Pseudodifferential Operators on Manifolds with Corners

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Abstract

We describe an algebra of pseudodifferential operators on a manifold with corners.

Contents

Introduction

It is well-known that parametrices of elliptic partial differential equations on C^∞ manifolds can be expressed by pseudodifferential operators. This implies the elliptic regularity in terms of the standard Sobolev spaces that are natural domains of pseudodifferential operators.

The same problem is extremely interesting on manifolds or, more generally, on the Thom-Mather stratified spaces, with piecewise smooth geometry, e.g., with conical points, edges, corners of higher order or non-compact 'exits'. Analytically this corresponds to operators with 'degenerate' symbols.

In recent years this area of problems found growing interest in the literature while the applications in differential geometry, topology and natural sciences are often classical. The vast variety of special investigations suggests a general approach for sufficiently wide classes of such problems.

The present paper is devoted to the corresponding pseudodifferential calculus. It is based on the articles of the first author [Sch89, Sch92].

Corners of higher order are also of interest as they occur, for instance, in problems of 'quarter plane type'. Our theory will be organised in such a way that it can be iterated for those cases.

Recall that by a "manifold" with conical singularities is meant a Hausdorff topological space B with a discrete subset B_0 of 'conical points', such that $B \setminus B_0$ is a C^∞ manifold and every point $v \in B_0$ has a neighbourhood O in B homeomorphic to the topological cone over a C^∞ compact closed manifold X. Thus,

$$O \overset{\cong}{\hookrightarrow} \frac{[0,1) \times X}{\{0\} \times X}, \tag{0.0.1}$$

the manifold X being referred to as the *base* of the cone close to v. Moreover, we require these local homeomorphisms to restrict to diffeomorphisms of open sets

$$O \setminus \{v\} \overset{\cong}{\hookrightarrow} (0,1) \times X.$$

Any two homeomorphisms h_1 and h_2 are said to be *equivalent* if the composition $h_2 \circ h_1^{-1}$ extends to a diffeomorphism of $[0,1) \times X$. This gives B a C^∞ structure with singular points.

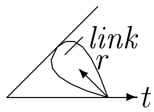

Fig. 1: A corner.

According to (0.0.1), local coordinates in a punctured neighbourhood of any conical point v in B split as (r, x), $r \in \mathbb{R}_+$ being the cone axis variable and $x \in X$.

The cone is a special case of a corner, the base of the latter being itself a manifold with conical points. Some elements of the cone theory are to be applied for the general corners again, cf. Schulze [Sch98].

A "manifold" with corners is a Hausdorff topological space C along with closed subspaces C_0 and C_1, such that C_0 is a discrete subset of C_1 consisting of the 'corners', $C_1 \setminus C_0$ is a C^∞ manifold of dimension 1 consisting of the edges which emanate from the corners, and $C \setminus C_1$ is a C^∞ manifold of dimension $n+2$. Every $v \in C_0$ has an open neighbourhood O in C homeomorphic to the topological cone over a C^∞ compact closed manifold with conical singularities B,

$$O \overset{\cong}{\hookrightarrow} \frac{[0,1) \times B}{\{0\} \times B}, \tag{0.0.2}$$

B being the *base* of the corner close to v (see Fig. 1). We require these homeomorphisms to restrict to diffeomorphisms of open sets

$$\begin{array}{rcl} O \cap (C_1 \setminus C_0) & \overset{\cong}{\hookrightarrow} & (0,1) \times B_0, \\ O \setminus C_1 & \overset{\cong}{\hookrightarrow} & (0,1) \times (B \setminus B_0). \end{array}$$

Once again we specify classes of *equivalent* homeomorphisms by requiring suitable compositions to preserve the differentiability up to $t = 0$. This gives C a C^∞ structure with corners near v.

By (0.0.2), local coordinates in a deleted neighbourhood of any corner $v \in C$ split as (t, p), $t \in \mathbb{R}_+$ being the corner axis variable and $p \in B$. In the theory of Thom-Mather stratified spaces B is known as the *link* of the stratum C through v, cf. [GM88].

A further assumption on C is that every point $p \in C_1 \setminus C_0$ has a neighbourhood O in C which is homeomorphic to a wedge

$$O \overset{\cong}{\hookrightarrow} \Omega \times \frac{[0,1) \times X}{\{0\} \times X}, \tag{0.0.3}$$

Ω being an open interval on the real axis and X a C^∞ compact closed manifold. We confine ourselves to those homeomorphisms (0.0.3) which restrict to

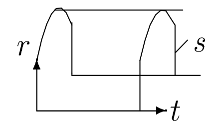

Fig. 2: A stretched corner.

diffeomorphisms

$$O \setminus C_1 \overset{\cong}{\hookrightarrow} \Omega \times (0,1) \times X,$$

and the various mappings are assumed to be compatible in an obvious way over the intersections of neighbourhoods. This leads to a C^∞ structure with edges near $C_1 \setminus C_0$ on C.

It is easy to introduce the categories of spaces with conical points, edges, corners and evident notions of morphisms, in particular, isomorphisms. For brevity we avoid here a lengthy discussion of axioms.

The space in Fig. 1 has one corner v, the base B of the corner has one conical point.

The analysis near v will take place in the coordinates (t, r, x) of the open stretched corner $\mathbb{R}_+ \times \mathbb{R}_+ \times X$ (cf. Fig. 2).

For a C^∞ manifold M, we denote by $\mathrm{Diff}^m(M)$ the space of all differential operators of order m with C^∞ coefficients on M. It is equipped with a natural Fréchet topology. In particular, we look at operators $A \in \mathrm{Diff}^m(C \setminus C_1)$ of the form

$$A = \frac{1}{t^m} \sum_{j=0}^{m} A_j(t)\, (tD_t)^j$$

in the coordinates from (0.0.2) on $O \setminus C_1$, where $A_j(t)$ is a C^∞ function on $[0,1)$ with values in $\mathrm{Diff}^{m-j}(B \setminus B_0)$. We require every $A_j(t)$ to be of Fuchs type on B, i.e., A takes the form

$$A = \frac{1}{(tr)^m} \sum_{j+k \le m} A_{jk}(t,r)\, (trD_t)^j\, (rD_r)^k \tag{0.0.4}$$

in the coordinates from (0.0.1) close to any point $v \in B_0$, with $A_{jk}(t,r)$ a C^∞ function on $[0,1) \times [0,1)$ with values in $\mathrm{Diff}^{m-(j+k)}(X)$. The factors t^{j-m} are no longer important away from the corners where t is non-zero. Hence, near any point on the smooth part of an edge in $C_1 \setminus C_0$ we can rewrite (0.0.4) in the form

$$A = \frac{1}{r^m} \sum_{j+k \le m} \tilde{A}_{jk}(t,r)\, (rD_t)^j\, (rD_r)^k$$

in the coordinates from (0.0.3), $\tilde{A}_{jk}(t,r)$ being a C^∞ function on $\Omega \times [0,1)$ with values in $\mathrm{Diff}^{m-(j+k)}(X)$.

Operators of the form (0.0.4) will be referred to as the *typical* differential operators on manifolds with corners. By the above, they bear both cone and edge degeneracy as well as more general corner degeneracy.

As but one example of corner-degenerate operators we show the Laplace-Beltrami operator with respect to the Riemannian metric on C that is of the form

$$\begin{aligned} dt^2 + t^2\, dp^2 &= dt^2 + t^2 \left(dr^2 + r^2\, dx^2\right) \\ &= (tr)^2 \left(\left(\frac{dt}{tr}\right)^2 + \left(\frac{dr}{r}\right)^2 + dx^2 \right) \end{aligned}$$

in the local coordinates from (0.0.2), the first equality being close to a conical point of the link B (cf. Fig. 1).

Now the program of the calculus is to introduce symbol structures, pseudodifferential operators and adequate Sobolev spaces, such that the elliptic operators are Fredholm and possess parametrices within the calculus. Here, the ellipticity of an operator A means the invertibility of certain symbols related to A, while the Fredholm property refers to the scale of Sobolev spaces above. Moreover, it is desirable to study subspaces with corner asymptotics, such that the elliptic regularity still remains valid therein.

The solution will necessary employ an analogous theory on manifolds with conical points and edges. The calculus for edges is known to recover the case of boundary value problems where the edge is the boundary and the model cone is the inner normal $\bar{\mathbb{R}}_+$, cf. Ch. 4 in [Sch98]. Here, for getting the Fredholm property additional edge conditions are posed, satisfying an analogue of the Lopatinskii condition. In general, they are of trace and potential type just as in Boutet de Monvel's theory, cf. [BdM71]. This should be combined with knowledge from the cone theory where an additional operator-valued Mellin symbolic level is required to be bijective along a weight line, cf. Kondrat'ev [Kon67]. Summarising, we deduce that there will actually arise several leading symbolic levels.

The goal of this paper is a calculus of pseudodifferential operators on manifolds with corners including ellipticity, Fredholm property and asymptotics of solutions. The style of exposition is dictated by the desire to formulate the theory on the whole and to confirm the expectation that in spite of complexity of tools there does actually exist an operator algebra containing the ideas of Vishik and Eskin's [VE65, VE67, VE66] and Boutet de Monvel's [BdM71] theories as well as higher order operator structures for conical and edge singularities.

Chapter 1 repeats some material on conical singularities and edges. The cone operators from Section 1.2 are model for the shape of corner operators of Chapter 3. The results of Section 1.4 are motivated by applications in the corner theory.

New ingredients such as parameter-dependent cone operators, corner asymptotics and Mellin symbols are given in Chapter 2.

Chapter 3 establishes a full corner calculus including the symbol structures and the concept of ellipticity.

In the literature there are several approaches to the analysis on manifolds with corners.

Boundary value problems in domains with conical points on the boundary were studied quite thoroughly by Kondrat'ev [Kon67]. Maz'ya and Plamenevskii [MP77] treat elliptic boundary value problems for differential equations on manifolds with singularities of a sufficiently general nature. As singularities they admit edges of different dimensions and their various intersections at non-zero angles. The same sets are regarded as carriers of discontinuities of the coefficients. However, the treatment falls short of providing a pseudodifferential algebra where the parametrices to elliptic elements are available. Melrose [Mel87, Mel96a] studies so-called b-pseudodifferential operators on manifolds with corners. While originating from geometry his theory does not apply, however, to many interesting elliptic operators, e.g., the Laplace operator in the corner $(\bar{\mathbb{R}}_+)^n$, $n \geq 3$. As far as we know the problem of representing parametrices of differential operators near corners in terms of symbols of operators in an algebra and of obtaining corner asymptotics by means of the parametrices was first treated by Schulze [Sch92]. The experience with simpler problems with singularities, e.g., mixed elliptic ones such as the Zaremba problem, shows that the analogous questions lead at once to corresponding algebras of rather generality. For proving an analogue of the Atiyah-Singer index theorem such an approach would be necessary, anyway. Plamenevskii and Senichkin [PS95] discuss the C^*-algebras generated by pseudodifferential operators of order zero with discontinuous symbols. The symbols may have discontinuities along some submanifolds of the unit sphere intersecting at non-zero angles. The purpose is to describe the spectrum of such algebras, i.e., the set of all equivalence classes of irreducible representations endowed with a natural topology (the so-called Jacobson topology). Thus, they confine ourselves to homogeneous symbols and L^2-spaces.

Note that C can in turn be regarded as the base of a "third order" corner, namely

$$\frac{[0,1) \times C}{\{0\} \times C},$$

etc. The theory for this singular configuration encompasses the problems of quarter plane type as well as boundary value problems in domains whose boundary bears "second order" corners. The axiomatic ideas contain formal procedures to obtain from a given operator algebra on some singular variety a new one in the cone over that base by means of a machinery called "conification". A suitable conified algebra on a corresponding infinite model cone then serves as a starting object for another procedure called "edgification". These concepts are elaborated in Schulze [Sch91, Sch98].

In carrying out the program for corners it turns out that even the functional analytic background of corner Sobolev spaces and subspaces with asymptotics, these being closely related to the character of smoothing operators, should first be prepared. Also other in a sense non-standard elements of the calculus, e.g., the parameter-dependent version of a given operator algebra and various operator conventions are to be established in advance. Our exposition is voluminous, for all these elements are needed here at the same time.

Finally, we touch a few aspects of the analysis on manifolds with edges which intersect each other at zero angles, cf. [RST99]. The underlying space looks locally like a wedge $\mathbb{R}^q \times B^\wedge$, where $B^\wedge = \mathbb{R}_+ \times B$ is the semicylinder over a manifold with singularities B. The Riemannian metric on $\mathbb{R}^q \times B^\wedge$ is of the form

$$dy^2 + dt^2 + \left(\frac{1}{\delta'(t)}\right)^2 dp^2 = \left(\frac{1}{\delta'(t)}\right)^2 \left(\left(\frac{d}{dt}e^{\delta(t)}\right)^2 (e^{-\delta(t)}dy)^2 + (d\delta(t))^2 + dp^2\right),$$

where dy^2 and dp^2 are Riemannian metrics on $\mathbb{R}^q$ and B, respectively, and $T = \delta(t)$ is a diffeomorphism of a neighbourhood of $t = 0$ to a neighbourhood of $T = -\infty$. In the case of transversal intersections we have $\delta(t) = \log(t)$. In contrast to this, $\delta(t)$ behaves like $-1/t^p$, $p > 0$, in the case of power-like cuspidal singularities. The "conification" on $B^\wedge$ and the "edgification" on $\mathbb{R}^q \times B^\wedge$ have to complete an operator algebra over B by functions of the vector fields

$$\frac{\partial}{\partial\delta(t)} = \frac{1}{\delta'(t)}\frac{\partial}{\partial t},$$
$$e^{\delta(t)}\frac{\partial}{\partial y_j}, \qquad j = 1, \ldots, q,$$

respectively. Under the coordinate $s = e^{\delta(t)}$ on the cone axis, the vector fields are written as $s\,\partial/\partial s$ and $s\,\partial/\partial y_j$, these latter occur in the case of transversal intersections. This is not surprising because any cuspidal singularity can topologically be transformed to a conical one. However, the change of variables $s = e^{\delta(t)}$ fails to preserve the C^∞ structure close to the edge $t = 0$. Hence it follows that "smooth coefficients" near $t = 0$ are pushed forward to "singular ones" near $s = 0$. The analysis on manifolds with cuspidal singularities thus reduces to that for the case of conical singularities, but with singular "coefficients".

What smoothness of symbols near the edge $t = 0$ is required, depends on the function spaces to be domains of operators in the algebra. In fact, the only natural requirement stems from the general observation that the domain of an operator should be a module over the space of its "coefficients". Since pseudodifferential operators on manifolds with singularities are intended to act in spaces with asymptotics, the coefficients themselves should bear appropriate asymptotic expansions. As usual, the asymptotic expansions correspond to Euler solutions to equations with coefficients constant in t, which are obtained from given equations by freesing the "coefficients" at $t = 0$. This behaviour of symbols near $t = 0$ is then inherited under any change of variables $s = e^{\delta(t)}$. Thus, our results apply as well to problems on manifolds with cuspidal corners, unless we leave the setting of Euler asymptotics. For more details we refer the reader to [ST99].

Chapter 1

Calculi for Lower Order Singularities

1.1 Motivation

This section prepares auxiliary material on cone pseudodifferential operators which is also a motivation for the methods for corners.

Recall that each manifold with conical singularities B gives rise through a blow up at every conical point to a C^∞ manifold with boundary $\mathcal{B}$, the boundary being the disjoint union of the links of B at conical points. More precisely, we have $\partial\mathcal{B} = X_1 \cup \ldots \cup X_N$, with X_ν a C^∞ compact closed manifold which is the cone base close to $v_\nu \in B_0$, for $\nu = 1, \ldots, N$. As is known, $\partial\mathcal{B}$ has a collar neighbourhood O in $\mathcal{B}$ along with a diffeomorphism

$$\kappa : \ O \to [0,1) \times \partial\mathcal{B}$$

which restricts to diffeomorphisms $\kappa_\nu : O_\nu \to [0,1) \times X_\nu$, for $\nu = 1, \ldots, N$, where O_ν is the corresponding collar neighbourhood of the component X_ν of $\partial\mathcal{B}$.

We may actually view B as the quotient space $(\ldots(\mathcal{B}/X_1)\ldots/X_N)$, thus specifying a blow-down mapping $b\colon \mathcal{B} \to B$. This mapping is C^∞ and restricts to a diffeomorphism of $\mathcal{B} \setminus \partial\mathcal{B}$ onto $B \setminus B_0$. As defined above, any C^∞ structure on $\mathcal{B}$ determines, via b, a unique C^∞ structure with conical points on B. We call $\mathcal{B}$ the *stretched* manifold associated with B. The analysis on B will be performed in the interior of $\mathcal{B}$.

Remark 1.1.1 *As is shown in Fig. 2, any manifold with corners C can consecutively be stretched to a C^∞ manifold with corners on the boundary $\mathcal{C}$, i.e., $b\colon \mathcal{C} \to C$.*

Without loss of generality we may assume that B_0 consists of a single point, i.e., we have only one cone base X, possibly with several connected components. It is convenient to develop the analysis also on the infinite semicylinder over X,

$$X^\wedge = \mathbb{R}_+ \times X.$$

Given a C^∞ manifold M, we write $H^s_{\mathrm{loc}}(M)$ for the space of all $u \in \mathcal{D}'(M)$ such that $h_* u \in H^s_{\mathrm{loc}}(\Omega)$, for every chart (O, h) on M, $\Omega \subset \mathbb{R}^N$ being the image of O by h. This is a Fréchet space in a canonical way. Moreover, $H^s_{\mathrm{comp}}(M)$ consists of all $u \in H^s_{\mathrm{loc}}(M)$ with compact support. We give it the topology of the inductive limit of Banach spaces. If M is compact, we write $H^s(M)$ instead of $H^s_{\mathrm{loc}}(M) = H^s_{\mathrm{comp}}(M)$.

If V is a locally convex space which is a module over an algebra A, we denote by $[a]V$ the completion of $\{av : v \in V\}$ in V, for any fixed $a \in A$. Further, if V_1 and V_2 are Fréchet spaces which are subspaces of a Hausdorff space V, then the non-direct sum $V_1 + V_2$ is also a Fréchet space in the topology induced by the isomorphism $V_1 + V_2 \cong V_1 \oplus V_2 / \Delta$ where $\Delta = \{(v, -v) : v \in V_1 \cap V_2\}$. Incidentally we shall employ analogous sums of more general topological vector spaces.

Let $\Omega \subset \mathbb{R}^N$ be open and $m \in \mathbb{R}$. Then $\mathcal{S}^m(\Omega \times \mathbb{R}^n)$ stands for the space of all C^∞ functions $a(x, \xi)$ on $\Omega \times \mathbb{R}^n$, such that $|D^\alpha_x D^\beta_\xi a(x, \xi)| \le c \, \langle \xi \rangle^{m - |\beta|}$ for all $(x, \xi) \in K \times \mathbb{R}^n$ and $\alpha \in \mathbb{Z}^N_+$, $\beta \in \mathbb{Z}^n_+$, with K any compact subset of Ω and c a constant depending on K, α, β. This is a Fréchet space in a natural way. By $\mathcal{S}^m_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ we denote the subspace of all classical elements. The convergence in $\mathcal{S}^m_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ amounts to the convergence in $C^\infty_{\mathrm{loc}}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ of all homogeneous components of the elements of a given sequence.

If Ω is an open set in $\mathbb{R}^n$, we write $\Psi^m(\Omega)$ for the space of all pseudodifferential operators on Ω, i.e., the space of all

$$A = \mathcal{F}^{-1}_{\xi \mapsto x} a(x, \xi) \mathcal{F}_{x \mapsto \xi} + S \tag{1.1.1}$$

with $a \in \mathcal{S}^m(\Omega \times \mathbb{R}^n)$ and S an operator with a C^∞ kernel on $\Omega \times \Omega$, $\mathcal{F}$ being the Fourier transform in $\mathbb{R}^n$.

Let $\Lambda = \mathbb{R}^d$ for some $d \in \mathbb{N}$. Then $\Psi^m(\Omega; \Lambda)$ denotes the space of all operator families $A(\lambda)$, $\lambda \in \Lambda$, of the form (1.1.1), with $a(x, \xi)$ replaced by $a(x, \xi, \lambda) \in \mathcal{S}^m(\Omega \times (\mathbb{R}^n \times \Lambda))$ and S replaced by $S(\lambda) \in \mathcal{S}(\Lambda, C^\infty_{\mathrm{loc}}(\Omega \times \Omega))$. Here, by $\mathcal{S}(\Lambda, V)$ is meant the Schwartz space of V-valued functions on the parameter space Λ.

By using invariance of all these classes under diffeomorphisms of open sets we can define $\Psi^m(M)$ and $\Psi^m(M; \Lambda)$ for any C^∞ manifold M. These spaces are Fréchet in a natural way.

The corresponding operator spaces with the subscript "cl" are introduced in an obvious way. The Fréchet topology in the spaces with "cl" is stronger than that induced by the spaces without "cl". In order to avoid too much comments on the topology we shall mainly deal with classical operators.

Suppose X is a C^∞ compact closed manifold. For any $A(\lambda) \in \Psi^m_{\mathrm{cl}}(X;\Lambda)$, we denote by $\sigma^m(A)(x,\xi,\lambda)$ the parameter-dependent principal homogeneous symbol of order m of $A(\lambda)$. It belongs to $\mathcal{S}^m_{\mathrm{hg}}((T^*X \times \Lambda) \setminus \{0\})$, the space of all C^∞ (positively) homogeneous functions of degree m on $(T^*X \times \Lambda) \setminus \{0\}$, where 0 indicates $(\xi,\lambda) = 0$. The mapping

$$\sigma^m : \ \Psi^m_{\mathrm{cl}}(X;\Lambda) \to \mathcal{S}^m_{\mathrm{hg}}((T^*X \times \Lambda) \setminus \{0\}) \tag{1.1.2}$$

is surjective.

Then, $A(\lambda)$ is said to be *parameter-dependent elliptic* if $\sigma^m(A)(x,\xi,\lambda) \neq 0$ for all $x \in X$ and $(\xi,\lambda) \neq 0$.

If $A(\lambda) \in \Psi^m_{\mathrm{cl}}(X;\Lambda)$ is a parameter-dependent elliptic operator, then there is an $R > 0$ such that $A(\lambda) : H^s(X) \to H^{s-m}(X)$ is an isomorphism for all $|\lambda| \geq R$ and every $s \in \mathbb{R}$.

On X we fix once and for all a Riemannian metric. Then we get the space $L^2(X)$ with respect to the corresponding Riemannian density dx. It is easy to see that $H^0(X) \cong L^2(X)$.

An example of an element in $\mathcal{S}^m_{\mathrm{hg}}((T^*X\times\Lambda)\setminus\{0\})$ is $(|\xi|^2+|\lambda|^2)^{m/2}$. In view of the surjectivity of (1.1.2) we find an $R^m(\lambda) \in \Psi^m_{\mathrm{cl}}(X;\Lambda)$ with the property that $\sigma^m(R^m)(x,\xi,\lambda) = (|\xi|^2 + |\lambda|^2)^{m/2}$. We now apply this argument again, with λ replaced by (λ, c), c being large enough, to obtain a family $R^m(\lambda)$ which induces isomorphisms $R^m(\lambda) \colon H^s(X) \to H^{s-m}(X)$ for all $\lambda \in \Lambda$ and $s \in \mathbb{R}$.

Let us now turn to some elementary facts on the Mellin transform, first for scalar functions on $\mathbb{R}_+$. The Mellin transform

$$\mathcal{M}u(z) = \int_0^\infty r^{-iz} u(r) \frac{dr}{r}, \quad z \in \mathbb{C},$$

gives rise to a continuous operator $C^\infty_{\mathrm{comp}}(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$, where $\mathcal{A}(\mathbb{C})$ is the space of all holomorphic functions on $\mathbb{C}$ with the Fréchet topology of uniform convergence on compact subsets. Set

$$\Gamma_\gamma = \{z \in \mathbb{C} : \ \Im z = \gamma\},$$

for any $\gamma \in \mathbb{R}$. Denote by $\mathcal{S}(\Gamma_\gamma)$ the Schwartz space on Γ_γ, i.e., the pull-back of $\mathcal{S}(\mathbb{R})$ under the diffeomorphism $\Gamma_\gamma \to \mathbb{R}$ given by $\varrho + i\gamma \mapsto \varrho$. Then $\mathcal{M}$ induces also continuous operators $\mathcal{M}_\gamma : C^\infty_{\mathrm{comp}}(\mathbb{R}_+) \to \mathcal{S}(\Gamma_\gamma)$ for all $\gamma \in \mathbb{R}$, where $\mathcal{M}_\gamma u = \mathcal{M}u\,|_{\Gamma_\gamma}$. The inverse is

$$\mathcal{M}_\gamma^{-1} f(r) = \frac{1}{2\pi} \int_{\Gamma_\gamma} r^{iz} f(z)\, dz, \quad r \in \mathbb{R}_+.$$

It is well-known that $\mathcal{M}_\gamma$ extends by continuity to a unitary isomorphism $L^2(\mathbb{R}_+, r^{2\gamma} dm) \to L^2(\Gamma_\gamma)$, where $dm = dr/r$.

For a fixed $\zeta \in \mathbb{C}$, let $(T^{\zeta} f)(z) = f(z+\zeta)$ mean the shift in the covariable by ζ. Then we have $\mathcal{M}(r^{\gamma} u) = T^{i\gamma} \mathcal{M}u$, for every $u \in C^{\infty}_{\mathrm{comp}}(\mathbb{R}_+)$.

It is clear that

$$rD_r = \mathcal{M}^{-1} z \mathcal{M},$$

when defined, for instance, on the subspace of all $u \in L^2(\mathbb{R}_+)$ such that $(rD_r)u \in L^2(\mathbb{R}_+)$. This gives rise to Mellin pseudodifferential operators

$$\mathrm{op}_{\mathcal{M}}(a)u\,(r) = \mathcal{M}^{-1}_{z\mapsto r}\left(\mathcal{M}_{r'\mapsto z}\, a(r,r',z)u(r')\right) \tag{1.1.3}$$

with Mellin amplitude functions $a(r,r',z)$ to be defined more precisely below. We also set

$$\mathrm{op}_{\mathcal{M},\gamma}(a) = r^{\gamma}\, \mathrm{op}_{\mathcal{M}}\left(T^{-i\gamma} a\right) r^{-\gamma},$$

for $\gamma \in \mathbb{R}$.

The notions around the Mellin transform have straightforward extensions to functions with values in a Fréchet space V. This will tacitly be used in the sequel. We set

$$\begin{array}{rcl} \mathcal{A}(\Omega, V) & = & \mathcal{A}(\Omega)\hat{\otimes}_{\pi} V, \\ \mathcal{S}(\mathbb{R}^q, V) & = & \mathcal{S}(\mathbb{R}^q)\hat{\otimes}_{\pi} V, \end{array}$$

$\hat{\otimes}_{\pi}$ being the completed projective tensor product, and so on.

Distributions u on $X^{\wedge}$ will often be regarded as being vector-valued and then written as $u(r)$ with the Mellin transform $\mathcal{M}u\,(z)$.

We define $\Psi^m_{\mathrm{cl}}(X;\Gamma_{-\gamma})$ to consist of all $A(z)$, z varying over $\Gamma_{-\gamma}$, such that $A(\varrho - i\gamma) \in \Psi^m_{\mathrm{cl}}(X;\mathbb{R}_{\varrho})$.

As explained above, for every $s \in \mathbb{R}$ we can choose a parameter-dependent elliptic operator $R^s(\varrho) \in \Psi^s_{\mathrm{cl}}(X;\mathbb{R})$ such that $R^s(z) : H^s(X) \to L^2(X)$ is an isomorphism for all $\varrho \in \mathbb{R}$.

Definition 1.1.2 *For $s, \gamma \in \mathbb{R}$, the space $\mathcal{H}^{s,\gamma}(X^{\wedge})$ is defined to be the completion of $C^{\infty}_{\mathrm{comp}}(X^{\wedge})$ with respect to the norm*

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^{\wedge})} = \left(\int_{\Gamma_{-\gamma}} \|R^s(\Re z)\, \mathcal{M}u(z)\|^2_{L^2(X)} dz \right)^{1/2}.$$

Clearly, the norm of $\mathcal{H}^{s,\gamma}(X^{\wedge})$ is independent, up to equivalent norms, of the concrete choice of the order reducing family $R^s(\varrho)$. These spaces are 'weighted' in the sense that $\mathcal{H}^{s,\gamma}(X^{\wedge}) = r^{\gamma}\mathcal{H}^s(X^{\wedge})$, for any $\gamma \in \mathbb{R}$, where $\mathcal{H}^s(X^{\wedge}) = \mathcal{H}^{s,0}(X^{\wedge})$. Moreover, it is easy to see that $\mathcal{H}^s(X^{\wedge}) \hookrightarrow H^s_{\mathrm{loc}}(X^{\wedge})$ for all $s \in \mathbb{R}$.

When identifying X locally with an open set on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we get a local diffeomorphism π of $X^{\wedge}$ to $\mathbb{R}^{n+1}$ given by $\pi(r,x) = rx$. In other words, we may regard $(r,x) \in X^{\wedge}$ as local "polar coordinates" in $\mathbb{R}^{n+1}$. This allows us to define the pull-back of $H^s(\mathbb{R}^{n+1})$ under π in a way independent of the identification above, provided that r is bounded away from 0. Now a familiar argument with a suitable open covering of X and a subordinate partition of unity leads to a global space $\pi^* H^s(\mathbb{R}^{n+1})$ on $X^\wedge$, whose restriction to every closed set away from $r = 0$ does not depend on any elements entered into the definition.

Both $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\pi^* H^s(\mathbb{R}^{n+1})$ are Banach spaces and modules over the algebra of all C^∞ functions on $\mathbb{R}_+$ which are constant close to $r = 0$ and $r = +\infty$. Pick a cut-off function $\omega \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ at $r = 0$, i.e., $\omega(r) \equiv 1$ near $r = 0$. Define

$$H^{s,\gamma}(X^\wedge) = [\omega]\, \mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]\, \pi^* H^s(\mathbb{R}^{n+1})$$

equipped with the topology of a non-direct sum. Obviously, $H^{s,\gamma}(X^\wedge)$ is independent of the particular choice of ω.

The scale $H^{s,\gamma}(X^\wedge)$, with $s, \gamma \in \mathbb{R}$, is intended for the analysis on infinite cones. On a compact closed manifold with conical points B we introduce a weighted Sobolev space $H^{s,\gamma}(\mathcal{B})$ by gluing together the weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ on collar neighbourhoods of conical points and the usual Sobolev space $H^s_{\mathrm{loc}}(\mathcal{B} \setminus \partial\mathcal{B})$ on the smooth part of B. In the sequel we shall often drop pull-backs if the identifications of objects with their pull-backs are evident.

The spaces $H^{s,\gamma}(X^\wedge)$ and $H^{s,\gamma}(\mathcal{B})$ can actually be thought of as Hilbert spaces after having chosen suitable scalar products. We only need fixed scalar products in the case $s = \gamma = 0$. In $H^{0,0}(X^\wedge)$ we take the L^2- scalar product with respect to the product measure $dmdx$, where $dm = \omega dr/r + (1-\omega) r^n dr$. In $H^{0,0}(\mathcal{B})$ we get a scalar product from $[\omega]\mathcal{H}^0(X^\wedge) + [1-\omega]L^2(\mathcal{B})$, the space $L^2(\mathcal{B})$ relying on a Riemannian metric in $\mathcal{B}$.

Now let $A \in \mathrm{Diff}^m(\mathcal{B} \setminus \partial\mathcal{B})$ be an arbitrary differential operator on the smooth part of B, which takes the form

$$A = \frac{1}{r^m} \sum_{j=0}^{m} A_j(r)\, (rD_r)^j$$

in the coordinates $(r, x) \in (0,1) \times X$ close to $\partial\mathcal{B}$, where $A_j(r)$ is a C^∞ function on $[0,1)$ with values in $\mathrm{Diff}^{m-j}(X)$. Then A extends to a continuous mapping $H^{s,\gamma}(\mathcal{B}) \to H^{s-m,\gamma-m}(\mathcal{B})$, for any $s, \gamma \in \mathbb{R}$.

The cone operator theory says that we have to control two leading symbols for the Fredholm property of A. The first of the two is the usual principal homogeneous symbol of order m of A in the interior of $\mathcal{B}$. As however $\sigma^m(A)$ blows up on the boundary of $\mathcal{B}$, we use instead the so-called *compressed interior symbol*, namely

$$^b\sigma^m(A)(r, x, \tilde{\varrho}, \xi) = \sum_{j=0}^{m} \sigma^{m-j}\left(A_j(r)\right)(x, \xi)\, \tilde{\varrho}^j \qquad (1.1.4)$$

defined everywhere on ${}^bT^*\mathcal{B}$, the compressed cotangent bundle of $\mathcal{B}$ (cf. Melrose [Mel96b]). To give (1.1.4) a precise sense away from a collar neighbourhood of $\partial\mathcal{B}$, we extend r to a positive C^∞ function on the interior of $\mathcal{B}$. In this way we obtain what is referred to as a *defining function* of the boundary of $\mathcal{B}$.

While ${}^b\sigma^m(A)$ takes its values in $\mathrm{Hom}(\mathbb{C})$, the second leading symbol is operator-valued. This is the *conormal symbol*

$$\sigma_{\mathcal{M}}(A)(z) = \sum_{j=0}^{m} A_j(0)\, z^j \tag{1.1.5}$$

which is regarded as an operator family $H^s(X) \to H^{s-m}(X)$, for any $s \in \mathbb{R}$, parametrised by the real part of $z \in \Gamma_{-\gamma}$.

Definition 1.1.3 *A typical operator $A \in \mathrm{Diff}^m(\overset{\circ}{\mathcal{B}})$ is said to be elliptic of order m, with respect to a weight $\gamma \in \mathbb{R}$, if*

1) ${}^b\sigma^m(A) \neq 0$ *on* ${}^bT^*\mathcal{B} \setminus \{0\}$;
2) $\sigma_{\mathcal{M}}(A)(z)\colon H^s(X) \to H^{s-m}(X)$ *is an isomorphism for all* $z \in \Gamma_{-\gamma}$ *and any fixed* $s \in \mathbb{R}$.

In view of the particular form of A we introduce the space $\mathcal{S}^m({}^bT^*\mathcal{B})$ of all symbols of order m in the interior of $\mathcal{B}$ that take the form

$$a(r, x, \varrho, \xi) = \tilde{a}(r, x, r\varrho, \xi)$$

in the coordinates $(r, x) \in (0,1) \times X$ near the boundary, where $\tilde{a}(r, x, \tilde{\varrho}, \xi)$ is C^∞ up to $r = 0$. Moreover, we use the symbol $\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{B})$ to designate the subspace of $\mathcal{S}^m({}^bT^*\mathcal{B})$ that originates from classical symbols just in the same way.

The symbols of $\mathcal{S}^m({}^bT^*\mathcal{B})$ give rise to pseudodifferential operators on $\mathcal{B}$ which still fit in the leading symbol structure described above.

Theorem 1.1.4 *The following conditions are equivalent:*

1) *A is elliptic of order m, with respect to a weight $\gamma \in \mathbb{R}$;*
2) *the operator $A\colon H^{s,\gamma}(\mathcal{B}) \to H^{s-m,\gamma-m}(\mathcal{B})$ is Fredholm, for each $s \in \mathbb{R}$.*

We now return to differential operators A on $C \setminus C_1$ mentioned at the introduction, cf. (0.0.4), where C is a compact closed manifold with corners C_0, and C_1 the skeleton of one-dimensional edges. Once again it is convenient to introduce a 'stretched manifold' $\mathcal{C}$ defined as a C^∞ compact manifold with corners on the boundary, along with a blow-down mapping $b : \mathcal{C} \to C$ such that the interior $\mathcal{C} \setminus \partial\mathcal{C}$ is mapped diffeomorphically onto $C \setminus C_1$; $b^{-1}(v) \cong \mathcal{B}$

for any $v \in C_0$, with $\mathcal{B}$ the stretched manifold of the link of C through v; and $b^{-1}(p) \cong X$ for all $p \in C_1 \setminus C_0$, where X is the base of the model cone of the wedge with edge through p.

The manifold $\mathcal{C}$ can be viewed as another compactification of $C \setminus C_1$. Therefore, we shall identify the smooth part of C with the interior of $\mathcal{C}$, the singularities of C being blown up to the boundary of $\mathcal{C}$. The boundary bears the structure of a fibre bundle under the blow-down mapping b. As mentioned, $\partial\mathcal{C}$ consists of a finite number of C^∞ hypersurfaces intersecting at non-zero angles. Choosing collar neighbourhoods of these hypersurfaces in $\mathcal{C}$ just amounts to endowing C with local fibre bundle structures near the boundary. More precisely, each point $v \in C_0$ has a neighbourhood O in C with the property that $b^{-1}(O) \cong [0,1) \times \mathcal{B}$, and every point $p \in C_1 \setminus C_0$ has a neighbourhood O in C such that $b^{-1}(O) \cong \Omega \times [0,1) \times X$, where Ω is an open interval on $\mathbb{R}$.

We also set $\mathcal{W} = b^{-1}(C \setminus C_0)$ which can be regarded as the stretched manifold of the (non-compact) manifold with edges $W = C \setminus C_0$.

In Section 2.2 we define corner Sobolev spaces $H^{s,\gamma}(\mathcal{C})$, for $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}^2$, analogously to the cone Sobolev spaces of Definition 1.1.2. Here we have a couple of weights $\gamma = (\gamma_0, \gamma_1)$, where γ_0 refers to the cone axis variable r and γ_1 to the corner axis variable t. Then, our operator A will extend to continuous operators $H^{s,\gamma}(\mathcal{C}) \to H^{s-m,\gamma-m}(\mathcal{C})$ for all $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}^2$, where $\gamma - m = (\gamma_0 - m, \gamma_1 - m)$. In order to get an analogue of Theorem 1.1.4, we have to identify the leading symbols of A, such that their bijectivity is responsible for the Fredholm property. We want to read off the adequate symbolic structure from (0.0.4).

First we have, of course, the usual principal homogeneous symbol of order m, namely $\sigma^m(A) \in \mathcal{S}^m_{\mathrm{hg}}(T^*(\mathcal{C} \setminus \partial\mathcal{C}) \setminus \{0\})$, which lives over the smooth part of C. We look for a substitute for $\sigma^m(A)$ which is well defined up to the singularities. To this end we introduce the space $\mathcal{S}^m({}^bT^*\mathcal{C})$ of all symbols a of order m in the interior of $\mathbb{C}$, such that

1) for any $v \in C_0$, we have $a(t,p,\tau,\vartheta) = \tilde{a}(t,p,t\tau,\vartheta)$ in the coordinates $(t,p) \in (0,1) \times (\mathcal{B} \setminus \partial\mathcal{B})$ near $b^{-1}(v)$, where $\tilde{a}(t,p,\tilde{\tau},\vartheta)$ is C^∞ up to $t = 0$;

2) for any $v \in C_0$, we have $a(t,r,x,\tau,\varrho,\xi) = \tilde{a}(t,r,x,tr\tau,r\varrho,\xi)$ in the coordinates $(t,r,x) \in (0,1) \times (0,1) \times X$ near $b^{-1}(v)$, where $\tilde{a}(t,r,x,\tilde{\tau},\tilde{\varrho},\xi)$ is C^∞ up to $t = r = 0$; and

3) for any $p \in C_1 \setminus C_0$, we have $a(t,r,x,\tau,\varrho,\xi) = \tilde{a}(t,r,x,r\tau,r\varrho,\xi)$ in the coordinates $(t,r,x) \in \Omega \times [0,1) \times X$ near $b^{-1}(p)$, where $\tilde{a}(t,r,x,\tilde{\tau},\tilde{\varrho},\xi)$ is C^∞ up to $r = 0$.

Moreover, we write $\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{C})$ for the subspace of $\mathcal{S}^m({}^bT^*\mathcal{C})$ that originates similarly from classical symbols.

An equivalent definition of $\mathcal{S}^m({}^bT^*\mathcal{C})$ is to extend both r and t to defining functions of the corresponding boundary hypersurfaces on the entire manifold $\mathcal{C}$ and require only the degenerate form 2) close to the boundary of $\mathcal{C}$, cf. Melrose [Mel87, Mel96a]. To any A of the form (0.0.4), we then assign the so-called *compressed interior symbol* by

$$ {}^b\sigma^m(A)(t,r,x,\tilde{\tau},\tilde{\varrho},\xi) = \sum_{j+k\leq 0} \sigma^{m-(j+k)}\left(A_{jk}(t,r)\right)(x,\xi)\,\tilde{\tau}^j\,\tilde{\varrho}^k \tag{1.1.6} $$

defined everywhere on ${}^bT^*\mathcal{C}$, the compressed cotangent bundle of $\mathcal{C}$ (cf. Melrose [Mel96b]).

To introduce an analogue of the conormal symbol (1.1.5) for A, we choose a cut-off function $\omega(r)$ at $r=0$ and write A in a neighbourhood of a corner $v \in C_0$ as

$$ A = \omega(r)\frac{1}{(tr)^m}\sum_{j+k\leq m} A_{jk}(t,r)\,(trD_t)^j\,(rD_r)^k + (1-\omega(r))\frac{1}{t^m}\sum_{j=0}^{m} A_j(t)\,(tD_t)^j $$

where

$$ \begin{array}{rcl} A_{jk}(t,r) & \in & C^\infty_{\mathrm{loc}}([0,1)\times[0,1), \mathrm{Diff}^{m-(j+k)}(X)), \\ A_j(t) & \in & C^\infty_{\mathrm{loc}}([0,1), \mathrm{Diff}^{m-j}(B\setminus B_0)). \end{array} $$

Then we set

$$ \sigma_{\mathcal{M}}(A)(\zeta) = \omega(r)\frac{1}{r^m}\sum_{j+k\leq m} A_{jk}(0,r)\,(r\zeta)^j\,(rD_r)^k + (1-\omega(r))\sum_{j=0}^{m} A_j(0)\,\zeta^j \tag{1.1.7} $$

regarded as a family of operators on $\mathcal{B}$, parametrised by the real part of ζ varying along a weight line $\Gamma_{-\gamma_1}$, $\gamma_1 \in \mathbb{R}$. Given any ζ, the symbol $\sigma_{\mathcal{M}}(A)(\zeta)$ induces continuous mappings $H^{s,\gamma_0}(\mathcal{B}) \to H^{s-m,\gamma_0-m}(\mathcal{B})$ for all $s \in \mathbb{R}$ and $\gamma_0 \in \mathbb{R}$.

As already mentioned at the beginning, a novelty for corners is the edge symbolic level along $C_1 \setminus C_0$, also being operator-valued. We shall adopt here the notation from the general theory of pseudodifferential operators on manifolds with edges, cf. Schulze [Sch98]. By assumption, the operator A is of the form

$$ A = \frac{1}{r^m}\sum_{j+k\leq m} \tilde{A}_{jk}(t,r)\,(rD_t)^j\,(rD_r)^k $$

in the coordinates (t,r,x) near any point p of the edge $C_1 \setminus C_0$, with certain $\tilde{A}_{jk} \in C^\infty_{\mathrm{loc}}(\Omega\times[0,1), \mathrm{Diff}^{m-(j+k)}(X))$. For p close to $v \in C_0$, this is, of course, compatible with (0.0.4), so that $\tilde{A}_{jk}(t,r) = t^{-m}A_{jk}(t,r)t^j$ with the above $A_{jk}(t,r)$. We set

$$ \sigma^m_{\mathrm{edge}}(A)(t,\tau) = \frac{1}{r^m}\sum_{j+k\leq m} \tilde{A}_{jk}(t,0)\,(r\tau)^j\,(rD_r)^k\,, \tag{1.1.8} $$

for $(t,\tau) \in \Omega \times \mathbb{R}$. Here, τ is treated as a covariable of t with respect to the Fourier transform along $\mathbb{R}$. The role of inserting $r = 0$ in the coefficients is the same as that of freezing coefficients on the boundary, for boundary value problems. We view (1.1.8) as an operator family $H^{s,\gamma_0}(X^\wedge) \to H^{s-m,\gamma_0-m}(X^\wedge)$, for any $s \in \mathbb{R}$ and $\gamma_0 \in \mathbb{R}$, parametrised by $(t,\tau) \in T^*(C_1 \setminus C_0) \setminus \{0\}$. It is called the principal homogeneous *edge symbol* of order m of A, the homogeneity referring to the group action

$$\kappa_\lambda u\,(r,x) = \lambda^{\frac{1+n}{2}}\, u(\lambda r, x), \quad \lambda > 0,$$

on the spaces $H^{s,\gamma_0}(X^\wedge)$. Obviously, κ_λ is a continuous function of $\lambda \in \mathbb{R}_+$ with values in $\mathcal{L}_\sigma(H^{s,\gamma}(X^\wedge))$, for every $s, \gamma \in \mathbb{R}$, where σ indicates the strong operator topology. Then, (1.1.8) satisfies

$$\sigma^m_{\mathrm{edge}}(A)(t,\lambda\tau) = \lambda^m\, \kappa_\lambda\, \sigma^m_{\mathrm{edge}}(A)(t,\tau)\, \kappa_\lambda^{-1} \tag{1.1.9}$$

for all $\lambda \in \mathbb{R}_+$.

The form of A close to the corner $v \in C_0$ suggests also to introduce a principal Mellin edge symbol of order m, namely

$${}^b\sigma^m_{\mathrm{edge}}(A)(t,\tilde\tau) = \frac{1}{r^m} \sum_{j+k \le m} A_{jk}(t,0)\, (r\tilde\tau)^j\, (rD_r)^k\,, \tag{1.1.10}$$

for $(t,\tilde\tau) \in \Omega \times \mathbb{R}$. This is also an operator family $H^{s,\gamma_0}(X^\wedge) \to H^{s-m,\gamma_0-m}(X^\wedge)$, for each $s \in \mathbb{R}$ and $\gamma_0 \in \mathbb{R}$, parametrised by t close to $t = 0$ and $\tilde\tau = \Re\zeta$. It fulfills a homogeneity property analogous to (1.1.9). Moreover, the equality holds

$$\sigma^m_{\mathrm{edge}}(A)(t,\tau) = \frac{1}{t^m}\, {}^b\sigma^m_{\mathrm{edge}}(A)(t, t\tau). \tag{1.1.11}$$

We have chosen the designation ${}^b\sigma^m_{\mathrm{edge}}(A)$ rather than, e.g., $\sigma^m_{M,\mathrm{edge}}(A)$ because this symbol is associated to $\sigma^m_{\mathrm{edge}}(A)$ just in the same way as the compressed interior symbol ${}^b\sigma^m(A)$ is to $\sigma^m(A)$.

The components of the triple $\big({}^b\sigma^m(A), \sigma_{\mathcal{M}}(A), \sigma^m_{\mathrm{edge}}(A)\big)$ obey obvious compatibility conditions.

Definition 1.1.5 *A typical operator $A \in \mathrm{Diff}^m(\mathring{\mathcal{C}})$ is said to be elliptic of order m, with respect to a weight $\gamma = (\gamma_0, \gamma_1)$ in $\mathbb{R}^2$, if*

1) ${}^b\sigma^m(A) \ne 0$ *on* ${}^bT^*\mathcal{C} \setminus \{0\}$*;*
2) $\sigma_{\mathcal{M}}(A)(\zeta)\colon H^{s,\gamma_0}(\mathcal{B}) \to H^{s-m,\gamma_0-m}(\mathcal{B})$ *is an isomorphism for all* $\zeta \in \Gamma_{-\gamma_1}$ *and any fixed* $s \in \mathbb{R}$*;*
3) $\sigma^m_{\mathrm{edge}}(A)(t,\tau)\colon H^{s,\gamma_0}(X^\wedge) \to H^{s-m,\gamma_0-m}(X^\wedge)$ *is an isomorphism for all* $(t,\tau) \in T^*(C_1 \setminus C_0) \setminus \{0\}$ *and any fixed* $s \in \mathbb{R}$.

If the condition 2) of Definition 1.1.5 is fulfilled for some real $s = s_0$, then it is fulfilled for all $s \in \mathbb{R}$. Analogously, the condition 3) for a fixed $s = s_0$ actually implies the same for all $s \in \mathbb{R}$.

Theorem 1.1.6 *The following conditions are equivalent:*

1) *A is elliptic of order m, with respect to a weight $\gamma = (\gamma_0, \gamma_1)$ in $\mathbb{R}^2$;*
2) *the operator $A\colon H^{s,\gamma}(\mathcal{C}) \to H^{s-m,\gamma-m}(\mathcal{C})$ is Fredholm, for each $s \in \mathbb{R}$.*

Similarly to the theory of cone pseudodifferential operators it is interesting to perform a parametrix construction within a suitable corner calculus and to obtain elliptic regularity with asymptotics close to $C_1 \setminus C_0$ in the sense of edge asymptotics and close to C_0 in an appropriate corner sense. In other words our goal is to introduce corner pseudodifferential operators and the corresponding distribution spaces with asymptotics.

Since away from C_0 the theory coincides with the calculus of operators on manifolds with edges, we have in general additional trace and potential conditions along $C_1 \setminus C_0$. They will also take part in the ellipticity and the edge symbolic structure. Even for a corner-degenerate differential operator A it may happen that A has to be filled to a matrix

$$\mathcal{A} = \begin{pmatrix} A & P \\ T & B \end{pmatrix} : \begin{matrix} H^{s,\gamma}(\mathcal{C}) \\ \oplus \\ H^{s,\delta}(C_1 \setminus C_0, W) \end{matrix} \to \begin{matrix} H^{s-m,\gamma-m}(\mathcal{C}) \\ \oplus \\ H^{s-m,\delta-m}(C_1 \setminus C_0, \tilde{W}) \end{matrix}$$

for obtaining the Fredholm property. A convenient choice of the weight δ will be specified below.

1.2 Cone calculus

The theory of pseudodifferential operators on manifolds with corners will employ some material from the cone theory. In this section we briefly discuss the necessary tools. For a thorough treatment we refer the reader to Schulze [Sch98].

We first recall the asymptotics of distributions close to a conical point of B or, equivalently, on the interior of $\mathcal{B}$ near $\partial\mathcal{B}$. This is formulated in terms of $X^\wedge$, X being the link of B through the conical point.

Given any $u \in H^{s,\gamma}(X^\wedge)$, the *discrete asymptotics* of $u(r, x)$ for $r \to \infty$ are defined as

$$u(r,x) \sim \sum_{\nu=1}^{\infty} \sum_{k=0}^{m_\nu} r^{ip_\nu} (\log r)^k \, c_{\nu k}(x), \tag{1.2.1}$$

where (p_ν) is a sequence of complex numbers such that $\Im p_\nu \to -\infty$ as $\nu \to \infty$, and the coefficients $c_{\nu k}(x)$ belong to finite-dimensional subspaces Σ_ν of $C^\infty(X)$,

$0 \le \nu \le m_\nu$. In this case we speak about the *discrete* asymptotic type as of u, with

$$\text{as} = (p_\nu, m_\nu, \Sigma_\nu)_{\nu=1,2,\dots},$$

the set $\pi_{\mathbb{C}}\,\text{as} = (p_\nu)_{\nu\in\mathbb{N}}$ being called a *carrier* of asymptotics.

Clearly, we assume $\Im p_\nu < -\gamma$. The precise meaning of (1.2.1) is that, for any cut-off function $\omega(r)$ and real $\epsilon > 0$, there is an N depending on ϵ, such that

$$u(r,x) - \omega(r) \sum_{\nu=1}^{N} \sum_{k=0}^{m_\nu} r^{ip_\nu} (\log r)^k \, c_{\nu k}(x) \in H^{s,\gamma+\epsilon}(X^\wedge).$$

Denote by $H^{s,\gamma}_{\text{as}}(X^\wedge)$ the space of all $u \in H^{s,\gamma}(X^\wedge)$ with asymptotics of the type "as". This is a Fréchet space in a natural way, the family of seminorms being given by the norms of $c_{\nu k}(x)$ in $C^\infty(X)$ and the norms of the remainders in $H^{s,\gamma+\epsilon}(X^\wedge)$.

The asymptotic types are related to a weight $\gamma \in \mathbb{R}$ and the infinite weight interval $\mathcal{I} = (-\infty, 0]$ below the weight line $\Gamma_{-\gamma}$. We write $w = (\gamma, \mathcal{I})$ for the weight data.

If occurring along edges, the asymptotics are parameter-dependent and lead to non-constant asymptotic types. More precisely, the numbers (p_ν) vary along with a point on the edge and may thus fill in compact subsets of the complex plane. These variable discrete asymptotics can be formulated as a particular case of *continuous asymptotics* defined as follows. Instead of (p_ν) we take an arbitrary sequence of compact sets (K_ν) in the complex plane, such that

$$\begin{aligned} K_\nu \subset \{z \in \mathbb{C} : \ \Im z < -\gamma\} \qquad & \text{for all } \ \nu; \\ \sup_{z\in K_\nu} \Im z \to -\infty \qquad & \text{as } \ \nu \to \infty. \end{aligned} \tag{1.2.2}$$

For a Fréchet space V, we denote by $\mathcal{A}'(K,V) = \mathcal{A}'(K)\hat{\otimes}_\pi V$ the space of all V-valued analytic functionals carried by K. Any $f \in \mathcal{A}'(K,V)$ can be applied to r^{iz} with respect to $z \in \mathbb{C}$, for each $r \in \mathbb{R}_+$. The result belongs to $C^\infty_{\text{loc}}(\mathbb{R}_+, V)$.

Pick a sequence $f_\nu \in \mathcal{A}'(K_\nu, C^\infty(X))$, $\nu = 1, 2, \dots$. By continuous asymptotics of $u \in H^{s,\gamma}(X^\wedge)$ for $r \to \infty$ are meant

$$u(r,x) \sim \sum_{\nu=1}^{\infty} \langle f_\nu, r^{iz} \rangle,$$

the asymptotic sum being understood similarly to (1.2.1). In contrast to (1.2.1) the functionals f_ν are in general not uniquely defined by the function u. Hence, for the calculus of continuous asymptotics we have to identify any two sequences $f_\nu \in \mathcal{A}'(K_\nu, C^\infty(X))$ and $\tilde{f}_\nu \in \mathcal{A}'(\tilde{K}_\nu, C^\infty(X))$ with the property that $\sum_{\nu=1}^{\infty} \langle f_\nu - \tilde{f}_\nu, r^{iz} \rangle \sim 0$. Moreover, we call a set $\sigma \subset \mathbb{C}$ a *carrier of asymptotics* if it is closed, has connected complement and meets each horizontal strip of a finite width in a bounded set. As but one example of this we show $\sigma = (\cup K_\nu)\hat{}$ where $\hat{\sigma}$, for a set $\sigma \subset \mathbb{C}$, means the complement of the union of all unbounded connected components of $\mathbb{C}\setminus\bar{\sigma}$. Then, we may still speak about the *continuous* asymptotic type as of u defined as as $= (\sigma, \Sigma)$, where σ is a carrier of asymptotics lying in the lower half-plane $\{z \in \mathbb{C} : \Im z < -\gamma\}$, and Σ the space of all equivalence classes of sequences $f_\nu \in \mathcal{A}'(K_\nu, C^\infty(X))$ with $K_\nu \subset \sigma$ satisfying (1.2.2).

We continue to write $H^{s,\gamma}_{\mathrm{as}}(X^\wedge)$ for the space of all $u \in H^{s,\gamma}(X^\wedge)$ with continuous asymptotics of type "as". This is a Fréchet space in a natural way, cf. [Dor98, 2.1.3].

It will be convenient to introduce asymptotic types also for finite weight intervals $\mathcal{I} = (-l, 0]$, where $l > 0$. For finite $\mathcal{I}$, by a discrete asymptotic type is meant any finite collection as $= (p_\nu, m_\nu, \Sigma_\nu)_{\nu=1,\dots,N}$, with p_ν a complex number in the strip $\{z \in \mathbb{C} : -\gamma - l < \Im z < -\gamma\}$, m_ν a non-negative integer, and Σ_ν a finite-dimensional subspace of $C^\infty(X)$. Moreover, by a continuous asymptotic type we mean any pair as $= (K, \Sigma)$, where K is a carrier of asymptotics contained in the strip $\{z \in \mathbb{C} : -\gamma - l < \Im z < -\gamma\}$, and Σ the space of all equivalence classes of functionals $f \in \mathcal{A}'(K, C^\infty(X))$, two functionals f and $\tilde{f}$ being equivalent if $\omega(r) \langle f - \tilde{f}, r^{iz} \rangle \in H^{\infty,\gamma+l-0}(X^\wedge)$. In case $\mathcal{I}$ is finite the definition of spaces with asymptotics is quite straightforward. Namely, given an asymptotic type as related to the weight data $w = (\gamma, \mathcal{I})$, we denote by $\mathcal{A}_{\mathrm{as}}(X^\wedge)$ the space of all potentials $u(r, x) = \omega(r) \langle f(x), r^{iz} \rangle$, with $f(x)$ a functional in the relevant subspace of $\mathcal{A}'(K, C^\infty(X))$, and $\omega(r)$ a fixed cut-off function. Set

$$H^{s,\gamma}_{\mathrm{as}}(X^\wedge) = \mathcal{A}_{\mathrm{as}}(X^\wedge) + H^{s,\gamma+l-0}(X^\wedge)$$

in the sense of non-direct sums of Fréchet spaces, $\mathcal{A}_{\mathrm{as}}(X^\wedge)$ being endowed with the topology induced by the embedding to $\mathcal{A}'(K, C^\infty(X))$, and $H^{s,\gamma+l-0}(X^\wedge)$ by the projective limit topology.

As defined above, the continuous asymptotic types are actually identified with their carriers. The reason for distinguishing the notation of these objects is that below for corners there are more complicated spaces of coefficients for the asymptotics. Then, the present notation corresponds to a simpler particular case.

To define weighted Sobolev spaces with asymptotics on the entire manifold B, we fix a cut-off function $\omega(r)$ supported in a collar neighbourhood of $\partial\mathcal{B}$, e.g., $\omega(r) \equiv 0$ for $r > 1/2$. Given any asymptotic type "as", being discrete or continuous, we set

$$\begin{aligned} H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}) &= [\omega]\, H^{s,\gamma}_{\mathrm{as}}(X^\wedge) + [1-\omega]\, H^{s}(\mathcal{B}), \\ \mathcal{S}^{\gamma}_{\mathrm{as}}(X^\wedge) &= [\omega]\, H^{\infty,\gamma}_{\mathrm{as}}(X^\wedge) + [1-\omega]\, \mathcal{S}(X^\wedge) \end{aligned}$$

where $\mathcal{S}(X^\wedge) = \mathcal{S}(\mathbb{R}_+) \hat{\otimes}_\pi C^\infty(X)$ and $\mathcal{S}(\mathbb{R}_+)$ stands for the restriction of $\mathcal{S}(\mathbb{R})$

to the half-axis.

Definition 1.2.1 *For weight data* $w = (\gamma, \delta, \mathcal{I})$*, the space* $\Psi_G(X^\wedge; w)$ *is defined to consist of all* $G \in \cap_{s\in\mathbb{R}}\mathcal{L}(H^{s,\gamma}(X^\wedge), H^{\infty,\delta}(X^\wedge))$ *with the property that*

$$\begin{aligned} G &: H^{s,\gamma}(X^\wedge) \to \mathcal{S}^{\delta}_{\mathrm{as}}(X^\wedge), \\ G^* &: H^{s,-\delta}(X^\wedge) \to \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge) \end{aligned}$$

for all $s \in \mathbb{R}$ *and some asymptotic types "as" and "$\tilde{\text{as}}$" related to weight data* $(\delta, \mathcal{I})$ *and* $(-\gamma, \mathcal{I})$*, respectively.*

Here G^* is the formal adjoint of G with respect to a fixed scalar product in $H^{0,0}(X^\wedge)$.

The operators of $\Psi_G(X^\wedge; w)$ are said to be *Green operators* on $X^\wedge$ with discrete or continuous asymptotics. This concept extends in a natural way to operators given on the whole manifold B. Namely, we write $\Psi_G(\mathcal{B}; w)$ for the space of all $G \in \cap_{s\in\mathbb{R}}\mathcal{L}(H^{s,\gamma}(\mathcal{B}), H^{\infty,\delta}(\mathcal{B}))$ that induce continuous mappings

$$\begin{aligned} G &: H^{s,\gamma}(\mathcal{B}) \to H^{\infty,\delta}_{\mathrm{as}}(\mathcal{B}), \\ G^* &: H^{s,-\delta}(\mathcal{B}) \to H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{B}) \end{aligned}$$

for all $s \in \mathbb{R}$ and certain asymptotic types "as" and "$\tilde{\text{as}}$" related to weight data $(\delta, \mathcal{I})$ and $(-\gamma, \mathcal{I})$, respectively, G^* being the formal adjoint with respect to a scalar product in $H^{0,0}(\mathcal{B})$.

The cone pseudodifferential operators are defined by means of operator-valued Mellin symbols which also reflect the asymptotics. The task is now to introduce the discrete and continuous asymptotic types of Mellin symbols.

If σ is a subset of the complex plane we call a $\chi \in C^\infty_{\mathrm{loc}}(\mathbb{C})$ a σ-excision function if $\chi(z) = 0$, when $\mathrm{dist}(z, \sigma) < \epsilon'$, and $\chi(z) = 1$, when $\mathrm{dist}(z, \sigma) > \epsilon''$, for some $0 < \epsilon' < \varepsilon'' < \infty$.

Fix a collection $T = (p_\nu, m_\nu, \mathcal{L}_\nu)_{\nu\in\mathbb{Z}}$, where p_ν are complex numbers satisfying $|\Im p_\nu| \to \infty$ as $|\nu| \to \infty$, m_ν non-negative integers, and $\mathcal{L}_\nu$ finite-dimensional subspaces of finite rank operators in $\Psi^{-\infty}(X)$. Any such collection T is called a *discrete* asymptotic type for Mellin symbols. The set $\pi_{\mathbb{C}}T = (p_\nu)_{\nu\in\mathbb{Z}}$ gives rise to what we have already called a carrier of asymptotics.

Definition 1.2.2 *By* $\mathcal{M}_T(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$ *is meant the space of all meromorphic functions* $a(z)$ *in* $\mathbb{C}$ *with values in* $\Psi^m_{\mathrm{cl}}(X)$*, such that*

1) *given a* $\pi_{\mathbb{C}}T$*-excision function* $\chi(z)$*, we have* $\chi(z)a(z)|_{\Gamma_{-\gamma}} \in \Psi^m_{\mathrm{cl}}(X; \Gamma_{-\gamma})$ *for all* $\gamma \in \mathbb{R}$*, uniformly in* γ *in compact intervals of* $\mathbb{R}$*;*
2) $a(z)$ *has poles at* p_ν *of multiplicities* $m_\nu + 1$ *with Laurent coefficients at* $(z - p_\nu)^{-(k+1)}$ *belonging to* $\mathcal{L}_\nu$*, for every* $0 \le k \le m_\nu$ *and* $\nu \in \mathbb{Z}$*.*

The spaces $\mathcal{M}_T(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$ are Fréchet in a natural way. If $\pi_{\mathbb{C}} T$ is empty we omit T and write simply $\mathcal{M}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$. It is immediate from the definition that

$$\mathcal{M}_T(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X)) = \mathcal{M}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X)) + \mathcal{M}_T(\mathbb{C}, \Psi^{-\infty}(X)) \tag{1.2.3}$$

as a non-direct sum of Fréchet spaces.

The *continuous* asymptotic types of Mellin symbols require more care and are first defined for *quasi-discrete* carriers σ, i.e., those of the form $\sigma = \cup_{\nu \in \mathbb{Z}} K_\nu$ where

$$\begin{aligned} \sup_{z \in K_{\nu+1}} \Im z < \inf_{z \in K_\nu} \Im z \qquad & \text{for all } \nu; \\ \sup_{z \in K_\nu} |\Im z| \to \infty \qquad & \text{as } |\nu| \to \infty. \end{aligned}$$

If $T = (\sigma, \Psi^{-\infty}(X))$, with σ a quasi-discrete carrier of asymptotics, then $\mathcal{M}_T(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$ is defined by (1.2.3), where $\mathcal{M}_T(\mathbb{C}, \Psi^{-\infty}(X))$ denotes the space of all $a(z) \in \mathcal{A}(\mathbb{C} \setminus \sigma, \Psi^{-\infty}(X))$ such that, given any σ-excision function $\chi(z)$, we have $\chi(z)a(z)|_{\Gamma_{-\gamma}} \in \Psi^{-\infty}(X; \Gamma_{-\gamma})$ for all $\gamma \in \mathbb{R}$, uniformly in γ in compact intervals of $\mathbb{R}$. It is clear that any carrier of asymptotics σ has the form $\sigma = \sigma_1 \cup \sigma_2$, for quasi-discrete σ_1 and σ_2. Thus, a continuous asymptotic type $T = (\sigma, \Psi^{-\infty}(X))$ with arbitrary carrier $\sigma \subset \mathbb{C}$ can be thought of as an equivalence class of pairs (T_1, T_2), where both T_1 and T_2 have quasi-discrete carriers. We identify any two pairs (T_1, T_2) and $(\tilde{T}_1, \tilde{T}_2)$ such that $\sigma_1 \cup \sigma_2 = \tilde{\sigma}_1 \cup \tilde{\sigma}_2$, where σ_j and $\tilde{\sigma}_j$ are the carriers of T_j and $\tilde{T}_j$, respectively. If $\sigma = \sigma_1 \cup \sigma_2$, we also write $T = T_1 + T_2$. We define $\mathcal{M}_T(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$ to be the sum of Fréchet spaces $\mathcal{M}_{T_1}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$ and $\mathcal{M}_{T_2}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$, where $T = T_1 + T_2$ and T_1, T_2 are of quasi-discrete type. It is independent of the choice of the decomposition $T = T_1 + T_2$.

We shall employ a parameter-dependent analogue of $\mathcal{M}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X))$ with a parameter $\eta \in \mathbb{R}^q$. Namely, $\mathcal{M}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X; \mathbb{R}^q))$ stands for the space of all $a(\eta, z) \in \mathcal{A}(\mathbb{C}, \Psi^m_{\mathrm{cl}}(X; \mathbb{R}^q))$ such that $a(\eta, \varrho - i\gamma) \in \Psi^m_{\mathrm{cl}}(X; \mathbb{R}^{q+1}_{\eta,\varrho})$ for all $\gamma \in \mathbb{R}$, uniformly in γ in compact intervals of $\mathbb{R}$. This space is also Fréchet in a natural way.

We are now in a position to introduce two pseudodifferential calculus on B. The first of the two is known as the *cone algebra* with discrete asymptotics, cf. [Sch98, 2.4].

Definition 1.2.3 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $\mathcal{I} = (-l, 0]$, l being a positive integer. Then, $\Psi^\mu(\mathcal{B}; w)$ is the space of all operators*

$$A = A_b + A_i + M + G \tag{1.2.4}$$

where

A_b *is a Mellin operator with holomorphic symbol in a collar neighbourhood of $\partial\mathcal{B}$, i.e., $A_b = \varphi_b r^{-\mu} \mathrm{op}_{\mathcal{M},\gamma}(h) \psi_b$ where $h(r, z) \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^\mu_{\mathrm{cl}}(X)))$;*

A_i *is a usual pseudodifferential operator on the smooth part of B cut off away from $\partial\mathcal{B}$, i.e., $A_i = \varphi_i \Psi \psi_i$ where $\Psi \in \Psi^{\mu}_{\mathrm{cl}}(\mathcal{B} \setminus \partial\mathcal{B})$;*

M *is the sum of smoothing Mellin operators with meromorphic symbols in a collar neighbourhood of $\partial\mathcal{B}$, i.e., $M = \varphi_b r^{-\mu} \sum_{j=0}^{l-1} r^j \mathrm{op}_{\mathcal{M},\gamma_j}(m_j)\psi_b$ where $m_j(z) \in \mathcal{M}_{T_j}(\mathbb{C}, \Psi^{-\infty}(X))$ and $\gamma - (m-\mu) - j \le \gamma_j \le \gamma$ satisfy $\pi_{\mathbb{C}} T_j \cap \Gamma_{-\gamma_j} = \emptyset$; and*

G *is a Green operator with discrete asymptotics on $\mathcal{B}$ related to the weight data w, i.e., $G \in \Psi_G(\mathcal{B}; w)$.*

The cone algebra with discrete asymptotics is an efficient tool of analysis on manifolds with conical points because the asymptotics are controlled within finite-dimensional spaces of singular functions. However, problems on manifolds with edges fall out this framework and require a calculus with continuous asymptotics. The *cone algebra* with continuous asymptotics is formed by all operators of the form (1.2.4), now both M and G bearing continuous asymptotics. A smoothing Mellin operator with continuous asymptotics is defined by

$$M = \varphi_b \, r^{-\mu} \sum_{j=0}^{l-1} r^j \left(\mathrm{op}_{\mathcal{M},\gamma_j^{(1)}}(m_j^{(1)}) + \mathrm{op}_{\mathcal{M},\gamma_j^{(2)}}(m_j^{(2)}) \right) \psi_b$$

where $m_j^{(\iota)}(z) \in \mathcal{M}_{T_j^{(\iota)}}(\mathbb{C}, \Psi^{-\infty}(X))$ and $\gamma - (m-\mu) - j \le \gamma_j^{(\iota)} \le \gamma$ satisfy $\pi_{\mathbb{C}} T_j^{(\iota)} \cap \Gamma_{-\gamma_j^{(\iota)}} = \emptyset$, for $\iota = 1, 2$.

It will cause no confusion if we use the same notation $\Psi^{\mu}(\mathcal{B}; w)$ for the cone algebra with continuous asymptotics. The precise meaning is always clear from the context.

The cut-off functions $\varphi_b, \psi_b \in C^{\infty}_{\mathrm{comp}}[0,1)$ are arbitrary as well as the functions $\varphi_i, \psi_i \in C^{\infty}_{\mathrm{comp}}(\mathcal{B} \setminus \partial\mathcal{B})$. We shall assume without loss of generality that (φ_b, φ_i) form a partition of unity on $\mathcal{B}$, and ψ_b, ψ_i cover φ_b, φ_i in the sense that $\psi_b \varphi_b = \varphi_b$ and $\psi_i \varphi_i = \varphi_i$, respectively. Moreover, we require A_b and A_i to be *compatible* in the sense that

$$\kappa_\sharp \, \Psi = r^{-\mu} \, \mathrm{op}_{\mathcal{M},\gamma}(h) \tag{1.2.5}$$

modulo $\Psi^{-\infty}((0,1) \times \partial\mathcal{B})$, where O is a collar neighbourhood of $\partial\mathcal{B}$ along with a diffeomorphism $\kappa \colon O \to [0,1) \times \partial\mathcal{B}$ and $\kappa_\sharp \Psi = \kappa_* \Psi \kappa^*$ is the operator push-forward of Ψ under κ.

Finally, we define the class $\Psi^{\mu}(X^\wedge; w)$ by inserting $\Psi \in \Psi^{\mu}_{\mathrm{cl,exit}}(X^\wedge)$ and $G \in \Psi_G(X^\wedge)$ in (1.2.4). Here $\Psi^{\mu}_{\mathrm{cl,exit}}(X^\wedge)$ stands for the space of all classical pseudodifferential operators on $X^\wedge$ satisfying the *exit condition* as $r \to \infty$. Let us recall this latter concept in case $X = \mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$. The general case then follows by a globalisation via a finite covering of X by open sets diffeomorphic to coordinate neighbourhoods on $\mathbb{S}^n$, and a subordinate partition of unity. For $X = \mathbb{S}^n$, we have $X^\wedge \cong \mathbb{R}^{n+1} \setminus \{0\}$, the diffeomorphism being by $(r, x) \mapsto rx$. The specific things concern "large" values of the variable $\tilde{x}$ and the covariable $\tilde{\xi}$, hence it suffices to look at symbols on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. For $\mu, \epsilon \in \mathbb{R}$, we denote by $\mathcal{S}^{\mu,\epsilon}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ the set of all $a \in \mathcal{S}^{\mu}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that

$$|D_{\tilde{x}}^{\alpha} D_{\tilde{\xi}}^{\beta} a(\tilde{x}, \tilde{\xi})| \leq c\ (1 + |\tilde{x}|)^{\epsilon - |\alpha|}\ (1 + |\tilde{\xi}|)^{\mu - |\beta|}$$

for all $(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, the constant c depending on $\alpha, \beta \in \mathbb{Z}_+^{n+1}$. Moreover, we write $\mathcal{S}_{\mathrm{cl}}^{\mu,\epsilon}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ for the space of all $a \in \mathcal{S}^{\mu,\epsilon}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ with the property that $a(\tilde{x}, \tilde{\xi})$ is classical in $\tilde{\xi}$ and there is a sequence a_j, $j \in \mathbb{Z}_+$, in $\mathcal{S}_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$, every $a_j(\tilde{x}, \tilde{\xi})$ being homogeneous of degree $\epsilon - j$ in $\tilde{x}$, for large $|\tilde{x}|$, such that

$$a(\tilde{x}, \tilde{\xi}) - \chi(\tilde{x}) \sum_{j=0}^{J} a_j(\tilde{x}, \tilde{\xi}) \in \mathcal{S}^{\mu, \epsilon - (J+1)}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$$

for all $j \in \mathbb{Z}_+$. (We also assume an analogous condition in ξ.) Now, the class of exit symbols is $\mathcal{S}_{\mathrm{cl}}^{\mu,0}((\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{R}^{n+1})$, the conditions being required for $|\tilde{x}| \geq \varepsilon$ with any $\varepsilon > 0$. The component $\sigma_{\mathrm{exit}}^0(a) = a_0$ is called the *principal* exit symbol of a.

The operator spaces $\Psi^\mu(\mathcal{B}; w)$ and $\Psi^\mu(X^\wedge; w)$ bear natural locally convex topologies of inductive limits of Fréchet spaces, cf. Sections 1.3.2 and 1.4.6 in [Sch98].

We shall not repeat here once again all elements of the cone theory. For references below let us only mention that each operator $A \in \Psi^\mu(\mathcal{B}; w)$ has two leading symbolic levels. More precisely, the principal homogeneous interior symbol of A, here in the compressed form ${}^b\sigma^m(A)$, lives away from the zero section of the compressed cotangent bundle of $\mathcal{B}$, cf. (1.1.4). At any conical point of B (here corresponding to $r = 0$) the operator A has also the sequence of conormal symbols

$$\sigma_{\mathcal{M}}^j(A)(z) = \frac{1}{j!}\,(\partial/\partial r)^j h\,(0, z) + m_j(z), \tag{1.2.6}$$

for $j = 0, 1, \ldots, l-1$, every $\sigma_{\mathcal{M}}^j(A)(z)$ being a family in $\mathcal{L}(H^s(M), H^{s-\mu}(M))$ holomorphic in a strip around $\Gamma_{-\gamma}$, cf. (1.1.5) for $j = 0$. These symbolic levels behave in a natural way under taking formal adjoints and composition of operators. An analogous remark also holds with $\mathcal{B}$ replaced by $X^\wedge$, where we have in addition the principal exit symbol of A as $r \to \infty$.

Remark 1.2.4 *Definition 1.1.3 extends to $\Psi^m(\mathcal{B}; w)$. Then Theorem 1.1.4 remains valid, the parametrices of elliptic operators lying in $\Psi^{-m}(\mathcal{B}; w^{-1})$ with $w^{-1} = (\gamma - m, \gamma, \mathcal{I})$.*

An analogous result is valid for the corresponding operator classes over the infinite cone $X^\wedge$, where the additional exit symbol for $r \to \infty$ is involved, cf. [Sch98, 2.4.3].

1.3 Edge calculus

In this section we present the material on pseudodifferential operators along the edges of dimension 1 that are emanated from corners.

A "manifold" with edges W is locally close to any point p of an edge E of the form

$$\Omega \times \frac{[0,1) \times X}{\{0\} \times X}$$

where Ω is a neighbourhood of p in E identified with an open interval in $\mathbb{R}$, and X a C^∞ compact closed manifold of dimension n, the base of the model cone. It is custom to pass to a stretched manifold $\mathcal{W}$ of W which is in this case a C^∞ manifold with boundary. Close to the boundary, $\mathcal{W}$ is of the form

$$\Omega \times [0,1) \times X,$$

and there is a canonical "projection" $b \colon \mathcal{W} \to W$ which restricts to a diffeomorphism $\mathcal{W} \setminus \partial\mathcal{W} \to W \setminus E$.

A global operator calculus with Fredholm property under ellipticity would insist on E being closed and compact (cf. [Sch98, 3.5.2]). Our application will concern $\mathcal{W} = b^{-1}(C \setminus C_0)$ where $E = C_1 \setminus C_0$ is not compact. In this case the main point is the local picture. Near corners the wedge calculus needs Mellin operators along $C_1 \setminus C_0$. This is the subject of the next section, whereas now the theory is based on the Fourier transform along the edges. We introduce a class of pseudodifferential operators on the smooth part of W, i.e., $\Psi^m(\mathcal{W}; w) \hookrightarrow \Psi^m_{\mathrm{cl}}(\mathcal{W} \setminus \partial\mathcal{W})$ where $w = (\gamma, \gamma - m, \mathcal{I})$ are double weight data. Moreover, we define a more general class $\Psi^m(\mathcal{W}; W, \tilde{W}; w)$ of matrix-valued operators

$$\mathcal{A} = \begin{pmatrix} A & P \\ T & B \end{pmatrix},$$

with $A \in \Psi^m(\mathcal{W}; w)$. The entries T and P play the role of additional trace and potential operators along the edge, respectively, while B is a usual pseudodifferential operator of type $W \to \tilde{W}$ and order μ on E.

Since the specific information is located near $\partial\mathcal{W}$ and the calculus is invariant under the diffeomorphisms obeying the cone bundle structure, we may first look at the theory on $\Omega \times X^\wedge$. To some extent this can be performed as a pseudodifferential calculus along Ω with operator-valued symbols. The technicalities may be found in Chapter 3 of [Sch98].

Recall the definition of abstract edge Sobolev spaces. Let V be a Banach space and $(\kappa_\lambda)_{\lambda>0}$ be a fixed representation of $\mathbb{R}_+$ in $\mathcal{L}(V)$, i.e., $\kappa_\lambda \in C(\mathbb{R}_+, \mathcal{L}_\sigma(V))$ satisfies $\kappa_1 = \mathrm{Id}$ and $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}$, for all $\lambda, \mu \in \mathbb{R}_+$. Choose a strictly positive C^∞ function $\tau \mapsto \langle \tau \rangle$ on $\mathbb{R}$ satisfying $\langle \tau \rangle = |\tau|$ for all $|\tau| \geq c$, the constant $c > 0$ being fixed, e.g., $c = 1$. We write $\mathcal{F}_{t\mapsto\tau}$ for the Fourier transform in $\mathbb{R}$.

Definition 1.3.1 *For $s \in \mathbb{R}$, the space $H^s(\mathbb{R}, \pi^* V)$ is defined to be the completion of $\mathcal{S}(\mathbb{R}, V)$ with respect to the norm*

$$\|u\|_{H^s(\mathbb{R},\pi^* V)} = \left(\int \langle \tau \rangle^{2s} \, \|\kappa_{\langle \tau \rangle}^{-1} \mathcal{F}_{t\mapsto\tau} u\|_V^2 \, d\tau \right)^{1/2}.$$

If $\kappa_\lambda \equiv \mathrm{Id}_V$ for all $\lambda \in \mathbb{R}_+$, we recover the usual Sobolev spaces $H^s(\mathbb{R}, V)$ of V-valued functions on $\mathbb{R}$. Yet another crucial example corresponds to the fibre space $V = H^{s,\gamma}(X^\wedge)$, with $s, \gamma \in \mathbb{R}$, and the group action on V given by $\kappa_\lambda u(r, x) = \lambda^{(1+n)/2} u(\lambda r, x)$, for $\lambda > 0$. Then, we obtain the *wedge Sobolev spaces* $H^{s,\gamma}(\mathbb{R} \times X^\wedge) = H^s(\mathbb{R}, \pi^* H^{s,\gamma}(X^\wedge))$ of smoothness s and weight γ. When localised to compact subsets of the wedge $\mathbb{R} \times X^\wedge$, the space $H^{s,\gamma}(\mathbb{R} \times X^\wedge)$ is known to coincide with the usual Sobolev space $H^s_{\mathrm{loc}}(\mathbb{R} \times X^\wedge)$. Moreover, we have

$$H^{0,-\frac{n+1}{2}}(\mathbb{R} \times X^\wedge) \cong L^2(\mathbb{R} \times X^\wedge, r^n dt dr dx), \tag{1.3.1}$$

the measure $r^n dr dx$ corresponding to the cone Riemannian metric in the fibre $X^\wedge$ of the wedge.

The operator $I = \mathcal{F}^{-1}_{\tau\to t} \kappa^{-1}_{\langle\tau\rangle} \mathcal{F}_{t\to\tau}$ induces an isomorphism of $H^s(\mathbb{R}, \pi^* V)$ onto $H^s(\mathbb{R}, V)$. This allows us to extend the definition of $H^s(\mathbb{R}, \pi^* \Sigma)$ to the vector subspaces $\Sigma \subset V$ that are not necessarily preserved under κ_λ. Namely, we set

$$H^s(\mathbb{R}, \pi^* \Sigma) \cong I^{-1} \, H^s(\mathbb{R}, \Sigma), \tag{1.3.2}$$

for any Banach space Σ continuously embedded to V. Then (1.3.2) is again a Banach space in the topology induced by the bijection I.

Remark 1.3.2 *For any two Banach spaces Σ_1 and Σ_2 continuously embedded to V, we have*

$$H^s(\mathbb{R}, \pi^*(\Sigma_1 + \Sigma_2)) = H^s(\mathbb{R}, \pi^* \Sigma_1) + H^s(\mathbb{R}, \pi^* \Sigma_2)$$

in the sense of non-direct sums of Fréchet spaces.

The definition of the edge Sobolev spaces $H^s(\mathbb{R}, \pi^* V)$ extends to Fréchet spaces V written as projective limits of Banach spaces V^ν, $\nu \in \mathbb{N}$, with continuous embeddings $V^{\nu+1} \hookrightarrow V^\nu$ and a strongly continuous action $(\kappa_\lambda)_{\lambda\in\mathbb{R}_+}$ on V_1 that restricts to a strongly continuous action on each V^ν, $\nu = 2, 3, \ldots$.

We then obtain natural embeddings $H^s(\mathbb{R}, \pi^* V^{\nu+1}) \hookrightarrow H^s(\mathbb{R}, \pi^* V^\nu)$ for all ν, and we set $H^s(\mathbb{R}, \pi^* V)$ to be the projective limit of the sequence $H^s(\mathbb{R}, \pi^* V^\nu)$, $\nu \in \mathbb{N}$. Note that Remark 1.3.2 remains valid in case Σ_1 and Σ_2 are projective limits of Banach spaces.

In particular, suppose "as" is an asymptotic type related to weight data $w = (\gamma, \mathcal{I})$ with a finite $\mathcal{I} = (-l, 0]$. Then,

$$\begin{array}{rcl} H^{s,\gamma}_{\text{as}}(\mathbb{R} \times X^\wedge) & = & H^s(\mathbb{R}, \pi^* H^{s,\gamma}_{\text{as}}(X^\wedge)), \\ H^{s,\gamma+l-0}(\mathbb{R} \times X^\wedge) & = & H^s(\mathbb{R}, \pi^* H^{s,\gamma+l-0}(X^\wedge)) \end{array}$$

will be regarded as wedge Sobolev space with *edge asymptotics* of type "as" and wedge Sobolev space of *edge flatness* l relative to the weight γ, respectively. By Remark 1.3.2 we get

$$H^{s,\gamma}_{\text{as}}(\mathbb{R} \times X^\wedge) = H^s(\mathbb{R}, \pi^* \mathcal{A}_{\text{as}}(X^\wedge)) + H^{s,\gamma+l-0}(\mathbb{R} \times X^\wedge),$$

where the space $H^s(\mathbb{R}, \pi^* \mathcal{A}_{\text{as}}(X^\wedge))$ consists just of the "singular functions" of edge asymptotics, namely

$$\mathcal{F}^{-1}_{\tau \to t} \left(\langle \tau \rangle^{\frac{1+n}{2}} \, \omega(r\langle \tau \rangle) \, \langle \mathcal{F}_{t \to \tau} \, f(t,x), (r\langle \tau \rangle)^{iz} \rangle \right) \tag{1.3.3}$$

where $f(t,x)$ runs over $H^s(\mathbb{R}, \mathcal{A}'(K, C^\infty(X)))$, K being the carrier of "as", cf. (1.3.2).

For infinite $\mathcal{I}$ we can write $u \sim \sum_{\nu=1}^\infty u_\nu$ where u_ν are singular functions associated with compact sets K_ν in the complex plane, satisfying (1.2.2). The interpretation of this asymptotic expansion is that, for any $\epsilon > 0$, there is an $N = N(\epsilon)$ such that $u - \sum_{\nu=1}^N u_\nu \in H^{s,\gamma+\epsilon}(\mathbb{R} \times X^\wedge)$.

We also need the "loc" and "comp" versions of our spaces. The definitions rely on the fact that the edge Sobolev spaces are modules over $C^\infty_{\text{comp}}(\mathbb{R})$. Let Ω be an open set in $\mathbb{R}$. Then, $H^s_{\text{loc}}(\Omega, \pi^* V)$ is defined to be the space of all $u \in \mathcal{D}'(\Omega, V)$ such that $\varphi u \in H^s(\mathbb{R}, \pi^* V)$ for each $\varphi \in C^\infty_{\text{comp}}(\Omega)$. Furthermore, $H^s_{\text{comp}}(\Omega, \pi^* V)$ stands for the space of all $u \in H^s_{\text{loc}}(\Omega, \pi^* V)$ with a compact support in Ω.

The invariance of our definitions under diffeomorphisms $\Omega \overset{\cong}{\to} \tilde{\Omega}$ and the property $H^{s,\gamma}(\mathbb{R} \times X^\wedge) \hookrightarrow H^s_{\text{loc}}(\mathbb{R} \times X^\wedge)$ give rise to the global weighted Sobolev spaces $H^{s,\gamma}_{\text{loc}}(\mathcal{W})$ and $H^{s,\gamma}_{\text{comp}}(\mathcal{W})$ on $\mathcal{W}$ as well as those with asymptotics $H^{s,\gamma}_{\text{as,loc}}(\mathcal{W})$ and $H^{s,\gamma}_{\text{as,comp}}(\mathcal{W})$.

In order to deal with asymptotics on manifolds, here on $\mathcal{W}$, it is necessary to put a so-called *shadow condition* on the asymptotic types "as". For discrete asymptotic types it means that from $(p, m, \Sigma) \in$ as it follows that $(p - j, m_j, \Sigma_j) \in$ as for all $j \in \mathbb{N}$, with $\Im p - j > -\gamma - l$, and certain $m_j \geq m$ and $\Sigma_j \supset \Sigma$. For continuous asymptotic types it says that $(\sigma - j) \cap \{z \in \mathbb{C}\colon -\gamma - l < \Im z < -\gamma\} \subset \sigma$ for all $j \in \mathbb{N}$, where σ is the carrier of "as".

By (1.3.1), the most natural scalar product on $\mathbb{R} \times X^\wedge$ is that of the space $H^{0,-\frac{n+1}{2}}(\mathbb{R} \times X^\wedge)$. By restricting the Lebesgue measure of $\mathbb{R}$ to Ω we also get a scalar product in $H^{0,-\frac{n+1}{2}}(\Omega \times X^\wedge)$. If we fix a Riemannian metric on E we obtain furthermore a Hilbert space $H^{0,-\frac{n+1}{2}}(\mathcal{W})$ with a fixed scalar product. On the other hand, we want to have $H^{0,0}(\mathcal{W})$ as a reference space for the formal adjoint, etc., for the shift in the weight exponent by $(n+1)/2$ no longer works for cuspidal edges. Were $\mathcal{W}$ compact, it would not cause any confusion. For a non-compact $\mathcal{W}$, merely the space $H^{0,0}_{\mathrm{loc}}(\mathcal{W})$ is invariantly defined. However, we have distinguished coordinates (t,p) near the corners where $\mathcal{W}$ ceases to be locally compact. This enables one to introduce a space $H^{0,0}(\mathcal{W})$ with a fixed Hilbert structure.

Let V and $\tilde{V}$ be Banach spaces and $(\kappa_\lambda)_{\lambda \in \mathbb{R}_+}$ and $(\tilde{\kappa}_\lambda)_{\lambda \in \mathbb{R}_+}$ fixed group actions on V and $\tilde{V}$, respectively.

Definition 1.3.3 *For an open set $\Omega \subset \mathbb{R}^Q$ and $m \in \mathbb{R}$, we denote by $\mathcal{S}^m(\Omega \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ the set of all C^∞ functions $a(t,\tau)$ on $\Omega \times \mathbb{R}$ with values in $\mathcal{L}(V,\tilde{V})$, such that*

$$\|\tilde{\kappa}^{-1}_{\langle\tau\rangle} \left(D^\alpha_t D^\beta_\tau a(t,\tau)\right) \kappa_{\langle\tau\rangle}\|_{\mathcal{L}(V,\tilde{V})} \le c_{K,\alpha,\beta} \, \langle\tau\rangle^{m-|\beta|}$$

for all $(t,\tau) \in K \times \mathbb{R}$ and $\alpha \in \mathbb{Z}^Q_+$, $\beta \in \mathbb{Z}_+$, where K is any compact subset of Ω.

The only choices of Ω occurring here are $\Omega \subset \mathbb{R}$ or $\Omega \times \Omega \subset \mathbb{R}^2$. In the latter case we write (t,t') for the variables in $\Omega \times \Omega$.

For a recent account of the theory of "twisted" pseudodifferential operators with operator-valued symbols we refer the reader to [Sch98, 1.3].

Suppose $\tilde{V}$ is a Fréchet space written as projective limit of Banach spaces $\tilde{V}^\nu$, $\nu \in \mathbb{N}$, with continuous embeddings $\tilde{V}^{\nu+1} \hookrightarrow \tilde{V}^\nu$ and a strongly continuous action $(\tilde{\kappa}_\lambda)_{\lambda \in \mathbb{R}_+}$ on $\tilde{V}_1$ which restricts to a strongly continuous action on every V^ν, $\nu = 2,3,\dots$. We then set $\mathcal{S}^m(\Omega \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ to be the projective limit of the sequence $\mathcal{S}^m(\Omega \times \mathbb{R}, \mathcal{L}(V,\tilde{V}^\nu))$, $\nu \in \mathbb{N}$. Analogous notation is used for subspaces of classical symbols, indicated by "cl".

More precisely, $\mathcal{S}^m_{\mathrm{cl}}(\Omega \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ is the subspace of $\mathcal{S}^m(\Omega \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ formed by the symbols a that possess asymptotic expansions $a \sim \chi(\tau)\sum_{j=0}^\infty a_j$ with $\chi(\tau)$ an excision function and $a_j \in C^\infty_{\mathrm{loc}}(\Omega \times (\mathbb{R} \setminus \{0\}), \mathcal{L}(V,\tilde{V}))$ homogeneous of degree $m-j$ with respect to the group actions in V and $\tilde{V}$ in the sense that

$$a_j(t,\lambda\tau) = \lambda^{m-j}\, \tilde{\kappa}_\lambda \, a_j(t,\tau)\, \kappa^{-1}_\lambda, \quad \lambda > 0,$$

for all $t \in \Omega$ and $\tau \in \mathbb{R} \setminus \{0\}$.

Any symbol $a \in \mathcal{S}^m((\Omega \times \Omega) \times \mathbb{R}, \mathcal{L}(V, \tilde{V}))$ induces a canonical operator on Ω by

$$\mathrm{op}(a)u\,(t) = \frac{1}{2\pi} \iint e^{i(t-t')\tau}\, a(t, t', \tau)\, u(t')\, dt' d\tau,$$

first regarded as a continuous mapping $C^\infty_{\mathrm{comp}}(\Omega, V) \to C^\infty_{\mathrm{loc}}(\Omega, \tilde{V})$.

Denote by $\Psi^m(\Omega; V, \tilde{V})$ the set of all operators $\mathrm{op}(a)$ with arbitrary double symbols $a \in \mathcal{S}^m((\Omega \times \Omega) \times \mathbb{R}, \mathcal{L}(V, \tilde{V}))$, and by $\Psi^m_{\mathrm{cl}}(\Omega; V, \tilde{V})$ the subspace of classical operators. Then, $\Psi^{-\infty}(\Omega; V, \tilde{V})$ can be identified, by the Schwartz Kernel Theorem, with $C^\infty_{\mathrm{loc}}(\Omega \times \Omega, \mathcal{L}(V, \tilde{V}))$. For $a \in \mathcal{S}^m((\Omega \times \Omega) \times \mathbb{R}, \mathcal{L}(V, \tilde{V}))$, we set

$$\sigma^m_{\mathrm{edge}}(a)(t, \tau) = a_0(t, t, \tau), \quad (t, \tau) \in T^*\Omega \setminus \{0\}, \tag{1.3.4}$$

and define the *principal homogeneous edge symbol* of an operator $A = \mathrm{op}(a)$ by $\sigma^m_{\mathrm{edge}}(A) = \sigma^m_{\mathrm{edge}}(a)$.

The kernels of operators $A = \mathrm{op}(a)$ live in the space of distributions on $\Omega \times \Omega$ with values in $\mathcal{L}(V, \tilde{V})$. Then, we may talk about the operators properly supported with respect to the (t, t')-variables. Every $A \in \Psi^m(\Omega; V, \tilde{V})$ extends to continuous operators $H^s_{\mathrm{comp}}(\Omega, \pi^* V) \to H^{s-m}_{\mathrm{loc}}(\Omega, \pi^* \tilde{V})$ for all $s \in \mathbb{R}$. We may write "comp" "loc" in both the domain and target space if A is properly supported.

In the sequel we use the letters W and $\tilde{W}$ to designate arbitrary C^∞ vector bundles over the edge E. When restricted to Ω, they are trivial, i.e.,

$$\begin{aligned} W &= \Omega \times \mathbb{C}^N, \\ \tilde{W} &= \Omega \times \mathbb{C}^{\tilde{N}}. \end{aligned}$$

Definition 1.3.4 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $m, \gamma \in \mathbb{R}$ and $\mathcal{I} = (-l, 0]$, $0 < l \le \infty$. Then, $\Psi^{-\infty}_G(\Omega \times X^\wedge; W, \tilde{W}; w)$ stands for the space of all operators*

$$\mathcal{G} \in \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma}_{\mathrm{comp}}(\Omega \times X^\wedge) \oplus H^s_{\mathrm{comp}}(\Omega, W), H^{\infty,\gamma-m}_{\mathrm{loc}}(\Omega \times X^\wedge) \oplus H^\infty_{\mathrm{loc}}(\Omega, \tilde{W}))$$

that map as

$$\begin{aligned} \mathcal{G} &: H^{s,\gamma}_{\mathrm{comp}}(\Omega \times X^\wedge) \oplus H^s_{\mathrm{comp}}(\Omega, W) \to H^{\infty,\gamma-m}_{\mathrm{as,loc}}(\Omega \times X^\wedge) \oplus H^\infty_{\mathrm{loc}}(\Omega, \tilde{W}), \\ \mathcal{G}^* &: H^{s,m-\gamma}_{\mathrm{comp}}(\Omega \times X^\wedge) \oplus H^s_{\mathrm{comp}}(\Omega, \tilde{W}) \to H^{\infty,-\gamma}_{\tilde{\mathrm{as}},\mathrm{loc}}(\Omega \times X^\wedge) \oplus H^\infty_{\mathrm{loc}}(\Omega, W) \end{aligned}$$

for all $s \in \mathbb{R}$ and some asymptotic types "as" *and* "$\tilde{\mathrm{as}}$" *related to weight data $(\gamma - m, \mathcal{I})$ and $(-\gamma, \mathcal{I})$, respectively.*

Note that $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$ with respect to a fixed scalar product in $H^{0,0}(\Omega \times X^\wedge) \oplus H^0(\Omega)$.

The operators of $\Psi^{-\infty}_G(\Omega \times X^\wedge; W, \tilde{W}; w)$ are called smoothing *Green operators* on the wedge $\Omega \times X^\wedge$. In contrast to the cone case, Green operators on the wedge are specified by their action along the edge, here along Ω. While being smoothing away from the edge, they may bear a finite order along Ω. In order to describe this new feature on the symbol level, we introduce Green edge symbols.

Definition 1.3.5 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $\mathcal{I} = (-l, 0]$, $0 < l \leq \infty$. Then, $\mathcal{S}^\mu_G((\Omega \times \Omega) \times \mathbb{R}; W, \tilde{W}; w)$ denotes the space of all symbols*

$$\mathfrak{g} \in \bigcap_{s \in \mathbb{R}} \mathcal{S}^\mu_{\mathrm{cl}}((\Omega \times \Omega) \times \mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^N, H^{\infty,\gamma-m}(X^\wedge) \oplus \mathbb{C}^{\tilde{N}}))$$

such that

$$\begin{array}{rcl} \mathfrak{g} & \in & \bigcap_{s \in \mathbb{R}} \mathcal{S}^\mu_{\mathrm{cl}}((\Omega \times \Omega) \times \mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^N, \mathcal{S}^{\gamma-m}_{\mathrm{as}}(X^\wedge) \oplus \mathbb{C}^{\tilde{N}})), \\ \mathfrak{g}^* & \in & \bigcap_{s \in \mathbb{R}} \mathcal{S}^\mu_{\mathrm{cl}}((\Omega \times \Omega) \times \mathbb{R}, \mathcal{L}(H^{s,m-\gamma}(X^\wedge) \oplus \mathbb{C}^{\tilde{N}}, \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge) \oplus \mathbb{C}^N)) \end{array}$$

*for certain asymptotic types "*as*" and "*$\tilde{\text{as}}$*" related to weight data $(\gamma - m, \mathcal{I})$ and $(-\gamma, \mathcal{I})$, respectively.*

Here are some comments. By $\mathfrak{g}^*$ we mean the pointwise formal adjoint of $\mathfrak{g}$ in the sense that

$$(\mathfrak{g}u, v)_{H^{0,0}(X^\wedge) \oplus \mathbb{C}^{\tilde{N}}} = (u, \mathfrak{g}^* v)_{H^{0,0}(X^\wedge) \oplus \mathbb{C}^N}$$

for all $u \in C^\infty_{\mathrm{comp}}(X^\wedge) \oplus \mathbb{C}^N$ and $v \in C^\infty_{\mathrm{comp}}(X^\wedge) \oplus \mathbb{C}^{\tilde{N}}$. We always assume that our groups act as the identity on the finite-dimensional complements, i.e., they are of the form $\kappa_\lambda \oplus \mathrm{Id}_{\mathbb{C}^N}$ and $\kappa_\lambda \oplus \mathrm{Id}_{\mathbb{C}^{\tilde{N}}}$, respectively, for any $\lambda \in \mathbb{R}_+$. Write $\Psi^\mu_G(\Omega \times X^\wedge; W, \tilde{W}; w)$ for the space of all operators of the form $\mathrm{op}(\mathfrak{g}) + \mathcal{G}$, where

$$\begin{array}{rcl} \mathfrak{g} & \in & \mathcal{S}^\mu_G((\Omega \times \Omega) \times \mathbb{R}; W, \tilde{W}; w), \\ \mathcal{G} & \in & \Psi^{-\infty}_G(\Omega \times X^\wedge; W, \tilde{W}; w). \end{array}$$

The entries of the upper left corners of the matrices $\mathfrak{g}$ and $\mathrm{op}(\mathfrak{g}) + \mathcal{G}$ are of basic interest in our theory. They form the spaces $\mathcal{S}^\mu_G((\Omega \times \Omega) \times \mathbb{R}; w)$ and $\Psi^\mu_G(\Omega \times X^\wedge; w)$, respectively.

The operators in $\Psi^\mu_G(\Omega \times X^\wedge; W, \tilde{W}; w)$ are said to be *Green operators* of order μ on the wedge $\Omega \times X^\wedge$. These are matrices

$$\mathcal{G} = \begin{pmatrix} G & P \\ T & B \end{pmatrix} : \begin{array}{c} H^s_{\mathrm{comp}}(\Omega, \pi^* H^{s,\gamma}(X^\wedge)) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega, W) \end{array} \to \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(\Omega, \pi^* H^{s-\mu,\gamma-m}_{\mathrm{as}}(X^\wedge)) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{W}) \end{array}$$

acting continuously for all $s \in \mathbb{R}$, with "as" an asymptotic type related to $w = (\gamma - m, \mathcal{I})$. The entries T and P are called *trace* and *potential* operators, respectively. Furthermore, we have $B \in \Psi^{\mu}_{\mathrm{cl}}(\Omega; W, \tilde{W})$, an $(\tilde{N} \times N)$-matrix with entries in $\Psi^{\mu}_{\mathrm{cl}}(\Omega)$.

According to the above general notation, every $\mathcal{G} \in \Psi^{\mu}_{G}(\Omega \times X^{\wedge}; W, \tilde{W}; w)$ has a principal homogeneous edge symbol of order μ which is a C^∞ operator family

$$\sigma^{\mu}_{\mathrm{edge}}(\mathcal{G})\,(t,\tau) : \begin{array}{c} H^{s,\gamma}(X^{\wedge}) \\ \oplus \\ \mathbb{C}^N \end{array} \to \begin{array}{c} H^{s-\mu,\gamma-m}(X^{\wedge}) \\ \oplus \\ \mathbb{C}^{\tilde{N}} \end{array} \tag{1.3.5}$$

living away from the zero section of $T^*\Omega$ and homogeneous of order μ in the sense that

$$\sigma^{\mu}_{\mathrm{edge}}(\mathcal{G})\,(t,\lambda\tau) = \lambda^{\mu} \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma^{\mu}_{\mathrm{edge}}(\mathcal{G})\,(t,\tau) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

for all $\lambda \in \mathbb{R}_+$. Clearly we have a more precise information on the range in (1.3.5) but it is not used in this context.

Now let us introduce another sort of classical operator-valued symbols, namely *smoothing Mellin symbols*

$$m(t,\tau) = \varphi_0(r\langle\tau\rangle)\, r^{-\mu} \sum_{j=0}^{l-1} r^j \sum_{\alpha \le j} \Big(\mathrm{op}_{\mathcal{M},\gamma^{(1)}_{j,\alpha}}(m^{(1)}_{j,\alpha}) + \mathrm{op}_{\mathcal{M},\gamma^{(2)}_{j,\alpha}}(m^{(2)}_{j,\alpha})\Big)\tau^{\alpha}\, \psi_0(r\langle\tau\rangle) \tag{1.3.6}$$

where $m^{(\iota)}_{j,\alpha}(t,z)$ is a C^∞ function of $t \in \Omega$ with values in $\mathcal{M}_{T^{(\iota)}_{j,\alpha}}(\mathbb{C}, \Psi^{-\infty}(X))$ and

$$\begin{array}{rcl} \gamma - (m-\mu) - j & \le & \gamma^{(\iota)}_{j,\alpha} \; \le \; \gamma, \\ \pi_{\mathbb{C}} T^{(\iota)}_{j,\alpha} \cap \Gamma_{-\gamma^{(\iota)}_{j,\alpha}} & = & \emptyset, \end{array}$$

for $\iota = 1, 2$. Moreover, φ_0 and ψ_0 are arbitrary cut-off functions on the semi-axis. We then have $m \in \mathcal{S}^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^{\wedge}), H^{\infty,\gamma-m}(X^{\wedge})))$ for all $s \in \mathbb{R}$. The principal edge symbol of m is a family in $\mathcal{L}(H^{s,\gamma}(X^{\wedge}), H^{s-\mu,\gamma-m}(X^{\wedge}))$ given by

$$\sigma^{\mu}_{\mathrm{edge}}(m)(t,\tau) = \varphi_0(r|\tau|)\, r^{-\mu} \sum_{j=0}^{l-1} (r\tau)^j \Big(\mathrm{op}_{\mathcal{M},\gamma^{(1)}_{j,j}}(m^{(1)}_{j,j}) + \mathrm{op}_{\mathcal{M},\gamma^{(2)}_{j,j}}(m^{(2)}_{j,j})\Big)\psi_0(r|\tau|)$$

for $(t,\tau) \in T^*\Omega \setminus \{0\}$.

Definition 1.3.6 *Let* $w = (\gamma, \gamma - m, \mathcal{I})$, *where* $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ *and* $\mathcal{I} = (-l, 0]$, l *being a positive integer. Then,* $\Psi^{\mu}_{M+G}(\Omega \times X^{\wedge}; w)$ *is the space of all operators* $\mathrm{op}(m) + G$, *where* m *is of the type* (1.3.6) *and* $G \in \Psi^{\mu}_{G}(\Omega \times X^{\wedge}; w)$.

As further and main ingredient we now add pseudodifferential operators in the interior of $\Omega \times X^\wedge$. We start with symbols on $\Omega \times O^\wedge$ where O is a coordinate neighbourhood on X identified with an open subset U of $\mathbb{R}^n$.

Denote by $\mathcal{S}^\mu_{\mathrm{cl}}({}^bT^*(\Omega \times U^\wedge))$ the space of all symbols a over $\Omega \times U^\wedge$ of the form

$$a(t,r,x,\tau,\varrho,\xi) = \tilde{a}(t,r,x,r\tau,r\varrho,\xi)$$

where $\tilde{a} \in \mathcal{S}^\mu_{\mathrm{cl}}((\Omega \times U^\wedge) \times \mathbb{R}^{2+n})$ is C^∞ up to $r = 0$. We will tacitly assume that our $\tilde{a}(t,r,x,\tilde{\tau},\tilde{\varrho},\xi)$ vanish for r large enough, for the symbols are cut off at the end. The symbols of $\mathcal{S}^\mu_{\mathrm{cl}}({}^bT^*(\Omega \times U^\wedge))$ are said to be *edge-degenerate*, cf. [Sch98, 3.1.2].

Given any edge-degenerate symbol $a(t,r,x,\tau,\varrho,\xi)$, we introduce the *compressed* principal homogeneous symbol of the operator $A = r^{-\mu}\,\mathrm{op}(a)$ over $\Omega \times U^\wedge$ by

$${}^b\sigma^\mu(A)(t,r,x,\tilde{\tau},\tilde{\varrho},\xi) = \tilde{a}_0(t,r,x,\tilde{\tau},\tilde{\varrho},\xi), \tag{1.3.7}$$

where $\tilde{a}_0$ is the principal homogeneous part of $\tilde{a}$ of order μ. The symbol ${}^b\sigma^\mu(A)$ is defined up to $r = 0$, and the ellipticity with respect to ${}^b\sigma^\mu(A)$ just amounts to that with respect to the symbol $\sigma^\mu(A)$ away from $r = 0$ along with the condition at $r = 0$, as above.

Theorem 1.3.7 *Let a be an edge-degenerate symbol of order μ, elliptic with respect to the compressed symbol up to $r = 0$. Then there exists an edge-degenerate symbol p of order $-\mu$, such that*

$$(r^\mu\, p) \circ_{(t,r,x)} \left(r^{-\mu}\, a\right) = 1 \mod \mathcal{S}^{-\infty}((\Omega \times U^\wedge) \times \mathbb{R}^{2+n}),$$

where "$\circ$" means the Leibniz product of the symbols, taken with respect to the indicated variables.

The same is true for the multiplication in reverse order or for the Leibniz product only with respect to (r,x).

Similarly to the calculus on a cone it is adequate to formulate operators globally along X and to consider the corresponding (t,r,τ,ϱ)-dependent operator families.

Let us fix a finite open covering of X by coordinate neighborhoods $(O_\nu)_{\nu\in\mathcal{N}}$ together with a system of charts $\kappa_\nu\colon O_\nu \to U_\nu$, U_ν being an open subset of $\mathbb{R}^n$, a subordinate partition of unity $(\varphi_\nu)_{\nu\in\mathcal{N}}$ on X, and functions $\psi_\nu \in C^\infty_{\mathrm{comp}}(U_\nu)$ satisfying $\varphi_\nu\psi_\nu = \varphi_\nu$, for every $\nu \in \mathcal{N}$. Given an arbitrary system of edge-degenerate symbols $a_\nu \in \mathcal{S}^\mu_{\mathrm{cl}}((\Omega \times U_\nu{}^\wedge) \times \mathbb{R}^{2+n})$, $\nu \in \mathcal{N}$, with the associated symbols $\tilde{a}_\nu(t,r,x,\tilde{\tau},\tilde{\varrho},\xi)$ which are C^∞ up to $r = 0$, we can form an operator family

$$\tilde{a}(t,r,\tilde{\tau},\tilde{\varrho}) = \sum_{\nu\in\mathcal{N}} \varphi_\nu \left(\left(\kappa_\nu^{-1}\right)_\sharp \mathrm{op}_{\mathcal{F}_x}\left(\tilde{a}_\nu\right)(t,r,\tilde{\tau},\tilde{\varrho})\right) \psi_\nu$$

where $\mathrm{op}_{\mathcal{F}_x}$ denotes the pseudodifferential action in $U_\nu \subset \mathbb{R}^n$ with respect to x. Then $\tilde{a}(t,r,\tilde{\tau},\tilde{\varrho}) \in C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}}))$. For obtaining a pseudodifferential operator on $\Omega \times X^\wedge$ we have to carry out the action also with respect to $t \in \Omega$ and $r \in \mathbb{R}_+$. Concerning the action in r we want to have a control up to $r = 0$. To this end we formulate for the edge-degenerate symbols an operator convention that allows us to apply the pseudodifferential calculus along $\mathbb{R}_+$ with respect to the Mellin transform (it is needed only for a neighborhood of $r = 0$). In other words we generalise some constructions from the cone calculus in a suitable parameter-dependent form.

The proof of the following result can be obtained by a kernel cut-off argument with respect to the parameter z.

Theorem 1.3.8 *Suppose $\gamma \in \mathbb{R}$. For every $\sigma(\tau,z) \in \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}\times\Gamma_{-\gamma})$ there exists an $h(\tau,z) \in \mathcal{M}(\mathbb{C},\Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}_\tau))$ such that*

$$h(\tau,z)\,|_{\Gamma_{-\gamma}} - \sigma(\tau,z) \in \Psi^{-\infty}(X;\mathbb{R}_\tau \times \Gamma_{-\gamma}).$$

It is worth noting that if $h_1(\tau,z), h_2(\tau,z) \in \mathcal{M}(\mathbb{C},\Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}_\tau))$ and there exists a $\gamma \in \mathbb{R}$ such that $(h_1(z,\tau) - h_2(z,\tau))\,|_{\Gamma_{-\gamma}} \in \Psi^{-\infty}(X;\mathbb{R}_\tau \times \Gamma_{-\gamma})$, then actually $h_1(\tau,z) - h_2(\tau,z) \in \mathcal{M}(\mathbb{C},\Psi^{-\infty}(X;\mathbb{R}_\tau))$. Since the kernel cut-off acts only on covariables, we have analogous results for the case when $\sigma(t,r,\tau,z)$ depends on $(t,r) \in \Omega \times \mathbb{R}_+$, up to $r = 0$. Then the other occurring objects also depend on $(t,r) \in \Omega \times \mathbb{R}_+$ in a suitable way.

Theorem 1.3.9 *For every $\tilde{a}(t,r,\tilde{\tau},\tilde{\varrho}) \in C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}}))$, there exists an $\tilde{h}(t,r,\tilde{\tau},z) \in C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C},\Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}_{\tilde{\tau}})))$ such that*

$$\mathrm{op}_{\mathcal{F}_r}(a)(t,\tau) = \mathrm{op}_{\mathcal{M},\gamma}(h)(t,\tau) \mod C^\infty_{\mathrm{loc}}(\Omega,\Psi^{-\infty}(X^\wedge;\mathbb{R}_\tau)),$$

where $a(t,r,\tau,\varrho) := \tilde{a}(t,r,r\tau,r\varrho)$ and $h(t,r,\tau,z) := \tilde{h}(t,r,r\tau,z)$. Moreover, such an $\tilde{h}(t,r,\tilde{\tau},z)$ is unique modulo $C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C},\Psi^{-\infty}(X;\mathbb{R}_{\tilde{\tau}})))$.

Proof. The proof of Theorem 1.3.9 is rather technical, however, the idea is easy. In a first step we set

$$\tilde{\sigma}_0(t,r,\tilde{\tau},\tilde{\varrho}) = \tilde{a}(t,r,\tilde{\tau},\tilde{\varrho}).$$

It is easy to check that there is an $\tilde{a}_1(t,r,\tilde{\tau},\tilde{\varrho}) \in C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \Psi^{\mu-1}_{\mathrm{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}}))$ such that

$$\mathrm{op}_{\mathcal{F}_r}(a)(t,\tau) = \mathrm{op}_{\mathcal{M}}(\sigma_0)(t,\tau) + \mathrm{op}_{\mathcal{F}_r}(a_1)(t,\tau) \mod C^\infty_{\mathrm{loc}}(\Omega,\Psi^{-\infty}(X^\wedge;\mathbb{R}_\tau)),$$

where

$$\begin{aligned} \sigma_0(t,r,\tau,z) &= \tilde{\sigma}_0(t,r,r\tau,z), \\ a_1(t,r,\tau,\varrho) &= \tilde{a}_1(t,r,r\eta,r\varrho). \end{aligned}$$

This allows us to start an iteration which yields a sequence

$$\tilde{\sigma}_j(t,r,\tilde{\tau},z) \in C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \Psi^{\mu-j}_{\mathrm{cl}}(X;\mathbb{R}_{\tilde{\tau}} \times \mathbb{R})), \quad j = 0,1,\ldots,$$

with $\mu - j \searrow -\infty$ as $j \to \infty$. The asymptotic sum

$$\tilde{\sigma}(t,r,\tilde{\tau},z) \sim \sum_{j=0}^{\infty} \tilde{\sigma}_j(t,r,\tilde{\tau},z)$$

in $C^\infty_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \Psi^{\mu}_{\mathrm{cl}}(X;\mathbb{R}_{\tilde{\tau}} \times \mathbb{R}))$ then gives a $\sigma(t,r,\tau,z) = \tilde{\sigma}(t,r,r\eta,z)$ such that

$$\mathrm{op}_{\mathcal{F}_r}(a)(t,\tau) = \mathrm{op}_{\mathcal{M}}(\sigma)(t,\tau) \mod C^\infty_{\mathrm{loc}}(\Omega, \Psi^{-\infty}(X^\wedge;\mathbb{R}_\tau)).$$

Finally, applying Theorem 1.3.8 in the (t,r)-dependent form and using the fact that $\mathrm{op}_{\mathcal{M},\gamma}(\sigma) = \mathrm{op}_{\mathcal{M}}(\sigma)$ for all $\gamma \in \mathbb{R}$, in the case of Mellin symbols which are holomorphic in z, we obtain the desired assertion.

□

This assertion can be viewed as a Mellin operator convention which expresses the Fourier pseudodifferential action in $r \in \mathbb{R}_+$ in terms of Mellin operators, modulo controlled remainders. Since there is such a remainder, the correspondence is not canonical, however the final operator algebra will be independent of the concrete choice, for there will be involved also $\Psi^{\mu}_{M+G}(\Omega \times X^\wedge; w)$ terms.

Pick a partition of unity $(\varphi_0, \varphi_\infty)$ on the semiaxis, $\varphi_0(r)$ being a cut-off function. Choose a system of C^∞ functions (ψ_0, ψ_∞) on $\mathbb{R}_+$ which covers $(\varphi_0, \varphi_\infty)$, i.e., $\psi_0(r)$ vanishes for large r, $\psi_\infty(r)$ vanishes near $r = 0$ and $\varphi_\nu \psi_\nu \equiv \varphi_\nu$, for $\nu = 0, \infty$. Combining Theorem 1.3.9 with the pseudolocality of pseudodifferential operators in parameter-dependent form, we obtain immediately

$$r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(a)(t,\tau) = a_0(t,\tau) + a_\infty(t,\tau) \mod C^\infty_{\mathrm{loc}}(\Omega, \Psi^{-\infty}(X^\wedge;\mathbb{R}_\tau)), \quad (1.3.8)$$

for every $\gamma \in \mathbb{R}$, where

$$\begin{aligned} a_0(t,\tau) &= \varphi_0(r\langle\tau\rangle)\, r^{-\mu}\, \mathrm{op}_{\mathcal{M},\gamma}(h)(t,\tau)\, \psi_0(r\langle\tau\rangle), \\ a_\infty(t,\tau) &= \varphi_\infty(r\langle\tau\rangle)\, r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(a)(t,\tau)\, \psi_\infty(r\langle\tau\rangle). \end{aligned}$$

Theorem 1.3.10 *Set $\sigma(t,\tau) = \varphi(r)\,(a_0(t,\tau) + a_\infty(t,\tau))\,\psi(r)$ where a and h are the operator-valued symbols of Theorem 1.3.9, and φ, ψ are arbitrary cut-off functions. Then $\sigma(t,\tau) \in \mathcal{S}^\mu(\Omega \times \mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^\wedge), H^{s-\mu,\gamma-\mu}(X^\wedge)))$, for each $s \in \mathbb{R}$. Moreover,*

$$\mathrm{op}_{\mathcal{F}_y}(\sigma) = \mathrm{op}_{\mathcal{F}_y}\left(\varphi\, r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(a)(t,\tau)\, \psi\right) \mod \Psi^{-\infty}(\Omega \times X^\wedge). \quad (1.3.9)$$

The proof of the first part of Theorem 1.3.10 is based on estimates of the norms of pseudodifferential operators by the symbols. The relation (1.3.9) is a consequence of (1.3.8).

For the symbol σ of Theorem 1.3.10, we set $\sigma^{\mu}_{\mathrm{edge}}(\sigma) = \sigma^{\mu}_{\mathrm{edge}}(a_0) + \sigma^{\mu}_{\mathrm{edge}}(a_\infty)$ where

$$\begin{aligned}
\sigma^{\mu}_{\mathrm{edge}}(a_0)(t,\tau) &= \varphi_0(r|\tau|)\, r^{-\mu}\, \mathrm{op}_{\mathcal{M},\gamma}(\tilde{h}(t,0,r\tau,z))\, \psi_0(r|\tau|),\\
\sigma^{\mu}_{\mathrm{edge}}(a_\infty)(t,\tau) &= \varphi_\infty(r|\tau|)\, r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(\tilde{a}(t,0,r\tau,r\varrho))\, \psi_\infty(r|\tau|),
\end{aligned}$$

for $(t,\tau) \in T^*\Omega \setminus \{0\}$. Then $\sigma^{\mu}_{\mathrm{edge}}(\sigma)(t,\tau)$ is a family of continuous operators $H^{s,\gamma}(X^\wedge) \to H^{s-\mu,\gamma-\mu}(X^\wedge)$, for any $s,\gamma \in \mathbb{R}$. It is homogeneous of order μ in the sense that

$$\sigma^{\mu}_{\mathrm{edge}}(\sigma)(t,\lambda\tau) = \lambda^{\mu}\, \kappa_\lambda\, \sigma^{\mu}_{\mathrm{edge}}(\sigma)(t,\tau)\kappa_\lambda^{-1}, \quad \lambda > 0,$$

for all $(t,\tau) \in T^*\Omega \setminus \{0\}$.

As defined in Theorem 1.3.10, $\sigma(t,\tau)$ is a parameter-dependent family of cone pseudodifferential operators on the infinite 'stretched' cone $X^\wedge$. The relation (1.3.9) actually shows that $\mathrm{op}(\sigma) \in \Psi^{\mu}_{\mathrm{cl}}(\Omega \times X^\wedge)$.

Theorem 1.3.11 *Let $\sigma(t,\tau)$ be given by Theorem 1.3.10. Then* $\mathrm{op}(\sigma)$ *extends to a continuous operator*

$$\mathrm{op}(\sigma):\ H^{s}_{\mathrm{comp}}(\Omega, \pi^* H^{s,\gamma}(X^\wedge)) \to H^{s-\mu}_{\mathrm{loc}}(\Omega, \pi^* H^{s-\mu,\gamma-\mu}(X^\wedge)),$$

for each $s,\gamma \in \mathbb{R}$.

Having disposed of this preliminary step, we are in a position to organise our local wedge operator algebra.

Definition 1.3.12 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $\mathcal{I} = (-l, 0]$, l being a positive integer. Then, $\Psi^{\mu}(\Omega \times X^\wedge; w)$ is the space of all operators*

$$A = A_0 + A_\infty + M + G$$

where $A_\nu = \mathrm{op}(a_\nu)$, $\nu = 0,\infty$, for arbitrary symbol $\tilde{a} \in C^{\infty}_{\mathrm{loc}}(\Omega \times \bar{\mathbb{R}}_+, \Psi^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^2)$, and $M + G \in \Psi^{\mu}_{M+G}(\Omega \times X^\wedge; w)$.

Finally, denote by $\Psi^{\mu}(\Omega \times X^\wedge; W, \tilde{W}; w)$ the space of all operators of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}, \tag{1.3.10}$$

with $A \in \Psi^{\mu}(\Omega \times X^\wedge; w)$ and $\mathcal{G} \in \Psi^{\mu}_{G}(\Omega \times X^\wedge; W, \tilde{W}; w)$.

The analogous operator classes for $\mathcal{I} = (-\infty, 0]$ are introduced to be the intersections of those for $\mathcal{I} = (-l, 0]$, over $l \in \mathbb{N}$.

Remark 1.3.13 *Every $\mathcal{A} \in \Psi^\mu(\Omega \times X^\wedge; W, \tilde{W}; w)$ can be regarded as being properly supported with respect to the variables $y \in \Omega$, modulo operators in $\Psi_G^{-\infty}(\Omega \times X^\wedge; W, \tilde{W}; w)$.*

The proof of this assertion is not obvious, as being a Green operator of order $-\infty$ requires not only being *smoothing* along the edge but also being *flattening* in the fibres over the edge. The idea is to rewrite the operators as those with symbols replaced by their derivatives of sufficiently large order in the covariable τ. This is possible provided that the kernel has no singularities on the diagonal of $\Omega \times \Omega$. It remains to use the general property of the operator algebra $\Psi^\mu(\Omega \times X^\wedge; W, \tilde{W}; w)$ that the differentiation of symbols in the covariable τ improves their mapping properties in the fibres. For a proof, see Proposition 3.4.36 in [Sch98].

Theorem 1.3.14 *Any $\mathcal{A} \in \Psi^\mu(\Omega \times X^\wedge; W, \tilde{W}; w)$ induces continuous operators*

$$\mathcal{A} \ : \ \begin{array}{c} H^s_{\mathrm{comp}}(\Omega, \pi^* H^{s,\gamma}(X^\wedge)) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega, W) \end{array} \to \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(\Omega, \pi^* H^{s-\mu,\gamma-m}(X^\wedge)) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{W}) \end{array},$$

$$\mathcal{A} \ : \ \begin{array}{c} H^s_{\mathrm{comp}}(\Omega, \pi^* H^{s,\gamma}_{\mathrm{as}}(X^\wedge)) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega, W) \end{array} \to \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(\Omega, \pi^* H^{s-\mu,\gamma-m}_{\tilde{\mathrm{as}}}(X^\wedge)) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{W}) \end{array}$$

for any $s \in \mathbb{R}$ and asymptotic type "as" related to $(\gamma, \mathcal{I})$, with some resulting asymptotic type "ãs" related to $(\gamma - m, \mathcal{I})$.

We emphasise that "ãs" depends on "as" and $\mathcal{A}$, but not on s. If $\mathcal{A}$ is properly supported in $y \in \Omega$ then it preserves the spaces with the subscripts 'comp' and 'loc'.

Let us highlight the leading symbolic levels of the wedge calculus. From $\Psi^\mu(\Omega \times X^\wedge; w) \hookrightarrow \Psi^\mu_{\mathrm{cl}}(\Omega \times X^\wedge)$ we get the 'compressed' *principal interior symbol* of order μ, i.e.,

$${}^b\sigma^\mu(A) \in \mathcal{S}^\mu_{\mathrm{hg}}({}^bT^*(\Omega \times X^\wedge) \setminus \{0\})$$

given by ${}^b\sigma^\mu(A) = \sigma^\mu(\tilde{a})$ (cf. Definition 1.3.12), the subscript "hg" referring to *homogeneous* symbols. Moreover, we have the *principal edge symbol* of order μ, i.e.,

$$\sigma^\mu_{\mathrm{edge}}(A) \in \bigcap_{s \in \mathbb{R}} \mathcal{S}^\mu_{\mathrm{hg}}(T^*\Omega \setminus \{0\}, \mathcal{L}(H^{s,\gamma}(X^\wedge), H^{s-\mu,\gamma-m}(X^\wedge)))$$

defined by $\sigma^\mu_{\text{edge}}(A) = \sigma^\mu_{\text{edge}}(a_0) + \sigma^\mu_{\text{edge}}(a_\infty) + \sigma^\mu_{\text{edge}}(m) + \sigma^\mu_{\text{edge}}(g)$. For more general $\mathcal{A} \in \Psi^\mu(\Omega \times X^\wedge; W, \tilde{W}; w)$, we set

$$\begin{aligned} {}^b\sigma^\mu(\mathcal{A}) &= {}^b\sigma^\mu(A), \\ \sigma^\mu_{\text{edge}}(\mathcal{A}) &= \begin{pmatrix} \sigma^\mu_{\text{edge}}(A) & 0 \\ 0 & 0 \end{pmatrix} + \sigma^\mu_{\text{edge}}(\mathcal{G}), \end{aligned} \tag{1.3.11}$$

$\mathcal{A}$ being given by (1.3.10).

When localised close to $r = 0$, the spaces $\Psi^\mu(\Omega \times X^\wedge; W, \tilde{W}; w)$ survive under natural operations such as compositions, adjoints and push-forwards under diffeomorphisms $\Omega \to \tilde{\Omega}$. These are compatible with the symbolic levels, which is rather evident anyway. Hence it follows that we may globalise our pseudodifferential operators to C^∞ vector bundles W and $\tilde{W}$ over the entire edge E, thus arriving at local "algebras" $\Psi^\mu(E \times X^\wedge; W, \tilde{W}; w)$ near the boundary on $\mathcal{W}$.

In order to develop the calculus globally over the manifold $\mathcal{W}$ we first need the global negligible operators.

Definition 1.3.15 *Let $m, \gamma \in \mathbb{R}$ and $w = (\gamma, \gamma - m, \mathcal{I})$, where $\mathcal{I} = (-l, 0]$, $0 < l \leq \infty$. Then, $\Psi_G^{-\infty}(\mathcal{W}; W, \tilde{W}; w)$ denotes the space of all operators*

$$\mathcal{G} \in \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma}_{\text{comp}}(\mathcal{W}) \oplus H^s_{\text{comp}}(E, W), H^{\infty,\gamma-m}_{\text{loc}}(\mathcal{W}) \oplus H^\infty_{\text{loc}}(E, \tilde{W}))$$

which map as

$$\begin{aligned} \mathcal{G} &: H^{s,\gamma}_{\text{comp}}(\mathcal{W}) \oplus H^s_{\text{comp}}(E, W) \to H^{\infty,\gamma-m}_{\text{as,loc}}(\mathcal{W}) \oplus H^\infty_{\text{loc}}(E, \tilde{W}), \\ \mathcal{G}^* &: H^{s,m-\gamma}_{\text{comp}}(\mathcal{W}) \oplus H^s_{\text{comp}}(E, \tilde{W}) \to H^{\infty,-\gamma}_{\tilde{\text{as}},\text{loc}}(\mathcal{W}) \oplus H^\infty_{\text{loc}}(E, W) \end{aligned}$$

for all $s \in \mathbb{R}$ and certain asymptotic types "as" and "$\tilde{\text{as}}$" related to weight data $(\gamma - m, \mathcal{I})$ and $(-\gamma, \mathcal{I})$, respectively.

By $\mathcal{G}^*$ we mean the formal adjoint of $\mathcal{G}$ with respect to a fixed scalar product in $H^{0,0}(\mathcal{W}) \oplus H^0(E)$. The entries of the upper left corner of the matrix $\mathcal{G}$ form the space $\Psi_G^{-\infty}(\mathcal{W}; w)$.

We now suppose $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $w = (\gamma, \gamma - m, \mathcal{I})$, with $\mathcal{I} = (-l, 0]$, $l \in \mathbb{N}$. For vector bundles W, $\tilde{W}$ over E, the space $\Psi^\mu(\mathcal{W}; W, \tilde{W}; w)$ is defined to consist of all

$$\mathcal{A} \in \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma}_{\text{comp}}(\mathcal{W}) \oplus H^s_{\text{comp}}(E, W), H^{s-\mu,\gamma-m}_{\text{loc}}(\mathcal{W}) \oplus H^{s-\mu}_{\text{loc}}(E, \tilde{W}))$$

of the form

$$\mathcal{A} = \varphi_b \, \mathcal{A}_b \, \psi_b + \varphi_i \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} \psi_i + \mathcal{G} \tag{1.3.12}$$

where $\mathcal{A}_b \in \Psi^\mu(E \times X^\wedge; W, \tilde{W}; w)$, $A_i \in \Psi^\mu_{\mathrm{cl}}(\mathring{\mathcal{W}})$, and $\mathcal{G} \in \Psi_G^{-\infty}(\mathcal{W}; W, \tilde{W}; w)$. Moreover, (φ_b, φ_i) is a partition of unity on $\mathcal{W}$ and (ψ_b, ψ_i) a system of C^∞ functions covering (φ_b, φ_i), both $\varphi_b(r)$ and $\psi_b(r)$ being cut-off functions supported in $[0, 1)$.

Remark 1.3.16 *Every $\mathcal{A} \in \Psi^\mu(\mathcal{W}; W, \tilde{W}; w)$ can be written as $\mathcal{A} = \mathcal{A}' + \mathcal{G}$ where $\mathcal{A}' \in \Psi^\mu(\mathcal{W}; W, \tilde{W}; w)$ is properly supported with respect to the variables in E, and $\mathcal{G} \in \Psi_G^{-\infty}(\mathcal{W}; W, \tilde{W}; w)$.*

Theorem 1.3.14 has a global analogue for $\mathcal{A} \in \Psi^\mu(\mathcal{W}; W, \tilde{W}; w)$. Moreover, the local symbol levels (1.3.11) can easily be generalised to the corresponding global ones.

Theorem 1.3.17 *Let*

$$\begin{array}{lll} \mathcal{A}_1 & \in & \Psi^{\mu_1}(\mathcal{W}; W^1, W^2; w_1), \qquad w_1 = (\gamma, \gamma - m_1, \mathcal{I}), \\ \mathcal{A}_2 & \in & \Psi^{\mu_2}(\mathcal{W}; W^2, W^3; w_2), \qquad w_2 = (\gamma - m_1, \gamma - m_1 - m_2, \mathcal{I}), \end{array}$$

one of the $\mathcal{A}_1$ and $\mathcal{A}_2$ being properly supported. Then the composition $\mathcal{A}_2\mathcal{A}_1$ is well defined in $\Psi^{\mu_1+\mu_2}(\mathcal{W}; W^1, W^3; w_2 \circ w_1)$, with $w_2 \circ w_1 = (\gamma, \gamma - m_1 - m_2, \mathcal{I})$, and

$$\begin{array}{rcl} {}^b\sigma^{m_1+m_2}(\mathcal{A}_2\mathcal{A}_1) & = & {}^b\sigma^{m_2}(\mathcal{A}_2)\, {}^b\sigma^{m_1}(\mathcal{A}_1), \\ \sigma^{m_1+m_2}_{\mathrm{edge}}(\mathcal{A}_2\mathcal{A}_1) & = & \sigma^{m_2}_{\mathrm{edge}}(\mathcal{A}_2)\, \sigma^{m_1}_{\mathrm{edge}}(\mathcal{A}_1). \end{array}$$

The theory of the operator classes $\Psi^\mu(\mathcal{W}; W, \tilde{W}; w)$ has a straightforward generalisation to upper left corners A acting between distribution sections of C^∞ vector bundles V and $\tilde{V}$ over $\mathcal{W}$. For simplicity we restrict ourselves to the case where A are still scalar operators. But it may happen that non-trivial W and $\tilde{W}$ are actually necessary, as is known, e.g., in the particular case of boundary value problems.

Definition 1.3.18 *Let $w = (\gamma, \gamma - m, \mathcal{I})$ where $m, \gamma \in \mathbb{R}$ and $\mathcal{I} = (-l, 0]$, l being a positive integer or ∞. An operator $\mathcal{A} \in \Psi^m(\mathcal{W}; W, \tilde{W}; w)$ is said to be elliptic if*

1) ${}^b\sigma^m(\mathcal{A}) \neq 0$ *on* ${}^bT^*\mathcal{W} \setminus \{0\}$;
2) $\sigma^m_{\mathrm{edge}}(\mathcal{A})(t, \tau) \colon H^{s,\gamma}(X^\wedge) \oplus W_t \to H^{s-m,\gamma-m}(X^\wedge) \oplus \tilde{W}_t$ *is an isomorphism for all* $(t, \tau) \in T^*E \setminus \{0\}$ *and some* $s \in \mathbb{R}$.

Here, W_t and $\tilde{W}_t$ are the fibres of W and $\tilde{W}$ over $t \in E$, respectively. By the cone theory, the condition 2) for one particular $s = s_0$ implies the same for all $s \in \mathbb{R}$.

Recall that in our application $\mathcal{W}$ is not necessarily compact while E is $C_1 \setminus C_0$. Hence one might better speak about interior "edge ellipticity". The ellipticity up to non-compact "exits" of E is just the contents of the "corner ellipticity" in Section 3.3. If

$$\mathcal{A} = \begin{pmatrix} A & P \\ T & B \end{pmatrix}$$

is elliptic, then T, P and B are called *elliptic edge conditions* for A. The fibre dimensions of W and $\tilde{W}$ depend in general on the weight γ.

Let

$$\begin{array}{rcllrcl} \mathcal{A} & \in & \Psi^m(\mathcal{W}; W, \tilde{W}; w), & & w & = & (\gamma, \gamma - m, \mathcal{I}), \\ \mathcal{P} & \in & \Psi^{-m}(\mathcal{W}; \tilde{W}, W; w^{-1}), & & w^{-1} & = & (\gamma - m, \gamma, \mathcal{I}), \end{array}$$

$\mathcal{A}$ or $\mathcal{P}$ being properly supported. Then, $\mathcal{P}$ is said to be a *parametrix* of $\mathcal{A}$ if

$$\begin{array}{rcl} \mathcal{P}\mathcal{A} - 1 & \in & \Psi_G^{-\infty}(\mathcal{W}; W; w^{-1} \circ w), \\ \mathcal{A}\mathcal{P} - 1 & \in & \Psi_G^{-\infty}(\mathcal{W}; \tilde{W}; w \circ w^{-1}) \end{array}$$

where 1 stands for the identity operator in the corresponding classes.

Theorem 1.3.19 *Every elliptic edge problem* $\mathcal{A} \in \Psi^m(\mathcal{W}; W, \tilde{W}; w)$ *has a properly supported parametrix* $\mathcal{P} \in \Psi^{-m}(\mathcal{W}; \tilde{W}, W; w^{-1})$.

The following result is a straightforward consequence of the existence of a parametrix $\mathcal{P}$ within the "algebra", for any elliptic operator $\mathcal{A}$ of the form (1.3.12).

Corollary 1.3.20 *Let* $\mathcal{A}$ *be elliptic. Every* $u \in H^{-\infty,\gamma}_{\mathrm{comp}}(\mathcal{W}) \oplus H^{-\infty}_{\mathrm{comp}}(E, W)$ *satisfying* $\mathcal{A}u = f$, *with* $f \in H^{s-m,\gamma-m}_{\mathrm{loc}}(\mathcal{W}) \oplus H^{s-m}_{\mathrm{loc}}(E, \tilde{W})$, $s \in \mathbb{R}$, *actually belongs to* $H^{s,\gamma}_{\mathrm{comp}}(\mathcal{W}) \oplus H^{s}_{\mathrm{comp}}(E, W)$. *If* $f \in H^{s-m,\gamma-m}_{\mathrm{loc},\tilde{\mathrm{as}}}(\mathcal{W}) \oplus H^{s-m}_{\mathrm{loc}}(E, \tilde{W})$, *for some asymptotic type* "$\tilde{\mathrm{as}}$", *then* $u \in H^{s,\gamma}_{\mathrm{comp,as}}(\mathcal{W}) \oplus H^{s}_{\mathrm{comp}}(E, W)$, *for a resulting asymptotic type* "as" *depending on* $\mathcal{A}$ *and* "$\tilde{\mathrm{as}}$".

If $\mathcal{W}$ is compact, then the ellipticity of $\mathcal{A}$ is equivalent to the Fredholm property of $\mathcal{A}$ in weighted Sobolev spaces on $\mathcal{W}$, cf. [Dor98, 3.3].

1.4 Edge asymptotics close to corners

The previous section dealt with pseudodifferential operators on manifolds with non-compact edges, based on the Fourier transform in t along the edges. The corner pseudodifferential operators close to $t = 0$ require an analogue of this calculus in terms of the Mellin transform in $t \in \mathbb{R}_+$. This yields in particular the edge contribution to the corner asymptotics for $t \to 0$. It is indispensable to formulate explicitly this Mellin edge calculus, though it needs a lot of material.

However, we shall see that many things can be red off from the Fourier version under employing information from the cone theory. On the other hand, we get this way a new interesting operator algebra with a rich symbol structure.

The concrete distribution and operator classes will be given on $\mathbb{R}_+ \times X^\wedge$ where now a neighbourhood of $t = 0$ is of main interest. We write (t, r, x) for the corresponding splitting of coordinates. Similarly to the above wedge theory we first look at an abstract formulation.

We start with a Banach space V and a group action $(\kappa_\lambda)_{\lambda>0}$ on V, i.e., $\kappa_\lambda \in C(\mathbb{R}_+, \mathcal{L}_\sigma(V))$. As above, we employ the function $\tau \mapsto \langle\tau\rangle$ for the covariable $\tau \in \mathbb{R}$ of $t \in \mathbb{R}_+$. Since we are in the Mellin set-up, τ will at the same time be interpreted as the real part of $\zeta = \tau + iv$. If $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+, V)$ then $\mathcal{M}u(\zeta) = \mathcal{M}_{t\mapsto\zeta}u$ is an entire function of $\zeta \in \mathbb{C}$ with values in V. Hence it follows that $\mathcal{M}u(\zeta)$ may also be regarded as a function on the line $\Gamma_{-\delta}$, for each $\delta \in \mathbb{R}$.

Definition 1.4.1 *Given any $s, \delta \in \mathbb{R}$, the space $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*V)$ is defined to be the completion of $C^\infty_{\mathrm{comp}}(\mathbb{R}_+, V)$ with respect to the norm*

$$\|u\|_{\mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*V)} = \left(\int_{\Gamma_{-\delta}} \langle\tau\rangle^{2s} \, \|\kappa^{-1}_{\langle\tau\rangle} \mathcal{M}_{t\mapsto\zeta}u\|^2_V \, d\zeta \right)^{1/2} .$$

Set $\mathcal{H}^s(\mathbb{R}_+, \pi^*V) = \mathcal{H}^{s,0}(\mathbb{R}_+, \pi^*V)$. Moreover, if $\kappa_\lambda \equiv \mathrm{Id}_V$ for all λ, we write $\mathcal{H}^{s,\delta}(\mathbb{R}_+, V)$ and $\mathcal{H}^s(\mathbb{R}_+, V)$, respectively.

Example 1.4.2 Let s and γ, δ be real numbers, and $V = H^{s,\gamma}(X^\wedge)$. Consider the group action on V given by $\kappa_\lambda u\,(r, x) = \lambda^{(1+n)/2} u(\lambda r, x)$, for $\lambda > 0$. Then,

$$H^{s,(\gamma,\delta)}(\mathbb{R}_+ \times X^\wedge) = \mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* H^{s,\gamma}(X^\wedge))$$

is said to be the *corner Sobolev space* of smoothness s and weight (γ, δ).

□

The final corner Sobolev spaces over $\mathbb{R}_+ \times \mathcal{B}$, $\mathcal{B}$ being the stretched link of C near the corner, will contain (as it ought to be) a further contribution from $\mathbb{R}_+ \times \mathring{\mathcal{B}}$.

Note that

$$H^{0,(-\frac{n+1}{2},\delta)}(\mathbb{R}_+ \times X^\wedge) \cong L^2(\mathbb{R}_+ \times X^\wedge, t^{-2\delta-1} r^n dt dr dx), \tag{1.4.1}$$

cf. (1.3.1).

Let us return to abstract spaces V. Suppose V is a projective limit of a sequence $(V^\nu)_{\nu\in\mathbb{N}}$ of Banach spaces with continuous embeddings $V^{\nu+1} \hookrightarrow V^\nu$ and a strongly continuous action $(\kappa_\lambda)_{\lambda\in\mathbb{R}_+}$ on V_1 which restricts to a strongly continuous action on each V^ν, $\nu = 2, 3, \ldots$. We define $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V)$ to be the projective limit of the sequence $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V^\nu)$, $\nu \in \mathbb{N}$. Thus, $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V)$ is a Fréchet space.

As usual, the Mellin transform $\mathcal{M}$ is regarded as a mapping from distributions on $\mathbb{R}_+$ to distributions on $\Gamma_0 = \mathbb{R}$, unless we do not observe at the same time analytic extensions. Analogously, the weighted Mellin transform $\mathcal{M}_\delta$, for $\delta \in \mathbb{R}$, takes distributions on $\mathbb{R}_+$ to distributions on Γ_δ, so that $\mathcal{M}_0 = \mathcal{M}$.

Remark 1.4.3 *The operator $I_\delta = \mathcal{M}^{-1}_{-\delta,\zeta\to t}\kappa^{-1}_{\langle\tau\rangle}\mathcal{M}_{-\delta,t\to\zeta}$ extends to an isomorphism of $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V)$ onto $\mathcal{H}^{s,\delta}(\mathbb{R}_+, V)$, for each $s, \delta \in \mathbb{R}$.*

For a V-valued distribution u on $\mathbb{R}_+$, we consider its pull-back under the diffeomorphism $t = e^\theta$ of $\mathbb{R}$ onto $\mathbb{R}_+$. Set

$$(T_\delta u)(\theta) = e^{-\delta\theta}\, u(e^\theta) \tag{1.4.2}$$

defined for $\theta \in \mathbb{R}$. It is a simple matter to check that

$$\mathcal{M}u\,(\tau - i\delta) = \mathcal{F}_{\theta\mapsto\tau}\,(T_\delta u)\,,$$

with the one-dimensional Fourier transform $\mathcal{F}_{\theta\mapsto\tau} v = \int e^{-i\tau\theta} v(\theta) d\theta$ on the right-hand side. We deduce that

$$u \mapsto \left(\int_{\mathbb{R}} \langle\tau\rangle^{2s}\, \|\kappa^{-1}_{\langle\tau\rangle}\, \mathcal{F}_{\theta\mapsto\tau}\,(T_\delta u)\,\|^2_V\, d\tau \right)^{1/2}$$

is actually the same norm on $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V)$. In other words, (1.4.2) induces an *isometry*

$$T_\delta : \ \mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V) \overset{\cong}{\to} H^s(\mathbb{R}, \pi^* V),$$

cf. Definition 1.3.1.

Remark 1.4.4 *In particular, $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V)$ are 'weighted' spaces in the sense that $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^* V) = t^\delta\, \mathcal{H}^s(\mathbb{R}_+, \pi^* V)$ for all $s, \delta \in \mathbb{R}$.*

Remark 1.4.3 enables us to extend the definition of $\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*\Sigma)$ to vector subspaces $\Sigma \subset V$ which are not necessarily preserved under κ_λ. More precisely, we set

$$H^{s,\delta}(\mathbb{R}_+, \pi^*\Sigma) \cong I_\delta^{-1}\, \mathcal{H}^{s,\delta}(\mathbb{R}_+, \Sigma), \tag{1.4.3}$$

for any Fréchet space Σ continuously embedded to V, Σ being a projective limit of Banach spaces. Just as in the case of edge spaces we get, for any two such $\Sigma_1, \Sigma_2 \hookrightarrow V$,

$$\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*(\Sigma_1 + \Sigma_2)) = \mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*\Sigma_1) + \mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*\Sigma_2)$$

in the sense of non-direct sums of Fréchet spaces.

It is a simple matter to see that

$$\|I_\delta u\|_{\mathcal{H}^{s,\delta}(\mathbb{R}_+,V)} = \|IT_\delta u\|_{H^s(\mathbb{R},V)}$$

for all $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+,V)$. Hence $I_\delta^{-1}\mathcal{H}^{s,\delta}(\mathbb{R}_+,\Sigma) = T_\delta^{-1}I^{-1}H^s(\mathbb{R},V)$, and so (1.4.2) induces an isometry

$$T_\delta :\ \mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*\Sigma) \overset{\cong}{\to} H^s(\mathbb{R},\pi^*\Sigma).$$

We are now in a position to introduce the subspaces of $H^{s,(\gamma,\delta)}(\mathbb{R}_+ \times X^\wedge)$ with asymptotics. To this end, pick an asymptotic type "as", discrete or continuous, related to weight data $w = (\gamma,\mathcal{I})$ with finite $\mathcal{I} = (-l,0]$. By the above,

$$\mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*H^{s,\gamma}_{\mathrm{as}}(X^\wedge)) = \mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*\mathcal{A}_{\mathrm{as}}(X^\wedge)) + \mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*H^{s,\gamma+l-0}(X^\wedge)).$$

We will also set

$$\begin{array}{rcl} H^{s,(\gamma,\delta)}_{\mathrm{as}}(\mathbb{R}_+ \times X^\wedge) &=& \mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*H^{s,\gamma}_{\mathrm{as}}(X^\wedge)), \\ H^{s,(\gamma+l-0,\delta)}(\mathbb{R}_+ \times X^\wedge) &=& \mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*H^{s,\gamma+l-0}(X^\wedge)). \end{array} \tag{1.4.4}$$

The spaces (1.4.4) are expressions of the edge asymptotics close to the corner $t=0$. The "singular functions" are just constituted by the elements of $\mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*\mathcal{A}_{\mathrm{as}}(X^\wedge))$. These are

$$\mathcal{M}^{-1}_{-\delta,\tau\to t}\left(\langle\tau\rangle^{\frac{1+n}{2}}\,\omega(r\langle\tau\rangle)\,\langle\mathcal{M}_{-\delta,t\to\tau}\,f(t,x),(r\langle\tau\rangle)^{iz}\rangle\right) \tag{1.4.5}$$

where $f(t,x)$ runs over $\mathcal{H}^{s,\delta}(\mathbb{R}_+,\mathcal{A}'(K,C^\infty(X)))$, K being the carrier of "as", cf. (1.3.3).

For infinite weight interval $\mathcal{I} = (-\infty,0]$, we can write $u \sim \sum_{\nu=1}^\infty u_\nu$ where u_ν are singular functions associated with compact sets K_ν in the plane of $z \in \mathbb{C}$, satisfying (1.2.2). The interpretation of the sum is that the difference $u - \sum_{\nu=1}^N u_\nu$ belongs to $H^{s,(\gamma+\epsilon,\delta)}(\mathbb{R}_+ \times X^\wedge)$ for each $\epsilon > 0$, with some N depending on ϵ.

Write $H^{s,\gamma}_{\mathrm{as}}(X^\wedge)$ as the projective limit of a sequence of Banach spaces V^ν, $\nu \in \mathbb{N}$, with the properties listed after Remark 1.3.2. Then $H^{s,(\gamma,\delta)}_{\mathrm{as}}(\mathbb{R}_+ \times X^\wedge)$ just amounts to the projective limit of the sequence $\mathcal{H}^{s,\delta}(\mathbb{R}_+,\pi^*V^\nu)$ where $\nu \in \mathbb{N}$.

Theorem 1.4.5 *For each $s \in \mathbb{R}$, $(\gamma,\delta) \in \mathbb{R}^2$ and asymptotic type* "as" *satisfying the shadow condition, one has*

$$\begin{array}{rcl} H^{s,(\gamma,\delta)}(\mathbb{R}_+ \times X^\wedge) &\hookrightarrow& H^{s,\gamma}_{\mathrm{loc}}(\mathbb{R}_+ \times X^\wedge), \\ H^{s,(\gamma,\delta)}_{\mathrm{as}}(\mathbb{R}_+ \times X^\wedge) &\hookrightarrow& H^{s,\gamma}_{\mathrm{as,loc}}(\mathbb{R}_+ \times X^\wedge). \end{array}$$

Proof. This follows from the invariance of $H^{s,\gamma}_{\mathrm{as,loc}}(\mathbb{R}_+ \times X^\wedge)$ under diffeomorphisms of $\mathbb{R}_+$ and the fact that

$$T_\delta : \ H^{s,(\gamma,\delta)}_{\mathrm{as}}(\mathbb{R}_+ \times X^\wedge) \stackrel{\cong}{\to} H^{s,\gamma}_{\mathrm{as}}(\mathbb{R} \times X^\wedge).$$

□

Our next objective is to perform the Mellin pseudodifferential calculus with operator-valued symbols. To this end, pick Banach spaces V and $\tilde{V}$ along with group actions

$$(\kappa_\lambda)_{\lambda\in\mathbb{R}_+},$$
$$(\tilde{\kappa}_\lambda)_{\lambda\in\mathbb{R}_+},$$

respectively.

Definition 1.4.6 *Denote $\mathcal{S}^m((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \mathbb{R}, \mathcal{L}(V, \tilde{V}))$ the space of all symbols $a(t,t',\tau)$ with the property that there are a neighbourhood Ω of $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$ in $\mathbb{R}^2$ and $\tilde{a}(t,t',\tau) \in \mathcal{S}^m(\Omega \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ whose restriction to $(\mathbb{R}_+ \times \mathbb{R}_+) \times \mathbb{R}$ is $a(t,t',\tau)$.*

Analogously we introduce the spaces $\mathcal{S}^m(\bar{\mathbb{R}}_+ \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ as well as those with the subscript "cl".

Furthermore, it will be convenient to write $\mathcal{S}^m((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_\delta, \mathcal{L}(V,\tilde{V}))$ for the pull-back of $\mathcal{S}^m((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \mathbb{R}, \mathcal{L}(V,\tilde{V}))$ under the mapping $\Gamma_\delta \to \mathbb{R}$ given by $\zeta = \tau + iv \mapsto \tau$, and similarly for the other spaces. We can also talk about corresponding classes of amplitude functions in case $\tilde{V}$ is a Fréchet space.

As usual, for $a(t,t',\zeta) \in \mathcal{S}^m((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_{-\delta}, \mathcal{L}(V,\tilde{V}))$, we set

$$\mathrm{op}_{\mathcal{M},\delta}(a) = t^\delta \, \mathrm{op}_{\mathcal{M}}(T^{-i\delta}a)\, t^{-\delta}$$

where $(T^{-i\delta}a)(t,t',\zeta) = a(t,t',\zeta - i\delta)$ and

$$\mathrm{op}_{\mathcal{M}}(a)u\,(t) = \mathcal{M}^{-1}_{\zeta\mapsto t}\left(\mathcal{M}^{-1}_{t'\mapsto\zeta} a(t,t',\zeta)u(t')\right)$$

cf. (1.1.3). Thus, $\mathrm{op}_{\mathcal{M}}(a) = \mathrm{op}_{\mathcal{M},0}(a)$. The operator $\mathrm{op}_{\mathcal{M},\delta}(a)$ is first regarded as acting from $C^\infty_{\mathrm{comp}}(\mathbb{R}_+, V)$ to $C^\infty_{\mathrm{loc}}(\mathbb{R}_+, \tilde{V})$.

The generalities are needed only as a background information, where we restrict ourselves, for convenience, to Hilbert spaces V and $\tilde{V}$.

Let us first complete the list of notation around the $\mathcal{H}^s$- spaces. Namely, the space $\mathcal{H}^s_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*V)$ is defined to consist of all functions u on $\mathbb{R}_+$ with values in V, such that $\varphi u \in \mathcal{H}^s(\mathbb{R}_+, \pi^*V)$ for each $\varphi \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+, V)$. Furthermore, $\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*V)$ is the subspace of $\mathcal{H}^s_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*V)$ consisting of all u with a bounded support in $t \in \mathbb{R}_+$. In the same manner we define the spaces $\mathcal{H}^s_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*\Sigma)$ and $\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*\Sigma)$, $\Sigma \subset V$ being not necessarily preserved under κ_λ.

Denote by $\Psi_M^m(\bar{\mathbb{R}}_+; V, \tilde{V})$ the space of all operators $A = \mathrm{op}_{\mathcal{M}}(a) + G$, with $a(t,t',\zeta)$ an arbitrary symbol in $\mathcal{S}^m((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_0, \mathcal{L}(V,\tilde{V}))$ and G any operator satisfying

$$\begin{array}{rcl} G & \in & \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*V), \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*\tilde{V})), \\ G^* & \in & \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*\tilde{V}), \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*V)). \end{array}$$

Here, by G^* is meant the formal adjoint of G with respect to fixed scalar products in $\mathcal{H}^0(\mathbb{R}_+, \pi^*V)$ and $\mathcal{H}^0(\mathbb{R}_+, \pi^*\tilde{V})$. We may introduce them, for instance, by

$$(u,v)_{\mathcal{H}^0(\mathbb{R}_+,\pi^*V)} = \int_{\Gamma_0} (\mathcal{M}u(\zeta), \mathcal{M}v(\zeta))_V \, d\zeta.$$

The notation $\Psi^m_{M,\mathrm{cl}}(\bar{\mathbb{R}}_+; V, \tilde{V})$ has an obvious meaning. Every $a(t,t',\zeta)$ of $\mathcal{S}^m_{\mathrm{cl}}((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_0, \mathcal{L}(V,\tilde{V}))$ has a principal homogeneous part of order m, $a_0(t,t',\tau)$, which is a C^∞ function on $(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times (\mathbb{R} \setminus \{0\})$ with values in $\mathcal{L}(V,\tilde{V})$, satisfying

$$a_0(t,t',\lambda\tau) = \lambda^m \, \kappa_\lambda \, a_0(t,t',\tau) \, \kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$, τ being interpreted as $\Re\zeta$ for $\zeta \in \Gamma_0$. For each excision function $\chi(\tau)$, we have

$$a(t,t',\zeta) - \chi(\tau)\, a_0(t,t',\tau) \in \mathcal{S}^{m-1}_{\mathrm{cl}}((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_0, \mathcal{L}(V,\tilde{V})).$$

We set

$$\begin{array}{rcl} \sigma^m_{\mathrm{edge}}(a)(t,\tau) & = & a_0(t,t,\tau), \\ {}^b\sigma^m_{\mathrm{edge}}(A)(t,\tau) & = & \sigma^m_{\mathrm{edge}}(a)(t,\tau) \end{array} \tag{1.4.6}$$

for any classical operator $A = \mathrm{op}_{\mathcal{M}}(a) + G$. This is obviously compatible with (1.3.4).

The distribution kernels of operators in $\Psi_M^m(\bar{\mathbb{R}}_+; V, \tilde{V})$ live on $\mathbb{R}_+ \times \mathbb{R}_+$ and take their values in $\mathcal{L}(V,\tilde{V})$. The notation "properly supported" refers to the variables (t,t').

Each $A \in \Psi_M^m(\bar{\mathbb{R}}_+; V, \tilde{V})$ induces continuous mappings

$$A: \ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*V) \to \mathcal{H}^{s-m}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*\tilde{V})$$

for all $s \in \mathbb{R}$. We may write "loc" or "comp" on both sides if A is properly supported.

Lemma 1.4.7 *We have* $\Psi^m_{M,\mathrm{cl}}(\bar{\mathbb{R}}_+; V, \tilde{V}) \hookrightarrow \Psi^m_{\mathrm{cl}}(\mathbb{R}_+; V, \tilde{V})$, *the right-hand side referring to the notation to* (1.3.4), *and*

$$\sigma^m_{\mathrm{edge}}(A)(t,\tau) = \frac{1}{t^m}\, {}^b\sigma^m_{\mathrm{edge}}(A)(t, t\tau).$$

Proof. Fix $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+, V)$. Changing the covariable by $\tau \to F(t,t')\tau$, we obtain

$$\begin{aligned}\mathrm{op}(a)u\,(t) &= \frac{1}{2\pi}\iint e^{i(t-t')\tau} a(t,t',\tau)u(t')\,dt'd\tau \\ &= \frac{1}{2\pi}\iint \left(\frac{t}{t'}\right)^{i\tau} (t'F(t,t'))\,a\,(t,t',F(t,t')\tau)\,u(t')\,\frac{dt'}{t'}d\tau\end{aligned}$$

where

$$F(t,t') = \frac{\log t - \log t'}{t-t'}.$$

Using Taylor's expansion yields

$$\frac{\log t' - \log t}{t'-t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\frac{1}{t^{k+1}}\,(t'-t)^k,$$

which establishes the formula, cf. (1.1.11).

□

If we replace V in Definition 1.4.1 by $V \oplus \mathbb{C}^N$, the latter being endowed with the group action $\kappa_\lambda(v \oplus c) = (\kappa_\lambda v) \oplus c$, we arrive at the spaces

$$\mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*(V \oplus \mathbb{C}^N)) = \mathcal{H}^{s,\delta}(\mathbb{R}_+, \pi^*V) \oplus \mathcal{H}^{s,\delta}(\mathbb{R}_+, \mathbb{C}^N).$$

Set

$$\begin{aligned} W &= \bar{\mathbb{R}}_+ \times \mathbb{C}^N, \\ \tilde{W} &= \bar{\mathbb{R}}_+ \times \mathbb{C}^{\tilde{N}}. \end{aligned}$$

Definition 1.4.8 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $m, \gamma \in \mathbb{R}$ and $\mathcal{I} = (-l, 0]$, $0 < l \le \infty$. Then, by $\Psi^{-\infty}_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w)$ is meant the space of all operators*

$$\mathcal{G} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*H^{s,\gamma}(X^\wedge)) & & \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*H^{\infty,\gamma-m}(X^\wedge)) \\ \oplus & , & \oplus \\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, W) & & \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \tilde{W}) \end{pmatrix}$$

which induce continuous mappings

$$\begin{aligned} \mathcal{G} \;:\; & \begin{matrix} \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*H^{s,\gamma}(X^\wedge)) \\ \oplus \\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, W) \end{matrix} \to \begin{matrix} \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*H^{\infty,\gamma-m}_{\mathrm{as}}(X^\wedge)) \\ \oplus \\ \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \tilde{W}) \end{matrix}, \\ \mathcal{G}^* \;:\; & \begin{matrix} \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^*H^{s,m-\gamma}(X^\wedge)) \\ \oplus \\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \tilde{W}) \end{matrix} \to \begin{matrix} \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^*H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge)) \\ \oplus \\ \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, W) \end{matrix} \end{aligned}$$

*for all $s \in \mathbb{R}$ and some asymptotic types "*as*" and "*$\tilde{\mathrm{as}}$*" related to weight data $(\gamma - m, \mathcal{I})$ and $(-\gamma, \mathcal{I})$, respectively.*

Note that $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$ with respect to a fixed scalar product in $\mathcal{H}^0(\mathbb{R}_+, \pi^* H^{0,0}(X^\wedge)) \oplus \mathcal{H}^0(\mathbb{R}_+)$.

Furthermore we introduce symbol spaces $\mathcal{S}^\mu_G((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \mathbb{R}; W, \tilde{W}; w)$, with $m - \mu \in \mathbb{Z}_+$, analogously to Definition 1.3.5, where Ω is everywhere to be replaced by $\bar{\mathbb{R}}_+$. Given any $\delta \in \mathbb{R}$, we write $\mathcal{S}^\mu_G((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_\delta; W, \tilde{W}; w)$ for the pull-back of $\mathcal{S}^\mu_G((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \mathbb{R}; W, \tilde{W}; w)$ under the mapping $\Gamma_\delta \to \mathbb{R}$ given by $\tau + iv \mapsto \tau$. We use the designation $\mathfrak{g}(t, t', \zeta)$ for elements of this space. The classes of t'-independent $\mathfrak{g}$ will be indicated by $\bar{\mathbb{R}}_+$ instead of $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$. We denote $\mathcal{S}^\mu_G(\bar{\mathbb{R}}_+ \times \Gamma_\delta; w)$ the space of upper left corners of the elements in $\mathcal{S}^\mu_G(\bar{\mathbb{R}}_+ \times \Gamma_\delta; W, \tilde{W}; w)$.

We then obtain operator classes $\Psi^\mu_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w)$ consisting of all $\mathrm{op}_\mathcal{M}(\mathfrak{g}) + \mathcal{G}$, where

$$\begin{aligned} \mathfrak{g} &\in \mathcal{S}^\mu_G((\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+) \times \Gamma_0; W, \tilde{W}; w), \\ \mathcal{G} &\in \Psi^{-\infty}_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w). \end{aligned}$$

The entries of the upper left corners of matrices $\mathrm{op}(\mathfrak{g}) + \mathcal{G}$ form the space $\Psi^\mu_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; w)$. They are called Mellin *Green operators* of order μ on the wedge $\bar{\mathbb{R}}_+ \times X^\wedge$. The meaning of the other entries in $\mathrm{op}(\mathfrak{g}) + \mathcal{G}$ is analogous to that in the Fourier approach, cf. Section 1.3.

By the above, it follows that every $\mathcal{G} \in \Psi^\mu_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w)$ has a principal homogeneous Mellin edge symbol of order μ. This is a C^∞ operator family

$${}^b\sigma^\mu_{\mathrm{edge}}(\mathcal{G})\,(t, \tau) : \begin{array}{c} H^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathbb{C}^N \end{array} \to \begin{array}{c} H^{s-\mu,\gamma-m}(X^\wedge) \\ \oplus \\ \mathbb{C}^{\tilde{N}} \end{array} \tag{1.4.7}$$

living away from the zero section of $T^*\bar{\mathbb{R}}_+$ and homogeneous of order μ in the sense that

$${}^b\sigma^\mu_{\mathrm{edge}}(\mathcal{G})\,(t, \lambda\tau) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} {}^b\sigma^\mu_{\mathrm{edge}}(\mathcal{G})\,(t, \tau) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

for all $\lambda \in \mathbb{R}_+$. Here we have smoothness in t up to $t = 0$.

Next we turn to the Mellin operators in t with smoothing operator-valued symbols on $X^\wedge$. Let

$$m(t, \zeta) = \varphi_0(r\langle\tau\rangle)\, r^{-\mu} \sum_{j=0}^{l-1} r^j \sum_{\alpha \le j} \Big(\mathrm{op}_{\mathcal{M}, \gamma^{(1)}_{j,\alpha}}(m^{(1)}_{j,\alpha}) + \mathrm{op}_{\mathcal{M}, \gamma^{(2)}_{j,\alpha}}(m^{(2)}_{j,\alpha}) \Big) \zeta^\alpha\, \psi_0(r\langle\tau\rangle) \tag{1.4.8}$$

where $m^{(\iota)}_{j,\alpha}(t, z)$ is a C^∞ function of $t \in \bar{\mathbb{R}}_+$ with values in $\mathcal{M}_{T^{(\iota)}_{j,\alpha}}(\mathbb{C}, \Psi^{-\infty}(X))$ and

$$\begin{aligned} \gamma - (m - \mu) - j \;\le\; \gamma^{(\iota)}_{j,\alpha} &\;\le\; \gamma, \\ \pi_\mathbb{C} T^{(\iota)}_{j,\alpha} \cap \Gamma_{-\gamma^{(\iota)}_{j,\alpha}} &\;=\; \emptyset, \end{aligned}$$

for $\iota = 1, 2$ (cf. (1.3.6)). Furthermore, φ_0 and ψ_0 are arbitrary cut-off functions on the semiaxis. We have $m \in \mathcal{S}^{\mu}_{\mathrm{cl}}(\bar{\mathbb{R}}_+ \times \Gamma_\delta, \mathcal{L}(H^{s,\gamma}(X^\wedge), H^{\infty,\gamma-m}(X^\wedge)))$ for all $\delta \in \mathbb{R}$. The principal Mellin edge symbol of m is a family of operators in $\mathcal{L}(H^{s,\gamma}(X^\wedge), H^{s-\mu,\gamma-m}(X^\wedge))$, given by

$$ {}^b\sigma^{\mu}_{\mathrm{edge}}(m)(t,\tau) = \varphi_0(r|\tau|)\, r^{-\mu} \sum_{j=0}^{l-1} (r\tau)^j \left(\mathrm{op}_{\mathcal{M},\gamma^{(1)}_{j,j}}(m^{(1)}_{j,j}) + \mathrm{op}_{\mathcal{M},\gamma^{(2)}_{j,j}}(m^{(2)}_{j,j}) \right) \psi_0(r|\tau|) $$

for $(t,\tau) \in T^*\bar{\mathbb{R}}_+ \setminus \{0\}$.

The operators $M = \mathrm{op}_{\mathcal{M}}(m)$ are said to be *smoothing Mellin operators* in the class of Mellin wedge pseudodifferential operators.

Definition 1.4.9 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $\mathcal{I} = (-l, 0]$, l being a positive integer. Then, $\Psi^{\mu}_{M,M+G}(\bar{\mathbb{R}}_+ \times X^\wedge; w)$ stands for the space of all operators $\mathrm{op}_{\mathcal{M}}(m) + G$, where m is of the type* (1.4.8) *and $G \in \Psi^{\mu}_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; w)$.*

It remains to add pseudodifferential operators in the interior of $\bar{\mathbb{R}}_+ \times X^\wedge$. We shall introduce the symbols first on $\mathbb{R}_+ \times O^\wedge$ where O is a coordinate neighbourhood on X with local coordinates $x \in U$, U being an open subset of $\mathbb{R}^n$.

Denote by $\mathcal{S}^{\mu}_{\mathrm{cl}}({}^bT^*(\bar{\mathbb{R}}_+ \times U^\wedge))$ the space of all symbols a on $\mathbb{R}_+ \times U^\wedge$ of the form

$$ a(t,r,x,\tau,\varrho,\xi) = \tilde{a}(t,r,x,tr\tau,r\varrho,\xi) $$

where $\tilde{a} \in \mathcal{S}^{\mu}_{\mathrm{cl}}((\bar{\mathbb{R}}_+ \times U^\wedge) \times \mathbb{R}^{2+n})$ is C^∞ both up to $t = 0$ and $r = 0$. We will tacitly assume that our $\tilde{a}(t,r,x,\tilde{\tau},\tilde{\varrho},\xi)$ vanish for r large enough, for the symbols are cut off at the end. The symbols of $\mathcal{S}^{\mu}_{\mathrm{cl}}({}^bT^*(\bar{\mathbb{R}}_+ \times U^\wedge))$ are said to be *corner-degenerate*.

For every corner-degenerate symbol $a(t,r,x,\tau,\varrho,\xi)$, we introduce the *compressed* principal homogeneous symbol of the operator $A = (tr)^{-\mu}\,\mathrm{op}(a)$ over $\mathbb{R}_+ \times U^\wedge$ by

$$ {}^b\sigma^{\mu}(A)(t,r,x,\tilde{\tau},\tilde{\varrho},\xi) = \tilde{a}_0(t,r,x,\tilde{\tau},\tilde{\varrho},\xi), \qquad (1.4.9) $$

where $\tilde{a}_0$ is the principal homogeneous part of $\tilde{a}$ of order μ. The symbol ${}^b\sigma^{\mu}(A)$ is defined up to the faces $t = 0$ and $r = 0$, and the ellipticity with respect to ${}^b\sigma^{\mu}(A)$ just amounts to that with respect to the symbol $\sigma^{\mu}(A)$ away from both $t = 0$ and $r = 0$, along with additional invertibility conditions at the faces, as above.

Theorem 1.4.10 *Suppose a is a corner-degenerate symbol of order μ, elliptic with respect to the compressed symbol up to $t = 0$ and $r = 0$. Then there is a corner-degenerate symbol p of order $-\mu$, such that*

$$ ((tr)^{\mu}\, p) \circ_{(t,r,x)} \left((tr)^{-\mu}\, a \right) = 1 \mod \mathcal{S}^{-\infty}((\mathbb{R}_+ \times U^\wedge) \times \mathbb{R}^{2+n}), $$

where "$\circ$" means the Leibniz product of the symbols with respect to the indicated variables.

Proof. Cf. Proposition 4.1.8 in [Dor98].

□

We now turn to operator conventions which assign to a system of local symbols on X a (t,τ)-dependent operator family.

Choose any finite open covering of X by coordinate neighborhoods $(O_\nu)_{\nu\in\mathcal{N}}$ together with a system of charts $\kappa_\nu\colon O_\nu \to U_\nu$, U_ν being an open subset of $\mathbb{R}^n$, a subordinate partition of unity $(\varphi_\nu)_{\nu\in\mathcal{N}}$ on X, and functions $\psi_\nu \in C^\infty_{\mathrm{comp}}(U_\nu)$ such that $\varphi_\nu\psi_\nu = \varphi_\nu$, for every $\nu \in \mathcal{N}$. Given an arbitrary system of corner-degenerate symbols $a_\nu \in \mathcal{S}^\mu_{\mathrm{cl}}((\bar{\mathbb{R}}_+ \times {U_\nu}^\wedge) \times \mathbb{R}^{2+n})$, $\nu \in \mathcal{N}$, with the associated symbols $\tilde{a}_\nu(t,r,x,\tilde{\tau},\tilde{\varrho},\xi)$ which are C^∞ up to $t=0$ and $r=0$, we can form an operator family

$$\tilde{a}(t,r,\tilde{\tau},\tilde{\varrho}) = \sum_{\nu\in\mathcal{N}} \varphi_\nu \left(\left(\kappa_\nu^{-1}\right)_\sharp \mathrm{op}_{\mathcal{F}_x}(\tilde{a}_\nu)(t,r,\tilde{\tau},\tilde{\varrho})\right)\psi_\nu$$

where $\mathrm{op}_{\mathcal{F}_x}$ stands for the pseudodifferential action in $U_\nu \subset \mathbb{R}^n$ with respect to x. Then

$$\tilde{a}(t,r,\tilde{\tau},\tilde{\varrho}) \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}})).$$

By carrying out the Fourier pseudodifferential action in t and r with symbol a we arrive at an operator in $\Psi^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times X^\wedge)$. The final operator conventions in the present set-up will be obtained in several steps. First we switch to the Mellin conventions with respect to t and r. Both reformulations of the original Fourier action will be glued together then by t-dependent cut-off functions in the r-variable.

Theorem 1.4.11 *For every $\tilde{a} \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}}))$ there exists a $\tilde{\sigma} \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}_{\tilde{\varrho}})))$ with the property that whenever $\delta \in \mathbb{R}$, we have*

$$\mathrm{op}_{\mathcal{F}_t}\mathrm{op}_{\mathcal{F}_r}(a) = \mathrm{op}_{\mathcal{M},\delta}\,\mathrm{op}_{\mathcal{F}_r}(\sigma) \mod \Psi^{-\infty}(\mathbb{R}_+ \times X^\wedge)$$

where

$$\begin{aligned} a(t,r,\tau,\varrho) &= \tilde{a}(t,r,tr\tau,r\varrho),\\ \sigma(t,r,\zeta,\varrho) &= \tilde{\sigma}(t,r,r\zeta,r\varrho). \end{aligned}$$

Proof. This is in fact a consequence of Theorem 1.3.9, up to minor modifications.

□

Clearly, we get $\mathrm{op}_{\mathcal{F}_r}(\tilde{\sigma}) \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^\mu_{\mathrm{cl}}(X^\wedge))$. Hence we may apply Theorem 1.3.9 once again, now in the r-variable. To formulate the result, we need appropriate spaces of entire functions of two variables with values in

$\Psi^m_{\text{cl}}(X)$. These are introduced similarly to Definition 1.2.2 and denoted by $\mathcal{M}(\mathbb{C}\times\mathbb{C},\Psi^m_{\text{cl}}(X))$.

More precisely, by $\mathcal{M}(\mathbb{C}\times\mathbb{C},\Psi^m_{\text{cl}}(X))$ we will mean the space of all holomorphic functions $a(\zeta,z)$ on $\mathbb{C}\times\mathbb{C}$, taking their values in $\Psi^m_{\text{cl}}(X)$, such that $a(\tau-i\delta,\varrho-i\gamma)\in\Psi^m_{\text{cl}}(X;\mathbb{R}^2_{\tau,\varrho})$ for all $\delta,\gamma\in\mathbb{R}$, uniformly in δ and γ in compact intervals of $\mathbb{R}$.

Theorem 1.4.12 *For any $\tilde{a}\in C^\infty_{\text{loc}}(\bar{\mathbb{R}}_+\times\bar{\mathbb{R}}_+,\Psi^\mu_{\text{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}}))$, there exists an $\tilde{h}\in C^\infty_{\text{loc}}(\bar{\mathbb{R}}_+\times\bar{\mathbb{R}}_+,\mathcal{M}(\mathbb{C}\times\mathbb{C},\Psi^\mu_{\text{cl}}(X)))$ with the property that whenever $\delta,\gamma\in\mathbb{R}$, we have*

$$\operatorname{op}_{\mathcal{F}_t}\operatorname{op}_{\mathcal{F}_r}(a)=\operatorname{op}_{\mathcal{M},\delta}\operatorname{op}_{\mathcal{M},\gamma}(h)\mod\Psi^{-\infty}(\mathbb{R}_+\times X^\wedge)$$

where

$$\begin{aligned} a(t,r,\tau,\varrho) &= \tilde{a}(t,r,tr\tau,r\varrho),\\ h(t,r,\zeta,z) &= \tilde{h}(t,r,r\zeta,z). \end{aligned}$$

Note that

$$\operatorname{op}_{\mathcal{F}_r}(\sigma)=\operatorname{op}_{\mathcal{M},\gamma}(h) \tag{1.4.10}$$

modulo $C^\infty_{\text{loc}}(\bar{\mathbb{R}}_+,\mathcal{M}(\mathbb{C},\Psi^{-\infty}(X^\wedge)))$. Moreover, both $\tilde{\sigma}$ and $\tilde{h}$ are independent of δ and γ and are actually unique modulo elements of order $-\infty$.

Let us fix a partition of unity $(\varphi_0,\varphi_\infty)$ on the semiaxis, $\varphi_0(r)$ being a cut-off function. Choose a system of C^∞ functions (ψ_0,ψ_∞) on $\mathbb{R}_+$, such that $\psi_0(r)$ vanishes for large r, $\psi_\infty(r)$ vanishes near $r=0$ and $\varphi_\nu\psi_\nu\equiv\varphi_\nu$, for $\nu=0,\infty$. Then

$$\begin{aligned} a_0(t,\zeta) &= \varphi_0(r\langle\tau\rangle)\,r^{-\mu}\operatorname{op}_{\mathcal{M},\gamma}(h)(t,\zeta)\,\psi_0(r\langle\tau\rangle),\\ a_\infty(t,\zeta) &= \varphi_\infty(r\langle\tau\rangle)\,r^{-\mu}\operatorname{op}_{\mathcal{F}_r}(\sigma)(t,\zeta)\,\psi_\infty(r\langle\tau\rangle) \end{aligned} \tag{1.4.11}$$

are operator families $H^{s,\gamma}(X^\wedge)\to H^{s-\mu,\gamma-\mu}(X^\wedge)$ parametrised by $t\in\mathbb{R}_+$ and $\zeta=\tau+iv$.

Theorem 1.4.13 *Set $\sigma(t,\zeta)=\varphi(r)\,(a_0(t,\zeta)+a_\infty(t,\zeta))\,\psi(r)$ where a_0 and a_∞ are defined as above, and φ, ψ are arbitrary cut-off functions. Then $\sigma(t,\zeta)\in\mathcal{S}^\mu(\bar{\mathbb{R}}_+\times\Gamma_{-\delta},\mathcal{L}(H^{s,\gamma}(X^\wedge),H^{s-\mu,\gamma-\mu}(X^\wedge)))$ for all $\delta,s\in\mathbb{R}$. Moreover,*

$$\operatorname{op}_{\mathcal{M}}(\sigma)=\operatorname{op}_{\mathcal{F}_t}\left(\varphi\,r^{-\mu}\operatorname{op}_{\mathcal{F}_r}(a)(t,\tau)\,\psi\right)\mod\Psi^{-\infty}(\mathbb{R}_+\times X^\wedge). \tag{1.4.12}$$

Proof. This follows by the same method as in Theorem 1.3.10, for the Mellin transform is the pull-back of the Fourier one under the diffeomorphism $t\mapsto\log t$ of $\mathbb{R}_+$ onto $\mathbb{R}$.

□

For the symbol σ of Theorem 1.4.13, we set

$$^b\sigma^\mu_{\mathrm{edge}}(\sigma) = {}^b\sigma^\mu_{\mathrm{edge}}(a_0) + {}^b\sigma^\mu_{\mathrm{edge}}(a_\infty)$$

where

$$\begin{aligned} {}^b\sigma^\mu_{\mathrm{edge}}(a_0)(t,\tau) &= \varphi_0(r|\tau|)\, r^{-\mu}\, \mathrm{op}_{\mathcal{M},\gamma}(\tilde{h}(t,0,r\tau,z))\, \psi_0(r|\tau|), \\ {}^b\sigma^\mu_{\mathrm{edge}}(a_\infty)(t,\tau) &= \varphi_\infty(r|\tau|)\, r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(\tilde{\sigma}(t,0,r\tau,r\varrho))\, \psi_\infty(r|\tau|), \end{aligned}$$

for $(t,\tau) \in T^*\bar{\mathbb{R}}_+ \setminus \{0\}$. Then ${}^b\sigma^\mu_{\mathrm{edge}}(\sigma)(t,\tau)$ is a family of continuous operators $H^{s,\gamma}(X^\wedge) \to H^{s-\mu,\gamma-\mu}(X^\wedge)$, for each $s,\gamma \in \mathbb{R}$. It is homogeneous of order μ, i.e.,

$$^b\sigma^\mu_{\mathrm{edge}}(\sigma)(t,\lambda\tau) = \lambda^\mu\, \kappa_\lambda\, {}^b\sigma^\mu_{\mathrm{edge}}(\sigma)(t,\tau)\kappa_\lambda^{-1}, \quad \lambda > 0,$$

for all $(t,\tau) \in T^*\bar{\mathbb{R}}_+ \setminus \{0\}$.

Definition 1.4.14 *Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $\mathcal{I} = (-l, 0]$, l being a positive integer. Then, $\Psi^\mu_M(\bar{\mathbb{R}}_+ \times X^\wedge; w)$ denotes the space of all operators*

$$A = A_0 + A_\infty + M + G$$

where $A_\nu = \mathrm{op}_{\mathcal{M}}(a_\nu)$, $\nu = 0, \infty$, for a symbol $\tilde{a} \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}^2)$, and $M + G \in \Psi^\mu_{M,M+G}(\bar{\mathbb{R}}_+ \times X^\wedge; w)$.

We also introduce the space $\Psi^\mu_M(\Omega \times X^\wedge; W, \tilde{W}; w)$ consisting of all operators of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G},$$

with $A \in \Psi^\mu_M(\bar{\mathbb{R}}_+ \times X^\wedge; w)$ and $\mathcal{G} \in \Psi^\mu_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w)$.

The restriction of Definition 1.4.14 to (t,ζ)-dependent a_ν and m could be dropped. We might actually allow the (t,t',ζ)-dependent symbols that are smooth up to $t = 0$ and $t' = 0$. However, it is a property of the calculus that the general case can be reduced to the (t,ζ)-dependent one, modulo $\Psi^{-\infty}_{M,G}(\bar{\mathbb{R}}_+ \times X^\wedge; w)$.

Theorem 1.4.15 *Every $\mathcal{A} \in \Psi^\mu_M(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w)$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^* H^{s,\gamma}(X^\wedge)) \\ \oplus \\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, W) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^* H^{s-\mu,\gamma-m}(X^\wedge)) \\ \oplus \\ \mathcal{H}^{s-\mu}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \tilde{W}) \end{array},$$

$$\mathcal{A} : \begin{array}{c} \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^* H^{s,\gamma}_{\mathrm{as}}(X^\wedge)) \\ \oplus \\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, W) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^* H^{s-\mu,\gamma-m}_{\tilde{\mathrm{as}}}(X^\wedge)) \\ \oplus \\ \mathcal{H}^{s-\mu}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \tilde{W}) \end{array}$$

for any $s \in \mathbb{R}$ and asymptotic type "as" related to $(\gamma, \mathcal{I})$, with some resulting asymptotic type "$\tilde{\text{as}}$" related to $(\gamma - m, \mathcal{I})$.

Note that "$\tilde{as}$" depends on "as" and $\mathcal{A}$, but not on s. If moreover $\mathcal{A}$ is properly supported in $t \in \bar{\mathbb{R}}_+$ then it preserves the spaces with the subscripts 'comp' and 'loc'.

As in the previous section we have two leading symbols of our operators. From $\Psi^\mu_M(\bar{\mathbb{R}}_+ \times X^\wedge; w) \hookrightarrow \Psi^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times X^\wedge)$ we get the 'compressed' principal homogeneous *interior symbol* of order μ, i.e.,

$$^b\sigma^\mu(A) \in \mathcal{S}^\mu_{\mathrm{hg}}(^bT^*(\bar{\mathbb{R}}_+ \times X^\wedge) \setminus \{0\})$$

defined by (1.4.9), where the subscript "hg" indicates *homogeneous* symbols. Moreover, we have the principal homogeneous Mellin *edge symbol* of order μ, i.e.,

$$\sigma^\mu_{\mathrm{edge}}(A) \in \bigcap_{s\in\mathbb{R}} \mathcal{S}^\mu_{\mathrm{hg}}(T^*\bar{\mathbb{R}}_+ \setminus \{0\}, \mathcal{L}(H^{s,\gamma}(X^\wedge), H^{s-\mu,\gamma-m}(X^\wedge)))$$

given by

$$\sigma^\mu_{\mathrm{edge}}(A) = {}^b\sigma^\mu_{\mathrm{edge}}(a_0) + {}^b\sigma^\mu_{\mathrm{edge}}(a_\infty) + {}^b\sigma^\mu_{\mathrm{edge}}(m) + {}^b\sigma^\mu_{\mathrm{edge}}(g).$$

For more general $\mathcal{A} \in \Psi^\mu_M(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w)$, we set quite analogously to (1.3.11)

$$\begin{aligned} ^b\sigma^\mu(\mathcal{A}) &= {}^b\sigma^\mu(A), \\ \sigma^\mu_{\mathrm{edge}}(\mathcal{A}) &= \begin{pmatrix} \sigma^\mu_{\mathrm{edge}}(A) & 0 \\ 0 & 0 \end{pmatrix} + \sigma^\mu_{\mathrm{edge}}(\mathcal{G}). \end{aligned} \tag{1.4.13}$$

Lemma 1.4.16 *We have*

$$\Psi^\mu_M(\bar{\mathbb{R}}_+ \times X^\wedge; W, \tilde{W}; w) \hookrightarrow \Psi^\mu(\mathbb{R}_+ \times X^\wedge; W, \tilde{W}; w),$$

the right-hand side referring to Definition 1.3.12, and

$$\sigma^m_{\mathrm{edge}}(\mathcal{A})(t,\tau) = {}^b\sigma^m_{\mathrm{edge}}(\mathcal{A})(t, t\tau).$$

Proof. Cf. Lemma 1.4.7.

□

To introduce pseudodifferential operators globally over $\mathcal{W} = b^{-1}(C \setminus C_0)$ we begin with the negligible operators. For this purpose we first introduce global analogues of spaces (1.4.4). Since we deal here with the local calculus close to a corner, $\mathcal{W}$ is of the form $\mathbb{R}_+ \times \mathcal{B}$.

The idea is to glue together the objects on $\mathbb{R}_+ \times X^\wedge$ near $r = 0$ and those in the interior part of $\mathbb{R}_+ \times \mathcal{B}$. We may double $\mathcal{B}$ through its boundary, thus obtaining a C^∞ compact closed manifold $2\mathcal{B}$ of dimension $1 + n$. Fix $2\mathcal{B}$ once and for all. Then Definition 1.1.2 gives us the spaces $\mathcal{H}^{s,\delta}((2\mathcal{B})^\wedge)$, for each $s, \delta \in \mathbb{R}$. Choose a cut-off function $\omega(r)$ supported on $[0, 1) \times X$. Then

$1-\omega(r)$ can be thought of as a function on the entire manifold $\mathcal{B}$ vanishing in a collar neighbourhood of $\partial\mathcal{B}$. Set

$$\begin{aligned}\mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}(\mathcal{B})) &= [\omega]\,\mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}(X^\wedge)) + [1-\omega]\,\mathcal{H}^s((2\mathcal{B})^\wedge),\\ \mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}_{\mathrm{as}}(\mathcal{B})) &= [\omega]\,\mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}_{\mathrm{as}}(X^\wedge)) + [1-\omega]\,\mathcal{H}^s((2\mathcal{B})^\wedge)\end{aligned} \tag{1.4.14}$$

where, for abbreviation, we drop the pull-backs under relevant diffeomorphisms.

Obviously, the first space of (1.4.14) is Banach, and the second one is Fréchet, both being independent of the concrete choice of ω. The operator of multiplication by any $\varphi\in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ acts continuously on these spaces. This allows one to introduce in a familiar way the corresponding "loc" and "comp" spaces.

Theorem 1.4.17 *For each $s\in\mathbb{R}$, $\gamma\in\mathbb{R}$ and asymptotic type "as," one has*

$$\begin{aligned}\mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}(\mathcal{B})) &\hookrightarrow H^{s,\gamma}_{\mathrm{loc}}(\mathcal{W}),\\ \mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}_{\mathrm{as}}(\mathcal{B})) &\hookrightarrow H^{s,\gamma}_{\mathrm{as,loc}}(\mathcal{W}).\end{aligned}$$

This theorem shows that the spaces $\mathcal{H}^s(\mathbb{R}_+,\pi^*H^{s,\gamma}(\mathcal{B}))$ living near corners can be glued together with weighted Sobolev spaces living on the stretched manifold with edges, $\mathcal{W}$. The same is still valid for the spaces with asymptotics. The underlying spaces away from the singularities are actually the usual Sobolev spaces.

Definition 1.4.18 *Let $m,\gamma\in\mathbb{R}$ and $w=(\gamma,\gamma-m,\mathcal{I})$, where $\mathcal{I}=(-l,0]$, $0<l\le\infty$. Then, $\Psi^{-\infty}_{M,G}(\mathcal{W};W,\tilde{W};w)$ denotes the space of all operators*

$$\mathcal{G}\in\bigcap_{s\in\mathbb{R}}\mathcal{L}\begin{pmatrix}\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+,\pi^*H^{s,\gamma}(\mathcal{B})) & & \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\pi^*H^{\infty,\gamma-m}(\mathcal{B}))\\ \oplus & , & \oplus\\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+,W) & & \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\tilde{W})\end{pmatrix}$$

which induce continuous mappings

$$\begin{array}{cccc}\mathcal{G}\ : & \begin{array}{c}\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+,\pi^*H^{s,\gamma}(\mathcal{B}))\\ \oplus\\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+,W)\end{array} & \to & \begin{array}{c}\mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\pi^*H^{\infty,\gamma-m}_{\mathrm{as}}(\mathcal{B}))\\ \oplus\\ \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\tilde{W})\end{array},\\ \mathcal{G}^*\ : & \begin{array}{c}\mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+,\pi^*H^{s,m-\gamma}(\mathcal{B}))\\ \oplus\\ \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+,\tilde{W})\end{array} & \to & \begin{array}{c}\mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\pi^*H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{B}))\\ \oplus\\ \mathcal{H}^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,W)\end{array}\end{array}$$

for all $s\in\mathbb{R}$ and some asymptotic types "as" and "$\tilde{\mathrm{as}}$" related to weight data $(\gamma-m,\mathcal{I})$ and $(-\gamma,\mathcal{I})$, respectively.

By $\mathcal{G}^*$ is meant the formal adjoint of $\mathcal{G}$ with respect to a fixed scalar product in $\mathcal{H}^0(\mathbb{R}_+, \pi^* H^{0,0}(\mathcal{B})) \oplus \mathcal{H}^0(\mathbb{R}_+)$. We also write $\Psi^{-\infty}_{M,G}(\mathcal{W}; w)$ for the space of upper left corners of the matrices $\mathcal{G}$.

Suppose $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $w = (\gamma, \gamma - m, \mathcal{I})$, with $\mathcal{I} = (-l, 0]$ being a finite weight interval. For $\mathcal{W} = \mathbb{R}_+ \times \mathcal{B}$ and the bundles W, $\tilde{W}$ over $\mathbb{R}_+$ as above, the space $\Psi^{\mu}_{M}(\mathcal{W}; W, \tilde{W}; w)$ is defined to consist of all operators of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G},$$

with $A \in \Psi^{\mu}_{\mathrm{cl}}(\overset{\circ}{\mathcal{W}})$ and $\mathcal{G}$ having a kernel in

$$\begin{pmatrix} C^{\infty}_{\mathrm{loc}}(\overset{\circ}{\mathcal{W}}) \hat{\otimes}_{\pi} C^{\infty}_{\mathrm{loc}}(\overset{\circ}{\mathcal{W}}) & C^{\infty}_{\mathrm{loc}}(\overset{\circ}{\mathcal{W}}) \hat{\otimes}_{\pi} C^{\infty}_{\mathrm{loc}}(\mathbb{R}_+, W^*) \\ C^{\infty}_{\mathrm{loc}}(\mathbb{R}_+, \tilde{W}) \hat{\otimes}_{\pi} C^{\infty}_{\mathrm{loc}}(\overset{\circ}{\mathcal{W}}) & C^{\infty}_{\mathrm{loc}}(\mathbb{R}_+, \tilde{W}) \hat{\otimes}_{\pi} C^{\infty}_{\mathrm{loc}}(\mathbb{R}_+, W^*) \end{pmatrix},$$

such that, for any cut-off functions $\tilde{\varphi}_b(t)$, $\tilde{\psi}_b(t)$ and $\varphi_b(r)$, $\psi_b(r)$ on the semiaxis, the operator

$$\begin{pmatrix} \tilde{\varphi}_b \varphi_b & 0 \\ 0 & \tilde{\varphi}_b \end{pmatrix} \mathcal{A} \begin{pmatrix} \tilde{\psi}_b \psi_b & 0 \\ 0 & \tilde{\psi}_b \end{pmatrix}$$

belongs to $\Psi^{\mu}_{M}(\bar{\mathbb{R}}_+ \times X^{\wedge}; W, \tilde{W}; w)$, and to $\Psi^{-\infty}_{M,G}(\mathcal{W}; W, \tilde{W}; w)$ whenever $\tilde{\varphi}_b$, $\tilde{\psi}_b$ have disjoint supports.

For $\mathcal{I} = (-\infty, 0]$, we define the corresponding operator spaces as intersections over integers $l > 0$ of those for $\mathcal{I} = (-l, 0]$.

Theorem 1.4.15 remains still valid for the operators $\mathcal{A} \in \Psi^{\mu}_{M}(\mathcal{W}; W, \tilde{W}; w)$, with $X^{\wedge}$ replaced by $\mathcal{B}$. Moreover, the local symbol levels (1.4.13) extend to the corresponding global ones, and Lemma 1.4.16 is true if $X^{\wedge}$ is replaced by $\mathcal{B}$.

The above framework again suggests the standard elements of the calculus. We content ourselves with the following material.

Theorem 1.4.19 *Suppose*

$$\begin{aligned} \mathcal{A}_1 &\in \Psi^{\mu_1}_{M}(\mathcal{W}; W^1, W^2; w_1), \qquad w_1 = (\gamma, \gamma - m_1, \mathcal{I}), \\ \mathcal{A}_2 &\in \Psi^{\mu_2}_{M}(\mathcal{W}; W^2, W^3; w_2), \qquad w_2 = (\gamma - m_1, \gamma - m_1 - m_2, \mathcal{I}), \end{aligned}$$

and one of the $\mathcal{A}_1$ and $\mathcal{A}_2$ is properly supported. Then the composition $\mathcal{A}_2 \mathcal{A}_1$ is well defined in $\Psi^{\mu_1+\mu_2}_{M}(\mathcal{W}; W^1, W^3; w_2 \circ w_1)$, with $w_2 \circ w_1 = (\gamma, \gamma - m_1 - m_2, \mathcal{I})$, and

$$\begin{aligned} {}^b\sigma^{m_1+m_2}(\mathcal{A}_2 \mathcal{A}_1) &= {}^b\sigma^{m_2}(\mathcal{A}_2)\, {}^b\sigma^{m_1}(\mathcal{A}_1), \\ {}^b\sigma^{m_1+m_2}_{\mathrm{edge}}(\mathcal{A}_2 \mathcal{A}_1) &= {}^b\sigma^{m_2}_{\mathrm{edge}}(\mathcal{A}_2)\, {}^b\sigma^{m_1}_{\mathrm{edge}}(\mathcal{A}_1). \end{aligned}$$

The symbols ${}^b\sigma^m(\mathcal{A})$ and ${}^b\sigma^m_{\mathrm{edge}}(\mathcal{A})$ fall short of controlling the invertibility modulo compact operators, since from ${}^b\sigma^m(\mathcal{A}) \equiv 0$ and ${}^b\sigma^m_{\mathrm{edge}}(\mathcal{A}) \equiv 0$ it does not follow that $\mathcal{A}$ is compact. What is still lacking is a rigorous control of $\mathcal{A}$ on the face of $\mathcal{W}$ that corresponds to $t = 0$. We will turn to this question in Chapter 2. For now we merely look at the ellipticity in the class $\Psi^m_M(\mathcal{W}; W, \tilde{W}; w)$.

Definition 1.4.20 *Let $w = (\gamma, \gamma - m, \mathcal{I})$ where $m, \gamma \in \mathbb{R}$ and $\mathcal{I} = (-l, 0]$, l being a positive integer or ∞. An operator $\mathcal{A} \in \Psi^m_M(\mathcal{W}; W, \tilde{W}; w)$ is said to be elliptic if*

1) ${}^b\sigma^m(\mathcal{A}) \neq 0$ *on* ${}^bT^*\mathcal{W} \setminus \{0\}$ *up to* $t = 0$;
2) ${}^b\sigma^m_{\mathrm{edge}}(\mathcal{A})(t, \tau) \colon H^{s,\gamma}(X^\wedge) \oplus W_t \to H^{s-m,\gamma-m}(X^\wedge) \oplus \tilde{W}_t$ *is an isomorphism for all* $(t, \tau) \in T^*\bar{\mathbb{R}}_+ \setminus \{0\}$ *and some* $s \in \mathbb{R}$.

By the cone theory, if the condition 2) is satisfied for one particular $s = s_0$, then it is automatically for all $s \in \mathbb{R}$.

Suppose

$$\begin{array}{rclcrcl} \mathcal{A} & \in & \Psi^m_M(\mathcal{W}; W, \tilde{W}; w), & & w & = & (\gamma, \gamma - m, \mathcal{I}), \\ \mathcal{P} & \in & \Psi^{-m}_M(\mathcal{W}; \tilde{W}, W; w^{-1}), & & w^{-1} & = & (\gamma - m, \gamma, \mathcal{I}), \end{array}$$

and one of $\mathcal{A}$ and $\mathcal{P}$ is properly supported. Then, $\mathcal{P}$ is called a *parametrix* of $\mathcal{A}$ if

$$\begin{array}{rcl} \mathcal{P}\mathcal{A} - 1 & \in & \Psi^{-\infty}_{M,G}(\mathcal{W}; W; w^{-1} \circ w), \\ \mathcal{A}\mathcal{P} - 1 & \in & \Psi^{-\infty}_{M,G}(\mathcal{W}; \tilde{W}; w \circ w^{-1}), \end{array}$$

1 standing for the identity operator in the corresponding Ψ^0_M-classes.

Theorem 1.4.21 *Every elliptic operator $\mathcal{A} \in \Psi^m_M(\mathcal{W}; W, \tilde{W}; w)$ possesses a properly supported parametrix $\mathcal{P} \in \Psi^{-m}_M(\mathcal{W}; \tilde{W}, W; w^{-1})$.*

The role of ellipticity and elliptic regularity with asymptotics in the operator class $\Psi^m_M(\mathcal{W}; W, \tilde{W}; w)$, with $\mathcal{W} = \mathbb{R}_+ \times \mathcal{B}$, is to describe the corresponding properties along $C_1 \setminus C_0$ close to C_0 in the final "corner" pseudodifferential calculus. In this sense the next corollary gives the edge contribution to asymptotics near the corners. Recall that the relevant singular functions are described by (1.4.5).

Corollary 1.4.22 *Suppose $\mathcal{A} \in \Psi^m_M(\mathcal{W}; W, \tilde{W}; w)$ is elliptic. If*

$$\begin{array}{rcl} u & \in & \mathcal{H}^{-\infty}_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^* H^{-\infty,\gamma}(\mathcal{B})) \oplus \mathcal{H}^{-\infty}_{\mathrm{comp}}(\bar{\mathbb{R}}_+, W), \\ \mathcal{A}u & \in & \mathcal{H}^{s-m}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^* H^{s-m,\gamma-m}(\mathcal{B})) \oplus \mathcal{H}^{s-m}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \tilde{W}), \end{array}$$

for some $s \in \mathbb{R}$, then $u \in \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^* H^{s,\gamma}(\mathcal{B})) \oplus \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \tilde{W})$. If moreover

$$\begin{array}{rcl} u & \in & \mathcal{H}^{-\infty}_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^* H^{-\infty,\gamma}(\mathcal{B})) \oplus \mathcal{H}^{-\infty}_{\mathrm{comp}}(\bar{\mathbb{R}}_+, W), \\ \mathcal{A}u & \in & \mathcal{H}^{s-m}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \pi^* H^{s-m,\gamma-m}_{\widetilde{\mathrm{as}}}(\mathcal{B})) \oplus \mathcal{H}^{s-m}_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \tilde{W}), \end{array}$$

for an asymptotic type "$\widetilde{\mathrm{as}}$", then $u \in \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \pi^* H^{s,\gamma}_{\mathrm{as}}(\mathcal{B})) \oplus \mathcal{H}^s_{\mathrm{comp}}(\bar{\mathbb{R}}_+, \tilde{W})$, for a resulting asymptotic type "as" depending on $\mathcal{A}$ and "$\widetilde{\mathrm{as}}$".

Chapter 2

Corner Mellin Operators

2.1 Parameter-dependent cone operators

The operators in $\Psi_M^\mu(\mathcal{W};w)$, with $\mathcal{W}=\mathbb{R}_+\times\mathcal{B}$, can be regarded as pseudodifferential operators with operator-valued Mellin symbols. More precisely, these are

$$A=\mathrm{op}_{\mathcal{M}}(a)$$

modulo $\Psi_{M,G}^{-\infty}(\mathcal{W};w)$, where $a=a(t,\zeta)$ is given by

$$\varphi_b(r)\,(a_0(t,\zeta)+a_\infty(t,\zeta)+m(t,\zeta)+g(t,\zeta))\,\psi_b(r)+\varphi_i(r)\,a_i(t,\zeta)\,\psi_i(r), \tag{2.1.1}$$

for $(t,\zeta)\in\mathbb{R}_+\times\Gamma_0$. Here, a_0 and a_∞ are described in front of Theorem 1.4.13, m is given by (1.4.8), $g\in\mathcal{S}_G^\mu(\bar{\mathbb{R}}_+\times\Gamma_0;w)$, and $a_i\in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\Psi^\mu_{\mathrm{cl}}(\mathcal{B}\setminus\partial\mathcal{B};\Gamma_0))$. Moreover, (φ_b,φ_i) is a partition of unity on $\mathcal{B}$, with φ_b supported in a collar neighbourhood of $\partial\mathcal{B}$, and (ψ_b,ψ_i) is a system of C^∞ functions on $\mathcal{B}$ covering (φ_b,φ_i).

An analogous description holds for $\Psi_M^\mu(\mathcal{W};W,\tilde{W};w)$. We shall first discuss the upper left corners.

The symbols (2.1.1) satisfy $a\in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+\times\Gamma_0,\Psi^\mu(\mathcal{B};w))$ where $\Psi^\mu(\mathcal{B};w)$ is the class of cone operators over $\mathcal{B}$ from Definition 1.2.3. The t-independent $a(\zeta)$ will be examples of parameter-dependent cone operator families in the sense of this section. We want to study the parameter-dependent ellipticity for getting order reducing families of operators over $\mathcal{B}$ analogous to $R^m(\lambda)$ of Section 1.1. Furthermore, we look at those families which are holomorphic in ζ. General considerations allow parameters to vary in a conical subset of a finite-dimensional vector space. For simplicity, here we take the one-dimensional parameter space $\tau\in\mathbb{R}$. In the sequel we then switch to $\tau+iv\in\Gamma_{-\delta}$, for any fixed $\delta\in\mathbb{R}$.

Definition 2.1.1 *Let* $w = (\gamma, \delta, \mathcal{I})$. *Then* $\Psi_G^{-\infty}(\mathcal{B}; w; \mathbb{R})$ *is the space of all operator families* $g(\tau) \in \cap_{s\in\mathbb{R}} C^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}(H^{s,\gamma}(\mathcal{B}), H^{\infty,\delta}(\mathcal{B})))$ *with the property that*

$$\begin{aligned} g(\tau) &\in \textstyle\bigcap_{s\in\mathbb{R}} \mathcal{S}(\mathbb{R}, \mathcal{L}(H^{s,\gamma}(\mathcal{B}), H^{\infty,\delta}_{\mathrm{as}}(\mathcal{B}))), \\ g^*(\tau) &\in \textstyle\bigcap_{s\in\mathbb{R}} \mathcal{S}(\mathbb{R}, \mathcal{L}(H^{s,-\delta}(\mathcal{B}), H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{B}))) \end{aligned}$$

for some asymptotic types "as" and "$\tilde{\mathrm{as}}$" related to weight data $(\delta, \mathcal{I})$ *and* $(-\gamma, \mathcal{I})$, *respectively.*

Here by g^* is meant the formal adjoint of g with respect to a fixed scalar product in $H^{0,0}(\mathcal{B})$.

The space of all $g(\tau) \in \Psi_G^{-\infty}(\mathcal{B}; w; \mathbb{R})$ with fixed asymptotic types "as" and "$\tilde{\mathrm{as}}$" is Fréchet in a natural way.

Let $w = (\gamma, \gamma - m, \mathcal{I})$, where $\gamma \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $\mathcal{I} = (-l, 0]$, l being finite or infinite. Then, $\Psi_G^\mu(X^\wedge; w; \mathbb{R})$ denotes the space of all "twisted" symbols

$$g(\tau) \in \bigcap_{s\in\mathbb{R}} \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^\wedge), H^{\infty,\gamma-m}(X^\wedge)))$$

such that

$$\begin{aligned} g(\tau) &\in \textstyle\bigcap_{s\in\mathbb{R}} \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^\wedge), \mathcal{S}^{\gamma-m}_{\mathrm{as}}(X^\wedge))), \\ g^*(\tau) &\in \textstyle\bigcap_{s\in\mathbb{R}} \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R}, \mathcal{L}(H^{s,m-\gamma}(X^\wedge), \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge))) \end{aligned}$$

for certain asymptotic types "as" and "$\tilde{\mathrm{as}}$" related to weight data $(\gamma - m, \mathcal{I})$ and $(-\gamma, \mathcal{I})$, respectively.

Furthermore, $\Psi^\mu_{M+G}(X^\wedge; w; \mathbb{R})$ stands for the space of all $m(\tau)+g(\tau)$ where $g(\tau) \in \Psi^\mu_G(X^\wedge; w; \mathbb{R})$ and $m(\tau)$ is given by (1.3.6), the t-dependence being dropped.

Note that $\Psi^\mu_G(X^\wedge; w; \mathbb{R})$ coincides with the subclass of (t, t')-independent elements of $\mathcal{S}^\mu_G((\mathbb{R}_+ \times \mathbb{R}_+) \times \mathbb{R}; w)$, the latter space being introduced in Definition 1.3.5. From (1.4.7) we obtain a principal homogeneous "Mellin edge symbol" of order μ,

$${}^b\sigma^\mu_{\mathrm{edge}}(m + g)(\tau) \in \bigcap_{s\in\mathbb{R}} \mathcal{S}^\mu_{\mathrm{hg}}(\mathbb{R} \setminus \{0\}, \mathcal{L}(H^{s,\gamma}(X^\wedge), H^{s-\mu,\gamma-m}(X^\wedge))).$$

Finally, the Fuchs type operators close to the boundary of $\mathcal{B}$ contribute to the calculus by

$$\begin{aligned} a_0(\tau) &= \varphi_0(r\langle\tau\rangle)\, r^{-\mu}\, \mathrm{op}_{\mathcal{M},\gamma}(h)(\tau)\, \psi_0(r\langle\tau\rangle), \\ a_\infty(\tau) &= \varphi_\infty(r\langle\tau\rangle)\, r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(a)(\tau)\, \psi_\infty(r\langle\tau\rangle) \end{aligned}$$

where

$$\begin{aligned} a(r, \tau, \varrho) &= \tilde{a}(r, r\tau, r\varrho), \\ h(r, \tau, z) &= \tilde{h}(r, r\tau, z), \end{aligned}$$

for

$$\begin{array}{rcl}\tilde{a}(r,\tilde{\tau},\tilde{\varrho}) & \in & C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}^2_{\tilde{\tau},\tilde{\varrho}})),\\ \tilde{h}(r,\tilde{\tau},z) & \in & C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+,\mathcal{M}(\mathbb{C},\Psi^\mu_{\mathrm{cl}}(X;\mathbb{R}_{\tilde{\tau}})))\end{array}$$

compatible in the sense that $\mathrm{op}_{\mathcal{F}_r}(a)=\mathrm{op}_{\mathcal{M},\gamma}(h)$ modulo $\Psi^{-\infty}(X^\wedge;\mathbb{R}_\tau)$.

Definition 2.1.2 *Suppose $w=(\gamma,\gamma-m,\mathcal{I})$, where $\gamma\in\mathbb{R}$, $m-\mu\in\mathbb{Z}_+$ and $\mathcal{I}=(-l,0]$, $l\in\mathbb{N}$. Then, $\Psi^\mu(\mathcal{B};w;\mathbb{R})$ denotes the space of all operator families*

$$a(\tau)=\varphi_b(r)\,(a_0(\tau)+a_\infty(\tau)+m(\tau)+g(\tau))\,\psi_b(r)+\varphi_i(r)\,a_i(\tau)\,\psi_i(r)+s(\tau),$$

where $a_0(\tau)$ and $a_\infty(\tau)$ are as above; $m(\tau)+g(\tau)$ lies in $\Psi^\mu_{M+G}(X^\wedge;w;\mathbb{R})$; $a_i(\tau)$ is a usual pseudodifferential operator with parameter in the interior of $\mathcal{B}$, i.e., $a_i\in\Psi^\mu_{\mathrm{cl}}(\mathcal{B}\setminus\partial\mathcal{B};\mathbb{R})$; and $s(\tau)$ is a negligible element in the calculus, i.e., $s\in\Psi^{-\infty}_G(\mathcal{B};w;\mathbb{R})$.

Note that here we impose a similar structure for $a_0(\tau)$ and $a_\infty(\tau)$, as that of (1.4.11). Since negligible remainders cause only a change of $g(\tau)$, we may actually assume that

$$\tilde{a}(r,\tilde{\tau},\tilde{\varrho})=\sum_{\nu\in\mathcal{N}}\varphi_\nu\left(\left(\kappa_\nu^{-1}\right)_\sharp\mathrm{op}_{\mathcal{F}_x}(\tilde{a}_\nu)\,(r,\tilde{\tau},\tilde{\varrho})\right)\psi_\nu$$

where $\tilde{a}_\nu(r,x,\tilde{\tau},\tilde{\varrho},\xi)\in\mathcal{S}^\mu_{\mathrm{cl}}((\bar{\mathbb{R}}_+\times O_\nu)\times\mathbb{R}^{2+n})$ are local symbols corresponding to a covering of X by coordinate patches.

Write $\Psi^\mu_{M+G}(\mathcal{B};w;\mathbb{R})$ for the subspace of $\Psi^\mu(\mathcal{B};w;\mathbb{R})$ consisting of all operator families $\varphi_b(r)\,(m(\tau)+g(\tau))\,\psi_b(r)+s(\tau)$, where $m(\tau)$, $g(\tau)$ and $s(\tau)$ are as in Definition 2.1.2. Moreover, we denote by $\Psi^\mu_G(\mathcal{B};w;\mathbb{R})$ the subclass where $m(\tau)$ vanishes.

In case $\mathcal{I}$ is infinite, the space $\Psi^\mu(\mathcal{B};w;\mathbb{R})$ is defined to be the intersection of corresponding spaces for $\mathcal{I}=(-l,0]$, $l=1,2,\ldots$.

The space $\Psi^\mu(\mathcal{B};w;\mathbb{R})$ is a parameter-dependent version of $\Psi^\mu(\mathcal{B};w)$ from Definition 1.2.3. The elements $a(\tau)\in\Psi^\mu(\mathcal{B};w;\mathbb{R})$ are a preliminary version of operator-valued Mellin symbols for corners. Our special parameter dependence involves the edge degeneracy in τ if τ is regarded as a Mellin covariable along the t-axis.

We have

$$\Psi^\mu(\mathcal{B};w;\mathbb{R})\hookrightarrow\Psi^\mu_{\mathrm{cl}}(\mathring{\mathcal{B}};\mathbb{R}),$$

the subspace $\Psi^\mu_{M+G}(\mathcal{B};w;\mathbb{R})$ getting into smoothing operators in the interior of $\mathcal{B}$ under this embedding. As is described in Section 1.3, cf. (1.3.12), each parameter-dependent operator $a(\tau)\in\Psi^\mu(\mathcal{B};w;\mathbb{R})$ bears two leading homogeneous symbols

$$\begin{array}{rcl}{}^b\sigma^\mu(a) & \in & \mathcal{S}^\mu_{\mathrm{hg}}(({}^bT^*\mathcal{B}\times\mathbb{R})\setminus\{0\}),\\ {}^b\sigma^\mu_{\mathrm{edge}}(a) & \in & \displaystyle\bigcap_{s\in\mathbb{R}}\mathcal{S}^\mu_{\mathrm{hg}}(\mathbb{R}\setminus\{0\},\mathcal{L}(H^{s,\gamma}(X^\wedge),H^{s-\mu,\gamma-m}(X^\wedge))),\end{array}$$

the latter being the principal homogeneous Mellin edge symbol.

Without loss of generality we may assume that

$$a_i(\tau) = r^{-\mu}\, \mathrm{op}_{\mathcal{F}_r}(a)(\tau)$$

modulo $\Psi^{-\infty}(O \setminus \partial\mathcal{B}; \mathbb{R})$, where O is the collar neighbourhood of $\partial\mathcal{B}$. Then, the concrete choice of the partitions of unity and covering systems involved in (2.1.1) affects $a(t)$ only modulo $\Psi^{\mu}_{G}(\mathcal{B}; w; \mathbb{R})$.

For the Mellin pseudodifferential calculus along the corner axis $\mathbb{R}_+$ with operator-valued symbols acting on $\mathcal{B}$ it is necessary to endow $\Psi^{\mu}(\mathcal{B}; w; \mathbb{R})$ with a natural locally convex topology. Let us start with $\Psi^{\mu}_{G}(\mathcal{B}; w; \mathbb{R})$. By definition, we have

$$\Psi^{\mu}_{G}(\mathcal{B}; w; \mathbb{R}) = \varphi_b\, \Psi^{\mu}_{G}(X^\wedge; w; \mathbb{R})\, \psi_b + \Psi^{-\infty}_{G}(\mathcal{B}; w; \mathbb{R}).$$

Denote by $\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R})$ the space of all $g(\tau) \in \Psi^{\mu}_{G}(X^\wedge; w; \mathbb{R})$ with fixed asymptotic types "as" and "$\tilde{\mathrm{as}}$", as above. Further, let $\mathcal{S}^{\mu}_{hg,\mathrm{as},\tilde{\mathrm{as}}}(\mathbb{R}\setminus\{0\}; w)$ stand for the space of all "twisted" homogeneous functions $\sigma(\tau)$ with the properties that

$$\begin{array}{rcl} \sigma(\tau) & \in & \bigcap_{s\in\mathbb{R}} \mathcal{S}^{\mu}_{\mathrm{hg}}(\mathbb{R} \setminus \{0\}, \mathcal{L}(H^{s,\gamma}(X^\wedge), \mathcal{S}^{\gamma-m}_{\mathrm{as}}(X^\wedge))), \\ \sigma^*(\tau) & \in & \bigcap_{s\in\mathbb{R}} \mathcal{S}^{\mu}_{\mathrm{hg}}(\mathbb{R} \setminus \{0\}, \mathcal{L}(H^{s,m-\gamma}(X^\wedge), \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge))). \end{array}$$

The space $\mathcal{S}^{\mu}_{\mathrm{hg},\mathrm{as},\tilde{\mathrm{as}}}(\mathbb{R} \setminus \{0\}; w)$ is Fréchet in a canonical way. The mapping $\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R}) \to \mathcal{S}^{\mu}_{\mathrm{hg},\mathrm{as},\tilde{\mathrm{as}}}(\mathbb{R} \setminus \{0\}; w)$ given by $g(\tau) \mapsto {}^b\sigma^{\mu}_{\mathrm{edge}}(g)(\tau)$ is surjective, and its null-space is $\Psi^{\mu-1}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R})$. Given any excision function $\chi(\tau)$, we have

$$g_1(\tau) := g(\tau) - \chi(\tau)\, {}^b\sigma^{\mu}_{\mathrm{edge}}(g)(\tau) \in \Psi^{\mu-1}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R})$$

for all $g(\tau) \in \Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R})$. We now apply this argument again, with $g(\tau)$ replaced by $g_1(\tau)$, and so on, to obtain mappings

$$\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R}) \to \otimes_{j=0}^{J}\, \mathcal{S}^{\mu-j}_{\mathrm{hg},\mathrm{as},\tilde{\mathrm{as}}}(\mathbb{R} \setminus \{0\}; w) \tag{2.1.2}$$

and

$$\begin{array}{rcl} \Psi^{\mu}_{G}(X^\wedge; w; \mathbb{R}) & \to & \bigcap_{s\in\mathbb{R}} \mathcal{S}^{\mu-(J+1)}(\mathbb{R}, \mathcal{L}(H^{s,\gamma}(X^\wedge), \mathcal{S}^{\gamma-m}_{\mathrm{as}}(X^\wedge))), \\ \Psi^{\mu}_{G}(X^\wedge; w; \mathbb{R}) & \to & \bigcap_{s\in\mathbb{R}} \mathcal{S}^{\mu-(J+1)}(\mathbb{R}, \mathcal{L}(H^{s,m-\gamma}(X^\wedge), \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge))), \end{array} \tag{2.1.3}$$

for $J \in \mathbb{Z}_+$. The first mapping of (2.1.3) is given by $g \mapsto g - \chi \sum_{j=0}^{J} {}^b\sigma^{\mu-j}_{\mathrm{edge}}(g_j)$ with $g_0 = g$, and the second one is the adjoint. Now, $\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R})$ is a Fréchet space under the projective limit topology with respect to the mappings (2.1.2) and (2.1.3). It is independent of the concrete choice of χ. Gluing together $\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(X^\wedge; w; \mathbb{R})$ and $\Psi^{-\infty}_{G,\mathrm{as},\tilde{\mathrm{as}}}(\mathcal{B}; w; \mathbb{R})$, we arrive at a Fréchet space $\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(\mathcal{B}; w; \mathbb{R})$. Then, $\Psi^{\mu}_{G}(\mathcal{B}; w; \mathbb{R})$ is given the inductive limit topology of the spaces $\Psi^{\mu}_{G,\mathrm{as},\tilde{\mathrm{as}}}(\mathcal{B};w;\mathbb{R})$, with "as" and "$\tilde{\mathrm{as}}$" varying over all asymptotic types related to w.

We now turn to $\Psi^{\mu}_{M+G}(\mathcal{B};w;\mathbb{R})$. In this space we look at the conormal symbolic mappings operating only on the Mellin terms. We have the operator-valued "twisted" homogeneous components ${}^b\sigma^{\mu-i}_{\mathrm{edge}}(m)(\tau)$, cf. (1.4.8), defined by

$$\varphi_0(r|\tau|)\, r^{-\mu}\sum_{j=0}^{l-1} r^j\tau^{j-i}\Big(\mathrm{op}_{\mathcal{M},\gamma^{(1)}_{j,j-i}}(m^{(1)}_{j,j-i})+\mathrm{op}_{\mathcal{M},\gamma^{(2)}_{j,j-i}}(m^{(2)}_{j,j-i})\Big)\psi_0(r|\tau|),$$

for $i=0,1,\ldots,l-1$, both $m^{(1)}_{j,j-i}$ and $m^{(2)}_{j,j-i}$ vanishing unless $i\le j$. Then we may form the sequence of conormal symbols $\sigma^j_{\mathcal{M}}\left({}^b\sigma^{\mu-i}_{\mathrm{edge}}(m)(\tau)\right)(z)$, $j\le l-1$, by

$$\sigma^j_{\mathcal{M}}\left({}^b\sigma^{\mu-i}_{\mathrm{edge}}(m)(\tau)\right)(z)=\tau^{j-i}\left(m^{(1)}_{j,j-i}(z)+m^{(2)}_{j,j-i}(z)\right), \tag{2.1.4}$$

cf. (1.2.6). These are elements of $\mathcal{M}_T(\mathbb{C},\Psi^{-\infty}(X))$ determining ${}^b\sigma^{\mu-i}_{\mathrm{edge}}(m)(\tau)$ up to an element in $\mathcal{S}^{\mu-i}_{\mathrm{hg,as},\tilde{\mathrm{as}}}(\mathbb{R}\setminus\{0\};w)$, for certain asymptotic types "as" and "$\tilde{\mathrm{as}}$". The sequence (2.1.4) is finite for all i and j, thus defining a mapping of $\Psi^{\mu}_{M+G}(\mathcal{B};w;\mathbb{R})$ to the product of l^2 copies of $\cup_T\mathcal{M}_T(\mathbb{C},\Psi^{-\infty}(X))$, where T varies over all asymptotic types for Mellin symbols. Using this mapping, one can specify $\Psi^{\mu}_{M+G}(\mathcal{B};w;\mathbb{R})$ within the non-direct sums of Cartesian products $\Psi^{\mu}_{G}(\mathcal{B};w;\mathbb{R})\times\cup_{T'}\mathcal{M}_{T'}(\mathbb{C},\Psi^{-\infty}(X))$, the 'prime' meaning that the union is over asymptotic types whose carriers do not meet fixed weight lines. Therefore, $\Psi^{\mu}_{M+G}(\mathcal{B};w;\mathbb{R})$ can be endowed with the topology of non-direct sum of Fréchet spaces.

Having disposed of this most subtle step, we give $\Psi^{\mu}(\mathcal{B};w;\mathbb{R})$ the topology of the inductive limit of Fréchet spaces in a natural way.

Definition 2.1.3 *By* $\mathrm{Symb}\,\Psi^{\mu}(\mathcal{B};w;\mathbb{R})$ *is meant the space of all pairs of symbols* $\left({}^b\sigma^{\mu}(a),{}^b\sigma^{\mu}_{\mathrm{edge}}(a)\right)$ *where* $a(\tau)$ *runs through* $\Psi^{\mu}(\mathcal{B};w;\mathbb{R})$.

The components of $\mathrm{Symb}\,\Psi^{\mu}(\mathcal{B};w;\mathbb{R})$ are easily checked to satisfy a natural compatibility condition.

Theorem 2.1.4 *Suppose* l *is a positive integer. As defined above, the sequence*

$$0\to\Psi^{\mu-1}(\mathcal{B};w;\mathbb{R})\to\Psi^{\mu}(\mathcal{B};w;\mathbb{R})\to\mathrm{Symb}\,\Psi^{\mu}(\mathcal{B};w;\mathbb{R})\to 0$$

is exact and splits.

As we have seen in the preceding sections, it is necessary to consider also matrix-valued families, here denoted by $\Psi^{\mu}(\mathcal{B};W,\tilde{W};w;\mathbb{R})$, with $\Psi^{\mu}(\mathcal{B};w;\mathbb{R})$ as the class of occurring upper left corners. Basically we need only generalise the Green objects.

Definition 2.1.5 *Let* $w=(\gamma,\delta,\mathcal{I})$. *Then* $\Psi_G^{-\infty}(\mathcal{B};W,\tilde{W};w;\mathbb{R})$ *is the space of all operator families* $\mathfrak{g}(\tau)\in\cap_{s\in\mathbb{R}}C^\infty_{\mathrm{loc}}(\mathbb{R},\mathcal{L}(H^{s,\gamma}(\mathcal{B})\oplus\mathbb{C}^N,H^{\infty,\delta}(\mathcal{B})\oplus\mathbb{C}^{\tilde{N}}))$ *such that*

$$\begin{array}{rcl}\mathfrak{g}(\tau)&\in&\bigcap_{s\in\mathbb{R}}\ \mathcal{S}(\mathbb{R},\mathcal{L}(H^{s,\gamma}(\mathcal{B})\oplus\mathbb{C}^N,H^{\infty,\delta}_{\mathrm{as}}(\mathcal{B})\oplus\mathbb{C}^{\tilde{N}})),\\ \mathfrak{g}^*(\tau)&\in&\bigcap_{s\in\mathbb{R}}\ \mathcal{S}(\mathbb{R},\mathcal{L}(H^{s,-\delta}(\mathcal{B})\oplus\mathbb{C}^{\tilde{N}},H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{B})\oplus\mathbb{C}^N))\end{array}$$

for some asymptotic types "as" *and* "$\tilde{\text{as}}$" *related to weight data* $(\delta,\mathcal{I})$ *and* $(-\gamma,\mathcal{I})$, *respectively.*

Here $\mathfrak{g}^*$ stands for the formal adjoint of $\mathfrak{g}$ with respect to fixed scalar products in the spaces $H^{0,0}(\mathcal{B})\oplus W$ and $H^{0,0}(\mathcal{B})\oplus\tilde{W}$.

Let $w=(\gamma,\gamma-m,\mathcal{I})$, where $\gamma\in\mathbb{R}$, $m-\mu\in\mathbb{Z}_+$ and $\mathcal{I}=(-l,0]$, l being finite or infinite. Then, by $\Psi^\mu_G(X^\wedge;W,\tilde{W};w;\mathbb{R})$ we mean the space of all symbols

$$\mathfrak{g}(\tau)\in\bigcap_{s\in\mathbb{R}}\mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R},\mathcal{L}(H^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^N,H^{\infty,\gamma-m}(X^\wedge)\oplus\mathbb{C}^{\tilde{N}}))$$

such that

$$\begin{array}{rcl}\mathfrak{g}(\tau)&\in&\bigcap_{s\in\mathbb{R}}\ \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R},\mathcal{L}(H^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^N,\mathcal{S}^{\gamma-m}_{\mathrm{as}}(X^\wedge)\oplus\mathbb{C}^{\tilde{N}})),\\ \mathfrak{g}^*(\tau)&\in&\bigcap_{s\in\mathbb{R}}\ \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R},\mathcal{L}(H^{s,m-\gamma}(X^\wedge)\oplus\mathbb{C}^{\tilde{N}},\mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(X^\wedge)\oplus\mathbb{C}^N))\end{array}$$

for certain asymptotic types "as" and "$\tilde{\text{as}}$" related to weight data $(\gamma-m,\mathcal{I})$ and $(-\gamma,\mathcal{I})$, respectively.

We also introduce the space $\Psi^\mu_G(\mathcal{B};W,\tilde{W};w;\mathbb{R})$ consisting of all operators of the form

$$\begin{pmatrix}\varphi_b(r)&0\\0&1\end{pmatrix}\mathfrak{g}(\tau)\begin{pmatrix}\psi_b(r)&0\\0&1\end{pmatrix}+\mathfrak{s}(\tau)$$

where $\mathfrak{g}(\tau)\in\Psi^\mu_G(X^\wedge;W,\tilde{W};w;\mathbb{R})$ and $\mathfrak{s}(\tau)\in\Psi_G^{-\infty}(\mathcal{B};W,\tilde{W};w;\mathbb{R})$.

The space $\Psi^\mu(\mathcal{B};W,\tilde{W};w;\mathbb{R})$ is defined as the set of all

$$\begin{pmatrix}a(\tau)&0\\0&0\end{pmatrix}+\mathfrak{g}(\tau)$$

with $a(\tau)\in\Psi^\mu(\mathcal{B};w;\mathbb{R})$ and $\mathfrak{g}(\tau)\in\Psi^\mu_G(\mathcal{B};W,\tilde{W};w;\mathbb{R})$. Confining ourselves to $a(\tau)\in\Psi^\mu_{M+G}(\mathcal{B};w;\mathbb{R})$, we obtain the space $\Psi^\mu_{M+G}(\mathcal{B};W,\tilde{W};w;\mathbb{R})$ in the same way.

For the infinite weight interval $\mathcal{I}$, the space $\Psi^\mu(\mathcal{B};W,\tilde{W};w;\mathbb{R})$ is defined to be the intersection of the classes over all $\mathcal{I}_l=(-l,0]$, $l=1,2,\ldots$.

We also mention obvious parameter independent analogues $\Psi_G(\mathcal{B};W,\tilde{W};w)$ and $\Psi^\mu(\mathcal{B};W,\tilde{W};w)$ of the above spaces.

The construction of a locally convex topology in $\Psi^\mu(\mathcal{B};W,\tilde{W};w;\mathbb{R})$ is completely analogous to that for the upper left corners.

Moreover, we can introduce the set $\operatorname{Symb}\Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$ of all pairs $({}^b\sigma^\mu(\mathfrak{a}), {}^b\sigma^\mu_{\text{edge}}(\mathfrak{a}))$ where $\mathfrak{a}(\tau)$ runs over $\Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$. Theorem 2.1.4 carries over to this more general context.

Our spaces of operator families have many natural properties with respect to various operations. We will sketch a few of them.

Theorem 2.1.6 *Suppose* $\mathfrak{a} \in \Psi^0(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$, *for* $w = (\gamma, \gamma, \mathcal{I})$. *Then we have*

$$\|\mathfrak{a}(\tau)\|_{\mathcal{L}(H^{0,\gamma}(\mathcal{B})\oplus\mathbb{C}^N, H^{0,\gamma}(\mathcal{B})\oplus\mathbb{C}^{\tilde{N}})} \le c$$

with $c > 0$ *a constant independent of* $\tau \in \mathbb{R}$.

Clearly there are much more precise norm growth estimates for general $\mathfrak{a} \in \Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$, but we do not need them here.

Theorem 2.1.7 *If* $\mathfrak{a}(\tau) \in \Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$, *for* $w = (\gamma, \gamma - m, \mathcal{I})$, *then* $\mathfrak{a}^*(\tau) \in \Psi^\mu(\mathcal{B}; \tilde{W}, W; w^*; \mathbb{R})$, *for* $w^* = (m - \gamma, -\gamma, \mathcal{I})$, *where* $\mathfrak{a}^*(\tau)$ *is the pointwise formal adjoint with respect to the scalar products in* $H^{0,0}(\mathcal{B}) \oplus \mathbb{C}^N$ *and* $H^{0,0}(\mathcal{B}) \oplus \mathbb{C}^{\tilde{N}}$. *Moreover,*

$$\begin{aligned} {}^b\sigma^\mu(\mathfrak{a}^*) &= \left({}^b\sigma^\mu(\mathfrak{a})\right)^*, \\ {}^b\sigma^\mu_{\text{edge}}(\mathfrak{a}^*) &= \left({}^b\sigma^\mu_{\text{edge}}(\mathfrak{a})\right)^*. \end{aligned}$$

Here, the formal adjoints of symbols on the right-hand sides are specified in the sense of relevant operator classes.

Theorem 2.1.8 *Let*

$$\begin{aligned} \mathfrak{a}_1(\tau) &\in \Psi^{\mu_1}(\mathcal{B}; W^1, W^2; w_1; \mathbb{R}), && w_1 = (\gamma, \gamma - m_1, \mathcal{I}), \\ \mathfrak{a}_2(\tau) &\in \Psi^{\mu_2}(\mathcal{B}; W^2, W^3; w_2; \mathbb{R}), && w_2 = (\gamma - m_1, \gamma - m_1 - m_2, \mathcal{I}), \end{aligned}$$

then the composition $(\mathfrak{a}_2\mathfrak{a}_1)(\tau)$ *is well defined in* $\Psi^{\mu_1+\mu_2}(\mathcal{B}; W^1, W^3; w_2 \circ w_1; \mathbb{R})$, *with* $w_2 \circ w_1 = (\gamma, \gamma - m_1 - m_2, \mathcal{I})$, *and*

$$\begin{aligned} {}^b\sigma^{m_1+m_2}(\mathfrak{a}_2\mathfrak{a}_1) &= {}^b\sigma^{m_2}(\mathfrak{a}_2)\, {}^b\sigma^{m_1}(\mathfrak{a}_1), \\ {}^b\sigma^{m_1+m_2}_{\text{edge}}(\mathfrak{a}_2\mathfrak{a}_1) &= {}^b\sigma^{m_2}_{\text{edge}}(\mathfrak{a}_2)\, {}^b\sigma^{m_1}_{\text{edge}}(\mathfrak{a}_1). \end{aligned}$$

We are now in a position to use the machinery of pseudodifferential operators to construct order reducing families within the calculus on $\mathcal{B}$.

Definition 2.1.9 *Let* $w = (\gamma, \gamma - m, \mathcal{I})$ *where* $m, \gamma \in \mathbb{R}$ *and* $\mathcal{I} = (-l, 0]$, $l \in \mathbb{N} \cup \{\infty\}$. *An operator* $\mathfrak{a}(\tau) \in \Psi^m(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$ *is called elliptic with parameter if*

1) ${}^b\sigma^m(\mathfrak{a}) \neq 0$ *on* $({}^bT^*\mathcal{B} \times \mathbb{R}) \setminus \{0\}$;

2) ${}^b\sigma^m_{\text{edge}}(\mathfrak{a})(\tau)\colon H^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^N \to H^{s-m,\gamma-m}(X^\wedge) \oplus \mathbb{C}^{\tilde{N}}$ *is an isomorphism for all* $\tau \in \mathbb{R} \setminus \{0\}$ *and some* $s \in \mathbb{R}$.

By the above it is clear that when 2) is satisfied for one $s = s_0$ then it is automatically for all $s \in \mathbb{R}$.

Let

$$\begin{array}{rclrcl} \mathfrak{a}(\tau) & \in & \Psi^m(\mathcal{B}; W, \tilde{W}; w; \mathbb{R}), & w & = & (\gamma, \gamma - m, \mathcal{I}), \\ \mathfrak{p}(\tau) & \in & \Psi^{-m}(\mathcal{B}; \tilde{W}, W; w^{-1}; \mathbb{R}), & w^{-1} & = & (\gamma - m, \gamma, \mathcal{I}), \end{array}$$

then $\mathfrak{p}(\tau)$ is said to be a *parametrix* of $\mathfrak{a}(\tau)$ if

$$\begin{array}{rcl} \mathfrak{p}(\tau)\mathfrak{a}(\tau) - 1 & \in & \Psi_G^{-\infty}(\mathcal{B}; W; w^{-1} \circ w; \mathbb{R}), \\ \mathfrak{a}(\tau)\mathfrak{p}(\tau) - 1 & \in & \Psi_G^{-\infty}(\mathcal{B}; \tilde{W}; w \circ w^{-1}; \mathbb{R}) \end{array}$$

where 1 stands for the identity operator in the corresponding classes.

Theorem 2.1.10 *Every elliptic operator* $\mathfrak{a}(\tau) \in \Psi^m(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$ *has a parametrix* $\mathfrak{p}(\tau) \in \Psi^{-m}(\mathcal{B}; \tilde{W}, W; w^{-1}; \mathbb{R})$.

The following results are straightforward consequences of the existence of a parametrix $\mathfrak{p}(\tau)$ within the calculus.

Corollary 2.1.11 *Suppose that* $\mathfrak{a}(\tau) \in \Psi^m(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$ *is elliptic, where* $w = (\gamma, \gamma - m, \mathcal{I})$. *Then there exists a constant* $c > 0$ *with the property that*

$$\mathfrak{a}(\tau): \ H^{s,\gamma}(\mathcal{B}) \oplus \mathbb{C}^N \to H^{s-m,\gamma-m}(\mathcal{B}) \oplus \mathbb{C}^{\tilde{N}}$$

is an isomorphism for all $s \in \mathbb{R}$, *provided* $|\tau| \geq c$.

If $\mathfrak{a}(\tau)$ is an isomorphism for a fixed $s = s_0$ and all $\tau \in \mathbb{R}$, then it is so for *all* $s \in \mathbb{R}$ and all $\tau \in \mathbb{R}$, and $\mathfrak{a}^{-1}(\tau) \in \Psi^{-m}(\mathcal{B}; \tilde{W}, W; w; \mathbb{R})$.

Corollary 2.1.12 *For every weight data* $w = (\gamma, \gamma - m, \mathcal{I})$, $\mathcal{I} = (-l, 0]$, *there exists a parameter-dependent elliptic operator* $a(\tau) \in \Psi^m(\mathcal{B}; w; \mathbb{R})$ *such that*

$$a(\tau): \ H^{s,\gamma}(\mathcal{B}) \to H^{s-m,\gamma-m}(\mathcal{B})$$

is bijective for all $s \in \mathbb{R}$ *and all* $\tau \in \mathbb{R}$.

As mentioned, τ will be interpreted as $\Re\zeta$, for $\zeta \in \mathbb{C}$. For a fixed $\delta \in \mathbb{R}$, we denote $\Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \Gamma_\delta)$ the space of all operator families $\mathfrak{a}(\zeta)$, $\zeta = \tau + i\delta$, such that $\mathfrak{a}(\tau + i\delta) \in \Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$. Analogously we use notation like $\Psi^\mu(\mathcal{B}; w; \Gamma_\delta)$, etc.

Recall that $\Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$ is topologised as the inductive limit of Fréchet spaces. Let $\mathfrak{a}(\zeta)$ be a holomorphic function in a strip $\Delta' < \Im\zeta < \Delta''$ with values in $\cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{B}) \oplus \mathbb{C}^N, H^{s-\mu,\gamma-m}(\mathcal{B}) \oplus \mathbb{C}^{\tilde{N}})$. We say that $\mathfrak{a}(\zeta)$ belongs to $\Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \Gamma_\delta)$ uniformly in $\delta \in [\delta', \delta'']$, where $\Delta' < \delta' < \delta'' < \Delta''$, if $\mathfrak{a}(\tau + i\delta)$ lies in the same Fréchet space within $\Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \mathbb{R})$ for all $\delta \in [\delta', \delta'']$, and the corresponding seminorms of $\mathfrak{a}(\tau + i\delta)$ are bounded uniformly in $\delta \in [\delta', \delta'']$.

Theorem 2.1.13 *Let $\delta \in \mathbb{R}$. For every $\mathfrak{a}(\zeta) \in \Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \Gamma_{-\delta})$ there is an*

$$\mathfrak{h}(\zeta) \in \bigcap_{s\in\mathbb{R}} \mathcal{A}(\mathbb{C}, \mathcal{L}(H^{s,\gamma}(\mathcal{B}) \oplus \mathbb{C}^N, H^{s-\mu,\gamma-m}(\mathcal{B}) \oplus \mathbb{C}^{\tilde{N}}))$$

such that

1) $\mathfrak{h}(\zeta) \in \Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \Gamma_\upsilon)$ *for all $\upsilon \in \mathbb{R}$, uniformly in υ on compact intervals of $\mathbb{R}$;*
2) $\mathfrak{h}(\zeta)\,|_{\Gamma_{-\delta}} - \mathfrak{a}(\zeta) \in \Psi_G^{-\infty}(\mathcal{B}; W, \tilde{W}; w; \Gamma_{-\delta})$.

The proof follows by a generalisation of kernel cut-off arguments as they have been used systematically in Schulze [Sch98, 2.2.2], cf. also Theorem 1.3.8. Here τ is treated as $\Re\zeta$ for the Mellin covariable ζ with respect to the Mellin transform $\mathcal{M}_{t\mapsto\zeta}$.

The kernel cut-off technique also allows one to choose the order reducing family of Corollary 2.1.12 in the form $a(\zeta)\,|_{\Gamma_{-\delta}}$ with fixed $\delta \in \mathbb{R}$, $a(\zeta)$ being an entire function with values in $\bigcap_{s\in\mathbb{R}} \mathcal{A}(\mathbb{C}, \mathcal{L}(H^{s,\gamma}(\mathcal{B}), H^{s-m,\gamma-m}(\mathcal{B})))$, such that $a(\zeta) \in \Psi^\mu(\mathcal{B}; w; \Gamma_\upsilon)$ for all $\upsilon \in \mathbb{R}$, uniformly in each finite interval. Moreover, for any δ' and δ'' we can find $a(\zeta)$ in such a way that $a(\zeta)\,|_{\Gamma_{-\delta}}$ is order reducing for all $-\delta \in [\delta', \delta'']$.

Let $\mathfrak{a}(\zeta) \in \Psi^m(\mathcal{B}; W, \tilde{W}; w; \Gamma_{-\delta})$ be elliptic and $\mathfrak{h}(\zeta)$ be associated with $\mathfrak{a}(\zeta)$ via Theorem 2.1.13. Then $\mathfrak{h}(\zeta)\,|_{\Gamma_\upsilon}$ is elliptic for every $\upsilon \in \mathbb{R}$. Given any $\delta' < \delta''$, there is a $c > 0$ such that $\mathfrak{h}(\zeta) \colon H^{s,\gamma}(\mathcal{B}) \oplus \mathbb{C}^N \to H^{s-m,\gamma-m}(\mathcal{B}) \oplus \mathbb{C}^{\tilde{N}}$ is an isomorphism for all ζ in the strip $\Im\zeta \in [\delta', \delta'']$ with $|\Re\zeta| \geq c$. Furthermore, there exists a sequence of complex numbers $(p_\nu)_{\nu\in\mathbb{Z}}$ such that $|\Im p_\nu| \to \infty$ as $|\nu| \to \infty$, and $\mathfrak{h}(\zeta)$ is an isomorphism for all $\zeta \in \mathbb{C} \setminus (p_\nu)_{\nu\in\mathbb{Z}}$. The inverse $\mathfrak{h}^{-1}(\zeta)$, first defined on $\mathbb{C} \setminus (p_\nu)_{\nu\in\mathbb{Z}}$, extends to a meromorphic operator family over the entire complex plane $\mathbb{C}$ with poles at p_ν of multiplicity $m_{\nu+1}$ and Laurent expansions

$$\mathfrak{h}^{-1}(\zeta) = \sum_{k=-(m_\nu+1)}^{-1} l_{\nu k}\,(\zeta - p_\nu)^k + \sum_{k=0}^{\infty} l_{\nu k}\,(\zeta - p_\nu)^k$$

close to p_ν, where $l_{\nu k} \in \Psi_G(\mathcal{B}; W, \tilde{W}; w)$ are operators of finite rank for all $\nu \in \mathbb{Z}$ and $-(m_\nu + 1) \leq k \leq -1$.

2.2 Corner Sobolev spaces

A further essential ingredient of the calculus are corner Sobolev spaces. They have already been announced in Section 1.1. Recall that $\mathcal{C}$ stands for the 'stretched manifold' related to the given space C with corners C_0. Moreover we have $\mathcal{W}$ that is the 'stretched manifold' of the space $C \setminus C_0$ with edges $C_1 \setminus C_0$.

Near any corner $v \in C_0$, the manifold $\mathcal{C}$ is locally of the form $[0,1) \times \mathcal{B}$ where $\mathcal{B}$ is the stretched base of the corner with conical singularities. It will be convenient also to look at the spaces $H^{s,\gamma}(\mathcal{B}^\wedge)$, for $s \in \mathbb{R}$ and $\gamma = (\gamma_0, \gamma_1) \in \mathbb{R}^2$, where $\mathcal{B}^\wedge = \mathbb{R}_+ \times \mathcal{B}$ is the infinite semicylinder over $\mathcal{B}$. Combining Corollary 2.1.12 with a weight reduction obtained by multiplication with a strictly positive function in the interior of $\mathcal{B}$ equal to $r^{s-\gamma_0}$ near $\partial\mathcal{B}$, we arrive, for every weight data $w = (\gamma_0, 0, \mathcal{I})$, at an elliptic operator $R^{s,\gamma_0}(\zeta) \in \Psi^s(\mathcal{B}; w; \Gamma_{-\gamma_1})$ which induces an isomorphism of $H^{s,\gamma_0}(\mathcal{B})$ onto $H^{0,0}(\mathcal{B})$, for all $\zeta \in \Gamma_{-\gamma_1}$. We might use this family to define the space $H^{s,\gamma}(\mathcal{B}^\wedge)$ quite analogously to $\mathcal{H}^{s,\gamma_1}(X^\wedge)$, cf. Definition 1.1.2, with $L^2(X)$ replaced by $H^{0,0}(\mathcal{B})$. Unfortunately, the spaces $H^{s,(\gamma_0,0)}(\mathcal{B}^\wedge)$ obtained in this way don't agree with $\mathcal{H}^s(\mathbb{R}_+, \pi^* H^{s,\gamma_0}(\mathcal{B}))$ close to the edge $E \cong (0,1)$. Thus, we could not glue them together with the wedge Sobolev spaces $H^{s,\gamma_0}_{\mathrm{loc}}(\mathcal{W})$ near $b^{-1}(v)$. This forces us to take as $H^{s,\gamma}(\mathcal{B}^\wedge)$ the scale $\mathcal{H}^s(\mathbb{R}_+, \pi^* H^{s,\gamma_0}(\mathcal{B}))$ itself, completed by the weight factor t^{γ_1}. Then the corner Sobolev spaces $H^{s,\gamma}(\mathcal{C})$, for $\gamma \in \mathbb{R}^2$, will be obtained by gluing together $H^{s,\gamma}(\mathcal{B}^\wedge)$ close to $b^{-1}(v)$ with the corresponding wedge Sobolev spaces on $\mathcal{W}$.

The various weight shifts that might play a role in the following definition would depend on $n+1$, the dimension of $\mathcal{B}$. These weight conventions would ensure that $H^{0,(0,0)}(\mathcal{C})$ be the pull-back of $L^2(\mathcal{C})$ under the blow-down mapping b.

For simplicity we assume in the sequel that C_0 consists of a single point v. This can be achieved formally by allowing $\mathcal{B}$ to have several connected components, i.e., $\mathcal{C}$ remains untouched by this assumption.

Let $s \in \mathbb{R}$, and $\gamma = (\gamma_0, \gamma_1)$ be a pair of real numbers. As explained above, we put

$$H^{s,\gamma}(\mathcal{B}^\wedge) = t^{\gamma_1}\, \mathcal{H}^s(\mathbb{R}_+, \pi^* H^{s,\gamma_0}(\mathcal{B})), \tag{2.2.1}$$

the spaces on the right-hand side being introduced in (1.4.14) (cf. also Example 1.4.2).

Lemma 2.2.1 *For any $s \in \mathbb{R}$ and $\gamma = (\gamma_0, \gamma_1) \in \mathbb{R}^2$, we have a continuous embedding*

$$H^{s,\gamma}(\mathcal{B}^\wedge) \hookrightarrow H^{s,\gamma_0}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathcal{B}).$$

Proof. This follows from (2.2.1) and Theorem 1.4.17. □

Hence we may glue together the spaces $H^{s,\gamma}(\mathcal{B}^\wedge)$ near the corners with the spaces $H^{s,\gamma_0}_{\mathrm{loc}}(\mathcal{W})$ living on the non-compact stretched manifold with edges $\mathcal{W} = b^{-1}(C \setminus C_0)$. Namely, given any $s \in \mathbb{R}$ and a pair of real numbers $\gamma = (\gamma_0, \gamma_1)$, we set

$$H^{s,\gamma}(\mathcal{C}) = [\omega]\, H^{s,\gamma}(\mathcal{B}^\wedge) + [1-\omega]\, H^{s,\gamma_0}_{\mathrm{loc}}(\mathcal{W}) \tag{2.2.2}$$

where $\omega(t)$ is a suitable cut-off function, and the right-hand side is equipped with the topology of the non-direct sum of Fréchet spaces. Recall that a neighbourhood of $b^{-1}(v)$, for $v \in C_0$, is identified with $[0,1) \times \mathcal{B}$, so we assume $\operatorname{supp} \omega \subset [0,1)$.

Note that the space (2.2.2) is independent of the concrete choice of $\omega(t)$. Moreover,

$$\big(\omega(t) t^\delta + (1-\omega(t))\big)\, H^{s,\gamma}(\mathcal{C}) = H^{s,(\gamma_0,\gamma_1+\delta)}(\mathcal{C})$$

for all $\delta \in \mathbb{R}$, $\omega(t)$ being a cut-off function satisfying $0 \le \omega(t) \le 1$.

Theorem 2.2.2 *For each $s'' \ge s'$ and $\gamma_0'' \ge \gamma_0'$, $\gamma_1'' \ge \gamma_1'$, there is a continuous embedding*

$$H^{s'',\gamma''}(\mathcal{C}) \hookrightarrow H^{s',\gamma'}(\mathcal{C}),$$

and this embedding is compact provided that $s'' > s'$ and $\gamma_0'' > \gamma_0'$, $\gamma_1'' > \gamma_1'$.

The spaces (2.2.2) can be endowed with Hilbert structures inducing the same topologies thereon. We only need someone in $H^0(\mathcal{C}) = H^{0,(0,0)}(\mathcal{C})$, it will be fixed once and for all. The adequate scalar product is first defined in local terms and then globally by using a partition of unity on $\mathcal{C}$. The obvious details are left to the reader.

Theorem 2.2.3 *When defined on $C^\infty_{\mathrm{comp}}(\mathring{\mathcal{C}}) \times C^\infty_{\mathrm{comp}}(\mathring{\mathcal{C}})$, the scalar product of $H^0(\mathcal{C})$ extends to a non-degenerate sesquilinear pairing*

$$(\cdot,\cdot): \ H^{s,\gamma}(\mathcal{C}) \times H^{-s,-\gamma}(\mathcal{C}) \to \mathbb{C}$$

for all $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}^2$, which allows the identification $H^{s,\gamma}(\mathcal{C})' \cong H^{-s,-\gamma}(\mathcal{C})$.

Given any $A \in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{C}), H^{s-\mu,\gamma-m}(\mathcal{C}))$, we define the *formal adjoint* A^* of A by

$$(Au, g)_{H^0(\mathcal{C})} = (u, A^* g)_{H^0(\mathcal{C})},$$

first for $u, v \in C^\infty_{\mathrm{comp}}(\mathring{\mathcal{C}})$. The formal adjoint actually induces a continuous mapping

$$A^* \in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,-\gamma+m}(\mathcal{C}), H^{s-\mu,-\gamma}(\mathcal{C})).$$

2.3 Corner asymptotics

We turn to the corner asymptotics of distributions belonging to the spaces (2.2.1). The asymptotics will consist of two parts, one for $r \to 0$ along the edges emanated from the corners and another for $t \to 0$. We have already treated edge asymptotics close to corners (cf. Section 1.4), our next concern will be the corner axis asymptotics. Similarly to the cone asymptotics it will be convenient to speak of asymptotic types.

Let $\gamma = (\gamma_0, \gamma_1)$ be a pair of real numbers and $w = (w_0, w_1)$ be a pair of weight data

$$\begin{aligned} w_0 &= (\gamma_0, (-l_0, 0]), \\ w_1 &= (\gamma_1, (-l_1, 0]) \end{aligned}$$

related to γ. By a *discrete* asymptotic type for corner asymptotics is meant any collection

$$\mathrm{as}_1 = (p_\nu, m_\nu, \Sigma_\nu)_{\nu=1,\ldots,N},$$

where p_ν are complex numbers in the strip $\{\zeta \in \mathbb{C} : -\gamma_1 - l_1 < \Im\zeta < -\gamma_1\}$, m_ν non-negative integers, and Σ_ν finite-dimensional subspaces of $H^{\infty,\gamma_0}_{\mathrm{as}_0}(\mathcal{B})$, for some "$\mathrm{as}_0$" independent of ν. We also allow $l_1 = \infty$, in which case $N = \infty$ and $\Im p_\nu \to -\infty$ as $\nu \to \infty$.

By the above, the set $\pi_{\mathbb{C}}\, \mathrm{as}_1 = (p_\nu)_{\nu=1,\ldots,N}$ is said to be a *carrier* of asymptotics. Recall that we have distinguished between the discrete and continuous cone asymptotics. In much the same way as in Section 1.2 we can introduce *continuous* asymptotic types for corner asymptotics. Moreover, we have a mapping $\mathrm{as}_1 \mapsto \mathrm{as}_0$ making any asymptotic type for corner asymptotics to that for cone ones. Hence various combinations can occur, for instance, discrete asymptotic types for both corner and cone asymptotics. Clearly, the reader who wants first to study discrete asymptotics may restrict himself to the latter case.

Definition 2.3.1 *Suppose $s \in \mathbb{R}$, $\gamma \in \mathbb{R}^2$, and "as_1" is as above, l_1 being finite. Then, $H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$ denotes the space of all $u \in H^{s,\gamma}(\mathcal{B}^\wedge)$ with the property that*

$$\omega(t) \left(u(t,p) - \sum_{\nu=1}^{N} \sum_{k=0}^{m_\nu} t^{ip_\nu} (\log t)^k\, c_{\nu k}(p) \right) \in H^{s,(\gamma_0, \gamma_1 + l_1 - 0)}(\mathcal{B}^\wedge),$$

for certain $c_{\nu k} \in \Sigma_\nu$ and any cut-off function ω.

In the case $l_1 = \infty$ we modify this definition by requiring that there exist $c_{\nu k} \in \Sigma_\nu$ such that to any cut-off function ω and $\epsilon > 0$ there corresponds an $N = N(\epsilon)$, with

$$\omega(t) \left(u(t,p) - \sum_{\nu=1}^{N} \sum_{k=0}^{m_\nu} t^{ip_\nu} (\log t)^k\, c_{\nu k}(p) \right) \in H^{s,(\gamma_0, \gamma_1 + \epsilon)}(\mathcal{B}^\wedge).$$

If l_1 is finite, we set $\mathcal{A}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$ to consist of all potentials

$$\omega(t) \sum_{\nu=1}^{N} \sum_{k=0}^{m_\nu} t^{ip_\nu} (\log t)^k \, c_{\nu k}(p)$$

where $c_{\nu k} \in \Sigma_\nu$ and ω is a fixed cut-off function. Then $\mathcal{A}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$ is a finite-dimensional subspace of $H^{\infty,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$, and we have

$$[\omega] \, H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge) = \mathcal{A}_{\mathrm{as}_1}(\mathcal{B}^\wedge) + [\omega] \, H^{s,(\gamma_0,\gamma_1+l_1-0)}(\mathcal{B}^\wedge), \tag{2.3.1}$$

the sum being direct. When combined with the original topology of $H^{s,\gamma}(\mathcal{B}^\wedge)$, for large $t \gg 1$, (2.3.1) gives a natural Fréchet space structure to $H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$. This in turn yields a projective limit Fréchet topology on $H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$ in the case $l_1 = \infty$.

The functions of $H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$ can also be characterised by their Mellin transforms as follows. For any cut-off function $\omega(t)$, the image $\mathcal{M}\left(\omega \, H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)\right)$ consists of all meromorphic functions $f(\zeta)$ in the strip $\Im\zeta > -\gamma_1 - l_1$ with values in $H^{s,\gamma_0}(\mathcal{B})$, that have poles at p_ν of multiplicities $m_\nu + 1$ and Laurent expansions

$$f(\zeta) = \sum_{k=0}^{m_\nu} c_{\nu k}(p) \, \frac{1}{(\zeta - p_\nu)^{k+1}} + f_\nu(\zeta)$$

near p_ν, where $c_{\nu k} \in \Sigma_\nu$ and f_ν is holomorphic at p_ν for all $\nu = 1, \ldots, N$. Moreover, if $\chi(\zeta)$ is a $\pi_{\mathbb{C}}\,\mathrm{as}_1$-excision function, then χf meets certain estimates like

$$\int_{\Gamma_{-\delta}} \|\chi(\zeta) f(\zeta)\|^2_{H^{s,\gamma_0}_\zeta(\mathcal{B})} d\zeta < \infty \tag{2.3.2}$$

for all $\delta < \gamma_1 + l_1$, uniformly in δ on compact intervals, $\| \cdot \|_{H^{s,\gamma_0}_\zeta(\mathcal{B})}$ being a family of equivalent norms on $H^{s,\gamma_0}(\mathcal{B})$ parametrised by ζ.

Next we consider the corner axis asymptotics along with the edge asymptotics for $r \to 0$. Analogously to (1.4.14) and (2.2.1) we use the spaces

$$\begin{array}{rcl} \mathcal{H}^{s,\gamma_1}(\mathbb{R}_+, \pi^* H^{s,\gamma_0}(\mathcal{B})) & = & t^{\gamma_1} \, \mathcal{H}^{s}(\mathbb{R}_+, \pi^* H^{s,\gamma_0}(\mathcal{B})), \\ \mathcal{H}^{s,\gamma_1}(\mathbb{R}_+, \pi^* H^{s,\gamma_0}_{\mathrm{as}_0}(\mathcal{B})) & = & t^{\gamma_1} \, \mathcal{H}^{s}(\mathbb{R}_+, \pi^* H^{s,\gamma_0}_{\mathrm{as}_0}(\mathcal{B})), \end{array}$$

for arbitrary $\gamma_0, \gamma_1 \in \mathbb{R}$ and asymptotic type "as_0" related to the weight data w_0. Obviously,

$$\mathcal{A}_{\mathrm{as}_1}(\mathcal{B}^\wedge) \hookrightarrow \mathcal{H}^{\infty,\gamma_1}(\mathbb{R}_+, \pi^* H^{\infty,\gamma_0}_{\mathrm{as}_0}(\mathcal{B})),$$

"as_0" corresponding to "as_1" by definition.

Definition 2.3.2 *Let $s \in \mathbb{R}$ and $\mathrm{as} = (\mathrm{as}_0, \mathrm{as}_1)$, "$\mathrm{as}_j$" being related to w_j, $j = 0, 1$. Then, $H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge)$ denotes the subspace of $H^{s,\gamma}(\mathcal{B}^\wedge)$ consisting of all u such that*

1) *$u \in H^{s,\gamma}_{\mathrm{as}_1}(\mathcal{B}^\wedge)$ in the sense of Definition 2.3.2;*
2) *given any cut-off function ω, there are $c_{\nu k} \in \Sigma_\nu$ with*

$$\omega(t)\left(u(t,p) - \sum_{\nu=1}^{N}\sum_{k=0}^{m_\nu} t^{ip_\nu}(\log t)^k\, c_{\nu k}(p)\right) \in \mathcal{H}^{s,\gamma_1+l_1-0}(\mathbb{R}_+, \pi^* H^{s,\gamma_0}_{\mathrm{as}_0}(\mathcal{B})).$$

If $l_1 = \infty$, the last equality needs handling with greater care. However, the argument after Definition 2.3.1 still works.

Suppose $\mathrm{as} = (\mathrm{as}_0, \mathrm{as}_1)$ where "as_j" is related to w_j, $j = 0, 1$. Just as in (2.3.1), we can write

$$[\omega]\, H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge) = \mathcal{A}_{\mathrm{as}_1}(\mathcal{B}^\wedge) + [\omega]\, \mathcal{H}^{s,\gamma_1+l_1-0}(\mathbb{R}_+, \pi^* H^{s,\gamma_0}_{\mathrm{as}_0}(\mathcal{B})) \tag{2.3.3}$$

provided $l_1 < \infty$. This makes $[\omega]\, H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge)$ a Fréchet space, for both summands are Fréchet. For $l_1 = \infty$, we endow $[\omega]\, H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge)$ with a projective limit Fréchet topology.

Now we are in a position to introduce the subspaces of global spaces (2.2.2) over $\mathcal{C}$ with asymptotics.

Definition 2.3.3 *Suppose $\mathcal{C}$ is a stretched manifold with corners, as above, and $\mathrm{as} = (\mathrm{as}_0, \mathrm{as}_1)$ where as_0 satisfies the shadow condition. For $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}^2$, we set*

$$H^{s,\gamma}_{\mathrm{as}}(\mathcal{C}) = [\omega]\, H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge) + [1-\omega]\, H^{s,\gamma_0}_{\mathrm{as}_0,\mathrm{loc}}(\mathcal{W}).$$

As usual, we endow $H^{s,\gamma}_{\mathrm{as}}(\mathcal{C})$ with the topology of the sum. Theorem 1.4.5 shows that $H^{s,\gamma}_{\mathrm{as}}(\mathcal{C})$ does not depend on the concrete choice of ω. If we fix ω and form

$$[\omega]\, H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge) + [1-\omega]\, H^{s,\gamma_0}_{\mathrm{as}_2,\mathrm{loc}}(\mathcal{W}),$$

with arbitrary $\mathrm{as} = (\mathrm{as}_0, \mathrm{as}_1)$ and as_2, then there is a resulting asymptotic type $\tilde{\mathrm{as}}_0$ with the property that the above sum just amounts to $H^{s,\gamma}_{\tilde{\mathrm{as}}}(\mathcal{C})$, for $\tilde{\mathrm{as}} = (\tilde{\mathrm{as}}_0, \mathrm{as}_1)$.

2.4 Corner Mellin symbols

The next step towards establishing a pseudodifferential calculus on manifolds with corners is to define the spaces of operator-valued Mellin symbols, here operating globally along the base $\mathcal{B}$ of the corner.

Definition 2.4.1 *Suppose $m, \mu, \delta \in \mathbb{R}$, $m - \mu \in \mathbb{Z}_+$ and $w = (\delta, \delta - m, \mathcal{I})$, where $\mathcal{I} = (-l, 0]$, $l \in \mathbb{N} \cup \{\infty\}$. Then, $\mathcal{M}(\mathbb{C}, \Psi^\mu(\mathcal{B}; W, \tilde{W}; w))$ stands for the space of all holomorphic functions $\mathfrak{h}(\zeta)$ with values in $\Psi^\mu(\mathcal{B}; W, \tilde{W}; w)$, such that*

$$\mathfrak{h}(\zeta)\,|_{\Gamma_{-\delta}} \in \Psi^\mu(\mathcal{B}; W, \tilde{W}; w; \Gamma_{-\delta})$$

for all $\delta \in \mathbb{R}$, uniformly in δ in every compact interval of $\mathbb{R}$.

Furthermore, we write $\mathcal{M}(\mathbb{C},\Psi^\mu(\mathcal{B};w))$ for the class of upper left corners of the elements in $\mathcal{M}(\mathbb{C},\Psi^\mu(\mathcal{B};W,\tilde{W};w))$.

Let us emphasise the link of Definition 2.4.1 to Theorem 2.1.13. The corner calculus employes not only holomorphic but also meromorphic Mellin symbols with values in $\Psi^\mu(\mathcal{B};W,\tilde{W};w)$. The poles and Laurent expansions are described in terms of asymptotic types.

More precisely, choose a collection $T=(p_\nu,m_\nu,\mathcal{L}_\nu)_{\nu\in\mathbb{Z}}$ of complex numbers p_ν satisfying $|\Im p_\nu|\to\infty$ as $|\nu|\to\infty$, non-negative integers m_ν, and finite-dimensional subspaces $\mathcal{L}_\nu$ of finite rank operators in $\Psi_G(\mathcal{B};W,\tilde{W};w)$, the asymptotic types of the operators being independent of ν. Such collections T are said to be *discrete* asymptotic types for corner Mellin symbols. As usual, we set

$$\pi_{\mathbb{C}}T=(p_\nu)_{\nu\in\mathbb{Z}}.$$

Definition 2.4.2 *By $\mathcal{M}_T(\mathbb{C},\Psi_G(\mathcal{B};W,\tilde{W};w))$ is meant the space of all meromorphic functions $\mathfrak{g}(\zeta)$ in $\mathbb{C}$ with values in $\Psi_G(\mathcal{B};W,\tilde{W};w)$, such that*

1) $\chi(\zeta)\mathfrak{g}(\zeta)\,|_{\Gamma_{-\delta}}\in\Psi_G^{-\infty}(\mathcal{B};W,\tilde{W};w;\Gamma_{-\delta})$ *for all $\delta\in\mathbb{R}$, uniformly in δ in compact intervals of $\mathbb{R}$, $\chi(\zeta)$ being any $\pi_{\mathbb{C}}T$-excision function;*
2) $\mathfrak{g}(\zeta)$ *has poles at p_ν of multiplicities $m_\nu+1$ with Laurent coefficients at $(\zeta-p_\nu)^{-(k+1)}$ belonging to $\mathcal{L}_\nu$, for every $0\le k\le m_\nu$ and $\nu\in\mathbb{Z}$.*

Denote $\mathcal{M}_T(\mathbb{C},\Psi_G(\mathcal{B};w))$ the set of upper left corners of the matrices in $\mathcal{M}_T(\mathbb{C},\Psi_G(\mathcal{B};W,\tilde{W};w))$.

Both $\mathcal{M}(\mathbb{C},\Psi^\mu(\mathcal{B};W,\tilde{W};w))$ and $\mathcal{M}_T(\mathbb{C},\Psi_G(\mathcal{B};W,\tilde{W};w))$ are inductive limits of Fréchet spaces in a canonical way. Thus, we can introduce the non-direct sums

$$\mathcal{M}_T(\mathbb{C},\Psi^\mu(\mathcal{B};W,\tilde{W};w))=\mathcal{M}(\mathbb{C},\Psi^\mu(\mathcal{B};W,\tilde{W};w))+\mathcal{M}_T(\mathbb{C},\Psi_G(\mathcal{B};W,\tilde{W};w)). \tag{2.4.1}$$

Each symbol $\mathfrak{a}(\zeta)\in\mathcal{M}_T(\mathbb{C},\Psi^\mu(\mathcal{B};W,\tilde{W};w))$ gives rise to a Mellin pseudodifferential operator

$$\mathrm{op}_{\mathcal{M},\delta}(\mathfrak{a})\,u=t^\delta\,\mathrm{op}_{\mathcal{M}}(T^{-i\delta}\mathfrak{a})\left(t^{-\delta}u\right),$$

for any $\delta\in\mathbb{R}$ with $\pi_{\mathbb{C}}T\cap\Gamma_{-\delta}=\emptyset$. Here u is regarded as a vector-valued function of $t\in\mathbb{R}_+$, and we first assume that $u(t)$ is a C^∞ function with a compact support in $\mathbb{R}_+$.

Theorem 2.4.3 *Let $\gamma=(\gamma_0,\gamma_1)$ and $w_0=(\gamma_0,\gamma_0-m,\mathcal{I}_0)$. Suppose that $\mathfrak{a}(\zeta)\in\mathcal{M}_T(\mathbb{C},\Psi^\mu(\mathcal{B};W,\tilde{W};w_0))$ satisfies $\pi_{\mathbb{C}}T\cap\Gamma_{-\gamma_1}=\emptyset$. Then, given any cut-off functions $\varphi(t)$ and $\psi(t)$, the operator $\varphi\,\mathrm{op}_{\mathcal{M},\gamma_1}(\mathfrak{a})\,\psi$ extends to a continuous mapping*

$$\begin{array}{ccc} H^{s,\gamma}(\mathcal{B}^\wedge) & & H^{s-\mu,(\gamma_0-m,\gamma_1)}(\mathcal{B}^\wedge) \\ \oplus & \to & \oplus \\ \mathcal{H}^{s,\gamma_1}(\mathbb{R}_+,W) & & \mathcal{H}^{s-\mu,\gamma_1}(\mathbb{R}_+,\tilde{W}) \end{array} \tag{2.4.2}$$

for all $s \in \mathbb{R}$.

This result generalises the first statement of Theorem 1.4.15 to arbitrary corner Sobolev spaces, i.e., including those with non-zero $\gamma_1 \in \mathbb{R}$. Indeed, the operator $\varphi \operatorname{op}_{\mathcal{M}}(\mathfrak{a})\, \psi$ belongs to $\Psi^{\mu}_{M}(\mathbb{R}_+ \times \mathcal{B}; W, \tilde{W}; w_0)$, as is easy to check. Moreover, to every asymptotic type $\mathrm{as} = (\mathrm{as}_0, \mathrm{as}_1)$ there corresponds an asymptotic type $\tilde{\mathrm{as}} = (\tilde{\mathrm{as}}_0, \tilde{\mathrm{as}}_1)$ such that (2.4.2) restricts to a continuous mapping

$$\varphi \operatorname{op}_{\mathcal{M},\gamma_1}(\mathfrak{a})\, \psi : \begin{array}{c} H^{s,\gamma}_{\mathrm{as}}(\mathcal{B}^\wedge) \\ \oplus \\ \mathcal{H}^{s,\gamma_1}_{\mathrm{as}_1}(\mathbb{R}_+, W) \end{array} \to \begin{array}{c} H^{s-\mu,(\gamma_0-m,\gamma_1)}_{\tilde{\mathrm{as}}}(\mathcal{B}^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma_1}_{\tilde{\mathrm{as}}_1}(\mathbb{R}_+, \tilde{W}) \end{array}$$

for all $s \in \mathbb{R}$.

Chapter 3

Corner Calculus

3.1 Corner operator algebra

The operators of the corner algebra on $\mathcal{C}$ are expected to be matrix-valued with "proper" pseudodifferential operators in the upper left corners and additional trace and potential conditions along the edges. They can be introduced entirely by a concise notation. However, we prefer here to look first at the upper left corners separately. The general class will then follow in Section 3.2.

To simplify notation, we assume as above that C_0 consists of a single corner v. On the other hand, the base $\mathcal{B}$ is allowed to have several connected components.

We start with the Green operators. Let $m \in \mathbb{R}$, $\gamma = (\gamma_0, \gamma_1) \in \mathbb{R}^2$, and $w = (w_0, w_1)$ be a pair of weight data

$$\begin{array}{rcl} w_0 & = & (\gamma_0, \gamma_0 - m, \mathcal{I}_0), \\ w_1 & = & (\gamma_1, \gamma_1 - m, \mathcal{I}_1) \end{array}$$

related to γ. Then $\Psi_G(\mathcal{C}; w)$ is the space of all $G \in \cap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{C}), H^{\infty,\gamma-m}(\mathcal{C}))$ which induce continuous mappings

$$\begin{array}{rcl} G & : & H^{s,\gamma}(\mathcal{C}) \to H^{\infty,\gamma-m}_{\mathrm{as}}(\mathcal{C}), \\ G^* & : & H^{s,-\gamma+m}(\mathcal{C}) \to H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{C}) \end{array}$$

for all $s \in \mathbb{R}$ and certain asymptotic types

$$\begin{array}{rcl} \mathrm{as} & = & (\mathrm{as}_0, \mathrm{as}_1), \\ \tilde{\mathrm{as}} & = & (\tilde{\mathrm{as}}_0, \tilde{\mathrm{as}}_1), \end{array}$$

where "as_j" is related to $(\gamma_j - m, \mathcal{I}_j)$ and "$\tilde{\mathrm{as}}_j$" to $(-\gamma_j, \mathcal{I}_j)$, G^* being the formal adjoint with respect to a scalar product in $H^{0,0}(\mathcal{C})$.

Definition 3.1.1 *Let $m - \mu \in \mathbb{Z}_+$ and $w = (w_0, w_1)$, where $\gamma \in \mathbb{R}^2$ and $\mathcal{I}_j = (-l_j, 0]$, l_j being a positive integer. Then, $\Psi^\mu(\mathcal{C}; w)$ is the space of all operators*

$$A = A_c + A_e + M + G \tag{3.1.1}$$

where

- A_c *is a Mellin operator with holomorphic symbol in a neighbourhood of v, i.e., $A_c = \varphi t^{-\mu} \mathrm{op}_{\mathcal{M},\gamma_1}(h)\psi$ where $h(t,\zeta) \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^\mu(\mathcal{B}; w_0)))$;*
- A_e *is a pseudodifferential operator on the manifold with edges $W = C \setminus C_0$, i.e., $A_e = (1-\varphi)\Psi\tilde{\psi}$ where $\Psi \in \Psi^\mu(\mathcal{W}; w_0)$ and $\tilde{\psi}$ covers $1-\varphi$;*
- M *is the sum of smoothing Mellin operators with meromorphic symbols in a neighbourhood of v, cf.* (2.4.1)*; and*
- G *is a Green operator on $\mathcal{C}$ related to the weight data w, as described above, i.e., $G \in \Psi_G(\mathcal{C}; w)$.*

Note that this definition is quite analogous to Definition 1.2.3. Recall that there is a diffeomorphism κ of a punctured neighbourhood $O \setminus \{v\}$ of v in C onto $(0,1) \times \mathcal{B}$. Similarly to (1.2.5) we may assume that A_c and A_e are *compatible* in the sense that

$$\kappa_\sharp \, \Psi = t^{-\mu} \, \mathrm{op}_{\mathcal{M},\gamma_1}(h)$$

modulo $\Psi_G^{-\infty}((0,1) \times \mathcal{B}; w_0)$, cf. Definition 1.3.15. Then any other choice of φ, ψ and $\tilde{\psi}$ modifies (3.1.1) only by some element of $\Psi_G(\mathcal{C}; w)$.

Theorem 3.1.2 *Every operator $A \in \Psi^\mu(\mathcal{C}; w)$ induces continuous mappings*

$$\begin{array}{rcl} A & : & H^{s,\gamma}(\mathcal{C}) \to H^{s-\mu,\gamma-m}(\mathcal{C}), \\ A & : & H^{s,\gamma}_{\mathrm{as}}(\mathcal{C}) \to H^{s-\mu,\gamma-m}_{\tilde{\mathrm{as}}}(\mathcal{C}) \end{array}$$

for all $s \in \mathbb{R}$ and asymptotic types "as"*, with some resulting asymptotic types* "$\tilde{\mathrm{as}}$" *depending on* "as"*.*

The inclusions

$$\begin{array}{rcl} \Psi^\mu(\mathcal{C}; w) & \hookrightarrow & \Psi^\mu_{\mathrm{cl}}(C \setminus C_1), \\ \Psi^\mu(\mathcal{C}; w) & \hookrightarrow & \Psi^\mu(C \setminus C_0; w_0), \\ t^{\mu-\gamma_1} \, \Psi^\mu(\mathcal{C}; w) \, t^{\gamma_1} & \hookrightarrow & \Psi^\mu_M([0,1) \times \mathcal{B}; w_0) \end{array} \tag{3.1.2}$$

give rise to the symbol levels over $\Psi^\mu(\mathcal{C}; w)$ from the corresponding larger classes, namely ${}^b\sigma^\mu(A)$, $\sigma^\mu_{\mathrm{edge}}(A)$ and ${}^b\sigma^m_{\mathrm{edge}}(A)$. In particular, for $A \in \Psi^\mu(\mathcal{C}; w)$ we set ${}^b\sigma^\mu_{\mathrm{edge}}(A) := {}^b\sigma^\mu_{\mathrm{edge}}(t^\mu A)$, the last symbol being over $\Psi^\mu_M([0,1) \times \mathcal{B}; w_0)$. Hence

$$\sigma^m_{\mathrm{edge}}(A)(t,\tau) = \frac{1}{t^m} \, {}^b\sigma^m_{\mathrm{edge}}(A)(t, t\tau), \tag{3.1.3}$$

which is traced back to (1.1.11).

We call ${}^b\sigma^\mu(A)$ the principal homogeneous *compressed interior symbol* of order μ, $\sigma^\mu_{\text{edge}}(A)$ the principal homogeneous *edge symbol* of order μ, and ${}^b\sigma^m_{\text{edge}}(A)$ the principal homogeneous *Mellin edge symbol* of order μ of A. Let us also define

$$\sigma_{\mathcal{M}}(A)(\zeta) := h(0,\zeta) + \left(m_0^{(1)}(\zeta) + m_0^{(2)}(\zeta)\right), \tag{3.1.4}$$

the *conormal symbol* of A at the corner, where h and $m_0^{(\imath)}$ are specified from (3.1.1).

By definition, $\sigma_{\mathcal{M}}(A)$ is an element of $\cup_T \mathcal{M}_T(\mathbb{C}, \Psi^\mu(\mathcal{B}; w_0))$. Thus, (3.1.4) will be regarded as a family of operators

$$\sigma_{\mathcal{M}}(A)(\zeta) : \ H^{s,\gamma_0}(\mathcal{B}) \to H^{s-\mu,\gamma_0-m}(\mathcal{B})$$

parametrised by $\zeta \in \Gamma_{-\gamma_1}$, for any $s \in \mathbb{R}$.

The operator classes of Definition 3.1.1 behave well under taking formal adjoints and compositions. We postpone this to the next section where these properties will be formulated for matrices with $A \in \Psi^\mu(\mathcal{C}; w)$ in the upper left corners.

Let us demonstrate the new structures by the example of typical differential operators on C. If A is such an operator of order m, then $A \in \Psi^m(\mathcal{C}; w)$ for $w = (w_0, w_1)$, where $w_j = (\gamma_j, \gamma_j - m, (-\infty, 0])$ with arbitrary $\gamma_j \in \mathbb{R}$. Close to $t = 0$ A has the form

$$A = \omega(r)\frac{1}{(tr)^m}\sum_{j+k\le m} A_{jk}(t,r)\,(trD_t)^j\,(rD_r)^k + (1-\omega(r))\frac{1}{t^m}\sum_{j=0}^m A_j(t)\,(tD_t)^j$$

where

$$\begin{array}{rcl} A_{jk}(t,r) & \in & C^\infty_{\text{loc}}([0,1)\times[0,1), \text{Diff}^{m-(j+k)}(X)), \\ A_j(t) & \in & C^\infty_{\text{loc}}([0,1), \text{Diff}^{m-j}(B\setminus B_0)). \end{array}$$

Set

$$h(t,\zeta) = \omega(r)\frac{1}{r^m}\sum_{j+k\le m} A_{jk}(t,r)\,(r\zeta)^j\,(rD_r)^k + (1-\omega(r))\sum_{j=0}^m A_j(t)\,\zeta^j,$$

then $h(t,\zeta)$ is a family of operators on $\mathcal{B}$, parametrised by the real part of $\zeta \in \Gamma_{-\gamma_1}$.

We next claim that $h \in C^\infty_{\text{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^m(\mathcal{B}; w_0)))$, for $A = t^{-m}\text{op}_{\mathcal{M},\gamma_1}(h)$ is then of the form (3.1.1) near the corner. To prove this, we only need to show that

$$h\,|_{\Gamma_{-\delta}} \in \Psi^m(\mathcal{B}; w_0; \Gamma_{-\delta})$$

for all $\delta \in \mathbb{R}$, uniformly in δ on compact intervals of $\mathbb{R}$. As usual, we choose a system (ψ_b, ψ_i) covering $(\omega, 1-\omega)$, with $\psi_b(r)$ supported in $[0,1)$. Then we have

$$h(t,\zeta) = \omega(r)\left(a_0(t,\zeta) + a_\infty(t,\zeta)\right)\psi_b(r) + (1-\omega(r))\left(\sum_{j=0}^{m} A_j(t)\zeta^j\right)\psi_i(r),$$

where

$$\begin{aligned} a_0(t,\zeta) &= \varphi_0(r\langle\tau\rangle)\, r^{-m}\, \mathrm{op}_{\mathcal{M},\gamma_0}\big(\textstyle\sum_{j+k\le m} A_{jk}(t,r)(r\zeta)^j z^k\big)\,\psi_0(r\langle\tau\rangle), \\ a_\infty(t,\zeta) &= \varphi_\infty(r\langle\tau\rangle)\, r^{-m}\big(\textstyle\sum_{j+k\le m} A_{jk}(t,r)(r\zeta)^j (rD_r)^k\big)\,\psi_\infty(r\langle\tau\rangle). \end{aligned}$$

We thus obtain $h(t,\zeta)$ in the form (2.1.1) which was a model for Definition 2.1.2. In other words, we conclude that $h \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^m(\mathcal{B}; w_0)))$, indeed.

It is a simple matter to check that A belongs to the algebra $\Psi^m(\mathcal{W}; w_0)$ over $W = C \setminus C_0$. In (1.1.8) we have already specified the leading edge symbol of A, $\sigma^m_{\mathrm{edge}}(A)$. By this example one can observe typical relations between various symbolic levels. Since $\sigma^m_{\mathrm{edge}}(A)$ takes its values in $\Psi^m(X^\wedge; w_0)$, we may look at the conormal symbol of $\sigma^m_{\mathrm{edge}}(A)$ with respect to the r-variable. It is equal to

$$\sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(A)\right)(z) = \frac{1}{t^m}\sum_{k=0}^{m} A_{0,k}(t,0) z^k,$$

which does not depend on τ. On the other hand, the conormal symbol of A with respect to t is

$$\sigma_{\mathcal{M}}(A)(\zeta) = h(0,\zeta),$$

the right-hand side taking its values in $\Psi^m(\mathcal{B}; w_0)$. It bears a leading conormal symbol at $\partial\mathcal{B}$, namely

$$\sigma_{\mathcal{M}}\left(\sigma_{\mathcal{M}}(A)\right)(z) = \sum_{k=0}^{m} A_{0,k}(0,0) z^k$$

whence

$$t^m\, \sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(A)\right)(z)\,|_{t=0} = \sigma_{\mathcal{M}}\left(\sigma_{\mathcal{M}}(A)\right)(z). \tag{3.1.5}$$

This is a crucial relation for understanding the interaction between edge and corner contributions to the asymptotics of solutions. The use of the variable t is allowed because (3.1.5) concerns a neighbourhood of $t = 0$. Otherwise we could invoke a defining function of the corresponding face of the stretched manifold $\mathcal{C}$.

Theorem 3.1.3 *For each operator $A \in \Psi^\mu(\mathcal{C}; w)$, the equality* (3.1.5) *is still valid with m replaced by μ.*

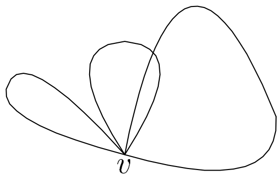

Fig. 3.1: The "branched space" of edges.

3.2 Trace and potential conditions

We now complete the operators of Definition 3.1.1 to matrices, where two entries have the meaning of potential and trace operators with respect to the edges, and the lower right corners are operators analogous to those from Definition 1.2.3, acting over the "branched space" formed by the edges (see Fig. 3.1). These edges $E = C_1 \setminus C_0$ are diffeomorphic to a system of open intervals, i.e., to a disjoint union of $\Omega_\nu \cong (0,1)$, $\nu = 1, \ldots, \mathcal{N}$, the diffeomorphisms extending to those of the closures. Letting Ω denote any component Ω_ν, we introduce the spaces

$$H^{s,\delta}(\Omega) = [\omega]\, \mathcal{H}^{s,\delta}(0,\infty) + [1-\omega]\, \mathcal{H}^{s,\delta}(-\infty,1),$$

for $s, \delta \in \mathbb{R}$, where $\omega(t)$ is a cut-off function with a support in $[0,1)$ and the spaces on the right are identified with their pull-backs under the diffeomorphism $\Omega \to (0,1)$.

As usual, we write $H^{s,\delta}(\Omega, \mathbb{C}^N) = H^{s,\delta}(\Omega) \otimes \mathbb{C}^N$ for the space of $\mathbb{C}^N$-valued functions.

Set

$$\begin{aligned} H^{s,\delta}(E,W) &= \oplus_{\nu=1}^{\mathcal{N}} H^{s,\delta}(\Omega_\nu, \mathbb{C}^{N_\nu}), \\ H^{s,\delta}(E,\tilde{W}) &= \oplus_{\nu=1}^{\mathcal{N}} H^{s,\delta}(\Omega_\nu, \mathbb{C}^{\tilde{N}_\nu}), \end{aligned}$$

for some choice of N_ν and $\tilde{N}_\nu$.

The asymptotic types of distributions in $H^{s,\delta}(E,W)$ near v are a straightforward generalisation of those in $H^{s,\delta}(\mathbb{R}_+)$. In this way we arrive at spaces with asymptotics $H^{s,\delta}_{\mathrm{as}}(E,W)$. We could, of course, distinguish between the weight data on different branches and allow them to be different. But we prefer the simplest form.

The matrix-valued analogue of the Green operators of Section 3.1 is as follows. Suppose $m \in \mathbb{R}$, $\gamma = (\gamma_0, \gamma_1) \in \mathbb{R}^2$, and $w = (w_0, w_1)$ is a pair of weight data

$$\begin{aligned} w_0 &= (\gamma_0, \gamma_0 - m, \mathcal{I}_0), \\ w_1 &= (\gamma_1, \gamma_1 - m, \mathcal{I}_1) \end{aligned}$$

related to γ. Then $\Psi_G(\mathcal{C};W,\tilde{W};w)$ denotes the space of all

$$\mathcal{G} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{C}) \oplus H^{s,\gamma_1}(E,W), H^{\infty,\gamma-m}(\mathcal{C}) \oplus H^{\infty,\gamma_1-m}(E,\tilde{W}))$$

which induce continuous operators

$$\begin{array}{lcl} \mathcal{G} & : & H^{s,\gamma}(\mathcal{C}) \oplus H^{s,\gamma_1}(E,W) \to H^{\infty,\gamma-m}_{\mathrm{as}}(\mathcal{C}) \oplus H^{\infty,\gamma_1-m}_{as}(E,\tilde{W}), \\ \mathcal{G}^* & : & H^{s,-\gamma+m}(\mathcal{C}) \oplus H^{s,-\gamma_1+m}(E,\tilde{W}) \to H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{C}) \oplus H^{\infty,-\gamma_1}_{\tilde{as}}(E,W) \end{array}$$

for all $s \in \mathbb{R}$ and certain asymptotic types (as, as) and $(\tilde{\mathrm{as}}, \tilde{as})$ depending on $\mathcal{G}$, $\mathcal{G}^*$ being the formal adjoint with respect to a fixed scalar product in $H^{0,0}(\mathcal{C}) \oplus H^{0,0}(E)$.

Definition 3.2.1 *Suppose $m - \mu \in \mathbb{Z}_+$, $\gamma \in \mathbb{R}^2$, and $w = (w_0, w_1)$ are weight data as above. Then, $\Psi^\mu(\mathcal{C};W,\tilde{W};w)$ stands for the space of all operators*

$$\mathcal{A} = \mathcal{A}_c + \mathcal{A}_e + \mathcal{M} + \mathcal{G}$$

where

- $\mathcal{A}_c$ *is a Mellin operator with a holomorphic symbol in a neighbourhood of v, i.e., $A_c = \varphi t^{-\mu} \mathrm{op}_{\mathcal{M},\gamma_1}(\mathfrak{h})\psi$ with $\mathfrak{h} \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^\mu(\mathcal{B};W,\tilde{W};w_0)))$;*
- $\mathcal{A}_e$ *is a pseudodifferential operator on the manifold with edges $C \setminus C_0$, i.e., $A_e = (1-\varphi)\Psi\tilde{\psi}$ where $\Psi \in \Psi^\mu(\mathcal{W};W,\tilde{W};w_0)$ and $\tilde{\psi}$ covers $1-\varphi$;*
- $\mathcal{M}$ *is the sum of smoothing Mellin operators with meromorphic symbols in a neighbourhood of v, cf.* (2.4.1)*; and*
- $\mathcal{G}$ *is a Green operator on $\mathcal{C}$ related to the weight data w, as described above, i.e., $G \in \Psi_G(\mathcal{C};W,\tilde{W};w)$.*

The important point to note here is the number of additional conditions along each connected component of the set of edges $C_1 \setminus C_0$. It depends on the component, so that the ranks of the bundles W and $\tilde{W}$ need not coincide over different connected components of $C_1 \setminus C_0$.

It follows by definition that $\Psi^\mu(\mathcal{C};w)$ just amounts to the space of all upper left corners of $\Psi^\mu(\mathcal{C};W,\tilde{W};w)$.

Theorem 3.2.2 *Every operator $\mathcal{A} \in \Psi^\mu(\mathcal{C};W,\tilde{W};w)$ induces continuous mappings*

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(\mathcal{C}) \\ \oplus \\ H^{s,\gamma_1}(E,W) \end{array} \to \begin{array}{c} H^{s-\mu,\gamma-m}(\mathcal{C}) \\ \oplus \\ H^{s-\mu,\gamma_1-m}(E,\tilde{W}) \end{array},$$

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}_{\mathrm{as}}(\mathcal{C}) \\ \oplus \\ H^{s,\gamma_1}_{as}(E,W) \end{array} \to \begin{array}{c} H^{s-\mu,\gamma-m}_{\tilde{\mathrm{as}}}(\mathcal{C}) \\ \oplus \\ H^{s-\mu,\gamma_1-m}_{\tilde{as}}(E,\tilde{W}) \end{array}$$

for all $s \in \mathbb{R}$ and asymptotic types "as" and "as", with some resulting asymptotic types "$\tilde{\text{as}}$" and "$\tilde{as}$".

Let us now look at the leading symbols within the calculus on $\mathcal{C}$, namely ${}^b\sigma^\mu(\mathcal{A})$, $\sigma^\mu_{\text{edge}}(\mathcal{A})$ and $\sigma_\mathcal{M}(\mathcal{A})$. They act as

$$\begin{array}{rcl} {}^b\sigma^\mu(\mathcal{A})(q,p) & : & \mathbb{C} \to \mathbb{C}, \\ \sigma^\mu_{\text{edge}}(\mathcal{A})(t,\tau) & : & H^{s,\gamma_0}(X^\wedge) \oplus W \to H^{s-\mu,\gamma_0-m}(X^\wedge) \oplus \tilde{W}, \\ \sigma_\mathcal{M}(\mathcal{A})(\zeta) & : & H^{s,\gamma_0}(\mathcal{B}) \oplus W \to H^{s-\mu,\gamma_0-m}(\mathcal{B}) \oplus \tilde{W} \end{array} \qquad (3.2.1)$$

pointwise over ${}^bT^*\mathcal{C} \setminus \{0\}$, $T^*(C_1 \setminus C_0) \setminus \{0\}$ and all of the weight line $\Gamma_{-\gamma_1}$, respectively.

Theorem 3.2.3 *As defined above, the leading symbols control the order of operators within $\Psi^\mu(\mathcal{C}; W, \tilde{W}; w)$, i.e.,*

$$\ker\left({}^b\sigma^\mu, \sigma^\mu_{\text{edge}}, \sigma_\mathcal{M}\right) = \Psi^{\mu-1}(\mathcal{C}; W, \tilde{W}; w).$$

Since lower order operators act through compact embeddings of weighted Sobolev spaces, we also have the following result completing Theorem 3.2.3.

Theorem 3.2.4 *Suppose $\mathcal{A} \in \Psi^{\mu-1}(\mathcal{C}; W, \tilde{W}; w)$. Then the mappings of Theorem 3.2.2 are compact.*

Combining Theorems 3.2.3 and 3.2.4 we see that the triple $\left({}^b\sigma^\mu, \sigma^\mu_{\text{edge}}, \sigma_\mathcal{M}\right)$ enables us to perform a standard parametrix construction on the symbol level, provided that the pseudodifferential operators behave naturally under composition.

Theorem 3.2.5 *Let $m_1 - \mu_1 \in \mathbb{Z}_+$, $m_2 - \mu_2 \in \mathbb{Z}_+$ and let $\mathcal{I}_j = (-l_j, 0]$, l_j being positive integers. If*

$$\begin{array}{ll} \mathcal{A}_1 \in \Psi^{\mu_1}(\mathcal{C}; W^1, W^2; w_1), & w_1 = (w_{1,0}, w_{1,1}), \\ \mathcal{A}_2 \in \Psi^{\mu_2}(\mathcal{C}; W^2, W^3; w_2), & w_2 = (w_{2,0}, w_{2,1}), \end{array}$$

then $\mathcal{A}_2\mathcal{A}_1 \in \Psi^{\mu_1+\mu_2}(\mathcal{C}; W^1, W^3; w_2 \circ w_1)$ with $w_2 \circ w_1 = (w_{2,0} \circ w_{1,0}, w_{2,1} \circ w_{1,1})$, and

$$\begin{array}{rcl} {}^b\sigma^{\mu_1+\mu_2}(\mathcal{A}_2\mathcal{A}_1) & = & {}^b\sigma^{\mu_2}(\mathcal{A}_2)\, {}^b\sigma^{\mu_1}(\mathcal{A}_1), \\ \sigma^{\mu_1+\mu_2}_{\text{edge}}(\mathcal{A}_2\mathcal{A}_1) & = & \sigma^{\mu_2}_{\text{edge}}(\mathcal{A}_2)\, \sigma^{\mu_1}_{\text{edge}}(\mathcal{A}_1), \\ \sigma_\mathcal{M}(\mathcal{A}_2\mathcal{A}_1) & = & (T^{i\mu_1}\sigma_\mathcal{M}(\mathcal{A}_2))\, \sigma_\mathcal{M}(\mathcal{A}_1). \end{array}$$

If $\mathcal{A}$ is a continuous operator in the sense of Theorem 3.2.2 for all $s \in \mathbb{R}$, then we can define the formal adjoint

$$\mathcal{A}^* : \begin{array}{c} H^{-s+\mu,-\gamma+m}(\mathcal{C}) \\ \oplus \\ H^{-s+\mu,-\gamma_1+m}(E, \tilde{W}) \end{array} \to \begin{array}{c} H^{-s,-\gamma}(\mathcal{C}) \\ \oplus \\ H^{-s,-\gamma_1}(E, W) \end{array} \qquad (3.2.2)$$

by

$$(\mathcal{A}u, g)_{H^{0,0}(\mathcal{C})\oplus H^{0,0}(E,\tilde{W})} = (u, \mathcal{A}^* g)_{H^{0,0}(\mathcal{C})\oplus H^{0,0}(E,W)}$$

first for all C^∞ sections u and g compactly supported away from the singularities. This extends then to a continuous operator (3.2.2) for all $s \in \mathbb{R}$.

Theorem 3.2.6 *Each operator* $\mathcal{A} \in \Psi^\mu(\mathcal{C}; W, \tilde{W}; w)$ *allows a formal adjoint* $\mathcal{A}^* \in \Psi^\mu(\mathcal{C}; \tilde{W}, W; w^*)$, *where* $w^* = (w_0^*, w_1^*)$ *and* $w_j^* = (-\gamma_j+m, -\gamma_j, \mathcal{I}_j)$. *Moreover,*

$$\begin{aligned} {}^b\sigma^\mu(\mathcal{A}^*) &= \left({}^b\sigma^\mu(\mathcal{A})\right)^*, \\ \sigma^\mu_{\mathrm{edge}}(\mathcal{A}^*) &= \left(\sigma^\mu_{\mathrm{edge}}(\mathcal{A})\right)^*, \\ \sigma_{\mathcal{M}}(\mathcal{A}^*) &= T^{i\mu}\left(\sigma_{\mathcal{M}}(\mathcal{A})\right)^*. \end{aligned}$$

Theorems 3.2.5 and 3.2.6 can be summarised by saying that $\Psi^0(\mathcal{C}; w)$ is a C^*-algebra over $\mathcal{C}$.

Let $\omega(t)$ be a cut-off function on $\mathbb{R}_+$, satisfying $0 \le \omega(t) \le 1$. Then $w^\delta(t) = \omega(t)t^\delta + (1-\omega(t))$ can be regarded as a function on all of $\mathcal{C}$. For any $\delta \in \mathbb{R}$, we have

$$w^\delta\, \Psi^\mu(\mathcal{C}; W, \tilde{W}; w)\, w^{-\delta} = \Psi^\mu(\mathcal{C}; W, \tilde{W}; (w_0, w_{1,\delta})),$$

where $w_{1,\delta} = (\gamma_1+\delta, \gamma_1+\delta-m, \mathcal{I}_1)$, and

$$\begin{aligned} {}^b\sigma^\mu(w^\delta\,\mathcal{A}\,w^{-\delta}) &= {}^b\sigma^\mu(\mathcal{A}), \\ \sigma^\mu_{\mathrm{edge}}(w^\delta\,\mathcal{A}\,w^{-\delta}) &= \sigma^\mu_{\mathrm{edge}}(\mathcal{A}), \\ \sigma_{\mathcal{M}}(w^\delta\,\mathcal{A}\,w^{-\delta}) &= T^{i\delta}\,\sigma_{\mathcal{M}}(\mathcal{A}). \end{aligned} \tag{3.2.3}$$

Thus, we may invoke the conjugation by $w^{-\gamma_1}(t)$ to reduce the matter to the weight exponent $\gamma_1 = 0$.

3.3 Elliptic operators

The ellipticity of operators in $\Psi^\mu(\mathcal{C}; W, \tilde{W}; w)$ will refer to the particular choice of μ, namely $\mu = m$.

Definition 3.3.1 *An operator* $\mathcal{A} \in \Psi^m(\mathcal{C}; W, \tilde{W}; w)$ *is said to be elliptic if each symbol of* (3.2.1) *is an isomorphism.*

Mention that in (3.2.1) it is sufficient to restrict oneself to any particular value $s \in \mathbb{R}$. By ellipticity on spaces of lower order singularities, the isomorhisms then take place for *all* $s \in \mathbb{R}$.

Suppose

$$\begin{aligned} \mathcal{A} &\in \Psi^m(\mathcal{C}; W, \tilde{W}; w), & w &= (w_0, w_1), \\ \mathcal{P} &\in \Psi^{-m}(\mathcal{C}; \tilde{W}, W; w^{-1}), & w^{-1} &= (w_0^{-1}, w_1^{-1}), \end{aligned}$$

where $w_j^{-1} = (\gamma_j - m, \gamma_j, \mathcal{I}_j)$. Then, $\mathcal{P}$ is called a *parametrix* of $\mathcal{A}$ if

$$\begin{array}{rcl} \mathcal{P}\mathcal{A} - 1 & \in & \Psi_G(\mathcal{C}; W; w^{-1} \circ w), \\ \mathcal{A}\mathcal{P} - 1 & \in & \Psi_G(\mathcal{C}; \tilde{W}; w \circ w^{-1}) \end{array}$$

where 1 stands for the identity operator in the corresponding classes.

Theorem 3.3.2 *Every elliptic edge problem* $\mathcal{A} \in \Psi^m(\mathcal{C}; W, \tilde{W}; w)$ *has a parametrix* $\mathcal{P} \in \Psi^{-m}(\mathcal{C}; \tilde{W}, W; w^{-1})$.

Theorem 3.3.2 is the main result of our corner operator theory. Therefore we sketch the main ideas of the proof. To construct a parametrix we have to invert the symbols. It employs the following result.

Lemma 3.3.3 *For each elliptic operator* $\mathcal{A} \in \Psi^m(\mathcal{C}; W, \tilde{W}; w)$ *there exists an operator* $\mathcal{R} \in \Psi^{-m}(\mathcal{C}; \tilde{W}, W; w^{-1})$ *such that*

$$\begin{array}{rcl} {}^b\sigma^{-m}(\mathcal{R}) & = & \left({}^b\sigma^m(\mathcal{A})\right)^{-1}, \\ \sigma^{-m}_{\text{edge}}(\mathcal{R}) & = & \left(\sigma^m_{\text{edge}}(\mathcal{A})\right)^{-1}, \\ \sigma_{\mathcal{M}}(\mathcal{R}) & = & T^{-im}\left(\sigma_{\mathcal{M}}(\mathcal{A})\right)^{-1}. \end{array}$$

The construction of $\mathcal{R}$ is rather complicated because of the various symbolic levels. However, we can proceed step by step. First we find an elliptic operator $\mathcal{R}_i \in \Psi^{-m}(\mathcal{C}; \tilde{W}, W; w^{-1})$ such that ${}^b\sigma^{-m}(\mathcal{R}_i) = ({}^b\sigma^m(\mathcal{A}))^{-1}$. Then the composition $\mathcal{A}_1 := \mathcal{R}_i\mathcal{A}$ belongs to $\Psi^0(\mathcal{C}; W; w^{-1} \circ w)$, is elliptic and satisfies ${}^b\sigma^0(\mathcal{A}_1) = 1$. Next we construct an elliptic $\mathcal{R}_e \in \Psi^0(\mathcal{C}; W; w^{-1} \circ w)$ with the properties that ${}^b\sigma^0(\mathcal{R}_e) = 1$ and $\sigma^0_{\text{edge}}(\mathcal{R}_e) = (\sigma^0_{\text{edge}}(\mathcal{A}_1))^{-1}$. Then the composition $\mathcal{A}_2 := \mathcal{R}_e\mathcal{A}_1$ belongs to $\Psi^0(\mathcal{C}; W; w^{-1} \circ w)$, is elliptic and satisfies ${}^b\sigma^0(\mathcal{A}_2) = 1$, $\sigma^0_{\text{edge}}(\mathcal{A}_2) = 1$. It remains to construct an elliptic operator $\mathcal{R}_c \in \Psi^0(\mathcal{C}; W; w^{-1} \circ w)$ such that ${}^b\sigma^0(\mathcal{R}_c) = 1$, $\sigma^0_{\text{edge}}(\mathcal{R}_c) = 1$ and $\sigma_{\mathcal{M}}(\mathcal{R}_c) = (\sigma_{\mathcal{M}}(\mathcal{A}_2))^{-1}$. Then we may set $\mathcal{R} = \mathcal{R}_c\mathcal{R}_e\mathcal{R}_i$. The details of the construction need technicalities of the Mellin pseudodifferential calculus, the scheme being to large extent analogous to the cone and edge theories, cf. [Sch98].

Combining Theorems 3.2.3 and 3.2.4 we readily deduce that $\mathcal{R}$ is a regulariser of $\mathcal{A}$, i.e., an inverse modulo compact operators. We then make use of a formal Neumann series argument to pass from $\mathcal{R}$ to a true parametrix $\mathcal{P}$ for $\mathcal{A}$.

Since the Green operators are compact in weighted Sobolev spaces on $\mathcal{C}$, it follows from Theorem 3.3.2 that each elliptic edge problem is Fredholm. It is to be expected that the ellipticity is also necessary for the Fredholm property, but such a theorem has been hardly obtained as yet. However, Theorem 3.2.5 implies that the ellipticity is necessary for the existence of a pseudodifferential parametrix.

As usual, a straightforward consequence of the existence of a left parametrix within the calculus is a *Regularity Theorem.*

Corollary 3.3.4 *Suppose $\mathcal{A}$ is elliptic. If*

$$\begin{array}{rcl} u & \in & H^{-\infty,\gamma}(\mathcal{C}) \oplus H^{-\infty,\gamma_1}(E,W), \\ f & \in & H^{s-m,\gamma-m}(\mathcal{C}) \oplus H^{s-m,\gamma_1-m}(E,\tilde{W}) \end{array}$$

and $\mathcal{A}u = f$, then u actually belongs to $H^{s,\gamma}(\mathcal{C}) \oplus H^{s,\gamma_1}(E,W)$. Moreover, if

$$\begin{array}{rcl} u & \in & H^{-\infty,\gamma}(\mathcal{C}) \oplus H^{-\infty,\gamma_1}(E,W), \\ f & \in & H^{s-m,\gamma-m}_{\tilde{\mathrm{as}}}(\mathcal{C}) \oplus H^{s-m,\gamma_1-m}_{\tilde{as}}(E,\tilde{W}) \end{array}$$

for some asymptotic types "$\tilde{\mathrm{as}}$" and "$\tilde{as}$", then $u \in H^{s;\gamma}_{\mathrm{as}}(\mathcal{C}) \oplus H^{s;\gamma_1}_{as}(E,W)$, for resulting asymptotic types "as" and "as" depending on $\mathcal{A}$ and "$\tilde{\mathrm{as}}$", "$\tilde{as}$".

Proof. Indeed, from $\mathcal{A}u = f$ it follows that

$$u = (1 - \mathcal{P}\mathcal{A})\, u + \mathcal{P}f.$$

Applying Theorem 3.2.2 yields the asserted quality of $\mathcal{P}f$, whereas the regularity of $(1-\mathcal{P}\mathcal{A})u$ is just expressed by the very definition of a Green operator. □

Let $\mathcal{A} \in \Psi^m(\mathcal{C}; W, \tilde{W}; w)$ be elliptic. Then the operator

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(\mathcal{C}) \\ \oplus \\ H^{s,\gamma_1}(E,W) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\mathcal{C}) \\ \oplus \\ H^{s-m,\gamma_1-m}(E,\tilde{W}) \end{array}$$

is Fredholm for all $s \in \mathbb{R}$. By Corollary 3.3.4, the index of $\mathcal{A}$ is independent of s. If $\mathcal{A}_0, \mathcal{A}_1 \in \Psi^m(\mathcal{C}; W, \tilde{W}; w)$ and

$$\begin{array}{rcl} {}^b\sigma^m(\mathcal{A}_0) & = & {}^b\sigma^m(\mathcal{A}_1), \\ \sigma^m_{\mathrm{edge}}(\mathcal{A}_0) & = & \sigma^m_{\mathrm{edge}}(\mathcal{A}_1), \\ \sigma_{\mathcal{M}}(\mathcal{A}_0) & = & \sigma_{\mathcal{M}}(\mathcal{A}_1), \end{array}$$

then the indices of $\mathcal{A}_0$ and $\mathcal{A}_1$ coincide, which is clear from Theorems 3.2.3 and 3.2.4. The problem of how to evaluate the index of $\mathcal{A}$ in terms of the leading symbols is a straightforward generalisation of the index problem for elliptic operators on smooth manifolds, cf. Gelfand [Gel60].

3.4 Calculi for higher order singularities

We are now in a position to single out three main features of our approach to constructing algebras of pseudodifferential operators on singular varieties. The first feature is using special coordinates near singularities to identify coordinate patches with cone bundles over Euclidean spaces $\mathbb{R}^q$, $q \geq 0$. Under these coordinates the singularity itself is moved to a variety of points at infinity. This enables one to write the operators in a unified way, namely as Fourier pseudodifferential operators along $\mathbb{R}^q$ with operator-valued symbols, in spite of the diversity of possible operator representations (Mellin, etc.). The analysis in weighted Sobolev spaces near singularities reduces to that in weighted Sobolev spaces over $\mathbb{R}^q$, with natural weight functions $e^{\gamma\langle y\rangle}$ and $\langle y\rangle^{\mu}$ to control the behaviour of functions at points at infinity. The choice of two weights actually fits in the analysis on varieties with cuspidal singularities, cf. [RST99], because $\delta'(r) \sim \langle\delta(r)\rangle^{p/(p-1)}$ as $r \to 0$, $t = \delta(r)$ being a local coordinate close to the cusp. The second feature is the idea of edge Sobolev spaces $H^s(\mathbb{R}^q, \pi^* V)$ whose definition relies on group actions in the fibres over the edge $\mathbb{R}^q$, cf. Definition 1.3.1. The property

$$H^s(\mathbb{R}^{q_1}, \pi^* H^s(\mathbb{R}^{q_2}, \pi^* V)) = H^s(\mathbb{R}^{q_1+q_2}, \pi^* V), \tag{3.4.1}$$

$H^s(\mathbb{R}^{q_2}, \pi^* V)$ being endowed with the group action $u \mapsto \kappa_\lambda(\lambda^{q_2/2} u(\lambda y))$, cf. [Sch98, p. 115], allows one to ensure compatibility of definitions over different strata. Finally, the third feature is invoking operator-valued Fourier symbols over $\mathbb{R}^q$ which admit asymptotic expansions in "twisted" homogeneous symbols, the "twisted" homogeneity referring to the group actions. This makes the theory of pseudodifferential operators on stratified varieties quite analogous to Boutet de Monvel's theory of pseudodifferential boundary value problems, cf. [BdM71].

More precisely, let V be a *smoothly stratified space with local cone bundle neighbourhoods*, cf. [GM88]. We shall introduce this notion by defining inductively for each $N \geq 1$ a category LCB(N) of smoothly stratified spaces with local cone bundle neighbourhoods with at most N strata. A smoothly stratified space with local cone bundle neighbourhoods is simply an object in LCB $= \cup_N$LCB(N). For each of these categories there will be natural notions of product with a smooth manifold and of the boundary of an object. Also, underlying each object in LCB(N) will be a stratified space with at most N strata. The category LCB(1) is the category of smooth manifolds and smooth mappings. Assume that LCB(j) has been defined for all $j < N$. For any closed finite-dimensional object B in LCB($N-1$) we denote by $C_t(B)$ the cone on the topological space underlying B. Its top point is v, and the open cone is identified with $(0,1] \times B$. If Ω is an open set in a Euclidean space, we give $\Omega \times C_t(B)$ a stratification in which $\Omega \times \{v\}$ is the bottom stratum and the others are of the form Ω times the open cone over strata of B. An LCB(N) *coordinate chart* for a stratified space V is a strata-preserving homeomorphism h from an open subset O of V to $\Omega \times C_t(B)$, where Ω is an open subset of a Euclidean space and B is a closed finite-dimensional object in LCB($N-1$). Two coordinate charts $h_\nu \colon O_\nu \to \Omega_\nu \times C_t(B_\nu)$, $\nu = 1, 2$, are said to be *compatible* if the composition $h_2 \circ h_1^{-1}$ induces local diffeomorphisms both from Ω_1 to Ω_2 and from

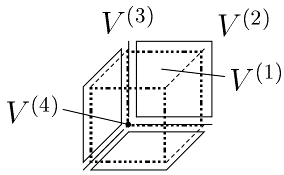

Fig. 3.2: A smooth stratification.

$\Omega_1 \times ((0,1] \times B_1)$ to $\Omega_2 \times ((0,1] \times B_2)$, the former being a "limit" of the latter. An object in LCB(N) is a stratified space with a maximal atlas of compatible LCB(N) coordinate charts covering the space. A *morphism* in this category is a strata-preserving mapping f which is C^∞ on each stratum and behaves properly with respect to local cone bundle structures. It is well known that any *Whitney stratified subspace* of a smooth manifold admits a natural structure of a smoothly stratified space with local cone bundle neighbourhoods, cf. *ibid.*

Write $V^{(1)}, \ldots, V^{(N)}$ for the strata of V, $V^{(j)}$ being of codimension n_j (see Fig. 3.2). A stratum need not be connected, hence we may assume without loss of generality that $0 = n_1 < \ldots < n_N \le \dim V$. By definition, each $V^{(j)}$ is a smooth manifold, and the closure of $V^{(j)}$ in V lies in $V^{(j)} \cup \ldots \cup V^{(N)}$ for all $j = 1, \ldots, N$.

We may formally write

$$V^{(j)} = V^{(J)} \times \frac{V^{(j)}}{V^{(J)}}, \tag{3.4.2}$$

for $1 \le j \le J \le N$, and our next goal is to give a precise meaning to this equality. We shall interpret it locally in a neighbourhood O of any point $p \in V^{(J)}$. If O is small enough, then $V^{(j)}$ has the structure of a cone bundle over $V^{(J)}$ within O, the fibre being $C_t(B)$. If non-empty, B is a smoothly stratified space with local cone bundle neighbourhoods with at most $(J-j)$ strata. We call B the link of $V^{(j)}$ over $V^{(J)}$, it is actually the same over each connected component of $V^{(J)}$. The fibre $C_t(B)$ is invariant under the group action $(t,x) \mapsto (\lambda t, x)$, $\lambda > 0$; this latter specifies a singular fibre structure by itself. To not exclude artificial stratifications, we allow the fibre $C_t(B)$ to bear a smooth structure, too. In this case $C_t(B)$ is locally identified with a Euclidean space (namely, $\mathbb{R}^{n_J - n_j}$) and endowed with the group action $z \mapsto \lambda z$, $\lambda > 0$. Now, by $V^{(j)}/V^{(J)}$ in (3.4.2) is just meant the local fibre of $V^{(j)}$ over $V^{(J)}$.

In case the strata meet each other at non-zero angles the typical differential operators on V are those of Fuchs-type. This suggests a canonical transformation of the coordinate along the cone axis of $C_t(B)$, which maps the vertex $t = 0$ to the point at infinity. More precisely, the new coordinates $(T = \log t, x)$ for small $t > 0$ allow one to identify $C_t(B)$ with the cylinder $\mathbb{R} \times B$ over B, the conical point corresponding to $T = -\infty$. Note that B itself has a finite covering by local cone bundle neighbourhoods and that a smooth manifold is locally a cone over the unit sphere. We are thus led to the problem of defining weighted Sobolev spaces in the wedge $\mathbb{R}^q \times C_t(B)$, which properly control the behaviour of functions up to points at infinity. We start with the scale $H^s(B)$, $s \in \mathbb{R}$, of usual Sobolev spaces on a C^∞ manifold B, which possesses a finite coordinate covering. Such a manifold has one stratum, the cone over B has two ones. We proceed by induction and assume that the relevant Sobolev spaces have been introduced for all B with at most $(J-j)$ strata. Any stratum contributes by additional weight exponents, so we have a scale of Sobolev spaces $H^{s,w'}(B)$ on B, parametrised by smoothness $s \in \mathbb{R}$ and a tuple of weights $w' = (w_1, \ldots, w_{J-j-1})$, every component being a weight function on $\mathbb{R}_+$ usually identified with a quadruple of real numbers. By the above, B has a finite covering (O_ν) by coordinate charts, each O_ν being diffeomorphic to a wedge by $h_\nu : O_\nu \to \Omega_\nu \times C_t(B_\nu)$. Pick a C^∞ partition of unity on B subordinate to this covering, (φ_ν). The space $H^{s,w'}(B)$ is glued together from the local spaces $H^s(\mathbb{R}^{q_\nu}, \pi^* H^{s,w'}(C_t(B_\nu)))$ in the sense that $u \in H^{s,w'}(B)$ if and only if $(h_\nu)_* (\varphi_\nu u) \in H^s(\mathbb{R}^{q_\nu}, \pi^* H^{s,w'}(C_t(B_\nu)))$ for all ν. We have $C_t(B) = \cup_\nu C_t(O_\nu)$, and the diffeomorphisms $1 \times h_\nu$ take $C_t(O_\nu)$ to the cones over $\mathbb{R}^{q_\nu} \times C_t(B_\nu)$. Note that we think of $C_t(B)$ as being infinite and we regard the infinity as a conical point. Choose the covering of $\bar{\mathbb{R}}_+$ by the intervals $[0,1)$ and $(1/2, \infty)$, and a partition of unity on $\bar{\mathbb{R}}_+$ subordinate to this covering, $(\omega, 1-\omega)$. We make use of the cut-off function ω to introduce a weight function along the cone axis,

$$w_{J-j}(t) = t^{-\gamma} (\log 1/t)^{\mu}\, \omega(t) + t^{\delta} (\log t)^{\nu} (1 - \omega(t)),$$

with $\gamma, \mu, \delta, \nu \in \mathbb{R}$, which is typical for the analysis on a cone. For the tuple $w = (w', w_{J-j})$, the space $H^{s,w}(C_t(B))$ is defined to consist of all functions $u(t,x)$, such that

$$\begin{array}{rcl} (\log t \times h_\nu)_* (\omega \varphi_\nu\, w_{J-j} u) & \in & H^s(\mathbb{R}^{1+q_\nu}, \pi^* H^{s,w'}(C_t(B_\nu))), \\ (1 \times h_\nu)_* ((1-\omega)\varphi_\nu\, w_{J-j} u) & \in & H^s(\mathbb{R}^{1+q_\nu}, \pi^* H^{s,w'}(C_t(B_\nu))) \end{array}$$

for all ν.

We might certainly express this in a unified way by introducing a diffeomorphism $T = \delta(t)$ of $\mathbb{R}_+$ onto $\mathbb{R}$ with the property that $\delta(t) = \log t$, for small t, and $\delta(t) = t$, for large t.

Lemma 3.4.1 *As defined above, the space $H^{s,w}(C_t(B))$ is invariant under the group action $(K_\lambda u)(t,x) = \lambda^N u(\lambda t, x)$, $\lambda > 0$.*

Proof. When topologising $H^{s,w}(C_t(B))$ under the natural norm, we get, by (3.4.1),

$$\|K_\lambda u\|^2_{H^{s,w}(C_t(B))} = \sum_\nu \|(\delta \times h_\nu)_* \varphi_\nu w_{J-j} K_\lambda u\|^2_{H^s(\mathbb{R}, \pi^* V_\nu)}$$

for all $\lambda > 0$, where $V_\nu = H^s(\mathbb{R}^{q_\nu}, \pi^* H^{s,w'}(C_t(B_\nu)))$. We will restrict our attention to those functions u which are supported in a fixed chart O_ν. Then we may omit, by abuse of notation, the index ν and both φ_ν and h_ν. Moreover, it will cause no confusion if we write $u(t)$ for u and $w(t)$ for w_{J-j}. It follows that

$$\|K_\lambda u\|^2_{H^{s,w}(C_t(B))} = \lambda^{2N} \int_{\mathbb{R}} \langle\tau\rangle^{2s} \|\kappa^{-1}_{\langle\tau\rangle} \mathcal{F}_{T\mapsto\tau} \delta_* \left(w(t)u(\lambda t)\right)\|^2_V \, d\tau,$$

(κ_λ) being the group action in the fibre V. Changing the variable by

$$T = \delta\left(\frac{1}{\lambda}\delta^{-1}(S)\right),$$

we obtain

$$\mathcal{F}_{T\mapsto\tau}\delta_*\left(w(t)u(\lambda t)\right) = \int_{\mathbb{R}} e^{-i\tau\delta\left(\frac{1}{\lambda}\delta^{-1}(S)\right)} w\left(\frac{1}{\lambda}\delta^{-1}(S)\right) u\left(\delta^{-1}(S)\right) d\delta\left(\frac{1}{\lambda}\delta^{-1}(S)\right)$$

whence

$$\begin{aligned} \mathcal{F}_{T\mapsto\tau}\delta_*\left(w(t)u(\lambda t)\right) &= \frac{1}{\lambda}\int_{\mathbb{R}} K\left(\lambda;\tau,\sigma\right) \mathcal{F}_{S\mapsto\sigma}\delta_*\left(wu\right) d\sigma, \\ K\left(\lambda;\tau,\sigma\right) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\tau\delta\left(\frac{1}{\lambda}\delta^{-1}(S)\right)+i\sigma S} \frac{(\delta' w)\left(\frac{1}{\lambda}\delta^{-1}(S)\right)}{(\delta' w)\left(\delta^{-1}(S)\right)} \, dS, \end{aligned}$$

the latter integral being understood in the sense of distributions. Applying Schwarz's inequality yields

$$\|\kappa^{-1}_{\langle\tau\rangle}\mathcal{F}_{T\mapsto\tau}\delta_*(w(t)u(\lambda t))\|^2_V \le \left(\frac{1}{\lambda^2}\int_{\mathbb{R}} \langle\sigma\rangle^{-2s}|K|^2 \left\|\kappa^{-1}_{\frac{\langle\tau\rangle}{\langle\sigma\rangle}}\right\|^2_{\mathcal{L}(V)} d\sigma\right) \|u\|^2_{H^{s,w}(C_t(B))},$$

and so

$$\|K_\lambda u\|^2_{H^{s,w}(C_t(B))} \le \left(\lambda^{2N-2} \iint_{\mathbb{R}^2} \left(\frac{\langle\tau\rangle}{\langle\sigma\rangle}\right)^{2s} \left\|\kappa^{-1}_{\frac{\langle\tau\rangle}{\langle\sigma\rangle}}\right\|^2_{\mathcal{L}(V)} |K|^2 d\tau d\sigma\right) \|u\|^2_{H^{s,w}(C_t(B))}.$$

To complete the proof it suffices to observe that the expression in the round brackets is dominated by

$$\lambda^{2N-2} \iint_{\mathbb{R}^2} \langle\tau-\sigma\rangle^{2Q} \left|K\left(\lambda;\tau,\sigma\right)\right|^2 d\tau d\sigma$$

which is due to a well-known property of abstract group actions and Peetre's inequality.

□

Having defined the space $H^{s,w}(C_t(B))$ along with the group action, we can invoke Definition 1.3.1 to introduce the space $H^s(\mathbb{R}^q, \pi^* H^{s,w}(C_t(B)))$, for any q, thus completing the step of induction. What is still lacking is an explicit value of N in dependence of q and the norm in $H^{s,w}(C_t(B))$. The criterion for the choice of N is that the space $H^s(\mathbb{R}^q, \pi^* H^{s,w}(C_t(B)))$ be locally equivalent to the usual Sobolev space H^s_{loc} away from the singularities of $\mathbb{R}^q \times C_t(B)$, "locally" meaning "on compact sets". As usual, taking $\nu = 0$ and $N = \delta + 1/2$ fills the bill.

The underlying idea of our "resolution of singularities" consists of moving the singular points to infinity and using the Fourier analysis in a Euclidean space under strong control by weight functions. This idea goes back at least as far as [Rab69].

We now turn to an arbitrary smoothly stratified space with local cone bundle neighbourhoods V. In order to introduce pseudodifferential operators on V, we proceed by induction. Suppose we have already defined a pseudodifferential calculus on all spaces with at most $N - 1$ strata. Given any $B \in \mathrm{LCB}(N-1)$, by a pseudodifferential operator on $C_t(B)$ we mean $\delta^\sharp \mathcal{A}$, where $\mathcal{A}$ is a pseudodifferential operator on $\mathbb{R}$ whose symbol takes its values in the algebra of pseudodifferential operators on B, $T = \delta(t)$ is a diffeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}$, as above, and $\delta^\sharp \mathcal{A} = \delta^* \mathcal{A} \delta_*$ is the operator pull-back of $\mathcal{A}$ under δ. Obviously, the operators obtained this way are of the Mellin type close to the conical point of $C_t(B)$. Let now $V \in \mathrm{LCB}(N)$. By definition, V has a finite covering (O_ν) by coordinate charts diffeomorphic to model wedges by $h_\nu : O_\nu \to \Omega_\nu \times C_t(B_\nu)$, where Ω_ν is an open subset of $\mathbb{R}^{q_\nu}$. We pick a C^∞ partition of unity on V subordinate to this covering, (φ_ν), and a system of C^∞ functions (ψ_ν) which covers (φ_ν). The pseudodifferential operators on V are of the form

$$\mathcal{A} = \sum_\nu \varphi_\nu \left(h_\nu^\sharp \mathcal{A}_\nu \right) \psi_\nu,$$

where $\mathcal{A}_\nu$ is a pseudodifferential operator along $\mathbb{R}^{q_\nu}$ with an operator-valued symbol taking its values in the algebra on $C_t(B_\nu)$, and $h_\nu^\sharp \mathcal{A}_\nu = h_\nu^* \mathcal{A}_\nu (h_\nu)_*$ is the pull-back of $\mathcal{A}_\nu$ under h_ν.

Another way of stating the pseudodifferential calculus on V is to cover V by local cylinder bundle neighbourhoods (O_ν). This requires diffeomorphisms $\Delta_\nu : O_\nu \to \Omega_\nu \times ((-\infty, T_\nu) \times B_\nu)$, for every ν, together with a precise control of the behaviour at $T = -\infty$. In the case of transversal intersections, we carry over the point at infinity to a finite point by the diffeomorphism $t = e^T$, thus transplanting the C^∞ structure from $t = 0$ to $T = -\infty$. For cuspidal intersections, other diffeomorphisms are used, whose explicit nature is prescribed by the geometry. Note that a cylinder $\mathbb{R} \times B$ bears the group action $(T, x) \mapsto (T + \log \lambda, x)$, for $\lambda \in \mathbb{R}$, which has actually been used in the definition of twisted Sobolev spaces. Under this approach, pseudodifferential operators in O_ν are of the form $\Delta^\sharp_\nu \mathcal{A}_\nu$, with $\mathcal{A}_\nu$ a pseudodifferential operator on $\mathbb{R}^{q_\nu+1}$ whose symbol takes the values in the algebra on B_ν. They can be treated in the framework of a calculus of pseudodifferential operators in $\mathbb{R}^Q$ with slowly varying operator-valued symbols, cf. [RST97]. However, it is more convenient to have specified the exit to infinity by choosing a relevant variable, here T.

The pseudodifferential operators to be introduced are intended to act in weighted Sobolev spaces on V as

$$\mathcal{A} : \bigoplus_{j=1}^{N} H^{s,(w_{j+1},\dots,w_N)}(V^{(j)}, F_j) \to \bigoplus_{j=1}^{N} H^{s-m,(w_{j+1},\dots,w_N)-m}(V^{(j)}, \tilde{F}_j) \quad (3.4.3)$$

for all $s \in \mathbb{R}$, where $w = (w_2, \dots, w_N)$ is a tuple of weights, w_j corresponding to the stratum $V^{(j)}$, and F_j, $\tilde{F}_j$ are C^∞ vector bundles over $V^{(j)}$. For $j = N$, the tuple $(w_{j+1}, \dots, w_N)$ is empty, which causes no confusion because $V^{(N)}$ is a C^∞ compact closed manifold. By (3.4.3), $\mathcal{A}$ can be specified as an $(N\times N)$-matrix of operators

$$\mathcal{A} = (A_{ij})_{\substack{i=1,\dots,N\\ j=1,\dots,N}},$$

with

$$A_{ij} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^{s,(w_{j+1},\dots,w_N)}(V^{(j)}, F_j), H^{s-m,(w_{i+1},\dots,w_N)-m}(V^{(i)}, \tilde{F}_i)).$$

The entries A_{ij} with $i > j$ have the meaning of *trace operators*, the "trace" standing for restriction from $V^{(j)}$ to $V^{(i)}$. On the other hand, the entries A_{ij} with $i < j$ have the dual meaning of *potential operators*. To handle such operators within an algebra, we should also add compositions of trace and potential operators. They contribute to the diagonal entries A_{jj} and are known as *Green operators* on the corresponding strata. In fact, all the entries are pseudodifferential operators and can be specified through their operator-valued symbols.

To this end, it is sufficient to describe the action of $\mathcal{A}$ close to any point $p \in V$. For definiteness, consider $p \in V^{(J)}$ where $1 \le J \le N$. Then we have $\mathcal{A} = \mathcal{F}^{-1}_{\eta\to y}\mathfrak{a}^{(J)}(y,\eta)\mathcal{F}_{y\to\eta}$ in local coordinates $y \in \mathbb{R}^{q_J}$ of $V^{(J)}$ near the point p, where

$$\mathfrak{a}^{(J)} = \begin{pmatrix} a^{(J)}_{11} & 0 & \dots & 0 & a_{1J} \\ 0 & a^{(J)}_{22} & \dots & 0 & a_{2J} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a^{(J)}_{J-1,J-1} & a_{J-1,J} \\ a_{J1} & a_{J2} & \dots & a_{J,J-1} & a^{(J)}_{JJ} \end{pmatrix} \quad (3.4.4)$$

acts as

$$\mathfrak{a}^{(J)} : \pi^* \bigoplus_{j=1}^{J} H^{s,(w_{j+1},\dots,w_J)}\left(\tfrac{V^{(j)}}{V^{(J)}}, F_j\right) \to \pi^* \bigoplus_{j=1}^{J} H^{s-m,(w_{j+1},\dots,w_J)-m}\left(\tfrac{V^{(j)}}{V^{(J)}}, \tilde{F}_j\right)$$

for all $s \in \mathbb{R}$, π standing for the canonical projection $T^*V^{(J)} \to V^{(J)}$. To encompass all strata, we identify these matrices within those with $(N \times N)$ entries in an obvious way.

Write $\kappa_{j,\lambda}^{(J)}$ for the group action in $H^{s,(w_{j+1},\dots,w_J)}(V^{(j)}/V^{(J)})$, and gather them to a matrix

$$\kappa_\lambda^{(J)} = \bigoplus_{j=1}^{J} \kappa_{j,\lambda}^{(J)}$$

to act in $\oplus_{j=1}^{J} H^{s,(w_{j+1},\dots,w_J)}(V^{(j)}/V^{(J)}, F_j)$. By the above, the usual choice for $\kappa_{j,\lambda}^{(J)}$ is

$$u(t) \mapsto \lambda^{\frac{n_J - n_j}{2}} u(\lambda t),$$

which meets (3.4.1). Analogously, we denote by $\tilde{\kappa}_\lambda^{(J)}$, $\lambda > 0$, the group action in $\oplus_{j=1}^{J} H^{s-m,(w_{j+1},\dots,w_J)-m}(V^{(j)}/V^{(J)}, \tilde{F}_j)$.

Thus, every stratum $V^{(J)}$ gives rise to group actions $\kappa_\lambda^{(J)}$ and $\tilde{\kappa}_\lambda^{(J)}$ in weighted Sobolev spaces in the fibres of V over $V^{(J)}$. Associated to these group actions are spaces of operator-valued symbols satisfying "twisted" symbol estimates, cf. Definition 1.3.3, and a concept of homogeneity. This allows one to repeat all the steps in the construction of the algebra of pseudodifferential operators on a manifold with smooth edges, as described in Section 1.3. It begins with typical differential operators on V whose form can be read off from that of the Laplace operator with respect to a Riemannian metric on the "smooth part" of V. Near $V^{(J)}$ the symbols of typical operators contain the covariables along $V^{(J)}$ through the aggregates $(r_1 \dots r_{J-1})\eta$ where $r_1, \dots, r_{J-1}$ are defining functions of the faces whose intersection gives $V^{(J)}$. Hence any differentiation in the covariables results not only in decreasing the order by 1 but also in an additional factor $(r_1 \dots r_{J-1})$ which vanishes on $V^{(J)}$ and thus leads to a gain in the weight. Since the gains in both order and weight provide compactness in the relevant weighted Sobolev spaces, the symbols of typical operators belong to the class of symbols of compact fibre variation introduced in [Luk72]. To include the broadest interesting operator classes we reduce our standing assumptions on the symbols under consideration to the only requirement of compact fibre variation.

When summing up over all strata, we arrive at the matrixes of operators which look like

$$\mathcal{A} = \begin{pmatrix} \sum_{J=1}^{N} A_{11}^{(J)} & A_{12} & \dots & A_{1,N-1} & A_{1N} \\ A_{21} & \sum_{J=2}^{N} A_{22}^{(J)} & \dots & A_{2,N-1} & A_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ A_{N-1,1} & A_{N-1,2} & \dots & \sum_{J=N-1}^{N} A_{N-1,N-1}^{(J)} & A_{N-1,N} \\ A_{N1} & A_{N2} & \dots & A_{N,N-1} & A_{NN}^{(N)} \end{pmatrix},$$

the stratum $V^{(J)}$ contributing by $A_{jj}^{(J)}$ to any entry (j,j) with $j \le J$. Only the summand $A_{jj}^{(j)}$ of the entry (j,j) is a usual pseudodifferential operator with scalar-valued symbol on $V^{(j)}$. The summands $A_{jj}^{(J)}$ with $j < J$ bear operator-valued symbols living on the cotangent bundle of $V^{(J)}$. They are known as *Green operators* on $V^{(j)}$ associated to the stratum $V^{(J)}$, cf. Definition 1.3.5 and elsewhere.

Denote $\Psi^m(V; F, \tilde{F}; w)$ the space of all operators (3.4.3) on V, as described above. Any operator $\mathcal{A} \in \Psi^m(V; F, \tilde{F}; w)$ has N principal homogeneous symbols of order m,

$$\sigma(\mathcal{A}) = (\sigma_J^m(\mathcal{A}))_{J=1,\ldots,N},$$

$\sigma_J^m(\mathcal{A})$ corresponding to the stratum $V^{(J)}$. For $(y,\eta) \in T^*V^{(J)}$, we actually have

$$\sigma_J^m(\mathcal{A})(y,\eta) = \lim_{\lambda\to\infty} \lambda^{-m}\, \tilde{\kappa}_{\lambda^{-1}}^{(J)}\, \mathfrak{a}^{(J)}(y,\lambda\eta)\, \kappa_{\lambda}^{(J)} \qquad (3.4.5)$$

where $\mathfrak{a}^{(J)}$ is a local symbol of $\mathcal{A}$ along $V^{(J)}$, cf. (3.4.4). The passage to the limit automatically includes freesing the coefficients at the vertex of each cone $V^{(j)}/V^{(J)}$, $1 \le j \le J$, in a special manner.

It is worth pointing out that formula (3.4.5) gives a symbol $\sigma_N^m(\mathcal{A})$ even if $\dim V^{(N)} = 0$. In this case $V^{(N)}$ consists of isolated points, hence its cotangent bundle is also discrete. Taking $\sigma_N^m(\mathcal{A})$ over a point $v \in V^{(N)}$ just amounts to freesing the "coefficients" of $\mathcal{A}$ at v.

If all the symbols $\sigma_1^m(\mathcal{A}), \ldots, \sigma_N^m(\mathcal{A})$ vanish identically, then the operator (3.4.3) is compact. Hence we may invoke the tuple $\sigma(\mathcal{A})$ to introduce a concept of elliptic operators on V.

Definition 3.4.2 *An operator $\mathcal{A} \in \Psi^m(V; F, \tilde{F}; w)$ is said to be elliptic if, for every $J = 1, \ldots, N$, the symbol $\sigma_J^m(\mathcal{A})$ is invertible away from the zero section of $T^*V^{(J)}$.*

The condition on the invertibility of $\sigma_N^m(\mathcal{A})$ away from the zero section of $T^*V^{(N)}$ is vague in case $V^{(N)}$ is zero-dimensional. To define it more exactly, we observe that what we really need is the Fredholm property of the complete operator-valued symbol over all of $T^*V^{(N)}$ along with its invertibility outside a compact subset of $T^*V^{(N)}$. Hence a proper substitute for the invertibility will be the Fredholm property of $\sigma_N^m(\mathcal{A})$ on all of $T^*V^{(N)}$, provided that $V^{(N)}$ is of dimension 0. If $\mathcal{A}$ is pseudodifferential close to $V^{(N)}$, the Fredholm property of $\sigma_N^m(\mathcal{A})$ is in turn equivalent to the invertibility of its Fourier symbol over a suitable horizontal line, which acts in weighted Sobolev spaces on the link of $V^{(1)}$ over $V^{(N)}$.

Let us have look from this viewpoint at the analysis on manifolds with conical points. Obviously, these correspond to smoothly stratified spaces V with local cone bundle neighbourhoods with at most 2 strata. In fact, $V^{(1)}$ is the "smooth" part of the manifold, and $V^{(2)}$ is the discrete set of conical points.

Assume for simplicity that $V^{(2)}$ consists of only one point v, and write X for the link of $V^{(1)}$ over $V^{(2)}$. When localised to $V^{(1)}$, the space $\Psi^m(V;F,\tilde{F};w)$ is nothing but $\Psi^m_{\mathrm{cl}}(V^{(1)};F_1,\tilde{F}_1)$, i.e., the space of classical pseudodifferential operators of type $F_1 \to \tilde{F}_1$ and order m, acting in the usual Sobolev spaces on $V^{(1)}$ as $H^s_{\mathrm{comp}}(V^{(1)},F_1) \to H^{s-m}_{\mathrm{loc}}(V^{(1)},\tilde{F}_1)$. Further, v possesses a neighbourhood O on V with local coordinates $h\colon O \to C_t(X)$. Since $V^{(2)}$ is zero-dimensional, the restriction of $\mathcal{A} \in \Psi^m(V;F,\tilde{F};w)$ to O is identified with its operator-valued symbol $\mathfrak{a}^{(2)}(y,\eta)$ living on $T^*V^{(2)} \cong \{0\}$. This symbol is a (2×2)-matrix acting as

$$\begin{array}{ccc} H^{s,w}(C_t(X),F_1) & & H^{s-m,w-m}(C_t(X),\tilde{F}_1) \\ \oplus & \to & \oplus \\ F_2 & & \tilde{F}_2 \end{array}$$

for all $s \in \mathbb{R}$, where $w(t) = t^{-\gamma}(\log 1/t)^{\mu}\omega(t) + t^{\delta}(\log t)^{\nu}(1-\omega(t))$ is a weight function on the cone axis. The symbol $\sigma_2^m(\mathcal{A})$ is obtained from $\mathfrak{a}^{(2)}$ by freesing the "coefficients" at $t=0$. Since both F_2 and $\tilde{F}_2$ are finite-dimensional, the Fredholm property of $\sigma_2^m(\mathcal{A})$ just amounts to that of its entry $(1,1)$, for which we write $\sigma_2^m(a_{11}^{(2)})$. As described above, the change of variables $T = \log t$ yields local coordinates $\Delta : O \to (-\infty,T) \times X$ near v, v itself corresponding to $\{-\infty\} \times X$. In these coordinates, we have $a_{11}^{(2)} = \Delta^{\sharp}\mathcal{F}^{-1}_{\tau\to T}a(T,\tau-i\gamma)\mathcal{F}_{T\to\tau}$ where $a(T,\tau-i\gamma)$ takes its values in $\cap_{s\in\mathbb{R}}\mathcal{L}(H^s(X,F_1),H^{s-m}(X,\tilde{F}_1))$. It follows that

$$\sigma_2^m(a_{11}^{(2)}) = \Delta^{\sharp}\mathcal{F}^{-1}_{\tau\to T}a(-\infty,\tau-i\gamma)\mathcal{F}_{T\to\tau},$$

and so $\sigma_2^m(a_{11}^{(2)})$ is Fredholm if and only if the family $a(-\infty,\tau-i\gamma)$ is invertible. Clearly, $a(-\infty,\tau-i\gamma)$ is the conormal symbol of $\mathcal{A}$ at v. We conclude that for spaces with point singularities Definition 3.4.2 reduces to the concept of ellipticity of Section 1.2.

We finish the paper by characterising Fredholm operators in the calculus on a smoothly stratified space with local cone bundle neighbourhoods.

Theorem 3.4.3 *Assume that $\mathcal{A} \in \Psi^m(V;F,\tilde{F};w)$ is elliptic. Then the operator (3.4.3) is Fredholm, for each $s \in \mathbb{R}$, and it has a parametrix in $\Psi^{-m}(V;\tilde{F},F;w^{-1})$.*

Proof. By assumption, the symbol $\sigma_1^m(\mathcal{A})$ is invertible outside the zero section of $T^*V^{(J)}$. Using the standard Leibniz product argument, we find an elliptic operator $\mathcal{R}_1 \in \Psi^{-m}(V;\tilde{F},F;w^{-1})$ such that

$$\sigma_1^{-m}(\mathcal{R}_1) = \left(\sigma_1^m(\mathcal{A})\right)^{-1}.$$

Since the symbol mappings behave naturally under composition of operators, it follows readily that $\mathcal{R}_1\mathcal{A} \in \Psi^0(V;F;w^{-1}\circ w)$ is an elliptic operator and

$$\begin{aligned}\sigma_1^0(\mathcal{R}_1\mathcal{A}) &= \sigma_1^{-m}(\mathcal{R}_1)\,\sigma_1^m(\mathcal{A})\\ &= 1.\end{aligned}$$

We now find an elliptic operator $\mathcal{R}_2 \in \Psi^0(V;F;w^{-1}\circ w)$ with the property that

$$\begin{aligned}\sigma_1^0(\mathcal{R}_2) &= 1,\\ \sigma_2^0(\mathcal{R}_2) &= \left(\sigma_2^0(\mathcal{R}_1\mathcal{A})\right)^{-1}.\end{aligned}$$

Arguing as above, we conclude that $\mathcal{R}_2\mathcal{R}_1\mathcal{A} \in \Psi^0(V;F;w^{-1}\circ w)$ is an elliptic operator and

$$\begin{aligned}\sigma_1^0(\mathcal{R}_2\mathcal{R}_1\mathcal{A}) &= 1,\\ \sigma_2^0(\mathcal{R}_2\mathcal{R}_1\mathcal{A}) &= 1.\end{aligned}$$

We continue by induction. Suppose $\mathcal{R}_1,\dots,\mathcal{R}_J$, $1 \le J < N$, have already been constructed. Find an elliptic operator $\mathcal{R}_{J+1} \in \Psi^0(V;F;w^{-1}\circ w)$ such that

$$\begin{aligned}\sigma_1^0(\mathcal{R}_{J+1}) &= 1,\\ \dots\ &\dots\ \dots\\ \sigma_J^0(\mathcal{R}_{J+1}) &= 1,\\ \sigma_{J+1}^0(\mathcal{R}_{J+1}) &= \left(\sigma_{J+1}^0(\mathcal{R}_J\dots\mathcal{R}_1\mathcal{A})\right)^{-1}.\end{aligned}$$

Then

$$\begin{aligned}\sigma_1^0(\mathcal{R}_{J+1}\mathcal{R}_J\dots\mathcal{R}_1\mathcal{A}) &= 1,\\ \dots\ &\dots\ \dots\\ \sigma_J^0(\mathcal{R}_{J+1}\mathcal{R}_J\dots\mathcal{R}_1\mathcal{A}) &= 1,\\ \sigma_{J+1}^0(\mathcal{R}_{J+1}\mathcal{R}_J\dots\mathcal{R}_1\mathcal{A}) &= 1,\end{aligned}$$

which completes the step of induction.

Set

$$\mathcal{R} = \mathcal{R}_N\dots\mathcal{R}_1,$$

then $\mathcal{R} \in \Psi^{-m}(V;\tilde{F},F;w^{-1})$ and

$$\sigma_J^0(\mathcal{R}\mathcal{A}-1) = 0$$

for all $J = 1,\dots,N$. Hence it follows that $\mathcal{R}\mathcal{A}-1$ is a compact operator, i.e., $\mathcal{R}$ is a left parametrix of $\mathcal{A}$.

In the same way we prove the existence of a right parametrix, thus showing that actually $\mathcal{R}$ is a parametrix.

□

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