Removable Singularities of CR Functions on Singular Boundaries

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Abstract

The problem of analytic representation of integrable *CR* functions on hypersurfaces with singularities is treated. The nature of singularities does not matter while the set of singularities has surface measure zero. For simple singularities like cuspidal points, edges, corners, etc., also the behaviour of representing analytic functions near singular points is studied.

Contents

1 Introduction

The theorem of analytic representation for *CR* functions on a hypersurface [AH72, Chi75] plays a crucial role in the theory of *CR* functions, cf. for instance [Khe85]. As but one consequence of such a representation we mention the Hartogs-Bochner theorem on the removability of compact singularities of holomorphic functions. Recall the theorem on analytic representation.

Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$, with $n > 1$, whose Dolbeault cohomology with coefficients in the sheaf of germs of holomorphic functions vanishes at step 1, i.e., $H^1(\mathcal{D}, \mathcal{O}) = 0$. This is the case, in particular, if $\mathcal{D}$ is a domain of holomorphy in $\mathbb{C}^n$.

Assume that $\mathcal{S}$ is a smooth (i.e., of class C^1) closed orientable hypersurface in $\mathcal{D}$, dividing $\mathcal{D}$ into two open sets $\mathcal{D}^+$ and $\mathcal{D}^-$. We write $\mathcal{S}$ in the form $\mathcal{S} = \{z \in \mathcal{D} \colon \rho(z) = 0\}$ where ρ is a smooth real-valued function in $\mathcal{D}$, such that $\nabla \rho \neq 0$ on $\mathcal{S}$. We set

$$\mathcal{D}^{\pm} = \{z \in \mathcal{D} \colon \pm\rho(z) > 0\}$$

and give $\mathcal{S}$ the orientation induced from $\mathcal{D}^-$. Thus, $\mathcal{D}^- \cup \mathcal{S}$ is an oriented manifold with boundary.

As usual, a function $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ is said to be a *CR function* on $\mathcal{S}$ if it satisfies

$$\int_{\mathcal{S}} f\, \bar{\partial} v = 0$$

for all differential forms v of bidegree $(n, n-2)$ with coefficients of class $C^\infty(\mathcal{D})$ and a compact support in $\mathcal{D}$.

In the terminology of De Rham's currents this just amounts to saying that the current $f[\mathcal{S}]^{0,1}$ is $\bar{\partial}$-closed in $\mathcal{D}$.

Theorem 1.1 ([AH72, Chi75]) *For any CR function $f \in L^1_{\mathrm{loc}}(\mathcal{S})$, there is a distribution h in $\mathcal{D}$ with the property that $\bar{\partial} h = f[\mathcal{S}]^{0,1}$.*

Denote $h^\pm$ the restriction of h to $\mathcal{D}^\pm$. From the equality $\bar{\partial} h = f[\mathcal{S}]^{0,1}$ it follows readily that $h^\pm$ is holomorphic in $\mathcal{D}^\pm$, i.e., $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$. Moreover, we have

$$f = h^+ - h^- \quad \text{on} \quad \mathcal{S}. \tag{1.1}$$

If $n = 1$ a solution $h^\pm$ to problem (1.1) is given by the Cauchy-type integral of f. Any two solutions then differ by a holomorphic function on the whole domain $\mathcal{D}$.

For $n > 1$, the boundary behaviour of $h^\pm$ is still completely determined by the local Bochner-Martinelli integral of the function f, cf. [Kyt95, Ch. 2]. More precisely, the equality (1.1) is interpreted as follows:

1) if $\mathcal{S} \in C^{k+1}$, $k \in \mathbb{Z}_+$, and $f \in C^{k,\lambda}_{\mathrm{loc}}(\mathcal{S})$, $0 < \lambda < 1$, then $h^{\pm} \in C^{k,\lambda}_{\mathrm{loc}}(\mathcal{S} \cup \mathcal{D}^{\pm})$ and (1.1) is fulfilled at each point of $\mathcal{S}$;

2) if $\mathcal{S} \in C^1$ and $f \in L^p_{\mathrm{loc}}(\mathcal{S})$, $p \geq 1$, then for each point $z^0 \in \mathcal{S}$ there is a neighbourhood U such that
$$\lim_{\varepsilon 0+} \int_{\mathcal{S} \cap U} \left| \left(h^+(\zeta + \varepsilon \nu(\zeta)) - h^-(\zeta - \varepsilon \nu(\zeta)) \right) - f(\zeta) \right|^p d\Lambda_{2n-1} = 0,$$
where $d\Lambda_{2n-1}$ is the $(2n-1)$-dimensional Lebesgue measure on $\mathcal{S}$, and $\nu(\zeta)$ the unit outward normal vector to $\mathcal{S}$ at a point $\zeta \in \mathcal{S}$.

Recall that by $C^{k,\lambda}_{\mathrm{loc}}(\mathcal{S})$ is meant the space of all k times differentiable functions on $\mathcal{S}$ whose derivatives of order k satisfy a Hölder condition of order λ on any compact subset of $\mathcal{S}$.

If $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ is a *CR* function on $\mathcal{S} \setminus \sigma$, σ being a closed set of zero measure in $\mathcal{S}$, then the theorem on analytic representation fails in general, cf. Examples 2.1 and 2.2 below. The main reason of this lies in the fact that the cohomology group $H^1(\mathcal{D} \setminus \sigma, \mathcal{O})$ may be non-trivial.

In case $\mathcal{S}$ is the boundary of a bounded domain $\mathcal{D}^-$, the problem of analytic representation just amounts to the problem of analytic continuation of a *CR* function f on $\mathcal{S} \setminus \sigma$ to all of $\mathcal{D}^-$. The latter problem goes back at least as far as [Lup87], cf. also the surveys [Sto93, ChS94], and it is well studied.

The problem of analytic representation has been investigated far worse. The case where σ is a holomorphically convex compact set in $\mathcal{D}$ and $n > 1$ is treated in [Kyt90]. In this paper it is shown that $H^1(\mathcal{D} \setminus \sigma, \mathcal{O})$ vanishes, and so the theorem on analytic representation is still valid. This allows one to describe certain sets removable for the *CR* functions.

Removable singularities of integrable *CR* functions on smooth *CR* manifolds $\mathcal{S}$ were studied in [Kyt89, KR95, AC94, MP99].

This paper is aimed at describing conditions on a locally integrable function f under which f still admits an analytic representation (1.1). No assumption on σ is required, moreover, the hypersurface $\mathcal{S}$ itself is allowed to have singularities in σ. For model singularities we also describe the behaviour of $h^{\pm}$ near the set σ.

The case where $\mathcal{S}$ bears a power-like cuspidal singularity at an isolated point is studied in [KMT99].

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2 Non-representable *CR* functions

We start with two examples.

Example 2.1 (cf. [Chi75]) Let $\mathcal{D}$ be the unit bidisk, i.e.,

$$\mathcal{D} = \{z \in \mathbb{C}^2 : \; |z_1| < 1, \; |z_2| < 1\},$$

and

$$\begin{array}{rcl} \mathcal{S} & = & \{z \in \mathcal{D} : \; \Im z_2 = 0\}, \\ \sigma & = & \{z \in \mathcal{S} : \; z_1 = 0\}. \end{array}$$

Set

$$\mathcal{D}^{\pm} = \{z \in \mathcal{D} : \; \pm \Im z_2 > 0\}.$$

Consider the function $f(z) = 1/z_1$. It is easy to see that $f \in L^1(\mathcal{S})$ is a *CR* function on $\mathcal{S} \setminus \sigma$.

Were Theorem 1.1 valid for f, we had $f = h^+ - h^-$ on $\mathcal{S} \setminus \sigma$, where $h^{\pm}$ are holomorphic functions in $\mathcal{D}^{\pm}$, respectively. Then

$$\lim_{\Im z_2 \to 0+} \int_{|z_1|=1/2} h^{\pm}(z_1, \pm i \Im z_2)\, dz_1 = 0,$$

by the Cauchy theorem, because $h^{\pm}(z_1, i\Im z_2)$ is holomorphic in z_1 in the unit disk, for each fixed $\Im z_2 \neq 0$. However,

$$\begin{array}{rcl} \displaystyle\int_{|z_1|=1/2} \frac{1}{z_1}\, dz_1 & = & 2\pi i \\ & \neq & 0, \end{array}$$

what contradicts Theorem 1.1.

□

This example shows that even in simple situations a variety σ of complex codimension 1 in $\mathcal{S}$ is not removable for integrable *CR* functions.

Example 2.2 (cf. [Kyt90, KMT99]) Let $\mathcal{D}$ be the unit bidisk, as in Example 2.1, and

$$\mathcal{S} = \{z \in \mathcal{D} : \; |z_1| = |z_2|\}.$$

The origin $O(0,0)$ is a singular point of $\mathcal{S}$. Indeed, $\mathcal{S} = \{z \in \mathcal{D} : \; \rho(z) = 0\}$ where

$$\rho(z) = z_1 \bar{z}_1 - z_2 \bar{z}_2,$$

and $\nabla \rho(z)$ vanishes at the only point $z = O$ on $\mathcal{S}$. Obviously, this is a conical point.

Consider the open sets $\mathcal{D}^{\pm} = \{z \in \mathcal{D} : \; \pm\rho(z) > 0\}$ and the holomorphic function

$$f(z) = \frac{1}{z_1 z_2}$$

away from the planes $z_j = 0$, for $j = 1, 2$. The restriction of f is a smooth *CR* function on $\mathcal{S} \setminus \{O\}$.

Furthermore, we have $f \in L^1(\mathcal{S})$. Indeed, parametrise $\mathcal{S}$ by

$$\begin{array}{rclcrcl} \Re z_1 & = & r \cos\varphi_1, & \quad & \Re z_2 & = & r \cos\varphi_2, \\ \Im z_1 & = & r \sin\varphi_1; & & \Im z_2 & = & r \sin\varphi_2, \end{array}$$

where $0 < r \leq 1$ and $0 \leq \varphi_1, \varphi_2 < 2\pi$. Then the Gramian of $\mathcal{S}$ has the form

$$G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

whence

$$\begin{array}{rcl} d\Lambda_3 & = & \sqrt{\det G}\, dr d\varphi_1 d\varphi_2 \\ & = & \sqrt{2} r^2\, dr d\varphi_1 d\varphi_2. \end{array}$$

It follows that

$$\begin{array}{rcl} \displaystyle\int_{\mathcal{S}} |f|\, d\Lambda_3 & = & \displaystyle\int_{\mathcal{S}} \frac{1}{|z_1 z_2|}\, d\Lambda_3 \\ & = & \displaystyle\sqrt{2} \int_0^1 dr \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \\ & = & \sqrt{2}\, (2\pi)^2 \end{array}$$

is finite, as desired.

Suppose that f meets the conclusion of Theorem 1.1, i.e., $f = h^+ - h^-$ on $\mathcal{S} \setminus \{O\}$, where $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$ are continuous up to $\mathcal{S} \setminus \{O\}$. By the Cauchy theorem in one dimension it follows that

$$\int_{\substack{|z_1|=1/2 \\ |z_2|=1/2}} h^\pm(z)\, dz_1 \wedge dz_2 = 0$$

while

$$\int_{\substack{|z_1|=1/2 \\ |z_2|=1/2}} \frac{1}{z_1 z_2}\, dz_1 \wedge dz_2 = (2\pi i)^2$$

is different from zero. The contradiction shows that f can not be represented as the difference of holomorphic functions on $\mathcal{D}^\pm$.

□

The latter example shows that if the hypersurface $\mathcal{S}$ bears singularities then even isolated points on $\mathcal{S}$ can be unremovable for integrable *CR* functions. All the points of $\mathcal{S}$ different from the origin are removable for integrable *CR* functions, for they lie on complex manifolds stratifying $\mathcal{S} \setminus \{O\}$, cf. Proposition 1 in [KR95].

As a rule, isolated points on a smooth hypersurface are removable for integrable *CR* functions, cf. [Kyt89]. Thus, the availability of singular points in $\mathcal{S}$ leads to new effects.

3 Analytic representation

From now on we consider the following situation. Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$, $n > 1$, such that $H^1(\mathcal{D}, \mathcal{O}) = 0$. Further, $\mathcal{S}$ is a closed subset of $\mathcal{D}$ dividing this domain into two open sets $\mathcal{D}^\pm$. Suppose $\mathcal{S}$ has the form $\mathcal{M} \cup \sigma$ where σ is a closed set of zero $(2n-1)$-dimensional measure in $\mathcal{S}$, and $\mathcal{M}$ is a smooth (i.e., of class C^1) orientable hypersurface in $\mathcal{D} \setminus \sigma$. Write

$$\mathcal{M} = \{z \in \mathcal{D} \setminus \sigma : \ \rho(z) = 0\}$$

with ρ a C^1 real-valued function in $\mathcal{D} \setminus \sigma$ satisfying $\nabla \rho \neq 0$ on $\mathcal{M}$, and we redefine

$$\mathcal{D}^\pm = \{z \in \mathcal{D} \setminus \sigma : \ \pm\rho(z) > 0\}.$$

We introduce a function space $L^1_{\mathrm{loc}}(\mathcal{S})$ to consist of all $f \in L^1_{\mathrm{loc}}(\mathcal{M})$ with the property that

$$\sup_{\varepsilon > 0} \int_{(\mathcal{M} \cap K) \setminus \{z:\ \mathrm{dist}(z,\sigma) < \varepsilon\}} |f(z)|\, d\Lambda_{2n-1} < \infty,$$

for each compact set $K \subset \mathcal{D}$, where $\mathrm{dist}(z, \sigma)$ is the distance of z to σ.

If $\mathcal{S}$ is a smooth submanifold of $\mathcal{D}$ then $L^1_{\mathrm{loc}}(\mathcal{S})$ coincides with the usual space of locally integrable functions on $\mathcal{S}$, for $\Lambda_{2n-1}(\sigma) = 0$.

Moreover, if $\mathcal{S}$ has locally finite $(2n-1)$-dimensional Lebesgue measure, i.e., $\Lambda_{2n-1}(\mathcal{S} \cap K) < \infty$ for all compact sets $K \subset \mathcal{D}$, then the bounded functions on $\mathcal{S}$ belong to $L^1_{\mathrm{loc}}(\mathcal{S})$.

Any function $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ defines a current $f[\mathcal{S}]$ of degree 1 in $\mathcal{D}$ by

$$\langle f[\mathcal{S}], \varphi \rangle = \lim_{\varepsilon \to 0} \int_{\mathcal{M} \setminus \{z:\ \mathrm{dist}(z,\sigma) < \varepsilon\}} f\varphi,$$

φ being a differential $(2n-1)$-form of class $C^\infty(\mathcal{D})$ with a compact support in $\mathcal{D}$.

Pick a non-negative real-valued function $\varrho(z)$ of class $C^\infty_{\mathrm{loc}}(\mathcal{D})$, such that $\sigma = \{z \in \mathcal{D} : \ \varrho(z) = 0\}$. A function with this property does exist, cf. for instance Lemma 1.4.13 of [Nar73].

Set

$$\mathcal{M}_\varepsilon = \{z \in \mathcal{M} : \ \varrho(z) > \varepsilon\},$$

for $\varepsilon > 0$. By Sard's theorem, cf. Theorem 1.4.16 of [Nar73], $\mathcal{M}_\varepsilon$ has a smooth boundary $\partial\mathcal{M}_\varepsilon$ for almost all $\varepsilon > 0$.

Given any compact set K in $\mathcal{D}$, the set of all stationary points of the function ϱ on $\mathcal{M} \cap K$ is compact and so is its image under ϱ. Hence it follows that the set of those $\varepsilon > 0$, for which $\partial\mathcal{M}_\varepsilon \cap K$ is a smooth submanifold, is actually open.

If $f \in L^1_{\mathrm{loc}}(\mathcal{M})$ then $f \in L^1_{\mathrm{loc}}(\partial\mathcal{M}_\varepsilon)$ for almost all $\varepsilon > 0$. Indeed, let K be a compact set in $\mathcal{D}$, and $0 < a < b$ be such that $\partial\mathcal{M}_\varepsilon \cap K$ is smooth for all $\varepsilon \in (a, b)$. By Fubini's theorem,

$$\int_{\{z\in\mathcal{M}\ a<\varrho(z)<b\}\cap K} |f(z)|\, d\Lambda_{2n-1} = \int_a^b d\varepsilon \int_{\partial\mathcal{M}_\varepsilon\cap K} |f(z)|\, \Delta(z)\, d\Lambda_{2n-2}$$

where Λ_{2n-2} is the $(2n-2)$ -dimensional Lebesgue measure on $\partial\mathcal{M}_\varepsilon \cap K$, and $\Delta = |d\Lambda_{2n-1}/d\varepsilon|$, cf. [Chi85, p. 248]. Since $\Delta \neq 0$ on $\partial\mathcal{M}_\varepsilon \cap K$ we deduce that

$$\int_{\partial\mathcal{M}_\varepsilon\cap K} |f(z)|\, d\Lambda_{2n-2} < \infty$$

for almost all $\varepsilon \in (a, b)$, as desired.

Given any function $f \in L^1_{\mathrm{loc}}(\mathcal{M})$ and compact set $K \subset \mathcal{D}$, we define

$$S_f(\varepsilon, K) = \int_{\partial\mathcal{M}_\varepsilon\cap K} |f(z)|\, d\Lambda_{2n-2},$$

cf. [KMT99, p. 8] [1].

Theorem 3.1 *Assume that $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ is a CR function on $\mathcal{S} \setminus \sigma$, satisfying*

$$S_f(\varepsilon, K) = o(1) \tag{3.1}$$

as $\varepsilon \to 0$, for each compact set $K \subset \mathcal{D}$. Then Theorem 1.1 holds true for f, more precisely, we have $f = h^+ - h^-$ on $\mathcal{S} \setminus \sigma$, where $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$ and the boundary behaviour of $h^\pm$ close to $\mathcal{S} \setminus \sigma$ is actually the same as that in Theorem 1.1.

4 The proof

As is mentioned, any function $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ defines a current $f[\mathcal{S}]$ of degree 1 in $\mathcal{D}$. We show that

$$\bar\partial \left(f[\mathcal{S}]^{0,1} \right) = 0 \tag{4.1}$$

in $\mathcal{D}$, provided that f meets the assumptions of Theorem 3.1.

Lemma 4.1 *Suppose f is a locally integrable CR function on $\mathcal{S} \setminus \sigma$. Then, for almost all $\varepsilon > 0$, we have*

$$\int_{\mathcal{M}_\varepsilon} f\, \bar\partial v = \int_{\partial\mathcal{M}_\varepsilon} f v$$

whenever v is a differential form of bidegree $(n, n-2)$ with smooth coefficients and compact support in $\mathcal{D}$.

[1] In [KMT99] we use another characteristic of f which differs from $S_f(\varepsilon, K)$ by the normalising factor $1/\mathrm{vol}(\partial\mathcal{M}_\varepsilon \cap K)$.

Proof. Fix $z^0 \in \mathcal{S} \setminus \sigma$. There is a ball $B = B(z^0, r)$ with centre z^0 and radius $r > 0$, such that f can be approximated in the $L^1(\mathcal{M} \cap B)$-norm by holomorphic polynomials. I.e., there is a sequence of holomorphic polynomials (p_ν) such that

$$\int_{\mathcal{M} \cap B} |f - p_\nu| \, d\Lambda_{2n-1} \to 0$$

as $\nu \to \infty$. This is in general a consequence of the approximation theorem of Baouendi and Treves, cf. [BT81]. For the case of hypersurfaces, see also Corollary 6.6 in [Kyt95].

We can assume, by decreasing r if necessary, that B does not intersect the set σ.

Suppose $a < \varrho(z) < b$ for all $z \in \mathcal{M} \cap B$, and v is a differential form of bidegree $(n, n-2)$ with smooth coefficients and a compact support in B. By Fubini's theorem,

$$\begin{aligned} \int_{\mathcal{M} \cap B} |f - p_\nu| \, d\Lambda_{2n-1} &= \int_a^b d\varepsilon \int_{\partial \mathcal{M}_\varepsilon \cap B} |f - p_\nu| \, \Delta \, d\Lambda_{2n-2} \\ &\to 0 \end{aligned}$$

when $\nu \to \infty$. Hence there is a subsequence (ν_j) such that

$$\int_{\partial \mathcal{M}_\varepsilon \cap B} |f - p_{\nu_j}| \, \Delta \, d\Lambda_{2n-2} \to 0$$

when $j \to \infty$, for almost all $\varepsilon \in (a, b)$. This in turn implies

$$\lim_{j \to \infty} \int_{\partial \mathcal{M}_\varepsilon \cap B} |f - p_{\nu_j}| \, d\Lambda_{2n-2} \to 0.$$

We thus conclude that the subsequence $\left(p_{\nu_j}\right)$ approximates f in the norm of $L^1(\partial \mathcal{M}_\varepsilon \cap B)$, for almost all $\varepsilon \in (a, b)$. By Stokes' formula,

$$\int_{\mathcal{M}_\varepsilon} p_{\nu_j} \, \bar{\partial} v = \int_{\partial \mathcal{M}_\varepsilon} p_{\nu_j} v$$

for all j. Letting $j \to \infty$ we obtain the assertion of the lemma for v supported in $B(z^0, r)$.

Since every v with compact support can be written as the sum of a finite number of forms supported in a sufficiently small balls, our claim follows in the general case.

□

Note that the proof of this lemma actually repeats the proof of Lemma 8.2 in [Kyt95].

We proceed to prove Theorem 3.1. By Lemma 4.1,

$$\begin{aligned}\langle \bar\partial \left(f[\mathcal{S}]^{0,1}\right), v\rangle &= \langle f[\mathcal{S}]^{0,1}, \bar\partial v\rangle \\ &= \lim_{\varepsilon\to 0+} \int_{\mathcal{M}_\varepsilon} f\, \bar\partial v \\ &= \lim_{\varepsilon\to 0+} \int_{\partial\mathcal{M}_\varepsilon} f\, v\end{aligned}$$

for all $(n, n-2)$-forms v with smooth coefficients and compact support in $\mathcal{D}$. Since

$$\begin{aligned}\left|\int_{\partial\mathcal{M}_\varepsilon} f v\right| &\leq c \int_{\partial\mathcal{M}_\varepsilon\cap K} |f|\, d\Lambda_{2n-2} \\ &= c\, S_f(\varepsilon, K)\end{aligned}$$

where c is a positive constant independent of ε, and $K = \operatorname{supp} v$, the equality (4.1) is a direct consequence of (3.1). Thus, the current $f[\mathcal{S}]^{0,1}$ is $\bar\partial$-closed in $\mathcal{D}$.

If $\mathcal{S}$ is a smooth hypersurface then (4.1) means that f is a *CR* function on $\mathcal{S}$. Hence the singularity of f on σ is removable, for every locally integrable *CR* function f on $\mathcal{S}$ satisfying (3.1).

In case $\mathcal{S}$ bears singularities on σ the equality (4.1) can be thought of as a definition of a *CR* function on a singular space.

Our next goal is to show that the theorem on analytic representation is valid for all functions f satisfying (4.1). This follows by the same method as in [Chi75], cf. also Section 6 in [Kyt95].

We first recall a $\bar\partial$-homotopy formula of [HL75]. Namely, let

$$F = \sum_{j=1}^{n} F_j\, (\partial/\partial \bar z_j)$$

be a vector field in $\mathbb{C}^n$ whose coefficients are distributions on all of $\mathbb{C}^n$, satisfying

$$\sum_{j=1}^{n} \frac{\partial F_j}{\partial \bar z_j} = \delta,$$

δ being the Dirac delta-function. Then, given any current T of bidegree (p, q) with compact support, we have

$$T = F\#\bar\partial T + \bar\partial\, (F\#T) \tag{4.2}$$

where $F\#T$ stands for the contraction of T by F.

More precisely, if

$$T = \sum_{I,J} T_{I,J}\, dz_I \wedge d\bar z_J,$$

the sum being over increasing multi-indices $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ of $1, \ldots, n$, then

$$F\#T = \sum_{I,J} \sum_{j \in J} (F_j * T_{I,J})\, \sigma(I, J, j)\, dz_I \wedge d\bar{z}_{J \setminus j}$$

where $*$ is the usual convolution of distributions, and $\sigma(I, J, j) = \pm 1$ is determined from the equality $d\bar{z}_j \wedge dz_I \wedge d\bar{z}_{J\setminus j} = \sigma(I, J, j) dz_I \wedge d\bar{z}_J$. If $q = 0$ then we set $F\#T = 0$, otherwise $F\#T$ is a current of bidegree $(p, q-1)$ in all of $\mathbb{C}^n$.

We take

$$F_j = \frac{(n-1)!}{2\pi^n} \frac{\bar{z}_j}{|z|^{2n}}, \quad j = 1, \ldots, n,$$

as F, then

$$F\# (\delta_z\, d\Lambda_{2n}) = U(\zeta, z),$$

δ_z being the Dirac delta-function at a point $z \in \mathbb{C}^n$, and $U(\zeta, z)$ the Bochner-Martinelli kernel in $\mathbb{C}^n$,

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}}\, d\zeta \wedge d\bar{\zeta}[j]$$

where

$$\begin{aligned} d\zeta &= d\zeta_1 \wedge \ldots \wedge d\zeta_n, \\ d\bar{\zeta}[j] &= d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{j-1} \wedge d\bar{\zeta}_{j+1} \wedge \ldots \wedge d\bar{\zeta}_n. \end{aligned}$$

Since $H^1(\mathcal{D}, \mathcal{O}) = 0$ and $\bar{\partial}\,(f[\mathcal{S}]^{0,1}) = 0$ in $\mathcal{D}$, the current $f[\mathcal{S}]^{0,1}$ is $\bar{\partial}$-exact in $\mathcal{D}$, by Dolbeault's theorem. Thus, there is a distribution h in $\mathcal{D}$, such that $f[\mathcal{S}]^{0,1} = \bar{\partial} h$. Hence it follows that h is holomorphic in $\mathcal{D} \setminus \mathcal{S}$, for $f[\mathcal{S}]^{0,1}$ is supported on $\mathcal{S}$.

Pick a point $z^0 \in \mathcal{S}$. Choose a function $\chi \in C^\infty_{\mathrm{comp}}(\mathcal{D})$ which is equal to 1 in a polycylindrical neighbourhood U of z^0. The current

$$T = \chi\, f[\mathcal{S}]^{0,1}$$

has a compact support in $\mathcal{D}$ and satisfies $\bar{\partial} T = 0$ in U. Write T by formula (4.2). Since

$$\begin{aligned} \bar{\partial} T &= \bar{\partial} \left(F\# \bar{\partial} T \right) \\ &= 0 \end{aligned}$$

in U and $F\#\bar{\partial}T$ has harmonic coefficients in U, we conclude, by Grothendieck's lemma, that $F\#\bar{\partial}T$ is $\bar{\partial}$-exact in U, i.e., $F\#\bar{\partial}T = \bar{\partial} u$ for some smooth function u. Hence it follows that

$$T = \bar{\partial}\, (F\#T + u)$$

in U.

Comparing the latter equality with

$$\begin{aligned}\bar{\partial} h &= f[\mathcal{S}]^{0,1} \\ &= T\end{aligned}$$

in U we see that the difference $h - (F\#T + u)$ is holomorphic in U. Hence the jump of h across $\mathcal{S} \cap U$ is completely determined by the boundary behaviour of $F\#T$ near $\mathcal{S} \cap U$.

As is shown in [Kyt95, Section 6],

$$-(F\#T)(z) = \int_{\mathcal{S}} \chi(\zeta) f(\zeta)\, U(\zeta, z) \qquad (4.3)$$

for all $z \notin \mathcal{S}$. Combining this with jump theorems for the Bochner-Martinelli integral, cf. Chapter 1 *ibid*, we complete the proof.

5 Analytic sets of singularities

Examples 2.1 and 2.2 show that the condition (3.1) cannot be relaxed in general. In Example 2.2 we have

$$\partial \mathcal{M}_\varepsilon = \{z \in \mathcal{D} : \ |z_1| = |z_2| = \varepsilon\}$$

and so, for the function

$$f(z) = \frac{1}{z_1 z_2},$$

we get

$$\begin{aligned}S_f(\varepsilon, K) &= \sqrt{2}\varepsilon^2 \int_0^{2\pi}\!\!\int_0^{2\pi} \frac{1}{\varepsilon^2}\, d\varphi_1 d\varphi_2 \\ &= \sqrt{2}\,(2\pi)^2,\end{aligned}$$

which is $O(1)$ as $\varepsilon \to 0$. Similar considerations apply to the function of Example 2.1.

However, for singular sets σ lying on analytic hypersurfaces in $\mathcal{D}$, Theorem 3.1 can be strengthen.

More precisely, suppose there exists a holomorphic function F in $\mathcal{D}$ whose zero set is σ, i.e., $\sigma = \{z \in \mathcal{D} : \ F(z) = 0\}$. Since the Hausdorff dimension of an analytic set does not exceed $2n - 2$, cf. for instance [Chi85, p. 25], we have $\Lambda_{2n-1}(\sigma) = 0$.

Lemma 5.1 *Let $f \in L^1_{\mathrm{loc}}(\mathcal{S} \setminus \sigma)$ be a CR function on $\mathcal{S} \setminus \sigma$. Suppose there is a real $R \geq 0$ such that $\mathrm{dist}(z,\sigma)^R f(z) \in L^1_{\mathrm{loc}}(\mathcal{S})$. Then there is an integer $N \geq 0$ such that $F^N f \in L^1_{\mathrm{loc}}(\mathcal{S})$ and $\bar{\partial}\left(F^N f [\mathcal{S}]^{0,1}\right) = 0$ in $\mathcal{D}$, i.e., $F^N f$ is a CR function on $\mathcal{S}$.*

Proof. Given any compact set $K \subset \mathcal{D}$, we have

$$\begin{aligned} |F(z)| &= |F(z) - F(\zeta)| \\ &\leq C\,|z-\zeta| \end{aligned}$$

uniformly in $z \in K$ and ζ on compact subsets of σ. Taking the infimum in $\zeta \in \sigma$ we get

$$|F(z)| \leq C\,\mathrm{dist}(z,\sigma) \tag{5.1}$$

for all $z \in K$, with C a constant depending only on K.

From (5.1) it follows that $F^N f \in L^1_{\mathrm{loc}}(\mathcal{S})$, for every integer N satisfying $N \geq R$.

Let v be a differential form of bidegree $(n, n-2)$ with smooth coefficients and a compact support in K. Then

$$\begin{aligned} \langle \bar{\partial}\left(F^N f[\mathcal{S}]^{0,1}\right), v\rangle &= \int_{\mathcal{S}} F^N f\,\bar{\partial}v \\ &= \int_{\mathcal{S}} F^N f\,\bar{\partial}\left((1-\chi_\varepsilon)v\right) + \int_{\mathcal{S}} F^N f\,\bar{\partial}\left(\chi_\varepsilon v\right) \end{aligned}$$

where χ_ε is a C^∞ function in $\mathcal{D}$ which is equal to 1 in $O_{(1/3)\varepsilon} \cap K$ and vanishes outside $O_{(2/3)\varepsilon}$. Here,

$$O_\varepsilon = \{z \in \mathcal{D} :\ \mathrm{dist}(z,\sigma) < \varepsilon\}$$

stands for the ε-neighbourhood of σ in $\mathcal{D}$.

We may take as χ_ε the convolution of the characteristic function of the set $O_{(1/2)\varepsilon} \cap K$ and a standard bell-shaped function. Then

$$|(\partial/\partial\bar{z}_j)\chi_\varepsilon| \leq \mathrm{const}\ \frac{1}{\varepsilon}$$

in all of $\mathbb{C}^n$, for each $j = 1, \ldots, n$, cf. for instance Lemma 1.1.12 in [Tar97]. By (5.1),

$$\begin{aligned} \Big|\int_{\mathcal{S}} F^N f\,\bar{\partial}\left(\chi_\varepsilon v\right)\Big| &\leq c\,\varepsilon^{N-R-1} \int_{\mathcal{S}\cap K} |\mathrm{dist}(z,\sigma)^R f|\,d\Lambda_{2n-1} \\ &\to 0 \end{aligned}$$

as $\varepsilon \to 0$, provided that $N > R+1$.

On the other hand, the integral

$$\int_{\mathcal{S}} F^N f \, \bar{\partial} ((1 - \chi_\varepsilon) v) = \int_{\mathcal{S}} f \, \bar{\partial} \left(F^N (1 - \chi_\varepsilon) v \right)$$

vanishes because the differential form $F^N(1-\chi_\varepsilon)v$ is supported in $\mathcal{D} \setminus \sigma$. This completes the proof.

□

We are now in a position to formulate the main result of this section. In case both $\mathcal{S}$ and σ are smooth manifolds and $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ this result is proved in [KR95], cf. Proposition 1.

Theorem 5.2 *Let $\sigma \subset \mathcal{S}$ be the zero set of a holomorphic function F in $\mathcal{D}$. Assume that $f \in L^1_{\mathrm{loc}}(\mathcal{S} \setminus \sigma)$ is a CR function on $\mathcal{S} \setminus \sigma$, such that $\mathrm{dist}(z, \sigma)^R f(z) \in L^1_{\mathrm{loc}}(\mathcal{S})$, for some $R \geq 0$. Then Theorem 3.1 on analytic representation holds for f.*

Proof. From Lemma 5.1 and the proof of the second part of Theorem 3.1 we deduce that the theorem on analytic representation is valid for $F^N f$. In other words, there are functions $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$ such that $F^N f = h^+ - h^-$ on the smooth part of $\mathcal{S}$, i.e., on $\mathcal{S} \setminus \sigma$. Hence

$$f = \frac{h^+}{F^N} - \frac{h^-}{F^N}$$

on $\mathcal{S} \setminus \sigma$. Obviously, this analytic representation behaves near $\mathcal{S} \setminus \sigma$ in the same way as that of Theorem 3.1, as desired.

□

Note that Example 2.2 does not contradict Theorem 5.2. Indeed, the point $O(0,0)$ lies in the analytic set $\sigma = \{z \in \mathcal{D} : \ z_1 = z_2\}$. On the set $\mathcal{S} \setminus \sigma$ we have

$$\frac{1}{z_1 z_2} = h^+ - h^- \tag{5.2}$$

where

$$\begin{aligned} h^+ &= \frac{1}{z_2 - z_1} \frac{1}{z_1}, \\ h^- &= \frac{1}{z_2 - z_1} \frac{1}{z_2}. \end{aligned}$$

Thus, the equality (5.2) holds almost everywhere on $\mathcal{S} \setminus \{O\}$, but it fails to be fulfilled in the sense of distributions on $\mathcal{S} \setminus \{O\}$.

Corollary 5.3 *Under the hypotheses of Theorem 5.2, if moreover $\mathcal{S}$ is a C^∞ hypersurface in $\mathcal{D}$ then the function f extends to a CR distribution $\tilde{f}$ on $\mathcal{S}$, i.e., σ is a removable set for all CR functions of finite order of growth close to σ.*

Proof. Consider the functions

$$H^{\pm} = \frac{h^{\pm}}{F^N}$$

holomorphic in $\mathcal{D}^{\pm}$. Using a Lojasiewicz's inequality, cf. Chapter 4 in [Mal66], we conclude that for any compact set $K \subset \mathcal{D}$ there are constants $C > 0$ and $E > 0$, such that

$$\begin{aligned} |F(z)| &\geq C \operatorname{dist}(z, \sigma)^E \\ &\geq C \operatorname{dist}(z, \mathcal{S})^E \end{aligned}$$

for all $z \in K$. It follows that

$$|H^{\pm}(z)| \leq \frac{c}{\operatorname{dist}(z, \mathcal{S})^{\gamma}}$$

for all $z \in K \cap \mathcal{D}^{\pm}$, with some constants c and γ, for the Bochner-Martinelli integral of f has a similar estimate near $\mathcal{S}$.

According to [Str84], the functions $H^{\pm}(z)$ possess weak boundary values on $\mathcal{S}$, which are *CR* functions $\tilde{f}^{\pm}$. Setting

$$\tilde{f} = \tilde{f}^{+} - \tilde{f}^{-}$$

finishes the proof.

□

If $\mathcal{S}$ is the boundary of a bounded domain in $\mathbb{C}^n$ Corollary 5.3 is proved in [Kyt89].

6 Hypersurfaces with cuspidal edges

In the rest part of the paper we study the boundary behaviour of the functions $h^{\pm}$ near the singular set σ. As is mentioned, this behaviour coincides completely with that of the Bochner-Martinelli integral of f. Hence we will be aimed at investigating the boundary behaviour of this integral close to σ. If $\mathcal{S}$ is piecewise smooth the behaviour of the Bochner-Martinelli integral near $\mathcal{S}$ is studied fairly well, cf. the book [Kyt95] and the references given there. However, for arbitrary singular sets σ, the problem is hard. We restrict ourselves to those classes of σ which are usually encountered in the analysis on singular spaces cf. [Sch98, RST00].

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Let the coordinates $w = (w_1, \ldots, w_{2n})$ of $\mathbb{R}^{2n}$ split as $w = (w', w_{q+1}, w'')$ where

$$\begin{aligned} w' &= (w_1, \ldots, w_q), \\ w'' &= (w_{q+2}, \ldots, w_{2n}). \end{aligned}$$

We require $0 \le q \le 2n-2$. Set

$$\begin{array}{rcl} w' & = & (y_1, \dots, y_q), \\ w_{q+1} & = & r, \\ w'' & = & (x_1, \dots, x_{d+1}), \end{array}$$

where $d = 2n-2-q$.

Suppose X is a compact connected submanifold of dimension d and class C^1 in $\mathbb{R}^{d+1}$, such that $0 \notin X$. For example, X may be a d-dimensional sphere with centre at the origin. Consider

$$\mathcal{C}_0 = \{(r, w'') \in \mathbb{R}^{d+2} : \ r \in [0, \varepsilon^0], \ w'' = \varphi(r)x, \, x \in X\}$$

where $\varphi \in C^1[0, \varepsilon^0]$ satisfies $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r \in (0, \varepsilon^0]$. Obviously, $\mathcal{C}_0$ is a smooth (i.e., of class C^1) submanifold of $\mathbb{R}^{d+2} \setminus \{0\}$. Moreover, 0 is a singular point of $\mathcal{C}_0$, for $\varphi'(0) < \infty$.

If $\varphi'(0) \ne 0$ then 0 is a conical point of $\mathcal{C}_0$. In case $\varphi'(0) = 0$ the point 0 is a cusp, cf. [RST00].

Denote by Λ_d the d-dimensional Lebesgue measure on X, and by Λ_{d+1} the $(d+1)$-dimensional Lebesgue measure on $\mathcal{C}_0$.

Lemma 6.1 *For some constants $c, C > 0$, we have*

$$c\,(\varphi(r))^d\, dr d\Lambda_d \le d\Lambda_{d+1} \le C\,(\varphi(r))^d\, dr d\Lambda_d. \tag{6.1}$$

Proof. We can locally give X a parametric representation

$$\left\{ \begin{array}{rcl} x_1 & = & F_1(\vartheta), \\ \dots & \dots & \dots \\ x_{d+1} & = & F_{d+1}(\vartheta), \end{array} \right.$$

where $\vartheta = (\vartheta_1, \dots, \vartheta_1)$ varies over an open set $U \subset \mathbb{R}^d$, and $F = (F_1, \dots, F_{d+1})$ is a C^1 mapping of U to $\mathbb{R}^{d+1}$ having maximal rank d in U. Then $\mathcal{C}_0$ has local parametric representation

$$(r, \varphi(r)F(\vartheta))$$

where $r \in [0, \varepsilon^0]$ and $\vartheta \in U$.

Let $G_{\mathcal{C}_0}$ be the Gramian of $\mathcal{C}_0$, i.e.,

$$G_{\mathcal{C}_0} = \begin{pmatrix} 1 + (\varphi')^2\,(F,F) & \varphi'\varphi\,\left(F, F'_{\vartheta_1}\right) & \dots & \varphi'\varphi\,\left(F, F'_{\vartheta_d}\right) \\ \varphi\varphi'\,\left(F'_{\vartheta_1}, F\right) & \varphi^2\,\left(F'_{\vartheta_1}, F'_{\vartheta_1}\right) & \dots & \varphi^2\,\left(F'_{\vartheta_1}, F'_{\vartheta_d}\right) \\ \dots & \dots & \dots & \dots \\ \varphi\varphi'\,\left(F'_{\vartheta_d}, F\right) & \varphi^2\,\left(F'_{\vartheta_d}, F'_{\vartheta_1}\right) & \dots & \varphi^2\,\left(F'_{\vartheta_d}, F'_{\vartheta_d}\right) \end{pmatrix}$$

where $F'_{\vartheta_j} = (\partial F_1/\partial\vartheta_j, \ldots, \partial F_{d+1}/\partial\vartheta_j)$, and $\left(F'_{\vartheta_j}, F'_{\vartheta_k}\right)$ is the scalar product of F'_{ϑ_j} and F'_{ϑ_k}, for $j, k = 1, \ldots, d$. Then

$$d\Lambda_{d+1} = \sqrt{\det G_{\mathcal{C}_0}}\, dr\, d\vartheta_1 \ldots d\vartheta_d.$$

On the other hand, the Gramian of X is

$$G_X = \left(\left(F'_{\vartheta_j}, F'_{\vartheta_k}\right)\right)_{\substack{j=1,\ldots,d\\ k=1,\ldots,d}}$$

whence

$$\begin{aligned} d\Lambda_{d+1} &= \frac{\sqrt{\det G_{\mathcal{C}_0}}}{\sqrt{\det G_X}}\, dr d\Lambda_d \\ &= \Phi(r,\vartheta)\,(\varphi(r))^d\, dr d\Lambda_d, \end{aligned}$$

the function $\Phi(r,\vartheta)$ being different from zero on $[0,\varepsilon^0] \times U$. Consequently, $0 < c \leq \Phi(r,\vartheta) \leq C$ on this set, and the lemma follows.

□

In the sequel we assume that $\mathcal{S} = Y \times \mathcal{C}_0$ where Y is a connected open set in $\mathbb{R}^q$, and $\mathcal{C}_0$ a hypersurface in $\mathbb{R}^{d+2}$ with a singularity at the origin, as above. Thus, $\mathcal{S}$ is a wedge with the edge $\sigma = Y \times \{0\}$. It is clear that the $(2n-1)$-dimensional Hausdorff measure of $\mathcal{S}$ is locally finite. Furthermore, we have $\mathcal{D} \setminus \mathcal{S} = \mathcal{D}^+ \cup \mathcal{D}^-$, the coordinate w_{q+1} being positive in the domain $\mathcal{D}^+$.

Similar considerations apply to $\mathcal{S}$ which are obtained locally by embedding $Y \times \mathcal{C}_0$ to $\mathcal{D}$. The estimates given below remain still valid in this more general situation. For simplicity of notation, we restrict our discussion to the local case $\mathcal{S} = Y \times \mathcal{C}_0$.

The case $q = 0$ and $d = 2n-2$ corresponds to a point singularity at the origin. It is treated in [KMT99].

We can assume, by shrinking $\mathcal{D}$ if necessary, that $f \in L^1(\mathcal{S})$. In fact, this means that f is an integrable function on the set $Y \times [0,\varepsilon^0] \times X$ with respect to the measure

$$d\Lambda_{2n-1} = \sqrt{\det G_{\mathcal{C}_0}}\, dy dr d\vartheta,$$

where $dy = dy_1 \ldots dy_n$. We require also that $\Lambda_{2n-1}(\mathcal{S}) < \infty$, i.e., Y be bounded in $\mathbb{R}^q$.

As $\mathcal{M}_\varepsilon$ we take $Y \times \mathcal{C}_\varepsilon$ where

$$\mathcal{C}_\varepsilon = \{(r, w'') \in \mathbb{R}^{d+2} : \ r \in [\varepsilon, \varepsilon^0],\ w'' = \varphi(r)x,\ x \in X\},$$

for $\varepsilon > 0$. Then $\partial\mathcal{M}_\varepsilon = Y \times \partial\mathcal{C}_\varepsilon$, with

$$\partial\mathcal{C}_\varepsilon = \{(\varepsilon, w'') \in \mathbb{R}^{d+2} : \ w'' = \varphi(\varepsilon)x,\ x \in X\}.$$

Hence analysis similar to that in the proof of Lemma 6.1 shows that the Lebesgue measure $d\Lambda_{2n-2}$ on $\partial\mathcal{M}_\varepsilon$ fulfills

$$c\,(\varphi(\varepsilon))^d\,dyd\Lambda_d \leq d\Lambda_{2n-2} \leq C\,(\varphi(\varepsilon))^d\,dyd\Lambda_d, \tag{6.2}$$

where dy is the Lebesgue measure on Y, $d\Lambda_d$ is that on X, and the constants $c, C > 0$ are independent of ε.

For a locally integrable function f on $\mathcal{S}\setminus\sigma$, we consider a slight modification of the characteristic $S_f(\varepsilon, K)$, namely

$$S_f(\varepsilon) = \int_{Y\times\partial\mathcal{C}_\varepsilon} |f(z)|\,dyd\Lambda_d$$

where $\varepsilon \in (0, \varepsilon^0]$.

Corollary 6.2 *Suppose $f \in L^1(\mathcal{S})$ is a CR function on $\mathcal{S}\setminus\sigma$. If*

$$S_f(\varepsilon) = o\left(\frac{1}{\varphi^d(\varepsilon)}\right) \quad as \quad \varepsilon \to 0$$

then Theorem 1.1 on analytic representation holds for f.

Proof. For the proof, it suffices to combine Theorem 3.1 and estimate (6.2).

□

In particular, the condition of Corollary 6.2 is satisfied for the *CR* functions f of class $L^\infty(\mathcal{S})$.

In the case of point singularities, i.e., $q = 0$ and $d = 2n-2$, Corollary 6.2 just amounts to Theorem 2.1 of [KMT99].

7 Auxiliary estimates

Let $f \in L^1(\mathcal{S})$. Consider the integral

$$\mathcal{P}_m(z) = \int_{\mathcal{S}} f(\zeta)\,\frac{d\Lambda_{2n-1}(\zeta)}{|\zeta - z|^m}, \quad z \in \mathcal{D}\setminus\mathcal{S},$$

where

$$\begin{aligned} z &= (w', w_{q+1}, w'') \\ &= (y, r, x), \end{aligned}$$

and $m > 0$.

We are interested in investigating the behaviour of the integral $\mathcal{P}_m(z)$ for z close to σ, depending on the behaviour of the function f near the singular set σ. We restrict ourselves to the case $z \in \mathcal{D}^+$, the estimates for $z \in \mathcal{D}^+$ are similar.

Let $\tilde{z}$ be the orthogonal projection of z to the subspace $\mathbb{R}^q \times \mathbb{R}$, namely, $\tilde{z} = (y, r, 0)$. If $z \in \mathcal{D}^+$ then $r > 0$ and

$$|\tilde{z}| \leq |z| \leq C\, |\tilde{z}|, \tag{7.1}$$

with C a constant independent of z. Indeed, we have

$$\begin{aligned} |z|^2 &= |y|^2 + r^2 + \varphi^2(r) \left(\frac{|x|}{\varphi(r)} \right)^2 \\ &\leq |y|^2 + (1+c)\, r^2 \end{aligned}$$

the constant $c > 0$ depending only on φ and X, for $\varphi \in C^1[0, \varepsilon^0]$ vanishes at 0 and X is compact. Hence it follows that

$$\begin{aligned} 1 &\leq \frac{|z|^2}{|\tilde{z}|^2} \\ &\leq \frac{|y|^2 + (1+c)\, r^2}{|y|^2 + r^2} \\ &\leq 1 + c, \end{aligned}$$

as desired.

From (7.1) we deduce that z and $\tilde{z}$ are equivalent when $z \to 0$. If we estimate $\mathcal{P}_m(z)$ in terms of $\varphi(r)$, for any z lying in the plane $x = 0$, then similar estimates will be valid for all $z = (y, r, x)$ in $\mathcal{D}^+$. Hence we assume in the sequel that $z = \tilde{z}$.

The estimates (5.1) yield

$$\begin{aligned} \mathcal{P}_m(z) &= \int_{\mathcal{S}} |f(\upsilon)| \frac{d\Lambda_{2n-1}(\upsilon)}{(|\upsilon' - y|^2 + (\upsilon_{q+1} - r)^2 + |\upsilon''|^2)^{m/2}} \\ &\leq C \int_0^{\varepsilon^0} \varphi^d(\tau)\, d\tau \int_{Y \times X} |f(\upsilon', \tau, \varphi(\tau) x)| \frac{d\upsilon'\, d\Lambda_d(x)}{((\tau - r)^2 + \varphi^2(\tau)|x|^2)^{m/2}} \end{aligned}$$

where $\tau = \upsilon_{q+1}$. Since X is compact and $0 \not\in X$ the module of $x \in X$ is uniformly bounded below by a constant $\delta > 0$. Taking into account the equality

$$\int_{Y \times X} |f(\upsilon', \tau, \varphi(\tau) x)|\, d\upsilon'\, d\Lambda_d(x) = S_f(\tau)$$

we thus arrive at an estimate

$$|\mathcal{P}_m(z)| \leq C \int_0^{\varepsilon^0} \frac{\varphi^d(\tau)}{((\tau - r)^2 + \delta^2 \varphi^2(\tau))^{m/2}}\, S_f(\tau)\, d\tau, \tag{7.2}$$

for any $z = (y, r, 0)$ in $\mathcal{D}^+$.

Replacing $\delta\,\varphi(\tau)$ by $\varphi(\tau)$, which does not cause any confusion, we now proceed to estimate the integral

$$\int_0^{\varepsilon^0} \frac{\varphi^d(\tau)}{((\tau - r)^2 + \varphi^2(\tau))^{m/2}}\, S_f(\tau)\, d\tau,$$

for $r > 0$.

To this end, we need auxiliary material. Let $\tau^* = \tau^*(r)$ stand for the minimum point of the function $(\tau - r)^2 + \varphi^2(\tau)$ over $\tau \in [0, \varepsilon^0]$.

Lemma 7.1 *The following formulas hold:*

$$\begin{aligned} \lim_{r \to 0+} \frac{r}{\tau^*(r)} &= 1 + (\varphi'(0))^2, \\ \lim_{r \to 0+} \frac{(\tau^* - r)^2 + \varphi^2(\tau^*)}{\varphi^2(r)} &= \frac{1}{1 + (\varphi'(0))^2}. \end{aligned}$$

Proof. At the point of minimum we clearly have $r - \tau^* = \varphi(\tau^*)\, \varphi'(\tau^*)$, whence

$$\frac{r}{\tau^*} - 1 = \frac{\varphi(\tau^*)}{\tau^*}\, \varphi'(\tau^*).$$

For $\tau \geq r$, the function $(\tau - r)^2 + \varphi^2(\tau)$ increases, and so $0 \leq \tau^*(r) \leq r$. It follows that $\tau^*(r) \to 0+$ as $r \to 0+$. Hence we conclude that

$$\begin{aligned} \lim_{r \to 0+} \frac{r}{\tau^*(r)} &= \lim_{r \to 0+} \left(1 + \frac{\varphi(\tau^*)}{\tau^*}\, \varphi'(\tau^*) \right) \\ &= 1 + (\varphi'(0))^2, \end{aligned}$$

as desired.

To prove the second formula, we consider separately two cases, namely $\varphi'(0) = 0$ and $\varphi'(0) > 0$.

If $\varphi'(0) = 0$, then the first equality gives

$$\lim_{r \to 0+} \frac{r}{\tau^*(r)} = 1.$$

We claim that

$$\lim_{r \to 0+} \frac{\varphi(r)}{\varphi(\tau^*(r))} = 1.$$

Indeed,

$$\frac{\varphi(r)}{\varphi(\tau^*(r))} = \frac{\varphi(r) - \varphi(\tau^*(r)) + \varphi(\tau^*(r))}{\varphi(\tau^*(r))}$$

$$
\begin{aligned}
&= 1 + \frac{\varphi(r) - \varphi(\tau^*(r))}{\varphi(\tau^*(r))} \\
&= 1 + \frac{(r - \tau^*(r))\, \varphi'(\theta(r))}{\varphi(\tau^*(r))} \\
&= 1 + \varphi'(\tau^*(r))\varphi'(\theta(r)),
\end{aligned}
$$

and this latter expression tends to 1 when $r \to 0+$, for $\tau^*(r) \to 0$ and $\theta(r) \to 0$ when $r \to 0+$.

Therefore

$$
\begin{aligned}
\lim_{r \to 0+} \frac{(\tau^* - r)^2 + \varphi^2(\tau^*)}{\varphi^2(r)} &= \lim_{r \to 0+} \frac{\varphi^2(\tau^*)((\varphi'(\tau^*))^2 + 1)}{\varphi^2(r)} \\
&= 1.
\end{aligned}
$$

On the other hand, if $\varphi'(0) > 0$ then

$$
\begin{aligned}
\lim_{r \to 0+} \frac{\varphi(\tau^*(r))}{\varphi(r)} &= \lim_{r \to 0+} \frac{\varphi\left(\frac{r}{1+(\varphi'(0))^2}\right)}{\varphi(r)} \\
&= \frac{1}{1 + (\varphi'(0))^2},
\end{aligned}
$$

which is due to L'Hospital's rule. Hence

$$
\begin{aligned}
\lim_{r \to 0+} \frac{(\tau^* - r)^2 + \varphi^2(\tau^*)}{\varphi^2(r)} &= \lim_{r \to 0+} \frac{\varphi^2(\tau^*)((\varphi'(\tau^*))^2 + 1)}{\varphi^2(r)} \\
&= \left(\lim_{r \to 0+} \frac{\varphi(\tau^*)}{\varphi(r)} \right)^2 \left(1 + (\varphi'(0))^2\right) \\
&= \frac{1}{1 + (\varphi'(0))^2},
\end{aligned}
$$

which establishes the formula.

□

Our next objective is to estimate the integral

$$
\mathcal{I}_s(r) = \int_0^{\varepsilon^0} \frac{d\tau}{((\tau - r)^2 + \varphi^2(\tau))^{s/2}}, \tag{7.3}
$$

for $s \in \mathbb{R}$.

Lemma 7.2 *As defined above, the integral $\mathcal{I}_s(r)$ meets the following estimates:*

1) *If* $2 \le s$, *then*

$$\mathcal{I}_s(r) = O\left(\frac{1}{\varphi^{s-1}(r)}\right) \quad as \quad r \to 0+\,.$$

2) *If* $1 \le s < 2$, *then*

$$\mathcal{I}_s(r) = O\left(\frac{|\log \varphi(r)|}{\varphi^{s-1}(r)}\right) \quad as \quad r \to 0+\,.$$

3) *If* $s < 1$, *then*

$$\mathcal{I}_s(r) = O\,(1) \quad as \quad r \to 0+\,.$$

Proof.
1) Let $s \ge 2$. Then

$$\mathcal{I}_s(r) = \frac{1}{\varphi^{s-2}(r)} \int_0^{\varepsilon^0} \frac{\varphi^{s-2}(r)\,d\tau}{((\tau-r)^2+\varphi^2(\tau))^{s/2}}.$$

Using Lemma 7.1, we obtain

$$\begin{aligned} |\mathcal{I}_s(r)| &\le \frac{c}{\varphi^{s-2}(r)} \int_0^{\varepsilon^0} \frac{\varphi^{s-2}(r)\,d\tau}{((\tau^*-r)^2+\varphi^2(\tau^*))^{\frac{s-2}{2}}((\tau-r)^2+\varphi^2(\tau))} \\ &\le \frac{C}{\varphi^{s-2}(r)} \int_0^{\varepsilon^0} \frac{d\tau}{(\tau-r)^2+\varphi^2(\tau)}, \end{aligned} \tag{7.4}$$

the constants c and C being independent of $r \in (0, \varepsilon^0]$.

Let us prove that

$$a \le \frac{(\tau-r)^2+\varphi^2(\tau)}{(\tau-r)^2+\varphi^2(r)} \le A \tag{7.5}$$

for all $r, \tau \in (0, \varepsilon^0]$, where a and A are positive constants independent of τ and r.

To do this, denote $y = \varphi(r)$. Since $\varphi'(r) > 0$ for all $r \in (0, \varepsilon^0]$, there exists an inverse function $r = \varphi^{-1}(y)$ which is differentiable in $y > 0$.

Put

$$\begin{cases} \tau - r &= w, \\ r &= r \end{cases}$$

and

$$\begin{cases} w &= \vartheta \tilde{w}, \\ y &= \vartheta \tilde{y} \end{cases}$$

where $\tilde{w}^2 + \tilde{y}^2 = 1$. Then

$$
\begin{aligned}
\frac{(\tau - r)^2 + \varphi^2(\tau)}{(\tau - r)^2 + \varphi^2(r)} &= \frac{w^2 + \varphi^2(w + \varphi^{-1}(y))}{w^2 + y^2} \\
&= \frac{\vartheta^2 \tilde{w}^2 + \varphi^2(\vartheta \tilde{w} + \varphi^{-1}(\vartheta \tilde{y}))}{\vartheta^2} \\
&= \tilde{w}^2 + \left(\frac{\varphi(\vartheta \tilde{w} + \varphi^{-1}(\vartheta \tilde{y}))}{\vartheta} \right)^2
\end{aligned}
$$

which gives

$$
\begin{aligned}
\frac{(\tau - r)^2 + \varphi^2(\tau)}{(\tau - r)^2 + \varphi^2(r)} &= \tilde{w}^2 + \left(\frac{\varphi(\vartheta \tilde{w} + \varphi^{-1}(\vartheta \tilde{y})) - \varphi(\varphi^{-1}(\vartheta \tilde{y})) + \varphi(\varphi^{-1}(\vartheta \tilde{y}))}{\vartheta} \right)^2 \\
&= \tilde{w}^2 + \left(\frac{\varphi'(\theta)\, \vartheta \tilde{w} + \vartheta \tilde{y}}{\vartheta} \right)^2 \\
&= \tilde{w}^2 + (\varphi'(\theta)\, \tilde{w} + \tilde{y})^2 \\
&\leq A,
\end{aligned}
$$

the constant A does not depend on τ and r.

Considering the reverse fraction, we obtain an estimate

$$
\frac{(\tau - r)^2 + \varphi^2(r)}{(\tau - r)^2 + \varphi^2(\tau)} \leq \frac{1}{a},
$$

with a a constant independent of τ and r. This yields (7.5). Finally, combining (7.4) and (7.5) we get

$$
\begin{aligned}
|\mathcal{I}_s(r)| &\leq \frac{C}{a} \frac{1}{\varphi^{s-2}(r)} \int_0^{\varepsilon^0} \frac{d\tau}{(\tau - r)^2 + \varphi^2(r)} \\
&\leq \frac{C}{a} \frac{1}{\varphi^{s-1}(r)} \arctan \frac{\tau - r}{\varphi(r)} \Big|_0^{\varepsilon^0} \\
&\leq \frac{C}{a} \pi \frac{1}{\varphi^{s-1}(r)},
\end{aligned}
$$

as desired.

2) Suppose $1 \leq s < 2$. Once again we make use of the estimate (7.5) to obtain

$$
\begin{aligned}
|\mathcal{I}_s(r)| &= \frac{1}{\varphi^{s-1}(r)} \int_0^{\varepsilon^0} \frac{\varphi^{s-1}(r)\, d\tau}{((\tau - r)^2 + \varphi^2(\tau))^{s/2}} \\
&\leq \frac{1}{a^{\frac{s-1}{2}}} \frac{1}{\varphi^{s-1}(r)} \int_0^{\varepsilon^0} \frac{d\tau}{\sqrt{(\tau - r)^2 + \varphi^2(r)}} \\
&= \frac{1}{a^{\frac{s-1}{2}}} \frac{1}{\varphi^{s-1}(r)} \log \left| (\tau - r) + \sqrt{(\tau - r)^2 + \varphi^2(r)} \right| \Big|_0^{\varepsilon^0}
\end{aligned}
$$

whence

$$\mathcal{I}_s(r) \;=\; O\left(\frac{|\log\left(\sqrt{r^2+\varphi^2(r)}-r\right)|}{\varphi^{s-1}(r)}\right)$$

when $r \to 0+$. This establishes the formula.

3) If $s < 1$ then

$$\begin{aligned} |\mathcal{I}_s(r)| &\le \max\left(\frac{1}{a^{\frac{s}{2}}},\frac{1}{A^{\frac{s}{2}}}\right)\int_0^{\varepsilon^0}\frac{d\tau}{((\tau-r)^2+\varphi^2(r))^{s/2}} \\ &\le \max\left(\frac{1}{a^{\frac{s}{2}}},\frac{1}{A^{\frac{s}{2}}}\right)\int_0^{\varepsilon^0}\frac{d\tau}{|\tau-r|^s}, \end{aligned}$$

and the proof is complete.

□

We are now in a position to prove the main result of this section giving sharp estimates of the potential $\mathcal{P}_m(z)$.

Theorem 7.3 *Suppose that* $f \in L^1(\mathcal{S})$ *and*

$$S_f(\varepsilon) = O\left(\frac{1}{\varphi^N(\varepsilon)}\right)$$

as $\varepsilon \to 0+$*, for some* $N \le d$.

1) *If* $2+d-m \le N$*, then*

$$|\mathcal{P}_m(z)| = O\left(\frac{1}{\varphi^{N+m-d-1}(|(r,x)|)}\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

2) *If* $1+d-m \le N < 2+d-m$*, then*

$$|\mathcal{P}_m(z)| = O\left(\frac{|\log\varphi(|(r,x)|)|}{\varphi^{N+m-d-1}(|(r,x)|)}\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

3) *If* $N < 1+d-m$*, then*

$$|\mathcal{P}_m(z)| = O\left(1\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

Proof. Indeed, let us continue the inequality (7.2), thus obtaining

$$\begin{aligned} |\mathcal{P}_m(z)| &\le C\int_0^{\varepsilon^0}\frac{\varphi^{d-N}(\tau)}{((\tau-r)^2+\delta^2\varphi^2(\tau))^{m/2}}\,d\tau \\ &\le C'\,\mathcal{I}_{N+m-d}(r), \end{aligned}$$

the last integral being introduced by (7.3). To complete the proof it remains to apply Lemma 7.2.

□

Remark 7.4 *It is easy to see that Theorem 7.3 actually remains valid for all* $z \in \mathcal{D}^-$.

8 Boundary behaviour of representing functions

Consider the Bochner-Martinelli integral $M(z)$ of f,

$$-M(z) = \int_{\mathcal{S}} f(\zeta) U(\zeta, z), \quad z \in \mathcal{D}^{\pm}.$$

Since $|M(z)| \leq c\ |\mathcal{P}_{2n-1}(z)|$, with c a constant independent of z, Theorem 7.3 for $m = 2n-1$ implies the following statements.

Corollary 8.1 *Under the assumptions of Theorem 7.3, the following estimates hold for* $z \in \mathcal{D} \setminus \mathcal{S}$*:*

1) *If* $-2n+3+d \leq N$, *then*

$$|M(z)| = O\left(\frac{1}{\varphi^{N+2n-2-d}(|(r,x)|)}\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

2) *If* $-2n+2+d \leq N < -2n+3+d$, *then*

$$|M(z)| = O\left(\frac{|\log \varphi(|(r,x)|)|}{\varphi^{N+2n-2-d}(|(r,x)|)}\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

3) *If* $N < -2n+2+d$, *then*

$$|M(z)| = O\left(1\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

The equality (4.3) enables us to apply Corollary 8.1 to highlight the boundary behaviour of the representing analytic functions $h^{\pm}(z)$ of Theorem 3.1.

Theorem 8.2 *Let* $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ *be a CR function on* $\mathcal{S} \setminus \sigma$ *satisfying*

$$S_f(\varepsilon, K) = O\left(\frac{1}{\varphi^N(\varepsilon)}\right)$$

as $\varepsilon \to 0$, *for any compact set* $K \subset \mathcal{D}$, *where* $N < 0$. *Then,*

1) *For* $-2n+3 \leq N$, *we have*

$$|h^{\pm}(z)| = O\left(\frac{1}{\varphi^{N+2n-2}(|(r,x)|)}\right) \quad \textit{as} \quad |(r,x)| \to 0.$$

2) *For $-2n+2 \leq N < -2n+3$, we have*

$$|h^{\pm}(z)| = O\left(\frac{|\log \varphi(|(r,x)|)|}{\varphi^{N+2n-2}(|(r,x)|)}\right) \quad as \quad |(r,x)| \to 0.$$

3) *For $N < -2n+2$, we have*

$$|h^{\pm}(z)| = O\,(1) \quad as \quad |(r,x)| \to 0.$$

In particular, $h^{\pm}(z)$ are of finite order of growth when $|(r,x)| \to 0$, provided that so is f.

For the case of point singularity $\sigma = \{z^0\}$ on $\mathcal{S}$, Corollary 8.1 and Theorem 8.2 are proved in [KMT99].

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