\mathcal{E}^\delta(Y)$ should not confuse the reader.

For $\Phi \in \mathcal{E}^\delta(Y)$, we define $\text{c-ord}(\Phi) = (n+1)/2 - \max\{\operatorname{Re} p;\, \Phi(p) \neq 0\}$. In particular, $\text{c-ord}(0) = \infty$. Note that $\text{c-ord}(\Phi) > \delta$ if $\Phi \in \mathcal{E}^\delta(Y)$. For $\Phi_i \in \mathcal{E}^\delta(Y)$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \dots$ satisfying $\text{c-ord}(\Phi_i) \to \infty$ as $i \to \infty$, the sum

$$\Phi = \sum_{i=1}^\infty \alpha_i \Phi_i, \tag{2.15}$$

is defined in $\mathcal{E}^\delta(Y)$ in an obvious fashion: Let $\Phi_i \in \mathcal{E}_{V_i}^\delta(Y)$, where $V_i \in \mathcal{C}^{\delta_i}$, $\delta_i \geq \delta$, and $\delta_i \to \infty$ as $i \to \infty$. Then $V = \bigcup_i V_i \in \mathcal{C}^\delta$, and $\Phi \in \mathcal{E}_V^\delta(Y)$ is defined by $\Phi(p) = \sum_{i=1}^\infty \alpha_i \Phi_i(p)$ for $p \in V$, where, for each $p \in V$, the sum on the right-hand side is finite.

LEMMA 2.13. *Let* $\Phi_i \in \mathcal{E}^\delta(Y)$ *for* $i = 1, 2, \dots,$ c-ord$(\Phi_i) \to \infty$ *as* $i \to \infty$*. Then* (2.15) *holds if and only if*

$$\text{c-ord}(\Phi - \sum_{i=1}^{N} \alpha_i \Phi_i) \to \infty \quad \text{as } N \to \infty. \tag{2.16}$$

Note that (2.16) already implies that c-ord$(\alpha_i \Phi_i) \to \infty$ as $i \to \infty$.

DEFINITION 2.14. Let $\Phi_i$, $i = 1, 2, \dots,$ be a sequence in $\mathcal{E}^\delta(Y)$ with the property that c-ord$(\Phi_i) \to \infty$ as $i \to \infty$. Then this sequence is called linearly independent if, for all $\alpha_i \in \mathbb{C}$,

$$\sum_{i=1}^{\infty} \alpha_i \Phi_i = 0$$

implies that $\alpha_i = 0$ holds for all $i$. A linearly independent sequence $\Phi_i$ for $i = 1, 2, \dots$ in $J$ for a linear subspace $J \subseteq \mathcal{E}^\delta(Y)$ is called a basis for $J$ if every vector $\Phi \in J$ can be represented in the form (2.15) with certain (then uniquely determined) coefficients $\alpha_i \in \mathbb{C}$.

Note that $\sum_{i=1}^{\infty} \alpha_i \Phi_i = 0$ in $\mathcal{E}^\delta(Y)$ if and only if c-ord$(\sum_{i=1}^{N} \alpha_i \Phi_i) \to \infty$ as $N \to \infty$ according to Lemma 2.13. We also obtain:

LEMMA 2.15. *Let* $\Phi_i$, $i = 1, 2, \dots,$ *be a sequence in* $\mathcal{E}^\delta(Y)$ *such that* c-ord$(\Phi_i) \to \infty$ *as* $i \to \infty$*. Further, let* $\{\delta_j\}_{j=1}^{\infty}$ *be a strictly increasing sequence such that* $\delta_j > \delta$ *for all* $j$ *and* $\delta_j \to \infty$ *as* $j \to \infty$*. Assume that the* $\Phi_i$ *are numbered in such a way that* c-ord$(\Phi_i) \le \delta_j$ *if and only if* $1 \le i \le e_j$*. Then the sequence* $\Phi_i$, $i = 1, 2, \dots,$ *is linearly independent provided that, for each* $j = 1, 2, \dots,$

$$\Phi_1, \dots, \Phi_{e_j} \text{ are linearly independent over the space } \mathcal{E}^{\delta_j}(Y).$$

We generalize the notion of a characteristic basis from Appendix 2.A, see Lemma 2.70.

DEFINITION 2.16. Let $J \subseteq \mathcal{E}^\delta(Y)$ be a linear subspace, $TJ \subseteq J$, and $\Phi_i$ for $i = 1, 2, \dots$ be a sequence in $J$. Then $\Phi_i$, $i = 1, 2, \dots,$ is called a characteristic basis of $J$ if there are numbers $m_i \in \mathbb{N} \cup \{\infty\}$ such that $T^{m_i} \Phi_i = 0$ when $m_i < \infty$, while the sequence $\{T^k \Phi_i;\, i = 1, 2, \dots, 0 \le k < m_i\}$ forms a basis for $J$.

The question of the existence of a characteristic basis obeying one more property is taken up in Proposition 2.21. We also need following notion:

DEFINITION 2.17. $\Phi \in \mathcal{E}^\delta(Y)$ is called a special vector if $\Phi \in \mathcal{E}^\delta_{V_p}(Y)$ for some $p \in \mathbb{C}$.

Thus, $\Phi \in \mathcal{E}^\delta_V(Y)$ is a special vector if there is a $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$ such that $\Phi(p') = 0$ for all $p' \in V$, $p' \notin p - \mathbb{N}$. Obviously, if $\Phi \neq 0$, then $p$ is uniquely determined by $\Phi$, by the additional requirement that $\Phi(p) \neq 0$. We denote this complex number $p$ by $\gamma(\Phi)$.

### 2.2.2 Simplest properties of asymptotic types

In the following, we fix a splitting of coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, close to $\partial X$, cf. the non-canonical isomorphism (2.13). Coordinate invariance is discussed in Proposition 2.33.

Definition 2.18. An asymptotic type, $P$, for distributions as $x \to \partial X$, of conormal order at least $\delta$, is represented — in the given splitting of coordinates close to $\partial X$ — by a linear subspace $J \subset \mathcal{E}^{\delta}_{V}(Y)$ for some $V \in \mathcal{C}^{\delta}$ such that the following three conditions are met:
(a) $TJ \subseteq J$;
(b) $\dim J^{\delta+j} < \infty$ for all $j \in \mathbb{N}$, where $J^{\delta+j} = J/(J \cap \mathcal{E}^{\delta+j}(Y))$;
(c) There is a sequence $\{p_j\}_{j=1}^{M} \subset \mathbb{C}$, where $M \in \mathbb{N} \cup \{\infty\}$, $\operatorname{Re} p_j < (n+1)/2 - \delta$, and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ when $M = \infty$, such that $V \subseteq \bigcup_{j=1}^{M} V_{p_j}$ and

$$J = \bigoplus_{j=1}^{M} \left( J \cap \mathcal{E}^{\delta}_{V_{p_j}}(Y) \right). \tag{2.17}$$

The empty asymptotic type, $\mathcal{O}$, is represented by the trivial subspace $\{0\} \subset \mathcal{E}^{\delta}(Y)$. The set of all asymptotic types of conormal order $\delta$ is denoted by $\underline{\mathrm{As}}^{\delta}(Y)$.

For $P \in \underline{\mathrm{As}}^{\delta}$ represented by $J \subset \mathcal{E}^{\delta}_{V}(Y)$, we introduce

$$\delta_P = \min\{\text{c-ord}(\Phi);\ \Phi \in J\}, \tag{2.18}$$

Notice that $\delta_P > \delta$ and $\delta_P < \infty$ if and only if $P \neq \mathcal{O}$.
Obviously, $\underline{\mathrm{As}}^{\delta}(Y) \subseteq \underline{\mathrm{As}}^{\delta'}(Y)$ if $\delta \geq \delta'$. We also set

$$\underline{\mathrm{As}}(Y) = \bigcup_{\delta \in \mathbb{R}} \underline{\mathrm{As}}^{\delta}(Y).$$

On asymptotic types $P \in \underline{\mathrm{As}}^{\delta}(Y)$, we have the shift operation $T^{-\varrho}$ for $\varrho \in \mathbb{C}$, namely $T^{-\varrho}P$ is represented by the space

$$T^{-\varrho}J = \left\{\Phi \in \mathcal{E}^{-\operatorname{Re}\varrho+\delta}_{T^{-\varrho}V}(Y);\ \Phi(p) = \bar{\Phi}(p+\varrho) \text{ for } p \in \mathbb{C} \text{ and some } \bar{\Phi} \in J\right\},$$

where $J \subset \mathcal{E}^{\delta}_{V}(Y)$ represents $P$.
Furthermore, for $J \subset \mathcal{E}^{\delta}_{V}(Y)$ as in Definition 2.18,

$$J_p = \{\Phi(p);\ \Phi \in J\} \subset [C^{\infty}(Y)]^{\infty}$$

for $p \in \mathbb{C}$ denotes the localization of $J$ at $p$. Note that $TJ_p \subseteq J_p$ and $\dim J_p < \infty$; thus, $J_p$ is a local asymptotic type in the sense of Definition 2.69.

DEFINITION 2.19. Let $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ and let $P \in \underline{\mathrm{As}}^{\delta}(Y)$ be represented by $J \subset \mathcal{E}^{\delta}_{V}(Y)$. Then $u$ is said to have asymptotics of type $P$ if there is a vector $\Phi \in J$ such that

$$u(x) \sim \sum_{p \in V} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!} \log^k t \, \phi^{(p)}_{l}(y) \quad \text{as } t \to +0, \tag{2.19}$$

where $\Phi(p) = (\phi^{(p)}_0, \phi^{(p)}_1, \dots, \phi^{(p)}_{m_p-1})$ for $p \in V$. The space of all these $u$ is denoted by $C^{\infty}_{P}(X)$.

Note the shift from $m_p$ to $m_p - 1$ that, for notational convenience, appeared in formula (2.19) compared to formula (1.2).
Thus, by representation of an asymptotic type we mean that $P$ which, in the philosophy of asymptotic algebras, see Definition 2.55, is the same as the linear subspace $C^{\infty}_{P}(X)/C^{\infty}_{\mathcal{O}}(X) \subset C^{\infty,\delta}_{\mathrm{as}}(X)/C^{\infty}_{\mathcal{O}}(X)$, is mapped onto $J$ by the isomorphism (2.13). Recall that this isomorphism depends on the chosen splitting of coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, close to $\partial X$.
We are now going to investigate common properties of linear subspaces $J \subset \mathcal{E}^{\delta}_{V}(Y)$ satisfying (a) to (c) in Definition 2.18. Let $\Pi_j : J \to J^{\delta+j}$ be the canonical surjection. For $j' > j$, there is a natural surjective map $\Pi_{jj'} : J^{\delta+j'} \to J^{\delta+j}$ such that $\Pi_{jj''} = \Pi_{jj'}\Pi_{j'j''}$ for $j'' > j' > j$ and

$$\big(J, \Pi_j\big) = \operatorname*{proj\,lim}_{j \to \infty} \big(J^{\delta+j}, \Pi_{jj'}\big). \tag{2.20}$$

Note that $T \colon J^{\delta+j} \to J^{\delta+j}$ is nilpotent, where $T$ denotes the map induced by $T \colon J \to J$. Furthermore, for $j' > j$, the diagram

$$\begin{array}{ccc} J^{\delta+j'} & \xrightarrow{\Pi_{jj'}} & J^{\delta+j} \\ T\big\downarrow & & \big\downarrow T \\ J^{\delta+j'} & \xrightarrow{\Pi_{jj'}} & J^{\delta+j} \end{array} \tag{2.21}$$

commutes and the action of $T$ on $J$ is that one induced by (2.20), (2.21).

PROPOSITION 2.20. *Let $J \subset \mathcal{E}^{\delta}_{V}(Y)$ be a linear subspace for some $V \in \mathcal{C}^{\delta}$. Then there is a sequence $\Phi_i$ for $i = 1, 2, \dots$ of special vectors with* c-ord$(\Phi_i) \to \infty$ *as $i \to \infty$ such that the vectors $T^k\Phi_i$ for $i = 1, 2, \dots$, $k = 0, 1, 2 \dots$ generate $J$ if and only if $J$ fulfils conditions* (a), (b), *and* (c).

In the situation just described, we write $J = \langle \Phi_1, \Phi_2, \dots \rangle$, see Appendix 2.A behind Lemma 2.70.

*Proof.* Let $J \subset \mathcal{E}^{\delta}_{V}(Y)$ fulfil conditions (a) to (c). Due to (c) we may assume that $V = V_p$ for some $p \in \mathbb{C}$. Suppose that the vectors $\Phi_1, \dots, \Phi_e \in J$ are already chosen (where $e = 0$ is possible). Then we choose the vector $\Phi_{e+1}$ among

the vectors $\Phi \in J$ which do not belong to $\langle \Phi_1, \dots, \Phi_e \rangle$ such that $\gamma(\Phi_{e+1})$ is minimal. We claim that $J = \langle \Phi_1, \Phi_2, \dots \rangle$. In fact, if $\Phi \in J$, then $\Phi \in \langle \Phi_1, \dots, \Phi_e \rangle$, where $e$ is so that $\gamma(\Phi_e) \le \gamma(\Phi)$, while $\gamma(\Phi_{e+1}) > \gamma(\Phi)$. Otherwise, $\Phi_{e+1}$ would not have been chosen in the $(e+1)$th step.
The other direction is obvious. □

For $j \ge 1$, let $(m_1^j, \dots, m_{e_j}^j)$ denote the characteristic of the space $J^{\delta+j}$.

PROPOSITION 2.21. *Let $J \subset \mathcal{E}_V^\delta(Y)$ be a linear subspace and assume that the vectors $\Phi_i$ for $i = 1, 2, \dots, e$, where $e \in \mathbb{N} \cup \{\infty\}$, constructed in* Proposition 2.20 *are a characteristic basis of $J$. Then the following conditions are equivalent:*
(a) *For each $j$, $\Pi_j \Phi_1, \dots, \Pi_j \Phi_{e_j}^j$ is an $(m_1^j, \dots, m_{e_j}^j)$-basis of $J^{\delta+j}$;*
(b) *For each $j$, $T^{m_1^j - 1}\Phi_1, \dots, T^{m_{e_j}-1}\Phi_{e_j}$ are linearly independent over the space $\mathcal{E}^{\delta+j}(Y)$, while $T^k \Phi_i \in \mathcal{E}^{\delta+j}(Y)$ when either $1 \le i \le e_j$, $k \ge m_i^j$ or $i > e_j$.*
*In particular, if* (a), (b) *are fulfilled, then, for any $j' > j$, $\Pi_{jj'}\Phi_1^{j'}, \dots, \Pi_{jj'}\Phi_{e_j}^{j'}$ is a characteristic basis of $J^{\delta+j}$, while $\Pi_{jj'}\Phi_{e_j+1}^{j'} = \dots = \Pi_{jj'}\Phi_{e'_j}^{j'} = 0$. Here, $\Phi_i^{j'} = \Pi_{j'}\Phi_i$ for $1 \le i \le e_{j'}$.*

*Proof.* This is a direct consequence of Lemmas 2.15, 2.71. □

Notice that, for a linear subspace $J \subset \mathcal{E}_V^\delta(Y)$ satisfying conditions (a) to (c) of Definition 2.18, a characteristic basis possessing the equivalent properties of Proposition 2.21 need not exist. An example is provided below.

DEFINITION 2.22. An asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ represented by the linear subspace $J \subset \mathcal{E}_V^\delta(Y)$ is called proper if $J$ admits a characteristic basis $\Phi_1$, $\Phi_2, \dots$ satisfying the equivalent conditions in Proposition 2.21. The set of all proper asymptotic types is denoted by $\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y) \subsetneq \underline{\mathrm{As}}^\delta(Y)$.

*Example* 2.23. We provide an example showing that the inclusion $\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y) \subseteq \underline{\mathrm{As}}^\delta(Y)$ is strict. Let the space $J = \langle \Phi_1, \Phi_2 \rangle \subset \mathcal{E}_{V_p}^\delta(Y)$ for some $p \in \mathbb{C}$, $\mathrm{Re}\, p < (n+1)/2 - \delta$ be generated by two vectors $\Phi_1$, $\Phi_2$ in the sense of Proposition 2.20. We further assume that $\Phi_1(p) = (\psi_0, \star)$, $\Phi_1(p-1) = (\psi_1, \star, \star)$, $\Phi_2(p) = 0$, and $\Phi_2(p-1) = (\psi_1, \star)$, where $\psi_0$, $\psi_1 \in C^\infty(Y)$ are not identically zero and $\star$ stands for arbitrary entries. See Figure 3. Then, the asymptotic type represented by $J$ is non-proper. In fact, assume that $\mathrm{Re}\, p \ge (n+1)/2 - \delta + 1$. Then $\Pi_2\Phi_1$, $T\Pi_2\Phi_1 - \Pi_2\Phi_2$ is a characteristic basis of $J^{\delta+2}$, and any other characteristic basis of $J^{\delta+2}$ is, up to a non-zero multiplicative constant, of the form

$$\Pi_2\Phi_1 + \alpha(T\Pi_2\Phi_1 - \Pi_2\Phi_2), \ \ \beta(T\Pi_2\Phi_1 - \Pi_2\Phi_2) + \gamma\Phi_1, \tag{2.22}$$

where $\alpha$, $\beta$, $\gamma \in \mathbb{R}$ and $\beta \ne 0$. But then the conclusion in Proposition 2.21 is violated, since both vectors in (2.22) have non-zero image under the projection $\Pi_{12}$, while $\Pi_1\Phi_1$ forms a characteristic basis of $J^{\delta+1}$.

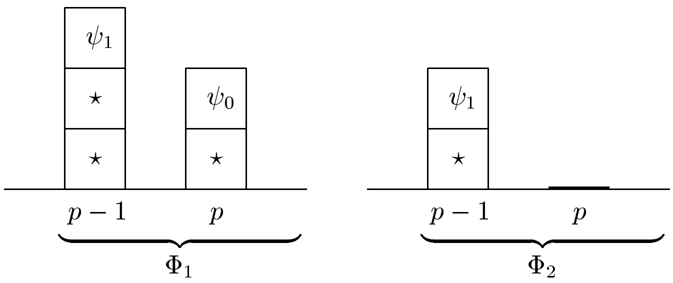



Figure 3: Example of a non-proper asymptotic type

We are now going to introduce the analogues of $L_F$ and $J^F$, see Appendix 2.A. For $\Phi \in \mathcal{E}^\delta(Y)$, $p \in \mathbb{C}$, and $\Phi(p) = (\phi_0^{(p)}, \phi_1^{(p)}, \dots, \phi_{m_p-1}^{(p)})$ we shall use, for any $q \in \mathbb{C}$, the notation

$$\Phi(p)[z-q] = \frac{\phi_0^{(p)}}{(z-q)^{m_p}} + \frac{\phi_1^{(p)}}{(z-q)^{m_p-1}} + \dots + \frac{\phi_{m_p-1}^{(p)}}{z-q} \in \mathcal{M}_q(C^\infty(Y)).$$

Here, $\mathcal{M}_p(C^\infty(Y))$, $\mathcal{A}_p(C^\infty(Y))$ have the same meaning as in Appendix 2.A, with $E$ replaced with the space $C^\infty(Y)$.

DEFINITION 2.24. For $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \in \mathrm{Symb}_M^\mu(Y)$, the linear subspace $L^\delta_{\mathfrak{S}^\mu} \subseteq C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$ is represented by the space that consists of all $\Phi \in \mathcal{E}^\delta(Y)$ such that there are functions $\widetilde{\phi}^{(p)}(z)$ for $p \in \mathbb{C}$, $\mathrm{Re}\, p > (n+1)/2 - \delta$ such that

$$\sum_{j=0}^{[(n+1)/2-\delta+\mu-\mathrm{Re}\, q]^-} \mathfrak{s}^{\mu-j}(z-\mu+j)\left(\Phi(q-\mu+j)[z-q] + \widetilde{\phi}^{(q-\mu+j)}(z)\right) \in \mathcal{A}_q(C^\infty(Y)) \quad (2.23)$$

holds for $q \in \mathbb{C}$, $\mathrm{Re}\, q < (n+1)/2 - \delta + \mu$. Here $[a]^-$ for $a \in \mathbb{R}$ is the largest integer strictly less than $a$, i.e., $[a]^- \in \mathbb{Z}$ and $[a]^- < a \le [a]^- + 1$.

Note that, if $\Phi \in \mathcal{E}^\delta_V(Y)$ for $V \in \mathcal{C}^\delta$, then condition (2.23) is only effective if

$$q \in \bigcup_{j=0}^{[(n+1)/2-\delta+\mu-\mathrm{Re}\, q]^-} T^{\mu-j} V,$$

since otherwise the sum appearing in (2.23) is zero.

*Remark* 2.25. Informally, if $\Phi \in \mathcal{E}^\delta(Y)$ belongs to the space representing $L^\delta_{\mathfrak{S}^\mu}$, and if $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ possesses asymptotics given by the vector $\Phi$ according to (2.19), then there is a $v \in C^\infty_{\mathcal{O}}(X)$ such that

$$\left(\sum_{j=0}^{\infty} \omega(t) t^{-\mu+j} \operatorname{op}_M^{(n+1)/2-\delta}\left(\mathfrak{s}^{\mu-j}(z)\right) \tilde{\omega}(t)\right)(u+v) \in C^\infty_{\mathcal{O}}(X).$$

Concerning the informality involved here, see the remarks in Appendix 2.B.

DEFINITION 2.26. For $P \in \underline{\mathrm{As}}^{\delta}(Y)$ being represented by $J \subset \mathcal{E}_V^{\delta}(Y)$ and $\mathfrak{S}^{\mu} \in \mathrm{Symb}_M^{\mu}(Y)$, the push-forward $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^{\mu})$ of $P$ under $\mathfrak{S}^{\mu}$ is the asymptotic type in $\underline{\mathrm{As}}^{\delta-\mu}(Y)$ represented by the linear subspace $K \subset \mathcal{E}_{T^{-\mu}V}^{\delta-\mu}(Y)$ consisting of all vectors $\Psi \in \mathcal{E}_{T^{-\mu}V}^{\delta-\mu}(Y)$ such that there is a $\Phi \in J$ and there are functions $\widetilde{\phi}^{(p)}(z) \in \mathcal{A}_p(C^{\infty}(Y))$ for $p \in V$ such that

$$\Psi(q)[z-q] = \sum_{j=0}^{[(n+1)/2-\delta+\mu-\mathrm{Re}\, q]^-} \Big[\mathfrak{s}^{\mu-j}(z-\mu+j)\Big(\Phi(q-\mu+j)[z-q]+\widetilde{\phi}^{(q-\mu+j)}(z-\mu+j)\Big)\Big]_q^*, \quad (2.24)$$

holds for all $q \in T^{\mu}V$, see (2.8).

*Remark* 2.27. As in Remark 2.67, for a holomorphic $\mathfrak{S}^{\mu} \in \mathrm{Symb}_M^{\mu}(Y)$ it is not necessary to refer to the holomorphic functions $\widetilde{\phi}^{(p)}(z) \in \mathcal{A}_p(C^{\infty}(Y))$ for $p \in V$ to define the push-forward $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^{\mu})$ in (2.24). We then also write $\mathcal{Q}(P;\mathfrak{S}^{\mu})$ instead of $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^{\mu})$.

Extending the notion of the push-forward from asymptotic types to arbitrary linear subspaces of $C_{\mathrm{as}}^{\infty,\delta}(X)/C_{\mathcal{O}}^{\infty}(X)$, the space $L_{\mathfrak{S}^{\mu}}^{\delta} \subseteq C_{\mathrm{as}}^{\infty,\delta}(X)/C_{\mathcal{O}}^{\infty}(X)$ for $\mathfrak{S}^{\mu} \in \mathrm{Symb}_M^{\mu}(Y)$ appears as the largest subspace of $C_{\mathrm{as}}^{\infty,\delta}(X)/C_{\mathcal{O}}^{\infty}(X)$ for which

$$\mathcal{Q}^{\delta-\mu}(L_{\mathfrak{S}^{\mu}}^{\delta};\mathfrak{S}^{\mu}) = \mathcal{Q}^{\delta-\mu}(\mathcal{O};\mathfrak{S}^{\mu}) \quad (2.25)$$

holds. In this sense, it characterizes the amount of asymptotics of conormal order at least $\delta$ annihilated by $\mathfrak{S}^{\mu} \in \mathrm{Symb}_M^{\mu}(Y)$.

DEFINITION 2.28. On the set $\underline{\mathrm{As}}^{\delta}(Y)$, there is a natural ordering defined by saying that $P \preccurlyeq P'$ for $P, P' \in \underline{\mathrm{As}}^{\delta}(Y)$ if and only if $J \subseteq J'$, where $J, J' \subset \mathcal{E}^{\delta}(Y)$ are representing spaces for $P$ and $P'$, respectively.

PROPOSITION 2.29. (a) *The ordered set* $(\underline{\mathrm{As}}^{\delta}(Y), \preccurlyeq)$ *is a lattice in which every non-empty subset* $\mathcal{S}$ *possesses a meet,* $\bigwedge \mathcal{S}$, *represented by* $\bigcap_{P\in\mathcal{S}} J_P$ *and every bounded subset* $\mathcal{T}$ *possesses a join,* $\bigvee \mathcal{T}$, *represented by* $\sum_{Q\in\mathcal{T}} J_Q$, *where* $J_P$ *and* $J_Q$ *represent the asymptotic types* $P$ *and* $Q$, *respectively. In particular,* $\bigwedge \underline{\mathrm{As}}^{\delta}(Y) = \mathcal{O}$.
(b) *For* $P \in \underline{\mathrm{As}}^{\delta}(Y)$, $\mathfrak{S}^{\mu} \in \mathrm{Symb}_M^{\mu}(Y)$, *we have* $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^{\mu}) \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$.

*Proof.* (a) is immediate from the definition of asymptotic type and (b) can be checked directly on the level of (2.24). □

*Remark* 2.30. To elements $\mathfrak{S}^{\mu} \in \mathrm{Symb}_M^{\mu}(Y)$ there is assigned a natural action $C_{\mathrm{as}}^{\infty,\delta}(X) \to C_{\mathrm{as}}^{\infty,\delta}(X)/C_{\mathcal{O}}^{\infty}(X)$. Its expression in the chosen splitting a coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, close to $\partial X$ is given by in (2.24).

Generalizing the notion of an asymptotic algebra from Definition 2.55 to the case that operators $F \in \mathfrak{M}$ act from $\mathfrak{F}$ to the quotient space $\mathfrak{F}/\mathfrak{F}_0$, see WITT [19], Proposition 2.29 states that the quadruple

$$\Big(\bigcup_{\mu\in\mathbb{R}} \mathrm{Symb}_M^\mu(Y), C_{\mathrm{as}}^{\infty,\delta}(X), C_{\mathcal{O}}^\infty(X), \underline{\mathrm{As}}^\delta(Y)\Big)$$

is in fact an asymptotic algebra in this generalized sense. Moreover, this asymptotic algebra is reduced and $\big(\mathrm{Symb}_M^0(Y), C_{\mathrm{as}}^{\infty,\delta}(X), C_{\mathcal{O}}^\infty(X), \underline{\mathrm{As}}^\delta(Y)\big)$ serves as a symbol algebra for $\big(\mathcal{C}^0(X,(\delta,\delta)), C_{\mathrm{as}}^{\infty,\delta}(X), C_{\mathcal{O}}^\infty(X), \underline{\mathrm{As}}^\delta(Y)\big)$, see again WITT [19].

This is the reason why a great deal of notion, see, in particular, Figure 2, and also the arguments of Appendix 2.A, carry over to the present situation.

THEOREM 2.31. *For a holomorphic* $\mathfrak{S}^\mu \in \mathrm{Ell}\,\mathrm{Symb}_M^\mu(Y)$*, that is elliptic with respect to the weight* $\delta$*, we have* $L_{\mathfrak{S}^\mu}^\delta \in \underline{\mathrm{As}}_{\mathrm{prop}}^\delta(Y)$.

*Proof.* Let $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}; j \in \mathbb{N}\} \subset \mathcal{M}_{\mathcal{O}}^\mu(Y)$. Assume that, for some $p \in \mathbb{C}$, $\mathrm{Re}\,p < (n+1)/2 - \delta$, $\Phi_0 \in L_{\mathfrak{s}^\mu(z)}$ at $z = p$ in the sense of Definition 2.59 and Lemma 2.66 (a). (Notice that $L_{\mathfrak{s}^\mu(z)}$ at $z = p$ is contained in the space $[C^\infty(Y)]^\infty$.) We then successively calculate the sequence $\Phi_0$, $\Phi_1$, $\Phi_2, \ldots$ from the relations, at $z = p$,

$$\mathfrak{s}^\mu(z-j)\Phi_j[z-p] + \mathfrak{s}^{\mu-1}(z-j+1)\Phi_{j-1}[z-p] + \cdots + \mathfrak{s}^{\mu-j}(z)\Phi_0[z-p] = 0, \;\; j = 0,1,2,\ldots, \tag{2.26}$$

see (2.24). In each step, we find a $\Phi_j \in [C^\infty(Y)]^\infty$ uniquely determined modulo $L_{\mathfrak{s}^\mu(z)}$ at $z = p-j$ such that (2.26) holds. In the end, we obtain that the vector $\Phi \in \mathcal{E}_{V_p}^\delta(Y)$ defined by $\Phi(p-j) = \Phi_j$ belongs to the linear subspace $J \subseteq \mathcal{E}^\delta(Y)$ representing $L_{\mathfrak{S}^\mu}^\delta$.

Conversely, each vector in $J$ is a finite sum of vectors $\Phi$ obtained in that way. Thus, upon choosing in each space $L_{\mathfrak{s}^\mu(z)}$ at $z = p$ a characteristic basis and then, for each characteristic basis vector $\Phi_0 \in [C^\infty(Y)]^\infty$, exactly one vector $\Phi \in \mathcal{E}_{V_p}^\delta(Y)$ as just constructed, we obtain a characteristic basis of $J$ in the sense of Definition 2.16 consisting completely of special vectors (since $L_{\mathfrak{s}^\mu(z)}$ at $z = p$ is zero for $z \in \mathbb{C}$, $\mathrm{Re}\,z < (n+1)/2 - \delta$, but for a set of $p$ belonging to $\mathcal{C}^\delta$). In particular, $J \subset \mathcal{E}_V^\delta(Y)$ for some $V \in \mathcal{C}^\delta$ and (a) to (c) of Definition 2.18 are satisfied. By its very construction, this characteristic basis fulfils condition (b) of Proposition 2.21. Therefore, the asymptotic type $L_{\mathfrak{S}^\mu}^\delta$ represented by $J$ is proper. □

Summarizing we have obtained:

PROPOSITION 2.32. *Let* $\mathfrak{S}^\mu \in \mathrm{Ell}\,\mathrm{Symb}_M^\mu(Y)$*. Then:*

(a) $L_{\mathfrak{S}^\mu}^\delta = \mathcal{Q}^\delta(\mathcal{O};(\mathfrak{S}^\mu)^{-1})$ *and* $L_{(\mathfrak{S}^\mu)^{-1}}^{\delta-\mu} = \mathcal{Q}^{\delta-\mu}(\mathcal{O};\mathfrak{S}^\mu)$;

(b) *There is an order-preserving bijection*

$$\begin{gathered}\{P \in \underline{\mathrm{As}}^{\delta}(Y);\, P \not\succeq L^{\delta}_{\mathfrak{S}^{\mu}}\} \to \{Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y);\, Q \not\succeq L^{\delta-\mu}_{(\mathfrak{S}^{\mu})^{-1}}\}, \\ P \mapsto \mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^{\mu}),\end{gathered} \tag{2.27}$$

*with its inverse being* $Q \mapsto \mathcal{Q}^{\delta}(Q;(\mathfrak{S}^{\mu})^{-1})$.

*Proof.* This is a direct consequence of Proposition 2.29 (b) and Proposition 2.61. The proof consists in a word-by-word repetition of the arguments given there. □

In its consequence, Proposition 2.32 enables us to perform explicit calculations on asymptotic types.
We conclude this section with the following basic observation:

PROPOSITION 2.33. *The notion of asymptotic type, as introduced above, is invariant under changes of coordinates.*

*Proof.* Let $\kappa\colon X \to X$ be a $C^\infty$-diffeomorphism and let $\kappa_*\colon C^\infty(X^\circ) \to C^\infty(X^\circ)$ be the corresponding push-forward on the level of functions, i.e., $(\kappa_* u)(x) = u(\kappa^{-1}(x))$ for $u \in C^\infty(X^\circ)$, where $\kappa^{-1}$ denotes the inverse $C^\infty$-diffeomorphism to $\kappa$. As is well-known, $\kappa_*$ restricts to $\kappa_*\colon C^{\infty,\delta}_{\mathrm{as}}(X) \to C^{\infty,\delta}_{\mathrm{as}}(X)$ for any $\delta \in \mathbb{R}$, see, e.g., SCHULZE [15].
We have to prove that, for each $P \in \underline{\mathrm{As}}^{\delta}(Y)$, there is a $\kappa_* P \in \underline{\mathrm{As}}^{\delta}(Y)$ so that the push-forward $\kappa_*$ restricts further to a linear isomorphism $\kappa_*\colon C^\infty_P(X) \to C^\infty_{\kappa_* P}(X)$, i.e., we have to show that there is a $\kappa_* P \in \underline{\mathrm{As}}^{\delta}(Y)$ so that $\kappa_*(C^\infty_P(X)) = C^\infty_{\kappa_* P}(X)$. Using Proposition 2.20, we finally arrive at proving that, for each $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ such that

$$u(x) \sim \sum_{j=0}^{\infty} \sum_{k+l=m_j-1} \frac{(-1)^k}{k!} \log^k t\, \phi^{(j)}_l(y) \ \text{ as } t \to +0, \tag{2.28}$$

where $\Phi \in \mathcal{E}^{\delta}_{V_p}(Y)$ for a certain $p \in \mathbb{C}$, $\mathrm{Re}\, p < (n+1)/2 - \delta$, and $\Phi(p-j) = (\phi^{(j)}_0, \phi^{(j)}_1, \dots, \phi^{(j)}_{m_j-1})$ for all $j \in \mathbb{N}$, see (2.19), the push-forward $\kappa_* u$ is again of the form (2.28), with some other $\kappa_*\Phi \in \mathcal{E}^{\delta}_{V_p}(Y)$ in place of $\Phi \in \mathcal{E}^{\delta}_{V_p}(Y)$. But this results from a direct computation, see, e.g., SCHULZE [15]. □

### 2.2.3 CHARACTERISTICS OF PROPER ASYMPTOTIC TYPES

We generalize the notion of characteristic, see Lemma 2.70 and the discussion thereafter, to proper asymptotic types. This will be the main ingredient for proving the analogue of Proposition 2.79 in Theorem 2.43.
Let $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ be represented by $J \subset \mathcal{E}^{\delta}_{V}(Y)$ and let $\Phi_1, \Phi_2, \dots$ by a characteristic basis of $J$ according to Definition 2.22. As before, let $(m^j_1, \dots, m^j_{e_j})$ be the characteristic of the space $J^{\delta+j}$. From Proposition 2.21 we conclude that

$e_1 \le e_2 \le \dots$ In the next lemma, we find suitable "paths through" the numbers $m_i^j$ for $j \ge j_i$, where $j_i = \min\{j;\, e_j \ge i\}$, i.e., an appropriate re-ordering of the tuples $(m_1^j, \dots, m_{e_j}^j)$.

LEMMA 2.34. *The numbering within the tuples* $(m_1^j, \dots, m_{e_j}^j)$ *can be chosen in such a way that, for each* $j \ge 1$, *there is a characteristic* $(m_1^j, \dots, m_{e_j}^j)$*-basis* $(\Phi_1^j, \dots, \Phi_{e_j}^j)$ *of* $J^{\delta+j}$ *such that, for all* $j' > j$,

$$\Pi_{jj'} \Phi_i^{j'} = \begin{cases} \Phi_i^j & \text{if } 1 \le i \le e_j, \\ 0 & \text{if } e_j + 1 \le i \le e_{j'} \end{cases}$$

*holds.*
*Furthermore, the scheme*

$$\begin{array}{rl} e_1 \text{ rows} & \left\{ \begin{array}{ccccc} m_1^1 & m_1^2 & m_1^3 & m_1^4 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ m_{e_1}^1 & m_{e_1}^2 & m_{e_1}^3 & m_{e_1}^4 & \dots \end{array} \right. \\ e_2 - e_1 \text{ rows} & \left\{ \begin{array}{ccccc} & m_{e_1+1}^2 & m_{e_1+1}^3 & m_{e_1+1}^4 & \dots \\ & \dots & \dots & \dots & \dots \\ & m_{e_2}^2 & m_{e_2}^3 & m_{e_2}^4 & \dots \end{array} \right. \\ e_3 - e_2 \text{ rows} & \left\{ \begin{array}{ccccc} & & m_{e_2+1}^3 & m_{e_2+1}^4 & \dots \\ & & \dots & \dots & \dots \\ & & m_{e_3}^3 & m_{e_3}^4 & \dots \end{array} \right. \\ & \left\{ \begin{array}{ccccc} & & & \vdots & \ddots, \end{array} \right. \end{array} \tag{2.29}$$

*where in the* $j$*th column the characteristic of the space* $J^{\delta+j}$ *appears, is uniquely determined up to permutation of the* $k$*th and the* $k'$*th row, where* $e_j+1 \le k,\ k' \le e_{j+1}$ *and some* $j$ $(e_0 = 0)$.

*Proof.* This is a reformulation of Proposition 2.21 in terms of the characteristics of the spaces $J^{\delta+j}$. Notice that one can recover the characteristic basis $\Phi_1, \Phi_2, \dots$ of $J$, that was initially given, from the property that $\Pi_j \Phi_i = \Phi_i^j$ holds for all $1 \le i \le e_j$, while $\Pi_j \Phi_i = 0$ for $i > e_j$. □

Performing the construction of the foregoing lemma for each space $J \cap \mathcal{E}_{V_{p_j}}^\delta(Y)$ in (2.17) separately, we see that the following definition makes perfect sense:

DEFINITION 2.35. Let $P \in \underline{\mathrm{As}}_{\mathrm{prop}}^\delta(Y)$ and let $J \subset \mathcal{E}_V^\delta(Y)$ represent $P$. If $\Phi_1, \Phi_2, \dots$ is a characteristic basis of $J$ according to Definition 2.22 and if the tuples $(m_1^j, \dots, m_{e_j}^j)$ are re-ordered according to Lemma 2.34, then the sequence

$$\operatorname{char} P = \left\{ \left( \gamma(\Phi_i); m_i^{j_i}, m_i^{j_i+1}, m_i^{j_i+2}, \dots \right) \right\}_{i=1}^{e} \tag{2.30}$$

is called the characteristic of $P$.

The characteristic $\operatorname{char} P$ of an asymptotic type $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ is unique up to permutation of the $k$th and the $k'$th entry, where $e_j + 1 \le k,\ k' \le e_{j+1}$ for some $j$. So far, it is an invariant associated with the representing space $J$; so it still depends on the chosen splitting of coordinates. Nevertheless, we have:

PROPOSITION 2.36. *The characteristic* $\operatorname{char} P$ *of an asymptotic type* $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ *is independent of the chosen splitting of coordinates* $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, *close to* $\partial X$.

*Proof.* Follow the proof of Proposition 2.33 to get the assertion. □

Now, let $\{(p_i; m_i^{j_i}, m_i^{j_i+1}, \dots)\}_{i=1}^{e} \subset \mathbb{C} \times \mathbb{N}^{\mathbb{N}}$ be any given sequence, where we additionally assume that $\operatorname{Re} p_i < (n+1)/2 - \delta$ for all $i$, $\operatorname{Re} p_i \to \infty$ as $i \to \infty$ when $e = \infty$, the $p_i$ are ordered so that $\operatorname{Re} p_i \ge (n+1)/2 - \delta - j$ holds if and only if $i \le e_j$ for a certain (then uniquely determined) sequence $e_1 \le e_2 \le \dots$ satisfying $e = \sup_j e_j$, and

$$1 \le m_i^{j_i} \le m_i^{j_i+1} \le m_i^{j_i+2} \le \dots,$$

where $j_i = \min\{j;\, e_j \ge i\}$ as above.

PROPOSITION 2.37. *Let the characteristic* $\{(p_i; m_i^{j_i}, m_i^{j_i+1}, \dots)\}_{i=1}^{e}$ *satisfying all the properties just mentioned be given. In case* $\dim Y = 0$, *we additionally assume that* $p_i \ne p_{i'}$ *for* $i \ne i'$ *and, for each* $i$, *there is an* $i' > i$ *such that* $p_{i'} = p_i - 1$ *and* $m_{i'}^{j_{i'}+k} \ge m_i^{j_i+k+1}$ *for* $k \ge 0$ $(j_{i'} = j_i + 1)$. *Then there exists a holomorphic* $\mathfrak{S}^{\mu} \in \operatorname{Symb}^{\mu}_{M}(Y)$ *that is elliptic with respect to the weight* $\delta \in \mathbb{R}$ *such that* $L^{\delta}_{\mathfrak{S}^{\mu}} \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ *has exactly this characteristic.*

*Proof.* In case $\dim Y = 0$, we choose an elliptic $\mathfrak{s}^{\mu}(z) \in \mathcal{M}^{\mu}_{\mathcal{O}}$ having zeros precisely at $z = p_i$ of order $m_i^{j_i}$ for all $i = 1, 2, \dots$ according to Proposition 2.7. In case $\dim Y > 0$, let $\{\phi_i\}_{i=1}^{e}$ be an orthonormal set in $C^{\infty}(Y)$ with respect to a fixed $C^{\infty}$-density $d\mu$ on $Y$. Let $\Pi_i$ for $i = 1, \dots, e$ be the orthogonal projection in $L^2(Y, d\mu)$ onto the subspace spanned by $\phi_i$. We then choose an elliptic $\mathfrak{s}^{\mu}(z) \in \mathcal{M}^{\mu}_{\mathcal{O}}(Y)$ such that, for every $p \in V_{p_i}$ and all $i$,

$$[\mathfrak{s}^{\mu}(z)]_p^{N_p} = \Bigl(1 - \sum_{p_{i'} - k = p} \Pi_{i'}\Bigr) + \sum_{p_{i'} - k = p} (z-p)^{m_{i'}^{j_{i'}+k}} \Pi_{i'}$$

where the sums are extended over all $i'$, $k$ such that $p_{i'} - k = p$, for some $N_p$ sufficiently large, while $\mathfrak{s}^{\mu}(q) \in L^{\mu}_{\mathrm{cl}}(Y)$ is invertible for all $q \in \mathbb{C} \setminus V$, again according to Proposition 2.7.
In both cases, we set $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j}(z)\}_{j=0}^{\infty}$ with $\mathfrak{s}^{\mu-j}(z) \equiv 0$ for $j > 0$. Then $\mathfrak{S}^{\mu} \in \operatorname{Symb}^{\mu}_{M}(Y)$ is elliptic with respect to the weight $\delta$, and the proper asymptotic type $L^{\delta}_{\mathfrak{S}^{\mu}}$ has characteristic $\{(p_i; m_i^{j_i}, m_i^{j_i+1}, \dots)\}_{i=1}^{e}$. □

### 2.2.4 Further properties of asymptotic types

Here, we study further properties of asymptotic types. First, asymptotic types can be decomposed into elementary building blocks.

PROPOSITION 2.38. (a) *An asymptotic type* $P \in \underline{\mathrm{As}}^\delta(Y)$ *is join-irreducible, i.e.,* $P \neq \mathcal{O}$ *and* $P = P_0 \vee P_1$ *for* $P_0, P_1 \in \underline{\mathrm{As}}^\delta(Y)$ *implies* $P = P_0$ *or* $P = P_1$*, if and only if there is a* $\Phi \in \mathcal{E}^\delta(Y)$*,* $\Phi \neq 0$*, such that the representing space,* $J$*, for* $P$*, in the given splitting of coordinates close to* $\partial X$*, has characteristic basis* $\Phi$*, i.e.,* $J = \langle \Phi \rangle$*. In particular, every join-irreducible asymptotic type is proper.*
(b) *The join-irreducible asymptotic types are join-dense in* $\underline{\mathrm{As}}^\delta(Y)$*.*

*Proof.* (a) Let $P \neq \mathcal{O}$. Assume that, for some $j \geq 1$, $J^{\delta+j}$ has characteristic of length larger 1. Then $J^{\delta+j} = K_0 + K_1$ for certain linear subspaces $K_i \subsetneq J^{\delta+j}$ satisfying $TK_i \subseteq K_i$, for $i = 0, 1$. Setting $J_i = \{\Phi \in J;\, \Pi_j \Phi \in K_i\}$, we get that $J = J_0 + J_1$, $J_i \subsetneq J$, and $TJ_i \subseteq J_i$ for $i = 0, 1$. Since this decomposition can be chosen compatible with (2.17), we obtain that a necessary condition for $P$ to be join-irreducible is that each space $J^{\delta+j}$ for $j \geq 1$ has characteristic of length at most 1, i.e., $J = \langle \Phi \rangle$ for some $\Phi \neq 0$. Vice versa, if $J = \langle \Phi \rangle$ for some $\Phi \neq 0$, then $P$ is join-irreducible, since the subspace $\langle T^k \Phi \rangle \subseteq J$ for $k \in \mathbb{N}$ are the only subspaces of $J$ that are invariant under the action by $T$.
(b) This follows directly from Proposition 2.20. □

Note that, by the foregoing proposition, also the proper asymptotic types are join-dense in $\underline{\mathrm{As}}^\delta(Y)$. We will utilize that fact in the definition of cone Sobolev spaces with asymptotics.
In constructing asymptotic types $P \in \underline{\mathrm{As}}^\delta(Y)$ obeying certain properties one often encounters the situation in which $P$ is successively constructed on strips $\{z \in \mathbb{C}; (n+1)/2 - \delta - \beta_h < \operatorname{Re} z < (n+1)/2 - \delta\}$ of finite width, where the sequence $\{\beta_h\}_{h=0}^\infty \subset \mathbb{R}_+$ is strictly increasing and $\beta_h \to \infty$ as $h \to \infty$. We will meet an example in Section 3.3.
To formulate the result, we need a further definition:

DEFINITION 2.39. Let $P, P' \in \underline{\mathrm{As}}^\delta(Y)$ be represented by $J \subset \mathcal{E}_V^\delta(Y)$ and $J' \subset \mathcal{E}_V^\delta(Y)$, respectively. Then, for $\vartheta \geq 0$, the asymptotic types $P$ and $P'$ are said to be equal up to the conormal order $\delta + \vartheta$ if $\Pi_\vartheta J = \Pi_\vartheta J'$, where $\Pi_\vartheta \colon J \to J / (J \cap \mathcal{E}^{\delta+\vartheta}(Y))$ is the canonical projection. Similarly, $P$ and $P'$ are said to be equal up to the conormal order $\delta + \vartheta - 0$ if they are equal up to the conormal order $\delta + \vartheta - \epsilon$, for any $\epsilon > 0$. (Similarly for the order relation $\preccurlyeq$ instead of equality.)

PROPOSITION 2.40. *Let* $\{P_\iota\}_{\iota \in \mathcal{I}} \subset \underline{\mathrm{As}}^\delta(Y)$ *be an increasing net of asymptotic types. Then the join* $\bigvee_{\iota \in \mathcal{I}} P_\iota$ *exists if and only if, for each* $j \geq 1$*, there is an* $\iota_j \in \mathcal{I}$ *such that* $P_\iota = P_{\iota'}$ *up to the conormal order* $\delta + j$ *for all* $\iota, \iota' \geq \iota_j$*.*

*Proof.* The condition is obviously sufficient.

Conversely, suppose that the join $\bigvee_{\iota\in\mathcal{I}} P_\iota$ exists. Let $P_\iota$ be represented by the subspace $J_\iota \subset \mathcal{E}^\delta_{V_\iota}(Y)$ for $V_\iota \in \mathcal{C}^\delta$. Since the join $\bigvee_{\iota\in\mathcal{I}} P_\iota$ exists, the carriers $V_\iota$ can be chosen in such way that $\bigcup_{\iota\in\mathcal{I}} V_\iota \subseteq V$ for some $V \in \mathcal{C}^\delta$. Thus $J_\iota \subset \mathcal{E}^\delta_V(Y)$ for all $\iota$. Now, for each $j \geq 1$, $\dim\bigl(\sum_{\iota\in\mathcal{I}} J_\iota^{\delta+j}\bigr) < \infty$, otherwise $\bigvee_{\iota\in\mathcal{I}} P_\iota$ does not exist. But since the net $\{J_\iota^{\delta+j}\}_{\iota\in\mathcal{I}}$ is increasing, this already implies that there is some $\iota_j \in \mathcal{I}$ such that $J_\iota^{\delta+j} = J_{\iota'}^{\delta+j}$ for $\iota$, $\iota' \geq \iota_j$, i.e., $P_\iota = P_{\iota'}$ up to the conormal order $\delta + j$ for $\iota$, $\iota' \geq \iota_j$. □

An equivalent condition is that the net $\{P_\iota\}_{\iota\in\mathcal{I}} \subset \underline{\mathrm{As}}^\delta(Y)$ of asymptotic types is bounded on each strip $\{z \in \mathbb{C}; (n+1)/2 - \delta - j \leq \operatorname{Re} z < (n+1)/2 - \delta\}$ of finite width.

## 2.3 PSEUDODIFFERENTIAL THEORY

Ee come to the proof of the analogue of Proposition 2.79. We need:

PROPOSITION 2.41. *Let* $P$, $P_0 \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, $Q \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$ *for* $\mu \in \mathbb{R}$. *Assume that* $P \wedge P_0 = \mathcal{O}$. *Then there is a holomorphic* $\mathfrak{S}^\mu \in \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$ *that is elliptic with respect to the weight* $\delta$ *such that* $L^\delta_{\mathfrak{S}^\mu} = P_0$ *and* $\mathcal{Q}(P; \mathfrak{S}^\mu) = Q$ *if and only if* $P$ *and* $Q$ *have the same characteristic shifted by* $\mu$, *i.e., we have* $\operatorname{char} P = \operatorname{char} Q - \mu$ (*with the obvious meaning of* $\operatorname{char} Q - \mu$).

*Proof.* As for Lemma 2.78, it is readily seen that $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, $Q \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$ have the same characteristic shifted by $\mu$ if there is a holomorphic $\mathfrak{S}^\mu \in \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$ such that $\mathcal{Q}(P; \mathfrak{S}^\mu) = Q$.
Now, suppose that $\operatorname{char} P = \operatorname{char} Q - \mu$. First, we deal with the case $P_0 = \mathcal{O}$. Let the asymptotic types $P$, $Q$ be represented by $J \subset \mathcal{E}^\delta_V(Y)$ and $K \subset \mathcal{E}^{\delta-\mu}_{T^{-\mu}V}(Y)$, respectively. Let $\{\Phi_i\}_{i=1}^e$ and $\{\Psi_i\}_{i=1}^e$ be characteristic bases of $J$ and $K$, respectively, corresponding to $\operatorname{char} P$ and $\operatorname{char} Q$.
We have to choose the sequence $\{\mathfrak{s}^{\mu-k}(z); k \in \mathbb{N}\} \subset \mathcal{M}^\mu_{\mathcal{O}}(Y)$ appropriately. By Proposition 2.7, it suffices to construct the finite parts $[\mathfrak{s}^{\mu-k}(z)]_{p'}^{N_{p'k}}$ for $p' \in V$, $k \in \mathbb{N}$, and $N_{p'k}$ sufficiently large in an appropriate way. Thereby, we can assume that $V = V_p$ for some $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$.
Let $e_1 \leq e_2 \leq \dots$, where $e = \sup_{j\in\mathbb{N}} e_j$, be such that $\gamma(\Phi_i) = \gamma(\Psi_i) - \mu = p - j$ holds for $e_{j-1} + 1 \leq i \leq e_j$ (and $e_0 = 0$). Then the finite parts $[\mathfrak{s}^{\mu-k}(z)]_{p-j}^{m^{j+k}}$ for all $j$, $k$ must be chosen so that, for each $j \in \mathbb{N}$,

$$\Phi_i(p-j)^{[\mathfrak{s}^\mu(z)]_{p-j}^{m^j}} + \Phi_i(p-j+1)^{[\mathfrak{s}^{\mu-1}(z)]_{p-j+1}^{m^j}} + \dots + \Phi_i(p)^{[\mathfrak{s}^{\mu-j}(z)]_p^{m^j}} = \Psi_i(p+\mu-j) \quad (2.31)$$

holds for all $1 \leq i \leq e_j$, where $m^j = \sup_{1\leq i\leq e_j} m_i^j$ and $\Phi_i(p-k) = 0$ if $e_k + 1 \leq i \leq e_j$. Here, $(m_1^j, \dots, m_{e_j}^j)$ is the characteristic of $J^{\delta+j}$. See Remark 2.67 for the notation used in (2.31).

Now, (2.31) can successively be solved for $[\mathfrak{s}^{\mu}(z-k)]^{m^j}_{p-j+k}$ for $j = 0, 1, 2, \ldots$ and $0 \le k \le j$. In fact, this can be done by choosing $[\mathfrak{s}^{\mu-k}(z)]^{m^j}_{p-j+k}$ for $k > 0$ arbitrarily (in particular, we may choose $\mathfrak{s}^{\mu-k}(z) \equiv 0$ for $k > 0$) and then finding $[s^{\mu}(z)]^{m^j}_{p-j}$ with the help of Lemma 2.8.
The general case, in which not necessarily $P_0 = \mathcal{O}$ holds, can be reduced to the case $P_0 = \mathcal{O}$ as in the proof of Lemma 2.78, see WITT [21], since the three rules from Lemma 2.60 used for that continues to hold in the present situation. □

*Remark* 2.42. (a) The proof of Proposition 2.41 shows that the holomorphic $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j};\, j \in \mathbb{N}\} \in \operatorname{Ell}\operatorname{Symb}^{\mu}_{M}(Y)$ satisfying $L^{\delta}_{\mathfrak{S}^{\mu}} = P_0$ and $\mathcal{Q}(P; \mathfrak{S}^{\mu}) = Q$ can always be chosen so that $\mathfrak{s}^{\mu-j}(z) \equiv 0$ for $j > 0$.
(b) Proposition 2.41 in connection with Theorem 2.31 also shows that the asymptotic types in $\underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ are precisely the asymptotic types which are of the form $L^{\delta}_{\mathfrak{S}^{\mu}}$ for some holomorphic $\mathfrak{S}^{\mu} \in \operatorname{Ell}\operatorname{Symb}^{\mu}_{M}(Y)$ that is elliptic with respect to the weight $\delta$. (Choose $P = Q = \mathcal{O}$ in Proposition 2.41.)

Now, we reach the final aim of this section.

THEOREM 2.43. *Let* $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ *and* $Q \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$*. Then there exists a* $\mathfrak{S}^{\mu} \in \operatorname{Symb}^{\mu}_{M}(Y)$ *that is elliptic with respect to the weight* $\delta$ *such that* $L^{\delta}_{\mathfrak{S}^{\mu}} = P$ *and* $L^{\delta-\mu}_{(\mathfrak{S}^{\mu})^{-1}} = Q$ *always when* $\dim Y > 0$ *and if and only if* $P \wedge T^{-\mu}Q = \mathcal{O}$ *when* $\dim Y = 0$.

*Proof.* The necessity of the condition $P \wedge T^{-\mu}Q = \mathcal{O}$ in case $\dim Y = 0$ is clear. Conversely, for the base manifold $Y$ being arbitrary, choose $P_1 \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$, $Q_1 \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$ having the same characteristics as $P$ and $Q$, respectively, and such that $P_1 \wedge T^{-\mu}Q_1 = \mathcal{O}$. As in the proof of Proposition 2.79, see WITT [21], it then suffices to construct holomorphic $\mathfrak{S}^0 \in \operatorname{Ell}\operatorname{Symb}^{0}_{M}(Y)$, $\mathfrak{T}^0 \in \operatorname{Ell}\operatorname{Symb}^{0}_{M}(Y)$ which are elliptic with respect to the weight $\delta$ such that $L^{\delta}_{\mathfrak{S}^0} = P_1$, $\mathcal{Q}^{\delta}(Q_1; \mathfrak{S}^0) = Q$ and $L^{\delta}_{\mathfrak{T}^0} = Q_1$, $\mathcal{Q}^{\delta}(P_1; \mathfrak{T}^0) = P$ hold. But this can be achieved using Proposition 2.41.
The rest of the proof is as for Proposition 2.79, see again WITT [21]. □

## 2.4 FUNCTION SPACES WITH ASYMPTOTICS

The introduction of cone Sobolev spaces with asymptotics is based on the Mellin transformation. For further details, we refer to JEANQUARTIER [5] for the Mellin transformation and to SCHULZE [15, 16] for function spaces with asymptotics, cf. also Remark 2.46.

### 2.4.1 WEIGHTED CONE SOBOLEV SPACES

Let $Mu(z) = \tilde{u}(z) = \int_0^\infty t^{z-1}u(t)\,dt$, $z \in \mathbb{C}$, for $u \in C_0^\infty(\mathbb{R}_+)$ denote the Mellin transform. The Mellin transformation is afterwards extended to larger distribution classes. In particular, $u$ will be allowed to be vector-valued. Recall the following properties of $M$:

$$M_{t\to z}\{(-t\partial_t - p)u\}(z) = (z-p)\tilde{u}(z),$$
$$M_{t\to z}\{t^{-p}u\}(z) = \tilde{u}(z-p), \qquad \text{for any } p \in \mathbb{C},$$

whenever both sides are defined, $M\colon L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2};(2\pi i)^{-1}dz)$ is an isometry, and

$$M_{t\to z}\left\{\frac{(-1)^k}{k!}\,t^{-p}\log^k t\,\chi_{(0,1]}(t)\right\}(z) = \frac{1}{(z-p)^{k+1}}. \tag{2.32}$$

Here, $\chi_{(0,1]}$ is the characteristic function of the interval $(0,1]$. From that we conclude that $h(z) = M_{t\to z}\left\{\frac{(-1)^k}{k!}\,\omega(t)t^{-p}\log^k t\right\}(z)$, where $\omega(t)$ is a cut-off function close to $t=0$, is a meromorphic function of $z$ having a pole at $z=p$, and the principal part of the Laurent expansion of this pole is given by the right-hand side of (2.32), i.e., $[h(z)]_p^* = [h(z)]_p^k = \frac{1}{(z-p)^{k+1}}$, see (2.8).

For $s$, $\delta \in \mathbb{R}$, let $\mathcal{H}^{s,\delta}(X)$ denote the space of all $u \in H^s_{\mathrm{loc}}(X^\circ)$ such that $M_{t\to z}\{\omega u\}(z) \in L^2_{\mathrm{loc}}\big(\Gamma_{(n+1)/2-\delta}; H^s(Y)\big)$ and the expression

$$\|u\|_{\mathcal{H}^{s,\delta}(X)} = \left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta}} \big\|R^s(z)M_{t\to z}\{\omega u\}(z)\big\|^2_{L^2(Y)}\right\}^{1/2} \tag{2.33}$$

is finite. See SCHULZE [16, Theorem 2.1.39]. Here, $R^s(z) \in L^s_{\mathrm{cl}}(Y;\Gamma_{(n+1)/2-\delta})$ is an order-reducing family, i.e., $R^s(z)$ is parameter-dependent elliptic and $R^s(z)\colon H^r(Y) \to H^{r-s}(Y)$ is an isomorphism for some $r \in \mathbb{R}$ (and then for all $r \in \mathbb{R}$) and all $z \in \Gamma_{(n+1)/2-\delta}$. For instance, if $f(z) \in \mathcal{M}^s_{\mathrm{as}}(Y)$ is elliptic and the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $f(z)$, then $f(z)$ is such an order-reduction.

### 2.4.2 Cone Sobolev spaces with asymptotics

We will use the latter observation for defining cone Sobolev spaces with asymptotics. Let $s$, $\delta \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$. By Theorem 2.43, there is an elliptic Mellin symbol $h^s_P(z) \in \mathcal{M}^s_{\mathcal{O}}(Y)$ such that the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $h^s_P(z)$ and, for $\mathfrak{S}^s = \{h^s_P(z),0,0,\dots\} \in \mathrm{Symb}^s_M(Y)$, $L^\delta_{\mathfrak{S}^s} = P$ holds.

DEFINITION 2.44. Let $s$, $\delta \in \mathbb{R}$, $\vartheta \geq 0$, and $P \in \underline{\mathrm{As}}^\delta(Y)$.
(a) For $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, the space $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ consists of all functions $u \in \mathcal{H}^{s,\delta}(X)$ such that $M_{t\to z}\{\omega u\}(z)$, which is a holomorphic function in $\{z \in \mathbb{C};\ \mathrm{Re}\,z > (n+1)/2-\delta\}$ taking values in $H^s(Y)$, possesses a meromorphic continuation to the half-space $\{z \in \mathbb{C};\ \mathrm{Re}\,z > (n+1)/2-\delta-\vartheta\}$,

$$h^s_P(z)M_{t\to z}\{\omega u\}(z) \in \mathcal{A}\big(\{z \in \mathbb{C}; \mathrm{Re}\,z > (n+1)/2-\delta+s-\vartheta\}; L^2(Y)\big),$$

and the expression

$$\sup_{\delta<\delta'<\delta+\vartheta}\left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta'+s}}\left\|h_P^s(z)M_{t\to z}\{\omega u\}(z)\right\|^2_{L^2(Y)}\,dz\right\}^{1/2}. \tag{2.34}$$

is finite.

(b) For a general $P\in\underline{\mathrm{As}}^\delta(Y)$, represented as the join $P=\bigvee_{\iota\in\mathcal{I}}P_\iota$ for a bounded family $\{P_\iota\}_{\iota\in\mathcal{I}}\subset\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, we define $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)=\sum_{\iota\in\mathcal{I}}\mathcal{H}^{s,\delta}_{P_\iota,\vartheta}(X)$.

It is readily seen that Definition 2.44 (a) is independent of the choice of the Mellin symbol $h_P^s(z)$, see Proposition 2.62 and Remark 2.63. Moreover, under the condition that (2.34) is finite, the limit

$$h_P^s(z)M_{t\to z}\{\omega u\}(z)\big|_{z=(n+1)/2-\delta'+i\tau}\to w(\tau)\ \text{ as } \delta'\to\delta+\vartheta-0$$

exists in $L^2(\mathbb{R}_\tau;L^2(Y))$. Thus, $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ is a Hilbert space with the norm

$$\|u\|_{\mathcal{H}^{s,\delta}_{P,\vartheta}(X)}=\left\{\|w\|^2_{L^2(\mathbb{R}_\tau;L^2(Y))}+\|u\|^2_{\mathcal{H}^{s,\delta}(X)}\right\}^{1/2}. \tag{2.35}$$

Definition 2.44 (b) is justified by Proposition 2.38 (b), since we obviously have $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)=\mathcal{H}^{s,\delta+\vartheta}(X)$ for $P\in\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$ and $\delta_P>\delta+\vartheta$. Again, this definition is seen to be independent of the choice of the representing family $\{P_\iota\}_{\iota\in\mathcal{I}}\subset\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, and it yields a Hilbert space $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$.

PROPOSITION 2.45. *Let $s$, $\delta\in\mathbb{R}$, $\vartheta\geq 0$, and $P\in\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$. Further, let $\mathfrak{S}^s=\{\mathfrak{s}^{s-j}(z)\ j=0,1,2,\dots\}\in\mathrm{Symb}^s_M(Y)$ be elliptic with respect to the weight $\delta$ and $L^\delta_{\mathfrak{S}^s}=P$, $L^{\delta-s}_{(\mathfrak{S}^s)^{-1}}=\mathcal{O}$. (Condition $L^{\delta-s}_{(\mathfrak{S}^s)^{-1}}=\mathcal{O}$ means that the Mellin symbols $\mathfrak{s}^{s-j}(z)$ are holomorphic when $\operatorname{Re}z>(n+1)/2-\delta$.) Then a function $u\in\mathcal{H}^{s,\delta}(X)$ belongs to the space $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ if and only if $M_{t\to z}\{\omega u\}(z)$ possesses a meromorphic continuation to the half-space $\{z\in\mathbb{C};\ \operatorname{Re}z>(n+1)/2-\delta-\vartheta\}$,*

$$\sum_{j=0}^M\mathfrak{s}^{s-j}(z-s+j)M_{t\to z}\{\omega u\}(z-s+j)\in\mathcal{A}\big(\{z\in\mathbb{C};\operatorname{Re}z>(n+1)/2-\delta+s-\vartheta\};L^2(Y)\big),$$

*and the expression*

$$\sup_{\delta<\delta'<\delta+\vartheta}\left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta'+s}}\left\|\sum_{j=0}^M\mathfrak{s}^{s-j}(z-s+j)M_{t\to z}\{\omega u\}(z-s+j)\right\|^2_{L^2(Y)}dz\right\}^{1/2}.$$

*is finite. Here, $M$ is any integer larger than $\vartheta$.*

*Proof.* This is an application of (a here adapted version of) Proposition 2.62 and Remark 2.63. Note that $\mathfrak{s}^{s-j}(z-s+j)M_{t\to z}\{\omega u\}(z-s+j) \in \mathcal{A}\big(\{z \in \mathbb{C}; \operatorname{Re} z > (n+1)/2-\delta+s-j\}; L^2(Y)\big)$ so that the condition is actually independent of the choice of the integer $M > \vartheta$. □

For $s$, $\delta \in \mathbb{R}$, $\vartheta > 0$, and $P \in \underline{\mathrm{As}}^{\delta}(Y)$, we will also employ the spaces

$$\mathcal{H}^{s,\delta}_{P,\vartheta-0}(X) = \bigcap_{\epsilon>0} \mathcal{H}^{s,\delta}_{P,\vartheta-\epsilon}(X). \tag{2.36}$$

These space $\mathcal{H}^{s,\delta}_{P,\vartheta-0}(X)$ are Fréchet-Hilbert spaces, i.e., Fréchet spaces the topology of which can be given with the help of a countable family of Hilbert semi-norms. We will also use standard notion like

$$\mathcal{H}^{\infty,\delta}_{P,\vartheta}(X) = \bigcap_{s\in\mathbb{R}} \mathcal{H}^{s,\delta}_{P,\vartheta-0}(X), \quad \mathcal{H}^{-\infty,\delta}_{P,\vartheta}(X) = \bigcup_{s\in\mathbb{R}} \mathcal{H}^{s,\delta}_{P,\vartheta-0}(X),$$
$$\mathcal{H}^{s,\delta}_{P,\vartheta+0}(X) = \bigcup_{\epsilon>0} \mathcal{H}^{s,\delta}_{P,\vartheta+\epsilon}(X), \qquad \text{etc.}$$

*Remark* 2.46. In the case that $P$ is a strongly discrete asymptotic type, the spaces $\mathcal{H}^{s,\delta}_{P,\vartheta-0}(X)$ are the function spaces with asymptotics employed by SCHULZE [16]. There the notation used is $\mathcal{H}^{s,\delta}_{P}(X)_\Delta$, with the half-open interval $\Delta = (-\vartheta, 0]$.

### 2.4.3 FUNCTIONAL-ANALYTIC PROPERTIES

We list properties of the spaces $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$.

PROPOSITION 2.47. *Let $s$, $s'$, $\delta$, $\delta' \in \mathbb{R}$, $\vartheta \geq 0$, $P \in \underline{\mathrm{As}}^{\delta}(Y)$, $P' \in \underline{\mathrm{As}}^{\delta'}(Y)$, and $\{P_\iota\}_{\iota\in\mathcal{I}} \subset \underline{\mathrm{As}}^{\delta}(Y)$ be a family of asymptotic types. Then*:
(a) $\mathcal{H}^{s,\delta}_{P,0}(X) = \mathcal{H}^{s,\delta}(X)$;
(b) $\mathcal{H}^{s,\delta}_{P,\vartheta}(X) = \mathcal{H}^{s,\delta-a}_{P,\vartheta+a}(X)$ *for any* $a > 0$;
(c) $\mathcal{H}^{s,\delta}_{\mathcal{O},\vartheta}(X) = \mathcal{H}^{s,\delta+\vartheta}(X)$;
(d) *We have*

$$\mathcal{H}^{s,\delta}_{P,\vartheta}(X) = \mathcal{H}^{s,\delta}_{\mathcal{O},\vartheta}(X) \oplus \Big\{\omega(t) \sum_{\substack{p\in V,\\ \operatorname{Re} p>(n+1)/2-\delta-\vartheta}} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!}\, t^{-p}\log^k t\, \phi^{(p)}_l(y); \quad \Phi(p) = (\phi^{(p)}_0, \dots, \phi^{(p)}_{m_p-1}) \text{ for some } \Phi \in J\Big\},$$

*where $J \subset \mathcal{E}^{\delta}_V(Y)$ is a linear subspace representing the asymptotic type $P$,* provided that $\operatorname{Re} p \neq (n+1)/2-\delta-\vartheta$ *holds for all* $p \in V$;

(e) *We have* $\mathcal{H}^{s,\delta}_{P,\vartheta}(X) \subseteq \mathcal{H}^{s',\delta'}_{P',\vartheta'}(X)$ *if and only if* $s \geq s'$, $\delta + \vartheta \geq \delta' + \vartheta'$, *and* $P \preccurlyeq P'$ *up to the conormal order* $\delta' + \vartheta'$;
(f) $\mathcal{H}^{s,\delta}_{\bigwedge_{\iota\in\mathcal{I}} P_\iota,\vartheta}(X) = \bigcap_{\iota\in\mathcal{I}} \mathcal{H}^{s,\delta}_{P_\iota,\vartheta}(X)$ *if the family* $\{P_\iota\}_{\iota\in\mathcal{I}}$ *is non-empty*;
(g) $\mathcal{H}^{s,\delta}_{\bigvee_{\iota\in\mathcal{I}} P_\iota,\vartheta}(X) = \sum_{\iota\in\mathcal{I}} \mathcal{H}^{s,\delta}_{P_\iota,\vartheta}(X)$ *if the family* $\{P_\iota\}_{\iota\in\mathcal{I}}$ *is bounded* (*where the sum sign stands for the non-direct sum of Hilbert spaces*);
(h) $C^\infty_P(X) = \bigcap_{s\in\mathbb{R},\,\vartheta\geq 0} \mathcal{H}^{s,\delta}_{P,\vartheta}(X)$;
(i) $C^\infty_P(X)$ *is dense in* $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$.

From (e) we get, in particular, that $\mathcal{H}^{s,\delta}_{P,\vartheta}(X) = \mathcal{H}^{s',\delta'}_{P',\vartheta'}(X)$ if and only if $s = s'$, $\delta + \vartheta = \delta' + \vartheta'$, and $P = P'$ up to the conormal order $\delta + \vartheta$. (b) and also (c), in view of (a), are special cases.

PROPOSITION 2.48. *For* $\delta \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^\delta(Y)$, *and any* $a \in \mathbb{R}$, *the family* $\{\mathcal{H}^{s,\delta}_{P,s-a}(X);\ s \geq a\}$ *of Hilbert spaces forms an interpolation scale with respect to the complex interpolation method.*

*Proof.* This follows immediate from the definition. □

PROPOSITION 2.49. *The spaces* $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ *are invariant under changes of coordinates in the sense of* Proposition 2.33.

*Proof.* Basically, this follows from the invariance of the spaces $C^\infty_P(X)$ under changes of coordinates, where the latter is just a reformulation of the fact that the asymptotic types in $\underline{\mathrm{As}}^\delta(Y)$ are coordinate invariant. See Proposition 2.33. □

### 2.4.4 MAPPING PROPERTIES AND ELLIPTIC REGULARITY

In the cone pseudodifferential calculus, one encounters operators of the form $\omega(t)t^{-\mu}\operatorname{op}_M^{(n+1)/2-\delta}(h)\omega(t)$, where $h(t,z) \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^\mu_{\mathrm{as}}(Y))$, see Appendix 2.B. Their mapping properties in the spaces $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ are stated next.

PROPOSITION 2.50. *Let* $h(t,z) \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^\mu_{\mathrm{as}}(Y))$ *and assume that the line* $\Gamma_{(n+1)/2-\delta}$ *is free of poles of* $\partial^j h(0,z)/\partial t^j$ *for all* $j = 0, 1, 2, \ldots$ *Then, for all* $P \in \underline{\mathrm{As}}^\delta(Y)$, $s \in \mathbb{R}$, $\vartheta \geq 0$,

$$\omega(t)t^{-\mu}\operatorname{op}_M^{(n+1)/2-\delta}(h)\,\tilde{\omega}(t)\colon \mathcal{H}^{s,\delta}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X), \tag{2.37}$$

*where* $\omega(t)$, $\tilde{\omega}(t)$ *are cut-off functions*, $\mathfrak{S}^\mu = \left\{\frac{1}{j!}\frac{\partial^j h}{\partial t^j}(0,z);\ j = 0, 1, 2, \ldots\right\} \in \mathrm{Symb}^\mu_M(Y)$, *and* $Q = \mathcal{Q}^{\delta-\mu}(P, \mathfrak{S}^\mu) \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$.

*Proof.* The previous definitions are designed to make this result holds. □

In particular, Proposition 2.50 implies the mapping property

$$A\colon \mathcal{H}^{s,\delta}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X) \tag{2.38}$$

stated in (2.54) for cone pseudodifferential operators $A \in \mathcal{C}^\mu(X, (\delta, \delta-\mu))$. Given $P \in \underline{\mathrm{As}}^\delta(Y)$, the minimal asymptotic type $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$ such that (2.38) holds, that exists according to Proposition 2.29 and Proposition 2.47 (f), shall be denoted by $\mathcal{Q}^{\delta-\mu}(P; A)$. Vice versa, in case $A$ is elliptic with respect to the weight $\delta$, given $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, the minimal asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ such that $u \in \mathcal{H}^{-\infty,\delta}(X)$, $Au \in \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X)$ implies $u \in \mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ shall be denoted by $\mathcal{P}^\delta(Q; A)$.

We shall employ the notion of push-forward also if more than one operator $A$ is involved, i.e., $\mathcal{Q}^{\delta-\mu}(P; A_1, \dots, A_m)$ denotes the minimal asymptotic type $Q$ for which $A_j \colon \mathcal{H}^{s,\delta}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X)$ for $1 \le j \le m$.

We conclude this section with the following result concerning Fuchsian differential operators.

THEOREM 2.51. *For $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$, $P \in \underline{\mathrm{As}}^\delta(Y)$, $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, we have $\mathcal{Q}^{\delta-\mu}(P; A) = \mathcal{Q}^{\delta-\mu}(P; \mathfrak{S}^\mu)$, where $\mathfrak{S}^\mu = \{\sigma^{\mu-j}_M(A)(z);\, j = 0, 1, \dots\} \in \mathrm{Symb}^\mu_M(Y)$, as well as, in case $A$ is elliptic with respect to the weight $\delta$, $\mathcal{P}^\delta(Q; A) = \mathcal{Q}^\delta(Q; (\mathfrak{S}^\mu)^{-1})$.*

*Proof.* In fact, $\mathcal{Q}^{\delta-\mu}(P; A) = \mathcal{Q}^{\delta-\mu}(P; \mathfrak{S}^\mu)$ follows from Proposition 2.50.

Furthermore, it is well-known that formal asymptotic solutions $u \in C^\infty_{\mathrm{as}}(X)$ to the equation $Au = f$ for $f \in C^\infty_R(X)$ and any $R \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$ can be constructed, see, e.g. MELROSE [13, Lemma 5.13]. More precisely, it can be shown that there is a right parametrix $B$ to $A$, $B \colon \mathcal{H}^{s-\mu,\delta-\mu}(X) \to \mathcal{H}^{s,\delta}(X)$ for all $s \in \mathbb{R}$, such that

$$AB = I + R, \quad R \colon \mathcal{H}^{-\infty,\delta-\mu}(X) \to C^\infty_{\mathcal{O}}(X),$$

i.e., $R$ is smoothing over $X^\circ$ and flattening to the infinite order close to $\partial X$. In fact, $B$ belongs to Schulze's cone pseudodifferential calculus of operators of order $-\mu$, i.e., $B \in \mathcal{C}^{-\mu}(X, (\delta-\mu, \delta))$, see Appendix 2.B. In particular, $B \in L^{-\mu}_{\mathrm{cl}}(X^\circ)$.

Now let $BA = I + R_0$. Obviously, $R_0$ is smoothing over $X^\circ$ such that $R_0 \colon \mathcal{H}^{s,\delta}(X) \to \mathcal{H}^{\infty,\delta-\mu}(X)$ for any $s \in \mathbb{R}$. Furthermore, $A(I + R_0) = ABA = (I + R)A$ so that

$$AR_0 = RA.$$

We conclude that $R_0 \colon \mathcal{H}^{s,\delta}(X) \to C^\infty_{P_0}(X)$, where $P_0 = \mathcal{Q}^\delta(\mathcal{O}; (\mathfrak{S}^\mu)^{-1})$. Hence, for $u \in \mathcal{H}^{-\infty,\delta}(X)$, $Au = f \in \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X)$, we get

$$u = Bf - R_0 u \in \mathcal{H}^{s,\delta}_{P,\vartheta}(X),$$

where $P = \mathcal{Q}^\delta(Q; (\mathfrak{S}^\mu)^{-1})$. Thus $\mathcal{P}^\delta(Q; A) = \mathcal{Q}^\delta(Q; (\mathfrak{S}^\mu)^{-1})$ as claimed. See also WITT [19, Remark after Proposition 5.5]. ☐

PROPOSITION 2.52. *Let $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ be elliptic with respect to the weight $\delta$. Then there is an order-preserving bijection*

$$\{P \in \underline{\mathrm{As}}^\delta(Y);\, P \succcurlyeq L^\delta_{\mathfrak{S}^\mu}\} \to \underline{\mathrm{As}}^{\delta-\mu}(Y), \;\; P \mapsto \mathcal{Q}(P; A), \tag{2.39}$$

*with its inverse given by* $Q \mapsto \mathcal{P}^\delta(Q; A)$*. In particular,* $L^\delta_{\mathfrak{S}^\mu}$ *corresponds to the empty asymptotic type,* $\mathcal{O}$*.*

*Proof.* This is implied by Proposition 2.32 and Theorem 2.51. Note that $L^{\delta-\mu}_{(\mathfrak{S}^\mu)^{-1}} = \mathcal{O}$, since the $\sigma^{\mu-j}_M(A)(z)$ for $j = 0, 1, 2, \ldots$ are holomorphic. □

Finally, we have the following locality principle:

PROPOSITION 2.53. *Let* $A \in \operatorname{Diff}^\mu_{\mathrm{Fuchs}}(X)$ *be elliptic with respect to the weight* $\delta$*,* $Q_0$*,* $Q_1 \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$*, and* $P_0 = \mathcal{P}^\delta(Q_0; A)$*,* $P_1 = \mathcal{P}^\delta(Q_1; A)$*. Then, for some* $\vartheta > 0$*,* $P_0 = P_1$ *up to the conormal order* $\delta + \vartheta$ *if* $Q_0 = Q_1$ *up to the conormal order* $\delta - \mu + \vartheta$*.*

*Proof.* This follows from $P_0 = \mathcal{Q}^\delta(Q_0; (\mathfrak{S}^{-\mu})^{-1})$, $P_1 = \mathcal{Q}^\delta(Q_1; (\mathfrak{S}^{-\mu})^{-1})$, where $\mathfrak{S}^\mu = \{\sigma^{\mu-j}_M(A)(z);\, j \in \mathbb{N}\} \in \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$. □

*Remark* 2.54. Combined with Theorem 2.31, Theorem 2.51 shows that each solution $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ to the equation $Au = f \in C^\infty_{\mathcal{O}}(X)$, where $A \in \operatorname{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is elliptic with respect to the weight $\delta$, can be written over finite weight intervals and modulo the corresponding flat class as a finite sum of functions of the form (2.19), where the $\Phi$ are taken from a characteristic basis of the linear subspace of $\mathcal{E}^\delta(Y)$ representing $\mathcal{P}^\delta(\mathcal{O}; A)$. If $\Phi(p) = (\phi_0, \ldots, \phi_{m-1})$ for such a vector $\Phi$, where $p = \gamma(\Phi)$, then we say, in slight abuse of notation, that $A$ admits an asymptotic series starting with the term $t^{-p} \log^{m-1} t\, \phi_0$. Since this is then the most singular term (when $\gamma(\Phi)$ is highest possible), if it coefficient can be shown to vanish, then the whole series must vanish up to the next appearance of a starting term for another asymptotic series.

## 2.A REVIEW OF THE GOHBERG-SIGAL THEORY

Here, results from WITT [19, 21] are recalled.
In fact, we provide the local theory of asymptotic types. This theory actually appears as coordinate-free version of part of GOHBERG-SIGAL's theory [4] of the inversion of meromorphic, operator-valued functions at a point. At the same time, it serves as the guiding example for considerations carried out in Section 2.2. Indeed, all what we have to do there is to replace the algebra $\mathcal{M}^{\mathrm{fin}}_p(E)$ and its group of invertible elements, $\mathcal{M}^{\mathrm{nor}}_p(E)$, see Definition 2.64 below, by the quasi-filtered algebra $\bigcup_{\mu\in\mathbb{R}} \operatorname{Symb}^\mu_M(Y)$ and its group of elliptic elements, $\bigcup_{\mu\in\mathbb{R}} \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$. For a dictionary of notation used here and in Section 2.2, see Figure 2.

### 2.A.1 ASYMPTOTIC ALGEBRAS

We first introduce the concept of an asymptotic algebra. Actually, we restrict ourselves to a special kind of asymptotic algebras, namely to so-called (asymptotic) symbol algebras, see Definition 2.57, which is sufficient for the

applications we have in mind. For a more general treatment, see WITT [19], and also Remark 2.30.
For $\mathfrak{F}$ being a linear space, $\mathfrak{F}_0$ being its linear subspace, let $\mathrm{Lat}(\mathfrak{F}/\mathfrak{F}_0)$ denote the lattice of all linear subspaces of the quotient space $\mathfrak{F}/\mathfrak{F}_0$.

DEFINITION 2.55. An asymptotic algebra is a quintuple $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$, where $\mathfrak{M}$ is a unital algebra, $\mathfrak{F}$ is a linear space, $\mathfrak{F}_0$ is its linear subspace, $p$ is a faithful representation of $\mathfrak{M}$ on $\mathfrak{F}$, and $\mathfrak{J}$ is an l.a.t. (lattice of asymptotic types) for the quadruple $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0)$. The latter means that $\mathfrak{J}$ is a sub-lattice of $\mathrm{Lat}(\mathfrak{F}/\mathfrak{F}_0)$ such that the following conditions are met:
(a) $\mathcal{O} \in \mathfrak{J}$ (where $\mathcal{O} = \mathfrak{F}_0/\mathfrak{F}_0$ is the empty asymptotic type);
(b) for each $F \in \mathfrak{M}$ and all $J \in \mathfrak{J}$, there is a $K \in \mathfrak{J}$ such that $J^F \subseteq K$ (where $J^F = (p(F)\pi^{-1}(J) + \mathfrak{F}_0)/\mathfrak{F}_0$ is the push-forward of $J \in \mathrm{Lat}(\mathfrak{F}/\mathfrak{F}_0)$ under the action by $F \in \mathfrak{M}$. Here, $\pi \colon \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_0$ is the canonical projection);
(c) $\bigcap_{\iota \in \mathcal{I}} J_\iota \in \mathfrak{J}$ for each non-empty family $\{J_\iota\}_{\iota \in \mathcal{I}} \subset \mathfrak{J}$.
The elements of $\mathfrak{J}$ are called asymptotic types.

In general, the subspace $\mathfrak{F}_0$ is not left invariant under the action by elements $F \in \mathfrak{M}$. In a sense, an asymptotic types measure this deviation. When meaning asymptotic types (in place of linear subspaces), we shall write $\preccurlyeq$, $\vee$, and $\wedge$ in $\mathfrak{J}$ to designate $\subseteq$, $+$, and $\cap$, respectively; thus, emphasizing the order structure on $\mathfrak{J}$. Also the representation $p$ enables us to identify $\mathfrak{M}$ with a unital subalgebra of $L(\mathfrak{F})$ (= all linear operators acting on $\mathfrak{F}$) and then to write $F$ instead of $p(F)$.

*Remark* 2.56. Property (c) already forces every non-empty subset $\mathcal{S} \subseteq \mathfrak{J}$ to possess a meet (= greatest lower bound) $\bigwedge \mathcal{S} = \bigcap_{J \in \mathcal{S}} J$ and every bounded subset $\mathcal{T} \subseteq \mathfrak{J}$ to possess a join (= least upper bound) $\bigvee \mathcal{T} = \bigwedge \{K;\, K \supseteq J \text{ for all } J \in \mathcal{T}\}$.

Given the quadruple $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0)$, there is a minimal sub-lattice of $\mathrm{Lat}(\mathfrak{F}/\mathfrak{F}_0)$, denoted by $\mathfrak{J}_0 = \mathfrak{J}_{0,\mathfrak{M}}$, satisfying (a), (c) of the previous definition and which is such that $J^F \in \mathfrak{J}$ holds whenever $F \in \mathfrak{M}$, $J \in \mathfrak{J}$.

DEFINITION 2.57. An asymptotic algebra $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is called a symbol algebra if $\mathfrak{J} = \mathfrak{J}_0$ and if it is reduced, i.e., $S \in \mathfrak{M}$ and $S(\mathfrak{F}) \subseteq \pi^{-1}(J)$ for some $J \in \mathfrak{J}$ imply $S = 0$.

The benefit of a symbol algebra is that each elliptic operator $F \in \mathfrak{M}$, i.e., each operator $F \in \mathfrak{M}$ for which there are $G, G' \in \mathfrak{M}$ such that $(FG - 1)(\mathfrak{F}) \subseteq \pi^{-1}(J)$, $(G'F - 1)(\mathfrak{F}) \subseteq \pi^{-1}(J')$ for certain $J, J' \in \mathfrak{J}$ is already invertible, with $G = G'$ being its inverse. Note that more general asymptotic algebras also occuring in singular analysis can often be reduced to symbol algebras, see WITT [19].
Let $\mathfrak{M}^{-1}$ denote the group of invertible elements of $\mathfrak{M}$ and $\mathfrak{A} = \{F \in \mathfrak{M};\, F\mathfrak{F}_0 \subseteq \mathfrak{F}_0\}$; $\mathfrak{A}$ is a subalgebra of $\mathfrak{M}$.

*Example* 2.58. (a) $\mathfrak{M} = \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$, $\mathfrak{F} = \mathcal{M}_p(E)$, $\mathfrak{F}_0 = \mathcal{A}_p(E)$, where $E$ is a Banach space. This example will be thoroughly discussed starting with

Section 2.A.2. We have that $\mathfrak{F}/\mathfrak{F}_0 \cong E^\infty$ is the space of all finite sequences in $E$, $\mathfrak{M}^{-1} = \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, and $\mathfrak{A} = \mathcal{A}_p(\mathcal{L}(E))$. The lattice $\mathfrak{J} = \mathfrak{J}_0$ is characterized in Definition 2.69.
(b) $\mathfrak{M} = \bigcup_{\mu\in\mathbb{R}} \mathrm{Symb}_M^\mu(Y)$, $\mathfrak{F}/\mathfrak{F}_0 = \mathcal{E}^\delta(Y)$ for some $\delta \in \mathbb{R}$, see (2.13), (2.14), and the l.a.t. $\mathfrak{J} = \mathfrak{J}_0$ is given in Definition 2.18.
(c) $\mathfrak{M} = \mathcal{C}^0(X, (\delta, \delta))$ for some $\delta \in \mathbb{R}$, see Appendix 2.B, $\mathfrak{F} = C_{\mathrm{as}}^{\infty,\delta}(X) = \bigcup_{P\in\underline{\mathrm{As}}^\delta(Y)} C_P^\infty(X)$ is the space of all smooth functions on $X^\circ$ obeying complete conormal asymptotic expansions as $x \to \partial X$ of conormal order larger than $\delta$, and $\mathfrak{F}_0 = C_{\mathcal{O}}^\infty(X)$ is the space of all smooth functions on $X$ vanishing to the infinite order on $\partial X$. By Borel's summation theorem, $\mathfrak{F}/\mathfrak{F}_0$ is the space of all formal complete conormal asymptotic expansions as $x \to \partial X$ of conormal order larger than $\delta$, and for the l.a.t. $\mathfrak{J}$ we can take the same $\mathfrak{J}$ as in (b) after having identified (in local coordinates) $C_{\mathrm{as}}^{\infty,\delta}(X)/C_{\mathcal{O}}^\infty(X) = \bigcup_{P\in\underline{\mathrm{As}}^\delta(Y)} C_P^\infty(X)/C_{\mathcal{O}}^\infty(X)$ with the space $\mathcal{E}^\delta(Y)$.

In this paper, we focus on (b) in the previous example and motivate the considerations by (a) in the same example. (a), (b) both provide symbol algebras; (b) in a slightly more general setting, see Remark 2.30.
The space $L_F$ introduced in the next definition measures the amount of asymptotics annihilated by $F \in \mathfrak{M}$.

DEFINITION 2.59. For $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ being an asymptotic algebra and $F \in \mathfrak{M}$, we set $L_F = (F^{-1}(\mathfrak{F}_0) + \mathfrak{F}_0)/\mathfrak{F}_0 \in \mathrm{Lat}(\mathfrak{F}/\mathfrak{F}_0)$.

That means that $L_F$ is the largest subspace of $\mathfrak{F}/\mathfrak{F}_0$ for which $(L_F)^F = \mathcal{O}^F$. Note that the space $L_F$ is defined without reference to the lattice $\mathfrak{J}$.
There are three rules which allow to manipulate asymptotic types in an effective manner.

LEMMA 2.60. *Let $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ be an asymptotic algebra.*
(a) *If $F \in \mathfrak{M}^{-1}$, then $L_F = \mathcal{O}^{F^{-1}}$.*
(b) *For $F, G \in \mathfrak{M}$, and $J \in \mathfrak{J}$, $\left(J^F\right)^G = J^{GF} \vee \mathcal{O}^G$.*
(c) *For $F \in \mathfrak{M}$, $G \in \mathfrak{M}^{-1}$, $(L_F)^G = L_{FG^{-1}} \vee L_{G^{-1}}$.*

*Proof.* The proofs of (a) to (c) are straightforward. ☐

For the rest of this section, we shall assume that $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is a symbol algebra. The major result in this context admitting in its consequence to operate on asymptotic types is stated next.

PROPOSITION 2.61. *Let $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ be a symbol algebra. Then, for $F \in \mathfrak{M}^{-1}$, $L_F$ is an asymptotic type. Furthermore, there is an order-preserving bijection*

$$\{J \in \mathfrak{J};\, J \succcurlyeq L_F\} \to \{K \in \mathfrak{J};\, K \succcurlyeq L_{F^{-1}}\},\quad J \mapsto J^F, \tag{2.40}$$

*with its inverse given by $K \mapsto K^{F^{-1}}$.*

*Proof.* By (a) of the previous lemma, $L_F = \mathcal{O}^{F^{-1}} \in \mathfrak{J}$. Moreover, (b) implies $\left(J^F\right)^{F^{-1}} = J \vee L_F$ and $\left(K^{F^{-1}}\right)^F = K \vee L_{F^{-1}}$ for any $J, K \in \mathfrak{J}$ yielding (2.40). ☐

From Lemma 2.60 and Proposition 2.61, we have the following useful consequence.

Proposition 2.62. *For $F, G \in \mathfrak{M}^{-1} \cap \mathfrak{A}$, $FG^{-1}, GF^{-1} \in \mathfrak{A}$ if and only if $L_F = L_G$.*

*Proof.* We show that, for $F \in \mathfrak{M}$, $G \in \mathfrak{M}^{-1} \cap \mathfrak{A}$, $L_F \preccurlyeq L_G$ if and only if $L_{FG^{-1}} = \mathcal{O}$. If, in addition, $F \in \mathfrak{M}^{-1}$, then this latter condition is equivalent to $GF^{-1} \in \mathfrak{A}$.
In fact, by Lemma 2.60 (c),

$$L_F \vee L_G = \left(L_{FG^{-1}}\right)^{G^{-1}}.$$

Thus $L_F \preccurlyeq L_G$ if and only if $L_G = \left(L_{FG^{-1}}\right)^{G^{-1}}$ and this holds, by the foregoing proposition, if and only if $L_{FG^{-1}} \preccurlyeq L_{G^{-1}} = \mathcal{O}$. ☐

*Remark* 2.63. For $(\mathfrak{M}, p, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ being a symbol algebra and $J \in \mathfrak{J}$, let $\mathfrak{F}_J$ be the space of all $u \in \mathfrak{F}$ having asymptotics of type $J$, i.e., $\mathfrak{F}_J = \pi^{-1}(J)$. Then $\mathfrak{F}_{\mathcal{O}} = \mathfrak{F}_0$ and $F\colon \mathfrak{F}_J \to \mathfrak{F}_{J^F}$ for $F \in \mathfrak{M}$, $J \in \mathfrak{J}$. Moreover, if $u \in \mathfrak{F}$, $Fu \in \mathfrak{F}_K$ for some $K \in \mathfrak{J}$, where $F \in \mathfrak{M}^{-1}$ then $u \in \mathfrak{F}_{K^{F^{-1}}}$, and Proposition 2.61 states that $F$ is an isomorphism from $\mathfrak{F}_J$ onto $\mathfrak{F}_{J^F}$ if $J \succcurlyeq L_F$.
Furthermore, if $J \in \mathfrak{J}$ is such that $J = \bigvee_{\iota \in \mathcal{I}} L_{G_\iota}$, for some bounded family $\{G_\iota\}_{\iota \in \mathcal{I}} \subset \mathfrak{M}^{-1} \cap \mathfrak{A}$, then $\mathfrak{F}_J = \sum_{\iota \in \mathcal{I}} \{u \in \mathfrak{F};\, G_\iota u \in \mathfrak{F}_0\}$. From Proposition 2.62 we recover that this characterization is actually independent of the choice of the family $\{G_\iota\}_{\iota \in \mathcal{I}}$. This — via the construction of suitable $G_\iota$ — is the way employed in the definition of function spaces with asymptotics in Sections 2.3, 2.4.

### 2.A.2 Finitely meromorphic functions

Now, we turn to a version of Gohberg-Sigal's theory [4]. See Remark 2.75 for a comparison.
Let $E$ be a Banach space. We shall consider the $m$-fold product $E^m$ for $m \in \mathbb{N}$, where $E^0 = \{0\}$. We identify $E^m$ with a subspace of $E^{m+1}$ via the map

$$E^m \to E^{m+1},\ \ (\phi_0, \ldots, \phi_{m-1}) \mapsto (0, \phi_0, \ldots, \phi_{m-1}).$$

Further, we set

$$E^\infty = \bigcup_{m \in \mathbb{N}} E^m. \tag{2.41}$$

Thus, $E^\infty$ is the linear space of all finite sequences in $E$, where sequences $(\phi_0, \ldots, \phi_{m-1})$ and $(\underbrace{0, \ldots, 0}_{h \text{ times}}, \phi_0, \ldots, \phi_{m-1})$ for $h \in \mathbb{N}$ are identified.

On $E^\infty$, we define the right shift operator $T$ by

$$T\colon E^\infty \to E^\infty,\ \ (\phi_0,\dots,\phi_{m-2},\phi_{m-1}) \mapsto (0,\phi_0,\dots,\phi_{m-2}). \tag{2.42}$$

By $\mathcal{M}_p(E)$ for $p \in \mathbb{C}$ we shall denote the space of all germs of $E$-valued meromorphic functions at $p$. Moreover, $\mathcal{A}_p(E)$ is the space of all germs of $E$-valued analytic functions at $p$.

DEFINITION 2.64. (a) $\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$ is the space of all germs of $\mathcal{L}(E)$-valued, finitely meromorphic functions at $p$, i.e., the space of all $F \in \mathcal{M}_p(\mathcal{L}(E))$ such that

$$F(z) = \frac{F_0}{(z-p)^\nu} + \frac{F_1}{(z-p)^{\nu-1}} + \cdots + \frac{F_{\nu-1}}{z-p} + F_\nu + \sum_{j\ge 1} F_{\nu+j}\,(z-p)^j \tag{2.43}$$

with finite-rank operators $F_0,\ F_1,\dots,\ F_{\nu-1} \in \mathcal{L}(E)$.
(b) $\mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ is the space of all germs of $\mathcal{L}(E)$-valued, normally meromorphic functions at $p$, i.e., the space of all $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$ such that $F(z) \in \mathcal{L}(E)$ is invertible for $z$ close to $p$, $z \neq p$, and $F_\nu \in \mathcal{L}(E)$ in the representation (2.43) is a Fredholm operator (then necessarily of index 0).

PROPOSITION 2.65 (BLEHER [1]). *$\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$ is an algebra and $\mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ is its group of invertible elements.*

### 2.A.3 LOCAL ASYMPTOTIC TYPES

We set $\mathfrak{M} = \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$, $\mathfrak{F} = \mathcal{M}_p(E)$, and $\mathfrak{F}_0 = \mathcal{A}_p(E)$ as in Example 2.58 (a) and identify $\mathcal{M}_p(E)/\mathcal{A}_p(E) \cong E^\infty$.

LEMMA 2.66. (a) *For $F \in \mathcal{M}_p^{\mathrm{fin}}(E)$, the space $L_F$ consists of all vectors $(\phi_0,\phi_1,\dots,\phi_{m-1}) \in E^\infty$ for which there is a $\tilde{\phi}(z) \in \mathcal{A}_p(E)$ such that*

$$F(z)\left(\frac{\phi_0}{(z-p)^m} + \frac{\phi_1}{(z-p)^{m-1}} + \cdots + \frac{\phi_{m-1}}{z-p} + \tilde{\phi}(z)\right) \in \mathcal{A}_p(E). \tag{2.44}$$

(b) *For $F \in \mathcal{M}_p^{\mathrm{fin}}(E)$, $J \subseteq E^\infty$ being a linear subspace,*

$$J^F = \big\{\big(F_0\phi_0, F_1\phi_0 + F_0\phi_1, \dots, F_{m+\nu-1}\phi_0 + F_{m+\nu-2}\phi_1 + \cdots + F_0\phi_{m+\nu-1}\big);\ (\phi_0,\dots,\phi_{m-1}) \in J,\ \phi_m,\phi_{m+1},\dots,\phi_{m+\nu-1} \in E\big\},$$

*where $F$ is given in the form* (2.43).

*Remark* 2.67. For $F \in \mathcal{A}_p(\mathcal{L}(E))$, i.e., we have $\nu = 0$ in (2.43), the operation $J \mapsto J^F$ is given directly on the level of the space $E^\infty$, namely

$$(\,)^F\colon E^\infty \to E^\infty,$$
$$(\phi_0,\dots,\phi_{m-1}) \mapsto \big(F_0\phi_0, F_1\phi_0 + F_0\phi_1, \dots, F_{m-1}\phi_0 + \cdots + F_0\phi_{m-1}\big), \tag{2.45}$$

and then $J^F = \{\Phi^F;\ \Phi \in J\}$. Moreover, $L_F$ is the kernel of the map $(\,)^F$. The map $(\,)^F$ when restricted to $E^m$ only depends on $F_0, F_1, \dots, F_{m-1}$. Therefore, we will occasionally write $(\,)^F = (\,)^{(F_0,F_1,\dots,F_{m-1})}$ on $E^m$.

LEMMA 2.68. *For $F \in \mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$, $L_F \subseteq E^\infty$ is a linear subspace that is invariant under the action by the right shift operator $T$, i.e., $TL_F \subseteq L_F$. Moreover, if $J \subseteq E^\infty$ is a linear subspace that is invariant under the action by $T$, then $J^F \subseteq E^\infty$ is a linear subspace that is also invariant under the action by $T$. In addition,* $\dim J < \infty$ *implies* $\dim J^F < \infty$.

Hence, we introduce local asymptotic types as follows:

DEFINITION 2.69. An asymptotic type, $J$, on $E$ is a finite-dimensional subspace of $E^\infty$ such that $TJ \subseteq J$. The set of all asymptotic types on $E$ is denoted by $\mathcal{J}(E)$.

Notice that, for $J \in \mathcal{J}(E)$, $J \subseteq E^m$ for some $m \in \mathbb{N}$ and, therefore, the right shift operator $T$ is nilpotent on $J$, since $T^m = 0$ on $E^m$. We will need the following fact from linear algebra.

LEMMA 2.70. *Let $J$ be a finite-dimensional linear space and $T\colon J \to J$ be a nilpotent linear operator. Then there are $\Phi_1, \dots, \Phi_e \in J$ such that*

$$\Phi_1, T\Phi_1, \dots, T^{m_1-1}\Phi_1, \dots, \Phi_e, T\Phi_e, \dots, T^{m_e-1}\Phi_e, \tag{2.46}$$

*where $m_j \in \mathbb{N}$, $m_j \geq 1$, is a linear basis of $J$, while $T^{m_j}\Phi_j = 0$ for $1 \leq j \leq e$. Furthermore, the numbers $m_1, \dots, m_e$ are uniquely determined up to permutation.*

*Proof.* Choose a linear basis in $J$ for which the associated matrix to $T$ is in Jordan form. □

Hence, the numbers $m_1, \dots, m_e$ appear as the sizes of Jordan blocks; $\dim J = m_1 + \cdots + m_e$. The tuple $(m_1, \dots, m_e)$ is called the characteristic of $J$ (with respect to the nilpotent operator $T$), $e$ is called the length of its characteristic, and $\Phi_1, \dots, \Phi_e$ is said to be a characteristic basis of $J$ (of characteristic $(m_1, \dots, m_e)$) or simply an $(m_1, \dots, m_e)$-basis of $J$. Note that the space $\{0\}$ has empty characteristic with length $e = 0$.
If $\Phi_1, \dots, \Phi_e$ is a characteristic basis of some linear subspace of $J$ that is invariant under the action by $T$, then this subspace will also be written as $\langle \Phi_1, \dots, \Phi_e \rangle$. More generally, for $\Phi_1, \dots, \Phi_e \in J$, by $\langle \Phi_1, \dots, \Phi_e \rangle$ we denote the minimal linear subspace of $J$ containing $\Phi_1, \dots, \Phi_e$ and being invariant under the action by $T$.
Notice that the next lemma there hints at an effective method of finding the characteristic and a characteristic basis upon constructing a suitable basis of $\ker T$.

LEMMA 2.71. *Let $J$ be a finite-dimensional linear space and $T\colon J \to J$ be a nilpotent linear operator as in the previous lemma. Suppose that the characteristic of $J$ equals $(m_1, \dots, m_e)$. Then $\Phi_1, \dots, \Phi_e \in J$ is an $(m_1, \dots, m_e)$-basis of $J$ if and onJy if $T^{m_1-1}\Phi_1, \dots, T^{m_e-1}\Phi_e$ is a linear basis of* $\ker T$.

We make a general remark concerning the appearance of asymptotic types.

*Remark* 2.72. Let $J \in \mathcal{J}(E)$ have characteristic $(m_1, m_2, \ldots, m_e)$. Let

$$\Phi_1 = \big(\phi_0^{(1)}, \ldots, \phi_{m_1-1}^{(1)}\big), \ldots, \Phi_e = \big(\phi_0^{(e)}, \ldots, \phi_{m_e-1}^{(e)}\big) \tag{2.47}$$

be an $(m_1, \ldots, m_e)$-basis of $J$. The vectors $\phi_0^{(1)}, \ldots, \phi_0^{(e)}$ are linearly independent, since $T^{m_j-1}\Phi_j = \big(\phi_0^{(j)}\big)$ for $1 \le j \le e$. We set

$$L_l = \operatorname{span}\{\phi_k^{(j)};\, m_j - k \ge \bar{m} - l + 1\}$$

for $1 \le l \le \bar{m} = \max_{1 \le j \le e} m_j$. The spaces $L_l$ are actually independent of the choice of $\Phi_1, \ldots, \Phi_e$, since $L_l$ is the projection of $J$ on the $l$th component of $E^{\bar{m}}$. In particular, $J \subseteq L_1 \times \cdots \times L_{\bar{m}} \subseteq E^{\bar{m}}$. In the latter relation, however, equality, in general, fails to hold.
Equality holds, i.e., we have

$$J = L_1 \times L_2 \times \cdots \times L_{\bar{m}} \tag{2.48}$$

if and only if, for all $1 \le j \le e$, $1 \le k \le m_j - 1$, $\phi_k^{(j)} \in \operatorname{span}\{\phi_0^{(h)};\, m_h \ge m_j - k\}$. Again, this is a condition that is independent of the choice of $\Phi_1, \ldots, \Phi_e$. This condition, in turn, is fulfilled if and only if $L_l = \operatorname{span}\{\phi_0^{(j)};\, m_j \ge \bar{m} - l + 1\}$ for $1 \le l \le \bar{m}$.

For $J \in \mathcal{J}(E)$, let $\ell(J)$ denote the length of its characteristic. Note that the linear independence of $\phi_0^{(1)}, \ldots, \phi_0^{(e)}$ implies that $\ell(J) = e \le \dim E$.

*Example* 2.73. For $\dim E = 1$, an asymptotic type is uniquely determined by a number $m \in \mathbb{N}$. Namely, $E = \mathbb{C}$ in this case and, if the asymptotic type $J$ has characteristic $(m)$, then $J = \mathbb{C}^m$. Moreover, for $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(\mathbb{C})) = \mathcal{M}_p(\mathbb{C})$, $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(\mathbb{C}))$ exactly means that $F(z) \not\equiv 0$. Then $L_F = \mathbb{C}^m$ if and only if $F(z)$ has a zero of order $m$ at $z = p$.

We immediately obtain:

LEMMA 2.74. (a) *For each tuple $(m_1, \ldots, m_e)$ with $e \le \dim E$, there is a $J \in \mathcal{J}(E)$ having characteristic $(m_1, \ldots, m_e)$.*
(b) *For $J$, $K \in \mathcal{J}(E)$,*

$$\ell(J \wedge K) \ge (\ell(J) + \ell(K) - \dim E)^+ . \tag{2.49}$$

*Furthermore, if $(m_1, \ldots, m_e)$, $(n_1, \ldots, n_f)$ are tuples with $e \le \dim E$, $f \le \dim E$, then there are $J$, $K \in \mathcal{J}(E)$ having $(m_1, \ldots, m_e)$ and $(n_1, \ldots, n_f)$, respectively, as characteristics such that in* (2.49) *equality holds.*

Moreover, for $J$, $K \in \mathcal{J}(E)$, it is easily seen that $\ell(J \wedge K) + \ell(J \vee K) = \ell(J) + \ell(K)$. Thus

$$\ell(J \vee K) \le \min\{\ell(J) + \ell(K), \dim E\},$$

and equality holds if and only if equality holds in (2.49).

*Remark* 2.75. In GOHBERG-SIGAL's paper [4], $p \in \mathbb{C}$ was called a characteristic value of $F \in \mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ if $\dim L_F > 0$. If additionally $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$, then we find an $(m_1, \dots, m_e)$-basis $\Phi_1, \dots, \Phi_e$ of $L_F$ as in (2.46). If $\Phi_1, \dots, \Phi_e$ are given by (2.47), then

$$\phi_0^{(1)}, \phi_1^{(1)}, \dots, \phi_{m_1-1}^{(1)}, \phi_0^{(2)}, \phi_1^{(2)}, \dots, \phi_{m_2-1}^{(2)}, \dots, \phi_0^{(e)}, \phi_1^{(e)}, \dots, \phi_{m_e-1}^{(e)},$$

was called a canonical system of eigenvectors and associated vectors for $F(z)$ at $z = p$. The numbers $m_j$ for $1 \le j \le e$ were called partial null multiplicities and $m_1 + m_2 + \cdots + m_e \,(= \dim L_F)$ was called the null multiplicity of the characteristic value $p$ of $F(z)$.

### 2.A.4 Singularity structure of inverses

Let $E'$ be the topological dual to $E$. For $\Phi \in E^\infty$, $\Psi \in (E')^\infty$, $\Phi = (\phi_0, \phi_1, \dots, \phi_{m-1})$, $\Psi = (\psi_0, \psi_1, \dots, \psi_{m-1})$, we define

$$\begin{aligned}&(\Phi \otimes \Psi)[z-p] \\ &\quad = \frac{\phi_0 \otimes \psi_0}{(z-p)^m} + \frac{\phi_1 \otimes \psi_0 + \phi_0 \otimes \psi_1}{(z-p)^{m-1}} + \cdots + \frac{\phi_{m-1} \otimes \psi_0 + \cdots + \phi_0 \otimes \psi_{m-1}}{z-p},\end{aligned}$$

where, for $\phi \in E$, $\psi \in E'$, $\phi \otimes \psi \in \mathcal{L}(E)$ denotes the rank-one operator $h \mapsto \langle \psi, h \rangle \phi$, $h \in E$, with $\langle\ ,\ \rangle$ being the dual pairing between $E$, $E'$.

PROPOSITION 2.76. *Let $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$, $J \in \mathcal{J}(E)$. Moreover, let $\Phi_1, \dots, \Phi_e \in J$ be an $(m_1, \dots, m_e)$-basis of $J$. Then $L_F \subseteq J$ if and only if there are $\Psi_1, \dots, \Psi_e \in (E')^\infty$ such that $T^{m_j}\Psi_j = 0$ for $1 \le j \le e$ and the principal part of the Laurent expansion of $F^{-1}(z)$ at $z = p$ equals*

$$\sum_{j=1}^{e} (\Phi_j \otimes \Psi_j)[z-p]. \tag{2.50}$$

*In that case, $\Psi_1, \dots, \Psi_e \in (E')^\infty$ are uniquely determined. Furthermore, $L_F = J$ if and only if $\Psi_1, T\Psi_1, \dots, T^{m_1-1}\Psi_1, \dots, \Psi_e, T\Psi_e, \dots, T^{m_e-1}\Psi_e$ are linearly independent.*

*Remark* 2.77. From the results of GOHBERG–SIGAL [4, Theorem 7.1] it follows that, for $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$, there is an $(m_1, \dots, m_e)$-basis $\Phi_1, \dots, \Phi_e$ of $L_F$ and an $(m_1, \dots, m_e)$-basis $\Psi_1, \dots, \Psi_e$ of $L_{F'}$ such that the principal part of the Laurent expansion of $F^{-1}(z)$ at $z = p$ has the form (2.50). In that respect, Proposition 2.76 is more general.

### 2.A.5 Characterization as a symbol algebra

LEMMA 2.78. *Let $J, K, L \in \mathcal{J}(E)$. Then there is an $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ satisfying $L_F = J$ and $K^F = L$ if and only if $K/(J \wedge K)$ and $L$*

*have the same characteristic. Here, the right shift operator induces a nilpotent operator on* $K/(J \wedge K)$*, since both* $K$ *and* $J \wedge K$ *are invariant under the action by* $T$.

From this lemma, which can be derived in a constructive way, we obtain the first main result of this appendix:

PROPOSITION 2.79. *Let* $J, K \in \mathcal{J}(E)$*. Then there is an* $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ *such that* $L_F = J$ *and* $L_{F^{-1}} = K$ *if and only if*

$$\ell(J) + \ell(K) \leq \dim E. \tag{2.51}$$

(This means there is no condition if $\dim E = \infty$, while in the other extreme case, $\dim E = 1$, we must have either $J = \mathcal{O}$ or $K = \mathcal{O}$ or both.)

*Sketch of proof.* Necessity of the condition $\ell(J) + \ell(K) \leq \dim E$ follows from the representation (2.50), which holds both for $F$, $F^{-1} \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, and from $F(z)F^{-1}(z) = 1$.
Now, let $\ell(J) + \ell(K) \leq \dim E$. We choose $J_1, K_1 \in \mathcal{J}(E)$ such that $J_1$ has the same characteristic as $J$, $K_1$ has the same characteristic as $K$, and $J_1 \wedge K_1 = \mathcal{O}$ holds. This is possible by Lemma 2.74 (b). By Lemma 2.78, there are $F, G \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ such that $L_F = J_1$, ${K_1}^F = K$, $L_G = K_1$, and ${J_1}^G = J$. By Lemma 2.60 (c), we eventually obtain

$$L_{FG^{-1}} = (L_F)^G = {J_1}^G = J, \;\; L_{GF^{-1}} = (L_G)^F = {K_1}^F = K,$$

showing that $FG^{-1} \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ is as required. □

As an immediate consequence, we obtain:

COROLLARY 2.80. We have

$$\mathcal{J}(E) = \big\{L_F;\, F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))\big\}.$$

In conclusion, we obtain the second main result of this appendix:

PROPOSITION 2.81. $\big(\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E)), \mathcal{M}_p(E), \mathcal{A}_p(E), \mathcal{J}(E)\big)$ *is a symbol algebra in the sense of* Definition 2.57.

*Remark* 2.82. The statement that $L_F$ characterizes the amount of asymptotics annihilated by $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$, while $\mathcal{O}^F$ contains the asymptotics produced by it, has to be read with some care. In fact, Proposition 2.79 shows that, already for $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, $L_F \wedge \mathcal{O}^F \neq \mathcal{O}$ is possible provided that $\dim E > 1$. As a simple example serves $F(z) = 1 + A(z-p)^{-1} \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, where $A \in \mathcal{L}(E)$ is a finite-rank operator and $A^2 = 0$ ($\dim E > 1$ if $A \neq 0$), with inverse $F^{-1}(z) = 1 - A(z-p)^{-1}$. We get

$$L_F = \mathcal{O}^F = AE.$$

Here, asymptotics of type $AE$ are annihilated, while at the same time exactly this sort of asymptotics is produced in a complementary direction.

## 2.B CONE PSEUDODIFFERENTIAL CALCULUS

We briefly introduce Schulze's cone pseudodifferential calculus, cf. SCHULZE [15, 16]. In its simplest form, the cone calculus is the union of all spaces $\mathcal{C}^\mu(X,(\gamma,\delta))$, where $\mu \in \mathbb{R}$ is the order of operators and $\gamma, \delta \in \mathbb{R}$ are conormal orders involved. Formally, the space $\mathcal{C}^\mu(X,(\gamma,\delta))$ consists of all operators $A$,

$$A = \omega(t)t^{-\gamma+\delta}\operatorname{op}_M^{(n+1)/2-\gamma}(h(t,z))\omega(t) + (1-\omega)A_\psi(1-\omega) + G, \tag{2.52}$$

where $\omega(t)$ is a cut-off function as above, $h(t,z) \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^\mu_{\text{as}}(Y))$, $A_\psi \in L^\mu_{\text{cl}}(X^\circ)$, and there is an asymptotic type $R \in \underline{\text{As}}^\delta(Y)$ depending on $A$ such that $G\colon \mathcal{H}^{s,\gamma}(X) \to C^\infty_R(X)$ for all $s \in \mathbb{R}$. The Mellin pseudodifferential operator $\operatorname{op}_M^{(n+1)/2-\gamma}(h(t,z))$, i.e., the definition of this operator is based on the Mellin instead of the Fourier transformation, is defined via the oscillatory integral

$$\operatorname{op}_M^{(n+1)/2-\gamma}(h(t,z))u(t) = \frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\gamma}}\int_0^\infty \left(\frac{t}{t'}\right)^{-z} h(t,z)u(t')\,\frac{dt'}{t'} \tag{2.53}$$

for $u \in C_0^\infty(\mathbb{R}_+; C^\infty(Y))$. The difficulties with this definition are plain. On the one hand, one has to explain what the space $C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^\mu_{\text{as}}(Y))$ is — meromorphic functions depending smoothly on an additional parameter lead to difficulties with varying poles — on the other hand $h(t,z)$ may have poles on the line $\Gamma_{(n+1)/2-\gamma}$. A possibility to overcome these difficulties is to consider the Taylor expansion $\sum_{j\ge 0} t^j h_j(z)$, where $h_j(z) = \frac{1}{j!}\frac{\partial^j h}{\partial t^j}(0,z) \in \mathcal{M}^\mu_{\text{as}}(Y)$, of the function $h(t,z)$ at $t=0$ instead of the function itself and then, at least at first, the formal series $\sum_{j\ge 0} t^j \operatorname{op}_M^{(n+1)/2-\gamma+\rho_j}(h_j(z))$, where $0 \le \rho_j \le j$ and the line $\Gamma_{(n+1)/2-\gamma+\rho_j}$ is free of poles of $h_j(z)$. Thereby, going over from the line of integration $\Gamma_{(n+1)/2-\gamma}$ to the line of integration $\Gamma_{(n+1)/2-\gamma+\rho_j}$, in (2.53), one makes a mistake which can be brought to the Green part, $G$. The functional-analytic details can be found in the references cited above.

The functions $h_j(z)$ for $j \in \mathbb{N}$ are uniquely determined by the operator $A$ (in the chosen splitting of coordinates $(t,y)$ close to $\partial X$) and form the sequence $\{\sigma_M^{\gamma-\delta-j}(A)(z);\, j \in \mathbb{N}\}$ of conormal symbols of $A$, where $\sigma_M^{\gamma-\delta-j}(A)(z) = h_j(z)$. This complete conormal symbol determines the manner in which asymptotics are mapped by $A$ (besides the Green part $G$ which also has an influence). For this reason, the behaviour of conormal symbols under compositions is quite essential for our purposes. But before stating it, we summarize further properties of the cone pseudodifferential calculus.

Let $C_{M+G}(X,(\gamma,\delta))$ (smoothing Mellin+Green) for $\gamma,\delta$ running through $\mathbb{R}$ form the sub-calculus of all smoothing operators, i.e., all operators $A$ of the form (2.52), where $h(t,z) \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^{-\infty}_{\text{as}}(Y))$ and $A_\psi \in L^{-\infty}(X^\circ)$. Moreover, let $\mathcal{C}_G(X,(\gamma,\delta)) \subset C_{M+G}(X,(\gamma,\delta))$ denote the space of Green operators which is given by additionally requiring $h_j(z) = 0$ for all $j$. The latter operators are entirely characterized by their mapping properties. Thus, definition (2.52) in that case reduces to the third summand, $G$.

Next, we discuss the symbolic structure of operators in $\mathcal{C}^\mu(X,(\gamma,\delta))$. Since $\mathcal{C}^\mu(X,(\gamma,\delta)) \subset L^\mu_{\mathrm{cl}}(X^\circ)$, we have the principal pseudodifferential symbol $\sigma^\mu_\psi(A)$, where $\sigma^\mu_\psi(A)(t,y,\tau,\eta) = t^{-\gamma+\delta}\tilde{\sigma}^\mu_\psi(A)(t,y,t\tau,\eta)$ for $\tilde{\sigma}^\mu_\psi(A)(t,y,\tilde{\tau},\eta) \in S^{(\mu)}_{\mathrm{cl}}([0,1)\times V\times\mathbb{R}^{n+1}_{\tilde{\tau},\eta})$, $V$ being a coordinate neighbourhood of $Y$. Between $\tilde{\sigma}^\mu_\psi(A)(t,y,\tilde{\tau},\eta)$ and $\sigma^{\gamma-\delta}_M(A)(z)$ the compatibility condition

$$\tilde{\sigma}^\mu_\psi(A)(t,y,\tilde{\tau},\eta)\big|_{t=0} = \sigma^\mu_\psi\big(\sigma^{\gamma-\delta}_M(A)(z)\big|_{z=(n+1)/2-\gamma-i\tilde{\tau}}\big)$$

holds. Here, $\sigma^\mu_\psi$ on the right-hand side denotes the parameter-dependent principal pseudodifferential symbol of $\sigma^{\gamma-\delta}_M(A)(z) \in L^\mu_{\mathrm{cl}}(Y;\Gamma_{(n+1)/2-\gamma})$.
For $A \in \mathcal{C}^\mu(X,(\gamma,\delta))$, we have:
(a) For each $P \in \underline{\mathrm{As}}^\gamma(Y)$, there is a $Q \in \underline{\mathrm{As}}^\delta(Y)$ such that

$$A\colon \mathcal{H}^{s,\gamma}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta}_{Q,\vartheta}(X) \tag{2.54}$$

for all $s\in\mathbb{R}$, $\vartheta \geq 0$. In particular, $A\colon \mathcal{H}^{s,\gamma}(X) \to \mathcal{H}^{s-\mu,\delta}(X)$.
(b) $A\colon \mathcal{H}^{s,\gamma}(X) \to \mathcal{H}^{s-\mu,\delta}(X)$ is a Fredholm operator for some $s\in\mathbb{R}$ (and then for all $s\in\mathbb{R}$) if and only if $A$ is elliptic in the sense that $A$ is elliptic as an operator in $L^\mu_{\mathrm{cl}}(X^\circ)$, $\tilde{\sigma}^\mu_\psi(A)(t,y,\tilde{\tau},\eta) \neq 0$ for all $(t,y,\tilde{\tau},\eta) \in \tilde{T}^*([0,1)\times V)$, and $\sigma^{\gamma-\delta}_M(A)(z)$ is invertible in $L^\mu_{\mathrm{cl}}(Y)$ for all $z\in\mathbb{C}$, $\mathrm{Re}\, z = (n+1)/2-\gamma$. Note that the second condition already forces $\sigma^{\gamma-\delta}_M(A)(z) \in L^\mu_{\mathrm{cl}}(Y;\Gamma_{(n+1)/2-\gamma})$ to be parameter-dependent elliptic.
(c) If $A$ is elliptic, then a parametrix $P \in \mathcal{C}^{-\mu}(X,(\delta,\gamma))$ exists, i.e., an operator $P$ such that $PA - I \in C_G(X,(\gamma,\gamma))$, $AP - I \in C_G(X,(\delta,\delta))$.
Furthermore, if $A \in \mathcal{C}^\mu(X,(\gamma_0,\delta))$ and $B \in \mathcal{C}^\nu(X,(\gamma,\gamma_0))$, then $AB \in \mathcal{C}^{\mu+\nu}(X,(\gamma,\delta))$ and the behaviour of the complete conormal symbol is regulated by the Mellin translation product (2.11), i.e.,

$$\sigma^{\gamma-\delta-l}_M(AB)(z) = \sum_{j+k=l} \sigma^{\gamma_0-\delta-j}_M(A)(z+\gamma-\gamma_0-k)\sigma^{\gamma-\gamma_0-k}_M(B)(z) \tag{2.55}$$

for all $l\in\mathbb{N}$.

## 3 Applications to semilinear equations

In this section, Theorem 1.1 is proved. To this end, multiplicatively closable and multiplicatively closed asymptotic types are investigated in Section 3.1. This allows the derivation of results concerning the action of nonlinear superposition operators on cone Sobolev spaces with asymptotics. In Section 3.2, the general scheme for establishing results of the type of Theorem 1.1 is then established. This scheme is specified to multiplicatively closable asymptotic types in Section 3.3; thus, completing the proof in this way.

### 3.1 Multiplicatively closed asymptotic types

Here, we investigate multiplicative properties of asymptotic types and the behaviour of cone Sobolev spaces $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ under the action by nonlinear superposition.

*Notation* 3.1. In connection with pointwise multiplication, it is useful to employ the following notation:

$$H^s_{P,\vartheta}(X) = \begin{cases} \mathcal{H}^{s,\delta}_{P,\delta_P-\delta+\vartheta}(X) & \text{if } \vartheta \geq 0, \\ \mathcal{H}^{s,\delta_P+\vartheta}(X) & \text{otherwise,} \end{cases}$$

where $P \in \underline{\mathrm{As}}^\delta(Y)$, $P \neq \mathcal{O}$, and $\delta < \delta_P$ in the first line. (Proposition 2.47 (b) yields that this definition is independent of the choice of $\delta$.) Thus, starting from $\delta_P$, the conormal order is improved by $\vartheta$ upon allowing asymptotics of type $P$. Similarly for $H^s_{P,\vartheta-0}(X)$.
Furthermore, we write $\{\vartheta\}$ if we mean either $\vartheta$ or $\vartheta - 0$. For instance, $\{\vartheta\} \geq 0$ means $\vartheta \geq 0$ if $\{\vartheta\} = \vartheta$ and $\vartheta > 0$ if $\{\vartheta\} = \vartheta - 0$.

#### 3.1.1 Multiplication of asymptotic types

The result admitting nonlinear superposition for function spaces with asymptotics is stated next.

Lemma 3.2. *Given $P \in \underline{\mathrm{As}}(Y)$, $Q \in \underline{\mathrm{As}}(Y)$, there is a minimal asymptotic type, $P \circ Q \in \underline{\mathrm{As}}(Y)$, such that*

$$C^\infty_P(X) \times C^\infty_Q(X) \to C^\infty_{P\circ Q}(X), \quad (u, v) \mapsto uv. \tag{3.1}$$

*Proof.* Suppose that the asymptotic types $P$, $Q$ are represented by subspaces $J \subset \mathcal{E}_V(Y)$ and $K \subset \mathcal{E}_W(Y)$, respectively, for suitable $V, W \in \mathcal{C}$. Then the asymptotic type $P \circ Q$ is carried by the set $V + W$, and it is represented by the linear subspace of $\mathcal{E}_{V+W}(Y)$ consisting of all $\Theta \in \mathcal{E}_{V+W}(Y)$ for which there are $\Phi \in J$, $\Psi \in K$ such that $\Theta(r) = \sum_{\substack{p+q=r,\\ p\in V,\, q\in W}} \Phi(p) \times \Psi(q)$ holds for all $r \in V + W$.
Here,

$$\begin{aligned} \Phi \times \Psi = \Bigg( &\binom{m+n}{m}\phi_0\psi_0, \binom{m+n-1}{m}\phi_0\psi_1 + \binom{m+n-1}{m-1}\phi_1\psi_0, \\ &\binom{m+n-2}{m}\phi_0\psi_2 + \binom{m+n-2}{m-1}\phi_1\psi_1 + \binom{m+n-2}{m-2}\phi_0\psi_2, \\ &\dots, \binom{1}{1}\phi_{m-1}\psi_n + \binom{1}{0}\phi_m\psi_{n-1}, \binom{0}{0}\phi_m\psi_n \Bigg) \end{aligned}$$

for $\Phi = (\phi_0, \phi_1, \dots, \phi_m)$, $\Psi = (\psi_0, \psi_1, \dots, \psi_n) \in [C^\infty(Y)]^\infty$, see (2.19). Note that $T(\Phi \times \Psi) = (T\Phi) \times \Psi + \Phi \times (T\Psi)$ and, for $\Phi_{V_p}(Y)$, $\Psi_{V_q}(Y)$, we have $\Phi \times \Psi \in \mathcal{E}_{V_{p+q}}(Y)$ showing that the linear subspace of $\mathcal{E}_{V+W}(Y)$ described above actually represents an asymptotic type. $\square$

The multiplication of asymptotic types possesses a unit, denoted by $\mathbf{1}$, that is represented by the space $\operatorname{span}\{(1)\} \subset \mathcal{E}_{\{0\}}(Y)$, where 1 is the function on $Y$ identically 1.

Definition 3.3. An asymptotic type $Q \in \underline{\mathrm{As}}(Y)$ is called multiplicatively closed if $Q \circ Q = Q$. An asymptotic type $Q \in \underline{\mathrm{As}}(Y)$ is called multiplicatively closable if it is dominated by a multiplicatively closed asymptotic type. In this case, the minimal multiplicatively closed asymptotic type dominating $Q$ is called the multiplicative closure of $Q$ and is denoted by $\widetilde{Q}$.

From the proof of Lemma 3.2,

$$\delta_{P \circ Q} \geq \delta_P + \delta_Q - (n+1)/2, \tag{3.2}$$

where equality holds if $P = Q$ or if $\dim Y = 0$. Especially, $\delta_Q = (n+1)/2$ if $Q$ is multiplicatively closed and $\delta_Q \geq (n+1)/2$ if $Q$ is multiplicatively closable. Furthermore, it is also seen $Q \succcurlyeq \mathbf{1}$ for any multiplicatively closed asymptotic type $Q$, see also Lemma 3.5 below.

### 3.1.2 The class $\underline{\mathrm{As}}^\sharp(Y)$ of multiplicatively closable asymptotic types

We study the class of asymptotic types that belong to bounded functions. It turns out that this class is intimately connected to the multiplication of asymptotic types.

Definition 3.4. (a) The class $\underline{\mathrm{As}}^b(Y)$ of bounded asymptotic types consists of all asymptotic types $Q \in \underline{\mathrm{As}}(Y)$ for which $\delta_Q \geq (n+1)/2$. Equivalently, a bounded asymptotic type $Q$ is represented by a subspace $J \subset \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}$, where $V \subset \{z \in \mathbb{C};\ \operatorname{Re} z \leq 0\}$.
(b) The class $\underline{\mathrm{As}}^\sharp(Y)$ consists of all bounded asymptotic types $Q$ represented by a subspace $J \subset \mathcal{E}(Y)$ such that $J_0 \subseteq \operatorname{span}\{(1)\}$ and $J_p = \{0\}$ for $\operatorname{Re} p = 0$, $p \neq 0$.

Lemma 3.5. *For $Q \in \underline{\mathrm{As}}(Y)$, the following conditions are equivalent:*
(a) *$Q$ is multiplicatively closable*;
(b) *the join $\bigvee_{k \geq 1} Q^k$ does exist, where $Q^k = \underbrace{Q \circ \cdots \circ Q}_{k \text{ times}}$ is the $k$-fold product*;
(c) $Q \in \underline{\mathrm{As}}^\sharp(Y)$.
*In case* (a) *to* (c) *are fulfilled, we have $\widetilde{Q} = \mathbf{1} \vee \bigvee_{k \geq 1} Q^k$.*

*Proof.* (a) and (b) are obviously equivalent. Moreover, (c) implies (b).
It remains to show that (a) also implies (c). If $Q$ is multiplicatively closable, then $\widetilde{Q}$ exists and $\delta_{\widetilde{Q}} = (n+1)/2$. In particular, $\widetilde{Q} \in \underline{\mathrm{As}}^b(Y)$. Let $\widetilde{Q}$ be represented by $J \subset \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}$, $V \subset \{z \in \mathbb{C};\ \operatorname{Re} z \leq 0\}$. Suppose that $\phi \in J_p$ for $p \in \mathbb{C}$, $\operatorname{Re} p = 0$, where $\phi \neq 0$. We immediately get $\phi^l \in J_{lp}$ for any $l \in \mathbb{N}$, $l \geq 1$. For $p \neq 0$, we obtain the contradiction $\{lp;\ l \in \mathbb{N}\} \subseteq V \in \mathcal{C}$. For $p = 0$ and $\phi$ not being constant, we obtain a contradiction to the fact that $\dim J_0 < \infty$. Thus, $\widetilde{Q} \in \underline{\mathrm{As}}^\sharp(Y)$ and, therefore, $Q \in \underline{\mathrm{As}}^\sharp(Y)$. □

LEMMA 3.6. *For each $Q \in \underline{\mathrm{As}}(Y)$, there are asymptotic types $Q^b \in \underline{\mathrm{As}}^b(Y)$ and $Q^\sharp \in \underline{\mathrm{As}}^\sharp(Y)$ which are maximal possessing the properties*

$$Q^b \preccurlyeq Q \text{ and } Q^\sharp \preccurlyeq Q, \tag{3.3}$$

*respectively. We have $Q^\sharp \preccurlyeq Q^b$.*

*Proof.* The proof is straightforward. □

### 3.1.3 NONLINEAR SUPERPOSITION

We investigate expressions like $F(x, v(x))$, where $F(x,\nu) \in C_R^\infty(X \times \mathbb{R})$ for some $R \in \underline{\mathrm{As}}(Y)$ and $v \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ with $s \geq 0$, $\vartheta > 0$, and $Q \in \underline{\mathrm{As}}^\sharp(Y)$. For later reference, we recall the following facts:

PROPOSITION 3.7. (a) *For $s > (n+1)/2$, $0 \leq s' \leq s$, $\gamma, \delta \in \mathbb{R}$, pointwise multiplication induces a bilinear continuous map*

$$\mathcal{H}^{s,\gamma}(X) \times \mathcal{H}^{s',\delta}(X) \to \mathcal{H}^{s',\gamma+\delta-(n+1)/2}(X). \tag{3.4}$$

(b) *For $s, \delta \in \mathbb{R}$, $\mathcal{H}^{s,\delta}(X) \subset L^\infty(X)$ if and only if $s > (n+1)/2$, $\delta \geq (n+1)/2$.*
(c) *For $s \geq 0$, $\gamma, \delta \geq (n+1)/2$, pointwise multiplication induces a bilinear continuous map*

$$\big(\mathcal{H}^{s,\gamma}(X) \cap L^\infty(X)\big) \times \big(\mathcal{H}^{s,\delta}(X) \cap L^\infty(X)\big) \to \mathcal{H}^{s,\gamma+\delta-(n+1)/2}(X) \cap L^\infty(X).$$

(d) *For $s \geq 0$, $\delta \in \mathbb{R}$, $p \in \mathbb{C}$, $c(y) \in C^\infty(Y)$, the multiplication operator*

$$\omega(t)t^{-p}c(y) \colon \mathcal{H}^{s,\delta}(X) \to \mathcal{H}^{s,\delta-\operatorname{Re} p}(X),$$

*where $\omega(t)$ is a cut-off function, is continuous.*
(e) *For $s \geq 0$, $v_1, \ldots, v_K \in \big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)$, and $F \in C^\infty(\mathbb{R}^K)$, we have*

$$F(v_1, \ldots, v_K) \in \big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X). \tag{3.5}$$

*The map $\Big(\big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)\Big)^K \to \big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)$ induced by (3.5) is continuous and sends bounded sets to bounded sets.*

*Proof.* A proof of (3.4) in case $s' = s$ was supplied by WITT [18, Lemma 2.7] using a result of DAUGE [2]. The other proofs are similar. □

*Remark* 3.8. Property (d) fails if logarithms appear and has to be replaced by

$$\omega(t)t^{-p}\log^k t\, c(y) \colon \mathcal{H}^{s,\delta}(X) \to \mathcal{H}^{s,\delta-\operatorname{Re} p-0}(X)$$

is continuous when $k \in \mathbb{N}$, $k \geq 1$.

First, Lemma 3.2 is sharpened:

PROPOSITION 3.9. *For $s > (n+1)/2$, $0 \le s' \le s$, $\vartheta > 0$, and $P, Q \in \underline{\mathrm{As}}(Y)$, pointwise multiplication induces a bilinear continuous map*

$$H^{s}_{P,\vartheta-0}(X) \times H^{s'}_{Q,\vartheta-0}(X) \to H^{s'}_{P\circ Q,\vartheta-0}(X). \tag{3.6}$$

*Proof.* Let $u \in H^{s}_{P,\vartheta-0}(X)$, $v \in H^{s'}_{Q,\vartheta-0}(X)$. Then $u = u_0 + u_1$, $v = v_0 + v_1$, where

$$u_0 = \sum_{j=0}^{M}\sum_{k=0}^{m_j} \omega(t) t^{-p_j} \log^k t\, c_{jk}(y), \quad v_0 = \sum_{j'=0}^{N}\sum_{k'=0}^{n_{j'}} \omega(t) t^{-q_{j'}} \log^{k'} t\, d_{j'k'}(y), \tag{3.7}$$

$\omega(t)$ is a cut-off function, the sequences $\{(p_j, m_j, c_{jk})\}$, $\{(q_{j'}, n_{j'}, d_{j'k'})\}$ are given by the asymptotic types $P$ and $Q$, respectively, see Definition 2.19, and $M$, $N$ are chosen so that $u_1 \in \mathcal{H}^{s,\delta_P+\vartheta-0}(X)$, $v_1 \in \mathcal{H}^{s',\delta_Q+\vartheta-0}(X)$. Since $u_0 \in \mathcal{H}^{\infty,\delta_P-0}(X)$, $v_0 \in \mathcal{H}^{\infty,\delta_Q-0}(X)$, we obtain

$$uv = u_0v_0 + u_1v_0 + u_0v_1 + u_1v_1,$$

where $u_1v_0 + u_0v_1 + u_1v_1 \in \mathcal{H}^{s',\delta_{P\circ Q}+\vartheta-0}(X)$ by (3.4) and

$$u_0v_0 = \sum_{j,j'=0}^{M,N}\sum_{k,k'=0}^{m_j,n_{j'}} \omega^2(t) t^{-(p_j+q_{j'})} \log^{k+k'} t\, c_{jk}(y) d_{j'k'}(y) \in H^{\infty}_{P\circ Q,\vartheta-0}(X),$$

for $\omega^2(t)$ is a cut-off function and the sequence

$$\Big\{\Big(r_{j''}, o_{j''}, \sum_{p_j+q_{j'}=r_{j''}}\ \sum_{k+k'=k''} c_{jk}d_{j'k'}\Big)\Big\},$$

where $o_{j''} = \max\{m_j + n_{j'};\, p_j + q_{j'} = r_{j''}\}$, is associated with an asymptotic type that is equal to $P \circ Q$ up to the conormal order $\delta_{P\circ Q} + \vartheta - 0$. This immediately gives $uv \in H^{s'}_{P\circ Q,\vartheta-0}(X)$. □

The significance of the class $\underline{\mathrm{As}}^b(Y)$ is uncovered by the next result.

PROPOSITION 3.10. *For $s \ge 0$, $\delta \in \mathbb{R}$, $\delta + \{\vartheta\} \ge (n+1)/2$, and $Q \in \underline{\mathrm{As}}^\delta(Y)$,*

$$\mathcal{H}^{s,\delta}_{Q,\{\vartheta\}}(X) \cap L^\infty(X) = \mathcal{H}^{s,\delta}_{Q^b,\{\vartheta\}}(X) \cap L^\infty(X). \tag{3.8}$$

*Proof.* Let $u \in \mathcal{H}^{s,\delta}_{Q,\{\vartheta\}}(X) \cap L^\infty(X)$ and write

$$u(x) = \sum_{j=0}^{M}\sum_{k=0}^{m_j} \omega(t) t^{-p_j} \log^k t\, c_{jk}(y) + u_1(x),$$

where the sequences $\{(p_j, m_j, c_{jk})\}$ is given by the asymptotic type $Q$ and $M$ is chosen so that $u_1 \in \mathcal{H}^{s,(n+1)/2-0}(X)$. Since $u \in L^\infty(X) \subset \mathcal{H}^{0,(n+1)/2-0}(X)$, we get that $\sum_{j=0}^{M}\sum_{k=0}^{m_j} \omega(t) t^{-p_j} \log^k t\, c_{jk}(y) \in \mathcal{H}^{0,(n+1)/2-0}(X)$ which implies $c_{jk}(y) = 0$ for $\operatorname{Re} p_j > 0$. Thus $u \in \mathcal{H}^{s,\delta}_{Q^b,\vartheta}(X)$. □

LEMMA 3.11. *For $s \geq 0$, $\vartheta > 0$, and $P$, $Q \in \underline{\mathrm{As}}^b(Y)$, pointwise multiplication induces a bilinear continuous map*

$$\left(H^s_{P,\vartheta-0}(X) \cap L^\infty(X)\right) \times \left(H^s_{Q,\vartheta-0}(X) \cap L^\infty(X)\right) \to H^s_{P\circ Q,\vartheta-0}(X) \cap L^\infty(X).$$

*Proof.* Represent $u = u_0 + u_1 \in H^s_{P,\vartheta-0}(X) \cap L^\infty(X)$, $v = v_0 + v_1 \in H^s_{Q,\vartheta-0}(X) \cap L^\infty(X)$ as in the proof of Proposition 3.9. Since $u_0$, $v_0 \in L^\infty(X)$ due to the assumption $P$, $Q \in \underline{\mathrm{As}}^b(Y)$, we get that $u_1 \in \mathcal{H}^{s,\delta_P+\vartheta-0}(X) \cap L^\infty(X)$, $v_1 \in \mathcal{H}^{s,\delta_Q+\vartheta-0}(X) \cap L^\infty(X)$ and, therefore, $u_1v_0 + u_0v_1 + u_1v_1 \in \mathcal{H}^{s,\delta_{P\circ Q}+\vartheta-0}(X) \cap L^\infty(X)$ in view of Proposition 3.7 (c). The assertion follows. □

A more precise statement is possible if $P$, $Q \in \underline{\mathrm{As}}^\sharp(Y)$.

LEMMA 3.12. *For $s \geq 0$, $\vartheta \geq 0$, and $P$, $Q \in \underline{\mathrm{As}}^\sharp(Y)$ satisfying $P \not\succcurlyeq \mathbf{1}$, $Q \not\succcurlyeq \mathbf{1}$, pointwise multiplication induces a bilinear continuous map*

$$\left(H^s_{P,\vartheta}(X) \cap L^\infty(X)\right) \times \left(H^s_{Q,\vartheta}(X) \cap L^\infty(X)\right) \to H^s_{P\circ Q,\vartheta}(X) \cap L^\infty(X). \quad (3.9)$$

*Especially, for $s \geq 0$, $\vartheta \geq 0$, and $Q \in \underline{\mathrm{As}}^\sharp(Y)$ being multiplicatively closed, $H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ is an algebra under pointwise multiplication.*

*Proof.* We may assume that $\vartheta > 0$. Write $u = u_0 + u_1 \in H^s_{P,\vartheta}(X) \cap L^\infty(X)$, $v = v_0 + v_1 \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ as in the proof of Proposition 3.9, where $u_0 = u_{00} + u_{01}$, $v_0 = v_{00} + v_{01}$, $u_{00} = \omega(t)c_{00}$, and $v_{00} = \omega(t)d_{00}$ with $c_{00}$, $d_{00}$ being constants and in the expressions for $u_{01}$, $v_{01}$ only appear exponents with $\mathrm{Re}\, p_j < 0$ and $\mathrm{Re}\, q_{j'} < 0$, respectively. Then

$$u_1v_{01} + u_{01}v_1 + u_1v_1 \in \mathcal{H}^{s,(n+1)/2+\vartheta+0}(X),$$

$u_{00}v \in H^s_{Q,\vartheta}(X) \subseteq H^s_{P\circ Q,\vartheta}(X)$, $uv_{00} \in H^s_{P,\vartheta}(X) \subseteq H^s_{P\circ Q,\vartheta}(X)$, and

$$u_{01}v_{01} \in H^\infty_{P\circ Q,\vartheta+0}(X),$$

which proves the assertion. □

The fact which has actually been used in the last proof is that Proposition 3.7 (d) applies to the function $\omega(t)1$ ($p = 0$, $c(y) \equiv 1$). This is also used in part (b) of the next result.

LEMMA 3.13. (a) *Let $s \geq 0$, $\vartheta > 0$, and $R$, $Q \in \underline{\mathrm{As}}(Y)$. Then pointwise multiplication induces a continuous map*

$$C^\infty_R(X) \times H^s_{Q,\vartheta-0}(X) \to H^s_{R\circ Q,\vartheta-0}(X). \quad (3.10)$$

(b) *If, in addition, $R \in \underline{\mathrm{As}}(Y)$ is so that the multiplicities of its highest singular values are one, i.e., $J_r \subseteq [C^\infty(Y)]^1$ for each $r \in V$, $\mathrm{Re}\, r = (n+1)/2 - \delta_R$, where $J \subset \mathcal{E}_V(Y)$ represents $R$, then pointwise multiplication induces a continuous map*

$$C^\infty_R(X) \times H^s_{Q,\vartheta}(X) \to H^s_{R\circ Q,\vartheta}(X).$$

*Proof.* (a) is immediate from Proposition 3.9. To get (b), we argue as in the proof of Lemma 3.12. □

PROPOSITION 3.14. *Let $s \geq 0$, $\vartheta \geq 0$, and $Q \in \underline{\mathrm{As}}^\sharp(Y)$ be multiplicatively closed. Then $v_1, \ldots, v_K \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ and $F \in C^\infty(\mathbb{R}^K)$ implies that*

$$F(v_1, \ldots, v_K) \in H^s_{Q,\vartheta}(X) \cap L^\infty(X). \tag{3.11}$$

*Proof.* We are allowed to assume that $\vartheta > 0$. Then $v \in H^s_{Q,\vartheta}(X)$ implies that $v\big|_{\partial X}$ is a constant, where $v\big|_{\partial X}$ means the factor in front of $t^0$ in the asymptotic expansion (1.2) (with $u$ replaced with $v$) of $v$ as $t \to +0$. Let $\beta_J = v_J\big|_{\partial X}$ for $1 \leq J \leq K$ be these constants. Using Taylor's formula, we obtain

$$\begin{aligned} F(v_1, \ldots, v_K) &= \sum_{|\alpha|<N} \frac{1}{\alpha!} (\partial^\alpha F)(\beta_1, \ldots, \beta_K)(v_1 - \beta_1)^{\alpha_1} \ldots (v_K - \beta_K)^{\alpha_K} \\ &+ N \sum_{|\alpha|=N} \int_0^1 \frac{(1-\sigma)^{N-1}}{\alpha!} (\partial^\alpha F)(\beta_1 + \sigma(v_1 - \beta_1), \ldots, \beta_K + \sigma(v_K - \beta_K))\, d\sigma \\ &\times (v_1 - \beta_1)^{\alpha_1} \ldots (v_K - \beta_K)^{\alpha_K}. \end{aligned} \tag{3.12}$$

By Lemma 3.12, $(v_1 - \beta_1)^{\alpha_1} \ldots (v_K - \beta_K)^{\alpha_K} \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ for any $\alpha \in \mathbb{N}^K$, thus the first summand on the right-hand side of (3.12) belongs to $H^s_{Q,\vartheta}(X) \cap L^\infty(X)$. On the other hand, choosing $N$ sufficiently large, we can arrange that $(v_1 - \beta_1)^{\alpha_1} \ldots (v_K - \beta_K)^{\alpha_K} \in \mathcal{H}^{s,(n+1)/2+\vartheta}(X) \cap L^\infty(X)$ for $|\alpha| \geq N$, since $v_J - \beta_J \in \mathcal{H}^{s,(n+1)/2+0}(X) \cap L^\infty(X)$ for $1 \leq J \leq K$. By (3.5), $\{(\partial^\alpha F)(\beta_1 + \sigma(v_1 - \beta_1), \ldots, \beta_K + \sigma(v_K - \beta_K))\, d\sigma;\ 0 \leq \sigma \leq 1\}$ is a bounded set in $\big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)$ for any $\alpha \in \mathbb{N}^K$. This shows that the second summand on the right-hand side of (3.12) belongs to $\mathcal{H}^{s,(n+1)/2+\vartheta}(X) \cap L^\infty(X)$. □

PROPOSITION 3.15. (a) *Let $s \geq 0$, $\vartheta > 0$. Further let $Q \in \underline{\mathrm{As}}^\sharp(Y)$ be multiplicatively closed and $R \in \underline{\mathrm{As}}(Y)$. Then $v_1, \ldots, v_K \in H^s_{Q,\vartheta-0}(X) \cap L^\infty(X)$ and $F \in C^\infty_R(X \times \mathbb{R}^K)$ implies that*

$$F(x, v_1, \ldots, v_K) \in H^s_{R\circ Q,\vartheta-0}(X). \tag{3.13}$$

(b) *If, in addition, $R$ satisfies the assumption of* Lemma 3.13 (b), *then $v_1, \ldots, v_K \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ and $F \in C^\infty_R(X \times \mathbb{R}^K)$ implies that*

$$F(x, v_1, \ldots, v_K) \in H^s_{R\circ Q,\vartheta}(X).$$

*Proof.* We prove (a), (b) is analogous. Since $C^\infty_R(X \times \mathbb{R}^K) = C^\infty_R(X)\hat{\otimes}_\pi C^\infty(\mathbb{R}^K)$, we can write

$$F(x, v) = \sum_{j=0}^{\infty} \alpha_j\, \varphi_j(x)\, F_j(v),$$

where $\{\alpha_j\}_{j=0}^\infty \in l^1$ and $\{\varphi_j\}_{j=0}^\infty \subset C_R^\infty(X)$ and $\{F_j\}_{j=0}^\infty \subset C^\infty(\mathbb{R}^K)$, respectively, are null sequences. By the preceeding proposition,

$$F_j(v_1,\dots,v_K) \to 0 \quad \text{as } j \to \infty \text{ in } H^s_{Q,\vartheta-0}(X).$$

By Lemma 3.13,

$$\varphi_j(x)F_j(v_1,\dots,v_K) \to 0 \quad \text{as } j \to \infty \text{ in } H^s_{R\circ Q,\vartheta-0}(X).$$

Thus

$$F(x,v_1,\dots,v_K) = \sum_{j=0}^\infty \alpha_j\,\varphi_j(x)F_j(v_1,\dots,v_K) \in H^s_{R\circ Q,\vartheta-0}(X),$$

where the sum on the right-hand side is absolutely convergent. □

## 3.2 The bootstrapping argument

We shall consider the equation

$$Au = \Pi(u), \tag{3.14}$$

where $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is an elliptic Fuchsian differential operator. Properties of the nonlinear operator $u \mapsto \Pi(u)$ are discussed below. The method proposed for deriving elliptic regularity for solutions to (3.14) essentially amounts to balancing two asymptotic types; one for the left-hand and the other one for the right-hand side of (3.14).

We shall assume: There are asymptotic types $\bar{P} \in \underline{\mathrm{As}}^\delta(Y)$, $\bar{Q} \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, numbers $a$, $b$, $s_0$, $\vartheta_0 \in \mathbb{R}$ with $a < \mu$, $b < \delta_{\bar{Q}} - \delta_{\bar{P}} + \mu$, $s_0 \geq a^+$, $\delta_{\bar{P}} + \{\vartheta_0\} \geq \delta$, and a subset $\mathcal{U} \subseteq H^{s_0}_{\bar{P},\{\vartheta_0\}}(X)$ such that the following conditions are met:

(A) $A$ is elliptic with respect to the conormal order $\delta$ and $\bar{P} \succcurlyeq \mathcal{P}^\delta(\bar{Q}; A)$, i.e., $u \in \mathcal{H}^{-\infty,\delta}(X)$, $Au \in C^\infty_{\bar{Q}}(X)$ implies $u \in C^\infty_{\bar{P}}(X)$;

(B) for $s \geq s_0$, $\vartheta \geq \vartheta_0$, we have

$$\Pi\colon \mathcal{U} \cap H^s_{\bar{P},\{\vartheta\}}(X) \to H^{s-a}_{\bar{Q},\{\vartheta\}-b}(X).$$

Note that $\{\vartheta_0\} - b + \delta_{\bar{Q}} \geq \delta - \mu$.

Proposition 3.16. *Under the conditions* (A), (B), *each solution* $u \in \mathcal{U} \subseteq H^{s_0}_{\bar{P},\{\vartheta_0\}}(X)$ *to* (3.14) *belongs to the space* $C^\infty_{\bar{P}}(X)$.

*Proof.* We are going to prove by induction on $j$ that

$$u \in H^{s_0+j(\mu-a)}_{\bar{P},\{\vartheta_0\}+j(\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}})}(X) \tag{3.15}$$

holds for all $j \in \mathbb{N}$. Since $\mu-a > 0$, $\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}} > 0$, this implies $u \in C^\infty_{\bar{P}}(X)$. By assumption, (3.15) holds for $j = 0$. Now suppose that (3.15) for some $j$ has already been proven. From (B) we conclude that $\Pi(u) \in H^{s_0+j(\mu-a)-a}_{\bar{Q},\{\vartheta_0\}+j(\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}})-b}(X)$. In view of (A), elliptic regularity gives $u \in H^{s_0+(j+1)(\mu-a)}_{\bar{P},\{\vartheta_0\}+(j+1)(\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}})}(X)$. □

*Example* 3.17. We provide an example for a nonlinearity $\Pi$ satisfying (B). Let $\Pi(u) = K_0(u)/K_1(u)$, where $K_0$, $K_1$ are polynomials of degree $m_0$ and $m_1$, respectively. Let $u \in H^s_{P,\vartheta-0}(X)$, where $s > (n+1)/2$, $\delta_P + \vartheta > (n+1)/2$, and $\vartheta > 0$. Further, we assume that the multiplicities of the highest singular values for $P$ are simple and the coefficient functions for these singular values nowhere vanish on $Y$. Then we have $K_0(u) \in H^s_{P_0,\vartheta-0}(X)$, $K_1(u) \in H^s_{P_1,\vartheta-0}(X)$ for resulting asymptotic types $P_0$, $P_1$. In particular, $P_0$ is dominated by $\mathbf{1} \vee \bigvee_{k=1}^{m_0} P^k$ and $P_1$ is dominated by $\mathbf{1} \vee \bigvee_{k=1}^{m_1} P^k$. Furthermore, it is readily seen that $v \in H^s_{P_1,\vartheta-0}(X)$ and $v \neq 0$ everywhere on $X^\circ$ implies that $1/v \in H^s_{Q_1,\vartheta'-0}(X)$ for some resulting asymptotic type $Q_1$. Hence, we are allowed to set $\bar{P} = P$, $\bar{Q} = P_0 \circ Q_1$, and

$$\mathcal{U} = \{u \in H^s_{P,\vartheta-0}(X);\ K_1(u) \neq 0 \text{ everywhere on } X^\circ\}.$$

Here, the condition $s > (n+1)/2$ can be replaced by $s \geq 0$. Then we additionally need $u \in L^\infty_{\mathrm{loc}}(X^\circ)$.

### 3.3 PROOF OF THE MAIN THEOREM

The main step consists in constructing asymptotic types $\bar{P}$, $\bar{Q}$ so that Proposition 3.16 applies. Thereby, upon choosing $\delta \in \mathbb{R}$ even smaller if necessary, we can assume that

$$\delta \leq \bar{\mu} + (n+1)/2$$

and that $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is elliptic with respect to the conormal order $\delta$. Set $\Delta = \delta_R + (\mu - \bar{\mu}) - (n+1)/2$. By assumption (1.4), $\Delta > 0$.

#### 3.3.1 CONSTRUCTION OF ASYMPTOTIC TYPES $P$, $Q$

First, we construct by induction on $h$ sequences $\{P_h\}_{h=0}^\infty \subset \underline{\mathrm{As}}^\delta(Y)$ and $\{Q_h\}_{h=0}^\infty \subset \underline{\mathrm{As}}^\sharp(Y)$ of asymptotic types as follows: Set $P_0 = \mathcal{P}^\delta(\mathcal{O}; A)$. Suppose that $P_0, \ldots, P_h$ and $Q_0, \ldots, Q_{h-1}$ for some $h$ have already been constructed. Then

$$Q_h = \big(\mathcal{Q}(P_h; B_1, \ldots, B_K)^\sharp\big)^\sim, \tag{3.16}$$

$$P_{h+1} = \mathcal{P}^\delta(R \circ Q_h; A). \tag{3.17}$$

LEMMA 3.18. *For each $h \geq 0$,*

$$P_h = P_{h+1} \text{ up to the conormal order } \delta_R + \mu + h\Delta - 0, \tag{3.18}$$

$$Q_{h+1} = Q_h \text{ up to the conormal order } \delta_R + (\mu - \bar{\mu}) + h\Delta - 0. \tag{3.19}$$

*In particular, the joins $P = \bigvee_{h=0}^\infty P_h$ and $Q = \bigvee_{h=0}^\infty Q_h$ exist.*

*Proof.* We set $Q_{-1} = \mathcal{O}$ and proceed by induction on $h$. (3.19) holds for $h = -1$, since $Q_0 \in \underline{\mathrm{As}}^\sharp(Y)$ and, therefore, $Q_0 = \mathcal{O}$ up to the conormal order $(n+1)/2 - 0$.

Suppose that $Q_h = Q_{h-1}$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) + (h-1)\Delta - 0$ for some $h \geq 0$ has already been proved. Then $R \circ Q_h = R \circ Q_{h-1}$ up to the conormal order $\delta_R + h\Delta - 0$ and $P_{h+1} = P_h$ up to the conormal order $\delta_R + \mu + h\Delta - 0$, since $P_h = \mathcal{P}^\delta(R \circ Q_h; A)$, $P_{h+1} = \mathcal{P}^\delta(R \circ Q_{h+1}; A)$.
Now suppose that $P_h = P_{h+1}$ up to the conormal order $\delta_R + \mu + h\Delta - 0$. We obtain $\mathcal{Q}(P_h; B_1, \dots, B_K) = \mathcal{Q}(P_{h+1}; B_1, \dots, B_K)$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) + h\Delta - 0$ and, therefore, $Q_h = Q_{h+1}$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) + h\Delta - 0$, since $Q_h = \big(\mathcal{Q}(P_h; B_1, \dots, B_K)^\sharp\big)^\sim$, $Q_{h+1} = \big(\mathcal{Q}(P_{h+1}; B_1, \dots, B_K)^\sharp\big)^\sim$.
This completes the inductive proof. □

LEMMA 3.19. *The asymptotic types* $P = \bigvee_{h=0}^\infty P_h \in \underline{\mathrm{As}}^\delta(Y)$, $Q = \bigvee_{h=0}^\infty Q_h \in \underline{\mathrm{As}}^\sharp(Y)$ *satisfy:*
(a) $\mathcal{Q}(P; B_1, \dots, B_K)^b = \mathcal{Q}(P; B_1, \dots, B_K)^\sharp$ *and* $Q = \big(\mathcal{Q}(P; B_1, \dots, B_K)^\sharp\big)^\sim$;
(b) $P = \mathcal{P}^\delta(R \circ Q; A)$;
(c) $Q$ *is multiplicatively closed.*
*Furthermore,* $P, Q$ *are minimal among all asymptotic types in* $\underline{\mathrm{As}}^\delta(Y)$ *and* $\underline{\mathrm{As}}^\sharp(Y)$*, respectively, satisfying* (a) *to* (c).

*Proof.* The assertions immediately follow from the description of the asymptotic types $P_h$, $Q_h$ given in the previous lemma.
Only $\mathcal{Q}(P; B_1, \dots, B_K)^b = \mathcal{Q}(P; B_1, \dots, B_K)^\sharp$ requires an argument. But $P = P_0$ up to the conormal order $\delta_R + \mu - 0$, so we get $\mathcal{Q}(P; B_1, \dots, B_K) = \mathcal{Q}(P_0; B_1, \dots, B_K)$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) - 0 = (n+1)/2 + \Delta - 0 > (n+1)/2$, and $\mathcal{Q}(P_0; B_1, \dots, B_K)^b = \mathcal{Q}(P_0; B_1, \dots, B_K)^\sharp$ is exactly the non-resonance condition (1.5). □

Note that, by the non-resonance condition (1.5) and Proposition 3.10,

$$B_J u \in \mathcal{H}^{s-\bar{\mu}, \delta-\bar{\mu}}_{\mathcal{Q}(P; B_1, \dots, B_K), \vartheta - 0}(X) \cap L^\infty(X) \subseteq \mathcal{H}^{s-\bar{\mu}, \delta-\bar{\mu}}_{\mathcal{Q}(P; B_1, \dots, B_K)^\sharp, \vartheta - 0}(X) \subseteq \mathcal{H}^{s-\bar{\mu}, \delta-\bar{\mu}}_{Q, \vartheta - 0}(X) \quad (3.20)$$

if $u \in \mathcal{H}^{s,\delta}_{P, \vartheta - 0}(X)$, $\delta - \bar{\mu} + \vartheta > (n+1)/2$, and $B_J u \in L^\infty(X)$.

### 3.3.2 END OF THE PROOF OF THEOREM 1.1

Since $B_J u \in L^\infty(X) \subset \mathcal{H}^{0,(n+1)/2-0}(X)$ for all $1 \leq J \leq K$, we have $F(x, B_1 u, \dots, B_K u) \in \mathcal{H}^{0, \delta_R - 0}(X)$ and

$$u \in H^\mu_{P_0, \delta_R + \mu - \delta_P - 0}(X) = H^\mu_{P, \delta_R + \mu - \delta_P - 0}(X)$$

by elliptic regularity.
To conclude the proof of Theorem 1.1, we apply Proposition 3.16 with $\Pi u = F(x, B_1 u, \dots,\, B_K u)$, $\bar{P} = P$, $\bar{Q} = R \circ Q$, where $P \in \underline{\mathrm{As}}^\delta(Y)$, $Q \in \underline{\mathrm{As}}^\sharp(Y)$ have

been constructed in Lemmas 3.18, 3.19, $s_0 = \mu$, $\{\vartheta_0\} = \delta_R + \mu - \delta_P - 0$, $a = \bar{\mu}$, $b = (n+1)/2 - \delta_P + \bar{\mu}$, and

$$\mathcal{U} = \big\{u \in H^{\mu}_{P,\delta_R+\mu-\delta_P-0}(X);\ B_J u \in L^{\infty}(X),\ 1 \le J \le K\big\}. \tag{3.21}$$

Then $a < \mu$, $b < \delta_{R\circ Q} - \delta_P + \mu$ for $\delta_{R\circ Q} = \delta_R$, $\Delta > 0$, and $\delta_P + \vartheta_0 = \delta_R + \mu > \bar{\mu} + (n+1)/2 \ge \delta$, i.e., $\delta_P + \{\vartheta_0\} \ge \delta$. Moreover, condition (A) is fulfilled.
To check condition (B), note that $u \in \mathcal{U} \cap H^s_{P,\vartheta-0}(X)$ for $s \ge \mu$, $\vartheta \ge \delta_R + \mu - \delta_P$ implies

$$F(x, B_1 u, \dots, B_K u) \in H^{s-\bar{\mu}}_{R\circ Q,\delta_P-\bar{\mu}-(n+1)/2+\vartheta-0}(X)$$

by (3.20) and Proposition 3.15.
Thus Proposition 3.16 applies to yield $u \in C^{\infty}_P(X)$.

*Remark* 3.20. From (3.21) it is seen that the asymptotic type $P \in \underline{\mathrm{As}}^{\delta}(Y)$ can be taken smaller, namely instead of $P = \mathcal{P}^{\delta}(R \circ Q; A)$ we can choose the asymptotic type

$$\bigvee \big\{P' \in \underline{\mathrm{As}}^{\delta}(Y);\ P' \preccurlyeq \mathcal{P}^{\delta}(R \circ Q; A),\ \mathcal{Q}(P'; B_1, \dots, B_K) \in \underline{\mathrm{As}}^{\sharp}(Y)\big\}.$$

In concrete problems, the resulting asymptotic type for $u$ can be even smaller, e.g., due to nonlinear interaction caused by the *special structure* of the nonlinearity.

## 3.A An example: The equation $\Delta u = Au^2 + B(x)u$ in three space dimensions

Let $\Omega$ be a bounded, smooth domain in $\mathbb{R}^3$ containing 0. We are going to study singular solutions to the equation

$$\Delta u = Au^2 + B(x)u \ \text{ on } \Omega \setminus \{0\}, \tag{3.22}$$

$$\gamma_0 u = c_0,\ \ u\big|_{\partial\Omega} = \phi, \tag{3.23}$$

where $\gamma_0 u = \lim_{x\to 0} |x| u(x)$, $A \in \mathbb{R}$, and $B \in C^{\infty}(\overline{\Omega})$ is real-valued. Since the quadratic polynomial $Au^2 + B(x)u$ rather than a general nonlinearity $F(x,u)$ enters, we may admit complex-valued solution $u$ to (3.22). In particular, $c_0 \in \mathbb{C}$.

*Remark* 3.21. By results in VÉRON [17], one has to expect that the existence of the limit $\lim_{x\to 0} |x| u(x)$ is typical for solutions $u = u(x)$ to (3.22).

On $\Omega \setminus \{0\}$, we introduce polar coordinates $(t, y) \in \mathbb{R}_+ \times S^2$, $t = |x|$, $y = x/|x|$. We further introduce the function spaces

$$\mathcal{X}^2 = \big\{c_0 t^{-1} + c_{11} \log t + u_0(x);\ c_0, c_{11} \in \mathbb{C},\ u_0 \in H^2(\Omega)\big\},$$
$$\mathcal{Y}^0 = \big\{d_0 t^{-2} + v_0(x);\ d_0 \in \mathbb{C},\ v_0 \in L^2(\Omega)\big\}$$

the definition of which is suggested by formal asymptotic analysis. On the space $\mathcal{X}^2$, we have the trace operators $\gamma_0$, $\gamma_1$, $\gamma_{11}$, where $\gamma_{11} u = \lim_{t\to+0} \big(u(x) - (\gamma_0 u) t^{-1}\big)/\log t$, $\gamma_1 u = \lim_{t\to+0} \big(u(x) - (\gamma_0 u) t^{-1} - (\gamma_{11} u) \log t\big)$.

PROPOSITION 3.22. *Suppose that $B(x) \geq 0$ for all $x \in \overline{\Omega}$. Then, for all $c_0 \in \mathbb{C}$, $\phi \in H^{3/2}(\partial\Omega)$ with $|c_0| + \|\phi\|_{H^{3/2}(\partial\Omega)}$ small enough, the boundary value problem (3.22), (3.23) admits a unique small solution $u \in \mathcal{X}^2$. This solution $u = u(x)$ obeys a complete conormal asymptotic expansion as $x \to 0$ which can successively be calculated. Especially,*

$$c_{11} = Ac_0^2, \tag{3.24}$$

*where $c_{11} = \gamma_{11}u$.*

*Proof.* Let us consider the nonlinear operator

$$\Psi\colon \mathcal{X}^2 \to \mathcal{Y}^0 \times \mathbb{C} \times H^{3/2}(\partial\Omega), \;\; u \mapsto \left(\Delta u - Au^2 - B(x)u, \gamma_0 u, u\big|_{\partial\Omega}\right).$$

It is readily seen that the linearization of $\Psi$ about $u = 0$ is an isomorphism between the indicated spaces. Thus, the existence of a unique small solution $u \in \mathcal{X}^2$ to (3.22), (3.23) is implied by the inverse function theorem. (3.24) likewise follows.
Furthermore, writing this solution in the form $u(x) = c_0t^{-1} + c_{11}\log t + u_0(x)$, where $u_0 \in H^2(\Omega)$, we get that $u_0$ fulfils the equation

$$\begin{aligned} c_1t^{-2} + \Delta u_0 &= A\left(c_0^2t^{-2} + 2c_0c_{11}t^{-1}\log t + c_{11}^2\log^2 t\right) \\ &+ 2A\left(c_0t^{-1} + c_{11}\log t\right)u_0 + Au_0^2 + B(x)\left(c_0t^{-1} + c_{11}\log t\right) + B(x)u_0 \end{aligned} \tag{3.25}$$

which can be brought into the form (1.1) with

$$\begin{aligned} F(x,\nu) = \big(2Ac_0c_{11}t^{-1}\log t + B(x)c_0t^{-1} + Ac_{11}^2\log^2 t + B(x)c_{11}\log t\big) \\ + \left(2Ac_0t^{-1} + 2Ac_{11}\log t + B(x)\right)\nu + A\nu^2, \end{aligned}$$

since $\Delta = t^{-2}\left((-t\partial_t)^2 - (-t\partial_t) + \Delta_{S^2}\right) \in \mathrm{Diff}^2_{\mathrm{Fuchs}}(\Omega \setminus \{0\})$, where $0 \in \Omega$ is considered as a conical singularity with cone base $S^2 = \{x \in \mathbb{R}^3;\, |x| = 1\}$, see Remark 1.2, and $\Delta_{S^2}$ is the Laplace-Beltrami operator on $S^2$. The conditions (1.4), (1.5) are obviously satisfied.
Thus, Theorem 1.1 applies to $u_0 \in H^2(\Omega) \subset L^\infty(\Omega)$ to yield that $u_0$, and, therefore, $u$, obeys a complete conormal asymptotic expansion. □

*Remark* 3.23. (a) Taking for $P$ the asymptotic type in $\underline{\mathrm{As}}^0(S^2)$ which comes out in the calculation of the conormal asymptotic expansion for $u$, i.e., we have $u \in C^\infty_P(\Omega \setminus \{0\})$, and for $Q$ the resulting asymptotic type in $\underline{\mathrm{As}}^{-2}(S^2)$ for the right-hand side of (3.22), we are in a situation in which Proposition 3.16 directly applies without having a boundedness assumption for $u$.
(b) Allowing more general $B \in C^\infty_R(\Omega \setminus \{0\})$ for some $R \in \underline{\mathrm{As}}^{1/2}(S^2)$ (ensuring that the term $Ac_0^2t^{-2}$ dominates on the right-hand side of (3.25)) rather than $B \in C^\infty_T(\Omega \setminus \{0\})$ one can perform the same analysis as before upon replacing the space $H^2(\Omega)$ in the definition of $\mathcal{X}^2$ accordingly.

## REFERENCES

[1] P. M. Bleher, *Operators that depend meromorphically on a parameter*, Vestnik Moskov. Univ. Ser. I Mat. Meh. 24 (1969), 30–36, (In Russian).

[2] M. Dauge, *Elliptic boundary value problems on corner domains*, Lecture Notes in Math., vol. 1341, Springer, Berlin, 1988.

[3] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, Cambridge Math. Textbooks, Cambridge Univ. Press, Cambridge, 1990.

[4] I. C. Gohberg and E. I. Sigal, *An operator generalization of the logarithmic residue theorem and the theorem of Rouché*, Math. USSR Sbornik 13 (1971), no. 4, 603–625.

[5] P. Jeanquartier, *Transformation de Mellin et développements asymptotiques*, Enseign. Math. 25 (1979), no. 3-4, 285–308.

[6] D. Kapanadze, B.-W. Schulze, and I. Witt, *Coordinate invariance of the cone algebra with asymptotics*, Advances in Partial Differential Equations (Parabolicity, Volterra Calculus, and Conical Singularities) (S. Albeverio, M. Demuth, E. Schrohe, and B.-W. Schulze, eds.), Oper. Theory Adv. Appl., vol. 138, Birkhäuser, Basel, 2002, pp. 307–358.

[7] S. Kichenassamy, *The blow-up problem for exponential nonlinearities*, Comm. Partial Differential Equations 21 (1996), 125–162.

[8] ———, *Fuchsian equations in Sobolev spaces and blow-up*, J. Differential Equations 125 (1996), 299–327.

[9] V. A. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Mosc. Math. Soc. 16 (1967), 227–313.

[10] X. Liu, *A cone pseudodifferential calculus on the half-line with respect to conormal asymptotics of a given type*, Ph.D. thesis, Department of Mathematics, Wuhan University, Wuhan, 2000.

[11] X. Liu and I. Witt, *Pseudodifferential calculi on the half-line respecting prescribed asymptotic types*, (Submitted).

[12] ———, *Nonlinear superposition in weighted Hölder-Zygmund spaces with asymptotics*, (In preparation).

[13] R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Res. Notes Math., A K Peters, Wellesley MA, Boston, 1993.

[14] S. Rempel and B.-W. Schulze, *Asymptotics for elliptic mixed boundary problems*, Math. Research, vol. 50, Akademie-Verlag, Berlin, 1989.

[15] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, Stud. Math. Appl., vol. 24, North-Holland, 1991.

[16] ______, *Boundary value problems and singular pseudo-differential operators*, Wiley Ser. Pure Appl. Math., J. Wiley, Chichester, 1998.

[17] L. Véron, *Singularities of solutions of second order quasilinear equations*, Pitman Research Notes Math. Series, vol. 353, Longman, Harlow, Essex, 1996.

[18] I. Witt, *Non-linear hyperbolic equations in domains with conical points*, Math. Research, vol. 84, Akademie Verlag, Berlin, 1995.

[19] ______, *Asymptotic algebras*, Microlocal Analysis and Asymptotic Analysis of PDE, RIMS Kokyuroku, vol. 1211, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 2001, pp. 21–33.

[20] ______, *On the factorization of meromorphic Mellin symbols*, Advances in Partial Differential Equations (Parabolicity, Volterra Calculus, and Conical Singularities) (S. Albeverio, M. Demuth, E. Schrohe, and B.-W. Schulze, eds.), Oper. Theory Adv. Appl., vol. 138, Birkhäuser, Basel, 2002, pp. 279–306.

[21] ______, *Local asymptotic types*, (Submitted).

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