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Chapter 3

Applications of Noncommutative Analysis to Operator Algebras on Singular Manifolds

3.1 Statement of the problem

In this chapter, we show how noncommutative analysis can be used to construct and study algebras of pseudodifferential operators on manifolds with singularities. Operator algebras are an important aspect in the theory of differential equations and elliptic theory on singular manifolds, and the theory of such algebras was comprehensively developed in [9], where the reader can also find an extensive literature on the topic of manifolds with singularities, of course not limited just to this one aspect. Here we restrict ourselves to manifolds with isolated singularities of one of the following types: conical singular points and cusps (of integer order). In general, noncommutative analysis permits one to consider operator algebras on much more general spaces, but we prefer to avoid technicalities and concentrate at the essential points instead.

First of all, let us explain what manifolds with singularities are and state the main problem, whose solution will be described in the subsequent exposition. This is the subject of the present section.

A *manifold with isolated singularities* is an object that looks like a usual smooth manifold everywhere except for a discrete set of isolated points. (Usually one deals with compact manifolds, and the discrete set is actually finite.) Let us transform this fuzzy description into a rigorous definition. It is convenient to give a two-step definition. At the first step, we just define the notion of manifold with singularities taking no care of what type these singularities are; at the second step, we supply relevant information determining the type (a cone or a cusp of specific order).

Definition 1 A *manifold with singularities* is a Hausdorff topological space M with distinguished *singular points* $\alpha_1, \ldots, \alpha_N \in M$ and with the following additional structures:

(1) the structure of a smooth manifold on

$$\overset{\circ}{M} = M \backslash \{\alpha_1, \ldots, \alpha_N\}; \tag{3.1}$$

(2) the "direct product" structure

$$U_j \cong \Omega_j \times [0,1)/\Omega_j \times \{0\} \tag{3.2}$$

in some neighborhood U_j of each point α_j, where Ω_j is a smooth compact manifold without boundary and the quotient implies that all points of the form $(\omega, 0) \in \Omega_j \times [0,1)$ are identified with one another, so that the space (3.2) looks like a cone (see Fig. 3.1), with the singular point α_j being just the vertex of the cone, $\alpha_j \cong \Omega_j \times \{0\}$. It is moreover assumed that the smooth structure naturally existing on the direct product

$$\overset{\circ}{U}_j = \Omega_j \times (0,1) \equiv U_j \backslash \{\alpha_j\}$$

is compatible with the smooth structure on $\overset{\circ}{M}$.

Thus, the definition says that each singular point has a neighborhood that looks like a cone. We point out that (3.2) is just a topological isomorphism and does not provide any information about the type of

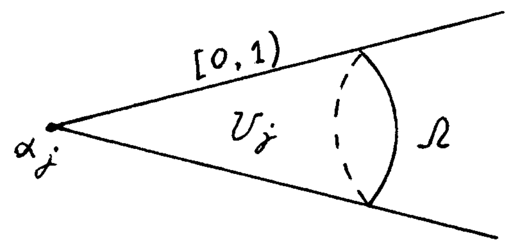

Figure 3.1: The neighborhood U_j of a singular point

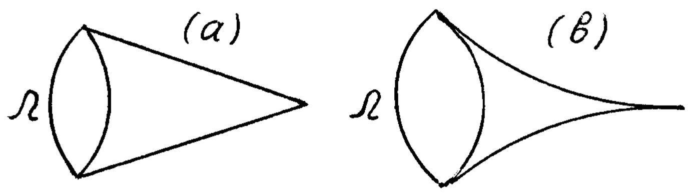

Figure 3.2: Topologically, a cusp (a) and a cone (b) with the same base Ω are equivalent (i.e. isomorphic). The isomorphism can be chosen in a way such that it respects the smooth structure outside the singular point

the singular point. (Topologically, cusps and cones are equivalent; see Fig. 3.2).

Now let us specify the type of singular points. This can be done in many various ways, e.g., by considering embeddings of M in Euclidean space or metrics on M degenerating at the singular points in a prescribed way, but the most direct method is as follows. Since our ultimate goal is to deal with differential equations on M, let us describe the type of singularity of M in terms of the *supply of differential operators that will be considered on* M.

More precisely, in the "smooth" part of M our differential operators may be arbitrary (except they must have smooth coefficients), whereas the behavior near the singular points must be controlled in a certain way. This method was described in [9].

Definition 2 Let M be a manifold with singularities $\alpha_1, \ldots, \alpha_N$. A *type of singularity* is a ring $\mathcal{D}$ of differential operators on M such that $\varphi\widehat{A} \in \mathcal{D}$ for any differential operator $\widehat{A}$ with smooth coefficients on M and any function $\varphi \in C_0^\infty(\overset{\circ}{M})$.

We consider only *conical singularities* and *cuspidal singularities* of integer order $k \geq 1$. While conical singularities and cuspidal singularities are substantily different in many respects, cuspidal singularities of various orders prove to be very much alike, and so we essentially deal with two distinct types of singularity.

Consider the ring $\mathcal{D}_0(M)$ of differential operators A that have the following form in the neighborhood U_j (3.2) of each singular point:

$$\widehat{A} = \sum_{|j| \leq m} \widehat{a}_j(r) \left(ir\frac{\partial}{\partial r} \right)^j, \tag{3.3}$$

where the $\widehat{a}_j(r)$ are differential operators with C^∞ coefficients on Ω_j smoothly depending on the parameter $r \in [0, 1)$ (up to the point $r = 0$). In other words, $\widehat{A}$ exhibits Fuchs type degeneration at the conical points.

Definition 3 A manifold M with isolated singularities $\alpha_1, \ldots, \alpha_N$ equipped with the ring $\mathcal{D}_0(M)$ is called a *manifold with conical singularities.*

Now let $k \geq 1$ be an integer. We consider the ring $\mathcal{D}_k(M)$ of differential operators of the form

$$\widehat{A} = \sum \widehat{a}_j(r) \left(ir^{k+1} \frac{\partial}{\partial r} \right)^j \tag{3.4}$$

in the neighborhoods U_j of the singular points α_j, $j = 1, \dots, N$. Here the $\widehat{a}_j(r)$ satisfy the same conditions as in the preceding definition.

Definition 4 A manifold M with isolated singularities equipped with the ring $\mathcal{D}_k(M)$ is called a *manifold with cuspidal singularities of order k.*

We refer the interested reader to [9] for the geometric motivation of these definitions as well as for an extended discussion of various other types of singularities.

The ring $\mathcal{D}_k(M)$ will be referred to as the *structure ring* of the singularity. The theory of differential equations on a manifold with singularities deals with operators belonging to the structure ring. However, it is well known (even in the theory of differential equations on smooth manifolds) that considering only differential operators is not sufficient. For example, parametrices (almost inverses) of elliptic differential operators are *pseudodifferential* rather than differential operators. Thus, the problem is to extend the structure ring $\mathcal{D}_k(M)$ to an algebra $PS\mathcal{D}_k(M)$ of pseudodifferential operators corresponding to the prescribed type of behavior near the singular points. Just as in the usual theory of pseudodifferential operators, this task is essentially *local*: we can construct pseudodifferential operators in coordinate charts and then patch the local definitions together using partitions of unity. Now that we consider a singular manifold M, the construction of pseudodifferential operators in the interior of M is clear in that it coincides with the usual one (e.g., see [2]–[5]). Accordingly, our attention will be focused on local models near singular points. The idea of the construction is quite obvious: since admissible differential operators have the form

$$\widehat{A} = A \left(\overset{2}{r}, \overset{1}{\overbrace{ir^{k+1} \frac{\partial}{\partial r}}} \right), \tag{3.5}$$

where the operator-valued symbol

$$A(r,p) = \sum_{|j|\leq m} a_j(r)p^j \tag{3.6}$$

is polynomial in p, we shall construct admissible pseudodifferential operators in the same form (3.5) but extend the class of symbols beyond polynomials. This is where noncommutative analysis works. To carry out the construction rigorously, we must do the following.

- Describe the operator algebra to which our would-be pseudodifferential operators will belong. (This will be the algebra of continuous operators in an appropriate scale of Hilbert spaces.)
- Describe the admissible class of symbols.
- Prove that the operators

  $$r, \quad ir^{k+1}\frac{\partial}{\partial r}$$

  are generators in this class of symbols.
- Finally, study whether operators with symbols of this class form an algebra.

  To this end, one uses the theory of ordered representations, which supplies the composition law in terms of symbols as a by-product.

Let us proceed to the implementation of the above program. The first two items actually belong in the statement of the problem and are considered in this section.

A. Function spaces. The choice of function spaces in which the operators to be constructed will act is a matter of crucial importance in that it affects all subsequent constructions and, if made wrongly, renders all the theory irrelevant. This choice has to be motivated from within the theory of differential equations itself. (Roughly speaking, these spaces must include solutions of the differential equations to be considered.) Fortunately, the problem of the appropriate choice of function spaces has been solved long ago. We refer the reader to [9] and the literature cited therein for the motivations, which in particular include a study of the asymptotics of solutions, and present here only the outcome, i.e. the definitions of function spaces. It is convenient to describe these spaces separately for $k = 0$ (conical singularities) and $k = 1$ (cuspidal singularities).

The case of conical singularities. Here our operators will act in the scales $\{H^{s,\gamma}(M)\}$ of *weighted Sobolev spaces* on M, where $s \in \mathbf{R}$ is the smoothness index and $\gamma \in \mathbf{R}$ is a given *weight exponent* of the scale. Thus, for each given $\gamma \in \mathbf{R}$ we have a scale of Hilbert spaces with index s verying from $-\infty$ to ∞. The space $H^{s,\gamma}(M)$ is defined as the completion of the space $C_0^\infty(\overset{\circ}{M})$ of smooth compactly supported functions on $\overset{\circ}{M}$ with respect to the norm

$$||u||_{s,\gamma} = \sum_{j=1}^{N} ||e_j u||_{s,\gamma} + ||e_0 u||_s, \tag{3.7}$$

where $1 = \sum\limits_{j=0}^{N} e_j$ is a smooth partition of unity on M subordinate to the cover

$$M = \overset{\circ}{M} \cup U_1 \cup \ldots \cup U_N, \tag{3.8}$$

$||e_0 u||_s$ is the usual Sobolev norm of order s on $\overset{\circ}{M}$ (since the support of e_0 is compact, the specific choice of $||\cdot||_s$ is irrelevant up to an equivalence of norms), and the norm $||e_j u||_{s,\gamma}$ is the *weighted Sobolev norm* on the model cone

$$K_j = [0,\infty) \times \Omega_j / \{0\} \times \Omega_j, \tag{3.9}$$

defined as follows:

$$||v||^2_{s,\gamma} = \int_0^\infty \int_\Omega \left| \left(1 - \left(r\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega \right)^{s/2} (r^{-\gamma} v) \right|^2 d\omega \frac{dr}{r}. \tag{3.10}$$

Here we (arbitrarily) equip Ω with some smooth Riemannian metric and associated volume element $d\omega$, and Δ_Ω is the Beltrami–Laplace operator on Ω corresponding to this metric. The function $e_j u$ can be interpreted, by virtue of the isomorphism (3.2), as a function on

$[0,1) \times \Omega_j / \{0\} \times \Omega_j$ and hence on the model cone K_j. The extension to the entire half-line $r \in [0,\infty)$ is made for convenience: the point is that the operator

$$1 - \left(r\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega$$

is essentially self-adjoint in

$$L^2\left(\Omega, \frac{dr}{r} d\omega \times [0,\infty)\right)$$

(but not in

$$L^2\left(\Omega \times [0,1), \frac{dr}{r} d\omega\right),$$

where one would be forced to pose some boundary condition at $r = 1$) and is also positive, so that all real powers

$$\left(1 - \left(r\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega\right)^{s/2}$$

are well defined.

The parameter of the scale $\{H^{s,\gamma}(M)\}_{s\in\mathbf{R}}$ is $s \in \mathbf{R}$, whereas the weight exponent γ is specific problems is chosen in some way and then is assumed to be fixed. We note the isomorphisms

$$H^{s,0}(M) \stackrel{\varphi^\gamma}{\to} H^{s,\gamma}(M) \tag{3.11}$$

given by the nultiplication by φ^γ, where φ is an arbitrary smooth function on $\overset{\circ}{M}$ with the following properties:

(i) $\varphi > 0$ everewhere on $\overset{\circ}{M}$;

(ii) $\varphi = r$ in each $\overset{\circ}{U}_j$, $j = 1, \ldots, N$.

In view of these isomorphisms, in what follows we consider only the case $\gamma = 0$ and denote the corresponding spaces by $H^s(M) \equiv H^{s,0}(M)$. This saves us one superscript in notation without any loss of generality, and an additional advantage is that all our symbols now need to be defined only for real values of the arguments (later we shall clarify this). The norm in $H^s(M)$ will be denoted by $||\cdot||_s$.

Thus, the operator algebra $\mathcal{A}_0$ in which we construct our pseudodifferential operators in the conical case is just the algebra of continuous operators in the scale $\{H^s(M)\}$: an operator A belongs to $\mathcal{A}_0$ if there exists an $m \in \mathbf{R}$ such that

$$A:\ H^s(M) \to H^{s-m}(M) \tag{3.12}$$

is continuous for all $s \in \mathbf{R}$. (Recall that when defining a continuous operator acting in a scale of spaces, one has two possibilities. First, one can consider a linear operator

$$A: \bigcup_s H^s(M) \to \bigcup_s H^s(M) \tag{3.13}$$

such that the restriction of A to each H^s gives a continuous operator in the spaces (3.12).

Second, one can take a linear operator

$$A:\ C_0^\infty(M) \bigcap_s H^s(M)$$

such that

$$||Au||_{s-m} \le C_s ||u||_s, \quad u \in C_0^\infty(M),$$

for every $s \in \mathbf{R}$ and then extend A by continuity to the entire $H^s(M)$ for each s. The extensons agree with each other and hence define a linear operator satisfying the estimates (3.12). Both methods give the same supply of operators.)

The case of cuspidal singularities. The passage from the global to the local definition of weighted Sobolev spaces $H_k^{s,\gamma}(M)$ on manifolds with cuspidal singularities of order $k \ge 1$ is pretty much the same as for manifolds with conical singularities. However, the definition of the "local" norm on the cone K_j is different. Naively, one would write something like

$$||v||^2_{s,\gamma,k} \int_0^\infty \int_{\Omega_j} \left| \left(1 - \left(r^{k+1} \frac{\partial}{\partial r} \right)^2 - \Delta_\Omega \right)^{s/2} (e^{-\gamma/kr^k} v) \right|^2 d\omega \frac{dr}{r^{k+1}}, \tag{3.14}$$

but this has to be interpreted somehow, for the operator

$$1 - \left(r^{k+1} \frac{\partial}{\partial r} \right)^2 - \Delta_\Omega$$

is not essentially self-adjoint in

$$L^2 \left(K_j, \frac{dr}{kr^{n+1}} d\omega \right),$$

but is only symmetric. This is actually due to the fact that the trajectories of the vector field

$$r^{k+1} \frac{\partial}{\partial r}$$

on the half-line $\mathbf{R}_+$ exit to infinity ($r = +\infty$) in finite time. Hence we must use a definition different from (3.14). Note that the operator

$$ir^{k+1} \frac{\partial}{\partial r}$$

can be represented in the form

$$ir^{k+1} \frac{\partial}{\partial r} = \mathcal{B}_k^{-1} \, p \, \mathcal{B}_k, \tag{3.15}$$

where $\mathcal{B}_k$ is the kth-order Borel–Laplace transform

$$[\mathcal{B}_k u](p) = \int_0^\infty \exp \left(\frac{-ip}{kr^k} \right) u(r) \frac{dr}{r^{k+1}} \tag{3.16}$$

and $\mathcal{B}_k^{-1}$ is the inverse transform

$$[\mathcal{B}_k^{-1} \psi](r) = \int_{-\infty+i\gamma}^{+\infty+i\gamma} \exp \left(\frac{i\gamma}{kr^k} \right) \psi(p) \, dp. \tag{3.17}$$

Thus, the operator

$$ir^{k+1} \frac{\partial}{\partial r}$$

is similar with the help of $\mathcal{B}_k$ to the operator of multiplication by p. Next, routine computations show that

$$\mathcal{B}_k : L^2\left(\Omega_j \times [0,\infty), e^{-2\gamma/kr^k}\frac{dr\,d\omega}{r^{k+1}}\right) \to L^2(\Omega_j \times \mathcal{L}_\gamma), \tag{3.18}$$

where $\mathcal{L}_\gamma = \{p \in \mathbf{C} \,|\, \operatorname{Im} p = \gamma\}$ is the *weight line* corresponding to the weight exponent γ, is an isometric embedding (but not an isomorphism). In view of all these considerations, we define the norm $||v||_{s,\gamma,k}$ as follows:

$$||v_{s,\gamma,k}||^2 = \int\limits_{\Omega_j}\int\limits_{\mathcal{L}_j} |(1+|p|^2 - \Delta_\Omega)^{s/2}(\mathcal{B}_k v)(\omega,p)|^2 d\omega\,dp. \tag{3.19}$$

Naturally, for integer $s/2$ this expression is the same as (3.14). Again, using obvious isomorphisms, in the following we only consider the case in which the weight exponent is zero. The corresponding spaces will be denoted by $H^s_k(M)$, and we often even omit the subscript k if it is clear from the context.

Remark 1 Since the globalization procedure is fairly standard and is described very well in any good textbook on pseudodifferential operators, we work only on the model cone (or cusp)

$$\widetilde{K}_j = \Omega_j \times [0,\infty)/\Omega_j \times \{0\} \tag{3.20}$$

in the remaining part of this chapter (except for Section 3.4) and study algebras of pseudodifferential operators on $\widetilde{K}_j$. Another somewhat subtle point is localization on Ω_j. To avoid technical complications and clarify the exposition, we work in local coordinates on Ω and do not paste the local representations together. So in fact we pretend that $\Omega = \mathbf{R}^{n-1}$ (noncompact!) but all our operators have symbols that are compactly supported in $\omega \in \Omega$ or at least decay sufficiently rapidly as $\omega \to \infty$ wherever necessary.

C. Symbol classes. Undoubtedly, the symbol class that is the most convenient for the use in applications of noncommutative analysis is

$S^\infty(\mathbf{R}^n)$. Recall that it consists of functions $f(y)$, $y \in \mathbf{R}^n$, satisfying the estimates

$$|f^{(\alpha)}(y)| \le C_\alpha(1+|y|)^m, \quad |\alpha| = 0, 1, 2, \dots, \tag{3.21}$$

with some $m = m(f)$ (see Chapter 1 for the definition of convergence in $S^\infty(\mathbf{R}^n)$). However, when one deals with pseudodifferential operators, the dependence of their symbols on the spatial variables is usually assumed to be uniform (no growth at infinity with respect to the spatial variables), and moreover, operators whose symbols *do grow* with respect to the spatial variables are not bounded in the spaces in questions, that is, do not belong to the algebra. This pertains to manifolds with singularities as well. To consider such operator algebras and symbol classes within the framework of noncommutative analysis conveniently, we slightly extend the definition of functions of noncommuting operators by allowing the symbols themselves to be operator-valued. Let us describe this extension. Let $\mathcal{A}$ be complete operator algebra with convergence, and let $\mathcal{F}$ be a symbol class. Next, let $\mathcal{B} \subset \mathcal{A}$ be a closed subalgebra. (In the applications considered in this chapter, $\mathcal{B}$ is commutative, but one need not assume this in the general definition.) We define classes of $\mathcal{B}$-valued n-ary symbols as

$$\mathcal{B}\mathcal{F}_n \stackrel{\text{def}}{=} \mathcal{B}\widehat{\otimes}\mathcal{F}^{\widehat{\otimes}n} \equiv \mathcal{B}\widehat{\otimes}\underbrace{\mathcal{F}\widehat{\otimes}\dots\widehat{\otimes}\mathcal{F}}_{n\text{ copies}} \tag{3.22}$$

(here the projective tensor product $\widehat{\otimes}$ is used; see the explanations in Chapter 1). Now if $f \in \mathcal{B}\mathcal{F}_n$ is a symbol and $A_1, \dots, A_n$ are $\mathcal{F}$-generators, then we define the operator

$$\widehat{f} = \overset{n+1}{f}(\overset{1}{A_1}, \dots, \overset{n}{A_n}) \tag{3.23}$$

in a natural way: if f is factorable,

$$f = B \otimes f_1 \otimes \dots \otimes f_N, \tag{3.24}$$

$B \in \mathcal{B}$, $f_j \in \mathcal{A}$, $j = 1, \dots, N$, then we set

$$\overset{n+1}{f}(\overset{1}{A_1}, \dots, \overset{n}{A_n}) = B \otimes f_n(A_n) \otimes \dots \otimes f_1(A_1). \tag{3.25}$$

(The place occupied by B in the product changes appropriately if we assign a different Feynman index to the symbol f). With this more general definition, the main theorem of Chapter 2, which expresses the composition law in terms of symbols via ordered representation operators remains valid. Namely, let $\overset{1}{A}_1, \dots, \overset{n}{A}_n$ be a given tuple of $\mathcal{F}$-generators in $\mathcal{A}$, and supose that there exist operators

$$l_j : \mathcal{BF}_n \to \mathcal{BF}_n \tag{3.26}$$

such that

$$A_j [\![\overset{n+1}{f} (\overset{1}{A}_1, \dots, \overset{n}{A}_n)]\!] = (\overset{n+1}{l_j f})(\overset{1}{A}_1, \dots, \overset{n}{A}_n). \tag{3.27}$$

Then

$$[\![\overset{n+1}{f} (\overset{1}{A}_1, \dots, \overset{n}{A}_n)]\!] [\![\overset{n+1}{g} (\overset{1}{A}_1, \dots, \overset{n}{A}_n)]\!] = \overset{n+1}{[\![} \overset{n+1}{f} (\overset{1}{l}_1, \dots, \overset{n}{l}_n)(g)]\!] (\overset{1}{A}_1, \dots, \overset{n}{A}_n) \tag{3.28}$$

for polynomial symbols f and arbitrary symbols g, and even for arbitrary f and g provided that the l_j are $\mathcal{F}$-generators. The proof *mutatis mutandis* reproduces the proof of the corresponding theorem in Chapter 2.

With this definition, we shall approach pseudodifferential operators. Namely, we shall consider operators of the form

$$\widehat{f} = f \left(\overset{2}{r}, \overset{2}{\omega}, \overset{1}{ir \frac{\partial}{\partial r}}, \overset{1}{-i \frac{\partial}{\partial \omega}} \right) \tag{3.29}$$

in the conical case and

$$\widehat{f} = f \left(\overset{2}{r}, \overset{2}{\omega}, \overset{1}{ir^{k+1} \frac{\partial}{\partial r}}, \overset{1}{-i \frac{\partial}{\partial \omega}} \right) \tag{3.30}$$

in the case of cusps of order k (the form of operator arguments will be slightly modified), where the symbols $f(r, \omega, p, q)$ satisfy the estimates

$$\begin{aligned} |f^{(\alpha\beta\gamma\delta)}_{r\omega pq}(r, \omega, p, q)| &\leq C_{\alpha\beta\gamma\delta}(1 + |p| + |q|)^m, \\ &|\alpha| + |\beta| + |\gamma| + |\delta| = 0, 1, 2, \dots, \end{aligned} \tag{3.31}$$

and we shall often use the following interpretation of this, which is in line with the above extended definition of functions of noncommuting operators.

Consider the algebra $\mathcal{B}$ of functions $g(r,\omega)$ bounded together with all derivatives. It is naturally interpreted as a subalgebra of the algebra $\mathcal{A}$ of continuous operators in the scale $\{H^s(\widetilde{K}_j)\}$ ($g(r,\omega)$ is identified with the operator of multiplication by $g(r,\omega)$). Then each symbol $f(r,\omega,p,q)$ satisfying (3.31) can be interpreted as an element $F \in \mathcal{B}S^\infty(\mathbf{R}^n_{p,q})$ as follows:

$$[F(p,q)](r,\omega) = f(r,\omega,p,q). \tag{3.32}$$

Now we set

$$f\left(\overset{2}{r},\overset{2}{\omega}, \overset{1}{ir^{k+1}\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial\omega}}\right) = \overset{2}{F}\left(\overset{1}{ir^{k+1}\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial\omega}}\right). \tag{3.33}$$

We note that if both sides of (3.33) can be defined independently, then the equality still takes place. In the following, we use the notation

$$f\left(\overset{2}{r},\overset{2}{\omega}, \overset{1}{ir^{k+1}\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial\omega}}\right)$$

but understand this as a function

$$\overset{2}{F}\left(\overset{1}{ir^{k+1}\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial\omega}}\right)$$

with an operator-valued symbol. This will not lead to errors or misunderstanding.

Let us summarize our task. We wish to extend the structure rings $D_k(\widetilde{K})$ of differential operators of the form (3.3) (or (3.4)) to wider sets of pseudodifferential operators of the form (3.29) (respectively, (3.30)) with symbols f satisfying the estimates (3.31). (This symbol space will be denoted by $S^{0,\infty}$ or by $\mathcal{B}S^\infty$, depending on whether f is interpreted as an ordinary function or an operator-valued symbol.) We intend to study whether these wider sets of pseudodifferential operators are algebras and, if so, find the corresponding composition laws. This is done in Section 3.3 for the case of cusps. In Section 3.4 we apply the results to construct regularizers for elliptic elements and to prove the finiteness theorem (the Fredholm property).

3.2 Operators on the Model Cone

Here we shall consider operators of the form

$$\widehat{f} = f\left(\overset{2}{r}, \overset{2}{\omega}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{1}{r^{k+1}\frac{\partial}{\partial r}}\right), \tag{3.34}$$

where $f \in S^{0,\infty}(\mathbf{R}_+ \times \Omega \times \mathbf{R}^n_{p,q})$, in the scale of spaces $H^s(\mathbf{R}_+ \times \Omega)$ (from now on, we omit the subscript j on Ω).

First, let us note that if we wish that the set of operators of the form (3.34) be an algebra then some modification is obviously needed. Indeed, let

$$\begin{aligned} f_1(r,\omega,p,q) &= p, \\ f_2(r,\omega,p,q) &= \sin r. \end{aligned} \tag{3.35}$$

Then the product $\widehat{f}_1\,\widehat{f}_2$ is

$$\widehat{f}_1\,\widehat{f}_2 = ir\frac{\partial}{\partial r}\sin r = \sin r \cdot ir\frac{\partial}{\partial r} + ir\cos r = \widehat{h},$$

where $h(r,\omega,p,q) = p\sin r + ir\cos r$ exhibits growth as $r \to \infty$, that is, $h \notin S^{0,\infty}$. The first idea is to allow symbols growing as $r \to \infty$, that is, consider the standard symbol class $S^\infty(\mathbf{R}_+ \times \Omega \times \mathbf{R}^n_{p,q})$. However, the operators

$$ir\frac{\partial}{\partial r}$$

and r cannot be simultaneously generators for this symbol class. This is shown by the following *reductio ad absurdum* argument (see [6]). Suppose that both

$$ir\frac{\partial}{\partial r}$$

and r are S^∞-generators. Consider the symbols

$$\begin{aligned} f(x,y) &= x - y - i, \\ g(x,y) &= (x-y-i)^{-1}. \end{aligned} \tag{3.36}$$

Both symbols are S^∞, and $fg = 1$. Thus

$$r = \overset{2}{r}\, f\left(\overset{3}{ir\frac{\partial}{\partial r}}, \overset{1}{ir\frac{\partial}{\partial r}} \right) g\left(\overset{3}{ir\frac{\partial}{\partial r}}, \overset{1}{ir\frac{\partial}{\partial r}} \right). \tag{3.37}$$

Moving indices apart, we obtain

$$r = \overset{2}{r}\, f\left(\overset{3}{ir\frac{\partial}{\partial r}}, \overset{1}{ir\frac{\partial}{\partial r}} \right) g\left(\overset{4}{ir\frac{\partial}{\partial r}}, \overset{0}{ir\frac{\partial}{\partial r}} \right).$$

Now we can isolate the middle factor in this expression and write

$$r = \overset{2}{[\![}\, \overset{2}{r}\, f\left(\overset{3}{ir\frac{\partial}{\partial r}}, \overset{1}{ir\frac{\partial}{\partial r}} \right)]\!]\, g\left(\overset{4}{ir\frac{\partial}{\partial r}}, \overset{0}{ir\frac{\partial}{\partial r}} \right). \tag{3.38}$$

Now this factor is

$$\overset{2}{r}\, f\left(\overset{3}{ir\frac{\partial}{\partial r}}, \overset{1}{ir\frac{\partial}{\partial r}} \right) = \left[ir\frac{\partial}{\partial r}, r \right] - i = 0, \tag{3.39}$$

and by substituting this into (3.38) we arrive at a contradiction: $r = 0$.

We see that the idea of extending the symbol class by allowing symbols growing in r does not make sense. But we have a different option: let us modify the definition of the operators themselves! Namely, let us replace the operator

$$ir\frac{\partial}{\partial r}$$

by

$$i\varphi(r)\frac{\partial}{\partial r},$$

where

$$\varphi(r) = \begin{cases} r, & 0 \le r \le 1/2, \\ 1, & r \ge 1, \end{cases} \tag{3.40}$$

and moreover, $\varphi(r)$ is real-valued and vanishes nowhere except for $r = 0$.

From the viewpoint of applications, this is adequate, since the operators on the model cone actually model the behavior of the operators on manifolds with the corresponding singularities only near the point $r = 0$. In a neighborhood of the point $r = \infty$, we can choose the behavior of our operators from the viewpoint of convenience. Using the new operator, we accordingly modify the definition of the Sobolev spaces $H^s(\widetilde{K})$ near $r = \infty$. Now we define the norm by setting

$$||u||_s = \int_0^\infty \int_\Omega \left| \left(1 - \left(\varphi(r)\frac{\partial}{\partial r} \right)^2 - \Delta_\Omega \right)^{s/2} u \right|^2 d\omega \frac{dr}{\varphi(r)}. \tag{3.41}$$

The operator

$$1 - \left(\varphi(r)\frac{\partial}{\partial r} \right)^2 - \Delta_\Omega$$

is essentially self-adjoint and positive in

$$L^2 \left(\widetilde{K}, d\omega \frac{dr}{\varphi(r)} \right),$$

so that arbitrary real powers of this operator are well defined.

Consequently, so are the spaces $H^s(\widetilde{K})$ with the new norm (3.41).

As follows from (3.7), only the norm of elements u supported in the interval $r \in [0, R]$ with some finite R is essential when we pass to Sobolev spaces on M. (Without loss of generality, we can assume that $R < 1/2$.) We claim that, for given R, the norm (3.41) is equivalent to the norm (3.10) (with $\gamma = 0$) on the set of functions u supported in $\Omega \times [0, R]$. This is obvious if $s = 2l > 0$ is an even positive number, since in this case the operators occurring in the de finitions of the norms are differential and (3.41) coincides with (3.10) for such functions u. To prove the desired equivalence for arbitrary s, we proceed as follows.

Let us denote the Sobolev space with the norm (3.10) (for $\gamma = 0$) by $H^s_{\rm old}(\widetilde{K})$. Let $\psi(r)$ be a smooth function such that

$$\psi(r) = \begin{cases} 1, & r \le R, \\ 0, & r \ge 1/2. \end{cases}$$

We claim that the operator of multiplication by $\psi(r)$ is continuous in the spaces

$$\psi(r) \; : \; H^s_{\rm old}(\widetilde{K}) \to H^s(\widetilde{K}), \tag{3.42}$$

$$\psi(r) \; : \; H^s(\widetilde{K}) \to H^s_{\rm old}(\widetilde{K}) \tag{3.43}$$

for $s = 2l$, $l \in \mathbf{Z}_+$. Indeed, consider, for example, the mapping (3.42). It can be decomposed as

$$H^s_{\rm old}(\widetilde{K}) \stackrel{\psi}{\to} H^s_{\rm old}(\widetilde{K}) \cap \{\operatorname{supp} u \in [0,1/2] \times \Omega\} \to H^s(\widetilde{K}).$$

The operator of multiplication by ψ is bounded in $H^s_{\rm old}(\widetilde{K})$, since ψ is bounded together with all derivatives (see Lemma 5 below), the second arrow in this deomposition is just an isometric embedding. The same reasoning pertains to the mapping (3.43). By interpolation theiry of linear operators in Hilbert scales, the operators (3.42) and (3.43) are continuous for every real $s \ge 0$. But $\psi u = u$ if $\operatorname{supp} u \subset \Omega \times [0,R]$, and hence we find that

$$c_s ||u||_{s,\rm old} \le ||u||_s \le C_s ||u||_{s,\rm old}, \quad s \ge 0 \tag{3.44}$$

with some constants $C_s, c_s > 0$. To prove that (3.44) remains valid for $s < 0$, we use the duality between $H^s(\widetilde{K})$ and $H^{-s}(\widetilde{K})$, as well as between $H^s_{\rm old}(\widetilde{K})$ and $H^{-s}_{\rm old}(\widetilde{K})$, given by the L^2 inner product. Let < 0, and let u be supported in $\Omega \times [0,R]$. Then

$$||u||_s = \max_{||\chi||_{-s}=1} |(\chi,u)| = \left\{ \min_{||(\chi,u)||=1} ||\chi||_{-s} \right\}^{-1},$$

and similarly for $||u||_{s,\rm old}$. We claim that, up to equivalence, it sufficies to take the minimum over elements χ supported in $[0,R]$. Indeed, we have

$$(\chi,u) = (\psi\chi,u),$$

and so this assertion is obvious. But for these χ we have the equivalence $||\chi||_{-s} \sim ||\chi||_{-s,\mathrm{old}}$, and hence we imply that $||u||_s \sim ||u||_{s,\mathrm{old}}$.

Now we intend to define functions of the class $S^{0,\infty}(\widetilde{K} \times \mathbf{R}^n) \cong \mathcal{B}S^\infty(\widetilde{K})$ of the operators $\overset{2}{\omega}, \overset{2}{r}$ (occurring only as arguments with respect to which the symbols are bounded) and

$$\overset{1}{i\frac{\partial}{\partial\omega}}, \overset{1}{i\varphi(r)\frac{\partial}{\partial r}}.$$

To prove that these functions are well defined, we state the following two lemmas.

Lemma 5 *The algebra $\mathcal{B}$ of operators of multiplication by functions $f(\omega, r)$ bounded together with all derivatives is a closed subalgebra of the algebra $\mathcal{A}$ of bounded operators in the scale $H^s(\widetilde{K})$ of Hilbert spaces with the norm* (3.41).

Lemma 6 *The operators*

$$-i\frac{\partial}{\partial\omega}$$

and

$$i\varphi(r)\frac{\partial}{\partial r}$$

are S^∞-generators in $\mathcal{A}$.

Proof of Lemma 5. We shall actually prove more. Namely, we claim that for any $f(\omega, r) \in \mathcal{B}$ the operator of multiplication by f is bounded in the spaces

$$f(\omega, r): \; H^s(\widetilde{K}) \to H^s(\widetilde{K}) \tag{3.45}$$

for every $s \in \mathbf{R}$. First, let us prove this for nonnegative integer s. For these s, there is an equivalence of norms

$$||u||_s \sim ||u||_{s-1} + \left|\left|\frac{\partial u}{\partial\omega}\right|\right|_{s-1} + \left|\left|\varphi(r)\frac{\partial u}{\partial r}\right|\right|_{s-1}. \tag{3.46}$$

We proceed by induction over s. For $s = 0$ the operator (3.45) is obviously bounded (this is the operator of multiplication by a bounded function in L^2). Now let the desired assertion be valid for all $s \le s_0$. For $s = s_0 + 1$ we have

$$\begin{aligned} ||fu||_s &\sim ||fu||_{s-1} + \left\|\frac{\partial(fu)}{\partial\omega}\right\|_{s-1} + \left\|\varphi(r)\frac{\partial(fu)}{\partial r}\right\|_{s-1} \\ &\le ||fu||_{s-1} + \left\|\frac{\partial f}{\partial\omega}\cdot u\right\|_{s-1} + \left\|f\frac{\partial u}{\partial\omega}\right\|_{s-1} \\ &\quad + \left\|\varphi(r)\frac{\partial f}{\partial r}\cdot u\right\|_{s-1} + \left\|f\cdot\varphi(r)\frac{\partial u}{\partial r}\right\|_{s-1}. \end{aligned} \tag{3.47}$$

Since the derivatives

$$\frac{\partial f}{\partial\omega}$$

and

$$\varphi(r)\frac{\partial f}{\partial r}$$

also belong to $\mathcal{B}$, it follows from the induction assumption that the right-hand side of (3.47) does not exceed

$$\text{const}\left(||u||_{s-1} + \left\|\frac{\partial u}{\partial\omega}\right\|_{s-1} + \left\|\varphi(r)\frac{\partial u}{\partial r}\right\|_{s-1}\right) \le \text{const}\,||u||_s. \tag{3.48}$$

Thus we have proved (3.45) for nonnegative integer s, and (3.45) for arbitrary nonnegative s now follows by interpolation theory. Next, let s be negative. Without loss of generality, we can assume that $f(\omega, r)$ is real-valued. Then the operator of multiplication by $f(\omega, r)$ is self-adjoint in

$$H^0 = L^2\left(\widetilde{K}, \frac{dr}{\varphi(r)}d\omega\right),$$

and the desired estimate follows readily from the fact that H^s and H^{-s} are dual with respect to the pairing defined by the H^0-inner product. The proof of Lemma 5 is complete.

Proof of Lemma 6. To prove this lemma, we use the following well-known criterion for an element A of an algebra $\mathcal{A}$ with convergence to be an S^∞-generator:

Proposition 7 ([8], Theorem IV.3 and Corollary IV.1) *An operator A is an S^∞-generator if and only if it is a generator of a tempered one-parameter group $e^{iAt} \in \mathcal{A}$ (that is, a group that grows as $t \to \pm\infty$ at most polynomially in t together with all of its derivatives).*

For the detailed proof of this proposition, we refer the reader to [8], Chapter III. Here we only give a brief explanation of why this proposition is true. Suppose that A is an S^∞-generator. This means that there is a continuous homomorphism

$$\mu : S^\infty(\mathbf{R}) \to \mathcal{A} \tag{3.49}$$

such that $\mu(x) = \mathcal{A}$. For any k, the mapping

$$\psi_k : \mathbf{R} \to S^\infty(\mathbf{R}), \quad \psi_k(t) = (ix)^k e^{itx}, \tag{3.50}$$

is continuously differentiable and of has most polynomial growth at infinity. Combining this with (3.49), we find that

$$\psi_k(A) = \left(\frac{\partial}{\partial t}\right)^k (e^{itA}) \tag{3.51}$$

is of tempered growth.

Conversely, if all $\psi_k(A)$ are of tempered growth, then the mapping (3.49) can be defined by the formula

$$\mu(f) = \frac{i}{2\pi} \int e^{itA} \tilde{f}(t)\, dt, \tag{3.52}$$

where $\tilde{f}(t)$ is the Fourier transform of f. Under our assumptions, one can justify the convergence of the integral (3.52) in suitably chosen seminorms. Thus, all we need to show when proving Lemma 6 is that the operators

$$-i\frac{\partial}{\partial \omega}$$

and

$$i\varphi(r)\frac{\partial}{\partial r}$$

generate groups of tempered growth in the spaces $H^s(\widetilde{K})$. This is however obvious, since both operators commute with

$$\left(1-\left(\varphi(r)\frac{\partial}{\partial r}\right)^2-\Delta_\Omega\right)^{s/2}$$

for arbitrary s and consequently are self-adjoint not only for $s=0$, but also for arbitrary s. Accordingly, the groups generated by these operators are unitary, which completes the proof of the lemma.

Let us summarize our results obtained so far in the form of a theorem.

Theorem 8 *For any symbol $f \in S^{0,\infty}(\widetilde{K}\times\mathbf{R}^n)$, the operator*

$$\widehat{f}=f\left(\overset{2}{r},\overset{2}{\omega},ir\overset{1}{\frac{\partial}{\partial r}},-i\overset{1}{\frac{\partial}{\partial\omega}}\right) \tag{3.53}$$

is well defined and belongs to the algebra $\mathcal{A}$ of bounded operators in the scale $\{H^s(\widetilde{K})\}_{s\in\mathbf{R}}$. In other words, there exists an $m\in\mathbf{R}$ such that

$$\widehat{f}:\, H^s(\widetilde{K})\to H^{s-m}(\widetilde{K}) \tag{3.54}$$

is bounded for any $s\in\mathbf{R}$.

This theorem does not relate the order k of growth of f as $|p|+|q|\to\infty$ with the order m of the corresponding operator $\widehat{f}$ in the scale $\{H^s(\widetilde{K})\}_{s\in\mathbf{R}}$. For general symbols $f\in S^{0,\infty}(\widetilde{K}\times\mathbf{R}^n)$, it is easy to obtain a coarse estimate $m\le k+n+\varepsilon$ for any $\varepsilon>0$ (cf. [8], Theorem IV.5), but proving anything beyon that would be extremely hard. However, one can say something more precise about the order of an operator $\widehat{f}$ with "classical" symbol

$$f\in L^m(\widetilde{K}\times\mathbf{R}^n)\subset S^{0,\infty}(\widetilde{K}\times\mathbf{R}^n),$$

where $L^m(\widetilde{K}\times\mathbf{R}^n)$ is the space of symbols $f(r,\omega,p,q)$ satisfying the estimates

$$\begin{aligned}|f^{(\alpha\beta\gamma\delta)}_{r,\omega,p,q}(r,\omega,p,q)| &\le C_{\alpha\beta\gamma\delta}(1+|p|+|q|)^{m-|\gamma|-|\delta|}, \\ &\qquad |\alpha|,|\beta|,|\gamma|,|\delta|=0,1,2,\dots.\end{aligned} \tag{3.55}$$

Theorem 9 *The operator*

$$\widehat{f} = f\left(\overset{2}{r}, \overset{2}{\omega}, ir\overset{1}{\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial \omega}}\right)$$

with symbol f *satisfying the estimates* (3.55) *is bounded in the spaces*

$$\widehat{f}:\ H^s(\widetilde{K}) \to H^{s-m}(\widetilde{K}). \tag{3.56}$$

In other words, the order of the operator does not exceed the order of its symbol.

Proof. We prove this theorem by reducing it to a special case of Theorems II.14 and IV.6 in [8]. To this end, we proceed from the cone to a more suitable space, namely, a cylinder.

Let us consider the change of variables $r = r(\tau)$ specified by the solution of the Cauchy problem for the ordinary differential equation

$$\begin{cases} \dot{r} = -\varphi(r), \\ r|_{\tau=0} = 1. \end{cases} \tag{3.57}$$

The solutions of the Cauchy problem (3.57) are globally defined on $\mathbf{R}_\tau$; by virtue of (3.40), one has

$$r(\tau) = -\tau + \text{const}\,, \quad \tau < 0, \tag{3.58}$$

$$r(\tau) = \text{const}\,\cdot e^{-\tau}, \quad \tau >> 0. \tag{3.59}$$

This change of variables takes $\widetilde{K}$ to the infinite cylinder $(-\infty, \infty) \times \Omega$ with coordinates (τ, ω) and the operator

$$i\varphi(r)\frac{\partial}{\partial r}$$

to

$$-i\frac{\partial}{\partial \tau}.$$

The spaces $H^s(\widetilde{K})$ become the usual Sobolev spaces on $(-\infty, \infty) \times \Omega$ with the norm

$$||u||_s^2 = \int\limits_{-\infty}^{\infty}\int\limits_{\Omega}\left|\left(1 - \frac{\partial^2}{\partial\tau^2} - \Delta_\Omega\right)^{s/2} u\right|^2 d\tau\, d\omega. \tag{3.60}$$

Finally, the symbol

$$g(\tau,\omega,p,q) = f(r(\tau),\omega,p,q) \tag{3.61}$$

satisfies the same uniform estimates

$$|g^{(\alpha\beta\gamma\delta)}_{\tau,\omega,p,q}(\tau,\omega,p,q)| \le C_{\alpha\beta\gamma\delta}(1+|p|+|q|)^{m-|\gamma|-|\delta|} \tag{3.62}$$

in the new variables, since all derivatives

$$\frac{\partial^k r}{\partial \tau^k}$$

are bounded on the entire axis. Now one has to prove that the operator

$$g\left(\overset{2}{\tau}, \overset{2}{\omega}, -i\overset{1}{\frac{\partial}{\partial\tau}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right)$$

with symbol satisfying the estimates (3.62) is bounded (of order $\le m$) in the spaces with the norms (3.60), but this is fairly standard. One version of proof can be found, say, in [8], Theorems II.14 and IV.6. The proof of Theorem 9 is complete.

Now we proceed to our main question: *Is the set of operators* (3.53) *with symbols* $f \in S^{0,\infty}(K\times\mathbf{R}^n)$ *an algebra?*

Noncommutative analysis gives the right tools to answer this question. Namely, the problem is reduced to constructing an ordered representation of the operators

$$\left(\overset{2}{r}, \overset{2}{\omega}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}\right)$$

and studying properties.

By virtue of the results of Chapter 2, we have the following equivalence (with some subtle points to be revealed later on in our exposition):

Functions with symbols $f \in \mathcal{F}_n$ of operators $A = (\overset{1}{A_1}, \dots, \overset{n}{A_n}) \in \mathcal{A}$ form an algebra	$\Leftrightarrow$	The n-tuple A has operators of left ordered representation $l = (l_1,\dots,l_n)$, and these operators are $\mathcal{F}$-generators in $\mathcal{F}_n$.

Thus we proceed to the construction of left ordered representation operators. According to our more general setting (see the discussion of operator-valued symbols at the end of in Section 3.1), we seek for operators

$$l_{-i\partial/\partial\omega} \;:\; \mathcal{B}S^\infty(\mathbf{R}^n) \to \mathcal{F}S^\infty(\mathbf{R}^n), \tag{3.63}$$

$$l \;:\; \mathcal{B}S^\infty(\mathbf{R}^n) \to \mathcal{B}S^\infty(\mathbf{R}^n) \tag{3.64}$$

such that

$$(\overset{2}{\overline{l_{-i\partial/\partial\omega}F}})\left(i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right) = -i\frac{\partial}{\partial\omega}[\![\overset{2}{F}\left(i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right)]\!] \tag{3.65}$$

and

$$\overset{2}{\overline{lF}}\left(i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right) = i\varphi(r)\frac{\partial}{\partial r}[\![\overset{2}{F}\left(i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right)]\!]. \tag{3.66}$$

(Note that $l_{-i\partial/\partial\omega}$ is actually a vector operator with $n-1$ components,

$$l_{-i\partial/\partial\omega} = (l_{-i\partial/\partial\omega_1}, \dots, l_{-i\partial/\partial\omega_{n-1}}).$$

In more customary notation, (3.65) and (3.66) can be rewritten as

$$(l_{-i\partial/\partial\omega}f)\left(\overset{2}{r}, \overset{2}{\omega}, i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right) = -i\frac{\partial}{\partial\omega}[\![f\left(\overset{2}{r}, \overset{2}{\omega}, i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right)]\!], \tag{3.67}$$

$$lf\left(\overset{2}{r}, \overset{2}{\omega}, i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right) = i\varphi(r)\frac{\partial}{\partial r}[\![\overset{2}{F}\left(\overset{2}{r}, \overset{2}{\omega} i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial\omega}}\right)]\!], \tag{3.68}$$

and $l_{-i\partial/\partial\omega}$ and l can be treated as operators acting in $S^{0,\infty}(\widetilde{K} \times \mathbf{R}^n) \sim \mathcal{B}S^\infty(\mathbf{R}^n)$, accordingly.

Let us construct l and $l_{-i\partial/\partial\omega}$. We have the commutation relations

$$\left[-i\frac{\partial}{\partial\omega}, \omega\right] = -i \tag{3.69}$$

$$\left[i\varphi(r)\frac{\partial}{\partial r}, r\right] = i\varphi(r). \tag{3.70}$$

All other commutators, not written out explicitly in (3.69) and (3.70), are zero. We apply the technique described in detail in Chapter 2 to compute the left ordered representation operators. First, we have

$$\overline{i\,\varphi(r)\frac{\partial}{\partial r}}^{3}\, f\left(\overset{2}{r},\overset{2}{\omega}, \overset{1}{i\varphi(r)\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial\omega}}\right)$$

$$= \overline{i\varphi(r)\frac{\partial}{\partial r}}^{3}\, f\left(\overset{4}{r},\overset{4}{\omega}, \overset{1}{i\varphi(r)\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial\omega}}\right)$$

$$+(\overset{2}{r} -(\overset{4}{r}\overline{i\varphi(r)\frac{\partial}{\partial r}}^{3}\, \frac{\delta f}{\delta r}\left(\overset{1}{r},\overset{5}{r};\overset{5}{\omega}, \overset{1}{i\varphi(r)\frac{\partial}{\partial r}}, \overset{1}{i\frac{\partial}{\partial\omega}}\right). \quad (3.71)$$

Transforming the isolated factor

$$(\overset{2}{r} - \overset{4}{r})\ \overset{3}{i\varphi(r)\frac{\partial}{\partial r}},$$

we obtain

$$(\overset{2}{r} - \overset{4}{r})\ \overset{3}{i\varphi(r)\frac{\partial}{\partial r}} = i\ \overset{3}{\varphi(r)}, \quad (3.72)$$

and finally

$$l = p + i\varphi(r)\frac{\partial}{\partial r}. \quad (3.73)$$

The computation of $l_{-i\partial/\partial\omega}$ is standard (in other notation, it has already been done in Chapter 2), and we have

$$l_{-i\partial/\partial\omega} = q - i\frac{\partial}{\partial\omega}. \quad (3.74)$$

Now we have to verify whether the operators (3.73) and (3.74) are S^∞-generators in the symbol class $S^{0,\infty}(\widetilde{K}\times\mathbf{R}^n)$. To this end, we compute the corresponding one-parameter groups. The operators (3.73) and (3.74) are first-order differential operators, and their one-parameter groups can readily be computed by solving a first-order linear partial differential equation. For $l_{-i\partial/\partial\omega}$, this equation reads

$$-i\frac{\partial u}{\partial t} = l_{-i\partial/\partial\omega}u \equiv qu - i\frac{\partial u}{\partial\omega}, \tag{3.75}$$

and so

$$e^{il_{-i\partial/\partial\omega}t}u(\omega, r, p, q) = e^{iqt}u(\omega + t, r, p, q). \tag{3.76}$$

The semi-norm of the right-hand side of (3.76) determined by the best possible constants $C_{\alpha\beta\gamma\delta}$ in (3.55) grows as $|t| \to \infty$ no faster than $|t|^{|\delta|}$, and we see that $e^{il_{-i\partial/\partial\omega}t}$is a group of tempered growth.

Now let us consider the group generated by l. It has the form

$$e^{ilt}u(\omega, r, p, q) = e^{ipt}u(\omega, \Phi(t, r), p, q), \tag{3.77}$$

where $\Phi(t, r)$ is the solution of the differential equation

$$\dot{\Phi} = \varphi(\Phi), \quad \Phi|_{t=0} = r. \tag{3.78}$$

For sufficiently small r, the solution of (3.78) has the form

$$\Phi(t, r) = e^t r \quad \text{for} \quad t \leq \ln\frac{1}{2r}. \tag{3.79}$$

This readily follows from formula (3.40) for $\varphi(r)$. Now if

$$\frac{\partial u}{\partial r}(\omega, 0, p, q) = c \neq 0,$$

then

$$\frac{\partial}{\partial r}[e^{ilt}u(\omega, r, p, q)]|_{r=0} = ce^t \tag{3.80}$$

grows exponentially as $t \to +\infty$. Thus the group e^{ilt} is of exponential growth, and the operator l is not an S^∞-generator. It follows that the product of two operators $\widehat{f}_1$ and $\widehat{f}_2$ with symbols $f_1, f_2 \in S^{0,\infty}(\widetilde{K} \times \mathbf{R}^n)$ is in general not representable in the form

$$\widehat{f}_1\widehat{f}_2 = f\left(\overset{2}{\omega}, \overset{2}{r}, \overset{1}{i\varphi(r)\frac{\partial}{\partial r}}, \overset{1}{i\frac{\partial}{\partial\omega}}\right) \tag{3.81}$$

with any symbol $f \in S^{0,\infty}(\widetilde{K}, \mathbf{R}^n)$. Thus we have proved the following theorem.

Theorem 10 *Operators*

$$f\left(\overset{2}{\omega}, \overset{2}{r}, + i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial \omega}}\right)$$

with symbols $f \in S^{0,\infty}(\widetilde{K} \times \mathbf{R}^n)$ *do not form an algebra.*

As the *reductio ad absurdum* argument carried out in (3.36)–(3.39) shows, the failure is caused by the fact that, with these operators, not only the behavior of symbols on the real line, but also their behavior in the complex plane is important (Eq. (3.39) can be stated in the more general form

$$\chi\left(ir\frac{\partial}{\partial r}\right) r = r\chi\left(ir\frac{\partial}{\partial r} + i\right), \tag{3.82}$$

and the argument p of the symbol is actually shifted by i. That is why in most of the papers about operators on manifolds with conical singularities, the requirement that the symbol must be analytic in the variable p is used. This, however, is a much more complicated theory, and we shall not touch it in the present chapter.

In the next section, we consider the theory for cuspidal singularities of order $k \geq 1$. Surprisingly, we shall see that the conical case is degenerate: for the case of cusps, operators with symbols in $S^{0,\infty}(\widetilde{K} \times \mathbf{R}^n)$ already form an algebra.

3.3 Operators on the Model Cusp of Order k

In this section we shall consider operators of the form

$$\widehat{f} = f\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial \omega}}, ir^{k+1}\overset{1}{\frac{\partial}{\partial r}}\right), \tag{3.83}$$

where $k \geq 1$ is an integer, with symbols $f \in S^{0,\infty}(\widetilde{K} \times \mathbf{R}^n)$ in the scale of Hilbert spaces $H^s_k(\widetilde{K})$. To ascribe rigorous meaning to the expression

(3.83), we need some modification of the operator

$$ir^{k+1}\frac{\partial}{\partial r}$$

for large r (which is fairly admissible, since the actual cusp corresponds to the point $r = 0$). Indeed, without this modification the operator

$$ir^{k+1}\frac{\partial}{\partial r}$$

does not generate a one-parameter group at all (the trajectories of the vector field

$$r^{k+1}\frac{\partial}{\partial r}$$

escape to infinity in finite time), and so S^∞-functions (other than polynomials) are in general undefined for this operator. We consider the function $\varphi_k(r)$ with the following properties:

(1) $\varphi_k(r)$ is everywhere positive for $r > 0$, and $\varphi_k(0) = 0$,

(2)

$$\varphi_k(r) = \begin{cases} r^{k+1}, & r \leq 1; \\ 2, & r \geq 1. \end{cases} \tag{3.84}$$

This is just a straightforward generalization of the construction in the preceding section.

Accordingly, we need to modify the spaces $H^s_k(\widetilde{K})$ themselves. We introduce a new norm $||\cdot||_{s,k}$ by setting

$$||u||^2_{s,k} = \int\limits_0^\infty \int\limits_\Omega \left| \left(1 - \left(\varphi_k(r)\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega \right)^{s/2} u \right|^2 \frac{dr}{\varphi_k(r)} d\omega. \tag{3.85}$$

This is well-defined, since the operator

$$1 - \left(\varphi_k(r)\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega$$

is positive and essentially self-adjoint in

$$L^2\left(\widetilde{K}, \frac{dr\,d\omega}{\varphi_k(r)}\right),$$

to the effect that all real powers of this operator are well defined. For functions supported in a neighborhood of $r = 0$ and for even positive s, this norm coincides with the one given by (3.55) (with $\gamma = 0$), and for any real s these two norms are equivalent by interpolation theory. The detailed argument is the same as for the conical case (cf. the preceding section), and we omit it altogether.

The operators

$$i\varphi_k(r)\frac{\partial}{\partial r}$$

and

$$-i\frac{\partial}{\partial \omega}$$

are self-adjoint in each space $H_k^s(\widetilde{K})$ of the newly defined scale and are continuous from $H_k^s(\widetilde{K})$ to $H_k^{s-1}(\widetilde{K})$. It follows that these operators are S^∞-generators in the algebra $\mathcal{A}_k$ of continuous operators in the scale $\{H_k^s(\widetilde{K})\}_{s\in\mathbf{R}}$. Thus, we arrive at the following theorem.

Theorem 11 *For any symbol $f \in S^{0,\infty}(\widetilde{K}\times\mathbf{R}^n)$, the operator*

$$\widehat{f} = f\left(\overset{2}{\omega}, \overset{2}{r}, \overset{1}{i\varphi_k(r)\frac{\partial}{\partial r}}, \overset{1}{-i\frac{\partial}{\partial \omega}}\right) \tag{3.86}$$

is a well-defined element of the algebra $\mathcal{A}_k$ of continuous operators in the scale $\{H_k^s(\widetilde{K})\}_{s\in\mathbf{R}}$.

Later on in this section, we shall prove an exact theorem on the order of operators with classical symbols.

Now let us find out whether operators of the form (3.86) form a subalgebra of $\mathcal{A}_k$. We proceed along the lines of the preceding section. The first thing we must do is to compute the left ordered representation operators for the operators $\overset{2}{\omega}$, $\overset{2}{r}$,

$$\overset{1}{i\varphi_k(r)\frac{\partial}{\partial r}},$$

and

$$-i\frac{\partial}{\partial\omega}.$$

The computations are actually the same as in the preceding section, with φ replaced by φ_k, and so we can readily write out the result:

$$l_{-i\partial/\partial\omega} = q - i\frac{\partial}{\partial\omega}, \quad \text{(the representation of } -i\frac{\partial}{\partial\omega}\text{)}, \tag{3.87}$$

$$l = p + i\varphi_k(r)\frac{\partial}{\partial r} \quad \text{(the representation of } i\varphi_k(r)\frac{\partial}{\partial r}\text{)} \tag{3.88}$$

In our extended setting of operator-valued symbols, we do not write out the representation operators corresponding to ω and r; their role is taken over by the (obvious) composition rule in the ring $\mathcal{B}$ of functions of (r,ω) bounded together with all of their derivatives.

The next step is to find out whether the operators l and $l_{-i\partial/\partial\omega}$ are S^∞-generators. The operator $l_{-i\partial/\partial\omega}$ is the same as in the conical case, so that the answer for this operator is already at hand: $l_{-i\partial/\partial\omega}$ is an S^∞-generator in $\mathcal{B}S^\infty(\mathbf{R}^n)$. Now let us proceed to the operator l. By analogy with (3.77)–(3.78), the group generated by this operator has the form

$$e^{ilt}u(\omega,r,p,q) = e^{ipt}u(\omega,\Phi_k(t,r),p,q), \tag{3.89}$$

where $\Phi_k(t,r)$ is the solution of the Cauchy problem

$$\dot\Phi_k = \varphi_k(\Phi_k), \quad \Phi_k|_{t=0} = r. \tag{3.90}$$

We claim that all derivatives of Φ_k (with respect to r and t) have at most polynomial growth in $|t|$ as $|t|\to\infty$. Indeed, let us study the behavior of Φ_k. Whenever both r and $\Phi_k(t,r)$ lie in the interval $[0,1]$, $\Phi_k(t,r)$ is the solution of the equation

$$\dot\Phi_k = (\dot\Phi_k)^{k+1}, \quad \Phi_k|_{t=0} = r. \tag{3.91}$$

This equation can readily be solved by separation of variables:

$$\Phi_k = \{(C-t)\cdot k\}^{-1/k} \tag{3.92}$$

where C is the constant of integration. It can be found from the initial conditions, and we obtain

$$\Phi_k(r,t) = \frac{r}{(1-ktr^k)^{1/k}} \tag{3.93}$$

provided that $r,\ \Phi_k(r,t) \le 1$. Let us estimate the r-derivatives of $\Phi_k(r,t)$. We do so for $k=1$, dropping the clumsy computations needed for $k>1$, the result being essentially the same. Thus, we have

$$\Phi_1(r,t) = \frac{r}{1-tr} \tag{3.94}$$

in the domain determined by the inequalities

$$0 \le r \le 1, \quad 0 \le \frac{r}{1-tr} \le 1. \tag{3.95}$$

We note that the second inequality in (3.95) can be rewritten in the form $r(1+t) \le 1$, that is, r may be arbitrary ≥ 0 if $t \le -1$ and

$$0 \le r \le \frac{1}{1+t} \quad \text{for} \quad t > -1. \tag{3.96}$$

The derivatives of the function (3.94) have the form

$$\frac{\partial^j \Phi_1(r,t)}{\partial r^j} = j!t^j(1-tr)^{-(j+1)}, \qquad j = 1, 2, \dots. \tag{3.97}$$

Let us give t some fixed value and find the maximum over r of the absolute value of the expression (3.97) in the domain determined by inequalities (3.95). For negative t, the maximum is attained at $r=0$ and is equal to $j!|t|^j$. For positive t, the maximum is attained at

$$r = \frac{1}{1+t}$$

and is equal to

$$j!t!\left(1 - \frac{t}{1+t}\right)^{-(j+1)} = j!t^j(1+t)^{j+1}. \tag{3.98}$$

Thus, in both cases we have polynomial growth at infinity. Now consider the case in which both r and $\Phi_k(r,t)$ are greater that 1. Then

$\Phi_k(r,t) = r + \psi_k(r,t)$, where the function $\psi_k(r,t)$ is the constant gained in the solution of the differential equation (3.90) on the interval $[1,2]$, where $\varphi_k(r) \neq r$, and obviously has all derivatives uniformly bounded. Finally, if, say $r > 1$ and $\Phi_k(r,t) > 1$ or vice versa, then the mapping $r \mapsto \Phi_k(r,t)$ is obtained as the composition of two mappings considered above (one takes $r \mapsto 1$ in some time, and the other $1 \mapsto \Phi_k(r,t)$ in the remaining time), and so again the polynomial estimate of growth as $t \to \infty$ is guaranteed. It follows that all derivatives of the function (3.89) admit estimates with polynomial growth as $|t| \to \infty$. All in all, we have proved the following theorem.

Theorem 12 *The operators $l_{-i\partial/\partial\omega}$ and l of the left ordered representation of*

$$\left(-i\frac{\partial}{\partial\omega}, i\varphi_k(r)\frac{\partial}{\partial r}\right),$$

given by (3.87), (3.88), *are S^∞-generators in the symbol space $S^{0,\infty}(\widetilde{K}\times \mathbf{R}^n) \cong \mathcal{B}S^\infty(\mathbf{R}^n)$.*

Using the standard composition theorem for tuples possessing left ordered representation, we obtain the following composition theorem.

Theorem 13 *The set $PS\mathcal{D}_k(\widetilde{K})$ of operators $\widehat{f}$ of the form*

$$\widehat{f} = f\left(\overset{2}{\omega}, \overset{2}{r}, i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}, -\overset{1}{i\frac{\partial}{\partial\omega}}\right) \tag{3.99}$$

with symbols $f \in S^{0,\infty}(\widetilde{K}\times\mathbf{R}^n)$ is a subalgebra of the algebra $\mathcal{A}_k$ of continuous operators in the Hilbert scale $\{H^s_k(\widetilde{K})\}_{s\in\mathbf{R}}$. The composition formula for two elements $\widehat{f}, \widehat{g} \in PS\mathcal{D}_k(\widetilde{K})$ of this subalgebra is given by

$$\widehat{f}\widehat{g} = \widehat{h},$$

where

$$h(\omega,r,p,q) = f\left(\overset{2}{\omega}, \overset{2}{r}, p + i\overset{1}{\varphi_k(r)\frac{\partial}{\partial r}}, q - \overset{1}{i\frac{\partial}{\partial\omega}}\right)(g) \overset{\text{def}}{=} f * g. \tag{3.100}$$

It remains to verify that the "twisted product" $f*g$ defined in (13) makes the symbol space $S^{0,\infty}(\widetilde{K}\times\mathbf{R})$ an algebra and that the mapping $f\mapsto\widehat{f}$ is a homomorphism of algebras. According to the theorem concerning the Jacobi condition in Chapter 2, it suffices to verify that the left ordered representation operators satisfy the Jacobi condition, that is,

$$[l_{-i\partial/\partial\omega},\omega]=-i,\quad [l,r]=+i\varphi(r),\tag{3.101}$$

and all the other commutators are zero. But this is already obvious from the explicit formulas (3.87), (3.88) for the left ordered representation operators.

In conclusion of this section, let us state and prove the sharp boundedness theorem for cuspidal pseudodifferential operators with classical symbols.

Definition 14 By $\mathcal{L}^m\subset PS\mathcal{D}_k(\widetilde{K})$ we denote the set of classical mth-order pseudodifferential operators of the form (3.99), that is, operators whose symbols satisfy the estimates

$$|f^{(\alpha\beta\gamma\delta)}_{\omega rpq}(\omega,r,p,q)|\le C_{\alpha\beta\gamma\delta}(1+|p|+|q|)^{m-|\gamma|-|\delta|}\tag{3.102}$$

for all α, β, γ, δ.

Remark 2 In fact, truly classical pseudodifferential operators form a subset of $\mathcal{L}^m$ — their symbols admit expansions

$$f\cong f_m+f_{m-1}+f_{m-2}+\dots\tag{3.103}$$

into asymptotic sums of homogeneous functions in (p,q) of orders m, $m-1$, $m-2$, ... as $|p|+|q|\to\infty$. However, as long as only the boundedness theorem is concerned, sharp estimates in Sobolev spaces can be obtained for the larger class $\mathcal{L}^m$, and so we do not restrict ourselves to the homogeneous case. Considering homogeneous operators will be important in the next section, where we deal with regularizers and the finiteness theorem.

Theorem 15 *Each operator $\widehat{f}\in\mathcal{L}^m$ is bounded in the spaces*

$$\widehat{f}:\ H^s_k(\widetilde{K})\to H^{s-m}_k(\widetilde{K})\tag{3.104}$$

for every $s\in\mathbf{R}$.

Proof. The proof reproduces the construction already used in Section 3.2. We consider the change of variables $r = r(t)$ specified by the solution of the Cauchy problem

$$\dot{r} = \varphi_k(r), \quad r|_{\tau=0} = 1. \tag{3.105}$$

Under this change of variables, $\widetilde{K}$ is taken to the infinite cylinder $\Omega \times (-\infty, \infty)$, the Sobolev spaces $H^s_k(\widetilde{K})$ are taken to the usual Sobolev spaces on the cylinder with the norms

$$||u||_s = \int_{-\infty}^{\infty} \int_{\Omega} \left| \left(1 - \frac{\partial^2}{\partial \tau^2} - \Delta_\Omega\right)^{s/2} u \right|^2 d\tau \, d\omega,$$

and the pseudodifferential operators $\widehat{f} \in \mathcal{L}^m$ pass into pseudodifferential operators on the cylinder with uniformly bounded coefficients; the boundedness of such operators is a trivial matter (see Theorems II.14 and IV.6 in [8]). The proof is complete.

In the following section, we apply the results obtained here to the construction of regularizers and the proof of finiteness theorems for classical elliptic pseudodifferential operators on manifolds with cuspidal singularities.

3.4 An Application to the Construction of Regularizers and Proof of the Finiteness Theorem

In this section, we use the left ordered representation operators of the cuspidal algebra, constructed in the preceding section, to write out regularizers and prove the Fredholm property for elliptic elements in the cuspidal algebra. First, we intend to globalize our definitions of cuspidal pseudodifferential operators. Being trivial, this operation was omitted in the preceding section, but here it is needed to state the theorem rigorously. Thus, let M be a compact manifold with cuspidal singular points $\alpha_1, \ldots, \alpha_N$ of order $k \geq 1$. (The case in which different cuspidal points have different orders can be treated with no additional difficulties, but the notation for this case would be extremely boring, and so we assume that the orders of all points are the same.)

Definition 16 A continuous operator

$$A: H_k^s(M) \to H_k^{s-m}(M)$$

of order m in the scale $\{H_k^s(M)\}_{s\in\mathbf{R}}$ of cuspidal Sobolev spaces is called a *pseudodifferential operator of order m on M* if the following conditions are satisfied.

(i) There exist smooth functions φ_j, ψ_j on M, $j = 1, \dots, N$, such that φ_j and ψ_j are supported in the coordinate neighborhood U_j of the conical point α_j, $\varphi_j \equiv 1$ near α_j, $\psi_j\varphi_j = \varphi_j$, and the operators

$$\varphi_j A(1-\psi_j), \quad (1-\psi_j)A\varphi_j \tag{3.106}$$

are compact operators of arbitrarily large negative order in the scale $\{H_k^s(M)\}_{s\in\mathbf{R}}$.

(ii) The operators

$$\widehat{A}_j = \psi_j A \varphi_j \tag{3.107}$$

belong to the algebra $PS\mathcal{D}_k(\widetilde{K}_j)$ of cusp pseudodifferential operators on the model cusp $\widetilde{K}_j = \Omega_j \times [0, +\infty)/\Omega_j \times \{0\}$ of order k; moreover,

$$\operatorname{ord} A_j = m.$$

(iii) For any two functions $\varphi, \psi \in C_0^\infty(\overset{\circ}{M})$, the operator $\varphi A \psi$ is a usual pseudodifferential operator of order m on the open manifold $\overset{\circ}{M}$ (acting in the local Sobolev spaces $H^s(\overset{\circ}{M})$).

We shall consider the narrower class of *classical* pseudodifferential operators $L^m(M)$. This class is defined as follows. Each operator $A \in L^m(M)$ becomes a classical pseudodifferential operator of order m on $\overset{\circ}{M}$ after the multiplication on the right and on the left by cutoff functions

$\varphi, \psi \in C_0^\infty(\overset{\circ}{M})$. As to the "cuspidal parts" $\widehat{A}_j$ (3.107) of the operator A, they must have the form

$$\widehat{A}_j = H_j\left(\overset{2}{\omega}, \overset{2}{r}, \overset{1}{-i\frac{\partial}{\partial\omega}}, \overset{1}{i\varphi_k(r)\frac{\partial}{\partial r}}\right), \tag{3.108}$$

where the symbols $H_j(\omega, r, q, p)$ admit asymptotic expansions in homogeneous functions for large $|p| + |q|$:

$$H_j(\omega, r, p, q) \sim \sum_{l=m}^{-\infty} H_j^{(l)}(\omega, r, p, q) \quad |p| + |q| \to \infty; \tag{3.109}$$

here each $H_j^{(l)}$ is homogeneous of order l:

$$H_j^{(l)}(\omega, r, \lambda p, \lambda q) = \lambda^l H_j(\omega, r, p, q), \tag{3.110}$$

$|p| + |q| \neq 0$, $\lambda \in \mathbf{R}_+$.

For operators of this form, there is a well-defined notion of principal symbol (see [9]). The principal symbol $\sigma(A)$ is a homogeneous function of order m (with respect to the natural action of the group $\mathbf{R}_+$ of positive integers) on the cotangent bundle $T^* \overset{\circ}{M}$ minus the zero section. Moreover, it satisfies the natural compatibility condition

$$\sigma(A)(\omega, r, q, p) = H_j^{(m)}(\omega, r, q, \varphi_k(r)^{-1}p) \tag{3.111}$$

in the coordinate neighborhoods U_j of the cusps α_j, $j = 1, \ldots, N$.

Finally, we introduce the notion of the *conormal symbol* of a pseudodifferential operator A. The conormal symbol at a singular point $\alpha_j \in M$ is defined as follows. Let

$$\widehat{A}_j = H_j\left(\overset{2}{\omega}, \overset{2}{r}, \overset{1}{-i\frac{\partial}{\partial\omega}}, \overset{1}{i\varphi_k(r)\frac{\partial}{\partial r}}\right) \tag{3.112}$$

be the pseudodifferential operator (3.108) on the model cusp corresponding to A for some choice of the functions φ_j and ψ_j satisfying the above conditions. Then the *conormal symbol* of A at the point α_j is the operator family

$$\sigma_{cj}(A)(p) = H_j\left(\overset{2}{\omega}, 0, -i\overset{1}{\frac{\partial}{\partial\omega}}, p\right) \tag{3.113}$$

in the Sobolev spaces $H^s(\Omega_j)$ on the base Ω_j of the cusp at the point α_j, depending on the parameter $p \in \mathbf{R}$. We point out that the definition (3.113) of the conormal symbol is independent of the particular choice of the functions φ_j and ψ_j. The proof of the independence is beyond the scope of our book, and we refer the reader, for example, to [9]. (We note that the considerations in [9] formally cover only the case in which the conormal symbol is analytic in the conormal variable p. However, the proof remains valid for nonanalytic symbols as well.)

Definition 17 A classical pseudodifferential operator A of order m on a manifold M with cuspidal singularities is said to be *formally elliptic* if its principal symbol does not vanish on $T^* \overset{\circ}{M} \setminus \{0\}$ and if $H_j^{(m)}(\omega, r, q, p)$ does not vanish for $|p| + |q| \neq 0$ for each cusp point α_j.

If A if formally elliptic, then the conormal symbols $\sigma_{cj}(A)(p)$ are families of pseudodifferential operators *elliptic with parameter* p on Ω in the sense of Agranovich–Vishik [1]. It follows that $\sigma_{cj}(A)(p)$ is invertible for $|p|$ sufficiently large.

Definition 18 A formally elliptic pseudodifferential operator A of order m on M is said to be *elliptic* if all conormal symbols $\sigma_{cj}(A)(p)$ are invertible for all $p \in \mathbf{R}$.

The validity of the conditions of Definition 18 for a formally elliptic operator A can usually be ensured as follows. In practically interesting situations, the conormal symbols are *analytic* in the variable p (for example, if A is a differential operator, then the $\sigma_{cj}(A)(p)$ are polynomials). In this case, the points p where the conormal symbol is not invertible form a discrete set in the domain of analyticity, with no such points at all in the domain

$$|\mathrm{Im}\, p| < \varepsilon, \quad |\mathrm{Re}\, p| > R \tag{3.114}$$

for sufficiently large R and sufficiently small ε. It follows that by slightly shifting the weight line $\{\operatorname{Im} p = 0\}$ into the complex plane, that is, by proceeding to the weight line $\{\operatorname{Im} p = \gamma\}$ with arbitrarily small γ, we ensure the invertibility of the conormal symbol on the entire weight line. As was already mentioned in the preceding sections, the theory in Sobolev spaces with weight exponent $\gamma \neq 0$ is unitarily equivalent to the theory with weight exponent $\gamma = 0$. Thus in the following we always assume that $\gamma = 0$ and hence all symbols on the model cusps are defined on the real line, $\{\operatorname{Im} p = 0\}$.

Let us now recall the notion of a Fredholm operator. Let

$$A : H_1 \to H_2 \tag{3.115}$$

be an bounded operator between Hilbert (or Banach, or topological vector) spaces H_1 and H_2.

Definition 19 The operator (3.115) is said to be *Fredholm* if the following conditions are satisfied.

(i) The null space Ker A is finite-dimensional, dim Ker $A < \infty$.

(ii) The range $R(A)$ has a finite codimension in H_2: $\dim H_2/R(A) < \infty$.

Remark 3 It follows from condition (ii) in this definition that $R(A)$ is closed in H_2. The quotient space $H_2/R(A)$ is denoted by Coker A and is called the *cokernel* of A. The difference

$$\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A \tag{3.116}$$

is called the *index* of a Fredholm operator A.

Remark 4 Elliptic operators on closed smooth manifolds prove to be Fredholm in Sobolev spaces. Index theory of elliptic operators is an important chapter of topology. In this section we illustrate just the first step in elliptic theory on manifolds with cuspidal points (cf. [9]). Namely, we prove the following theorem.

Theorem 20 *Let A be an elliptic classical pseudodifferential operator of order m on a manifold M with cuspidal singularities. Then A is Fredholm in the Sobolev spaces*

$$A: \, H^s(M) \to H^{s-m}(M) \tag{3.117}$$

for any $s \in \mathbf{R}$.

Proof. We shall carry out the proof by constructing right and left almost inverses of A, that is, operators R and R' such that the operators

$$\begin{aligned} Q &= AR - I, \\ Q' &= R'A - I, \end{aligned} \tag{3.118}$$

where I is the identity operator, are compact in the Sobolev spaces

$$\begin{aligned} Q &: \, H^{s-m}(M) \to H^{s-m}(M), \\ Q' &: \, H^s(M) \to H^s(M) \end{aligned} \tag{3.119}$$

for any $s \in \mathbf{R}$. The Fredholm property follows from the existence of left and right almost inverses (also widely known as regularizers) in a standard way. This is the subject of Fredholm theory.

Thus, let us proceed to the construction of regularizers. To simplify the exposition, we shall consider only the practically most important case in which the cuspidal operators A_j are actually *differential* rather than pseudodifferential operators. This saves us a lot of boresome, purely technical pages which otherwise would be filled with estimates of various remainders. First, we note that if the conormal symbol $\sigma_{cj}(A)(p)$ is invertible for all $p \in \mathbf{R}$, then the inverse

$$\widehat{D}_j(p) = [\sigma_{cj}(A)(p)]^{-1} \tag{3.120}$$

is a pseudodifferential operator with parameter p in the sense of Agranovich–Vishik on Ω, and

$$\operatorname{ord} \widehat{D}_j = -m. \tag{3.121}$$

This follows from the Agranovich–Vishik theory [1].

We start by constructing the right regularizer. As is customary in elliptic theory, regularizers can be constructed locally and then patched together with the help of partitions of unity. The construction of the right (and left) regularizer in the interior part of the manifold is fairly standard (e.g., see [7]) and will not be considered here. We only explain how to construct the local part of the regularizers near the cuspidal points. Thus, the problem is as follows.

Find an operator

$$\widehat{R} = R\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi_k(r)\overset{1}{\frac{\partial}{\partial r}}\right) \tag{3.122}$$

such that

$$\widehat{A}_j\widehat{R} \equiv [\![H_j\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi_k(r)\overset{1}{\frac{\partial}{\partial r}}\right)]\!][\![R\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi_k(r)\overset{1}{\frac{\partial}{\partial r}}\right)]\!] = 1+Q \tag{3.123}$$

for small r,[1] *where Q is a compact operator in relevant Sobolev spaces.*

To accomplish the construction, we note that the operator $\widehat{A}_j$ can be represented in the form

$$\begin{aligned}\widehat{A}_j &= H_j\left(\overset{2}{\omega}, 0, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi_k(r)\overset{1}{\frac{\partial}{\partial r}}\right) + \overset{2}{r}\,\frac{\delta H_j}{\delta r}\left(\overset{2}{\omega}; 0, \overset{2}{r}; -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi_k(r)\overset{1}{\frac{\partial}{\partial r}}\right)\\ &\equiv \sigma_{cj}(A)\left(i\varphi_k(r)\frac{\partial}{\partial r}\right) + r\widehat{Z}_j,\end{aligned} \tag{3.124}$$

where we have denoted

$$\widehat{Z}_j = \frac{\delta H_j}{\delta r}\left(\overset{2}{\omega}; 0, \overset{2}{r}; -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi_k(r)\overset{1}{\frac{\partial}{\partial r}}\right) \tag{3.125}$$

for brevity. We seek the solution $\widehat{R}$ of Eq. (3.122) in the form

$$\widehat{R} = \widehat{D}_j\left(i\varphi(r)\frac{\partial}{\partial r}\right) + R_1\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}\right). \tag{3.126}$$

[1] That is, the symbol of the first term on the right-hand side is actually equal to 1 for sufficiently small r.

Since

$$\sigma_{cj}(A)(p)\widehat{D}_j(p) \equiv 1, \tag{3.127}$$

it follows that

$$\sigma_{cj}(A)\left(i\varphi(r)\frac{\partial}{\partial r}\right)\widehat{D}_j\left(i\varphi(r)\frac{\partial}{\partial r}\right) = 1. \tag{3.128}$$

With regard for this, for the operator

$$\widehat{R}_1 = R_1\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}\right) \tag{3.129}$$

we obtain the equation

$$\widehat{H}_j\,\widehat{R}_1 = -r\widehat{Z}_j\widehat{D}_j\left(i\varphi(r)\frac{\partial}{\partial r}\right) + \widehat{Q}, \tag{3.130}$$

where $\widehat{Q}$ is a compact operator.

The product $\widehat{Z}_j\widehat{D}_j$ is a cusp pseudodifferential operator of order zero. This follows from the composition formula involving the operators of left ordered representation of the tuple

$$\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}\right)$$

(Theorem 13). Let us denote the symbol of this operator by $X(\omega, r, q, p)$:

$$\widehat{Z}_j\,\widehat{D}_j\left(i\varphi(r)\frac{\partial}{\partial r}\right) = X\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}\right). \tag{3.131}$$

Now we are in a position to write out the equation for the symbol R_1. Using the left ordered representation, from (3.130) we obtain

$$H_j\left(\overset{2}{\omega}, \overset{2}{r}, q - i\overset{1}{\frac{\partial}{\partial\omega}}, p + i\varphi(r)\overset{1}{\frac{\partial}{\partial r}}\right) R_1(\omega, r, q, p) = rX(\omega, r, q, p)$$

$$+ \text{(the symbol of a compact operator).} \tag{3.132}$$

Equation (3.132) is to be solved for small r. Let us expand H_j in the Taylor series in powers of

$$-i\frac{\partial}{\partial\omega}$$

and

$$i\varphi(r)\frac{\partial}{\partial r}.$$

This Taylor series proves to be finite. (This is the place where we use the fact that $\widehat{H}_j$ is a differential operator; otherwise we would have to deal with complicated remainders.) We have

$$H_j\left(\overset{2}{\omega},\overset{2}{r},q-i\overset{1}{\frac{\partial}{\partial\omega}},p+i\varphi\overset{1}{(r)}\frac{\partial}{\partial r}\right)=H_j(\omega,r,q,p) \tag{3.133}$$

$$+\sum_{1\le|\alpha|+\beta\le m}\frac{1}{\alpha!\beta!}\frac{\partial^{|\alpha|+\beta}H_j}{\partial q^\alpha\partial p^\beta}(\omega,r,q,p)\left(-i\frac{\partial}{\partial\omega}\right)^\alpha\left(i\varphi(r)\frac{\partial}{\partial r}\right)^\beta.$$

Next, in H_j we isolate the principal part $H_j^{(m)}(\omega,r,q,p)$, homogeneous of order m, and write

$$H_j\left(\overset{2}{\omega},\overset{2}{r},q-i\,\frac{\overset{2}{\partial}}{\partial\omega},p+i\varphi\overset{1}{(r)}\frac{\partial}{\partial r}\right)=H_j^{(m)}(\omega,r,q,p)+P(\omega;r,q,p)$$

$$+\sum_{1\le|\alpha|+\beta<m}\frac{1}{\alpha!\beta!}\frac{\partial^{|\alpha|+\beta}H_j}{\partial q^\alpha\partial p^\beta}\left(-i\frac{\partial}{\partial\omega}\right)^\alpha\left(i\varphi(r)\frac{\partial}{\partial r}\right)^\beta, \tag{3.134}$$

where $P(\omega,r,q,p)$ is a polynomial of degree $\le m-1$ in (q,p). Now let us solve Eq. (3.132) by the method of successive approximations, obtaining a remainder decaying as $|q|+|p|\to\infty$ sufficiently rapidly. In the course of solution, we shall divide by $H_j^{(m)}(\omega,r,q,p)$. By virtue of the ellipticity condition,

$$H_j^{(m)}(\omega,r,q,p)\neq 0 \quad \text{for} \quad |q|+|p|=0, \tag{3.135}$$

but for $|q|=|p|=0$ the symbol necessarily vanishes. To avoid the singularity, we use a cutoff function $\chi(p,q)$ such that $\chi(p,q)\in C^\infty$, $\chi(p,q)=0$ for $p^2+q^2\le 1$, and $\chi(p,q)=1$ for $p^2+q^2\ge 2$.

Thus, we apply the method of successive approximations as follows. We take

$$R_1^{(0)} = H_j^{(m)}(\omega, r, q, p)^{-1} r X(\omega, r, q, p)\chi(q, p) \tag{3.136}$$

and substitute this into Eq. (3.132). Then for the disrepency $R_1 - R_1^{(0)}$ we obtain a certain equation, which we again solve approximately by dividing by $H_j^{(m)}$. Let the new approximate solution be denoted by $R_1^{(1)}$. Then for the difference $R_1 - R_1^{(1)}$ we obtain a new equation, and so on. Finally, the formula for $R_1^{(n)}$ reads

$$\begin{aligned} R_1^{(n)} &= \chi(q,p) H_j^{(m)}(\omega, r, q, p)^{-1} \left[r X(\omega, r, q, p) \right. \\ &\quad - P(\omega, r, q, p) R^{(n-1)}(\omega, r, q, p) \\ &\quad - \sum_{1 \le |\alpha| + \beta < m} \frac{\partial^{|\alpha|+\beta} H_j}{\partial q^\alpha \partial p^\beta}(\omega, r, p, q) \left(-i \frac{\partial}{\partial \omega} \right)^\alpha \left(i\varphi(r) \frac{\partial}{\partial r} \right)^\beta \\ &\quad \left. \times R^{(n-1)}(\omega, r, q, p) \right], \\ &\qquad\qquad n = 1, 2, 3, \ldots . \end{aligned} \tag{3.137}$$

Note that $X(\omega, r, q, p)$ is a classical symbol of order zero, that is, it satisfies the estimates

$$\left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|} X}{\partial \omega^\alpha \partial r^\beta \partial q^\gamma \partial p^\delta} \right| \le C_{\alpha\beta\gamma\delta}(1 + |q| + |p|)^{-|\gamma| - \delta}. \tag{3.138}$$

The following lemma holds for our successive approximations.

Lemma 21 *Let $R_1^{(j)}$, $j = 0, 1, 2, \ldots$, be the successive approximations given by formulas (3.136), (3.137). Then*

(i) $R_1^{(n)} - R_1^{(n-1)} \in rL^{-m-n}$*;*

(ii) $H_j\left(\overset{2}{\omega}, \overset{2}{r}, q - i \overset{1}{\frac{\partial}{\partial \omega}}, p + i\varphi(r) \overset{1}{\frac{\partial}{\partial r}} \right) R_1^{(n)} - rX \in rL^{-(n+1)}$, $n = 0, 1, 2, \ldots$.

Here L^n is the set of classical symbols of order n.

Proof. We proceed by induction over n. Let

$$Q = P(\omega, r, q, p) + \sum_{1\le|\alpha|+\beta\le m} \frac{\partial^{|\alpha|+\beta} H_j}{\partial q^\alpha \partial p^\beta}(\omega, r, p, q)\left(-i\frac{\partial}{\partial\omega}\right)^\alpha \left(i\varphi(r)\frac{\partial}{\partial r}\right)^\beta . \tag{3.139}$$

Then the algorithm of our successive approximation method can be rewritten as

$$H_j^{(m)} R_1^{(n)} + \chi\widehat{Q} R_1^{(n-1)} = r\chi X, \quad n = 0, 1, 2, \ldots, \tag{3.140}$$

$$R^{(-1)} = 0. \tag{3.141}$$

Now, by subtracting two successive equations (3.140), we obtain

$$R_1^{(n+1)} - R_1^{(n)} = [H_j^{(m)}]^{-1}\chi\widehat{Q}(R_1^{(n)} - R_1^{(n-1)}), \tag{3.142}$$

so that

$$\operatorname{ord}(R_1^{(n+1)} - R_1^{(n)}) \le \operatorname{ord}(R_1^{(n)} - R_1^{(n-1)}) - 1. \tag{3.143}$$

(The product $[H_j^{(m)}]^{-1}\chi\widehat{Q}$ has the order -1 in (p, q) as $|p| + |q| \to \infty$.) Next,

$$R_1^{(0)} - R_1^{(-1)} = R_1^{(0)} = [H_j^{(m)}]^{-1} r\chi X \tag{3.144}$$

is of order $-m$, so that we obtain, in view of (3.143)

$$R_1^{(n)} - R_1^{(n-1)} \in L^{-n-m} \quad \text{for all} \quad n. \tag{3.145}$$

Note that the first difference $R_1^{(1)} - R_1^{(0)}$ contains the factor r. Let us prove that all subsequent differences also contain this factor. In fact, this can be observed from (3.142). The product $\chi[H_j^{(m)}]^{-1}$ is a smooth function, and $\widehat{Q}$ is a differential operator in

$$\frac{\partial}{\partial\omega}$$

and

$$\frac{\partial}{\partial r}$$

of rather specific structure: together with each differentiation

$$\frac{\partial}{\partial r}$$

it contains the factor $\varphi(r)$, which is r^{k+1} near the origin, It follows that any differentiations in $\widehat{Q}$ applied to the difference $R_1^{(n)} - R_1^{(n-1)}$ do not kill the factor r (moreover, each differentiation

$$\frac{\partial}{\partial r}$$

with the factor $\varphi(r) \sim r^{1+k}$ attached increases the power of r by k). Thus, we have proved (i). To prove (ii), we note that

$$H_j\left(\overset{2}{\omega}, \overset{2}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{1}{\varphi(r)}\frac{\partial}{\partial r}\right) R_1^{(n)} = H_j^{(m)} R_1^{(n)} + \widehat{Q} R_1^{(n)} \tag{3.146}$$

$$= \left\{H_j^{(m)} R_1^{(n+1)} + \chi\widehat{Q} R_1^{(n)}\right\} + (1-\chi)\widehat{Q} R_1^{(n)} + H_j^{(m)}(R_1^{(n)} - R_1^{(n-1)}).$$

Using (3.140) and (3.142), we find that this expression is equal to

$$rX + (1-\chi)(\widehat{Q}\widehat{R}_1^{(n)} - rX) + \chi\widehat{Q}(R_1^{(n)} - R_1^{(n-1)}). \tag{3.147}$$

The first term in the expression (3.147) is just the right-hand side of the equation that we solve asymptotically; the second term contains the factor r (by virtue of the preceding argument) and vanishes for $p^2 + q^2 \geq 2$; the third term contains the factor r (again by virtue of the preceding argument) and decays at infinity as $(|p| + |q|)^{-(n+1)}$ by virtue of (i) and with regard for the order of $\widehat{Q}$. The proof of Lemma 21 is complete.

By combining the local regularizers constructed according to this lemma in neighborhoods of cuspidal points with the standard pseudodifferential regularizer in the interior part of the manifold (this is, as usual, accomplished with the help of partitions of unity) we obtain a regularizer of the operator A. To prove that this is indeed a regularizer, we need the following assertion.

Lemma 22 *The operator*

$$B = \overset{2}{r} F\left(\overset{2}{r}, \overset{2}{\omega}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}\right), \tag{3.148}$$

where $F \in L^{-\varepsilon}$ for some $\varepsilon > 0$ and $F = 0$ for $r > R_0$, is compact in the spaces

$$B: H_k^s(\widetilde{K}) \to H_k^s(\widetilde{K}) \tag{3.149}$$

on the model cusp $\widetilde{K}$ for any $s \in \mathbf{R}$.

Proof. This readily follows from the fact that the composition of mappings

$$H^s \xrightarrow{j} H^{s-\varepsilon} \xrightarrow{\lambda(r)} H^{s-\varepsilon}, \tag{3.150}$$

where j is the natural embedding and $\lambda(r)$ is a smooth compactly supported function such that

$$\lambda(r) = r \quad \text{for small} \quad r \tag{3.151}$$

is a compact operator.

Our next task is to construct the left regularizer. Again, we do this only locally in neighborhoods of singular points, the passage to the global regularizer being trivial. The construction of the left regularizer has the following peculiarity. If we seek it in the form

$$\widehat{R} = R\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{1}{\varphi_k(r)\frac{\partial}{\partial r}}\right), \tag{3.152}$$

then it will be hard to compute the symbol of the product $\widehat{R}\widehat{A}$ as the result of action of some operator on the function $R(\omega, r, q, p)$, since the right ordered representation of the tuple

$$\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{1}{\varphi(r)\frac{\partial}{\partial r}}\right)$$

is itself hard to compute. Hence we seek $\widehat{R}$ in the form

$$\widehat{R} = R\left(\overset{2}{\omega}, \overset{1}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)\frac{\partial}{\partial r}}\right), \tag{3.153}$$

with the *inverse order* of the operators r and

$$i\varphi(r)\frac{\partial}{\partial r}.$$

Then, by the general theorems on ordered representation operators (see Chapter 2), the product $\widehat{R}\widehat{A}$ can be represented in the form

$$[\![\widehat{R}\left(\overset{2}{\omega}, \overset{1}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)\frac{\partial}{\partial r}}\right)]\!][\![A\left(\overset{2}{\omega}, \overset{2}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{1}{\varphi_k(r)\frac{\partial}{\partial r}}\right)]\!]$$

$$= h\left(\overset{2}{\omega}, \overset{1}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)\frac{\partial}{\partial r}}\right), \tag{3.154}$$

where the symbol $h(\omega, r, q, p)$ can be computed according to the rule

$$h(\omega, r, q, p) = A\left(\overset{1}{l}_\omega, \overset{1}{l}_r, \overset{2}{l}_{-i\partial/\partial\omega}, \overset{2}{l}_{i\varphi_k(r)\partial/\partial r}\right)(R(\omega, r, q, p)), \tag{3.155}$$

where l_ω, l_r, $l_{-i\partial/\partial\omega}$, $l_{i\varphi_k(r)\partial/\partial r}$ are the operators of right ordered representation of the tuple

$$\left(\overset{2}{\omega}, \overset{1}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)\frac{\partial}{\partial r}}\right).$$

Lemma 23 *The operators of right ordered representation of the tuple*

$$\left(\overset{2}{\omega}, \overset{1}{r}, -i\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)\frac{\partial}{\partial r}}\right)$$

have the form

$$l_\omega = \omega + i\frac{\partial}{\partial q}, \tag{3.156}$$

$$l_{-i\frac{\partial}{\partial\omega}} = q, \tag{3.157}$$

$$l_r = r, \tag{3.158}$$

$$l_{i\varphi_k(r)\partial/\partial r} = p - i\varphi_k(r)\frac{\partial}{\partial r}. \tag{3.159}$$

Proof. Formulas (3.156) and (3.157) are known from Chapter 2. Formula (3.158) is obvious, since

$$[\![f\left(\overset{2}{\omega}, \overset{1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)]\!] r = f\left(\overset{1}{\omega}, \overset{1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\overset{1}{r}\,. \tag{3.160}$$

Finally, we have

$$\begin{aligned}
&[\![f\left(\overset{2}{\omega}, \overset{1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)]\!] \left(i\varphi_k(r)\frac{\partial}{\partial r}\right)\\
&= f\left(\overset{2}{\omega}, \overset{1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\left(i\overset{0}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\\
&= f\left(\overset{2}{\omega}, \overset{-1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, \frac{i\overset{2}{\varphi_k(r)}\partial}{\partial r}\right)\left(i\overset{0}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\\
&+\left[f\left(\overset{2}{\omega}, \overset{1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\right.\\
&\left.-f\left(\overset{2}{\omega}, \overset{-1}{r}, -i\,\overset{1}{\frac{\partial}{\partial\omega}}, i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\right]\left(i\overset{0}{\varphi_k(r)}\frac{\partial}{\partial r}\right).
\end{aligned} \tag{3.161}$$

Let us transform the second term on the right-hand side in (3.161) using difference derivatives:

$$\left[f\left(\overset{2}{\omega},\overset{1}{r},-i\,\overset{1}{\frac{\partial}{\partial\omega}},i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)-f\left(\overset{2}{\omega},\overset{-1}{r},-i\,\overset{1}{\frac{\partial}{\partial\omega}},i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right)\right]$$
$$\times\left(i\overset{0}{\varphi_k(r)}\frac{\partial}{\partial r}\right)=(\overset{1}{r}-\overset{-1}{r})\left(i\varphi_k(r)\,\overset{0}{\frac{\partial}{\partial r}}\right) \qquad (3.162)$$
$$\times\frac{\delta f}{\delta r}\left(\overset{2}{\omega},-\overset{1}{r},\overset{-1}{r},-i\,\overset{1}{\frac{\partial}{\partial\omega}},i\overset{2}{\varphi_k(r)}\frac{\partial}{\partial r}\right).$$

Now

$$(\overset{1}{r}-\overset{-1}{r})\,i\overset{0}{\varphi_k(r)}\frac{\partial}{\partial r}=-i\overset{0}{\varphi_k}(r), \qquad (3.163)$$

and we arrive at (3.159).

The proof of Lemma 23 is complete. Starting from this place, the construction of the left regularizer goes perfectly along the same lines as the construction of the right regularizer, and we omit the details altogether.

The Fredholm property of the operator A, claimed in Theorem 20, follows from the existence of right and left regularizers.

In this section, we have shown that elliptic operators on manifolds with cuspidal singularities possess the Fredholm property. The proof, which was carried out only for differential operators, can be generalized to pseudodifferential operators, but this requires some lengthy estimates.

Remark 5 Once we deal only with differential elliptic operators, the proof of Theorem 20 can readily be generalized to operators on manifolds with conical singularities. However, for the general case of pseudodifferential operators on manifolds with conical singularities, a completely different technique is required.

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