# Green Integrals on Manifolds with Cracks

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**Abstract**

We prove the existence of a limit in $H^m(D)$ of iterations of a double layer potential constructed from the Hodge parametrix on a smooth compact manifold with boundary, $X$, and a crack $S \subset \partial D$, $D$ being a domain in $X$. Using this result we obtain formulas for Sobolev solutions to the Cauchy problem in $D$ with data on $S$, for an elliptic operator $A$ of order $m \geq 1$, whenever these solutions exist. This representation involves the sum of a series whose terms are iterations of the double layer potential. A similar regularisation is constructed also for a mixed problem in $D$.

# Contents

# 1 Introduction

This paper is based on the following simple observation. Consider an operator equation $Au = f$ with a bounded operator $A : H_0 \to H_1$ in Hilbert spaces. Suppose each element $u \in H_0$ can be written in the form $u = \pi_0 u + \pi_1 Au$ where $\pi_0$ is a projection onto the kernel of $A$ in $H_0$. Then it is to be expected that under reasonable conditions the element $\pi_1 f$ defines a solution to the equation $Au = f$.

For the Cauchy-Riemann operator $A = \bar{\partial}$ in $\mathbb{C}^n$, $n > 1$, the double layer potential involved in the regularisation formula is just the Martinelli-Bochner integral. In this case, results similar to ours were obtained by Romanov [6].

**Theorem 1.1** *Let $D$ be a bounded domain in $\mathbb{C}^n$, $n > 1$, with a connected boundary of class $C^1$, and $Mu$ stand for the Martinelli-Bochner integral of a function $u \in H^1(D)$. Then the limit $\lim_{N\to\infty} M^N$ exists in the strong operator topology of $H^1(D)$, and it is equal to $\pi_0$, a projection onto the (closed) subspace of holomorphic functions in $H^1(D)$.*

By using this result Romanov [6] obtained an explicit formula for a solution $u \in H^1(D)$ to $\bar{\partial} u = f$, where $D$ is a pseudoconvex domain with a smooth boundary, and $f$ a $\bar{\partial}$-closed $(0,1)$-form with coefficients in $H^1(D)$.

# 2 Preliminary results

Let $X$ be a $C^\infty$ manifold of dimension $n$ with a smooth boundary $\partial X$. We tacitly assume that it is enclosed into a smooth closed manifold $\tilde{X}$ of the same dimension.

For any smooth $\mathbb{C}$-vector bundles $E$ and $F$ over $X$, we write $\mathrm{Diff}^m(X; E, F)$ for the space of all linear partial differential operators of order $\le m$ between sections of $E$ and $F$.

Denote $E^*$ the conjugate bundle of $E$. Any Hermitian metric $(.,.)_x$ on $E$ gives rise to a sesquilinear bundle isomorphism $*_E : E \to E^*$ by the equality $\langle *_E v, u\rangle_x = (u, v)_x$ for all sections $u$ and $v$ of $E$.

Pick a volume form $dx$ on $X$, thus identifying dual and conjugate bundles. For $A \in \mathrm{Diff}^m(X; E, F)$, denote by $A' \in \mathrm{Diff}^m(X; F^*, E^*)$ the transposed operator and by $A^* \in \mathrm{Diff}^m(X; F, E)$ the formal adjoint operator. We obviously have $A^* = *_E^{-1} A' *_F$, cf. [9, 4.1.4] and elsewhere.

For an open set $O \subset X$, we write $L^2(O, E)$ for the Hilbert space of all measurable sections of $E$ over $O$ with a finite norm $(u, u)_{L^2(O,E)} = \int_O (u, u)_x dx$. When no confusion can arise, we also denote $H^m(O, E)$ the Sobolev space of distribution sections of $E$ over $O$, whose weak derivatives up to order $m$ belong to $L^2(O, E)$.

Given any open set $O$ in $\overset{\circ}{X}$, the interior of $X$, we let $\mathcal{S}_A(O)$ stand for the space of weak solutions to the equation $Au = 0$ in $O$. We also denote by $\mathcal{S}_A^m(O)$ the closed subspace of $H^m(O, E)$ consisting of all weak solutions to $Au = 0$ in $O$.

Write $\sigma(A)$ for the principal homogeneous symbol of order $m$ of the operator $A$, $\sigma(A)$ living on the cotangent bundle $T^*X$ of $X$. From now on we assume that $\sigma(A)$ is injective away from the zero section of $T^*X$. Hence it follows that the Laplacian $\Delta = A^*A$ is an elliptic differential operator of order $2m$ on $X$.

Let $\sigma$ be a compact subset in $\overset{\circ}{X}$. In fact, we assume that $\sigma$ lies on a smooth closed hypersurface $S$ in $X$. Our goal will be to construct the Hodge theory of the Dirichlet problem for the Laplacian $\Delta$ on the manifold $V = \overset{\circ}{X} \setminus \sigma$ with a crack along $\sigma$.

Crack problems are usually treated in the framework of analysis on manifolds with edges, cf. Schulze [7]. One thinks of the boundary of $\sigma$ on $S$ as an edge of $V$, the cross-section being a 2-dimensional plane with a cut along a ray. The relevant function spaces are therefore weighted Sobolev spaces $H^{s,w}((V, \partial\sigma), E)$ of smoothness $s$ and weight $w$, both $s$ and $w$ being real numbers. Recall that if $s \in \mathbb{Z}_+$ it coincides with the completion of sections of $E$ over $V$, $C^\infty$ up to the boundary and vanishing near $\partial\sigma$, with respect to the norm

$$\|u\|_{H^{s,w}((V,\partial\sigma),E)} = \left( \sum_{\nu} \int \sum_{|\alpha| \le s} \mathrm{dist}(x, \partial\sigma)^{2(|\alpha|-w)} |D^\alpha(\varphi_\nu u)|^2 dx \right)^{1/2},$$

where $(\varphi_\nu)$ is a partition of unity subordinate to a suitable finite open covering $(O_\nu)$ of $X$.

However, we will deal with the very particular case $H^{m,m}((V, \partial\sigma), E)$ which allows us to restrict ourselves to the usual Sobolev spaces on $X$.

Namely, let $H^m((V, \partial\sigma), E)$ be the closure of all sections of $E$ over $V$, $C^\infty$ up to the boundary and vanishing close to $\partial\sigma$, in $H^m(V, E)$.

**Theorem 2.1** *If the boundary of $\sigma$ is smooth, then $H^{m,m}((V, \partial\sigma), E)$ and $H^m((V, \partial\sigma), E)$ coincide as topological vector spaces.*

**Proof.** Obviously, it is sufficient to show that the $H^{m,m}((V, \partial\sigma), E)$ - and $H^m(V, E)$ -norms are equivalent on sections of $E$ over $V$, $C^\infty$ up to the boundary and vanishing close to $\partial\sigma$. Without loss of generality we can consider those sections $u$ whose supports are contained in the domain $O_\nu$ of some chart on $X$.

If $O_\nu$ does not meet $\partial\sigma$ then $\mathrm{dist}(x, \partial\sigma)$ is strictly positive in $O_\nu$. Hence the $H^{m,m}((V, \partial\sigma), E)$ - and $H^m(V, E)$ -norms are equivalent on sections of $E$ with a support in $O_\nu$.

In the case $O_\nu \cap \partial\sigma \neq \emptyset$ we choose local coordinates $x = (x_1, \ldots, x_n)$ in $O_\nu$, such that $O_\nu \cap \sigma$ is the half-plane $\{x_n = 0,\, x_{n-1} \leq 0\}$. Write $x = (x', x_{n-1}, x_n)$ where $x' = (x_1, \ldots, x_{n-2})$. We restrict ourselves to sections $u = u(x', x_{n-1}, x_n)$ supported in $Q \times B$, with $Q$ a rectangle in $\mathbb{R}^{n-2}$, and $B$ a disk with centre 0 and radius $R \gg 1$.

Since

$$\|u\|^2_{H^{m,m}((V,\partial\sigma),E)} = \int \sum_{|\alpha| \leq m} \operatorname{dist}(x, \partial\sigma)^{2(|\alpha|-m)} |D^\alpha u|^2 dx,$$

the $H^m(V, E)$-norm is obviously dominated by the $H^{m,m}((V, \partial\sigma), E)$-norm whence

$$H^{m,m}((V, \partial\sigma), E) \hookrightarrow H^m((V, \partial\sigma), E).$$

On the other hand, the summands involving the derivatives of order $m$ in the norms $\|u\|_{H^{m,m}((V,\partial\sigma),E)}$ and $\|u\|_{H^m(V,E)}$ coincide. To handle lower order summands, we fix a multi-index $\alpha \in \mathbb{Z}^n_+$ with $0 \leq |\alpha| \leq m - 1$. Introduce polar coordinates

$$\left\{ \begin{array}{rcl} x_{n-1} & = & r \cos\varphi, \\ x_n & = & r \sin\varphi \end{array} \right.$$

in $B$, and set $U(r) = D^\alpha u\, (x', r\cos\varphi, r\sin\varphi)$. Then

$$\int \operatorname{dist}(x, \partial\sigma)^{2(|\alpha|-m)} |D^\alpha u|^2 dx = \int_Q dx' \int_{-\pi}^{\pi} d\varphi \int_0^R |r^{|\alpha|-m} U(r)|^2\, r dr.$$

We next make use of a Hardy-Littlewood inequality for measurable functions on the semiaxis with values in a normed space. Namely,

$$\|r^{p-1} \int_0^r f(\varrho) d\varrho\|_{L^q(\mathbb{R}_+)} \leq \left( \frac{1}{q'} - p \right)^{-1} \|r^p f(r)\|_{L^q(\mathbb{R}_+)},$$

where $1 \leq q \leq \infty$, $1/q + 1/q' = 1$ and $p < 1/q'$. Take $f(r) = (\partial/\partial r) U(r)$ and observe that

$$\begin{array}{rcl} |f'(r)| & = & |D^{\alpha+1_{n-1}} u\, \cos\varphi + D^{\alpha+1_n} u\, \sin\varphi| \\ & \leq & |D^{\alpha+1_{n-1}} u| + |D^{\alpha+1_n} u|, \end{array}$$

$1_j$ being the multi-index from $\mathbb{Z}^n_+$ which is 1 in the $j$-th place and 0 in each other one. Repeated application of the Hardy-Littlewood inequality therefore yields

$$\int \operatorname{dist}(x, \partial\sigma)^{2(|\alpha|-m)} |D^\alpha u|^2 dx \leq c\, \|D^\alpha u\|^2_{H^{m-|\alpha|}(V,E)},$$

with $c$ a constant independent of $u$.

Summarising we conclude that the $H^{m,m}((V,\partial\sigma),E)$-norm is majorised by the $H^m(V,E)$-norm on functions vanishing near $\partial\sigma$. This completes the proof. $\square$

More generally, given an open set $O \subset X$ and a closed set $\sigma \subset X$, we denote $H^m((O,\sigma),E)$ the closure of all sections of $E$ over $O$, $C^\infty$ up to the boundary and vanishing near $\sigma$, in $H^m(O,E)$. If $\sigma = \partial O$, we obtain what is usually referred to as

$$\overset{\circ}{H}{}^m(O,E).$$

Fix a Dirichlet system $B_j$, $j = 0,1,\ldots,m-1$, of order $m-1$ on the boundary of $V$. More precisely, each $B_j$ is a differential operator of type $E \to F_j$ and order $m_j \le m-1$ in a neighbourhood $U$ of $\partial X \cup S$. Moreover, the symbols $\sigma(B_j)$, if restricted to the conormal bundle of $\partial X \cup S$, have ranks equal to the dimensions of $F_j$.

Set $t(u) = \oplus_{j=0}^{m-1} B_j u$, for $u \in H^m(V,E)$. It follows from the results of Hedberg [1] that

$$\overset{\circ}{H}{}^m(V,E) = \{u \in H^m(X,E) :\ t(u) = 0 \ \text{ on } \ \partial X \cup \sigma\}, \tag{2.1}$$

$\partial X \cup \sigma$ being the boundary of $V$.

**Corollary 2.2** *Suppose $\partial\sigma$ is smooth. Then we have a topological isomorphism*

$$\overset{\circ}{H}{}^m(V,E) \cong \{u \in H^{m,m}((V,\partial\sigma),E) :\ t(u) = 0 \ \ \textit{on} \ \ \partial X \cup \overset{\circ}{\sigma}\},$$

*the space on the right-hand side being endowed with the norm induced from $H^{m,m}((V,\partial\sigma),E)$.*

**Proof.** By Theorem 2.1 it suffices to show that $\overset{\circ}{H}{}^m(V,E)$ consists of all $u \in H^m((V,\partial\sigma),E)$ such that $t(u) = 0$ on $\partial V$.

On the one hand, if $u \in \overset{\circ}{H}{}^m(V,E)$ then $u \in H^m((V,\partial\sigma),E)$ and $t(u) = 0$ on $\partial V$, as is easy to see.

On the other hand, if $u \in H^m((V,\partial\sigma),E)$ and $t(u) = 0$ on $\partial V$ then $u \in H^m((V,\partial V),E)$, as follows from [1]. This just amounts to the desired assertion. $\square$

# 3 Hodge theory on manifolds with cracks

Let $H^{-m}(V,E)$ denote the dual space of $\overset{\circ}{H}{}^m(V,E)$ with respect to the pairing in $L^2(V,E)$. This is not a canonical definition, we rather follow the notation of [9, 1.4.9].

For every $u \in H^m(V,E)$, the correspondence

$$v \mapsto \int_V (Au, Av)_x \, dx$$

is a continuous conjugate linear functional on $\mathring{H}^m(V,E)$. Thus, the Laplacian $\Delta = A^*A$ extends to a mapping $H^m(V,E) \to H^{-m}(V,E)$.

The following boundary value problem is a straightforward generalisation of the classical Dirichlet problem, cf. [9, 9.2.4].

**Problem 3.1** *Given an $F \in H^{-m}(V,E)$, find a section $u \in H^m(X,E)$ such that*

$$\left\{ \begin{array}{rcll} \Delta u & = & F & \text{in} \ \ V, \\ t(u) & = & 0 & \text{on} \ \ \partial V. \end{array} \right.$$

Another way of stating the problem is to say, "Study the restriction of $\Delta$ to $\mathring{H}^m(V,E)$."

If $u \in \mathring{H}^m(V,E)$ and $\Delta u = 0$, then $Au = 0$ in $V$. In the sequel, $\mathcal{H}(V)$ stands for

$$\mathring{H}^m(V,E) \cap \mathcal{S}_A(V).$$

Furthermore, we let $\mathcal{H}^\perp(V)$ consist of all sections $F \in H^{-m}(V,E)$ satisfying

$$\int_V (F, v)_x \, dx = 0$$

for any $v \in \mathcal{H}(V)$.

**Lemma 3.2** *Problem 3.1 is Fredholm. The difference of any two solutions lies in $\mathcal{H}(V)$. The problem is solvable if and only if $F \in \mathcal{H}^\perp(V)$. Moreover, there is a constant $c > 0$ such that for any solution $u \in \mathcal{H}^\perp(V)$ to Problem 3.1, we have*

$$\|u\|_{H^m(X,E)} \le c \, \|F\|_{H^{-m}(V,E)}. \tag{3.1}$$

**Proof.** By definition, the equality $\Delta u = F$ means that

$$\int_V (Au, Av)_x \, dx = \int_V (F, v)_x \, dx \tag{3.2}$$

for all $v \in \mathring{H}^m(V,E)$. We are thus looking for a section $u \in \mathring{H}^m(V,E)$ satisfying (3.2).

It readily follows from (3.2) that the null-space of Problem 3.1 is just $\mathcal{H}(V)$. Since

$$\mathring{H}^m(V,E) \hookrightarrow H^m(X,E)$$

and $\sigma$ is a set of zero measure in $\overset{\circ}{X}$, we deduce that

$$\mathcal{H}(V) \hookrightarrow \overset{\circ}{H}{}^m(\overset{\circ}{X}, E) \cap \mathcal{S}_A(\overset{\circ}{X}),$$

the space on the right-hand side being $\mathcal{H}(\overset{\circ}{X})$. Taking into account that the boundary of $X$ is smooth, we deduce that $\mathcal{H}(V)$ is a finite-dimensional subspace of $C^\infty(X, E)$.

That the condition $F \in \mathcal{H}^\perp(V)$ is necessary for the problem to be solvable, follows from (3.2) immediately. Let us prove the sufficiency.

To this end, we invoke the classical Gårding inequality. Namely, as $A$ has injective symbol, we have

$$\|u\|^2_{H^m(X,E)} \leq C \int_X (Au, Au)_x \, dx + c \, \|u\|^2_{L^2(X,E)} \tag{3.3}$$

for all $u \in \overset{\circ}{H}{}^m(V, E)$, the constants $C$ and $c$ being independent of $u$ (cf. for instance [10]).

A familiar argument shows that there is a constant $C > 0$ with the property that

$$\|u\|^2_{H^m(X,E)} \leq C \int_X (Au, Au)_x \, dx,$$

for each $u \in \overset{\circ}{H}{}^m(V, E) \cap \mathcal{H}^\perp(V)$. Indeed, we argue by contradiction. If there is no such constant then we can find a sequence $(u_\nu)$ in $\overset{\circ}{H}{}^m(V, E) \cap \mathcal{H}^\perp(V)$, such that

$$\begin{array}{rcl} \|u_\nu\|_{H^m(X,E)} & = & 1, \\ \|Au_\nu\|_{L^2(X,F)} & < & 2^{-\nu}. \end{array}$$

As the unit ball in a separable Hilbert space is weakly compact, we can assume that $(u_\nu)$ converges weakly to a section $u_\infty \in \overset{\circ}{H}{}^m(V, E) \cap \mathcal{H}^\perp(V)$. It follows that

$$\begin{array}{rcl} \displaystyle\int_X (u_\infty, A^* v)_x \, dx & = & \displaystyle\lim_{\nu \to \infty} \int_X (u_\nu, A^* v)_x \, dx \\ & = & \displaystyle\lim_{\nu \to \infty} \int_X (Au_\nu, v)_x \, dx \\ & = & 0 \end{array}$$

for all $v \in C^\infty_{\mathrm{comp}}(\overset{\circ}{X}, E)$, i.e. $u_\infty \in \mathcal{H}(V)$. We thus conclude that $u_\infty = 0$. But the Gårding inequality yields

$$1 \leq C \, 2^{-\nu} + c \, \|u_\nu\|_{L^2(X,E)}$$

for all $\nu$. Since the inclusion $\overset{\circ}{H}{}^m(V,E) \hookrightarrow L^2(X,E)$ is compact, and thus $u_\nu$ converges strongly to $u_\infty$ in $L^2(X,E)$, we get

$$\|u_\infty\|_{L^2(X,E)} \geq 1/c,$$

which contradicts $u_\infty = 0$.

We have thus proved that the Hermitian form

$$\int_X (Au, Av)_x \, dx$$

defines a scalar product in the Hilbert space $\overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V)$, the corresponding norm being equivalent to the original one. Now the Riesz Theorem enables us to assert that for every $F \in H^{-m}(V,E)$ there exists a unique section

$$u \in \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V)$$

satisfying

$$\int_V (F, v)_x \, dx = \int_X (Au, Av)_x \, dx$$

for all $v \in \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V)$.

Obviously, every $v \in \overset{\circ}{H}{}^m(V,E)$ can be written in the form $v = v_1 + v_2$, with

$$\begin{array}{rcl} v_1 & \in & \mathcal{H}(V), \\ v_2 & \in & \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V). \end{array}$$

It follows that if $F \in \mathcal{H}^\perp(V)$ then $u$ satisfies (3.2) for all $v \in \overset{\circ}{H}{}^m(V,E)$, as desired.

Finally, since for any section $F \in H^{-m}(V,E)$ "orthogonal" to $\mathcal{H}(V)$ there is a unique solution to Problem 3.1 in

$$\overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V),$$

the estimate (3.1) follows from the Open Map Theorem.

□

We are now in a position to derive a Hodge decomposition for the Dirichlet problem in $V$.

**Theorem 3.3** *There are bounded linear operators*

$$\begin{array}{rrcl} H: & H^{-m}(V,E) & \to & \mathcal{H}(V), \\ G: & H^{-m}(V,E) & \to & \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V) \end{array}$$

*such that*

1) *$H$ is the $L^2(V,E)$-orthogonal projection onto the space $\mathcal{H}(V)$, with a kernel $K_H(x,y) = \sum_j h_j(x) \otimes *_E h_j(y)$ where $(h_j)$ is an orthogonal basis of $\mathcal{H}(V)$;*

2) *$AH = 0$ and $GH = HG = 0$;*

3)
$$\begin{array}{rcll} G\Delta u & = & u - Hu & \text{for all} \ \ u \in \mathring{H}^m(V,E), \\ \Delta GF & = & F - HF & \text{for all} \ \ F \in H^{-m}(V,E). \end{array}$$

**Proof.** As already mentioned in the proof of Lemma 3.2, $\mathcal{H}(V)$ is a finite-dimensional subspace of $C^\infty(X,E)$. Denote $H$ the $L^2(V,E)$-orthogonal projection onto $\mathcal{H}(V)$. Fix an orthogonal basis $(h_j)$ for $\mathcal{H}(V)$. Then $H$ has the kernel
$$K_H(x,y) = \sum_j h_j(x) \otimes *_E h_j(y),$$
because
$$(HF)(x) = \sum_j \left( \int_V (F(y), h_j(y))_y \, dy \right) h_j(x)$$
for all $F \in L^2(V,E)$. Since $H$ is a smoothing operator it extends to all of $H^{-m}(V,E)$, too, by
$$(HF)(x) = \langle K_H(x,\cdot), F \rangle_V,$$
for $x \in V$. Clearly,
$$H: \ H^{-m}(V,E) \to \mathcal{H}(V) \hookrightarrow \mathring{H}^m(V,E)$$
is bounded and $AH = 0$.

Pick $F \in H^{-m}(V,E)$. Since $K_H(x,y)^* = K_H(y,x)$ we get
$$\begin{array}{rcl} \displaystyle\int_V (F - HF, v)_x \, dx & = & \displaystyle\int_V (F - HF, Hv)_x \, dx \\ & = & \displaystyle\int_V \left( HF - H^2 F, v \right)_x \, dx \\ & = & \displaystyle\int_V (HF - HF, v)_x \, dx \\ & = & 0 \end{array}$$
for all $v \in \mathcal{H}(V)$, i.e.,
$$F - HF \in \mathcal{H}^\perp(V).$$
Therefore, Lemma 3.2 implies that there exists a solution $u \in \mathring{H}^m(V,E)$ to $\Delta u = F - HF$ in $V$. Setting
$$GF = u - Hu$$

we obtain
$$F = HF + \Delta GF$$
for all $F \in H^{-m}(V,E)$.

As $u - Hu \in \mathcal{H}^{\perp}(V)$ we see from (3.1) that
$$G:\ H^{-m}(V,E) \to \overset{\circ}{H}{}^{m}(V,E) \cap \mathcal{H}^{\perp}(V)$$
is bounded. By definition, $HGF = Hu - H^2u = 0$ and $GHF = 0$.

On the other hand, we easily obtain the $L^2(V,E)$ -orthogonal decomposition
$$\begin{aligned} u &= Hu + (u - Hu) \\ &= Hu + G\Delta u \end{aligned}$$
for all $u \in \overset{\circ}{H}{}^{m}(V,E)$. This completes the proof.

□

When restricted to $L^2(V,E)$, the operator $G$ is selfadjoint. In fact, given any $F, v \in H^{-m}(V,E)$, we have
$$\begin{aligned} (GF, v) &= (GF, Hv + \Delta Gv) \\ &= (GF, \Delta Gv) \\ &= (\Delta GF, Gv) \\ &= (F, Gv)\,, \end{aligned}$$
$(\cdot,\cdot)$ meaning the scalar product in $L^2(V,E)$. Hence it follows that the Schwartz kernel of $G$,
$$K_G(\cdot,\cdot) \in \overset{\circ}{H}{}^{m}(V,E) \otimes \overset{\circ}{H}{}^{m}(V,E^*) \hookrightarrow \mathcal{D}'(V \times V, E \otimes E^*),$$
is Hermitian, i.e., $K_G(x,y)^* = K_G(y,x)$ for all $x, y \in V$.

**Lemma 3.4** *The operator $T = GA^*$ extends to a continuous linear mapping*
$$L^2(V,F) \to \overset{\circ}{H}{}^{m}(V,E).$$

**Proof.** For any fixed $f \in L^2(V,F)$, the integral
$$\int_V (f, Av)_x \, dx$$
defines a continuous linear functional on $\overset{\circ}{H}{}^{m}(V,E)$. Hence, the (formal) adjoint $A^*$ extends to a mapping $L^2(V,F) \to H^{-m}(V,E)$, which is obviously continuous. Since $G$ maps $H^{-m}(V,E)$ continuously to $\overset{\circ}{H}{}^{m}(V,E)$, the lemma follows. □

As is easy to check by Stokes' formula, the Schwartz kernel of $T$ is

$$K_T(x,y) = (A^*(y,D))' K_G(x,y),$$

the 'prime' meaning the transposed operator.

Using $T$, we may rewrite the Hodge decomposition of Theorem 3.3 in the form

$$u = Hu + TAu \tag{3.4}$$

over $V$, for each $u \in \overset{\circ}{H}{}^m(V,E)$.

We now introduce the Hermitian form

$$h(u,v) = \int_V (Hu, Hv)_x \, dx + \int_V (Au, Av)_x \, dx$$

defined for $u, v \in \overset{\circ}{H}{}^m(V,E)$

**Theorem 3.5** *The Hermitian form $h(\cdot,\cdot)$ is a scalar product in $\overset{\circ}{H}{}^m(V,E)$ defining a norm equivalent to the original one. The operator $H$ is also an orthogonal projection from $\overset{\circ}{H}{}^m(V,E)$ onto $\mathcal{H}(V)$ with respect to $h(\cdot,\cdot)$. Moreover,*

$$h(Tf, u) = \int_V (f, Au)_x \, dx$$

*for all $f \in L^2(V,F)$ and $u \in \overset{\circ}{H}{}^m(V,E)$.*

**Proof.** The coefficients of $A$ are $C^\infty$ up to the boundary of $X$, and so $Au \in L^2(V,F)$ for all $u \in \overset{\circ}{H}{}^m(V,E)$. Moreover, it follows from (3.4) that $h(u,u) = 0$ implies $u \equiv 0$ in $X$. Hence $h(\cdot,\cdot)$ is a scalar product on $\overset{\circ}{H}{}^m(V,E)$.

Since $H$ is a smoothing operator, the original norm of $\overset{\circ}{H}{}^m(V,E)$ is not weaker than $\sqrt{h(\cdot,\cdot)}$.

Further, (3.4) and Lemma 3.4 show that there exists a constant $c > 0$ such that

$$\|u\|_{H^m(V,E)} \le c \left( \|Hu\|_{H^m(V,E)} + \|Au\|_{L^2(V,F)} \right)$$

for all $u \in \overset{\circ}{H}{}^m(V,E)$.

On the other hand, since $H$ is a finite rank operator, there is a constant $C > 0$ such that

$$\|Hu\|_{H^m(V,E)} \le C \, \|Hu\|_{L^2(V,E)}$$

for all $u \in \overset{\circ}{H}{}^m(V,E)$. This proves the equivalence of the topologies.

Suppose $f \in C^{\infty}_{\mathrm{comp}}(V,F)$ and $u \in \overset{\circ}{H}{}^{m}(V,E)$. By Theorem 3.3, we get $HTf = 0$. Moreover,

$$\begin{aligned}\int_V (HA^*f, v)_x\,dx &= \int_V (f, AHv)_x\,dx\\ &= 0\end{aligned}$$

for all $v \in L^2(V,E)$, whence $HA^*f = 0$. Thus,

$$\begin{aligned}h(Tf,u) &= \int_V (AG(A^*f), Au)_x\,dx\\ &= \int_V (\Delta G(A^*f), u)_x\,dx\\ &= \int_V (A^*f - H(A^*f), u)_x\,dx\\ &= \int_V (f, Au)_x\,dx.\end{aligned}$$

As $C^{\infty}_{\mathrm{comp}}(V,F)$ is dense in $L^2(V,F)$, we obtain the desired assertion on the integral $T$.

Finally, for any $u, v \in \overset{\circ}{H}{}^{m}(V,E)$, we have

$$\begin{aligned}h(Hu, &= h(u,v) - h(TAu,v)\\ &= h(u,v) - \int_V (Au,Av)_x\,dx\\ &= \int_V (Hu,Hv)_x\,dx,\end{aligned}$$

i.e., $H$ is a selfadjoint operator in $\overset{\circ}{H}{}^{m}(V,E)$ with respect to the scalar product $h(\cdot,\cdot)$, and $H^2 = H$, as desired.

□

# 4 Green formulas on manifolds with cracks

In this section we discuss Green formulas for sections of $E$ on open subsets of $V$. To this end, we choose a Green operator $G_A(\cdot,\cdot)$ for $A$ on $X$, cf. [9, 9.2.1]. Given an oriented hypersurface $S \subset X$, we denote $[S]^A$ the kernel over $X \times X$ defined by

$$\left\langle [S]^A, g \otimes u \right\rangle_{X\times X} = \int_S G_A(g,u)$$

for all $g \in C^{\infty}(X,F^*)$ and $u \in C^{\infty}(X,E)$ whose supports meet each other in a compact set.

In particular, the kernel $[\partial V]^A$ is supported by the hypersurface $\partial X \cup \sigma$. However, $\sigma$, if regarded as a part of the boundary of $V$, has two sides in $X$ with opposite orientations. When applied to sections $g$ and $u$ whose derivatives up to order $m-1$ are continuous in a neighbourhood of $\sigma$, the kernel $[\partial V]^A$ does not include any integration over $\sigma$ because the integrals over the sides with opposite orientations cancel. In general, the continuity up to the boundary in $V$ does not assume that the limit values from both sides of $\sigma$ coincide in the interior of $\sigma$ on $S$. Hence, $[\partial V]^A$ actually includes, along with the integral over $\partial X$, the integral over $\sigma$ of the difference of the limit values of $G_A(g,u)$ on $S$.

Away from the singularities of $V$, i.e., $\partial\sigma$, the Green operator $G$ behaves like the Green function of an elliptic boundary value problem, cf. [5]. The edge $\partial\sigma$ is well known to cause additional singularities of the kernel of $G$ on $(V \times \partial\sigma) \cup (\partial\sigma \times V)$.

Given any section $u \in H^m(V,E)$ vanishing in a neighbourhood of $\partial\sigma$, we set

$$\begin{array}{rcl} (Mu)(x) & = & -GA^* \left([\partial V]^A u\right) \\ & = & -\displaystyle\int_{\partial V} G_A(K_T(x,y), u(y)) \end{array}$$

for $x \in V$.

**Theorem 4.1** *As defined above, the operator $M$ extends to a continuous mapping of $H^m(V,E)$, and*

$$u = Hu + TAu + Mu \qquad (4.1)$$

*for all $u \in H^m(V,E)$.*

**Proof.** Given any $u \in H^m(V,E)$, we define $Mu$ from the equality (4.1), namely

$$Mu = u - Hu - TAu.$$

Note that $H$ is a smoothing operator in the sense that it extends naturally to a continuous mapping

$$H^{-\infty}(V,E) \to \overset{\circ}{H}{}^{\infty}(V,E),$$

where $\overset{\circ}{H}{}^{\infty}(V,E)$ is the projective limit of the family $\overset{\circ}{H}{}^{s}(V,E)$, $s \in \mathbb{Z}_+$, and $H^{-\infty}(V,E)$ the dual space under the pairing induced from $L^2(V,E)$. Hence it follows, by Lemma 3.4, that $M$ is a well-defined continuous mapping of $H^m(V,E)$.

We shall have established the theorem if we prove that the operator $M$ defined from (4.1) is actually an appropriate extension of the operator $M$

given before Theorem 4.1. This is an easy consequence of Stokes' formula. Indeed, pick a $u \in H^m(V, E)$ vanishing near $\partial\sigma$. Combining Stokes' formula and Theorem 3.3, we get

$$\begin{aligned}(u - Hu - TAu, v)_{L^2(V,E)} &= (u, v - Hv)_{L^2(V,E)} - (Au, AGv)_{L^2(V,F)} \\ &= (u, v - Hv - \Delta Gv)_{L^2(V,E)} - \int_{\partial V} G_A(*_F(AGv), u) \\ &= \left(-T\left([\partial V]^A u\right), v\right)_{L^2(V,E)}\end{aligned}$$

for all $v \in C^\infty_{\mathrm{comp}}(V, E)$. This shows that $Mu = -T\left([\partial V]^A u\right)$ in (the interior of) $V$, as desired.

□

We now consider the inhomogeneous Dirichlet problem for the Laplacian $\Delta$ on $V$.

To this end, we first give a rigorous meaning to the boundary condition $t(u) = u_0$ on $\partial V$. If $\partial\sigma$ is sufficiently smooth, $t$ induces a topological isomorphism

$$\frac{H^m(V, E)}{\overset{\circ}{H}{}^m(V, E)} \overset{\cong}{\to} \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial V, F_j),$$

which is due to (2.1). Hence we can more generally interpret $t$ as the quotient mapping

$$t: \; H^m(V, E) \to \frac{H^m(V, E)}{\overset{\circ}{H}{}^m(V, E)}, \tag{4.2}$$

the quotient on the right substituting the space of Dirichlet data on $\partial V$. We make use of the Hilbert structure in $H^m(V, E)$ to construct a continuous right inverse $t^{-1}$ for $t$.

**Problem 4.2** *Given $F \in H^{-m}(V, E)$ and $u_0 \in H^m(V, E)/\overset{\circ}{H}{}^m(V, E)$, find a section $u \in H^m(V, E)$ such that*

$$\left\{\begin{array}{rcll} \Delta u &=& F & \text{in } V, \\ t(u) &=& u_0 & \text{on } \partial V. \end{array}\right.$$

**Lemma 4.3** *Problem 4.2 is solvable if and only if $F \in \mathcal{H}^\perp(V)$. Moreover, for each $F \in \mathcal{H}^\perp(V)$,*

$$u = GF + M\left(t^{-1}u_0\right)$$

*is the solution to Problem 4.2 belonging to $H^m(V, E) \cap \mathcal{H}^\perp(V)$ and thus satisfying*

$$\|u\|_{H^m(V,E)} \le c\left(\|F\|_{H^{-m}(V,E)} + \|u_0\|_{\frac{H^m(V,E)}{\overset{\circ}{H}{}^m(V,E)}}\right). \tag{4.3}$$

**Proof.** The necessity of the condition $F \in \mathcal{H}^{\perp}(V)$ is obvious. What is left is to show that under this condition $u = GF + M\,(t^{-1}u_0)$ is the solution to Problem 4.2 in $H^m(V,E) \cap \mathcal{H}^{\perp}(V)$.

Indeed, Theorem 3.3 shows that $GF \in \overset{\circ}{H}{}^m(V,E)$ is "orthogonal" to $\mathcal{H}(M)$ and satisfies $\Delta(GF) = F$.

On the other hand, given any Dirichlet data $u_0$, we find a $U \in H^m(V,E)$ such that $t(U) = u_0$ on $\partial V$. Note that $MU$ is actually independent of the particular choice of $U$, for if $U', U'' \in H^m(V,E)$ satisfy

$$\begin{array}{rcl} t(U') & = & u_0, \\ t(U'') & = & u_0 \end{array}$$

then $U' - U'' \in \overset{\circ}{H}{}^m(V,E)$ whence

$$\begin{array}{rcl} MU' & = & MU'' + M\,(U' - U'') \\ & = & MU'' + (U' - U'') - H\,(U' - U'') - G\Delta\,(U' - U'') \\ & = & MU'', \end{array}$$

the last equality being a consequence of Theorem 3.3. Using Theorem 4.1 we get

$$\begin{array}{rcl} \Delta\, MU & = & \Delta\,(U - HU - G\Delta U) \\ & = & \Delta U - (\Delta U - H\,\Delta U) \\ & = & 0 \end{array}$$

and

$$\begin{array}{rcl} t\,(MU) & = & t\,(U - HU - G\Delta U) \\ & = & t\,(U) \\ & = & u_0. \end{array}$$

Finally, the section $MU$ is "orthogonal" to $\mathcal{H}(V)$ because so are both $U - HU$ and $G\Delta U$.

Summarising we conclude that $u = GF + MU$ gives a canonical solution to Problem 4.2, as desired. The estimate (4.3) is a consequence of the Open Map Theorem.

$\square$

Let $D$ be a relatively compact domain (i.e. open connected subset) in $\overset{\circ}{X}$ with a smooth boundary $(S =)\ \partial D$ containing $\sigma$.

For

$$\begin{array}{rcl} u & \in & H^m(D,E), \\ f & \in & L^2(D,F), \end{array}$$

we consider the integrals

$$\begin{array}{rcl} H_D u & = & H\left(\chi_D u\right), \\ T_D f & = & T\left(\chi_D f\right), \\ M_D u & = & -T\left([\partial D]^A u\right) \end{array}$$

in $V$, where $\chi_D$ is the characteristic function of $D$ in $X$. Analysis similar to that in the proof of Theorem 4.1 actually shows that

$$\chi_D u = H_D u + T_D A u + M_D u \tag{4.4}$$

over $V$, for every $u \in H^m(D, E)$.

**Lemma 4.4** *As defined above, the integrals $H_D$, $T_D$ and $M_D$ induce bounded operators*

$$\begin{array}{rcrcl} H_D & : & H^m(D,E) & \to & H^m((D,\sigma),E), \\ T_D & : & L^2(D,F) & \to & H^m((D,\sigma),E), \\ M_D & : & H^m(D,E) & \to & H^m(D,E). \end{array}$$

**Proof.** We first observe that the space $H^m((D,\sigma),E)$ coincides with the restriction of $\overset{\circ}{H}{}^m(V,E)$ to $D$.

Since $H$ extends to a continuous mapping $H^{-\infty}(V,E) \to \mathcal{H}(V)$, the boundedness of $H_D$ is clear.

Suppose $f \in L^2(D,F)$. Then $\chi_D f$ is naturally regarded as the extension of $f$ to a section in $L^2(X,F)$ by zero. We have $T_D f = T\left(\chi_D f\right)$, and so the mapping $T_D : L^2(D,F) \to H^m((D,\sigma),E)$ is continuous, which is due to Lemma 3.4.

Finally, in order to complete the proof it is sufficient to invoke the equality $M_D = \mathrm{Id} - H_D - T_D A$ in $D$.

□

# 5 Examples

**Example 5.1** Let $X$ be a bounded domain with smooth boundary in $\mathbb{R}^n$, $n > 1$, and $A$ an operator with injective symbol in a neighbourhood $\tilde{X}$ of $\bar{X}$. Assume that $A$ fulfills the uniqueness condition for the Cauchy problem in the small on $\tilde{X}$. Write $G$ for the Green function of the Dirichlet problem for the Laplacian $\Delta = A^* A$ in $X$. In [3], a scalar product $h_D(\cdot,\cdot)$ on $H^m(D,E)$ is constructed, such that the corresponding norm is equivalent to the original one and the operator $T_D$ is adjoint to $A\colon H^m(D,E) \to L^2(D,F)$ with respect to $h_D(\cdot,\cdot)$, i.e.

$$h_D(T_D f, v) = \int_D (f, Av)_x \, dx$$

for all $f \in L^2(D,F)$ and $v \in H^m(D,F)$. This implies that the iterations of the double layer potential $M_D$ in $H^m(D,F)$ converge to the projection onto the subspace $H^m(D,E) \cap \mathcal{S}_A(D)$. This case corresponds to the Hodge decomposition for the Dirichlet problem in $X$ with empty crack $\sigma$ and $\mathcal{H}(V)$ being trivial.

$\square$

Recall that $H^m((D,\sigma),E)$ just amounts to the subspace of $H^m(D,E)$ consisting of all $u$ with $t(u)=0$ on $\sigma$.

**Example 5.2** Under the assumptions of Example 5.1, let moreover $X$ have a crack along a closed piece $\sigma$ of a smooth hypersurface in $X$. We denote $G$ the Green function of the Dirichlet problem for the Laplacian $\Delta$ in $V = X \setminus \sigma$. In our paper [8], a scalar product $h_D(\cdot,\cdot)$ on $H^m((D,\sigma),E)$ is constructed, defining an equivalent topology on this space and such that the operator $T_D$ is actually adjoint to $A: \ H^m((D,\sigma),E) \to L^2(D,F)$ with respect to $h_D(\cdot,\cdot)$, i.e.,

$$h_D(T_D f, v) = \int_D (f, Av)_x \, dx$$

for all $f \in L^2(D,F)$ and $v \in H^m((D,\sigma),E)$. When combined with a general result of functional analysis, this implies that the limit of iterations $M_D^N$ in the strong operator topology of $\mathcal{L}(H^m((D,\sigma),E))$ is equal to zero. This case corresponds to the Hodge decomposition for the Dirichlet problem in $X$ with a crack along $\sigma$ and $H=0$.

$\square$

In the next section we will prove similar results for the integrals $T_D$ and $M_D$ in our more general setting.

# 6 Construction of the scalar product $h_D(\cdot,\cdot)$

We first apply Lemma 4.3 to $X \setminus D$, a $C^\infty$ manifold with boundary. Namely, write $\mathcal{S}^m_\Delta(\hat{X} \setminus D)$ for the subspace of $H^m(X \setminus D, E)$ consisting of all $u$, such that $\Delta u = 0$ in the interior of $X \setminus D$ and $t(u) = 0$ on $\partial X$. By Lemma 4.3, we get a topological isomorphism

$$\mathcal{S}^m_\Delta(\hat{X} \setminus D) \cap \mathcal{H}^\perp(X \setminus D) \xrightarrow{t_+} \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial D, F_j)$$

given by $u \mapsto t(u)\,|_{\partial D}$. Finally, composing the inverse $t_+^{-1}$ with the trace operator

$$H^m(D,E) \xrightarrow{t_-} \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial D, F_j)$$

we arrive at a continuous linear mapping

$$H^m(D,E) \ni u \mapsto \mathcal{E}(u) \in \mathcal{S}^m_\Delta(\hat{X} \setminus D) \cap \mathcal{H}^\perp(X \setminus D). \tag{6.1}$$

For $u \in H^m(D,E)$, we now set

$$e(u)(x) = \begin{cases} u(x) & \text{if } x \in D, \\ \mathcal{E}(u)(x) & \text{if } x \in X \setminus \bar{D}. \end{cases}$$

Since $t(\mathcal{E}(u)) = t(u)$ on $\partial D$, it follows that $e(u) \in H^m(X,E)$. Furthermore, we have

$$e(u) \in \mathring{H}^m(V,E)$$

for all $u \in H^m((D,\sigma),E)$.

**Theorem 6.1** *The Hermitian form $h_D(u,v) = h(e(u),e(v))$ is a scalar product in $H^m((D,\sigma),E)$ defining a topology equivalent to the original one.*

**Proof.** Theorem 3.5 implies the existence of a positive constant $c$ with the property that

$$\begin{aligned} \|u\|^2_{H^m(D,E)} &\le \|e(u)\|^2_{H^m(V,E)} \\ &\le c\, h(e(u),e(u)) \end{aligned}$$

for all $u \in H^m((D,\sigma),E)$.

On the other hand,

$$\begin{aligned} h_D(u,u) &\le C\, \|e(u)\|^2_{H^m(V,E)} \\ &\le 2C \left( \|u\|^2_{H^m(D,E)} + \|\mathcal{E}(u)\|^2_{H^m(X\setminus D,E)} \right) \end{aligned}$$

for all $u \in H^m((D,\sigma),E)$, with $C$ a constant independent of $u$. Using Lemma 4.3 and the continuity of the trace operator we see that

$$\begin{aligned} \|\mathcal{E}(u)\|_{H^m(X\setminus D,E)} &\le c \sum_{j=0}^{m-1} \|B_j u\|_{H^{m-m_j-1/2}(\partial D,F_j)} \\ &\le C\, \|u\|_{H^m(D,E)} \end{aligned}$$

for all $u \in H^m(D,E)$, the constants $c$ and $C$ need not be the same in different applications. This finishes the proof.

□

**Theorem 6.2** *Assume that $u \in H^m(D,E)$ and $f \in L^2(D,F)$. For every $v \in H^m((D,\sigma),E)$, it follows that*

$$\begin{aligned} h_D(T_D f, v) &= \int_D (f, Av)_x\, dx, \\ h_D((H_D + M_D)u, v) &= \int_{X\setminus D} (A\mathcal{E}(u), A\mathcal{E}(v))_x\, dx + \int_X (He(u), He(v))_x\, dx. \end{aligned}$$

**Proof.** Suppose $f \in C^\infty_{\mathrm{comp}}(D,F)$. Then $T_D f \in \overset{\circ}{H}{}^m(V,E)$. Let us show that

$$e\,(Tf\,|_D) = Tf.$$

For this purpose, it is sufficient to check that the restriction of $Tf$ to $X \setminus \bar{D}$ lies in $\mathcal{S}^m_\Delta(\widehat{X \setminus D}) \cap \mathcal{H}^\perp(X \setminus D)$.

However, $Tf = G\,(A^* f)$ and therefore, as we have already seen in the proof of Theorem 3.5,

$$\begin{aligned}\Delta Tf &= A^* f - HA^* f\\ &= A^* f\end{aligned}$$

on $X$. Since $f$ has a compact support in $D$ we readily deduce that $\Delta Tf = 0$ in $X \setminus \bar{D}$.

Note that $\mathcal{H}(X \setminus D) \subset \mathcal{H}(V)$. Indeed, every element $u \in \mathcal{H}(X \setminus D)$ can be extended by zero from $X \setminus D$ to all of $X$ as a solution to $Au = 0$ on $X$. Since $G\,(A^* f)$ is "orthogonal" to $\mathcal{H}(V)$ we conclude that $Tf\,|_{X\setminus\bar{D}} \in \mathcal{H}^\perp(X \setminus D)$, as desired.

Further, if $v \in H^m((D,\sigma),E)$ then $e(v) \in \overset{\circ}{H}{}^m(V,E)$ and Theorem 3.5 implies

$$\begin{aligned}h_D\,(T_D f, v) &= h\,(Tf, e(v))\\ &= \int_X (f, Ae(v))_x \, dx\\ &= \int_D (f, Av)_x \, dx.\end{aligned}$$

Since $C^\infty_{\mathrm{comp}}(D,F)$ is dense in $L^2(D,F)$ and the operator $T_D$ is bounded, this formula actually holds for all $f \in L^2(D,F)$.

Finally, (4.4) implies that

$$\begin{aligned}h_D\,((H_D + M_D)u, v) &= h_D\,(u - T_D A u, v)\\ &= \int_{X\setminus D} (A\mathcal{E}(u), A\mathcal{E}(v))_x \, dx + \int_X (He(u), He(v))_x \, dx\end{aligned}$$

for all $v \in H^m((D,\sigma),E)$, as desired.

□

# 7 Iterations of potentials

**Corollary 7.1** *The operators*

$$\begin{aligned}T_D A &: H^m((D,\sigma),E) \to H^m((D,\sigma),E),\\ H_D + M_D &: H^m((D,\sigma),E) \to H^m((D,\sigma),E)\end{aligned}$$

*are selfadjoint and non-negative with respect to* $h_D(\cdot,\cdot)$*, and the norms of* $T_D A$ *and* $H_D + M_D$ *do not exceed* 1.

**Proof.** This follows immediately from Theorems 6.1 and 6.2.

□

Similarly to $\mathcal{H}^{\perp}(V)$, we denote $\mathcal{H}^{\perp}(D)$ the space of all $F \in H^{-m}(D,E)$ such that

$$\int_D (F,v)_x \, dx = 0$$

for any $v \in \mathcal{H}(D)$. It is easy to see that $H^m((D,\sigma),E) \cap \mathcal{H}^{\perp}(D)$ just amounts to the orthogonal complement of $\mathcal{H}(D)$ in $H^m((D,\sigma),E)$ with respect to the scalar product $h_D(\cdot,\cdot)$ thereon. Indeed, we have $\mathcal{H}(D) \hookrightarrow \mathcal{H}(V)$ because every element $v \in \mathcal{H}(D)$ may be extended by zero from $D$ to $X$ as a solution to $Av = 0$ on $X$. It follows that $\mathcal{E}(v) = 0$ for all $v \in \mathcal{H}(D)$, whence $Ae(v) = 0$ on $X$ and

$$\begin{aligned} h_D(u,v) &= \int_X (He(u), He(v))_x \, dx \\ &= \int_D (u,v)_x \, dx, \end{aligned}$$

as desired.

Lemma 4.4 allows one to consider iterations of $T_D A$ and $H_D + M_D$ in the space $H^m((D,\sigma),E)$. Given a closed subspace $\Sigma$ of $H^m((D,\sigma),E)$, we write $\pi_\Sigma$ for the orthogonal projection of $H^m((D,\sigma),E)$ onto $\Sigma$ with respect to the scalar product $h_D(\cdot,\cdot)$.

**Corollary 7.2** *In the strong operator topology in* $\mathcal{L}(H^m((D,\sigma),E))$*, we have*

$$\begin{aligned} \lim_{N\to\infty} (T_D A)^N &= \pi_{\ker(H_D+M_D)}, \\ \lim_{N\to\infty} (H_D + M_D)^N &= \pi_{H^m((D,\sigma),E)\cap \mathcal{S}_A(D)}. \end{aligned}$$

*Moreover, in the strong operator topology of* $\mathcal{L}(L^2(D,F))$*,*

$$\lim_{N\to\infty} (\mathrm{Id} - AT_D)^N = \pi_{\ker T_D}.$$

**Proof.** It follows from Corollary 7.1 that

$$\begin{aligned} \lim_{N\to\infty} (T_D A)^N &= \pi_{\ker(\mathrm{Id}-T_D A)}, \\ \lim_{N\to\infty} (H_D + M_D)^N &= \pi_{\ker(\mathrm{Id}-H_D-M_D)}, \\ \lim_{N\to\infty} (\mathrm{Id} - AT_D)^N &= \pi_{\ker AT_D} \end{aligned}$$

in the strong operator topology of $\mathcal{L}(H^m((D,\sigma),E))$ or $\mathcal{L}(L^2(D,F))$, respectively (see, for instance, §2 of [3] or [4] for compact operators). Theorem 6.2 and (4.4) imply

$$\begin{aligned} \ker(\mathrm{Id} - T_D A) &= \ker(H_D + M_D), \\ \ker T_D A &= H^m((D,\sigma),E) \cap \mathcal{S}_A(D), \\ \ker A T_D &= \ker T_D, \end{aligned}$$

showing the corollary.

□

Obliviously, if the coefficients of $A$ are real analytic and $\sigma$ has at least one interior point on $\partial D$ then $H^m((D,\sigma),E) \cap \mathcal{S}_A(D) = \{0\}$. If moreover every connected component of $X \setminus \bar{D}$ meets the boundary of $V$, i.e., $\partial X \cup \sigma$, then $H_D = 0$ and

$$\ker M_D = \mathring{H}^m(D,E).$$

Indeed, according to Theorem 6.2, if $u \in H^m((D,\sigma),E)$ and $M_D u = 0$ then $A\mathcal{E}(u) = 0$ in $X \setminus \bar{D}$ and $t(\mathcal{E}(u)) = 0$ on $\partial X \cup \sigma$. Hence it follows that $\mathcal{E}(u) \equiv 0$ in $X \setminus \bar{D}$, and so $t(\mathcal{E}(u)) = 0$ on $\partial D$. From this we conclude that $t(u) = 0$ on $\partial D$ whence $u \in \mathring{H}^m(D,E)$. Conversely, if $u \in \mathring{H}^m(D,E)$ then $M_D u = 0$, as desired.

**Theorem 7.3** *In the strong operator topology of $\mathcal{L}(H^m((D,\sigma),E))$, we have*

$$\mathrm{Id} = H_D + \pi_{\ker(H_D+M_D)} + \sum_{\nu=0}^{\infty} (T_D A)^\nu M_D, \tag{7.1}$$

$$\mathrm{Id} = \pi_{H^m((D,\sigma),E)\cap\mathcal{S}_A(D)} + \sum_{\nu=0}^{\infty} (H_D + M_D)^\nu T_D A. \tag{7.2}$$

*Moreover, in the strong operator topology of $\mathcal{L}(L^2(D,F))$,*

$$\mathrm{Id} = \pi_{\ker T_D} + \sum_{\nu=0}^{\infty} A\,(H_D + M_D)^\nu T_D. \tag{7.3}$$

**Proof.** Write

$$\mathrm{Id} = (\mathrm{Id} - A T_D)^N + \sum_{\nu=0}^{N-1} (\mathrm{Id} - A T_D)^\nu A T_D, \tag{7.4}$$

for every $N = 1, 2, \ldots$. It is easily seen from (4.4) that

$$\begin{aligned} (\mathrm{Id} - A T_D)^\nu A T_D &= A\,(\mathrm{Id} - T_D A)^\nu T_D \\ &= A\,(H_D + M_D)^\nu T_D. \end{aligned}$$

Using Corollary 7.2 we can pass to the limit in (7.4), when $N \to \infty$, thus obtaining (7.3). The proofs for (7.1) and (7.2) are similar.

□

# 8 Cauchy problem

We first introduce the space of Cauchy data on $\sigma$, for our differential operator $A$. Since $A$ is given the domain $H^m(D,E)$, the space of zero Cauchy data is $H^m((D,\sigma),E)$. Recall that $H^m((D,\sigma),E)$ is proved to be the restriction of $\overset{\circ}{H}{}^m(V,E)$ to $D$.

Similarly to (4.2) we define the space of Cauchy data on $\sigma$ as the quotient space

$$\frac{H^m(D,E)}{H^m((D,\sigma),E)},$$

$t$ being thought of as the quotient mapping

$$t: \ H^m(D,E) \to \frac{H^m(D,E)}{H^m((D,\sigma),E)}. \tag{8.1}$$

Once again we use the Hilbert structure in $H^m(D,E)$ to construct a continuous right inverse $t^{-1}$ for $t$.

If the boundary of $\sigma$ on $\partial D$ is sufficiently smooth then the quotient space in (8.1) can be identified with $\oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\sigma,F_j)$.

Consider the following Cauchy problem, for the operator $A$ and the Dirichlet system $(B_j)_{j=0,\dots,m-1}$.

**Problem 8.1** *Given $f \in L^2(D,F)$ and $u_0 \in H^m(D,E)/H^m((D,\sigma),E)$, find $u \in H^m(D,E)$ satisfying*

$$\begin{cases} Pu &= f \quad \textit{in} \quad D, \\ t(u) &= u_0 \quad \textit{on} \quad \sigma. \end{cases}$$

This problem is ill-posed if $\sigma$ is different from the whole boundary. Using Theorem 7.3 we obtain approximate solutions to the problem. To this end, we observe that

$$\frac{H^m(V,E)}{\overset{\circ}{H}{}^m(V,E)} \hookrightarrow \frac{H^m(D,E)}{H^m((D,\sigma),E)}$$

is a well-defined mapping "onto", which substitutes restriction of sections over $\partial V$ to $\sigma$. Pick a $U \in H^m(V,E)$ such that $t(U) = u_0$ on $\sigma$. Lemma 4.3 yields that $MU \in H^m(V,E)$ satisfies $\Delta MU = 0$ in $V$ and $t(MU) = u_0$ on $\sigma$, the last property being sufficient. Problem 8.1 thus reduces to that with zero boundary conditions.

**Problem 8.2** *Given any $f \in L^2(D,F)$, find a section $u \in H^m((D,\sigma),E)$ such that $Au = f$ in $D$.*

Note that for the problem to be solvable it is necessary that $f \perp \ker T_D$. Indeed,

$$\begin{aligned}\int_D (f,g)_x\,dx &= \int_D (Au,g)_x\,dx\\ &= h_D(u, T_D g)\\ &= 0\end{aligned}$$

for all $g \in L^2(D,F)$ satisfying $T_D g = 0$, the second equality being due to Theorem 6.2.

**Theorem 8.3** *Suppose $f \in L^2(D,F)$. Problem 8.2 is solvable if and only if $f \perp \ker T_D$ and the series*

$$Rf = \sum_{\nu=0}^{\infty} (H_D + M_D)^\nu\, T_D f$$

*converges in $H^m((D,\sigma),E)$. Moreover, if these conditions hold then $Rf$ is a solution to Problem 8.2.*

**Proof.** As mentioned above, the necessity follows from Theorems 6.1 and 7.3.

Conversely, let both conditions of the theorem be fulfilled. Then (7.3) implies

$$f = \sum_{\nu=0}^{\infty} A\,(H_D + M_D)^\nu\, T_D f.$$

Since the series $Rf$ converges in $H^m((D,\sigma),E)$ we conclude that $f = ARf$, as desired.

□

In the case considered in Example 5.2 a similar result has been proved in [8].

Corollary 7.2 shows that the solution $u = Rf$ lies in the orthogonal complement of the subspace $H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ in $H^m((D,\sigma),E)$ with respect to the scalar product $h_D(\cdot,\cdot)$. Clearly, Problem 8.2 possesses at most one solution belonging to this orthogonal complement. The partial sums $R_N f$ of the series $Rf$ may be regarded as approximate solutions to Problem 8.2, provided that $f \perp \ker T_D$. In fact, it follows easily from Corollary 7.2 and Theorem 7.3 that $R_N f$ belongs to the orthogonal complement in question, for each $N = 0, 1, \ldots$, and

$$\lim_{N\to\infty} \|f - (\pi_{\ker T_D} f + AR_N f)\|_{L^2(D,F)} = 0$$

for all $f \in L^2(D,F)$. Indeed,

$$\| f - (\pi_{\ker T_D} f + AR_N f) \|_{L^2(D,F)} = \| \sum_{\nu=N+1}^{\infty} A\, M_D^{\nu}\, T_D f \|_{L^2(D,F)},$$

and the last expression is the rest of a converging series.

If $A$ is included into an elliptic complex

$$C^\infty_{\mathrm{loc}}(X,E) \xrightarrow{A} C^\infty_{\mathrm{loc}}(X,F) \xrightarrow{B} C^\infty_{\mathrm{loc}}(X,G)$$

then the condition $f \perp \ker T_D$ in Theorem 8.3 may be replaced by

1) $Bf = 0$ in $D$;

2) $f \perp \ker T_D \cap \mathcal{S}_B(D)$.

Write

$$n(g) = \oplus_{j=0}^{m-1} *_{F_j}^{-1} C_j *_F (g)$$

for the formal adjoint of $t$ with respect to the Green formula for $A$ in $D$, cf. [9, 9.2.3]. Set

$$\mathcal{H}^1(D,\sigma) = \{ g \in L^2(D,F) : \; A^* g = 0, \; Bg = 0, \text{ and } n(g) = 0 \text{ on } \partial D \setminus \sigma \}.$$

We call $\mathcal{H}^1(D,\sigma)$ the *harmonic space* in the Cauchy problem with data on $\sigma$. By the ellipticity assumption, the elements of $\mathcal{H}^1(D,\sigma)$ are of class $C^\infty$ in $D$.

**Lemma 8.4** $\ker T_D \cap \mathcal{S}_B(D) = \mathcal{H}^1(D,\sigma)$.

**Proof.** Let $g \in \ker T_D \cap \mathcal{S}_B(D)$. From Theorem 6.2 it follows that $A^* g = 0$ in the sense of distributions on $D$. By the ellipticity of $B \oplus A^*$ we conclude that $g \in C^\infty_{\mathrm{loc}}(D,F)$.

We next claim that $n(g) = 0$ weakly on $\partial D \setminus \sigma$. To prove this, we denote by $D_\varepsilon$ the set of all $x \in D$ such that $\mathrm{dist}(x, \partial D) > \varepsilon$. For $\varepsilon > 0$ small enough, $D_\varepsilon$ is also a domain with $C^\infty$ boundary. We shall have established the equality $n(g) = 0$ on $\partial D \setminus \sigma$ if we show that

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} (t(u), n(g))_x \, ds_\varepsilon = 0$$

for all $u \in C^\infty(\bar{D}, E)$ vanishing near $\sigma$. Here, $ds_\varepsilon$ is the area element of the surface $\partial D_\varepsilon$.

Since $g$ is $C^\infty$ in $D$, we get

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} (t(u), n(g))_x \, ds_\varepsilon &= \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (Au, g)_x \, dx \\
&= \int_D (Au, g)_x \, dx \\
&= h_D(u, T_D g) \\
&= 0,
\end{aligned}$$

the first equality being due to Stokes' formula and the equality $A^*g = 0$, the second equality being a consequence of the fact that $g \in L^2(D, F)$, and the third one being due to Theorem 6.2. We have thus proved that $\ker T_D \cap \mathcal{S}_B(D)$ is a subset of $\mathcal{H}^1(D, \sigma)$.

Let us prove the opposite inclusion. Pick a $g \in \mathcal{H}^1(D, \sigma)$. By ellipticity we conclude that $g \in C^\infty_{\mathrm{loc}}(D, F)$. For every $u \in C^\infty_{\mathrm{loc}}(\bar{D}, E)$ vanishing near $\sigma$, we have

$$\begin{aligned} h_D\,(u, T_D g) &= \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (Au, g)_x \; dx \\ &= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} (t(u), n(g))_x \; ds_\varepsilon \\ &= 0. \end{aligned}$$

Since such sections $u$ are dense in $H^m((D, \sigma), E)$, it follows that $T_D g = 0$, as desired.

□

It is worth pointing out that the space $\mathcal{H}^1(D, \sigma)$ fails to be finite-dimensional in general.

# 9 Applications to Zaremba problem

In this section we assume that $\sigma$ is the closure of an open subset in $\partial D$ with smooth boundary.

Let $H^{-m}((D, \partial D \setminus \sigma), E)$ be the dual space for $H^m((D, \sigma), E)$ with respect to the pairing in $L^2(D, E)$. It coincides with the completion of $C^\infty_{\mathrm{comp}}(D \cup \sigma, E)$ with respect to the norm

$$\|F\|_{H^{-m}((D, \partial D \setminus \sigma), E)} = \sup_{v \in C^\infty_{\mathrm{comp}}(\bar{D} \setminus \sigma, E)} \frac{|\int_D (F, v)_x \, dx|}{\|v\|_{H^m((D,\sigma),E)}}.$$

Recall that for $s \geq 0$ we write $H^{-s}(\partial D \setminus \sigma, F_j)$ for the dual of $\overset{\circ}{H}{}^s(\partial D \setminus \sigma, F_j)$ with respect to the pairing in $L^2(\partial D \setminus \sigma, F_j)$, cf. Section 3. One can prove that

$$H^{-s}(\partial D \setminus \sigma, F_j) \overset{\text{top.}}{\cong} \frac{H^{-s}(\partial D, F_j)}{H^{-s}_\sigma(\partial D, F_j)}$$

where $H^{-s}_\sigma(\partial D, F_j)$ is the subspace of $H^{-s}(\partial D, F_j)$ consisting of the elements with a support in $\sigma$.

By the above, the sesquilinear form

$$\int_{\partial D} (u_1, t(v))_x \; ds$$

is well defined for all

$$\begin{array}{rcl} u_1 & \in & \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j), \\ v & \in & H^m((D,\sigma),E). \end{array}$$

We are now in a position to consider the following generalised Zaremba problem in $D$.

**Problem 9.1** *Given*

$$\begin{array}{rcl} F & \in & H^{-m}((D,\partial D \setminus \sigma),E), \\ u_1 & \in & \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j), \end{array}$$

*find* $u \in H^m(D,E)$ *such that*

$$\left\{ \begin{array}{rclcl} \Delta u & = & F & \mathit{in} & D, \\ t(u) & = & 0 & \mathit{on} & \sigma, \\ n(Au) & = & u_1 & \mathit{on} & \partial D \setminus \sigma. \end{array} \right.$$

The equation $\Delta u = F$ has to be understood in the sense of distributions in $D$, while the boundary conditions are interpreted in the following weak sense: Find $u \in H^m((D,\sigma),E)$ satisfying

$$\int_D (Au, Av)_x \, dx = \int_D (F, v)_x \, dx + \int_{\partial D} (u_1, t(v))_x \, ds \tag{9.1}$$

for all $v \in H^m((D,\sigma),E)$.

We emphasise that the trace of $n(Au)$ on $\partial D \setminus \sigma$ is not defined for any $u \in H^m((D,\sigma),E)$, because the order of $n \circ A$ is equal to $2m-1$. To cope with this, a familiar way is to assign an operator $L$ with a dense domain $\mathrm{Dom}\, L \hookrightarrow H^m((D,\sigma),E)$ to Problem 9.1, such that $n(Au)$ is well defined for all $u \in \mathrm{Dom}\, L$. In fact, $\mathrm{Dom}\, L$ is defined to be the completion of $C^\infty_{\mathrm{comp}}(\bar{D}\setminus\sigma, E)$ with respect to the graph norm of $u \mapsto (u, n(Au))$ in $H^m((D,\sigma),E) \oplus \mathfrak{N}$, where

$$\mathfrak{N} = \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j).$$

For more details, see Roitberg [5] and elsewhere. Then, (9.1) defines a continuous operator $\mathrm{Dom}\, L \to H^{-m}((D,\partial D\setminus\sigma),E) \oplus \mathfrak{N}$ by $Lu = (\Delta u, n(Au))$.

If $A$ is the gradient operator in $\mathbb{R}^n$, then (9.1) is just the classical Zaremba problem in $D$.

**Lemma 9.2** *Suppose* $F = 0$ *and* $u_1 = 0$*. Then* $u \in H^m(D,E)$ *is a solution to Problem 9.1 if and only if* $u \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$*.*

**Proof.** Obviously, any $u \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ is a solution of Problem 9.1 with $F = 0$ and $u_1 = 0$.

Conversely, let $u$ be a solution to Problem 9.1 with $F = 0$ and $u_1 = 0$. Substituting $v = u$ to (9.1) implies $Au = 0$ whence $u \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$, as desired.

□

Lemma 9.2 shows that Problem 9.1 is not Fredholm in general, for the space $H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ need not be finite-dimensional.

For any $v \in \overset{\circ}{H}{}^m(V,E)$, the restriction $v\,|_D$ belongs to $H^m((D,\sigma),E)$. Hence to each $F \in H^{-m}((D,\partial D \setminus \sigma),E)$ we can assign an element $\tilde{F} \in H^{-m}(V,E)$ by

$$(\tilde{F}, v) = (F, v\,|_D)$$

for all $v \in \overset{\circ}{H}{}^m(V,E)$. We will write $\tilde{F}$ simply $\chi_D F$ when no confusion can arise. Therefore, the integral

$$G\,(\chi_D F) = \int_D \left(F, *_E^{-1} K_G(x,\cdot)\right)_y dy$$

defines an element of $H^m((D,\sigma),E)$, for any $F \in H^{-m}((D,\partial D \setminus \sigma),E)$.

Furthermore, since $u \mapsto t(u) \oplus n(Au)$ is a Dirichlet system of order $2m-1$ on $\partial D$, for every data $u_1 \in \mathfrak{N}$ there exists a $U \in \operatorname{Dom} L$ with the property that $n(AU) = u_1$ on $\partial D \setminus \sigma$ (see for instance Lemma 9.2.17 of [9]). This means that

$$\int_D (AU, Av)_x\, dx = \int_D (\Delta U, v)_x\, dx + \int_{\partial D} (u_1, t(v))_x\, ds$$

for all $v \in H^m((D,\sigma),E)$. Set

$$\begin{array}{rcl} P_{\mathrm{sl}} u_1\,(x) & = & -\displaystyle\int_{\partial D} G_{A^*}(K_G(x,\cdot), AU) \\ & = & \displaystyle\int_{\partial D} \left(n(AU), t\left(*_E^{-1} K_G(x,\cdot)\right)\right)_y ds, \end{array}$$

for $x \in D$.

This integral is well defined and it does not depend on the particular choice of $U$. Indeed, since $\Delta U \in H^{-m}((D,\partial D \setminus \sigma),E)$ we conclude, by Stokes' formula, that

$$\begin{array}{rcl} P_{\mathrm{sl}} u_1 & = & GA^*\,(\chi_D AU) - G\,(\chi_D \Delta U) \\ & = & T_D AU - G\,(\chi_D \Delta U)\,, \end{array}$$

which is in $H^m((D,\sigma),E)$.

Let now $U \in \operatorname{Dom} L$ be such that $n(AU) = 0$ on $\partial D \setminus \sigma$. Then using Theorem 3.3 yields

$$\begin{aligned}
\int_X \Big( - \int_{\partial D} G_{A^*}(K_G(x,\cdot), AU), v \Big)_x dx &= \int_X (GA^* \chi_D AU - G\chi_D \Delta U, v)_x \, dx \\
&= \int_X (A^* \chi_D AU - \chi_D \Delta U, Gv)_x \, dx \\
&= \int_D (AU, AGv)_x \, dx - \int_D (\Delta U, Gv)_x \, dx \\
&= 0
\end{aligned}$$

for all $v \in C^\infty_{\mathrm{comp}}(X,E)$, because $Gv \in H^m((D,\sigma),E)$. Hence, $P_{\mathrm{sl}} u_1$ is independent of the choice of $U$.

**Theorem 9.3** *Problem 9.1 is solvable if and only if*
1)
$$\int_D (F,v)_x \, dx + \int_{\partial D} (u_1, t(v))_x \, ds = 0$$
*for all* $v \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$;
2) *the series*

$$R(F,u_1) = \sum_{\nu=0}^{\infty} (H_D + M_D)^\nu \left( G(\chi_D F) + P_{\mathrm{sl}} u_1 \right)$$

*converges in the* $H^m(D,E)$*-norm.*
*If 1) and 2) hold then* $R(F,u_1)$ *is a solution to Problem 9.1.*

**Proof.** Let Problem 9.1 be solvable and let $u \in H^m((D,\sigma),E)$ be a solution. Then

$$\begin{aligned}
T_D Au &= G(\chi_D \Delta u) - \int_{\partial D} G_{A^*}(K_G(x,\cdot), Au) \\
&= G(\chi_D F) + P_{\mathrm{sl}} u_1,
\end{aligned}$$

and so the series $R(F,u_1) = RAu$ converges in the $H^m(D,E)$-norm, which is due to Corollary 7.2.

Conversely, assume that 1) and 2) are fulfilled. Let us prove that the series $R(F,u_1)$ satisfies (9.1). Indeed, by Theorem 6.1

$$\begin{aligned}
\int_D (AR(F,u_1), Av)_x \, dx &= h_D\left(G(\chi_D F) + P_{\mathrm{sl}} u_1, v\right) \\
&= h_D\left(G(\chi_D F) + GA^*(\chi_D AU) - G(\chi_D \Delta U), v\right)
\end{aligned}$$

with a section $U \in \operatorname{Dom} L$ such that $n(AU) = u_1$ on $\partial D \setminus \sigma$.

Using Theorem 3.3 we see that $e(G\tilde{F}) = G\tilde{F}$ for all $\tilde{F} \in H^{-m}(V, E)$ satisfying $\tilde{F} - H\tilde{F} = 0$ in $X \setminus \bar{D}$. We next apply this equality with

$$\tilde{F} = \chi_D F + A^* (\chi_D AU) - \chi_D \Delta U.$$

We have

$$\begin{aligned}
\int_X \left(\tilde{F}, v\right)_x dx &= \int_D (F - \Delta U, v)_x \, dx \\
&= \int_D (F, v)_x \, dx + \int_{\partial D} (n(AU), t(v))_x \, ds \\
&= 0
\end{aligned}$$

for all $v \in H^m((D, \sigma), E) \cap \mathcal{S}_A(D)$, the last equality being a consequence of condition 1). Hence it follows readily that $H\tilde{F} = 0$ in $V$, and so $\tilde{F} - H\tilde{F} = 0$ in $X \setminus \bar{D}$.

Therefore, $e(G\tilde{F}) = G\tilde{F}$ and we get

$$\begin{aligned}
\int_D (AR(F, u_1), Av)_x \, dx &= h\left(e(G\tilde{F}), e(v)\right) \\
&= \int_X \left(AG\tilde{F}, Ae(v)\right)_x dx + \int_X \left(HG\tilde{F}, He(v)\right)_x dx \\
&= \int_X \left(\tilde{F}, G\Delta e(v)\right)_x dx \\
&= \int_X \left(\tilde{F}, e(v) - He(v)\right)_x dx \\
&= \int_D (F - \Delta U, e(v))_x \, dx + \int_D (AU, Ae(v))_x \, dx \\
&= \int_D (F, v)_x \, dx + \int_{\partial D} (n(AU), t(v))_x \, ds
\end{aligned}$$

for all $v \in H^m((D, \sigma), E)$. Here, the fifth equality is due to condition 1) and the fact that $He(v) \in H^m((D, \sigma), E) \cap \mathcal{S}_A(D)$, and the last equality is a consequence of Stokes' formula. We have arrived at (9.1), thus proving the theorem.

□

Corollary 7.2 implies that the solution $R(F, u_1)$ to Problem 9.1 lies in the orthogonal complement of $H^m((D, \sigma), E) \cap \mathcal{S}_A(D)$ in $H^m((D, \sigma), E)$ with respect to the scalar product $h_D(\cdot, \cdot)$. Moreover, $R(F, u_1)$ is the unique solution belonging to this subspace. The partial sums $R_N(F, u_1)$ of the series $R(F, u_1)$ may be regarded as approximate solutions to Problem 9.1, provided that $F$

and $u_1$ meet condition 1) of Theorem 9.3. The sequence $R_N(F,u_1)$ has the following property:

$$\begin{aligned} & \left| \int_D (AR_N(F,u_1), Av)_x \, dx - \int_D (F,v)_y \, dy - \int_{\partial D} (u_1, t(v))_x \, ds \right| \\ & \qquad \leq c \, \| M_D^{N+1} \left( G(\chi_D F) + P_{\mathrm{sl}} u_1 \right) \|_{H^m(D,E)} \|v\|_{H^m(D,E)} \end{aligned} \tag{9.2}$$

for all $v \in H^m((D,\sigma),E)$, with $c$ a constant independent of $N$ and $v$. Indeed, as we have seen above (cf. (7.4)),

$$T_D A R_N(F,u_1) = (G(\chi_D F) + P_{\mathrm{sl}} u_1) - M_D^{N+1} \left( G(\chi_D F) + P_{\mathrm{sl}} u_1 \right)$$

whence

$$\begin{aligned} & \int_D (AR_N(F,u_1), Av)_x \, dx \\ & \quad = h_D \left( T_D A R_N(F,u_1), v \right) \\ & \quad = h_D \left( (G(\chi_D F) + P_{\mathrm{sl}} u_1) - M_D^{N+1} \left( G(\chi_D F) + P_{\mathrm{sl}} u_1 \right), v \right) \\ & \quad = \int_D (F,v)_x \, dx + \int_{\partial D} (u_1, t(v))_x \, ds - h_D \left( M_D^{N+1} \left( G(\chi_D F) + P_{\mathrm{sl}} u_1 \right), v \right). \end{aligned}$$

The scalar product $h_D(\cdot,\cdot)$ defines an equivalent norm, hence the estimate (9.2) holds. Since $G(\chi_D F) + P_{\mathrm{sl}} u_1$ is orthogonal to $H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ we see that

$$\lim_{N \to \infty} \| M_D^{N+1} \left( G(\chi_D F) + P_{\mathrm{sl}} u_1 \right) \|_{H^m(D,E)} = 0.$$

Of course, if Problem 9.1 is Fredholm then the series $R(F,u_1)$ converges for all data $F$ and $u_1$.

In the setting of Example 5.2 such a theorem was proved in [8]. We can also treat the inhomogeneous Zaremba problem.

**Problem 9.4** *Given*

$$\begin{array}{rcl} F & \in & H^{-m}((D, \partial D \setminus \sigma), E), \\ u_0 & \in & \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\sigma, F_j), \\ u_1 & \in & \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j), \end{array}$$

*find* $u \in H^m(D,E)$ *such that*

$$\left\{ \begin{array}{rcll} \Delta u & = & F & \text{in } D, \\ t(u) & = & u_0 & \text{on } \sigma, \\ n(Au) & = & u_1 & \text{on } \partial D \setminus \sigma. \end{array} \right.$$

Indeed, using the potential $M_D t^{-1} u_0$ as in Section 4, we easily reduce it to Problem 9.1.

# References

[1] L. I. Hedberg and T. H. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **33** (1983), no. 4, 161–187.

[2] L. I. Hedberg, *Approximation by harmonic functions and stability of the Dirichlet problem*, Exposition Math. **11** (1993), 193–259.

[3] M. Nacinovich and A. Shlapunov, *On iterations of the Green integrals and their applications to elliptic differential complexes*, Math. Nachr. **180** (1996), 243–286.

[4] M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskii, Ill-Posed Problems of Mathematical Physics and Analysis, Nauka, Moscow, 1980, 286 pp.

[5] Ya. A. Roitberg, Elliptic Boundary Value Problems in Generalized Functions, Kluwer Academic Publishers, Dordrecht NL, 2000.

[6] A. V. Romanov, *Convergence of iterations of Bochner-Martinelli operator, and the Cauchy-Riemann system*, Soviet Math. Dokl. **19** (1978), no. 5.

[7] B.-W. Schulze, *Crack problems in the edge pseudo-differential calculus*, Applic. Analysis **45** (1992), 333–360.

[8] B.-W. Schulze, A. A. Shlapunov, and N. Tarkhanov, Regularisation of Mixed Boundary Problems, Preprint 99/9, Univ. Potsdam, Potsdam, 1999, 32 pp.

[9] N. Tarkhanov, The Cauchy Problem for Solutions of Elliptic Equations, Akademie-Verlag, Berlin, 1995.

[10] M. I. Vishik, *On strongly elliptic systems of differential equations*, Mat. Sb. **29 (71)** (1951), no. 3.

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