On Surgery in Elliptic Theory

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Abstract

We prove a general theorem on the behavior of the relative index under surgery for a wide class of Fredholm operators, including relative index theorems for elliptic operators due to Gromov–Lawson, Anghel, Teleman, Booß-Bavnbek–Wojciechowski, et al. as special cases. In conjunction with additional conditions (like symmetry conditions), this theorem permits one to compute the analytical index of a given operator. In particular, we obtain new index formulas for elliptic pseudodifferential operators and quantized canonical transformations on manifolds with conical singularities.

Keywords: elliptic operators, index theory, surgery, relative index, manifold with singularities.

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Contents

Introduction

1. Recently there have been a number of results concerning elliptic operators with symmetry conditions on manifolds with singularities. These results in particular include index theorems for general elliptic pseudodifferential operators as well as quantized contact transformations (e.g., see [33, 15, 16, 26, 28]). The main tools for obtaining these results in the cited papers include gluing together two copies of the manifold and extending the operators to the double. Furthermore, the double is a smooth closed manifold, on which the Atiyah–Singer index theory [8] (for the case of pseudodifferential operators) or the Epstein–Melrose–Leichtnam–Nest–Tsygan theory [14, 23] (for the case of quantized canonical transformations) can be applied, and the main problem is to express the difference between twice the index of the original operator and the index of the extended operator on the double via invariants of the conormal symbol. Gluing a copy of the manifold followed by the extension of the operator or, at least, the symbol to the double (or, more generally, to a manifold containing the original manifold as a part) is in fact quite an old idea. Indeed, this technique was successfully applied to the index problem for boundary value problems in early Soviet papers on elliptic theory (e.g., see Agranovich [1] and Dezin [11] as well as Agranovich's review [2], where more detailed references can be found) and was later widely used by many authors. We mention the papers [21, 22, 34] by Hsiung and Stong, who used gluing by an orientation-reversing automorphism of the boundary to obtain the index of the signature operator, as well as the papers [18, 19] by Gilkey and Smith. Naturally, the list can be continued.

2. The fact that the extension to a wider manifold (in particular, to the double) proves to be fruitful in index problems is a consequence of the so-called *index locality principle* or a *relative index theorem.* The present paper just deals with a general interpretation and some specific applications of this principle, and we start from recalling it.

First, let us consider the simplest case. Let M be a compact smooth manifold without boundary divided into two parts M_- and M_+ by a smooth hypersurface S (Fig. 1a), and let D be an elliptic differential operator acting between spaces of sections of some vector bundles E and F over M. We cut away M_- along S and attach another manifold M'_- to M_+, to the effect that $M'_- \bigcup\limits_S M_+$ is again a smooth compact manifold M' without boundary (Fig. 1b). Needless to say, to define a smooth structure on the manifold produced by gluing, we must choose some direct product structure in a collar neighborhood of S in each of the manifolds considered here. Such neighborhoods $U \approx (-1,1) \times S$ are hatched in Fig. 1. Next, we extend the bundles $E|_{M_+}$ and $F|_{M_+}$ to bundles over the entire M' and the operator $D|_{M_+}$ to an elliptic operator D' on the entire M' in spaces of sections of these new bundles. (We assume that this is possible.) The difference

$$\operatorname{ind}(D', D) = \operatorname{ind} D' - \operatorname{ind} D \tag{0.1}$$

of indices of D' and D is called the *relative index* of these operators. Now the following question is of interest to us: Does the relative index (0.1) depend on the structure of D' and D on the set M_+, where they coincide? More precisely, let us perform the same cut-and-paste operation on the right half of the manifold, thus replacing M_+ by some other part $\widetilde{M}_+$. We obtain new manifolds $\widetilde{M}$ and $\widetilde{M}'$ (Fig. 1c, d). Carrying out the corresponding manipulations with bundles and operators, we arrive at new operators $\widetilde{D}$ and $\widetilde{D}'$, which still coincide on the right half of the manifolds, that is, on $\widetilde{M}_+$ in this case, while on the left half they coincide with D and D', respectively. Now the question is, are the relative indices the same for these two pairs of operators? In other words, is it true that

$$\operatorname{ind}(D', D) = \operatorname{ind}(\widetilde{D}', \widetilde{D})? \tag{0.2}$$

The answer is "yes," which trivially follows from the so-called *local index formula* (e.g., see [17]): the index of an elliptic operator D on a closed manifold M is given by the expression

$$\operatorname{ind} D = \int\limits_M \alpha(x), \tag{0.3}$$

where the "local density" $\alpha(x)$ at a point $x \in M$ depends only on the principal symbol $\sigma(D)$ and its derivatives in the fiber over x; it remains to compare the corresponding integrals for the four cases shown in Fig. 1. (This sort of argument was used, for example, in [9, Chap. 25].)

Equation (0.2) is none other than the locality principle, or the relative index theorem, for the above-mentioned simplest case. However, the proof becomes nontrivial if, still

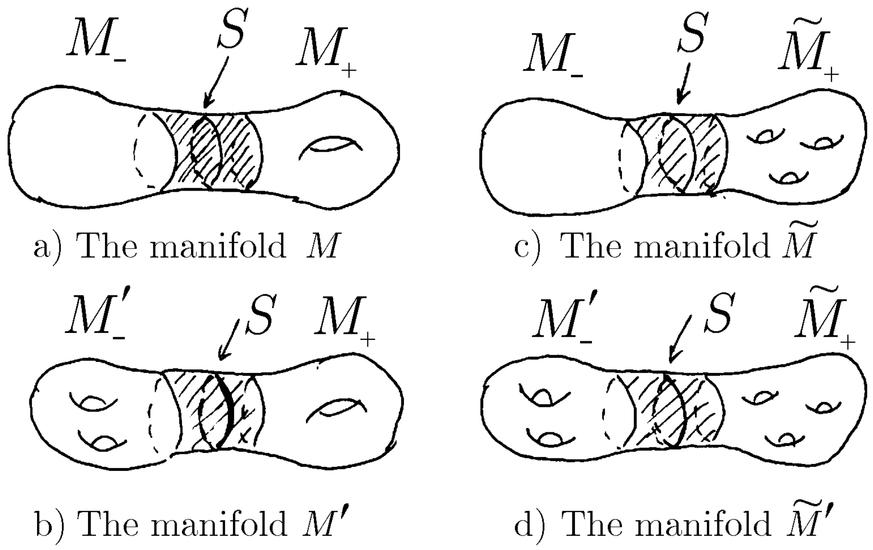

Figure 1: A simple surgery

assuming that S is compact, we reject the assumption that M is a smooth compact manifold without boundary. Then we no longer have formula (0.3), and known proofs of the locality principle (in some particular cases) are based on techniques related to the fundamental solution of the heat equation or the formula

$$\operatorname{ind} D = \operatorname{Trace}(1 - RD) - \operatorname{Trace}(1 - DR), \tag{0.4}$$

where R is an almost-inverse of D modulo trace class operators. A method based on Eq. (0.4) was used, for example, by Gromov and Lawson [20] in the proof of a relative index theorem for Dirac operators on complete noncompact Riemannian manifolds and later by Anghel [5] in a generalization of that theorem to arbitrary essentially self-adjoint supersymmetric Fredholm elliptic first-order operators. The same idea was used by us in the preceding versions [30, 27] of this paper. The heat kernel method was used by Donnelly [12] in the proof of a relative index theorem for the signature operator and also by Bunke [10]. We also mention the book [9] by Booß-Bavnbek and Wojciechowski and the paper [35], where Teleman used a subtle homotopy technique to prove the relative index theorem (0.2) for signature operators on Lipschitz manifolds (for the case in which $M_+ = \widetilde{M}_+$ in our notation). We do not try to give an exhaustive list of related publications. Let us only mention that the well-known Agranovich and Agranovich–Dynin theorems [1, 13, 4] that express the relative index of two boundary value problems with the same boundary conditions and different operators (coinciding on the boundary) or with the same operator but different boundary conditions can essentially be interpreted as a statement of the locality principle for boundary value problems.

3. The technique of attaching a copy of the manifold and passing to the double has also been successfully used in the index theory of pseudodifferential operators on manifolds with singularities. The index theorem for operators with a symmetric conormal symbol was obtained by this method in [33]. Later, the ideas of [33] (combined with the use of an orientation-reversing automorphism of the boundary) were applied in [16] to obtain an index theorem for operators that satisfy a symmetry condition involving this automorphism.[1] This theorem applies to the Cauchy–Riemann operator on a two-dimensional surface as well as (just as in [21, 22]) to the signature operator provided that the base of the cone possesses the above-mentioned automorphism. In [15], an index theorem for two-dimensional surfaces is obtained under a symmetry condition involving an arbitrary diffeomorphism of the base of the cone. Note that in all above-mentioned papers the symmetry condition is imposed on the *entire* conormal symbol.

The aim of the present paper is to prove the locality principle (0.2) in a sufficiently general case so as to ensure that it can be applied to elliptic pseudodifferential operators as well as elliptic Fourier integral operators on manifolds with singularities. Thus, we must introduce a new class of Fredholm operators for which the locality principle holds and which includes both pseudodifferential operators and Fourier integral operators. Note that the class of abstract elliptic operators introduced by Atiyah [7] (which served as a starting point for the development of KK-theory) cannot be used here, since the commutators of Fourier integral operators with operators of multiplication by functions are not compact in general.

That is why we introduce a different class of operators in § 1 and prove a relative index theorem for this class.

Section 2 deals with applications to index theorems for pseudodifferential operators and Fourier integral operators on manifolds with singularities. The main novelty in these theorems compared with those proved earlier in [33, 26, 28] is that the symmetry condition is imposed on the *principal symbol* of the conormal symbol rather than the entire conormal symbol. This is of course quite natural. The contribution of conical singular points to the index is then described via the *spectral flow* of a family of conormal symbols. This notion, which is close to the notion of "divisor flow" [25], is a generalization of the spectral flow of a family of self-adjoint operators [6].

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[1]In the earlier paper [21], the same gluing as in [16] was applied in the case of manifolds with boundary.

1 Bottleneck Spaces and a General Locality Principle for the Index

1.1 Bottleneck spaces and elliptic operators

In this subsection we define the following two important notions:

(a) an abstract analog of localization of a section of a vector bundle in M_-, M_+, or the collar neighborhood $U \approx (-1,1)\times S$, i.e., an analog of *support* of a section;

(b) an abstract analog of an *operator whose integral kernel is supported near the diagonal.*

Definition 1.1. (a) A *bottleneck space* is a separable Hilbert space H equipped with the structure of a module over the commutative topological algebra $C^\infty([-1,1])$ (the action is continuous, and the unit function $\mathbf{1}\in C^\infty([-1,1])$ acts as the identity operator in H).

(b) The *support* of an element $h\in H$ is the closed set

$$\operatorname{supp} h = \bigcap \varphi^{-1}(0) \subset [-1,1], \tag{1.1}$$

where $\varphi^{-1}(0)$ is the preimage of the point 0 and the intersection is taken over all elements $\varphi\in C^\infty([-1,1])$ such that $\varphi h=0$. For an arbitrary subset $F\subset[-1,1]$, by $H(F)$ we denote the closure in H of the set of elements $h\in H$ supported in F.

Since in $C^\infty([-1,1])$ there is a partition of unity subordinate to any given locally finite open cover, it follows that the notion of support thus introduced possesses natural properties. In particular, if $F_1,F_2\subset[-1,1]$ are subsets such that $\overline{F_1}\cap\overline{F_2}=\emptyset$, then $H(F_1\cup F_2)=H(F_1)\oplus H(F_2)$ (here the direct sum is not necessarily orthogonal).

Example 1.2. The simplest example of a bottleneck space is the Hilbert space

$$H = L^2([-1,1];W)$$

of square integrable functions on $[-1,1]$ ranging in a Hilbert space W. The product of an element $h\in H$ by $\varphi\in C^\infty([-1,1])$ is defined in a natural way: $(\varphi h)(t)=\varphi(t)\,h(t)$.

Example 1.3. Let M be a manifold (possibly, noncompact and with singularities and/or boundary), and let $\chi: M\to[-1,1]$ be a continuous function that is smooth in the smooth part of M and satisfies the following condition: $\overline{\chi^{-1}((-1,1))}$ is a compact set that does not contain any singularities and has an empty intersection with the boundary. The Sobolev spaces $H^s(M)$ (with arbitrary weight functions near singularities and at infinity) can be equipped with a natural structure of bottleneck spaces as follows: one sets

$$(\varphi h)(x) = \varphi(\chi(x))\,h(x), \quad x\in M \tag{1.2}$$

for $h \in H^s(M)$ and $\varphi \in C^\infty([-1,1])$. Since χ is locally constant outside a compact set and near singularities, it follows that the operator of multiplication by $\varphi(\chi(x))$ is continuous in $H^s(M)$ regardless of the choice of the weights. If M has a nonempty boundary, then the Sobolev spaces $H^s(\partial M)$ also bear a natural structure of bottleneck spaces. This structure is given by the same formula (1.2), where now $h \in H^s(\partial M)$ and $x \in \partial M$.

Example 1.4. Using the construction described in the preceding example, one can readily transform the Sobolev spaces on each of the manifolds shown in Fig. 1 into bottleneck spaces. Namely, one takes a function $\chi(x)$ such that $\chi(x) = -1$ to the left of U, $\chi(x) = 1$ to the right of U, and $\chi(x)$ increases from -1 to 1 in U and depends only on the variable t there.

The construction of Example 1.3 is the main construction in applications. To describe it unambiguously, it suffices to specify the manifold M, the Sobolev spaces, and the function χ.

Definition 1.5. Let H_1 and H_2 be bottleneck spaces, and let $F \subset [-1,1]$ be a given subset. We say that H_1 *coincides* with H_2 on F and write $H_1 \overset{F}{=} H_2$ if an isomorphism $H_1(F) \approx H_2(F)$ (not necessarily isometric) is given. In this case, we also say that H_1 and H_2 are *modifications* of each other on $[-1,1]\backslash F$ and write $H_1 \overset{[-1,1]\backslash F}{\longleftrightarrow} H_2$. A square

$$\begin{array}{ccc} H_1 & \overset{B}{\longleftrightarrow} & H_2 \\ {\scriptstyle A}\updownarrow & & \updownarrow{\scriptstyle C} \\ H_3 & \overset{D}{\longleftrightarrow} & H_4 \end{array}$$

of modifications is said to *commute* if the corresponding diagram

$$\begin{array}{ccc} H_1(F) & \approx & H_2(F) \\ \wr\wr & & \wr\wr \\ H_3(F) & \approx & H_4(F), \end{array} \qquad F = [-1,1]\backslash(A \cup B \cup C \cup D),$$

of isomorphisms commutes.

The notion of an operator homotopic to an operator whose integral kernel is supported in an arbitrarily narrow neighborhood of the diagonal is formalized in the context of bottleneck spaces in the following definition.

Definition 1.6. (a) A *proper operator* in bottleneck spaces H and G is a continuous (with respect to the uniform operator topology) family of bounded linear operators

$$D_\delta : H \to G, \quad \delta > 0, \tag{1.3}$$

such that for each $\varepsilon > 0$ there exists a $\delta_0 > 0$ with the following property:

$$\operatorname{supp} D_\delta h \subset U_\varepsilon(\operatorname{supp} h) \qquad \forall h \in H \tag{1.4}$$

for every $\delta < \delta_0$. Here $U_\varepsilon(F)$ is the ε-neighborhood of F.

(b) An *elliptic operator* in bottleneck spaces H and G is a proper operator (1.3) such that D_δ is Fredholm for each δ and has an almost-inverse $D_\delta^{[-1]}$ such that the family $D_\delta^{[-1]}$ is also a proper operator.

As a rule, we omit the parameter δ in the notation of a proper operator.

Remark 1.7. One can readily see that a product of proper operators (if it is defined) is again a proper operator. In bottleneck spaces given by the construction of Example 1.3 on compact manifolds (possibly, with conical singularities) or on manifolds with boundary, elliptic pseudodifferential operators or, respectively, operators corresponding to elliptic boundary value problems are elliptic in the sense of Definition 1.6. (More precisely, they cam be included in families depending on the parameter δ and satisfying the conditions imposed in this definition.) This is a trivial consequence of the fact that pseudodifferential operators, as well as parametrices of boundary value problems, are pseudolocal. One obtains the desired assertions by multiplying the integral kernels of these operators by a family of cutoff functions with supports shrinking to the diagonal.

Now let

$$A_i : H_i \to G_i, \quad i = 1, 2,$$

be proper operators in bottleneck spaces, and let $F \subset [-1, 1]$ be an *open* subset (in the topology of $[-1, 1]$).

Definition 1.8. We say that A_1 *coincides* with A_2 on F and write $A_1 \overset{F}{=} A_2$ if $H_1 \overset{F}{=} H_2$, $G_1 \overset{F}{=} G_2$, and the following condition is satisfied for each compact subset $K \subset F$: there exists a $\delta_0 > 0$ such that

$$A_{1\delta} h = A_{2\delta} h \quad \text{for} \quad \delta < \delta_0 \text{ and } \operatorname{supp} h \subset K. \tag{1.5}$$

(This is well defined, since $h \in H_1(F) \approx H_2(F)$ and, F being open, $A_{1\delta}h, A_{2\delta}h \in G_1(F) \approx G_2(F)$ for small δ.) In this case, we also say that A_1 is obtained from A_2 by a *modification* on $[-1, 1]\backslash F$ and write $A_1 \xleftrightarrow{[-1,1]\backslash F} A_2$. The notion of a *commutative square* of modifications is defined in an obvious way.

The following lemma is an important (though trivial) generalization of the theorem stating that two arbitrary almost-inverses of a Fredholm operator differ by a compact operator.

Lemma 1.9. *Let $A_1 \overset{F}{=} A_2$ be elliptic operators in bottleneck spaces. Then $A_1^{[-1]} \overset{F}{=} A_2^{[-1]} + K$, where K is a proper compact operator. In particular, for $A_1 = A_2 = A$ we find that two arbitrary almost-inverses of A differ by a proper compact operator.*

Proof. Let

$$A_1 A_1^{[-1]} = 1 + K_1,$$
$$A_2^{[-1]} A_2 = 1 + K_2,$$

where K_1 and K_2 are proper compact operators. Then

$$A_2^{[-1]} A_1 A_1^{[-1]} = A_2^{[-1]} + A_2^{[-1]} K_1$$
$$A_2^{[-1]} A_1 A_1^{[-1]} \overset{F}{=} A_2^{[-1]} A_2 A_1^{[-1]} = A_1^{[-1]} + K_2 A_1^{[-1]},$$

and so

$$A_2^{[-1]} \overset{F}{=} A_1^{[-1]} + \{K_2 A_1^{[-1]} - A_2^{[-1]} K_1\}.$$

The proof is complete, since the operator in braces is compact. □

Lemma 1.10. *Suppose that* $[-1,1] = \bigcup_j F_j$ *is an open cover and* $A_1 \overset{F_j}{=} A_2$ *for all* j, *where* A_1 *and* A_2 *are proper operators. Then* $A_{1\delta} = A_{2\delta}$ *for sufficiently small* δ.

Proof. Take a finite partition of unity subordinate to the cover $\{F_j\}$. Then the assertion of the lemma directly follows from Definition 1.8. □

1.2 The general locality principle (the relative index theorem)

Now we are in a position to state and prove our main result.

Theorem 1.11. *Suppose that the following commutative diagram of modifications of elliptic operators in bottleneck spaces holds:*

$$\begin{array}{ccc} D & \overset{\{-1\}}{\longleftrightarrow} & D_- \\ {\scriptstyle\{1\}}\updownarrow & & \updownarrow{\scriptstyle\{1\}} \\ D_+ & \overset{\{-1\}}{\longleftrightarrow} & D_{+-} \end{array} .$$

Then

$$\operatorname{ind}(D) - \operatorname{ind}(D_-) = \operatorname{ind}(D_+) - \operatorname{ind}(D_{+-}).$$

Proof. Let the operators involved in the theorem act in the following spaces:

$$D : H \to G \qquad D_- : H_- \to G_-$$
$$D_+ : H_+ \to G_+ \quad D_{+-} : H_{+-} \to G_{+-}.$$

Then, by definition, there are commutative modification diagrams

$$\begin{array}{ccc} H & \overset{\{-1\}}{\longleftrightarrow} & H_- \\ {\scriptstyle\{1\}}\updownarrow & & \updownarrow{\scriptstyle\{1\}} \\ H_+ & \overset{\{-1\}}{\longleftrightarrow} & H_{+-} \end{array}, \qquad \begin{array}{ccc} G & \overset{\{-1\}}{\longleftrightarrow} & G_- \\ {\scriptstyle\{1\}}\updownarrow & & \updownarrow{\scriptstyle\{1\}} \\ G_+ & \overset{\{-1\}}{\longleftrightarrow} & G_{+-} \end{array}. \tag{1.6}$$

Lemma 1.12. *Suppose that the diagrams* (1.6) *are trivial, that is, the modifications are identical:*

$$H = H_- = H_+ = H_{+-}, \qquad G = G_- = G_+ = G_{+-}.$$

Then the assertion of the theorem holds.

Proof. Indeed, in this case we have

$$\operatorname{ind}(D) - \operatorname{ind}(D_-) - \operatorname{ind}(D_+) + \operatorname{ind}(D_{+-}) = \operatorname{ind}(DD_-^{[-1]}D_{+-}D_+^{[-1]}).$$

However,

$$DD_-^{[-1]}D_{+-}D_+^{[-1]} \overset{(-1,1]}{\equiv\!\equiv} DD^{[-1]}D_+D_+^{[-1]} \equiv 1,$$

$$DD_-^{[-1]}D_{+-}D_+^{[-1]} \overset{[-1,1)}{\equiv\!\equiv} DD_-^{[-1]}D_-D^{[-1]} \equiv DD^{[-1]} \equiv 1,$$

where $\equiv$ stands for equality modulo proper compact operators, and consequently,

$$DD_-^{[-1]}D_{+-}D_+^{[-1]} \equiv 1$$

by Lemma 1.10. □

Now the assertion of the theorem in the general case is a consequence of the following lemma.

Lemma 1.13. *The isomorphisms of subspaces occurring in the definition of modifications in the diagrams* (1.6) *can be extended to isomorphisms*

$$\begin{array}{ccc} H & = & H_- \\ \| & & \| \\ H_+ & = & H_{+-} \end{array}, \qquad \begin{array}{ccc} G & = & G_- \\ \| & & \| \\ G_+ & = & G_{+-} \end{array} \tag{1.7}$$

such that the diagrams (1.7) commute and the isomorphisms themselves commute with the new action of $C^\infty([-1,1])$ *given by the formula*

$$\varphi * h = (\varphi \circ \chi)h, \tag{1.8}$$

where $\chi : [-1,1] \to [-1,1]$ *is a smooth monotone function equal to* -1 *on* $[-1,-1/2]$ *and* $+1$ *on* $[1/2,1]$.

Indeed, the operators D, D_-, D_+, and D_{+-} remain proper with respect to the new action. Moreover, they all act in the same spaces and satisfy an appropriate modification diagram. Now an application of Lemma 1.12 completes the proof of the theorem. □

Proof of Lemma 1.13. It suffices to carry out the proof for the first diagram. We have the direct sum expansions

$$
\begin{aligned}
H &= H^{(-)} \oplus H((-1,1)) \oplus H^{(+)},\\
H_- &= H_-^{(-)} \oplus H((-1,1)) \oplus H_-^{(+)},\\
H_+ &= H_+^{(-)} \oplus H((-1,1)) \oplus H_+^{(+)},\\
H_{+-} &= H_{+-}^{(-)} \oplus H((-1,1)) \oplus H_{+-}^{(+)},
\end{aligned}
$$

where

$$
\begin{aligned}
H^{(-)} &= H([-1,-1/2)) \ominus (H([-1,-1/2)) \cap H((-1,1))),\\
H^{(+)} &= H([1/2,1)) \ominus (H([1/2,1)) \cap H((-1,1))),
\end{aligned}
$$

etc., and $A \ominus B$ stands for the orthogonal complement of B in A.

Without loss of generality, we can assume that the spaces $H^{(-)}$, $H^{(+)}$, etc. are infinite-dimensional. (Otherwise, we add direct summands of the form l^2 to these spaces and the identity operators $id : l^2 \to l^2$ to all operators in question.) Next, by virtue of the diagram (1.6), we have the isomorphisms

$$
H^{(+)} = H_-^{(+)},\ H_+^{(+)} = H_{+-}^{(+)},\ H^{(-)} = H_+^{(-)},\ H_-^{(-)} = H_{+-}^{(-)}.
$$

It remains to choose arbitrary isomorphisms $H^{(-)} \approx H_-^{(-)}$ and $H^{(+)} \approx H_+^{(+)}$; such isomorphisms between separable Hilbert spaces always exist. The resulting isomorphisms (1.7) commute with the action (1.8), since $\varphi * h = \varphi(-1)h$ for elements h of the spaces $H^{(-)}, \dots, H_{+-}^{(-)}$ and $\varphi * h = \varphi(1)h$ for elements h of the spaces $H^{(+)}, \dots, H_{+-}^{(+)}$.

The proof of the lemma is complete. □

Remark 1.14. Formula (0.2) is a special case of the theorem. This follows from the construction in Example 1.4 with regard for Remark 1.7.

2 Applications to Index Formulas for Operators with Symmetry Conditions

In this section, we give applications of the theory to index formulas for operators on manifolds with conical singularities. The main definitions concerning operators on manifolds with conical singularities can be found in [32], where further references are also available.

2.1 An index theorem for pseudodifferential operators on manifolds with conical singularities

We need a generalization to arbitrary conormal families of the version [31] of the notion of spectral flow [6].

1. The multiplicity of a singular point of an operator family. Let $H_{1,2}$ be Hilbert spaces, and let $D(p)\colon H_1 \to H_2$ be a family of bounded linear operators with parameter p ranging in an open subset $\mathcal{U}$ of the complex plane. We say that this family is *strongly finitely meromorphic* if it is finitely meromorphic and has a finitely meromorphic inverse. (In other words, $D(p)$ is meromorphic in $p \in \mathcal{U}$, and the principal parts of the Laurent series at each of the poles are finite rank operators; the same must be valid for $D^{-1}(p)$.) The *singular points* of $D(p)$ are the poles of $D(p)$ and the poles of $D^{-1}(p)$. The *multiplicity* of a singular point p_0 of the family $D(p)$ is the number (necessarily integer)

$$m_D(p_0) = \operatorname{Trace} \operatorname*{Res}_{p=p_0} \left\{ D^{-1}(p) \frac{\partial D(p)}{\partial p} \right\}. \tag{2.1}$$

2. The spectral flow of a family of conormal symbols. The facts given in this subsection are closely related to the results of [25].

Let Ω be a smooth compact manifold without boundary, and let

$$D(p)\colon H^s(\Omega, E) \to H^{s-m}(\Omega, F)$$

be a family of mth-order pseudodifferential operators on Ω holomorphic in a parameter $p \in \mathbf{C}$ and elliptic with parameter p in the sense of Agranovich–Vishik [3] in some two-way sector of nonzero central angle containing the real axis. Then this family is strongly finitely meromorphic in that sector. Such families are referred to as *conormal symbols* on Ω in what follows.

Let $D_t = D_t(p)$ be a family of conormal symbols on Ω continuously depending on a parameter $t \in [0,1]$. We intend to define the notion of *spectral flow* $\operatorname{sf} D_t \equiv \operatorname{sf}_{t=0,1} D_t$ of the family D_t. Intuitively, $\operatorname{sf} D_t$ is the number of singular points (with regard for multiplicities) of D_t that cross the real axis upward (more precisely, pass from the open lower half-plane to the closed upper half-plane) as the parameter t varies from 0 to 1. (The points moving in the opposite direction are naturally treated as giving a negative contribution.) This definition naturally generalizes the notion of spectral flow of a family A_t of normally elliptic operators and passes into the latter for $D_t = p - A_t$. The problem however is to make the definition rigorous. The notion of spectral section is not available unless $D_t \neq p - A_t$, and we use a construction different from that in [31].

To simplify the formulas, we momentarily assume that that $D_t(p)$ has no singular points on the real axis for $t = 0$ and $t = 1$. (This restriction will be removed later.) We split the interval $[0,1]$ into subintervals by some points

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

and choose real numbers γ_i, $i = 1, \dots, N$, such that the operator $D_t(p)$ is invertible on the weight line $\operatorname{Im} p = \gamma_i$ for all $t \in [t_{i-1}, t_i]$. Furthermore, we require that $\gamma_1 = \gamma_N = 0$. The existence of such a partition and the numbers γ_i follows from ellipticity with parameter in the sector of nonzero central angle and the continuous dependence of our conormal symbols on t.

Definition 2.1. The *spectral flow* of the family $\{D_t\}_{t\in[0,1]}$ of conormal symbols is the number

$$\operatorname*{sf}_{t=0,1} D_t = \sum_{i=1}^{N-1} g_i, \tag{2.2}$$

where

$$g_i = \begin{cases} -\sum\limits_{\operatorname{Im} p_j \in (\gamma_i, \gamma_{i+1})} m_{D_{t_i}}(p_j) & \text{if } \gamma_i < \gamma_{i+1}, \\ \sum\limits_{\operatorname{Im} p_j \in (\gamma_{i+1}, \gamma_i)} m_{D_{t_i}}(p_j) & \text{if } \gamma_i \geq \gamma_{i+1}. \end{cases}$$

Here the p_j are the singular points of the conormal symbol in the indicated strip. (Note that there are finitely many such points, and so the sum is well defined.)

Instead of $\operatorname*{sf}_{t=0,1} D_t$, we usually write $\operatorname{sf} D_t$.

Theorem 2.2. *The following assertions hold.*

1. *The spectral flow is well defined by Definition* 2.1 (*that is, is independent of the partition of the interval* $[0,1]$ *and the numbers* γ_j).

2. *The spectral flow* $\operatorname{sf} D_t$ *depends only on the homotopy class of the path* D_t (*with fixed beginning and end*). *Moreover,* $\operatorname{sf} D_t$ *is also invariant under deformations* (*homotopies*) *of* D_t *such that the beginning and end are not fixed but the conormal symbols* D_0 *and* D_1 *do not have singular points on the real axis for any value of the homotopy parameter.*

Now we can generalize Definition 2.1 to the case in which $D_0(p)$ and $D_1(p)$ may have singular points on the real axis. It follows from the condition of ellipticity with parameter that there exists an $\varepsilon > 0$ such that the open strip $-\varepsilon < \operatorname{Im} p < 0$ does not contain any singular points of $D_0(p)$ and $D_1(p)$. Consider the family $D_{t,\tau}(p) = D_t(p - i\tau)$ of conormal symbols depending on an additional real parameter $\tau \in (0, \varepsilon)$. Then $D_{0,\tau}$ and $D_{1,\tau}$ have no singular points on the real axis, and it follows from Theorem 2.2, item 2 that the spectral flow $\operatorname{sf} D_{t,\tau}$ is independent of $\tau \in (0, \varepsilon)$. By definition, we set

$$\operatorname{sf} D_t = \operatorname{sf} D_{t,\varepsilon/2}.$$

3. The relationship between the index and the spectral flow. These two notions are related by the following important theorem. We omit the proof, which is not directly related to the subject of this paper.

Theorem 2.3. *Let*

$$D_t(p)\colon H^s(\Omega, E) \to H^s(\Omega, F), \qquad t \in [0, 1]$$

be a family of conormal symbols on Ω such that D_0 and D_1 have no singular points on the real axis. Let $\chi\colon (-\infty, \infty) \to [0, 1]$, $\chi(-\infty) = 0$, $\chi(\infty) = 1$, be a smooth function whose derivatives decay exponentially at infinity. We set

$$D(\tau, p) = D_{\chi(\tau)}(p)$$

and consider the operator

$$\widehat{D} \overset{\text{def}}{=} D\left(\overset{2}{\tau}, -i\, \overset{1}{\frac{\partial}{\partial \tau}}\right) : H^s(C, E) \longrightarrow H^s(C, F)$$

in the Sobolev spaces[2] *on the infinite cylinder $C = \Omega \times (-\infty, \infty)$. Then*

$$\operatorname{sf} D_t = -\operatorname{ind} \widehat{D}.$$

4. The index theorem for elliptic pseudodifferential operators. Now we use the preceding results to prove an index theorem for elliptic pseudodifferential operators on manifolds with conical singularities. This theorem includes previously known theorems given in [33, 16, 15] as special cases.

Let N be a compact closed manifold with conical singularities. Without loss of generality, we assume that there is only one conical point α (the base Ω of the corresponding cone is not assumed to be connected). Next, let

$$\widehat{D}\colon H^s(N, E) \longrightarrow H^{s-m}(N, F)$$

[2] We denote the lifts of the bundles E and F to C by the same letters. The weight line in the definition of Sobolev spaces is assumed to be $\{\operatorname{Im} p = 0\}$, and so we omit weights in the notation of spaces and write H^s instead of $H^{s,0}$.

be an elliptic pseudodifferential operator in Sobolev spaces on N with conormal symbol $D_0(p)$.

Suppose that $g\colon \Omega \to \Omega$ is a diffeomorphism and

$$\mu_E : E|_\Omega \longrightarrow g^*(E|_\Omega), \qquad \mu_F : F|_\Omega \longrightarrow g^*(F|_\Omega)$$

are bundle isomorphisms. Let the following condition be satisfied.

Condition A. *The conormal symbols $D_0(p)$ and $\mu_F^{-1} g^* D_0(-p)(g^*)^{-1}\mu_E$ are homotopic in the class of conormal symbols.*[3]

We take some homotopy and denote it by D_{0t},

$$D_{00}(p) = D_0(p), \qquad D_{01}(p) = \mu_F^{-1} g^* D_0(-p)(g^*)^{-1}\mu_E.$$

Let us construct a closed manifold $\mathcal{N}$ and bundles $\mathcal{E}$ and $\mathcal{F}$ over it as follows. By cutting away a small neighborhood of the conical point in N, we obtain a manifold $\widetilde{N}$ with boundary $\partial\widetilde{N} = \Omega$. Let us attach two copies of $\widetilde{N}$ to the opposite ends of the cylinder $\Omega \times [0,1]$, using the identity mapping for gluing at the left end and the mapping g at the right end. We lift the bundles $E|_\Omega$ and $F|_\Omega$ to $\Omega\times[0,1]$ in the natural way and glue them to the corresponding bundles over the two copies of $\widetilde{N}$ using the identity isomorphism at the left end and the isomorphisms μ_E and μ_F at the right end. The resulting bundles over $\mathcal{N}$ will be denoted by $\mathcal{E}$ and $\mathcal{F}$, respectively.

Theorem 2.4. *The following index formula holds under the above-mentioned conditions:*

$$\operatorname{ind}\widehat{D} = \frac{1}{2}\left\{\operatorname{ind}\mathcal{D} + \operatorname{sf} D_{0t}\right\},$$

where the elliptic operator $\mathcal{D}\colon H^s(\mathcal{N},\mathcal{E}) \longrightarrow H^{s-m}(\mathcal{N},\mathcal{F})$ on the closed manifold $\mathcal{N}$ will be described in the proof of the theorem.

Proof. Using the change of variables $r = e^{-t}$, where r is the distance to the conical point, in a neighborhood of α, we represent N as a manifold with a cylindrical end. Next, D is homotopic in the class of elliptic pseudodifferential operators with the same conormal symbol to an operator whose coefficients are independent of t for sufficiently large t. Without loss of generality, we assume that D itself satisfies this condition. Consider the following two configurations (see Fig. 2). The original configuration consists of two copies of N, each equipped with the operator $\hat{D}$, and the infinite cylinder $(-\infty,\infty)\times\Omega$ equipped with the operator with operator-valued symbol $D_{0\chi(t)}(p)$, where $\chi\colon(-\infty,\infty)\to[0,1]$ increases from 0 to 1 and is constant outside a compact set. The final configuration consists of the manifold $\mathcal{N}$ equipped with the operator $\mathcal{D}$ whose principal symbol coincides with the principal symbol of $\widehat{D}$ on the copies of $\mathcal{N}$ and with the principal symbol of the

[3]The case of symmetry with respect to a point $p_0 \neq 0$ can be treated in a completely similar way.

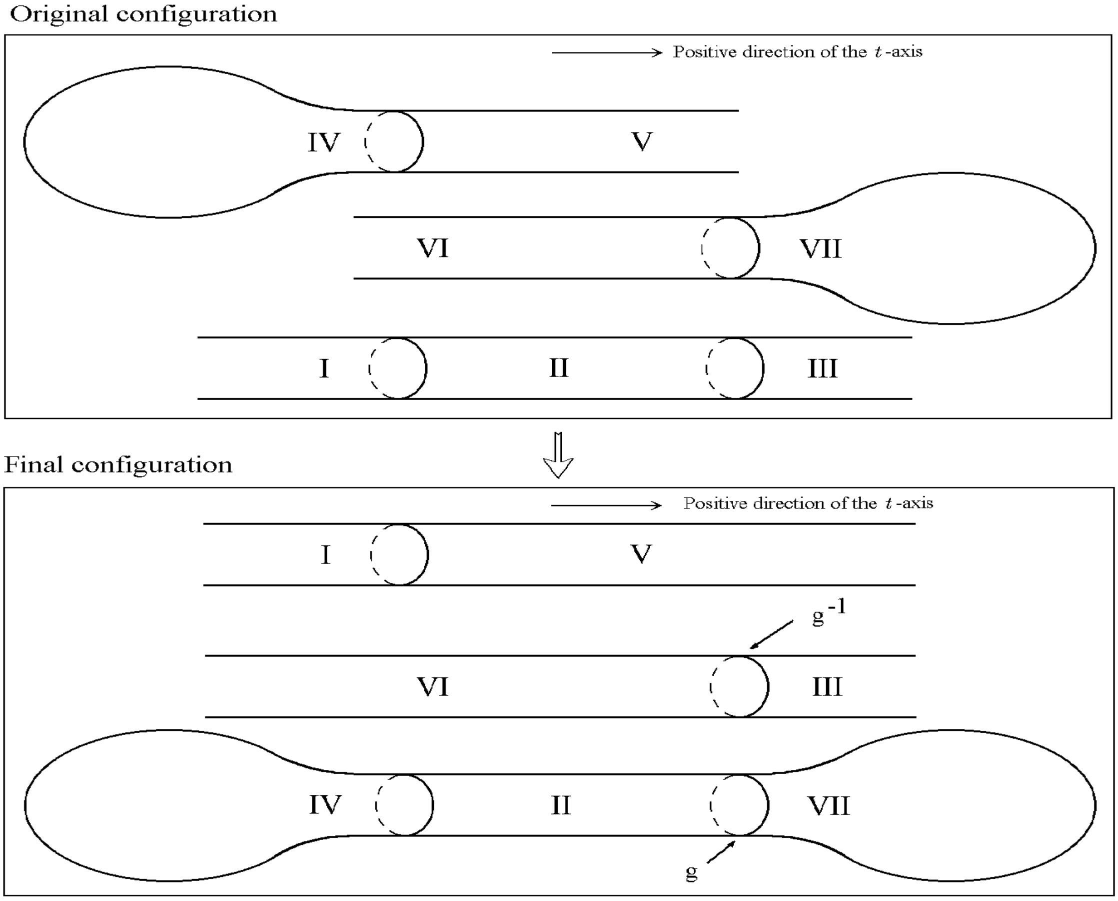

Figure 2: Proof of Theorem 2.4

homotopy D_{0t} on the cylinder $\Omega \times [0,1]$ and two infinite cylinders $(-\infty,\infty)\times\Omega$ equipped with the operators $D_0(-i\partial/\partial t)$ and $D_0(i\partial/\partial t)$, whose coefficients are independent of the variable $t \in (-\infty,\infty)$. The final configuration is obtained from the original one by a transition similar to that from M to M' in Fig. 1 if we denote the union of components I, III, IV, and VI by M_+ and the union of the remaining components by M_-. (Note that M'_- differs from M_- only in the arrangement of the components.) Now consider a new part $\widetilde{M}_+$ obtained from M_+ by the replacement of components IV and VI by half-infinite cylinders. (The coefficients of the new operators are independent of t on components IV+V and VI+VII.) In the new version (which corresponds to the transition from $\widetilde{M}$ to $\widetilde{M}'$ in Fig. 1) the original and final configurations are the same, and so the relative index is zero. We equip the Sobolev spaces in which all our operators act with the structure of bottleneck spaces in the same way as this was done in Example 1.4 for the simplest configurations shown in Fig. 1. According to Remark 1.7, we can assume that all these operators are elliptic in the sense of Definition 1.6. Now it follows from Theorem 1.11 that the relative index in the first case is also zero, that is, the index of the original and the final configuration is the same. Now by applying Theorem 2.3, we arrive at the desired assertion. The proof is complete. □

If the homotopy has the form $D_{0\tau}(p) = B(p + i\tau)$, $\tau \in [\gamma_1, \gamma_2]$, and g is the identity mapping, we recover the result of [33]. Using the results of [29], one can readily transfer the assertion of Theorem 2.4 to the case of general spectral boundary value problems with symmetry conditions (the *existence* of such formulas was proved in [31]).

2.2 An index theorem for Fourier integral operators on manifolds with conical singularities.

The index problem for Fourier integral operators (or quantized contact transformations) was posed by Weinstein [36, 37] and recently solved (for operators on smooth manifolds) in [14, 23].

In this section we state and prove an index theorem for Fourier integral operators on manifolds with conical singularities. This theorem is actually an analog of Theorem 2.4. To simplify the presentation, we restrict ourselves to the case in which the diffeomorphism g occurring in Condition A is the identity diffeomorphism. One can readily modify the argument to the case in which g is nontrivial.

1. Fourier integral operators on manifolds with singularities. First, we give the definition of Fourier integral operators on manifolds with conical singularities. We are rather brief on the subject; a more detailed exposition can be found in [26, 28] (cf. also [24]).

Let N_1 and N_2 be manifolds with conical singularities. We consider the *cotangent bundles* T^*N_1 and T^*N_2 of N_1 and N_2, defined as the *compressed cotangent bundles* of the manifolds $N_1^\wedge$ and $N_2^\wedge$ obtained from N_1 and N_2 by the blowup of the conical points (see [24]). Note that $N_1^\wedge$ and $N_2^\wedge$ are manifolds with boundary. Thus, T^*N_1 and T^*N_2 are also manifolds with boundary. They are equipped with a natural symplectic form that has a singularity at the boundary; in the conical coordinates (see [32]), the symplectic form is given by

$$\omega^2 = -\frac{dp \wedge dr}{r} + dq \wedge d\varphi,$$

where φ is a coordinate system on the base of the cone, r is the radial coordinate, p is the conormal variable, and q is the dual momenta of φ.

Let

$$g : T^*N_1 \backslash \{0\} \to T^*N_2 \backslash \{0\}$$

be an $\mathbf{R}_+$-homogeneous canonical transformation (that is, a mapping smooth up to the boundary and preserving the symplectic form).

Let

$$L(g) = \{(z, g(z))\} \subset T^*N_1 \backslash \{0\} \times T^*N_2 \backslash \{0\}$$

be the graph of g; this set is a Lagrangian submanifold of $T^*N_1\backslash\{0\} \times T^*N_2\backslash\{0\}$ with respect to the symplectic form $\pi_2^*\omega_2^2 - \pi_1^*\omega_1^2$, where π_j is the natural projection of the product on the jth factor and ω_j^2 is the symplectic form on T^*N_j.

Let Λ be the Maslov line bundle over $L(g)$ and $a : L(g) \to \Lambda$ a nonvanishing section homogeneous of order m.

Using coordinates lifted from T^*N_1 on $L(g)$, we can treat a as a section of a bundle over T^*N_1. We define a Fourier integral operator

$$T(g,a) = H^{s,\gamma}(N_1) \to H^{s-m,\gamma}(N_2)$$

in the standard way as the operator with integral kernel $\mathcal{K}_{L_g}(a)$, where $\mathcal{K}_{L_g}$ is the Maslov canonical operator on L_g (the homogeneous version). The details of the construction can be found in [26, 28]. Under an appropriate choice of elements of this construction, the conormal symbol $T_0(p)$ of the operator $T(g,a)$ is well defined as an operator family in the spaces

$$T_0(p) : H^s(\Omega_1) \to H^{s-m}(\Omega_2),$$

where Ω_1 and Ω_2 are the bases of the cones on the manifolds N_1 and N_2.

We say that the operator $T(g,a)$ is *elliptic* (on a given weight line $\{\operatorname{Im} p = \gamma\}$) if the conormal symbol $T_0(p)$ is invertible everywhere on the weight line. In what follows, we assume that $\gamma = 0$ to simplify the notation.

2. The index theorem. We introduce the following condition.

Condition B. The conormal symbols $T_0(p)$ and $T_0(-p)$ are homotopic in the class of conormal symbols of formally elliptic Fourier integral operators.

Theorem 2.5. *Let $T(g,a)$ be an elliptic Fourier integral operator whose conormal symbol satisfies Condition B.*

Then

$$\operatorname{ind} T(g,a) = \frac{1}{2}\{\operatorname{ind} \mathcal{T} + \operatorname{sf} T_{0t}\},$$

where T_{0t} is a homotopy joining $T_0(p)$ with $T_0(-p)$ and

$$\mathcal{T} : H^s(\mathcal{N}_1) \to H^s(\mathcal{N}_2)$$

is some Fourier integral operator on the closed manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$ obtained from N_1 and N_2 by the same construction as $\mathcal{N}$ is obtained from N in the proof of Theorem 2.4.

Needless to say, the operator $\mathcal{T}$ is obtained from T by surgery of the same type as in Theorem 2.4.

Proof. The proof of Theorem 2.5 reproduces that of Theorem 2.4 with some obvious modifications. The most important of these modifications are as follows.

1. The gluing is carried out separately for N_1 and N_2.

2. One must show that all Fourier integral operators in question are elliptic in the corresponding bottleneck spaces. In other words, we must include them in the corresponding families continuously depending on a parameter δ. This can be done by scaling the variable t on the cylindrical ends. For example, let us explain the corresponding construction for the operator $T(g, a)$ itself. To be definite, we make the following assumptions:

   (a) the variable t on the cylindrical ends of N_1 and N_2 varies from -1 to ∞;

   (b) the bottleneck space structure on the Sobolev spaces on both N_1 and N_2 is obtained by the standard construction of Example 1.3 with the function $\chi\colon N_j \to [-1, 1]$ defined as follows: $\chi = 1$ outside the cylindrical end and also on the cylindrical end for $t \leq 1/3$; $\chi = -1$ on the cylindrical end for $t \geq 2/3$; χ decreases from 1 to -1 as t varies from $1/3$ to $2/3$.

   For convenience, we adopt the convention that $t = -1$ outside the cylindrical end. Then t is a globally defined function on N_j, $j = 1, 2$. We can lift this function to T^*N_j by using the natural projection $\pi : T^*N_j \to N_j$. In the following, we write simply t instead of π^*t. Since g is a canonical transformation of cotangent bundles to manifolds with conical singularities, it follows by a simple argument that there exists a constant $R > 0$ such that

   $$|t(g(y)) - t(y)| < R \quad \forall y \in T^*N_1.$$

   Since the singularities of the integral kernel $K(x, x')$, $(x, x') \in N_1 \times N_2$, of $T(g, a)$ lie in the projection of the graph $L(g)$ of g on $N_1 \times N_2$, we have

   $$\operatorname{sing\,supp} K(x, x') \subset \{(x, x') \mid |t(x) - t(x')| < R\}.$$

   Consider diffeomorphisms $\varphi_b\colon N_j \to N_j$, $b > 0$, determined by the following conditions:

   (a) $\varphi_b(x) = x$ if $t(x) = -1$ or $t(x) > 2b$ (thus, φ_b is nontrivial only on some compact subset of the cylindrical end);

   (b) on the cylindrical end, the mapping φ_b affects only the t-component in the direct product structure (that is, $\varphi_b(\omega, t) = (\omega, \psi_b(t))$, where ω is a point of the base of the cone);

   (c) the function ψ_b is monotonic and satisfies

   $$\psi_b(t) = bt \quad \text{for } t \in [0, 1];$$

   (d) φ_b depends continuously on b and is the identity mapping for $b = 1$.

Now we can define a proper operator T_δ by multiplying the integral kernel of

$$(\varphi^*_{1/\delta})^{-1} T(g,a) \varphi^*_{1/\delta}$$

by an appropiate cutoff function $\chi_\delta(x,x')$ on $N_1 \times N_2$ such that $\chi_\delta(x,x') \equiv 1$ for $\delta = 1$ and moreover, for $\delta < 1$ one has $\chi_\delta(x,x') = 1$ if (x,x') belongs to the singular support of this kernel or if both $t(x)$ and $t(x')$ are sufficiently large. For sufficiently small δ, this cutoff function can be chosen to satisfy the condition

$$\chi_\delta(x,x') = 0 \quad \text{whenever } t(x) \in [1/3, 2/3] \text{ and } |t(x) - t(x')| < 2R\delta,$$

whence we see that T_δ is a proper operator. Next, let $T^{[-1]}(g,a)$ be an almost inverse of $T(g,a)$. (One can take $T^{[-1]}(g,a) = T(g^{-1}, g^*(a^{-1}))$, choosing the conormal symbol of $T^{[-1]}(g,a)$ to be the inverse of that of $T(g,a)$.) Then $T^{[-1]}_\delta$ can be constructed by the same procedure as T_δ, whence we see that T_δ is actually an elliptic operator in bottleneck spaces.

With these modifications, the remaining part of the proof is the same as in the case of pseudodifferential operators. □

We conclude the paper with the following remark.

Remark 2.6. The index ind $\mathcal{T}$ of a Fourier integral operator on a pair of smooth manifolds without boundary is given by the Leichtnam–Nest–Tsygan formula [23], and we see that Theorem 2.5 expresses the index of a Fourier integral operators on manifolds with conical singularities under symmetry conditions via that formula and the spectral flow of the family of conormal symbols.

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