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Chapter 11

Noncommutative Analysis and High-Frequency Asymptotics

Contents

In this chapter, we show how noncommutative analysis can be applied to a problem of interest in the theory of electromagnetic waves and plasma physics. From the physical viewpoint, the problem describes electromagnetic wave propagation in spatially inhomogeneous plasma. From the viewpoint of mathematics, this is a problem of constructing the high-frequency asymptotics of the solution of an equation with a nonsmooth right-hand side. The usual WKB scheme of constructing asymptotic solutions fails for this equation, since, owing to the presence of the nonsmooth right-hand side, each subsequent term of the asymptotic expansion is less smooth (or, if the reader prefers it, more singular) than the preceding one, thus rendering the expansion useless. The remedy is to seek *mixed asymptotics* of the solution to this problem, that is, asymptotics simultaneously with respect to the large parameter and smoothness. The subsequent terms of such an asymptotic expansion become more and more smooth and decay as the large parameter tends to infinity more and more rapidly. This asymptotic expansion is already adequate to the problem in question and permits one to obtain experimentally verifiable results, in particular, those concerning the so-called *transient rays* (forerunners).

The physical statement of the problem can be found in [1]. The construction of mixed asymptotics in general by operator methods was considered in [2, 3] and other papers (e.g., see [4]). The specific application to this radiophysical problem was developed in [5, 6].

11.1 Statement of the Problem

We consider the following physical problem. In infinite space occupied by spatially inhomogeneous plasma, a point source of electromagnetic waves is switched on at time $t = 0$. The source frequency is high but can slowly vary in time, and so can the source amplitude. The problem is to find the high-frequency asymptotics of the wave field generated by the source.

Mathematically, the problem is described by the system of Maxwell equations for the electromagnetic field incorporating terms that take into account the presence of plasma. To reduce extensive computations and focus ourselves on essential points, we adopt a simplified model assuming a scalar (rather than a vector) field and dealing with one-dimensional rather than three-dimensional space. Then the problem is described by the equation

$$-\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^2 u}{\partial x^2} - \lambda^2 b^2(x) u = \lambda \delta(x) \theta(t) r(t) e^{-i\lambda q(t)}, \tag{11.1}$$

where u is the unknown wave field, c is the light velocity in vacuum, $\lambda b(x)$ is the so-called *plasma frequency*, $\lambda q'(t)$ is the instantaneous source frequency, $\lambda r(t)$ is the source amplitude, $\delta(x)$ is the Dirac function concentrated at the origin, and $\theta(t)$ is the Heaviside step function,

$$\theta(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0. \end{cases}$$

A discussion concerning the derivation of this model can be found in [1]. Here we adopt the model as being given and study the mathematical and physical consequences that can be derived from it. Let us make a few remarks on the model (11.1).

A. Since the equation is linear, we see that the factor λ on the right-hand side in (11.1) is not very essential. It has been inserted to ensure a convenient normalization of the solution.

B. We assume that the plasma frequency $\lambda b(x)$ and the source frequency $\lambda q'(t)$ are of the same order of magnitude; namely, they both are $O(\lambda)$, and λ is regarded mathematically as a large parameter in our problem. It can be thought of as the "mean plasma frequency" or "mean source frequency."

C. For physically meaningful solutions of Eq. (11.1), this equation with the step function $\theta(t)$ on the right-hand side is equivalent to the following Cauchy problem with initial data prescribed at $t = 0$:

$$-\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^2 u}{\partial x^2} - \lambda^2 b^2(x) u = \lambda \delta(x) r(t) e^{-\lambda q(t)}, \quad t > 0, \tag{11.2}$$

$$u|_{t=0} = u_t|_{t=0} = 0. \tag{11.3}$$

In turn, by virtue of Duhamel's principle, the Cauchy problem (11.2)–(11.3) is equivalent to the following Cauchy problem with zero right-hand side in the equation but with nonhomogeneous initial conditions:

$$-\frac{\partial^2 v}{\partial t^2} + c^2 \frac{\partial^2 v}{\partial x^2} - \lambda^2 b^2(x) v = 0, \quad t > \tau; \tag{11.4}$$

$$v|_{t=\tau} = 0, \quad v_t|_{t=\tau} = \delta(x) r(\tau) e^{-i\lambda q(\tau)}. \tag{11.5}$$

More precisely, the solution of problem (11.2)–(11.3) (and hence of (11.1)) can be reconstructed from that of problem (11.4)–(11.5) by the formula

$$u(x,t) = -\int_0^t v(x,t,\tau)\, d\tau. \tag{11.6}$$

Thus, physically, it is equation (11.1) that describes the model, but mathematically, we shall mostly deal with system (11.4)–(11.5) and then return to the solution of the original problem by integration.

The model (11.1) produces an interesting phenomenon, which is confirmed in experiments: the presence of the so-called *transient rays*. One might expect that the wave field is asymptotically zero outside the *illuminated region* swept by the geometric-optics rays issuing from the point $x = 0$, where the source is located, for every $t \geq 0$.

However, this is not the case. Some time before the rays of geometric optics reach a given point $x_0 \neq 0$, it already becomes illuminated, as the so-called "forerunners" of the main wave arrive at this point first. These forerunners propagate at a higher velocity, but as $\lambda \to \infty$ their amplitude is of smaller order than that of the main wave. Thus, an observer at a point x_0 sees the following picture. First, it is completely dark at x_0. Then, at some time $t_0 = t_0(x_0)$, the "forerunners" arrive, and there is some kind of dawn. Finally, at $t_1 = t_1(x_0)$ the main wave arrives, and the point becomes completely illuminated. (We should mention, by the way, that the forerunners continue to arrive after $t = t_1(x_0)$. However, their amplitude is smaller in order than that of the main wave, and so in the leading approximation the forerunners can be neglected in the region $t > t_1(x_0)$.) The rays along which the forerunners propagate are called *transient rays*, and the ratio of the amplitude of the main wave to that of the forerunners is called the *diffraction coefficient*. In physics, a customary way to compute the diffraction coefficient is as follows. One replaces (11.1) by a simpler exactly solvable model and then computes the diffraction coefficient from the exact solution of the simpler problem. It is assumed that the diffraction coefficient for the solution of the actual problem differs only slightly from the one thus obtained. For example, Lewis [1] showed that for the case of homogeneous plasma ($b(x) = b = \text{const}$) Eq. (11.1) has the exact solution

$$\begin{aligned} u(x,t) &= \frac{i}{4\pi}\int_0^t d\tau \int_{-\infty}^{\infty} dk\, \frac{r(t)}{\sqrt{c^2k^2+b^2}} e^{i\lambda(kx-q(\tau))} \\ &\quad \times \left\{ e^{i\lambda\sqrt{c^2k^2+b^2}(t-\tau)} - e^{-i\lambda\sqrt{c^2k^2+b^2}(t-\tau)} \right\} \end{aligned} \tag{11.7}$$

and then obtained the heuristic value of the diffraction coefficient by the asymptotic analysis of the integral on the right-hand side in (11.7). In contrast, our computations in subsequent sections result in a rigorous evaluation of the diffraction coefficient.

Now let us explain intuitively why the transient rays occur at all. It is well known that the presence of the plasma frequency term $\lambda^2 b^2(x)$ in Eq. (11.1) results in *dispersion*: waves with different frequencies propagate at different velocities. To see this, let us assume for simplicity that $b(x) = \text{const}$ and substitute the plane wave

$$\psi(x,t) = e^{i\lambda(\omega t - kx)},$$

which propagates at the velocity

$$\tilde{c}(\omega) = \frac{\omega}{k},$$

into the homogeneous equation corresponding to (11.1). Then we obtain the *dispersion relation*

$$\omega^2 - c^2k^2 - b^2 = 0,$$

whence, for given ω,

$$k = \frac{\sqrt{\omega^2 - b^2}}{c}$$

and

$$\tilde{c}(\omega) = c\frac{\omega}{\sqrt{\omega^2 - b^2}}.$$

Thus, waves with frequency $\lambda\omega \le \lambda b$ less than the plasma frequency do not propagate at all (at least in the semiclassical approximation), and waves of given frequency $\lambda\omega > \lambda b$ propagate at the velocity $\tilde{c}(\omega)$ that decreases from $+\infty$ for $\omega = b$ to the light velocity in vacuum, $\tilde{c}(\infty) = c$, for $\omega = \infty$. In particular, the waves generated by the source at time $t \ge 0$ propagate at the velocity $\tilde{c}(q'(t))$. However, the source is switched on by jump at $t = 0$. The Fourier expansion of $\theta(t)$ contains components with all possible frequencies $\lambda\omega$. Those with $\omega < q'(0)$ propagate faster than the source field and form the above-mentioned forerunners. Those with $\omega > q'(0)$ propagate slower and, their amplitude being one order of magnitude less than that of the main field, can be neglected against the background of the main wave.

This heuristic explanation also shows that the usual high-frequency (WKB) expansion that can be formally written out for the solution of our problem will fail to represent the solution adequately at least in the regions where the transient rays are important: it does not take account of the frequencies occurring in the Fourier expansion of the nonsmooth right-hand side and hence ignores the transient rays.

It follows that we must devise some procedure which would permit us to take the high-frequency components of the singularities into account. A procedure of this kind is well known for the case in which we deal with singularities alone: it is given by the usual theory of regularizers for partial differential equations of hyperbolic type. Using a regularizer, we satisfy the equation modulo an operator with smooth kernel applied to the right-hand side. The new right-hand side thus obtained is smooth, and the high-frequency components in its Fourier expansion decay rapidly. However, this method does not take account of the large parameter λ present in the equation and hence also does not provide a correct asymptotic solution of the problem in question. What we need is the *mixed asymptotics* with respect to smoothness and the large parameter. In the next section, we briefly explain the general noncommutative analysis approach to obtaining mixed asymptotics, and in the subsequent sections we apply this method to our problem.

We conclude this introductive section by Figure 11.1 schematically showing the arrangement of geometric-optics and transient rays.

11.2 Mixed Asymptotics: the General Scheme

In the present section, we very briefly describe the general scheme used in noncommutative analysis to tackle with mixed asymptotics. A *mixed asymptotics* is an asymptotic expansion that combines two or more types of usual asymptotics. Suppose that we need to find a mixed asymptotic solution of the equation

$$Lu = v \tag{11.8}$$

in a Hilbert space H. Noncommutative analysis can be applied to this problem provided one can implement the following scheme.

1°. We represent each type of asymptotic expansions involved by an unbounded self-adjoint operator in H.

For example, if we deal with the asymptotics with respect to a large parameter λ, then elements of H depend on λ and we represent this type of asymptotics by the unbounded operator of multiplication by λ. If we seek the asymptotics with respect to

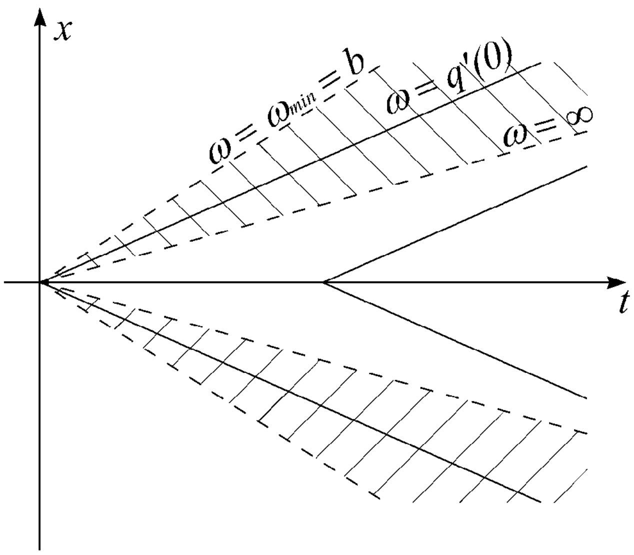

Figure 11.1: Solid lines, geometric-optics rays; dotted lines, transient rays; hatched region, the domain spanned by transient rays issuing from the source at $t = 0$

smoothness, then the corresponding operator will be, say, the differentiation operator $-i\frac{\partial}{\partial x}$ (or the n-tuple of these operators, one for each of the variables). If we seek the asymptotics with respect to growth as infinity, then one possibly would wish to take the operators of multiplication by the independent variables x_j as the operators representing this kind of asymptotics.

2°. Let $A_1, \ldots, A_m$ be the operators assigned at step **1°** to each kind of the asymptotics involved. We suppose that the operator

$$\Delta = 1 + \sum_{j=1}^{m} A_j^2$$

is self-adjoint in H and define the scale of Hilbert spaces $H^s = H_A^s$ associated with the operator $\sqrt{\Delta}$ in the usual way. Namely, for integer $s > 0$, $H^s = H_A^s$ is the domain of $\Delta^{s/2}$ equipped with the norm

$$||u||_s = ||\Delta^{s/2} u||_0,$$

where $||\cdot||_0$ is the norm in H. For negative s, H^s is the dual of H^s with respect to the pairing given by the inner product in H_0. Then one has the dense embeddings

$$\ldots \subset H_2 \subset H_1 \subset H_0 \subset H_{-1} \subset H_{-2} \subset \ldots,$$

and all A_j, as well as $\sqrt{\Delta}$, extend to continuous operators in the spaces

$$\begin{aligned} &A_j : H^s \to H^{s-1}, \qquad j = 1, \ldots, m, \qquad s \in \mathbf{Z}, \\ &\sqrt{\Delta} : H^s \to H^{s-1}, \qquad s \in \mathbf{Z}. \end{aligned}$$

3°. Now the notion of mixed asymptotics of the solution of Eq. (11.8) can be defined as follows.

Definition 11.1 A *mixed asymptotic solution* of Eq. (11.8) with respect to the operators $A_1, \ldots, A_m$ is a sequence u_n of elements of H such that

$$Lu_n - v \in H_A^n, \quad n = 0, 1, 2, \ldots.$$

Definition 11.2 A *mixed asymptotics* of a solution u of Eq. (11.8) with respect to the operators $A_1, \ldots, A_m$ is a sequence of elements $u_n \in H$ such that

$$u_n - u \in H_A^n, \quad n = 0, 1, 2, \ldots.$$

Proving that a mixed asymptotic solution is a mixed asymptotics of the solution (possibly, after some shift in the indices of the sequence u_n) can be reduced to proving that the inverse L^{-1} is bounded in the scale H_A^n, since

$$u_n - u = L^{-1}(Lu_n - v).$$

The boundedness of the inverse L^{-1} is usually proved by methods completely different from those used to construct an asymptotic solution. We shall not consider these methods in general, since they are equation specific.

Now we shall explain how to construct an asymptotic solution. To this end, one proceeds as follows.

4°. Take some additional operators $B_1, \ldots, B_s$ such that

(i) $B_1, \ldots, B_s$ are generators in the scale $\{H_A^n\}$ for suitably chosen symbol classes;

(ii) the tuple $(\overset{1}{A_1}, \ldots, \overset{m}{A_m}, \overset{m+1}{B_1}, \ldots, \overset{m+s}{B_s})$ has a left ordered representation $l_1, \ldots, l_{m+s}$;

(iii) L can be represented in the form

$$L = f(\overset{1}{A_1}, \ldots, \overset{m}{A_m}, \overset{m+1}{B_1}, \ldots, \overset{m+s}{B_s})$$

with a symbol f in a suitable class.

5°. Now the problem of finding L^{-1} is reduced to the solution of the corresponding equation in the algebra of symbols:

$$f(\overset{1}{l}_1, \dots, \overset{m+s}{l}_{m+s})g(y_1, \dots, y_{m+s}) = 1, \tag{11.9}$$

where $g(y_1, \dots, y_{m+s})$ is the symbol of the desired operator L^{-1}.

6°. Finally, we assume that the symbol $f(y_1, \dots, y_{m+s})$ and the operators $l_1, \dots, l_{m+s}$ are such that the operator $f(l_1, \dots, l_{m+s})$ can be represented in the form

$$f(\overset{1}{l}_1, \dots, \overset{m+s}{l}_{m+s}) = H\left(\overset{2}{y}, -i\,\frac{\overset{1}{\partial}}{\partial y}\right),$$

where the function $H(y,p)$, referred to as the *Hamiltonian* of the operator L with respect to the operators $A_1, \dots, A_m$, $B_1, \dots, B_s$, is *asymptotically homogeneous* in the variables $(y_1, \dots, y_m, p_{m+1}, \dots, p_s)$, that is, admits the asymptotic expansion

$$H(y,p) \sim \sum H_j(y,p) \quad \text{as} \quad |y_1| + \dots + |y_m| + |p_{m+1}| + |p_s| \to \infty,$$

where each $H_j(y,p)$ is homogeneous of order $r-j$:

$$\begin{aligned} &H_j(\mu y_1, \dots, \mu y_m, y_{m+1}, \dots, y_s, p_1, \dots, p_m, \mu p_{m+1}, \dots, \mu p_s) \\ &\quad = \mu^{r-j} H_j(y,p), \quad \mu > 0, \end{aligned}$$

where r is the *order* of H. Then we can apply the canonical operator method to obtain an asymptotic solution of Eq. (11.9) modulo symbols decaying as rapidly as desired as $|y_1| + \dots + |y_m| \to \infty$:

$$f(\overset{1}{l}_1, \dots, \overset{m+s}{l}_{m+s})g_N(y_1, \dots, y_{m+s}) = 1 + R_N(y),$$

where the symbol $R_N(y)$, as well as its derivatives, decays as $(|y_1|) + \dots + |y_m|)^{-N}$ at infinity with respect to the first m variables.

Then the operator

$$R_N(\overset{1}{A}_1, \dots, \overset{m}{A}_m, \overset{m+1}{B}_1, \dots, \overset{m+s}{B}_s)$$

has a high negative order (tending to $-\infty$ as $N \to \infty$) in the scale H_A^n, and so the sequence

$$u_n = g_n(\overset{1}{A}_1, \dots, \overset{m}{A}_m, \overset{m+1}{B}_1, \dots, \overset{m+s}{B}_s)v$$

is an asymptotic solution of Eq. (11.8).

A similar scheme applies if we have a Cauchy problem instead of Eq. (11.8).

We note that the exposition in this section is purely schematic. In specific applications of this approach, which is due to Maslov [2, 3] (see also [4]), one has to verify various additional conditions to fill in the gaps in the above-mentioned scheme.

11.3 The Asymptotic Solution of Main Problem

We intend to find the asymptotic solution of problem (11.4)–(11.5) with respect to smoothness and the large parameter $\alpha \to \infty$. Accordingly, the choice of the operators A in the general scheme described in Section 11.2 is obvious:

$$A_1 = -i\frac{\partial}{\partial x}, \quad A_2 = \lambda.$$

The space in which our operators will act will be defined as follows:

$$H = L_2(\mathbf{R}_x \times [1,\infty)_\lambda).$$

Then A_1 and A_2 (with appropriate domains) are self-adjoint in H, and so is $\Delta = A_1^2 + A_2^2 + 1$. This permits us to accomplish the construction of step **2°** and use the interpretation of the asymptotics given in step **3°**.

Now we proceed to step **4°**. We must find an additional operator B such that the operator occurring in (11.4) is representable in the form of a function of $\overset{1}{A}_1$, $\overset{1}{A}_2$, $\overset{2}{B}$. (We can assign the same Feynman indices to A_1 and A_2, since $[A_1, A_2] = 0$.) The choice of B is however obvious, $B = x$, and we have the following representation of the operator L occurring on the left-hand side in (11.4)

$$L = -\frac{\partial^2}{\partial t^2} - c^2 A_1^2 - b^2(B)A_1^2 = -\frac{\partial^2}{\partial t^2} - f(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}), \tag{11.10}$$

where

$$f(p,\lambda,x) = c^2p^2 + b^2(x)\lambda^2. \tag{11.11}$$

(We denote the arguments of the symbol by the letters p, x, λ; it will be always clear from the context whether x and λ are variables (arguments of the symbol) or the corresponding operators B and A_2 of multiplication by x and λ.)

We must find the asymptotic solution of the Cauchy problem (11.4)–(11.5) in the scale H^s of Hilbert spaces of functions $u(x,\lambda)$, $x \in \mathbf{R}$, $\lambda \in [1, +\infty)$, with finite norm

$$||u||_s = \left\| \left(1 - \frac{\partial^2}{\partial x^2} + \lambda^2\right)^{s/2} u \right\|_{L^2(\mathbf{R}\times[1,+\infty))}.$$

We seek the solution in the form of an operator

$$\widehat{\Phi} = \Phi(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B}, t, \tau) \tag{11.12}$$

applied to the right-hand side of the nontrivial initial condition in (11.5):

$$v(t,\tau) = \widehat{\Phi}\left(\delta(x) r(\tau) e^{-i\lambda q(\tau)}\right). \tag{11.13}$$

The initial conditions for the symbol $\Phi(p,\lambda,x,t,\tau)$ are clear from (11.5):

$$\begin{cases} \Phi(p,\lambda,x,t,\tau)|_{t=\tau} = 0, \\ \Phi_t(p,\lambda,x,t,\tau)|_{t=\tau} = 1. \end{cases} \tag{11.14}$$

To obtain an equation for Φ, we must compute the left ordered representation of the triple $(\overset{1}{A}_1, \overset{1}{A}_2, \overset{2}{B})$. This is however obvious. The left ordered representation for the operators

$$\left(\overset{2}{x}, -i\,\overset{1}{\frac{\partial}{\partial x}}\right)$$

has already been computed in Chapter 9. The operator A_2 of multiplication by λ commutes with the other operators, and hence is represented by the multiplication by the corresponding variable. Finally, we have

$$l_{A_1} = p - \frac{\partial}{\partial x}, \quad l_{A_2} = \lambda, \quad l_B = x.$$

Then from (11.4) we obtain the following equation for Φ:

$$-\frac{\partial^2\Phi}{\partial t^2} - c^2\left(p - i\frac{\partial}{\partial x}\right)^2 \Phi - \lambda^2 b^2(x)\Phi = 0. \tag{11.15}$$

As was explained in step **6°** in Section 11.2, we need the asymptotic solution of this equation (with the initial conditions (11.14)) as $\rho = \sqrt{\lambda^2 + p^2} \to \infty$. Let us transform Eq. (11.15) to a form in which the large parameter ρ occurs appropriately for the application of the standard WKB method (or the canonical operator, according to which is suitable). We isolate the large parameter in Eq. (11.15) by setting

$$p = \omega_1 \rho \quad \text{and} \quad \lambda = \omega_2 \rho.$$

Then $\omega_1^2 + \omega_2^2 = 1$, and Eq. (11.15) acquires the form

$$-\rho^{-2}\frac{\partial^2\Phi}{\partial t^2} - c^2\left(\omega_1 - i\rho^{-1}\frac{\partial}{\partial x}\right)^2 \Phi - \omega_2^2 b^2(x)\Phi = 0. \tag{11.16}$$

The operator on the left-hand side in (11.16) is a ρ^{-1}-pseudodifferential operator, and we can seek the solution of Eq. (11.16) with the initial condition (11.14) by the WKB-method or its global version, Maslov's canonical operator. According to this method, the solution is sought in the form

$$\Phi(p, \lambda, x, t, \tau) = \mathcal{K}_{\mathcal{L}_+}[\varphi_+] + \mathcal{K}_{\mathcal{L}_-}[\varphi_-], \tag{11.17}$$

where $\mathcal{K}_{\mathcal{L}_+}$ and $\mathcal{K}_{\mathcal{L}_-}$ are the canonical operators on two Lagrangian manifolds $\mathcal{L}_+$ and $\mathcal{L}_-$ to be constructed in what follows. The presence of two terms in (11.17) corresponds to the fact that Eq. (11.16) is of the second order.

At $t = \tau$, the function Φ_t is represented in the form

$$\Phi_t\big|_{t=\tau} = 1 = e^{i\rho S_0(\omega_1,\omega_2,x)} a_0(\omega_1, \omega_2, x)$$

with zero phase function $S_0(\omega_1, \omega_2, x) = 0$ and unit amplitude $a_0(\omega_1, \omega_2, x) = 1$. The function Φ itself vanishes at $t = \tau$ and can be represented in a similar form but with zero amplitude. Thus both functions correspond to the nonsingular Lagrangian manifold

$$\mathcal{L}_0 = \{(x, q) \mid q = 0\} \tag{11.18}$$

in the phase space $\mathbf{R}^{2n}_{x,q}$ with the symplectic form $dq \wedge dx$. Hence for small $t - \tau$ the components of the solution (11.17) will also be described by canonical operators with nonsingular charts, and hence Φ will be represented in the form

$$\begin{aligned} \Phi(p, \lambda, x, t, \tau) &= e^{i\rho S_+(\omega_1,\omega_2,x,t,\tau)} a_+(\omega_1, \omega_2, x, t, \tau) \\ &\quad + e^{i\rho S_-(\omega_1,\omega_2,x,t,\tau)} a_-(\omega_1, \omega_2, x, t, \tau). \end{aligned}$$

The initial Lagrangian manifold (11.18) evolves into $\mathcal{L}_+$ and $\mathcal{L}_-$ along the trajectories of Hamiltonian vector fields corresponding to the two Hamiltonians $H_\pm(\omega_1, \omega_2, x, q)$ associated with Eq. (11.16). The simplest way to see what these Hamiltonians are is to substitute the test function

$$\psi(x, t) = e^{i\rho S(x,t)} a(x, t) \tag{11.19}$$

into Eq. (11.16) and write out the resulting Hamilton–Jacobi equation for S (which is obtained by matching the coefficients of the highest power of ρ in the resulting expansions.) The substitution of the function (11.19) into Eq. (11.16) yields

$$\left\{ \left(\frac{\partial S}{\partial t} - i\rho^{-1} \frac{\partial}{\partial t} \right)^2 - c^2 \left(\omega_1 + \frac{\partial S}{\partial x} - i\rho^{-1} \frac{\partial}{\partial x} \right)^2 - \omega_2^2 b^2(x) \right\} a(x, t) = 0,$$

or, after matching the terms with like powers of ρ,

$$\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\omega_1 + \frac{\partial S}{\partial x}\right)^2 - \omega_2^2 b^2(x) = 0 \tag{11.20}$$

(the Hamilton–Jacobi equation)

$$\left\{\left[\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - \left(\omega_1 + \frac{\partial S}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2}\right] - i\rho^{-1}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\right\} a(x,t) = 0 \tag{11.21}$$

(the complete transport equation).

Once Eq. (11.20) is solved, we can seek the solution of (11.21) in the form of an asymptotic series in powers of ρ^{-1}. We shall return to the transport equation later on in this section.

For now, we see that Eq. (11.20) splits into two Hamilton–Jacobi equations

$$\begin{cases} \dfrac{\partial S}{\partial t} + H_\pm\left(\omega_1, \omega_2, x, \dfrac{\partial S}{\partial x}\right) = 0, \\ S|_{t=\tau} = 0 \end{cases} \tag{11.22}$$

with Hamiltonians

$$H_\pm(\omega_1, \omega_2, x, q) = \pm\sqrt{c^2(\omega_1 + q)^2 + b^2(x)\omega_2^2}. \tag{11.23}$$

The corresponding Hamiltonian systems are

$$\frac{dx}{dt} = \frac{c^2(\omega_1 + q)}{\sqrt{c^2(\omega_1 + q)^2 + b^2(x)\omega_2^2}}, \quad \frac{dq}{dt} = -2\frac{b'(x)b(x)\omega_2^2}{\sqrt{c^2(\omega_1 + q)^2 + b^2(x)\omega_2^2}} \tag{11.24}$$

for H_+, and the same system with the opposite signs of the right-hand sides for H_-. The corresponding phase functions $S_\pm$ are given by the Poincaré–Cartan integrals along the trajectories of the respective Hamiltonian systems:

$$\begin{aligned} S_\pm(\omega_1, \omega_2, x, t, \tau) &= \int_{(x_0,\tau)}^{(x_0(t,\tau),t)} (q\dot{x} - H_\pm)\, dt \\ &= \int_\tau^t \frac{c^2\omega_1(\omega_1 + q) + b^2\omega_2^2}{\sqrt{c^2(\omega_1 + q)^2 + b^2(x)\omega_2^2}}\, dt\Bigg|_{x=x^{0\pm}(x,t,\tau)}, \end{aligned} \tag{11.25}$$

where the $x^{0\pm}(x,t,\tau)$ are the inverse functions for the x-component of the solution

$$x = x^\pm(x_0, t, \tau), \quad q = q^\pm(x_0, t, \tau)$$

of the respective Hamiltonian system (11.24) with the initial conditions

$$x|_{t=\tau} = x_0, \quad q|_{t=\tau} = 0. \tag{11.26}$$

(Needless to say, these functions also depend on ω_1, ω_2.)

After the preliminary explanations, let us find an asymptotic solution of Eq. (11.15) with the initial conditions (11.14). Since the coefficients of the equation are independent of t, it follows that the solution has the form

$$\Phi(p, \lambda, x, t, \tau) \equiv \overset{\circ}{\Phi} (p, \lambda, t - \tau), \tag{11.27}$$

that is, depends only on the difference $t - \tau$ but not on t and τ separately, and it suffices to solve the Cauchy problem for $\overset{\circ}{\Phi}$ with initial data prescribed at $t = 0$ (that is, solve the original Cauchy problem for $\tau = 0$). From now on, we omit the circle over Φ and denote the right-hand side of (11.27) simply by Φ.

Let $\mathcal{L}_+(t)$, $t \geq 0$, be the Lagrangian manifold obtained from the manifold $\mathcal{L}_0$ (11.18) by applying the phase flow in time t of the Hamiltonian system (11.24) corresponding to the Hamiltonian H_+ (the upper sign in (11.23)). Thus, $\mathcal{L}_+(t)$ is the curve in $\mathbf{R}^2_{q,p}$ depending on the parameters (ω_1, ω_2) and described by the parametric equations

$$x = x^+(x_0, t), \quad q = q^+(x_0, t) \tag{11.28}$$

(t is fixed), where

$$(x^+(x_0, t) = x^+(x_0, 0, t),$$

$(q^+(x_0, t) = q^+(x_0, 0, t))$ is the solution of the Hamiltonian system (11.24) with the initial data

$$x^+(x_0, 0) = x_0, \quad q^+(x_0, 0) = 0.$$

Next, on the manifold $\mathcal{L}_0$ we consider the nondegenerate real measure

$$\mu_0 = dx.$$

This measure is transferred by the Hamiltonian flow to $\mathcal{L}_+(t)$ for each value of t, and so we obtain the following measure $\mu_+(t)$ on each $\mathcal{L}_+(t)$:

$$\mu_+(t) = dx_0,$$

where x_0 is the parameter in the parametric representation (11.28) of $\mathcal{L}_+(t)$.

Now let $\mathcal{K}^{1/\rho}_{(\mathcal{L}_+(t), \mu_+(t))}$ be the canonical operator with small parameter $1/\rho$ on the manifold $\mathcal{L}_+(t)$ with measure $\mu_+(t)$. Let $\varphi_+(x_0, t)$ be an arbitrary smooth function. We interpret x_0 as a coordinate on $\mathcal{L}_+(t)$ for each given t, and so we can regard $\varphi_+(x_0, t)$

as a function on $\mathcal{L}_+(t)$; thus we can apply the canonical operator $\mathcal{K}^{1/\rho}_{(\mathcal{L}_+(t),\mu_+(t))}$ to it and obtain a well-defined function

$$U_+(x,t,p) = \mathcal{K}^{1/\rho}_{(\mathcal{L}_+(t),\mu_+(t))}\varphi_+. \tag{11.29}$$

(Again, the function $U_+(x,t,\rho)$ depends also on the parameters ω_1, ω_2, $\omega_1^2+\omega_2^2=1$, which we do not bother to write out explicitly.) Let us substitute the function (11.29) into Eq. (11.16). Since the family $\mathcal{L}_+(t)$ of Lagrangian manifolds is associated with the Hamiltonian of Eq. (11.16), it follows from the general theory of the canonical operator that

$$\left[-\rho^{-2}\frac{\partial^2}{\partial t^2} - c^2\left(\omega_1 - i\rho^{-1}\frac{\partial}{\partial x}\right)^2 - \omega_2^2 b^2(x)\right] U_+(x,t,\rho)$$
$$= -i\rho^{-1}\mathcal{K}^{1/\rho}_{(\mathcal{L}_+(t),\mu_+(t))}\mathcal{P}_+\varphi_+ \mod O(\rho^{-2}),$$

where $\mathcal{P}_+$ is the *transport operator*:

$$\mathcal{P}_+\varphi_+ = 2\left.\sqrt{c^2(\omega_1+q)^2 + b^2(x)\omega_2^2}\right|_{\mathcal{L}_+(t)} [V(H_+)]\varphi_+, \tag{11.30}$$

$V(H_+)$ being the derivative along the Hamiltonian vector field (11.24) corresponding to the Hamiltonian H_+ in the extended phase space with coordinates (t,x,q). In the coordinates (x_0,t), one has

$$V(H_+)\varphi_+ = \frac{\partial\varphi_+}{\partial t}(x_0,t). \tag{11.31}$$

Remark 11.3 For general Hamiltonians, the transport operator involves the sum $V(H_+)+\mathcal{F}$, where $\mathcal{F}$ is a smooth function computable from the Hamiltonian, rather than the Hamiltonian vector field alone. The absence of the zero order term $\mathcal{F}$ in our case is due to the fact that the operator in Eq. (11.16) is self-adjoint.

Thus if we take $\varphi_+(x_0,t)$ independent of t, $\varphi_+(x_0,t) = \varphi_{+0}(x_0)$, where φ_{+0} is an arbitrarily given initial function, then we obtain

$$\left[-\rho^{-2}\frac{\partial^2}{\partial t^2} - c^2\left(\omega_1 - i\rho^{-1}\frac{\partial}{\partial x}\right)^2 - \omega_2^2 b^2(x)\right] U_+(x,t,\rho) = O(\rho^{-2}),$$

that is, the function (11.29) is an asymptotic solution of Eq. (11.16) modulo $O(\rho^{-2})$.

Now we can carry all the above constructions for the second Hamiltonian H_- (the lower sign in (11.23)), thus obtaining a family $\mathcal{L}_-(t)$ of Lagrangian manifolds equipped with the measure $\mu_-(t)$ and the corresponding canonical operator $\mathcal{K}^{1/\rho}_{(\mathcal{L}_+(t),\mu_+(t))}$. This canonical operator provides the second independent solution of Eq. (11.16) modulo $O(\rho^{-2})$, and now our task is to satisfy the two initial conditions (11.14).

To this end, we recall once again that for small t each of the Lagrangian manifolds $\mathcal{L}_+(t)$ and $\mathcal{L}_-(t)$ can be covered by a single nonsingular chart, and accordingly, the canonical operators for these t have the form

$$\mathcal{K}^{1/\rho}_{(\mathcal{L}_\pm(t),\mu_\pm(t))}\varphi_\pm = e^{i\rho S_\pm(\omega_1,\omega_2,x,t)}\left[\mathcal{J}_\pm^{-1/2}\varphi_1\right]\Big|_{x_0=x^{0\pm}(x,t)}, \tag{11.32}$$

where the $S_\pm(\omega_1,\omega_2,x,t)$ are the functions given by (11.25) for $\tau=0$, $x^{0\pm}(x,t)$ is the inverse function for $x=x^\pm(x_0,t)$ (see (11.28)), and $\mathcal{J}$ is the Jacobian

$$\mathcal{J}_\pm = \mathcal{J}_\pm(x_0,t) = \frac{\partial x^\pm(x_0,t)}{\partial x_0}.$$

The continuous branch of the square root of $\mathcal{J}_\pm$ in (11.32) is chosen such that $\sqrt{\mathcal{J}_\pm}|_{t=0} = 1$.

Now we take

$$\Phi = \mathcal{K}^{1/\rho}_{(\mathcal{L}_+,\mu_+)}\varphi_+ + \mathcal{K}^{1/\rho}_{(\mathcal{L}_-,\mu_-)}\varphi_- \tag{11.33}$$

and choose the functions $\varphi_+|_{t=0}$ and $\varphi_-|_{t=0}$ so as to satisfy the initial conditions (11.14).

Since

$$S_\pm|_{t=0} = 1, \qquad \frac{\partial S_\pm}{\partial t}\bigg|_{t=0} = -H_\pm(\omega_1,\omega_2,x,0) = \mp\sqrt{c^2\omega_1^2 + b^2(x)\omega_2^2},$$

$$\mathcal{J}_+|_{t=0} = \mathcal{J}_-|_{t=0} = 1,$$

we obtain

$$\begin{aligned} 0 &= \Phi|_{t=0} = \varphi_+(x,0) + \varphi_-(x,0), \\ 1 &= \Phi_t|_{t=0} = i\rho\sqrt{c^2\omega_1^2 + b^2(x)\omega_2^2}(\varphi_-(x,0) - \varphi_+(x,0)) + O(\rho^{-1}). \end{aligned} \tag{11.34}$$

We shall satisfy the initial conditions (11.34) in the leading term (the higher-order terms will determine the initial conditions for the subsequent terms of the asymptotic expansion of the solution, which we do not consider here). Thus, we neglect $O(\rho^{-1})$ on the right-hand side of the second equation in (11.34) and obtain

$$\varphi_\pm|_{t=0} = \pm\frac{i}{2\sqrt{c^2p^2 + b^2(x)\lambda^2}} \tag{11.35}$$

(we have taken into account the fact that $\rho\omega_1 = p$ and $\rho\omega_2 = \lambda$). Now the functions $\varphi_\pm(x_0,t)$ are determined by (11.35) and by the transport equation

$$\frac{\partial}{\partial t}\varphi_\pm(x_0,t) = 0, \tag{11.36}$$

which follows from (11.30)–(11.31), and we obtain the asymptotic solution in the form (11.33). Finally, we take the operator (11.12) with the symbol (11.33), apply it to the right-hand side of (11.5) as in (11.13) and use Duhamel's principle (11.6) to obtain the asymptotic solution of the original problem (11.10)–(11.11) in the form

$$u(x,\lambda,t) = -\int_0^t \Phi\left(-i\overset{1}{\frac{\partial}{\partial x}}, \lambda, \overset{2}{x}, t-\tau\right)[\delta(x)r(\tau)e^{-i\lambda q(\tau)}]d\tau. \tag{11.37}$$

(We omit the Feynman index over λ, singe λ commutes with all other operators occurring in (11.37).) Here the symbol $\Phi(p,\lambda,x,t)$ is given by formula (11.33) with the functions $\varphi_\pm$ satisfying the transport equations (11.36) and the initial conditions (11.35), and the Lagrangian manifolds $\mathcal{L}_\pm(t)$, as well as the measures $\mu_\pm(t)$ on these manifolds, were described in the preceding.

11.4 Analysis of the Asymptotic Solution

Now we shall analyze the asymptotic solution (11.37). Note that, according to the general formulas for functions of the operator $\widehat{p} = -i\dfrac{\partial}{\partial x}$, one has

$$\left(f(\overset{2}{x},\overset{1}{\widehat{p}})\psi\right)(x) = \left(\frac{i}{2\pi}\right)^{1/2}\int e^{ipx} f(x,p)\widetilde{\psi}(p)\,dp,$$

where

$$\widetilde{\psi}(p) = \left(-\frac{i}{2\pi}\right)^{1/2}\int e^{-ipx}\psi(x)\,dx$$

is the Fourier transform of the function $\psi(x)$. Accordingly, we can rewrite (11.37) in the form

$$u(x,\lambda,t) = -\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\Phi(p,\lambda,x,t-\tau)r(\tau)e^{i(px-\lambda q(\tau))}\,d\tau\,dp. \tag{11.38}$$

Next, let us assume that t is sufficiently small, so that the expression (11.33) for Φ via the canonical operators contains only nonsingular charts on the entire integration interval $\tau\in[0,t]$ in (11.38). Then (11.38) can be rewritten in the form

$$u(x,\lambda,t) \;=\; \frac{1}{4\pi i}\int_0^t\int_{-\infty}^{\infty}\left\{e^{i(px-\lambda q(\tau)+S_+(p,x,\lambda,t-\tau))}\right.$$

$$\times \left[\mathcal{J}_+ \left(c^2 \left(p + \frac{\partial S_+}{\partial x} \right)^2 + \lambda^2 b^2(x) \right) \right]^{-1/2} \tag{11.39}$$

$$- e^{i(px - \lambda q(\tau) + S_-(p,x,\lambda,t-\tau))}$$

$$\times \left[\mathcal{J}_- \left(c^2 \left(p + \frac{\partial S_-}{\partial x} \right)^2 + \lambda^2 b^2(x) \right) \right]^{-1/2} \Big\} r(\tau)\, d\tau\, dp.$$

(Here we have transformed the expression for the amplitude using the fact that the Hamiltonians $H_\pm$ are constant along the trajectories of the corresponding Hamiltonian systems.)

Our analysis starts from the expression (11.39). First of all, let us discuss the model case in which $b(x)$ is independent of x, that is, $b(x) \equiv b = \text{const}$. Then one can readily compute the solutions of the Hamilton–Jacobi equations (11.22) with zero initial data. Indeed, the Hamiltonians are independent of x, and hence the momentum is an integral of the corresponding Hamiltonian systems:

$$q = \frac{\partial S}{\partial x} = \left. \frac{\partial S}{\partial x} \right|_{t=0} = 0.$$

It follows that the Hamilton–Jacobi equation (11.22) becomes

$$\frac{\partial S}{\partial t} + H_\pm(\omega_1, \omega_2, x, 0) = 0,$$

or

$$\frac{\partial S}{\partial t} \pm \sqrt{c^2\omega_1^2 + b^2\omega_2^2} = 0, \tag{11.40}$$

whence we obtain

$$S_\pm = \mp t\sqrt{c^2\omega_1^2 + b^2\omega_2^2}.$$

Next, from the Hamiltonian system (11.24) we obtain

$$\dot{x} = \pm \frac{c^2\omega_1}{\sqrt{c^2\omega_1^2 + b^2\omega_2^2}}, \quad x = x_0 \pm \frac{tc^2\omega_1}{\sqrt{c^2\omega_1^2 + b^2\omega_2^2}},$$

and so the Jacobian is constant and never vanishes:

$$\mathcal{J}_\pm = \frac{\partial x^\pm}{\partial x_0} \equiv 1.$$

Thus, the solution (11.39) becomes

$$\begin{aligned} u(x,\lambda,t) &= \frac{1}{4\pi i}\int_0^t\int_{-\infty}^{\infty} e^{i(px-\lambda q(\tau))} \\ &\quad\times\frac{e^{-i(t-\tau)\sqrt{c^2p^2+b^2\lambda^2}}-e^{i(t-\tau)\sqrt{c^2p^2+b^2\lambda^2}}}{\sqrt{c^2p^2+b^2\lambda^2}}\,dp\,d\tau \\ &= \frac{i}{4\pi}\int_0^t\int_{-\infty}^{\infty} e^{i\lambda(kx-q(\tau))} \\ &\quad\times\frac{e^{i\lambda(t-\tau)\sqrt{c^2k^2+b^2}}-e^{-i\lambda(t-\tau)\sqrt{c^2k^2+b^2}}}{\sqrt{c^2b^2+b^2}}\,dk\,d\tau, \end{aligned}$$

where we have passed to the integration over the *wave number* $k = p/\lambda$. This just coincides with the expression (11.3) obtained for the solution of the model problem by Lewis [1].

Remark 11.4 This is just a mere occasion that the WKB method has produced the exact solution in this case. This often happens for equations with constant coefficients, when the right-hand sides of higher-order transport equations vanish and one can truncate the chain of transport equations after the first step, thus obtaining the exact solution.

Now let us analyze the general case, in which the coefficient $b(x)$ determining the plasma frequency may be variable. We shall analyze the asymptotic solution by expanding it in powers of λ^{-1} by the stationary phase method, which yields the asymptotics of the wave field in regions that are of interest to us.

The expression (11.39) contains two terms, one corresponding to the Hamiltonian H_+ and the second, to the Hamiltonian H_-. We shall consider only the first term; the second term can be analyzed in a similar way. Thus, we consider the integral

$$u_+(x,\lambda,t) = \frac{1}{4\pi i}\int_0^t\int_{-\infty}^{\infty}\frac{e^{i\lambda\{kx-q(\tau)+S_+(k,x,1,t-\tau)\}}}{\sqrt{J_+\left(c^2\left(k+\frac{\partial S_+}{\partial x}\right)^2+b^2(x)\right)}}\,dk\,d\tau; \qquad (11.41)$$

here we have again introduced the wave number $k = p/\lambda$. The integral (11.41) is taken over the infinite domain

$$\{0 \le \tau \le t,\ -\infty < k < \infty\},$$

and the application of the stationary phase method (11.40) to it can be justified if the stationary points of the phase function

$$L(k, \tau, t, x) = kx - q\tau + S_+(k, x, 1, t - \tau) \tag{11.42}$$

with respect to the variables (k, τ) all lie in some bounded domain. If this is the case, then we can use a partition of unity to reduce the integral (11.41) to an integral over a bounded domain plus an integral over an infinite domain where there are no stationary points. The second integral is $O(\lambda^{-\infty})$, which can be shown with the help of integration by parts, and the first integral satisfies the conditions under which the stationary phase method can be applied.

Thus we must analyze the stationary points of the phase (11.42) of the integral (11.41). Since the integration domain has a boundary ($\tau = 0$ and $\tau = t$), we must take into account two types of stationary points:

A. "*Interior*" *stationary points*, that is, points at which

$$\frac{\partial L}{\partial k} = \frac{\partial L}{\partial \tau} = 0.$$

(Of course, it may happen that an interior stationary point lies on the boundary. Then it gives half the usual contribution.)

B. *Boundary stationary points*. These are points on the boundary $\tau = 0$ or $\tau = t$ at which

$$\frac{\partial L}{\partial k} = 0.$$

Let us analyze both types of stationary points. For $S_+(k, x, 1, t - \tau)$ we have the expression

$$S_+(k, x, 1, t - \tau) \tag{11.43}$$
$$= \left\{ \frac{1}{\sqrt{c^2k^2 + b^2(x_0)}} \int_0^{t-\tau} [c^2k(k + q(x_0, \xi)) + b^2(x(x_0, \xi))] d\xi \right\}_{x^0 = x^{0+}(x, t-\tau)},$$

where the integral is taken over the trajectory of the Hamiltonian system corresponding to H_+ issuing at $\xi = 0$ from the point x^0 and entering the point x at time $\xi = t - \tau$. For interior stationary points, the equations read

$$\begin{cases} \dfrac{\partial L}{\partial k} \equiv x + \dfrac{\partial S_+}{\partial q}(k, x, 1, t - \tau) = 0, \\ \dfrac{\partial L}{\partial \tau} \equiv -q'(\tau) - \dfrac{\partial S_+}{\partial t}(k, x, 1, t - \tau) = 0. \end{cases} \tag{11.44}$$

Let us prove that system (11.44) has no solutions for $|k| > K_0$, where K_0 is sufficiently large. Since the function $S_+(q,x,\lambda,t)$ is first-order homogeneous in (q,λ), it follows that system (11.44) can be rewritten in the form

$$x + \frac{\partial S_+}{\partial q}(1,x,1/k,t-\tau) = 0,$$
$$-q'(\tau)/k - \frac{\partial S_+}{\partial t}(1,x,1/k,t-\tau) = 0.$$

In the limit as $k \to \infty$, this system becomes

$$x + \frac{\partial S_+}{\partial q}(1,x,0,t-\tau) = 0,$$
$$\frac{\partial S_+}{\partial t}(1,x,0,t-\tau) = 0.$$

However, straightforward verification shows that

$$S_+(1,x,0,t-\tau) = c(t-\tau).$$

(Indeed, this corresponds to the values $\omega_1 = 1$ and $\omega_2 = 0$.) Then $\dot{q} = 0$ in the Hamiltonian system (11.45), and accordingly, $q(t) \equiv 0$. Thus (11.43) is reduced to

$$S_+ = \frac{1}{\sqrt{c^2}} \int_0^{t-\tau} c^2 d\xi = c(t-\tau),$$

as was shown above.

Hence we have

$$\frac{\partial S_+}{\partial t}(1,x,0,t-\tau) = c \neq 0$$

and, by a continuity argument,

$$\frac{\partial S_+}{\partial t}(\omega_1,x,\omega_2,t-\tau) \neq 0$$

for $\omega_1^2 + \omega_2^2 = 1$, ω_2 being sufficiently small. It remains to note that $k = \omega_1/\omega_2$.

Now let us study the boundary stationary points. At these points, one must have

$$\frac{\partial L}{\partial k} \equiv x\frac{\partial S_+}{\partial q}(k,x,1,t-\tau) = 0 \tag{11.45}$$

and either $\tau = 0$ or $\tau = t$. For $\tau = t$, we have $S_+ = 0$ and

$$\frac{\partial S_+}{\partial q} = 0,$$

and so (11.45) implies $x = 0$. Thus, the only case in which there are boundary stationary points on the surface $\tau = t$ is when we are at the location of the source. This stationary points exists for all k and hence is degenerate; we see that the stationary phase method does not apply for $x = 0$. But for $x \neq 0$ there are no stationary points at all. Now let us consider the surface $\tau = 0$. To find out whether there are stationary points for large k, which would prevent us from applying the stationary phase method, we again pass to the limit as $k \to \infty$. Straightforward computation shows that

$$\lim_{k\to\infty} \frac{\partial S}{\partial q}(k, x, 1, t - \tau) = c(t - \tau),$$

and so the limit equation obtained from (11.45) reads, for $\tau = 0$,

$$x + ct = 0. \tag{11.46}$$

Accordingly, if x does not lie on the light cone[1] $x \pm ct = 0$, then all boundary stationary points lie in a bounded domain. We have proved the following theorem.

Theorem 11.5 *The stationary phase method can be applied to the expression* (11.39) *of the solution $u(x, \lambda, t)$ of problem* (11.2)–(11.3) *provided that $x \neq 0$ and x does not lie on the light cone $x \pm ct = 0$.*

For x satisfying the assumptions of the theorem, we can proceed with the application of the stationary phase method.

Then the following situations can occur.

1°. The integral (11.41) has an interior stationary point $k = k(x)$, $\tau = \tau(x)$. This corresponds to the so-called *illuminated region*. The asymptotics has the form

$$u_+(x, \lambda, t) = \lambda^{-1} e^{i\lambda L(k(x),x,1,t-\tau(x))} a(x) + O(\lambda^{-3/2}),$$

where $a(x)$ is some amplitude factor determined by the second derivatives of L at the stationary point and by the Jacobian.

2°. The integral (11.41) has no interior stationary points but has a boundary stationary point. In this case,

$$u_+(x, \lambda, t) = \lambda^{-3/2} \frac{e^{i\lambda L(k(x),x,1,t)} a(x)}{\dfrac{\partial L}{\partial t}(k(x), x, 1, t)} + O(\lambda^{-2}), \tag{11.47}$$

where $k = k(x)$ is the equation of the boundary stationary point at $\tau = 0$ and $a(x)$ is defined in the same way as in (11.26). We see that the intensity of the wave field is

[1] For S_-, Eq. (11.46) becomes $x - ct = 0$.

by a factor of $\lambda^{1/2}$ less than in the preceding case. This is the so-called umbral region, and formula (11.47) describes the contribution of *transient rays*. The factor

$$\kappa = \frac{1}{\dfrac{\partial L}{\partial t}(k(x), x, 1, t)} = \frac{1}{\sqrt{c^2k^2 + b^2(0)} - q'(0)}$$

is known as the diffraction coefficient. Traditionally, it is represented in the form

$$\kappa = \frac{1}{\omega_s - \omega_0},$$

where $\omega_0 = q'(0)$ is the instantaneous frequency of the source (ar $t = 0$) and $\omega_s = \sqrt{c^2k^2 + b^2(0)}$ is the frequency corresponding to the given transient ray.

3°. The inner stationary point and the boundary stationary point merge. In this case, the boundary stationary point is necessarily degenerate. We do not consider this more complicated case but point out that the asymptotic expansion can also be obtained by a sharpened version of the stationary phase method, which can be found in [7].

4°. There are neither interior nor boundary stationary points. In this case, the point x lies in the deep shadow region, where there are no forerunners or the main field, and the solution is exponentially small as $\lambda \to \infty$. Note that the exponential (rather than the power-law) decay of the solution must be proved by completely different methods.

Thus we have completed the asymptotic analysis of the simplest cases for the solution of the equation describing the propagation of electromagnetic waves in plasma. For large t, where the solution is given by the general canonical operator involving singular charts, the analysis of the solution is more complicated, but the results are physically the same.

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