Eta Invariant and Parity Conditions

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Abstract

We give a formula for the η-invariant of odd order operators on even-dimensional manifolds, and for even order operators on odd-dimensional manifolds. Geometric second order operators are found with nontrivial η-invariants. This solves a problem posed by P. Gilkey.

Keywords: eta invariant, parity conditions, K-theory, linking coefficients, Dirac operators, spectral flow, elliptic operators

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Contents

Introduction

1. P. Gilkey noticed [1] that the Atiyah–Patodi–Singer η-invariant is a homotopy invariant in the class of elliptic self-adjoint differential operators, provided the following condition is satisfied

$$\operatorname{ord} A + \dim M \equiv 1 \pmod 2 . \tag{1}$$

More precisely, in this case the fractional part $\{\eta(A)\} \in \mathbb{R}/\mathbb{Z}$ of the spectral η-invariant defines a homotopy invariant, which is determined by the principal symbol of the operator.

There arises the problem of computation of the η-invariant in topological terms and the question of finding examples, which show the nontriviality of the invariant.

On even-dimensional manifolds, operators with a nontrivial η-invariant were constructed in [2]. It turned out that geometrical first order Dirac type operators can only have dyadic η-invariants

$$\{\eta(A)\} \in \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}.$$

Moreover, if the manifold is orientable then the invariant is a half-integer. An interesting example is given by the Dirac operator on (nonorientable !) pin^c-manifolds (see

[2]), e.g., on an even-dimensional real projective space $\mathbb{RP}^{2n}$. For a number of manifolds, the η-invariant of the Dirac operator was computed in [2, 3], where applications to geometry are considered.

In this paper, we give a formula for the η-invariant of operators, which satisfy (1). A geometric second order operator with a nonzero fractional part of the η-invariant is constructed on some odd-dimensional manifolds. This solves the problem posed in [1].

2. Let us briefly describe the contents of the paper. The computational problem for the fractional part of the η-invariant is stated in the first section. The following three sections deal with the case of even order operators (for definiteness). Section 2 starts with the expression of the fractional part of the η-invariant in terms of the spectral flow modulo n. The following Section 3 expresses this spectral flow in topological terms. Then we give an expression for the η-invariant in terms of the *linking index* in K-theory, which is more useful for applications. In Section 5 we indicate the changes, which are necessary to obtain a formula for the fractional part of the η-invariant for odd order operators. The last section contains a second order operator with a non-zero fractional part of η-invariant on the nonorientable product $\mathbb{RP}^{2n} \times \mathbb{S}^1$.

Two appendices are placed at the end of the paper. The first contains an expression of the spectral flow in terms of the index. In particular, we obtain the expression of the spectral flow for periodic families as the index of a single operator (see [4]). In the second appendix, we give a formula for the action of antipodal involution on the K-group of the Thom space for a real vector bundle.

3. A preliminary version [5] contained the formula for the fractional part of the η-invariant. It was obtained by means of the reduction of the spectral invariant $\eta(A)$ to (a priori) homotopy invariant of the spectral subspace, which is generated by eigenvectors of A that correspond to the nonnegative eigenvalues (see [6, 7, 8]).

The present paper contains a direct proof of the formula for the η-invariant. This proof is independent of the above mentioned construction and the corresponding elliptic theory in subspaces, which are defined by pseudodifferential projections.

4. We are grateful to Prof. P. Gilkey for advice and a number of valuable remarks he made in the discussion concerning the results of this paper.

1 Statement of the problem

Let M be a smooth closed n-dimensional manifold and let

$$A : C^{\infty}(M, E) \longrightarrow C^{\infty}(M, E)$$

be an elliptic self-adjoint operator of a positive order. The spectral η-function of A is defined

$$\eta(s, A) = \frac{1}{2}\left(\sum \operatorname{sgn} \lambda_i \, |\lambda_i|^{-s} + \dim \ker A\right), \tag{2}$$

where the summation is taken over nonzero eigenvalues λ_i of A with respect to their multiplicities. The series in (2) is absolutely convergent in the half-plane, which is defined by the inequality $\operatorname{Re} s > \dim M / \operatorname{ord} A$. The spectral function $\eta(s, A)$ extends analytically to the complex plane with isolated singularities. Moreover, the residue at the point $s = 0$ is equal to zero ([4], [9]). The value of the η-function at the origin is called the *η-invariant* of operator A:

$$\eta(A) = \eta(0, A).$$

The spectral η-invariant *is not* a homotopy invariant of the operator: for a smooth family of operators A_t with parameter t, the function $\eta(A_t)$ is a piecewise smooth function with integer jumps at those values of t, where some eigenvalue of operator A_t changes its sign. At the same time, the family of fractional parts

$$\{\eta(A_t)\} \in \mathbb{R}/\mathbb{Z}$$

is smooth. It was noticed in [1] that for differential operators, which satisfy condition (1), the fractional part $\{\eta(A)\}$ is a homotopy invariant of the operator. Condition (1) will be referred to as the *parity condition.*

Problem. Compute the fractional invariant $\{\eta(A)\}$ under the parity condition in terms of the principal symbol of the operator A.

It is useful to enlarge the class of differential operators, in order to apply the methods of algebraic topology to the problem.

Definition 1 A pseudodifferential operator A is called *admissible* [1], if it is a classical operator (see e.g. [10], [11]), and the components of its complete symbol

$$\sigma(A)(x, \xi) \sim \sum_{j \geq 0} a_{d-j}(x, \xi)$$

are $\mathbb{R}_*$-homogeneous in the following sense:

$$a_k(x, -\xi) = (-1)^k a_k(x, \xi). \tag{3}$$

The basic properties of the η-invariant, which are used in the computation of the fractional part $\{\eta(A)\}$, are given in the following proposition (e.g., see [4, 1]).

Proposition 1 *Let A be an admissible elliptic self-adjoint operator, which satisfies the parity condition. Then*

1. $\{\eta(A)\} \in \mathbb{R}/\mathbb{Z}$ *defines a homotopy invariant of the operator A; more precisely, for a smooth operator family A_t, the following equality is valid*

$$\eta(A_1) - \eta(A_0) = \mathrm{sf}\,(A_t)_{t\in[0,1]}\,,$$

where $\mathrm{sf}\,(A_t)_{t\in[0,1]}$ *denotes the spectral flow (see* [4]*) of the family of self-adjoint operators A_t;*

2. $\eta(\triangle_E) = 0$, *where $\triangle_E$ denotes an admissible operator*

$$\triangle_E : C^\infty(M, E) \to C^\infty(M, E)\,,$$

with the principal symbol of the Laplacian, which acts on the sections of a vector bundle E on an odd-dimensional manifold. A similar statement is valid for powers of this operator: $\eta\left(\triangle_E^l\right) = 0$;

3. $\eta(-A) = -\eta(A)$, *if A is invertible;*

In the following sections, we compute the fractional part of the η-invariant for even order operators on odd-dimensional manifolds. Odd order operators can be treated similarly. The changes in the constructions and the proofs are given in Section 5.

2 Expression of the η-invariant in terms of the spectral flow

1. Let A be an admissible elliptic self-adjoint operator of even order $d = 2l$, which satisfies the parity condition, i.e. the manifold is odd-dimensional.

We introduce the stable homotopy equivalence relation on this set of operators.

Definition 2 Operators with the principal symbol, which is the direct sum of a positive and a negative definite symbols are called *trivial.* Two operators A_1 and A_2 are *stably homotopic,* iff for some trivial operators A', A'' the direct sums

$$A_1 \oplus A' \quad \text{and } A_2 \oplus A''$$

are homotopic.

The set of equivalence classes of stably homotopic admissible self-adjoint even order operators on M is denoted by $\mathcal{S}^{ev}(M)$. This set is a group with respect to the direct sum of operators.

This group $\mathcal{S}^{ev}(M)$ can be described in terms of K-theory. Indeed, the principal symbol $\sigma(A)(x,\xi)$ is a hermitian matrix at each of the points (x,ξ) of the cosphere bundle S^*M. Let us denote by $\Pi_+(\sigma(A))(x,\xi)$ the orthogonal projection onto the subspace generated by eigenvectors of the matrix $\sigma(A)(x,\xi)$ with positive eigenvalues. For an even order operator A, we obtain

$$\sigma(A)(x,-\xi)=\sigma(A)(x,\xi).$$

Consequently, the vector bundle $\operatorname{Im}\Pi_+(\sigma(A))$ lifts to the projectivization

$$P^*M=S^*M/\{(x,\xi)\sim(x,-\xi)\}.$$

Let us define the mapping

$$\begin{array}{ccc}\mathcal{S}^{ev}(M) & \stackrel{\chi^{ev}}{\longrightarrow} & K(P^*M)/p^*K(M),\\ [A] & \mapsto & [\operatorname{Im}\Pi_+(\sigma(A))],\end{array}$$

where $p^*:K(M)\to K(P^*M)$ is induced by the projection

$$p:P^*M\to M.$$

Lemma 1 *χ^{ev} is an isomorphism.*

Proof. Let us construct the inverse

$$K(P^*M)/p^*K(M)\stackrel{\chi^{-1}}{\longrightarrow}\mathcal{S}^{ev}(M).$$

Consider an element $[L]\in K(P^*M)$, where $L\in\operatorname{Vect}(P^*M)$ is a vector bundle. Let us embed it as a subbundle in the trivial $p^*\mathbb{C}^N$. The orthogonal projection onto L is denoted by $\Pi=\Pi(x,\xi)$

$$\Pi:p^*\mathbb{C}^N\longrightarrow p^*\mathbb{C}^N,\ \operatorname{Im}\Pi\simeq L.$$

Let us define the mapping χ^{-1} by the formula

$$\chi^{-1}[L]=[A],$$

where

$$A:C^\infty\left(M,\mathbb{C}^N\right)\to C^\infty\left(M,\mathbb{C}^N\right)$$

is an arbitrary elliptic self-adjoint admissible even order operator with the principal symbol

$$\sigma(A)(x,\xi) = |\xi|^{2l}(2\Pi(x,\xi) - 1) : \pi^*\mathbb{C}^N \longrightarrow \pi^*\mathbb{C}^N,$$

and $\pi : S^*M \to M$ is the projection. Let us show that this mapping is well-defined. Indeed, consider two realizations of L as a subbundle

$$L \subset p^*\mathbb{C}^{N_1}, L \subset p^*\mathbb{C}^{N_2}.$$

Then these subbundles are homotopic, i.e. there exists a homotopy of projections

$$\Pi_t : p^*\mathbb{C}^{N_1+N_2} \longrightarrow p^*\mathbb{C}^{N_1+N_2},$$

over P^*M, such that

$$\operatorname{Im}\Pi_0 = L \oplus 0 \subset p^*\mathbb{C}^{N_1+N_2}, \ \operatorname{Im}\Pi_1 = 0 \oplus L \subset p^*\mathbb{C}^{N_1+N_2}.$$

The homotopy of projections Π_t can be lifted up to a homotopy of admissible self-adjoint operators A_t. Therefore, the equivalence class of A is independent of the realization of the bundle L as a subbundle $L \subset p^*\mathbb{C}^N$. This shows that the mapping χ^{-1} is well-defined. It is the inverse to χ^{ev} by construction. Lemma 1 is proved.

□

Remark 1 If we drop the requirement of admissibility in the definition of the group $\mathcal{S}^{ev}(M)$, then the corresponding group $\mathcal{S}(M)$ (which is the group of stable homotopy classes of elliptic self-adjoint operators) was defined in [4], where it is shown that a similar mapping

$$\chi : \mathcal{S}(M) \longrightarrow K(S^*M)/K(M)$$

is an isomorphism. It is also known that the latter group is isomorphic to $K_c^1(T^*M)$ (the elements have compact supports) under the coboundary mapping

$$K(S^*M) \xrightarrow{\delta} K_c^1(T^*M).$$

In what follows, we denote by $[\sigma(A)]$ the corresponding element of the odd K-group

$$[\sigma(A)] = \delta\,[\operatorname{Im}\Pi_+\sigma(A)] \in K_c^1(T^*M).$$

On an odd-dimensional M, the group $K(P^*M)/p^*K(M)$ is a torsion group. More precisely, the following result is valid.

Lemma 2 (Gilkey [1]) *Suppose that* $\dim M$ *is odd. Then* $K^*(P^*M)/p^*K^*(M)$ *is a 2-torsion group, i.e. the orders of its elements are powers of two.*

2. The last two lemmas imply that for some $N \geq 1$ there exists a homotopy of admissible operators

$$B_t : C^\infty\left(M, 2^N E\right) \longrightarrow C^\infty\left(M, 2^N E\right), \quad t \in [0,1];$$

$$B_0 = 2^N A = \underbrace{A \oplus A \oplus \ldots \oplus A}_{2^N \text{times}}, \quad B_1 = \triangle^l_{E'} \oplus \left(-\triangle^l_{E''}\right), \tag{4}$$

where operator B_1 is a direct sum of powers of invertible operators

$$\triangle_{E'} : C^\infty\left(M, E'\right) \to C^\infty\left(M, E'\right),$$

$$\triangle_{E''} : C^\infty\left(M, E''\right) \to C^\infty\left(M, E''\right),$$

with positive definite leading symbols, and the following vector bundle isomorphism is valid

$$2^N E \simeq E' \oplus E''.$$

Applying Proposition 1 to the homotopy B_t, we get

$$\eta\left(B_1\right) - \eta\left(B_0\right) = \mathrm{sf}\left(B_t\right)_{t\in[0,1]}.$$

At the end points we have

$$\eta\left(B_1\right) = \eta\left(\triangle^l_{E'}\right) - \eta\left(\triangle^l_{E''}\right) = 0, \quad \eta\left(B_0\right) = 2^N \eta\left(A\right).$$

Thus, the fractional part of the η-invariant of A is expressed as

$$\left\{\eta\left(A\right)\right\} = \left\{-\frac{1}{2^N}\mathrm{sf}\left(B_t\right)_{t\in[0,1]}\right\}.$$

In other words,

$$\left\{\eta\left(A\right)\right\} = -\frac{1}{2^N} \bmod 2^N\text{-}\mathrm{sf}\left(B_t\right)_{t\in[0,1]}, \tag{5}$$

where $\bmod 2^N$ denotes the residue of an integer modulo 2^N.

Remark 2 Unfortunately, the formula for the spectral flow with values in a finite cyclic group (see [4]) does not apply to the present case, since the operator B_t does not have the form, which was assumed in that paper. In the following section, we compute (5) in terms of the principal symbol $\sigma\left(B_t\right)$ of the operator family B_t.

3 Computation of the spectral flow modulo n

Let n be a fixed natural number. Consider a smooth family of elliptic self-adjoint operators $A_t, t \in [0,1]$, such that

$$A_0 = nA, \quad A_1 = \triangle^l_{E_1} \oplus \left(-\triangle^l_{E_2}\right).$$

The residue modulo n of the spectral flow of the family (A_t) is denoted by

$$\operatorname{mod} n\text{-sf}\,(A_t)_{t\in[0,1]} \in \mathbb{Z}_n \tag{6}$$

(cf. [4]).

The spectral flow for families of this form can be computed by means of the Atiyah–Singer index formula for families. Namely, the spectral flow (6) can be computed in terms of the difference construction

$$[\sigma(A_t)] \in K_c(T^*M, \mathbb{Z}_n) \tag{7}$$

in K-theory with coefficients $\mathbb{Z}_n$. The group on the right-hand side of (7) is defined as

$$K_c(T^*M, \mathbb{Z}_n) = K_c(T^*M \times M_n, T^*M \times pt), \tag{8}$$

here M_n denotes the Moore space for the group $\mathbb{Z}_n$. Let us represent a generator of the reduced group $\widetilde{K}^0(M_n) = \mathbb{Z}_n$ by a line bundle γ. Let us also fix a trivialization

$$n\gamma \overset{\beta}{\simeq} \mathbb{C}^n.$$

It follows from (8) that the elements of the group $K_c(T^*M, \mathbb{Z}_n)$ can be represented as families of elliptic symbols on M, which are parametrized by the Moore space.

Let us define the difference construction

$$\begin{gathered}[\sigma(A_t)] = \left[\operatorname{Im}\Pi_+(\sigma(A_{1-t}))_{t\in[0,1]} \otimes \gamma, \operatorname{Im}\Pi_+(\sigma(A_{1-t}))_{t\in[0,1]}\right] \in \\ K(B^*M \times M_n, B^*M \times pt \cup S^*M \times M_n) \simeq \\ \simeq K_c(T^*M \times M_n, T^*M \times pt),\end{gathered} \tag{9}$$

where the vector bundles $\operatorname{Im}\Pi_+(\sigma(A_{1-t}))$ are located over the cospheres of M with radius t. The vector bundle isomorphism over the space $S^*M \times M_n$ (i.e. for $t=1$) is defined by means of the following equivalences

$$\begin{aligned}\operatorname{Im}\Pi_+(\sigma(A_0)) \otimes \gamma &= n \operatorname{Im}\Pi_+(\sigma(A)) \otimes \gamma \simeq \\ &\simeq \operatorname{Im}\Pi_+(\sigma(A)) \otimes n\gamma \overset{1\otimes\beta}{\simeq} \operatorname{Im}\Pi_+(\sigma(A)) \otimes \mathbb{C}^n \simeq \\ &\simeq n \operatorname{Im}\Pi_+(\sigma(A)) = \operatorname{Im}\Pi_+(\sigma(A_0)).\end{aligned}$$

We are now ready to prove the formula for mod-n spectral flow.

Theorem 1 *The following equality is valid*

$$\bmod n\text{-sf}\,(A_t)_{t\in[0,1]} = -p_!\,[\sigma\,(A_t)]\,, \tag{10}$$

where

$$p_! : K_c\,(T^*M, \mathbb{Z}_n) \longrightarrow K\,(pt, \mathbb{Z}_n) = \mathbb{Z}_n \tag{11}$$

is the direct image mapping in K-theory with coefficients $\mathbb{Z}_n$, which is induced by the mapping $p : M \longrightarrow pt$.

Proof. Let us consider the family $(A_t)_{t\in[0,1]}$ as a family over the parameter space $[0,1]\times M_n$. Denote by $A_t\otimes 1_\gamma$ the corresponding family with coefficients in γ. Consider the composition of these self-adjoint elliptic families:

$$B_t = \begin{cases} A_t\otimes 1_\gamma & t>0 \\ A_{-t} & t<0 \end{cases}, \qquad t\in[-1,1]\,.$$

At $t=0$ the families are glued by the isomorphism

$$A_0\otimes 1_\gamma = nA\otimes 1_\gamma \simeq A\otimes 1_{n\gamma} \overset{1\otimes\beta}{\simeq} A\otimes 1_{\mathbb{C}^n} = nA = A_0.$$

We obtain immediately

$$\text{sf}\,(B_t)_{t\in[-1,1]} = \text{sf}\,(A_t)_{t\in[-1,1]}\,([\gamma]-1) \in \widetilde{K}\,(M_n)\,. \tag{12}$$

On the other hand, the spectral flow for families of this form, i.e. for families that start and end on the direct sums of positive and negative definite operators, is equal to the index of an elliptic operator (see Proposition 5 and the Corollary that follows it in the Appendix A). Thus, the following equality is valid

$$\text{sf}\,(B_t)_{t\in[-1,1]} = -\operatorname{ind} U.$$

The element $[\sigma\,(A_t)]$, which we defined in (9), coincides with $[\sigma\,(U)]$ by construction. Hence, (12) and the Atiyah–Singer index theorem for families give the desired

$$\bmod n\text{-sf}\,(A_t)\,([\gamma]-1) = -p_!\,[\sigma\,(A_t)] \in \widetilde{K}\,(M_n) = \mathbb{Z}_n.$$

The formula for spectral flow modulo n is proved.

□

Let us apply the formula for the spectral flow to the family (4). We obtain the following result for the fractional part of the η-invariant.

Theorem 2 *Let A be an admissible elliptic self-adjoint operator of a positive even order on an odd-dimensional manifold.*

1. *The following formula is valid*

$$\{\eta(A)\} = \frac{1}{2^N} p_! [\sigma_t] \in \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}, \tag{13}$$

where σ_t *is an arbitrary homotopy of invertible self-adjoint even symbols*

$$\sigma_t(x, -\xi) = \sigma_t(x, \xi), \ \sigma_0 = 2^N \sigma(A), \sigma_1 = 1_{\pi^* E_1} \oplus (-1_{\pi^* E_2}) \tag{14}$$

on the cospheres S^*M. *The direct image mapping*

$$p_! : K_c(T^*M, \mathbb{Z}_{2^N}) \to K(pt, \mathbb{Z}_{2^N}) \simeq \mathbb{Z}_{2^N}$$

is induced by $p: M \to pt$.

2. *Consider the element*

$$i_*^\infty [\sigma_t] \in K_c\left(T^*M, \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}\right),$$

which is obtained under the natural inclusion of coefficient groups

$$i^\infty : \mathbb{Z}_{2^N} \subset \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}.$$

This element is independent of the choice of the homotopy σ_t.

The K*-group with coefficients* $\mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z}$ *is defined as the direct limit as* $N \to \infty$

$$\varinjlim K_c(T^*M, \mathbb{Z}_{2^N}) = K_c\left(T^*M, \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}\right).$$

It is induced by the mappings

$$K_c(T^*M, \mathbb{Z}_{2^N}) \to K_c(T^*M, \mathbb{Z}_{2^{N+1}}) \to \ldots$$

Proof. The first assertion follows directly from (5) and Theorem 1.

Let us prove the second assertion. Consider two homotopies σ_t, σ_t' of the form (14). We should prove the equality

$$i_*^{N'} [\sigma_t] = i_*^{N'} [\sigma_t'] \in K_c(T^*M, \mathbb{Z}_{2^{N+N'}})$$

for some N' and the natural inclusion

$$i^{N'} : \mathbb{Z}_{2^N} \to \mathbb{Z}_{2^{N+N'}}.$$

Consider the superposition of homotopies

$$\sigma_t \cup \sigma_{-t} \cup \sigma'_t. \tag{15}$$

In other words, as the parameter $t \in [0,3]$ increases, we start with the homotopy σ_t, then do it in the opposite direction, and finally, carry out the second homotopy σ'_t. Obviously, this superposition is equivalent to the homotopy σ'_t. Hence, we obtain an equality for the difference constructions

$$[\sigma_t \cup \sigma_{-t} \cup \sigma'_t] = [\sigma'_t] \in K_c\left(T^*M, \mathbb{Z}_{2^N}\right).$$

By the definition (9) of these elements, we obtain

$$[\sigma_t \cup \sigma_{-t} \cup \sigma'_t] - [\sigma_t] = [\sigma_{-t} \cup \sigma'_t] \otimes ([\gamma] - 1),$$

$$[\sigma_{-t} \cup \sigma'_t] \in K_c\left(T^*M\right).$$

The family $\sigma_{-t} \cup \sigma'_t$ is even with respect to the cotangent variables. We conclude that the element $[\sigma_{-t} \cup \sigma'_t]$ lies in the range of the composition

$$K^1(P^*M)/p^*K^1(M) \longrightarrow K^1(S^*M)/\pi^*K^1(M) \longrightarrow K_c\left(T^*M\right).$$

By virtue of Lemma 2, we have for some N'

$$2^{N'}[\sigma_{-t} \cup \sigma'_t] = 0.$$

Consequently,

$$i_*^{N'}[\sigma'_t] - i_*^{N'}[\sigma_t] = 2^{N'}[\sigma_{-t} \cup \sigma'_t] \otimes ([\gamma] - 1) = 0.$$

This gives the desired

$$i_*^{\infty}[\sigma_t] = i_*^{\infty}[\sigma'_t].$$

□

4 Eta invariant and linking index in K-theory

The formula (13) can be written in the form that is more suitable for applications. We shall show that the fractional part of twice the η-invariant is determined by the element of the group $K_c^1(T^*M)$ (see Remark 1). This element is defined by the principal symbol of the operator. The η-invariant can be calculated as the *linking index* in K-theory.

1. Let us define the *linking pairing*

$$\langle,\rangle : \operatorname{Tor} K_c^{i+1}\left(T^*M\right) \times \operatorname{Tor} K^i\left(M\right) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{16}$$

in the following way (cf. [12]).

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{i}{\longrightarrow} \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

induces a long exact sequence in K-theory

$$\cdots \to K_c(T^*M) \to K_c(T^*M,\mathbb{Q}) \to K_c(T^*M,\mathbb{Q}/\mathbb{Z}) \stackrel{\partial}{\to} K_c^1(T^*M) \stackrel{i_*}{\to} K_c^1(T^*M,\mathbb{Q}) \to \cdots \tag{17}$$

Here K-theories with coefficients $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are defined as the direct limits (see [4])

$$\begin{array}{ll} K^*(X,\mathbb{Q}) = \lim\limits_{N\to\infty} K^*(X,\mathbb{Z}), & \mathbb{Z} \stackrel{\times N}{\to} \mathbb{Z}, \\ K^*(X,\mathbb{Q}/\mathbb{Z}) = \lim\limits_{N\to\infty} K^*(X,\mathbb{Z}_N), & \mathbb{Z}_n \subset \mathbb{Z}_{mn}. \end{array} \tag{18}$$

Let us note the natural isomorphism

$$K^*(X,\mathbb{Q}) \simeq K^*(X)\otimes\mathbb{Q}.$$

Consider two torsion elements

$$x \in \operatorname{Tor} K_c^{i+1}(T^*M), \quad y \in \operatorname{Tor} K^i(M).$$

Then $i_*x = 0$ and the exact sequence (17) implies that for some $x' \in K_c^i(T^*M,\mathbb{Q}/\mathbb{Z})$ the following equality holds $\partial x' = x$.

Definition 3 The number

$$\langle x, y\rangle = p_!(x'y), \tag{19}$$

is called the *linking index* of elements x and y. Here

$$p_! : K_c(T^*M,\mathbb{Q}/\mathbb{Z}) \to K_c(pt,\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}.$$

Lemma 3 *The product*

$$x'y \in K_c(T^*M,\mathbb{Q}/\mathbb{Z})$$

is determined by x and y, it is independent of the choice of x'. Therefore, the linking index is well-defined.

Proof. The sequence (17) implies that the arbitrariness in the choice of x' is generated by the elements of the group $K_c^i(T^*M)\otimes\mathbb{Q}$. However, the product

$$\left(K_c^i(T^*M)\otimes\mathbb{Q}\right) \times \operatorname{Tor} K^i(M) \longrightarrow K_c^0(T^*M)\otimes\mathbb{Q}$$

is trivial. This proves the lemma.

□

Theorem 3 *The following equality holds*

$$\{2\eta(A)\} = \langle [\sigma(A)], [1-\Lambda^n(M^n)] \rangle, \tag{20}$$

where A is an admissible elliptic self-adjoint even order operator, $\Lambda^n(M^n)$ is the orientation bundle of the odd-dimensional manifold M.

Remark 3 It follows from this result that the invariant $\{2\eta(A)\}$ is determined by the element $[\sigma(A)] \in K_c^1(T^*M)$.

Proof of Theorem 3. By virtue of (13), the fractional part of the η-invariant is equal to

$$\{\eta(A)\} = \frac{1}{2^N} p_! [\sigma_t], \quad [\sigma_t] \in K_c(T^*M, \mathbb{Z}_{2^N}).$$

The element $[\sigma_t]$ satisfies the relation (see (17))

$$\partial [\sigma_t] = [\sigma(A)] \in K_c^1(T^*M).$$

Moreover, this element is invariant under the action of antipodal involution

$$\alpha : T^*M \to T^*M, \quad \alpha(x,\xi) = (x,-\xi)$$

on the K-group. Thus,

$$\{2\eta(A)\} = \frac{1}{2^N} p_! (1+\alpha^*)[\sigma_t]. \tag{21}$$

We show in Appendix B that the involution α^* acts on the K-group as the product with the orientation bundle:

$$\alpha^* x = (-1)^n [\Lambda^n(M^n)] x.$$

Substituting the last expression in (21), we get

$$\{2\eta(A)\} = \frac{1}{2^N} p_! ([\sigma_t](1-\Lambda^n(M^n))).$$

The right hand side of this expression coincides with the linking index (see (19), (18)). The theorem is proved.

□

2. The formula for the fractional part of the η-invariant can be written in the form, which resembles the index formula in K-theory. In contrast to the index, which is computed under the collapsing map to the point, the η-invariant is computed in terms of the map to the projective space.

The orientation bundle $\Lambda^n(M^n)$ is a line bundle with the structure group $\mathbb{Z}_2$. Consider the classifying mapping

$$f : M^n \longrightarrow \mathbb{B}\mathbb{Z}_2 = \mathbb{RP}^{\infty},$$

i.e. the mapping $f : M^n \to \mathbb{RP}^{2N}$, for which an isomorphism is valid

$$\Lambda^n(M^n) \simeq f^*\gamma,$$

here γ is the line bundle over the projective space $\mathbb{RP}^{2N}$. The mapping f is uniquely defined up to homotopy. The reduced K-group of the projective space $\mathbb{RP}^{2N}$ is a cyclic group

$$\widetilde{K}(\mathbb{RP}^{2N}) = \mathbb{Z}_{2^N},$$

the generator is $1 - [\gamma]$. The Poincare duality for torsion subgroups (16) implies that the value of the linking pairing with this element defines an isomorphism $\simeq$

$$\begin{array}{rcl} K^1_c(T^*\mathbb{RP}^{2N}) = \operatorname{Tor} K^1_c(T^*\mathbb{RP}^{2N}) & \simeq & \mathbb{Z}_{2^N} \in \mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z} \\ x & \mapsto & \langle x, [1-\gamma]\rangle. \end{array} \qquad (22)$$

Proposition 2 *The fractional part of twice the η-invariant is equal to*

$$\{2\eta(A)\} = f_!\,[\sigma(A)], \qquad (23)$$

$$f_!\,[\sigma(A)] \in K^1_c(T^*\mathbb{RP}^{2N}) \simeq \mathbb{Z}_{2^N} \subset \mathbb{Z}\left[\frac{1}{2}\right]\Big/\mathbb{Z}.$$

Proof. From the definitions of the linking index and the mapping f, we obtain

$$\{2\eta(A)\} = p_!\left([\sigma(A)] \times [1 - \Lambda^n(M^n)]\right) = p_!\left([\sigma(A)] \times [1 - f^*\gamma]\right).$$

Here $\times$ denotes the product

$$\operatorname{Tor} K^{i+1}_c(T^*M) \times \operatorname{Tor} K^i(M) \longrightarrow K_c(T^*M, \mathbb{Q}/\mathbb{Z})$$

from Lemma 3. By the functoriality of the direct image map $p_!$, we have

$$\{2\eta(A)\} = p'_! f_!\left([\sigma(A)] \times f^*[1-\gamma]\right) = p'_!\left(f_!\,[\sigma(A)] \times [1-\gamma]\right) = \langle f_!\,[\sigma(A)]\,, [1-\gamma]\rangle,$$

where $p' : \mathbb{RP}^N \to pt$. By virtue of (22), the last expression coincides with the desired (23). The Proposition is proved.

□

Corollary 1 *On an orientable manifold, the η-invariant is at most a half-integer. In the nonorientable case the following estimate for its denominator is valid:*

$$\left\{2^{k+1}\eta\left(A\right)\right\}=0, \tag{24}$$

on an $n=2k+1$-dimensional manifold.

Proof. By the approximation theorem, we can choose the classifying mapping

$$f:M\to\mathbb{RP}^{\,n}.$$

The reduced K-groups for the projective spaces have the form

$$\widetilde{K}\left(\mathbb{RP}^{\,2k}\right)\simeq\widetilde{K}\left(\mathbb{RP}^{\,2k+1}\right)\simeq\mathbb{Z}_{2^k}.$$

Consequently,

$$2^k\left[1-\Lambda^n\left(M^n\right)\right]=0.$$

Hence, we obtain the desired

$$\left\{2^{k+1}\eta(A)\right\}=\left\langle\left[\sigma(A)\right],\left[2^k\left(1-\Lambda^n\left(M^n\right)\right)\right]\right\rangle=\left\langle\left[\sigma(A)\right],0\right\rangle=0.$$

□

Remark 4 For Dirac type operators on even-dimensional manifolds, similar estimates were obtained in [2].

5 Odd order operators

1. Let

$$A:C^\infty\left(M,E\right)\to C^\infty\left(M,E\right)$$

be an admissible elliptic self-adjoint operator.

Let us introduce the stable homotopy equivalence relation on this set of operators.

Definition 4 The direct sums

$$A\oplus\left(-A\right),$$

for an elliptic self-adjoint operator A are called *trivial operators.* Two operators A_1 and A_2 are said to be *stably homotopic,* if for some trivial operators A', A'' the direct sums

$$A_1\oplus A' \quad \text{and } A_2\oplus A''$$

are homotopic.

The group of stable homotopy equivalence classes of odd order $d = 2l + 1$ elliptic operators on M is denoted by $\mathcal{S}^{odd}(M)$. It is a group with respect to the direct sum.

Principal symbols of odd order operators satisfy the equality

$$\sigma(A)(x, -\xi) = -\sigma(A)(x, \xi).$$

This gives

$$\operatorname{Im}\Pi_+(\sigma(A))(x, -\xi) = [\operatorname{Im}\Pi_+(\sigma(A))(x, \xi)]^\perp, \tag{25}$$

for the bundle defined by the positive spectral projection $\Pi_+(\sigma(A))$.

In contrast to the case of even order operators, the vector bundles on S^*M, which satisfy (25), can not be described in terms of some bundles over the projectivization P^*M, as it was done in Section 2. However, the following result is valid. Its proof follows from [13].

Theorem 4 *The group $\mathcal{S}^{odd}(M)$ is a 2-torsion group.*

Proof. For the sake of completeness of the presentation, let us recall the proof of this result. It suffices to show that for an arbitrary elliptic self-adjoint operator A of odd order for some N there exists a homotopy

$$2^N A \sim 2^N(-A)$$

(this gives the desired triviality $2^{N+1}A \sim 2^N(A \oplus -A)$).

Denote by L the vector bundle $\operatorname{Im}\Pi_+\sigma(A) \subset \pi^*E$ on the cospheres. We obtain

$$\alpha^* L = L^\perp, \tag{26}$$

since A has odd order. Here $L^\perp$ is the orthogonal complement of the bundle L. Similarly to the proof of the theorem for even order operators, it is sufficient to construct a homotopy of subbundles

$$2^N L \sim 2^N L^\perp, \tag{27}$$

which satisfy (26).

We claim that modulo 2-torsion the elements of $K(S^*M)$ are invariant under the involution α^*. Indeed, the projection $S^*M \to P^*M$ on an even-dimensional M induces an isomorphism

$$K(P^*M) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \longrightarrow K(S^*M) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

in K-theory, modulo 2-torsion (this can be proved using the Mayer-Vietoris principle).

Hence, (26) implies the existence of an isomorphism

$$\sigma : 2^N L \longrightarrow 2^N L^\perp.$$

Let us extend it to the whole space $2^N\pi^*E \supset 2^N L$ in accordance with the decomposition

$$\widetilde{\sigma} : 2^N\pi^*E = 2^N L \oplus 2^N\alpha^* L \to 2^N\alpha^* L \oplus 2^N L = 2^N\pi^* E$$

as

$$\widetilde{\sigma}(\xi) = \sigma(\xi) \oplus \sigma(-\xi).$$

Consider an even vector bundle isomorphism

$$\widetilde{\sigma} \oplus \widetilde{\sigma}^{-1} : 2^{N+1}\pi^*E \to 2^{N+1}\pi^*E.$$

It takes $2^{N+1}L$ to $2^{N+1}\alpha^*L$ and it is homotopic to the identity, since it is the sum of mutually inverse isomorphisms. If we denote such homotopy by

$$\sigma_t, \quad \sigma_0 = 1, \sigma_1 = \widetilde{\sigma} \oplus \widetilde{\sigma}^{-1},$$

then the desired homotopy of subbundles (27), which satisfy (26), has the form

$$\sigma_t(2^{N+1}L).$$

This proves the Theorem.

□

Similarly to the even order case, Theorem 4 implies the existence of a homotopy

$$B_t,\ t \in [0,1]; B_0 = 2^N A, B_1 = A_0 \oplus (-A_0), \tag{28}$$

of odd order operators, where A_0 is invertible. We obtain as before $\eta(B_1) = 0$, and the η-invariant is expressed in terms of the spectral flow

$$\{\eta(A)\} = -\frac{1}{2^N} \operatorname{mod} 2^N\text{-sf}\,(B_t)_{t\in[0,1]}.$$

In order to compute the right-hand side of this equality, let us extend the family B_t up to a family, which satisfies the conditions of the Theorem on the spectral flow modulo n. To this end, it suffices to construct a homotopy of invertible operators

$$\begin{pmatrix} A_0 & 0 \\ 0 & -A_0 \end{pmatrix} \sim \begin{pmatrix} |A_0| & 0 \\ 0 & -|A_0| \end{pmatrix}.$$

The desired homotopy can be defined, for example, by the formula (see [14])

$$\begin{pmatrix} A_0 & 0 \\ 0 & -|A_0| \end{pmatrix} + (|A_0| - A_0)\begin{pmatrix} \sin^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \cos^2\varphi \end{pmatrix}, \quad \varphi \in \left[0, \frac{\pi}{2}\right]. \tag{29}$$

Now we can apply the spectral flow Theorem 1 to the superposition of homotopies (28) and (29). It is possible to obtain the expression for the fractional part of the η-invariant for odd order operators along the lines of (13).

Let us finally note that the formulas of Theorem 3 and Proposition 2, which express the η-invariant as a linking index, remain valid for odd order operators on even-dimensional manifolds.

2. Example (*pin*^{c} Dirac operator, see [2]). Consider the even-dimensional real projective space $\mathbb{RP}^{2n}$. A straightforward computation shows that its orientation bundle is isomorphic to the line bundle γ. Therefore, the orientation bundle is the generator of the reduced K-group

$$1-\left[\Lambda^{2n}\left(\mathbb{RP}^{2n}\right)\right]\in\widetilde{K}\left(\mathbb{RP}^{2n}\right).$$

On the other hand, $\mathbb{RP}^{2n}$ has a *pin*^{c} structure, while the principal symbol of the self-adjoint *pin*^{c} Dirac operator D on it defines a generator of the isomorphic group

$$[\sigma(D)]\in K_c^1\left(T^*\mathbb{RP}^{2n}\right)=\operatorname{Tor}K_c^1\left(T^*\mathbb{RP}^{2n}\right)\simeq\mathbb{Z}_{2^N}.$$

The symbol $\sigma(D)$ can be written explicitly.

Consider the set of Clifford matrices $e_0, e_1, \ldots, e_{2n}$:

$$e_k e_j + e_j e_k = 2\delta_{kj}$$

of dimension $2^n\times 2^n$. For a vector $v=(v_0,\ldots,v_{2n})\in\mathbb{R}^{2n+1}$, define a linear operator

$$e(v)=\sum_{i=0}^{2n} v_i e_i : \mathbb{C}^{2^n}\longrightarrow\mathbb{C}^{2^n}.$$

Consider the self-adjoint symbol on the unit sphere $\mathbb{S}^{2n}\subset\mathbb{R}^{2n+1}$

$$\sigma(D)(x,\xi)=ie(x)e(\xi):\mathbb{C}^{2^n}\longrightarrow\mathbb{C}^{2^n},$$

where ξ is a tangent vector at a point $x\in\mathbb{S}^{2n}$. The symbol $\sigma(D)(x,\xi)$ is invariant under the involution

$$(x,\xi)\longrightarrow(-x,-\xi).$$

Thus, it defines an elliptic symbol on the projective space $\mathbb{RP}^{2n}$ — this is the symbol of the *pin*^{c} Dirac operator.

The nondegeneracy of the linking pairing (16) implies that the above described generators satisfy the equality

$$\left\langle 2^{n-1}[\sigma(D)],\left[1-\Lambda^{2n}\left(\mathbb{RP}^{2n}\right)\right]\right\rangle=\frac{1}{2}.$$

Hence, the η-invariant of the pin^c Dirac operator D has a "large" fractional part:

$$\{2^n \eta(D)\} = \langle 2^{n-1}[\sigma(D)], \left[1 - \Lambda^{2n}\left(\mathbb{RP}^{2n}\right)\right]\rangle = \frac{1}{2}.$$

This example shows that the estimate (24) of the denominator of the η-invariant is sharp on even-dimensional manifolds. The fractional part $\{\eta(D)\}$ was first computed in [2], where the following equality was obtained

$$\{\eta(D)\} = \frac{1}{2^{n+1}}. \tag{30}$$

6 The problem of P. Gilkey

In this section, we construct an even order operator on an odd-dimensional manifold, such that the η-invariant has a nontrivial fractional part.

Consider the product $\mathbb{RP}^{2n} \times \mathbb{S}^1$. The coordinates are denoted by x, φ, the dual coordinates in the cotangent spaces are ξ, τ. Let us define a second-order elliptic differential operator $\mathcal{D}$ on this manifold.

On the cylinder $\mathbb{RP}^{2n} \times [0, \pi]$, consider the expression

$$\mathcal{D} = \begin{pmatrix} 2\sin\varphi\left(-i\frac{\partial}{\partial\varphi}\right)D - i\cos\varphi D & \triangle_x e^{-i\varphi} + \left(-i\frac{\partial}{\partial\varphi}\right)e^{i\varphi}\left(-i\frac{\partial}{\partial\varphi}\right) \\ \triangle_x e^{i\varphi} + \left(-i\frac{\partial}{\partial\varphi}\right)e^{-i\varphi}\left(-i\frac{\partial}{\partial\varphi}\right) & 2\sin\varphi\left(i\frac{\partial}{\partial\varphi}\right)D + i\cos\varphi D \end{pmatrix},$$

where D is the pin^c Dirac operator on the projective space (see previous section), while $\triangle_x$ denotes the Laplacian

$$\triangle_x = D^2.$$

The expression $\mathcal{D}$ is self-adjoint and elliptic. Indeed, the self-adjointness is obvious, while the ellipticity is a consequence of the fact that the principal symbol

$$\sigma(\mathcal{D}) = \begin{pmatrix} 2\tau\sin\varphi\sigma(D)(x,\xi) & \xi^2 e^{-i\varphi} + e^{i\varphi}\tau^2 \\ \xi^2 e^{i\varphi} + e^{-i\varphi}\tau^2 & -2\tau\sin\varphi\sigma(D)(x,\xi) \end{pmatrix}$$

satisfies the equation

$$\sigma(\mathcal{D})^2(\xi,\tau) = \left(\xi^2 + \tau^2\right)^2$$

(i.e. $\mathcal{D}$ is a square root of the square of the Laplacian).

The principal symbol of $\mathcal{D}$ has the following property

$$\sigma(\mathcal{D})|_{\varphi=\pi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma(\mathcal{D})|_{\varphi=0} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{31}$$

Let F be the vector bundle on $\mathbb{RP}^{2n} \times \mathbb{S}^1$, which is obtained from the trivial bundle $\mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n}$ on the cylinder $\mathbb{RP}^{2n} \times [0,\pi]$ by means of the following transition function, which is defined on its bases

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It follows from (31) that the principal symbol $\sigma(\mathcal{D})$ acts on the bundle F:

$$\sigma(\mathcal{D}) : \pi^* F \longrightarrow \pi^* F, \quad \pi : S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1) \to \mathbb{RP}^{2n} \times \mathbb{S}^1.$$

Denote by

$$\mathcal{D} : C^\infty\left(\mathbb{RP}^{2n} \times \mathbb{S}^1, F\right) \longrightarrow C^\infty\left(\mathbb{RP}^{2n} \times \mathbb{S}^1, F\right)$$

a second-order elliptic self-adjoint differential operator, which is obtained as a result of smoothing the coefficients of the operator $\mathcal{D}$ on the product $\mathbb{RP}^{2n} \times \mathbb{S}^1$.

The following theorem answers a question of [1].

Theorem 5 *The following equality is valid*

$$\{2\eta\left(\mathcal{D}\right)\} = \frac{1}{2^{n-1}}.$$

Corollary 2 *There exist even order operators on odd-dimensional manifolds with an arbitrary dyadic η-invariants.*

The proof of the theorem is given at the end of the section. It is based on several auxiliary statements, which we now describe.

Unfortunately, the direct application of the formulas (20) or (23) to $\mathcal{D}$ is rather cumbersome (it is equivalent to the direct computation of the direct image in K-theory).

Our computation of the η-invariant is based on the following observation. The principal symbol of $\mathcal{D}$ defines the element

$$[\sigma\left(\mathcal{D}\right)] \in K_c^1\left(T^*\left(\mathbb{RP}^{2n} \times \mathbb{S}^1\right)\right),$$

which is representable as the symbol of the "cross product" [4] of an elliptic self-adjoint operator on the projective space and an elliptic operator on the circle with a nonzero index.

Consider the pseudodifferential elliptic operator D_1 of order 1

$$D_1 = \frac{1}{2}\left[e^{-i\varphi}\left(Q + |Q|\right) + e^{i\varphi}\left(|Q| - Q\right)\right], \quad Q = -i\frac{d}{d\varphi},\ \varphi \in [0, 2\pi], \tag{32}$$

on the circle $\mathbb{S}^1$ of length 2π. The expression

$$D_1 e^{ik\varphi} = \begin{cases} ke^{i(k-1)\varphi}, & k \geq 0, \\ -ke^{i(k+1)\varphi}, & k < 0 \end{cases}$$

gives: $\operatorname{ind} D_1 = 2$.

Proposition 3 *The following equality holds*

$$[\sigma(\mathcal{D})] = [\sigma(D)][\sigma(D_1)], \tag{33}$$

where

$$[\sigma(D)] \in K_c^1\left(T^*\mathbb{RP}^{2n}\right), \quad [\sigma(D_1)] \in K_c\left(T^*\mathbb{S}^1\right).$$

Proof. Recall that the product $[\sigma(D)][\sigma(D_1)]$ is realized as the so-called *cross product* $\sigma(D)\#\sigma(D_1)$ of elliptic symbols

$$\sigma(D)\#\sigma(D_1) = \begin{pmatrix} \sigma(D)\otimes 1 & 1\otimes\sigma(D_1^*) \\ 1\otimes\sigma(D_1) & -\sigma(D)\otimes 1 \end{pmatrix}. \tag{34}$$

In our case D_1 is not a differential operator. Hence, its principal symbol $\sigma(D_1)$ is not a polynomial. Therefore, the symbol $\sigma(D)\#\sigma(D_1)$ is not smooth (while the operator $D\#D_1$ is not pseudodifferential).

A trick to overcome this difficulty was introduced in [9]. The symbol (34) is replaced by a homotopic smooth symbol. To this end, let us replace the nonpseudodifferential operator $|Q|$ in the expression (32) for D_1 by a pseudodifferential operator $\sqrt{D^2+Q^2}$ on the product of manifolds. The resulting elliptic symbol

$$\begin{pmatrix} \sigma(D)(x,\xi) & \frac{1}{2}\left[e^{i\varphi}(\tau+1)+e^{-i\varphi}(1-\tau)\right] \\ \frac{1}{2}\left[e^{-i\varphi}(\tau+1)+e^{i\varphi}(1-\tau)\right] & -\sigma(D)(x,\xi) \end{pmatrix}, \ \xi^2+\tau^2=1 \tag{35}$$

is linearly homotopic to (34).

In order to compare the elements in (33), let us make one more homotopy. Note that the symbols on the antidiagonal of the matrix (35) are constant in the variables (ξ,τ) at the points $\varphi=0$ and π. Therefore, the symbol (35) is linearly homotopic to the symbol

$$\sigma'(x,\varphi,\xi,\tau) = \begin{pmatrix} |\sin\varphi|\sigma(D)(x,\xi) & \frac{1}{2}\left[e^{i\varphi}(\tau+1)+e^{-i\varphi}(1-\tau)\right] \\ \frac{1}{2}\left[e^{-i\varphi}(\tau+1)+e^{i\varphi}(1-\tau)\right] & -|\sin\varphi|\sigma(D)(x,\xi) \end{pmatrix}.$$

It turns out that the symbol σ' can be transformed to $\sigma(\mathcal{D})$ under the quadratic coordinate transformation

$$\Phi(\xi,\tau) = (2\tau\xi, \tau^2-\xi^2),$$

so that

$$\sigma'(x,\varphi,\Phi(\xi,\tau)) = \sigma(\mathcal{D})(x,\varphi,\xi,\tau), \quad \varphi\in[0,\pi].$$

Geometrically, this coordinate transformation is a double covering of the sphere such that the big circles, which pass through the north pole, are covered twice. Let us

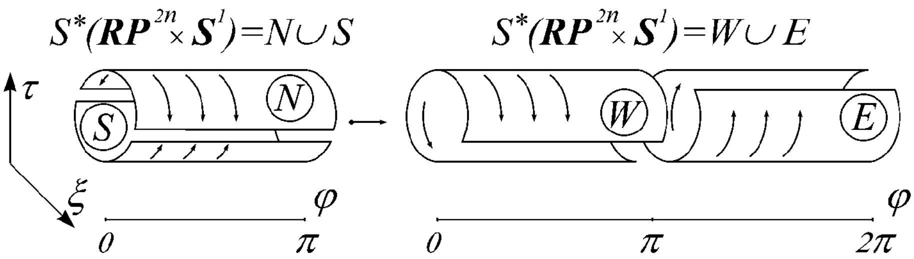

Figure 1: Two decompositions of $S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1)$

decompose the spherical bundle $S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1)$ into pieces, in order to obtain a one-to-one mapping.

Consider the bundle $S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1_\pi)$ with the circle of length π. The symbol $\sigma(\mathcal{D})$ is defined on this space. We decompose this cospherical bundle into two submanifolds

$$N = S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1_\pi) \cap \{\tau > 0\}, S = S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1_\pi) \cap \{\tau < 0\}$$

according to the sign of τ.

The space $S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1_{2\pi})$, where the symbol σ' is defined, corresponds to the circle of length 2π. This time we take a different decomposition

$$W = S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1_{2\pi}) \cap \{\tau \neq -1, \varphi \in (0,\pi)\},$$

$$E = S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1_{2\pi}) \cap \{\tau \neq 1, \varphi \in (\pi,2\pi)\}.$$

The pieces N, W and S, E are pairwise homeomorphic (see Fig. 1):

$$\Phi : N \longrightarrow W, \quad \Phi(x,\varphi,\xi,\tau) = (x,\varphi,2\tau\xi,\tau^2-\xi^2)$$

$$\Psi : S \longrightarrow E, \quad \Psi(x,\varphi,\xi,\tau) = (x,\varphi+\pi,-2\tau\xi,\xi^2-\tau^2).$$

In addition, the symbols satisfy the relations

$$\sigma' \circ \Phi = \sigma(\mathcal{D}), \quad \sigma' \circ \Psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma(\mathcal{D}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{36}$$

On the boundaries of N and S, the symbol $\sigma(\mathcal{D})$ depends only on the points of the base $\mathbb{RP}^{2n} \times \mathbb{S}^1$. Therefore, the following decomposition is valid

$$[\sigma(\mathcal{D})] = [\sigma(\mathcal{D})|_N] + [\sigma(\mathcal{D})|_S] \in K^1_c(T^*(\mathbb{RP}^{2n} \times \mathbb{S}^1). \tag{37}$$

Here we employ the isomorphism

$$K(S^*(\mathbb{RP}^{2n}\times\mathbb{S}^1))/K(\mathbb{RP}^{2n}\times\mathbb{S}^1)\simeq K_c^1(T^*(\mathbb{RP}^{2n}\times\mathbb{S}^1)).$$

Similarly, the symbol σ' depends only on x,φ on the boundary of W,E. Consequently,

$$[\sigma'] = [\sigma'|_N] + [\sigma'|_S]\,. \tag{38}$$

Substituting the result of transformations (37), (38) in (36), we obtain the desired equality

$$[\sigma(\mathcal{D})] = [\sigma'].$$

□

Let $M_{1,2}$ be two smooth closed manifolds. Consider two elements

$$[\sigma_1]\in \mathrm{Tor}K_c^1\left(T^*M_1\right),[\sigma_2]\in K_c^0\left(T^*M_2\right).$$

The product $M_1\times M_2$ is denoted by M. Let us compute the linking index of the product

$$[\sigma_1]\,[\sigma_2]\in \mathrm{Tor}\,K_c^1\left(T^*M\right)$$

with the orientation bundle of $M_1\times M_2$.

Proposition 4 *Suppose that M_2 is orientable. Then the following formula is valid*

$$\left\langle [\sigma_1]\,[\sigma_2]\,,\left[1-\Lambda^n\left(M^n\right)\right]\right\rangle = \left\langle [\sigma_1]\,,\left[1-\Lambda^k\left(M_1\right)\right]\right\rangle p_!\,[\sigma_2]\,,\quad \dim M_1 = k. \tag{39}$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} K_c^0\left(T^*M_1,\mathbb{Q}/\mathbb{Z}\right)\times K_c^0\left(T^*M_2\right) & \longrightarrow & K_c^0\left(T^*M,\mathbb{Q}/\mathbb{Z}\right)\\ \uparrow{\scriptstyle j\times 1} & & \uparrow{\scriptstyle j}\\ \mathrm{Tor}\,K_c^1\left(T^*M_1\right)\times K_c^0\left(T^*M_2\right) & \longrightarrow & \mathrm{Tor}\,K_c^1\left(T^*M\right), \end{array}$$

where j denotes the multiplication (see Lemma 3) from the right by $[1-\Lambda^k(M_1)]$. The horizontal maps denote products in K-theory. By virtue of the orientability of the second factor, we have

$$1-\left[\Lambda^k\left(M_1\right)\right] = 1-\left[\Lambda^n\left(M\right)\right].$$

Hence, (39) follows from this diagram, when we apply the direct image mapping

$$p_!: K_c^0\left(T^*M,\mathbb{Q}/\mathbb{Z}\right)\to\mathbb{Q}/\mathbb{Z}.$$

This proves Proposition 4.

□

Remark 5 Formula (39) is an analog of the well-known property of the η-invariant (see [4]), namely, the *η-invariant of a cross product of operators is the product of the η-invariant of the first factor times the index of the second one.*

Proof of Theorem 5. By virtue of (20), we have

$$\{2\eta(\mathcal{D})\} = \left\langle [\sigma(\mathcal{D})], [1-\Lambda^{2n+1}(\mathbb{RP}^{2n}\times\mathbb{S}^1)]\right\rangle.$$

The formula (39) for the product and the decomposition (33) lead to

$$\{2\eta(\mathcal{D})\} = \left\langle [\sigma(D)], [1-\Lambda^{2n}(\mathbb{RP}^{2n})]\right\rangle \operatorname{ind} D_1.$$

Consequently

$$\{2\eta(\mathcal{D})\} = \{2\eta(D)\} \operatorname{ind} D_1 = \frac{1}{2^n}2 = \frac{1}{2^{n-1}}.$$

In the second equality we use (30).

□

Appendix A. Spectral flow and index

Let $(A_t)_{t\in[0,1]}$ be a family of elliptic self-adjoint operators on a compact closed manifold, which starts and ends on the direct sums of positive- and negative-definite operators:

$$A_0 = \triangle_{E_0'} \oplus \left(-\triangle_{E_0''}\right), \; A_1 = \triangle_{E_1'} \oplus \left(-\triangle_{E_1''}\right).$$

The spectral flow for families of this form is equal to the index of an elliptic operator on M.

Let us denote by

$$U : C^\infty(M, E_0') \longrightarrow C^\infty(M, E_1') \tag{40}$$

a pseudodifferential operator with principal symbol $\sigma(U)$, which is defined as the solution of the Cauchy problem

$$\frac{d}{dt}u_t = \left[\frac{d}{dt}\Pi_+(\sigma(A_t)), \Pi_+(\sigma(A_t))\right]u_t; \quad u_0 = 1 \tag{41}$$

at $t = 1$:

$$\sigma(U) = u_1.$$

One can show that the difference construction

$$[\sigma(U)] \in K(B^*M, S^*M)$$

for (40) has the form

$$[\sigma(U)] = \left[\operatorname{Im}\Pi_+(\sigma(A_t))_{t\in[0,1]}, \pi_0^* E_1'\right], \qquad \pi_0 : B^*M \to M,$$

for the vector bundle

$$\operatorname{Im}\Pi_+(\sigma(A_t))_{t\in[0,1]}, \pi_0^* E_1' \in \operatorname{Vect}(B^*M).$$

Here the base of the bundle $\operatorname{Im}\Pi_+(\sigma(A_t))$ is the cosphere space of radius t. The bundles $\operatorname{Im}\Pi_+(\sigma(A_t))_{t\in[0,1]}$ and $\pi_0^* E_1'$ coincide over the subspace $S^*M \subset B^*M$.

Proposition 5 *The following equality holds*

$$\operatorname{sf}(A_t)_{t\in[0,1]} = -\operatorname{ind} U. \tag{42}$$

Proof. Denote by P_t the family of nonnegative spectral projections for A_t. The expression

$$P_t = \frac{1}{2|A_t|}(|A_t| + A_t)$$

(for invertible A_t) together with the results of R. Seeley [15] imply that the principal symbol of this projection coincides with the projection $\Pi_+(\sigma(A_t))$. Denote by U_t a smooth family of pseudodifferential operators, which corresponds to the family of principal symbols defined by (41). It follows that the symbol u_t is unitary and it defines an isomorphism of subbundles

$$u_t : \operatorname{Im}\Pi_+(\sigma(A_0)) \longrightarrow \operatorname{Im}\Pi_+(\sigma(A_t)).$$

Hence, the operator

$$\Pi_+(A_t)\,U_t : \operatorname{Im}\Pi_+(A_0) \longrightarrow \operatorname{Im}\Pi_+(A_t)$$

has the fredholm property (its almost inverse is $\Pi_+(A_0)\,U_t^{-1}$). The desired equality (42) is a consequence of the following more general relation

$$\operatorname{sf}(A_\tau)_{\tau\in[0,t]} = -\operatorname{ind}\left(\Pi_+(A_t)\,U_t : \operatorname{Im}\Pi_+(A_0) \longrightarrow \operatorname{Im}\Pi_+(A_t)\right), \tag{43}$$

which connects the index and the spectral flow.

In the case of general position, the proof of (43) meets no essential difficulties. Indeed, for the variation of the parameter t, both sides of the formula have the same jumps. These occur only for those values of the parameter, for which some eigenvalue of A_t changes its sign (for the right-hand side the sign change leads to a discontinuous change of the spectral projection $\Pi_+(A_t)$ and, consequently, of the index). For example, if some eigenvalue at $t=t_0$ passes from the negative to the positive side then we obtain

$$\text{sf }(A_\tau)_{\tau\in[0,t+\varepsilon]} - \text{sf }(A_\tau)_{\tau\in[0,t-\varepsilon]} = \text{sf }(A_\tau)_{\tau\in[t-\varepsilon,t+\varepsilon]} = 1,$$

by virtue of the additivity of the spectral flow. Similarly for the index

$$\begin{gathered}\operatorname{ind}\left(\Pi_+(A_{t+\varepsilon})\,U_{t+\varepsilon}\Pi_+(A_0)\right) - \operatorname{ind}\left(\Pi_+(A_{t-\varepsilon})\,U_{t-\varepsilon}\Pi_+(A_0)\right) = \\ \operatorname{ind}\left(\Pi_+(A_{t+\varepsilon})\,U_{t+\varepsilon}U_{t-\varepsilon}^{-1}\Pi_+(A_{t-\varepsilon})\right) = -1\end{gathered}$$

(in this notation the spaces, where the operators act, are omitted — we assume that they act in the subspaces defined by the corresponding projections). We have used the logarithmic property of the index.

The proof of the statement in the general case can be obtained, if one uses the definition of the spectral flow in terms of spectral projections (see [16] and also [17]). This proves Proposition 5.

□

Remark 6 There is a generalization of this result to the case of families of self-adjoint operators over some parameter space $[0,1]\times X$ for a compact space X. The notion of the spectral flow in this case was introduced in [18].

Appendix B. Antipodal involution and orientability

Let V be a real vector bundle over a compact space X. In the present appendix, we investigate the action of the antipodal involution

$$\alpha : V \longrightarrow V, \qquad \alpha(v) = -v$$

on the group $K_c^*(V)$.

Theorem 6 *The following formula is valid*

$$\alpha^* = (-1)^n \Lambda^n(V) : K_c^*(V) \longrightarrow K_c^*(V), \quad n = \dim V.$$

Proof. It suffices to prove the statement for the group $K_c^0(V)$ of an even-dimensional bundle V, $\dim V = 2k$. The elements of this group can be realized in terms of the difference construction

$$[E, F, \sigma] \in K_c^0(V), \qquad E, F \in \operatorname{Vect}(X),$$

where

$$\sigma : \pi^* E \longrightarrow \pi^* F, \quad \pi : SV \to X$$

is an isomorphism over the spheres SV (we assume that V is equipped with a scalar product).

M. Karoubi showed in [19, 20] that the group $K_c^0(V)$ is generated by the elements with quadratic transition functions, which we describe below.

Denote by $Cl(V)$ the Clifford algebra bundle, associated with the bundle V. Consider the quadruples of the form

$$(E, c, f_1, f_2), \tag{44}$$

where (E, c) is a Clifford module over $Cl(V)$, i.e. a complex vector bundle E and a homomorphism of algebra bundles

$$c : Cl(V) \longrightarrow \mathrm{End}(E),$$

while the involutions

$$f_{1,2} \in \mathrm{End}(E), \quad f_1^2 = f_2^2 = 1$$

anticommute with the Clifford structure

$$f_{1,2}c(v) + c(v) f_{1,2} = 0, \quad v \in V \subset Cl(V).$$

According to [19], the group $K_c^0(V)$ is generated by the difference constructions of the form

$$[\ker(f_1 - 1), \ker(f_2 - 1), (1 - c(v) f_2)(1 + c(v) f_1)]. \tag{45}$$

It is clear from this expression that the antipodal involution α acts on the quadruples (44) as

$$\alpha^*(E, c, f_1, f_2) = (E, -c, f_1, f_2).$$

Let us show that the quadruple

$$(E, -c, f_1, f_2)$$

differs from

$$(E, c, f_1, f_2) \otimes \Lambda^{2k}(V)$$

by a vector bundle isomorphism.

Let us take a local frame $e_1, e_2, ..., e_{2k}$ in V and consider the element

$$\beta = i^k c(e_1) \dots c(e_{2k}).$$

Then $\beta^2 = 1$, β anticommutes with the Clifford structure c and commutes with the involutions $f_{1,2}$

$$\beta c(v) + c(v)\beta = 0, \quad f_i\beta = \beta f_i. \tag{46}$$

Under the change of the frame, the element β is multiplied by the sign of the transition matrix, i.e. globally it defines a vector bundle isomorphism

$$\beta : E \longrightarrow E \otimes \Lambda^{2k}(V).$$

The commutation relations (46) take the form

$$\beta^{-1}\left(c(v)\otimes 1_{\Lambda^{2k}(V)}\right)\beta = -c(v), \quad \beta^{-1}\left(f_i \otimes 1_{\Lambda^{2k}(V)}\right)\beta = f_i.$$

Hence, the quadruples

$$\alpha^*(E, c, f_1, f_2) \quad \text{and} \quad (E, c, f_1, f_2)\otimes \Lambda^{2k}(V)$$

are isomorphic. This proves the theorem, since elements (45) generate the whole group $K_c^0(V)$.

□

Corollary 3 *The same formula*

$$\alpha^* = (-1)^n \Lambda^n(V) : K_c^*(V, \mathbb{Z}_n) \longrightarrow K_c^*(V, \mathbb{Z}_n)$$

holds in K-theory with coefficients $\mathbb{Z}_n$.

Indeed, K-theory with coefficients in $\mathbb{Z}_n$ is defined by means of the Moore space M_n

$$K_c^*(V, \mathbb{Z}_n) = K_c^*(V \times M_n, V \times pt).$$

Applying the theorem just proved to the product $X \times M_n$ and the pull-back of V, we obtain the desired result in K-theory with coefficients.

Remark 7 For a smooth manifold X with the cotangent bundle $V = T^*X$, expression (45) defines a class of elliptic symbols, such that an arbitrary symbol reduces to a symbol from this class by a stable homotopy.

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