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QUANTIZATION METHODS
in
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Part II

Quantization by the Method of Ordered Operators (Noncommutative Analysis)

Contents

0.1 Preliminary Remarks

1. In part I, starting from a physical analogy, we carried out a quantization of a classical system with linear phase space $\mathbf{R}^{2n}$ with coordinates q and p. This quantization procedure can be interpreted in two different ways:

(i) from the algebra of classical observables (that is, functions) equipped with the Poisson bracket $\{f, g\}$, we proceed to the algebra of quantum observables (operators), whose commutator passes in the semiclassical limit into the Poisson bracket in the following sense:

$$[\hat{f}, \hat{g}] = -ih\widehat{\{f, g\}} + O(h^2); \tag{0.1}$$

(ii) we replace the basis classical observables $q_1, \dots, q_n$, $p_1, \dots, p_n$ by quantum operators with the commutation relations

$$\begin{aligned} [\tilde{q}_k, \tilde{q}_j] &= [\tilde{p}_k, \tilde{p}_j] = 0 \\ [\tilde{p}_j, \tilde{q}_k] &= -ih\delta_{jk} \end{aligned} \tag{0.2}$$

(which is of course a special case of relations (0.1) satisfied *exactly*) and then pass from the "general" classical observables to the corresponding quantum observables according to the law

$$f(q_1, \dots, q_n, p_1, \dots, p_n) \mapsto \hat{f} = f(\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n); \qquad (0.3)$$

in other words, we replace the arguments $q_1, \dots, q_n$, $p_1, \dots, p_n$ of each classical observable $f(q_1, \dots, q_n, p_1, \dots, p_n)$ by their quantum analogs $\hat{q}_1, \dots, \hat{q}_n$, $\hat{p}_1, \dots, \hat{p}_n$. (Needless to say, one has to deal with the problem that this substitution is not uniquely determined, since the quantum observables $\hat{q}_1, \dots, \hat{q}_n$, $\hat{p}_1, \dots, \hat{p}_n$, as shown by (0.1), do not commute.)

After this substitution, we obtain a quantization satisfying property (0.1)! Thus, the second interpretation shows that one can quantize the algebra of classical observables with the Poisson bracket by considering functions of operators of the form (0.3), and this alternative method of quantization proves to be *equivalent* to the method based on wave packets.

2. Now suppose that we intend to quantize some classical mechanics, that is, an algebra of classical observables, which is more complicated than the simplest algebra $(F(\mathbf{R}^{2n}), \{\,\cdot,\,\cdot\})$ of functions on $\mathbf{R}^{2n}_{q,p}$ with the standard Poisson bracket

$$\{f, g\} = \sum_{j=1}^{n} \left\{ \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} \right\}. \qquad (0.4)$$

In this more complicated classical mechanics, the phase space need not be linear, and the Poisson bracket may be degenerate (in this case, it does not correspond to any symplectic structure on the phase space). Generally speaking, the fact that the phase space is not linear is not an obstruction to quantization (at least, to asymptotic quantization), since one may hope to solve the quantization problem locally (in this case, the phase space can be assumed to be linear) and then take global effects into account by pasting local quantizations together. If the Poisson bracket is nondegenerate, then it is generated by some symplectic structure ω^2 on the phase space M. Locally, this structure can always be reduced to the canonical form

$$\omega^2 = dp \wedge dq \equiv \sum_{j=1}^{n} dp_j \wedge dq_j \qquad (0.5)$$

by Darboux' theorem. Then the Poisson bracket acquires the form (0.4) and the quantization problem can be solved locally by the wave packet method. The situation is quite different if the Poisson bracket is degenerate. Then the quantization problem is nontrivial even locally, and we do not have any analog of the wave packet transform at hand. In that situation, it is natural to use the second approach to construct local quantizations. We can proceed as follows. In the phase space M of our classical system, we choose some local coordinates $x_1, \dots, x_m$ (where m is the dimension of M). Let the Poisson bracket be given by

$$\{x_j, x_k\} = F_{jk}(x_1, \dots, x_m), \tag{0.6}$$

where the F_{jk} are some smooth functions defined in the domain covered by the local coordinates and satisfying a system of identities readily following from the skew-symmetry and the Jacobi identity for the Poisson bracket. Let us quantize relations (0.6) by finding operators $A_1, \dots, A_n$ satisfying the commutation relations

$$[A_j, A_k] = -ihF_{jk}(A_1, \dots, A_k) + O(h^2) \tag{0.7}$$

and define a (local) quantum algebra by setting

$$\hat{f} = f(A_1, \dots, A_m) \tag{0.8}$$

for functions supported in this coordinate domain. Once the local quantization problem is solved, we can proceed to "pasting" the local solutions obtained in coordinate neighborhoods together.

3. All the preceding is no more than an intuitive idea of construction of a quantization method, since none of the steps involved has been described or even outlined so far. Moreover, it seems to be obvious (and actually it is true) that solving commutation relations of the form (0.7) with any nontrivial right-hand sides F_{jk} is some kind of art rather than routine work. In any case, our considerations already show that the notion of functions (0.8) of operators is the basis of any possible implementation of the above-suggested quantization scheme. (Note that the right-hand sides of relations (0.7) are also functions of this sort.) Thus, to develop this approach successfully, we must first have a technique permitting one to work with such functions. Fortunately, this technique is already available. It was developed by Maslov

[4] more than 25 years ago and is known as the *operator method* or, in a more up-to-date terminology, *noncommutative analysis*. In this part of the book we give the essentials of noncommutative analysis and then use it provide a complete quantization scheme for several special classes of commutation relations (or the corresponding classical mechanics, if you prefer that). We also describe a fairly large class of *exactly soluble* commutation relations.

4. Let us describe the structure of this part in more detail. In Chapter 1 we present *noncommutative analysis* itself, that is, the theory of functions of several operators that in general do not commute with each other. In Chapter 2 we consider the simplest class of classical mechanics with exactly soluble commutation relations and carry out the quantization scheme for this class of relations. In Chapter 3 we briefly outline the main points of the quantization procedure for relations that cannot be solved explicitly. We point out that in both cases (approximately and exactly soluble relations) the quantization itself is asymptotic rather than exact. Indeed, even if we manage to solve equations (0.7) exactly (that is, without the remainder $O(h^2)$ as in the second case), this does not imply that the relations

$$[\hat{f}, \hat{g}] = -ih\widehat{[f, g]}$$

hold for arbitrary quantized classical observables f and g. (As was already indicated in Part I, this is impossible even for the simplest classical system with a linear phase space and the standard Poisson bracket.) On the other hand, we shall see that, in contrast with the quantization procedure, noncommutative analysis need not be asymptotic (even though there are asymptotic versions of it in literature). Hence the small parameter h only seldom occurs in Chapter 1, which deals with noncommutative analysis itself.

Chapter 1

Noncommutative Analysis: Main Ideas, Definitions, and Theorems

The main goal of noncommutative analysis is assigning an exact meaning to the expression $f(A_1, \dots, A_m)$, where $A_1, \dots, A_m$ are given linear operators in a linear space E and $f(x_1, \dots, x_m)$ is a given function, which is referred to as a *symbol*. The second goal is to develop rules helping one to handle such functions. (We shall see that in many cases, working with functions of noncommuting operators is no more complicated than working with usual functions; the term "noncommutative analysis" was introduced in [5] to emphasize the analogy between calculations involving functions of noncommuting operators and usual calculations in differential and integral calculus.) As to definitions, even for $m = 1$ it is not a trivial problem to assign a meaning to the expression $f(A)$ unless $f(x)$ is a polynomial. The solutions of this problem in various particular cases are known in the literature as "functional calculi" (see § 1.1). In the presence of several arguments, there is an additional difficulty: if some of the operators $A_1, \dots, A_m$ do not commute with each other, then even for a polynomial $f(x_1, \dots, x_m)$ the result of the substitution of the operators $A_1, \dots, A_m$ for the numerical arguments $x_1, \dots, x_m$ is not uniquely determined, and one must have some method for keeping track of the *order* in which the operators act. Strange as it may seem, this is largely a problem of notation. A solution of this problem, based on an elegant idea used by R. Feynman [1] in a special case as long ago as in 1951, was given by Maslov [4]. The corresponding material will be given in § 1.2. Once this notation is introduced, we proceed to various formulas of the noncommutative operator calculus. The most important formulas are gathered in § 1.3. These formulas form a list of rules (somewhat similar to tables of derivatives and integrals of elementary functions in the ordinary calculus) which prove useful in the derivation of more advanced results in Chapters 2 and 3. Sections 1.4 and 1.5 deal, respectively, with the main techniques useful in dealing with functions of noncommuting operators and with composition laws for functions of given operator tuples.

1.1 Functions of One Operator (Functional Calculi)

Various constructions of functions $f(A)$ of a single operator A for specific classes of symbol $f(x)$ and operators A are known in the literature. Let us give just a few examples.

a) *Functional calculus of entire functions of bounded operators.* If

$$A : E \to E$$

is a bounded operator in a Banach space E and $f(x)$ is an entire function with power series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{f_k}{k!} x^k, \tag{1.1}$$

then one sets

$$f(A) = \sum_{k=0}^{\infty} \frac{f_k}{k!} A^k.$$

(In fact, $f(x)$ need not be entire. It suffices to require that the series (1.1) be convergent in a disk of radius larger than the spectral radius spr (A).)

b) *Holomorphic functional calculus.* This is a finer version of the functional calculus for entire functions. If A is again a bounded operator in a Banach space and $f(z)$ is a function holomorphic in a neighborhood of the spectrum $\sigma(A)$, then one sets

$$f(A) = \frac{1}{2\pi i} \oint f(\lambda)(\lambda - A)^{-1} d\lambda, \tag{1.2}$$

where the integral is taken over the contour lying in the domain where f is holomorphic and surrounding the spectrum $\sigma(A)$ counterclockwise.

c) *Functional calculus of self-adjoint operators.* If $A = A^*$ is a self-adjoint (possibly, unbounded) linear operator in a Hilbert space H and f is a continuous (for simplicity) function defined in a neighborhood of the spectrum $\sigma(A)$, then one sets

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda, \tag{1.3}$$

where $\{dE_x\}$ is the spectral resolution of unity corresponding to A. The operator $f(A)$ is bounded if the function $f(x)$ is bounded.

d) *Fourier–Laplace functional calculus.* If $iA : E \to E$ is the generator of a strongly continuous one-parameter group of linear operators in a Banach space E, then for functions $f(x)$ with continuous sufficiently rapidly decreasing Fourier transform

$$\tilde{f}(p) = \left(\frac{-i}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$$

one sets

$$f(A) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iAp} \tilde{f}(p)\, dp, \tag{1.4}$$

where e^{iAp} is the one-parameter group generated by iA:

$$\begin{cases} \dfrac{\partial}{\partial p}(e^{iAp}) = iAe^{iAp}, \\ e^{iAp}\Big|_{p=0} = 1. \end{cases}$$

The precise description of the class of functions $f(x)$ for which definition (1.4) is valid depends on the properties of the group e^{iAp}. If this is a group of exponential growth, then, generally speaking, the symbol $f(x)$ must satisfy some analyticity conditions. If this is a group of at most power-law growth (in the finite-dimensional case, it suffices to require that all eigenvalues of A be real; the operator may have Jordan chains), then the function (1.4) is well-defined for symbols of the class $S^\infty(\mathbf{R})$, that is, functions $f(x)$ such that the estimates

$$\left|\frac{\partial^\alpha f}{\partial x^\alpha}\right| \le C_\alpha(1+|x|)^m. \quad \alpha = 0, 1, 2, \ldots$$

hold for some given $m > 0$.

The above list of various versions of functional calculus is by no means complete, but we shall be merciful. Note that, despite the abundance of versions, they all have some important common features:

- being applied to a given symbol $f(x)$ and a given operator A, all versions that are well defined for the pair $(f(x), A)$ give the same result;[1]
- functions $f(A)$ with various symbols $f(x)$ commute with each other and with the operator A itself;
- if $f(x) \equiv x$, then $f(A) = A$;
- if $f(x)$ and $g(x)$ are admissible symbols for a given operator A, then the symbol $h(x) = f(x)\,g(x)$ is also admissible, and moreover,
$$f(A)\,g(A) = h(A).$$

With regard for the fact that noncommutative analysis is a general rather than special theory, we give an axiomatic definition of functions of one operator. This permits us to consider various special cases in a unified way. All we need in any specific example is to prove the validity of the axioms.

[1] Unless we consider certain pathological examples.

Definition 1 Let $\mathcal{F}$ be a unital algebra of functions $f(x)$ of the variable x ranging in a subset of the complex plane, and let $A: E \to E$ be a linear operator in a linear space E. We say that A is an *$\mathcal{F}$-generator* if there exists a homomorphism

$$\mu_A : \mathcal{F} \to \operatorname{End} E \tag{1.5}$$

such that $\mu_A[x] = A$. (Here $x \in \mathcal{F}$ is the function identically equal to x; we assume that $\mathcal{F}$ contains this function and hence all polynomials of x.)

Assuming that the homomorphism (1.5) is given, we write

$$\mu_A(f) \equiv f(A).$$

Exercise. Prove that the homomorphisms μ_A corresponding to functional calculi in examples a)–d) and defined on appropriate classes of functions satisfy the conditions of Definition 1.

The condition $\mu_A[x] = A$ uniquely determines the homomorphism μ_A on the subalgebra $\mathcal{P} \subset \mathcal{F}$ of polynomials, but if $\mathcal{P} \neq \mathcal{F}$, then the uniqueness of μ_A cannot be guaranteed. Usually, $\mathcal{F}$ and End E are equipped with some natural notion of convergence; in this case, we always require that the mapping (1.5) must be continuous. Unfortunately, even this requirement does not guarantee that μ_A is unique. However, the following remarkable theorem holds.

Theorem 2 ([5]) *Let $\mathcal{F}$ be a symbol algebra with convergence. Suppose that for each symbol $f \in \mathcal{F}$ the difference derivative*

$$\frac{\delta f}{\delta x}(x, y) \equiv \frac{f(x) - f(y)}{x - y}$$

belongs to the projective tensor product $\mathcal{F} \hat{\otimes} \mathcal{F}$. Then there exists a unique continuous homomorphism μ_A for any $\mathcal{F}$-generator A.

The proof of this theorem will be given in the next section, since it is surprisingly based on the theory of functions of several operators.

1.2 Functions of Several Operators

Suppose that a function $f(x_1, \dots, x_m)$ and m operators $A_1, \dots, A_m$ are given. We wish to define what it means to substitute these operators for the arguments $x_1, \dots, x_m$ into the function $f(x_1, \dots, x_m)$:

$$f(x_1, \dots, x_m) \mapsto f(A_1, \dots, A_m). \tag{1.6}$$

For simplicity, we assume that all operators $A_1, \dots, A_m$ are $\mathcal{F}$-generators with the same symbol class $\mathcal{F}$. (Considering the case in which the classes of unary symbols are different for different operators A_j is no more complicated in principle, but it *does* complicate the notation dramatically, and so we avoid it.) First, we consider the case in which the symbol $f(x_1, \dots, x_m)$ is factorable:

$$f(x_1, \dots, x_m) = f(x_1) \dots f_m(x_m), \tag{1.7}$$

where

$$f_1(x), \dots, f_m(x) \in \mathcal{F}.$$

In this case, it is clear that one carries out the substitution (1.6) by replacing the jth factor $f_j(x)$ by the operator $f_j(A_j)$. But here comes the difficulty: the operators $f_j(A_j)$ and $f_k(A_k)$ do not commute in general, and hence the order of factor becomes essential. In fact, there is no distinguished order of factors, and hence the choice of some given order must be included in the statement of the problem as additional information. We denote the order of operators by *Feynman indices*. (This notation was introduced in the large-scale use by Maslov [4].) Specifically, the order in which the operators act, that is, the arrangement of the corresponding operator factors in an operator expression, will be determined by numbers over these operators: the smaller a number, the closer is the corresponding operator to the right in the product. In other words, operators with smaller numbers act *before* operators with larger numbers. For example,

$$\overset{1}{A}\overset{2}{B} = BA;$$

$$(\overset{1}{A} + \overset{2}{B})^2 = A^2 + 2BA + B^2 \neq A^2 + AB + BA + B^2 = (A + B)^2,$$

$$\overset{1}{C}\overset{3}{C}\overset{2}{B} = CBC,$$

etc. In the general case, we write

$$f_1(\overset{j_1}{A_1})\dots f_n(\overset{j_m}{A_m}) = f_{\alpha_1}(A_{\alpha_1})\dots f_{\alpha_m}(A_{\alpha_m}),$$

where $\{\alpha_1,\dots,\alpha_m\}$ is the permutation of $\{1,\dots,m\}$ such that

$$j_{\alpha_1}, j_{\alpha_2}, \dots j_{\alpha_m} = m, m-1, \dots, 1.$$

Now suppose that a symbol $f(x_1,\dots,x_m)$ is a *sum* of factorable symbols of the form (1.7):

$$f(x_1,\dots,x_m) = \sum_l f_{e1}(x_1)\dots f_{em}(x_m), \tag{1.8}$$

where the sum is finite and $f_{ej}(x)\in\mathcal{F}$. Then the definition naturally extends to such symbols by linearity:

$$f\left(\overset{j_1}{A_1},\dots,\overset{j_m}{A_m}\right) = \sum_l \left(f_{e1}(\overset{j_1}{A_1})\dots f_{em}(\overset{j_m}{A_m})\right),$$

where the factors in each summand on the right-hand side are arranged in ascending order of the corresponding Feynman indices. One can readily verify that this is well defined (that is, independent of the representation of $f(x_1,\dots,x_m)$ in the form (1.8).

Thus we have defined the expression $f\left(\overset{j_1}{A_1},\dots,\overset{j_m}{A_m}\right)$ for elements $f\in\mathcal{F}\otimes\dots\otimes\mathcal{F}$ of the *algebraic tensor product* of m copies of the space $\mathcal{F}$ of unary symbols. If the mapping $f\mapsto f(A_j)$ is continuous on $\mathcal{F}$ for each j (which is always assumed), then the mapping

$$\begin{aligned}\mu\equiv\mu_{\overset{j_1}{A_1},\dots,\overset{j_m}{A_m}} : \underbrace{\mathcal{F}\otimes\dots\otimes\mathcal{F}}_{m\text{ copies}} &\to Op\\ f(x_1,\dots,x_m) &\mapsto f(\overset{j_1}{A_1},\dots,\overset{j_m}{A_m})\end{aligned}$$

thus constructed extends by continuity to a mapping (denoted by the same letter)

$$\mu : \underbrace{\mathcal{F}\hat{\otimes}\dots\hat{\otimes}\mathcal{F}}_{m\text{ copies}} \to Op$$

of the *projective tensor product* $\mathcal{F}\hat{\otimes}\dots\hat{\otimes}\mathcal{F}$ of m copies of the space $\mathcal{F}$ of unary symbols. For the case in which $\mathcal{F}$ is a Banach space, the projective tensor product is defined as follows. (For simplicity, we consider the case $m = 2$.) On $\mathcal{F} \otimes \mathcal{F}$ we introduce the norm

$$||f||^{\wedge} = \inf \sum_l ||f_l||\,||g_l||, \tag{1.9}$$

where the infimum is taken over all possible representations of $f(x, y)$ in the form

$$f = \sum_l f_l \otimes g_l,$$

that is,

$$f(x, y) = \sum_l f_l(x) g_l(y)$$

(the sum is finite). Now $\mathcal{F}\hat{\otimes}\mathcal{F}$ is defined as the completion of the tensor product $\mathcal{F} \otimes \mathcal{F}$ with respect to the norm (1.9). If $\mathcal{F}$ is a more general *convergence space* (for example, $S^\infty(\mathbf{R})$), then the definition of the projective tensor product is more complicated, and we do not reproduce it here. Instead, we refer the reader to the book [5]. Here we only note that

$$S^\infty(\mathbf{R}^1)\hat{\otimes}\dots\hat{\otimes}S^\infty(\mathbf{R}^1) = S^\infty(\mathbf{R}^n),$$

where the symbol space $S^\infty(\mathbf{R}^n)$, which is the main symbol space used in this book, is defined as follows:

$$S^\infty(\mathbf{R}^n) = \bigcup_m S^m(\mathbf{R}^n),$$

and $S^m(\mathbf{R}^n)$ is the Fréchet space of smooth functions $f(x)$, $x \in \mathbf{R}^n$, with finite seminorms

$$||f||_{m,k} \sup_{s\in\mathbf{R}^n} \sum_{|\alpha|=k} |f^{(\alpha)}(x)|\,(1 + |x|)^{-m}.$$

Let us establish the main rules of "arithmetics" of Feynman indices. The proofs of all these rules are similar: for symbols belonging to the algebraic tensor product $\mathcal{F} \otimes \dots \otimes \mathcal{F}$, they are derived from the fact that the mapping $f \mapsto f(A_j)$ is a homomorphism for each A_j, and the passage to general symbols $f \in \mathcal{F}\hat{\otimes}\ldots\hat{\otimes}\mathcal{F}$ is carried out by continuity.

Rule 1 (index shifting).

$$f(\overset{j_1}{A_1}), \ldots, \overset{j_m}{A_m}) = f(\overset{k_1}{A_1}), \ldots, \overset{k_m}{A_m}),$$

if the sequences $j_1, \ldots, j_m$ and $k_1, \ldots, k_m$ are arranged on the real line in the same order.

With regard for this rule, in the following we allow arbitrary real numbers as Feynman indices.

Examples. $f(\overset{1}{A}, \overset{5}{B}) = f(\overset{2}{A}, \overset{4}{B}) = f(\overset{e}{A}, \overset{\pi}{B})$. But, in general,

$$\overset{0}{A}\overset{100}{B} = \overset{1}{A}\overset{2}{B} \neq \overset{2}{A}\overset{2}{B}.$$

Rule 2 (moving indices apart).

$$f(\ldots, \overset{j}{A}, \ldots, \overset{k}{A}, \ldots) = f(\ldots, \overset{j}{A}, \ldots, \overset{j}{A}, \ldots)$$

if none of the Feynman indices in $f(\ldots, \overset{j}{A}, \ldots, \overset{k}{A}, \ldots)$ except for the indices j and k themselves lies on the interval $[j, k]$ or $[k, j]$.

Examples. $f(\overset{1}{A}, \overset{2}{A})\overset{3}{B} = f(\overset{1}{A}, \overset{1}{A})\overset{3}{B} = f(\overset{1}{A}, \overset{1}{A}, \overset{2}{B})$, but

$$f(\overset{1}{A}, \overset{3}{A})\overset{2}{B} \neq f(\overset{1}{A}, \overset{1}{A})\overset{2}{B}$$

in the general case.

We point out that the notation $f(\ldots, \overset{j}{A}, \ldots, \overset{j}{A}, \ldots)$ is understood as follows: the arguments x and y of the function $f(\ldots, x, \ldots, y, \ldots)$ are identified, that is, the function is *restricted to the diagonal* $x = y$. The new symbol thus obtained has one argument less than f. Into this symbol, we substitute the same operators as into f, and the "identified" argument is replaced by the operator A with the Feynman index j. With regard for Rule 2, Rule 1 can be extended to include sequences of Feynman indices in which j_k may coincide with j_l provided that $A_k = A_l$.

Rule 3 (extraction of a factor). Let $j_1, \dots, j_s$ and $k_1, \dots, k_r$ be two sequences of Feynman indices, and suppose that there exists an interval $[a, b]$ such that

$$k_1, \dots, k_r \in [a, b]$$
$$j_1, \dots, j_s \notin [a, b].$$

Then

$$f(\overset{j_1}{A_1}, \dots, \overset{j_s}{A_s}) g(\overset{k_1}{B_1}, \dots, \overset{k_r}{B_r}) = f(\overset{j_1}{A_1}, \dots, \overset{j_s}{A_s}) \overset{k_1}{C}, \tag{1.10}$$

where

$$C = g(\overset{k_1}{B_1}, \dots, \overset{k_r}{B_r}).$$

We shall write such identities in a more compact form as

$$f(\overset{j_1}{A_1}, \dots, \overset{j_s}{A_s}) g(\overset{k_1}{B_1}, \dots, \overset{k_r}{B_r}) = f(\overset{j_1}{A_1}, \dots, \overset{j_s}{A_s}) \overset{k_1}{[\![} g(\overset{k_1}{B_1}, \dots, \overset{k_r}{B_r})]\!],$$

where the *autonomous brackets* $[\![\,]\!]$ introduced in [4] are interpreted as follows: first, one computes the expression in the brackets, and then it is used as a new operator in subsequent computations. If one needs to assign a Feynman index to this new operator, then the index is written over the left autonomous bracket.

Now let us prove Theorem 2 of the preceding section. Suppose that there are two distinct homomorphisms

$$\mu_1, \mu_2 : \mathcal{F} \to \operatorname{End} E$$

such that

$$\mu_1[x] = \mu_2[x] = A.$$

We can use either of them to construct functions of operators; we write

$$\mu_1(f) = f(A),$$
$$\mu_2(f) = f(C).$$

Thus, A is the same operator as C, but when constructing functions of these operators, we use the homomorphism μ_1 for A and μ_2 for C. In the new notation, we must show that

$$f(A) = f(C)$$

for any symbol $f \in \mathcal{F}$.

Let us carry out the computations using the above rules. We have

$$f(A) - f(C) = f(\overset{1}{A}) - f(\overset{2}{C}) = (f(\overset{1}{A}) - f(\overset{2}{C})\frac{\delta f}{\delta y}(\overset{1}{A}, \overset{2}{C}).$$

So far, we have only used the identical transformation of the symbol

$$f(x) - f(y) = \frac{\delta f}{\delta x}(x, y)(x - y).$$

Now we use the fact that $\frac{\delta f}{\delta x} \in \mathcal{F}\hat{\otimes}\mathcal{F}$ is an admissible symbol, and so

$$\frac{\delta f}{\delta x}(x, y)(z - \omega) \in \mathcal{F}\hat{\otimes}\mathcal{F}\hat{\otimes}\mathcal{F}\hat{\otimes}\mathcal{F}$$

is also an admissible symbol. Moving indices apart by Rule 2 and extracting a factor by Rule 3, we obtain

$$\begin{aligned} f(A) - f(C) &= (\overset{1}{A} - \overset{2}{C})\frac{\delta h}{\delta y}(\overset{0}{A}, \overset{4}{C}) \\ &= \overset{1}{[\![}\overset{1}{A} - \overset{2}{C}]\!]\frac{\delta f}{\delta y}(\overset{0}{A}, \overset{4}{C}) \\ &= \overset{1}{0}\ \frac{\delta f}{\delta y}(\overset{0}{A}, \overset{4}{C}) = 0, \end{aligned}$$

and the proof of the theorem is complete.

Our next theorem asserts that if two operators A and B commute, then $f(\overset{1}{A}, \overset{2}{B}) = f(\overset{2}{A}, \overset{1}{B})$. Needless to say, the same assertion remains valid if the symbol contains additional arguments that are replaced by operators whose Feynman indices do not lie on the interval $[1, 2]$. Moreover, the theorem gives an explicit expression for the difference $f(\overset{1}{A}, \overset{2}{B}) - f(\overset{2}{A}, \overset{1}{B})$ for the case in which $[A, B] \neq 0$.

Theorem 3 *Suppose that $f(x, y) \in \mathcal{F}\hat{\otimes}\mathcal{F}$ and $\mathcal{F}$ is a proper symbol space in the sense that $\frac{\delta f}{\delta x} \in \mathcal{F}\hat{\otimes}\mathcal{F}$ for each $f \in \mathcal{F}$. Then*

$$f(\overset{1}{A}, \overset{2}{B}) - f(\overset{2}{A}, \overset{1}{B}) = \overset{3}{[}\ A, B]\frac{\delta^2 f}{\delta x\, \delta y}(\overset{1}{A}, \overset{5}{A}; \overset{2}{B}, \overset{4}{B}).$$

Remark 1 The commutator $[A, B]$ is not an $\mathcal{F}$-generator in general. But this is not needed for the validity of the assertion of the theorem, since the symbol into which the commutator $[A, B]$ is substituted is *linear* in the corresponding argument.

Proof of Theorem 2. We have

$$\begin{aligned} f(\overset{1}{A}, \overset{2}{B}) - f(\overset{2}{A}, \overset{1}{B}) &= f(\overset{1}{A}, \overset{2}{B}) - f(\overset{3}{A}, \overset{2}{B}) \\ &= (\overset{1}{A} - \overset{3}{A})\frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A}; \overset{2}{B}) \\ &= (\overset{1}{A} - \overset{3}{A})\frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A}; \overset{4}{B}) + \overset{3}{A}\left(\frac{\delta f}{\delta x}(\overset{1}{A}, \overset{5}{A}; \overset{2}{B}) - \frac{\delta f}{\delta x}(\overset{1}{A}, \overset{5}{A}; \overset{4}{B})\right). \end{aligned}$$

Now the second term is equal to

$$\begin{aligned} \overset{3}{A}(\overset{2}{B} - \overset{4}{B})\frac{\delta^2 f}{\delta x\, \delta y}(\overset{1}{A}, \overset{5}{A}; \overset{2}{B}, \overset{4}{B}) &= \overset{3}{A}(\overset{2,5}{B} - \overset{3,5}{B})\frac{\delta^2 f}{\delta x\, \delta y}(\overset{1}{A}, \overset{5}{A}; \overset{2}{B}, \overset{4}{B}) \\ &= \overset{3}{[\![} \overset{3}{A}(\overset{2,5}{B} - \overset{3,5}{B})]\!]\frac{\delta^2 f}{\delta x\, \delta y}(\overset{1}{A}, \overset{5}{A}; \overset{2}{B}, \overset{4}{B}) = \overset{3}{[}A, B]\frac{\delta^2 f}{\delta x\, \delta y}(\overset{1}{A}, \overset{5}{A}; \overset{2}{B}, \overset{4}{B}), \end{aligned}$$

as desired.

With regard for Theorem 3, from now on we allow the same Feynman indices over two operators A and B not only if $A = B$ (which is covered by the rule for moving indices apart), but also if $[A, B] = 0$. By the above theorem, such an operator expression has a unique interpretation as follows: we use a "small perturbation" to make the Feynman indices over A and B distinct. The value of the resulting operator expression is independent of the details of the perturbation.

1.3 Main Formulas of Operator Calculus

In the preceding section, we defined functions of several noncommuting operators and proved their simplest properties. In this section, we obtain main formulas, which are most often used in computations involving functions of noncommuting operators.

Theorem 4 (the commutation formula) *One has*

$$[A, f(B)] = \overset{2}{[} A, B] \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{B}).$$

Remark 2 It is assumed here and in the subsequent formulas that all symbols occurring in these formulas belong to the corresponding symbol spaces and the operator arguments are generators in the corresponding classes; we do not indicate this explicitly each time.

The proof is by straightforward computation:

$$\begin{aligned}
[A, f(B)] &= \overset{2}{A} (f(\overset{1}{B}) - f(\overset{8}{B})) = \overset{2}{A} (\overset{1}{B} - \overset{3}{B}) \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{2}{B}) \\
&= \overset{2}{A} (\overset{1,5}{B} - \overset{2,5}{B}) \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{B}) \\
&= \overset{2}{[} A, B] \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{B}),
\end{aligned}$$

as desired.

This formula has a little more complicated version. Suppose that four operators A, B, C, and D satisfy the commutation relation

$$AB = CA + D. \tag{1.11}$$

Theorem 5 *If* (1.11) *holds, then*

$$A f(B) = f(C)A + \overset{2}{D} \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{C}). \tag{1.12}$$

Proof. We have

$$\begin{aligned}
A f(B) - f(C)A &= \overset{2}{A} (f(\overset{1}{B}) - f(\overset{3}{C})) = \overset{2}{A} (\overset{1}{B} - \overset{3}{C}) \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{C}) \\
&= \overset{2}{A} (\overset{1,5}{B} - \overset{2,5}{C}) \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{C}) = \overset{2}{D} \frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{C}),
\end{aligned}$$

as desired.

Theorem 6 (the Daletskii–Krein formula) *Let $A(t)$ be a family of operators smoothly depending on a parameter t. Then*

$$\frac{d}{dt}(f(A(t))) = \frac{d\overset{2}{A}(t)}{dt}\frac{\delta f}{\delta x}(\overset{1}{A}(t), \overset{3}{A}(t)).$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt}f(A(t)) &= \lim_{\Delta t\to 0}\frac{f(A)(t+\Delta t)) - f(A(t))}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{\overset{1}{A}(t+\Delta t) - \overset{3}{A}(t)}{\Delta t}\frac{\delta f}{\delta x}(\overset{1}{A(t+\Delta t)}, \overset{3}{A(t)}) \\
&= \lim_{\Delta t\to 0}\frac{\overset{1,5}{A}(t+\Delta t) - \overset{3,5}{A}(t)}{\Delta t}\frac{\delta f}{\delta x}(\overset{1}{A(t+\Delta t)}, \overset{3}{A(t)}) \\
&= \lim_{\Delta t\to 0}\overset{2}{[\![}\frac{A(t+\Delta t) - A(t)}{\Delta t}]\!]\frac{\delta f}{\delta x}(\overset{1}{A(t+\Delta t)}, \overset{3}{A(t)}) \\
&= \frac{d\overset{2}{A}(t)}{dt}\frac{\delta f}{\delta x}(\overset{1}{A(t)}, \overset{3}{A(t)}).
\end{aligned}$$

The proof is complete.

The Daletskii–Krein formula has a more general version, which we give without proof.

Theorem 7 *Let L be an arbitrary derivation of the algebra* End E *(that is,*

$$L(AB) = L(A)B + AL(B)$$

for any $A, B \in \operatorname{End} E$). Then

$$L(f(A)) = \overset{2}{L(A)}\,\frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A}).$$

Needless to say, the Daletskii–Krein formula is obtained as a special case of this more general formula if we replace the algebra End E by End $(C^\infty((0,1),E))$ and consider the derivation $L = \frac{d}{dt}$ of the latter.

Our next theorem deals with a typical situation of perturbation theory: how does the function $f(A)$ behave as the operator A is subjected to a perturbation, $A \Rightarrow A+\varepsilon B$, where $\varepsilon \to 0$ is the small parameter of the perturbation? For example, if $f(x) = x^{-1}$, and so

$$f(A+\varepsilon B) = (A+\varepsilon B)^{-1},$$

we try to compute the inverse of $A+\varepsilon B$ assuming that A^{-1}, the inverse of A, is known. Then, under suitable functional-analytic conditions, $(A+\varepsilon B)^{-1}$ is close to A^{-1}:

$$(A+\varepsilon B)^{-1} = A^{-1} + \varepsilon C_1 + \varepsilon' C_2 + \ldots;$$

the subject of perturbation theory is the computation of the corrections C_1, C_2, ...etc. This is of course well known (e.g., see [3]). Noncommutative analysis offers a new insight into the problem and, in particular, new expansions for the terms of the perturbation theory series. Thus, the problem is to expand $f(A+\varepsilon B)$ in an asymptotic power series in ε:

$$f(A+\varepsilon B) \cong \sum_{j=0}^{\infty} \varepsilon^j D_j; \tag{1.13}$$

the series is asymptotic in the sense that

$$f(A+\varepsilon B) - \sum_{j=0}^{N} \varepsilon^j D_j = O(\varepsilon^{N+1})$$

for each $N \geq 0$; we write $C(\varepsilon) = O(\varepsilon^k)$ is $C(\varepsilon)$ is an operator family such that $\varepsilon^{-k} C(\varepsilon)$ is a continuous operator family on $[0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ (in particular, its value at $\varepsilon = 0$ can be defined by continuity). Note that the coefficients D_0 and D_1 in (1.13) are already known:

$$D_0 = f(A)$$

(which is obvious, by setting $\varepsilon = 0$ on both sides in (1.13)), and

$$D_1 = \overset{2}{B} \frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A}),$$

which follows by applying the Daletskii–Krein formula (Theorem 7) with $A(t) = A + tB$.

Theorem 8 *Suppose that f is an element of a proper symbol class $\mathcal{F}$. (Recall that this means that $\frac{\delta f}{\delta x} \in \mathcal{F}\hat{\otimes}\mathcal{F}$ for any $A+\varepsilon B$ is an $\mathcal{F}$-generator for any $\varepsilon \in [0, \varepsilon_0)$). Then one has the asymptotic expansion*

$$f(A+\varepsilon B) \cong \sum_{j=0}^{\infty} \varepsilon^j \frac{\delta^j f}{\delta x^j}(\overset{1}{A}, \overset{3}{A}, \dots \overset{2j+1}{A}) \underbrace{\overset{2}{B}\overset{4}{B} \dots \overset{2j}{B}}_{j \text{ factors}}. \tag{1.14}$$

Moreover, one has the following explicit expression for the remainder in the series (1.14):

$$f(A+\varepsilon B) - \sum_{j=0}^{\infty} \varepsilon^j \frac{\delta^j f}{\delta x^j}(\overset{1}{A}, \overset{3}{A}, \dots \overset{2j+1}{A}) \; \overset{2}{B}\overset{4}{B} \dots \overset{2j}{B}$$

$$+ \varepsilon^N \frac{\delta^N j f}{\delta x^N}(\overset{1}{A}, \dots \overset{2N-1}{A}, \overset{2N+1}{A+\varepsilon B}) \; \overset{2}{B}\overset{4}{B} \dots \overset{2N}{B} \tag{1.15}$$

Proof. It suffices to prove formula (1.15), known in noncommutative analysis as the *Newton formula* [4]. We have

$$f(A+\varepsilon B) - f(A) = \varepsilon \; \overset{2}{B} \; \frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A+\varepsilon B}) \tag{1.16}$$

(the derivation of this equation is the same as in the proof of the Daletskii–Krein formula.) Thus we have proved (1.15) for $N = 1$. We proceed by induction over N. If (1.15) has already bean proved for some N, then we can represent the remainder on the right-hand side in the form

$$\begin{aligned}
&\varepsilon^N \frac{\delta^N f}{\delta x^N}(\overset{1}{A}, \overset{3}{A}, \dots, \overset{2N+1}{A+\varepsilon B}) \; \overset{2}{B}\overset{4}{B} \dots \overset{2N}{B} \\
&= \varepsilon^N \frac{\delta^N f}{\delta x^N}(\overset{1}{A}, \overset{3}{A}, \dots, \overset{2N+3}{A}) \; \overset{2}{B}\overset{4}{B} \dots \overset{2N}{B} \\
&+ \varepsilon^N \left\{ \frac{\delta^N f}{\delta x^N}(\overset{1}{A}, \dots, \overset{2N+1}{A+\varepsilon B}) - \frac{\delta^N f}{\delta x^N}(\overset{1}{A}, \dots, \overset{2N+3}{A}) \right\} \overset{2}{B}\overset{4}{B} \dots \overset{2N}{B} \\
&= \varepsilon^N \frac{\delta^N f}{\delta x^N}(\overset{1}{A}, \overset{3}{A}, \dots, \overset{2N+1}{A}) \; \overset{2}{B}\overset{4}{B} \dots \overset{2N}{B} \\
&\quad + \varepsilon^N \varepsilon \; \overset{2N+2}{B} \left(\frac{\delta}{\delta x} \frac{\delta^N f}{\delta x^N} \right) (\overset{1}{A}, \dots, \overset{2N+1}{A}, \overset{2N+3}{A+\varepsilon B}) \; \overset{2}{B}\overset{4}{B} \dots \overset{2N}{B}
\end{aligned}$$

$$= \varepsilon^N \frac{\delta^N f}{\delta x^N}(\overset{1}{A}, \overset{3}{A}, \ldots, \overset{2N+1}{A}) \overset{2}{B}\overset{4}{B} \ldots \overset{2N}{B}$$

$$+\varepsilon^{N+1} \frac{\delta^{N+1} f}{\delta x^{N+1}}(\overset{1}{A}, \ldots, \overset{2N+1}{A}, \overset{2N+1}{A+\varepsilon B}) \overset{2}{B}\overset{4}{B} \ldots \overset{2N+2}{B}$$

(we have transformed the term in braces in the same way as in (1.16)), which completes the inductive step. The proof of the theorem is complete.

Let us write out a few of the first terms in the expansion (1.14):

$$f(A+\varepsilon B) == f(A) + \varepsilon \overset{2}{B} \frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A}) + \varepsilon^2 \overset{2}{B}\overset{4}{B} \frac{\delta^2 f}{\delta x^2}(\overset{1}{A}, \overset{3}{A}, \overset{5}{A}) + \ldots .$$

Thus, each subsequent term in this expansion depends on more operator arguments than the preceding term. It would be desirable to simplify this formula further; this is possible if we assume that the commutators of A and B are in some sense "small." Note that if $[A, B] = 0$, then we can use the shifting indices rule and hence make all Feynman indices over all occurrences of A in this formula the same (the same, of course, pertains to B). Since the easy-to-verify formula

$$\frac{\delta^N f}{\delta x^N}(x, \ldots, x) = \frac{1}{N!} f^{(N)}(x)$$

holds, we see that for the case in which $[A, B] \equiv 0$, formula (1.14) is reduced to the usual Taylor expansion:

$$f(A+\varepsilon B) \cong \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(A) B^k.$$

Now we assume that the commutator of A and B is small:

$$[A, B] = O(h), \tag{1.17}$$

where h is a small parameter (this is a typical situation in quantum mechanics). Then we can use the commutation formula (Theorem 4) to shift the indices appropriately. For example,

$$\begin{aligned} B^2 \frac{\delta f}{\delta x}(\overset{1}{A}, \overset{3}{A}) &= \overset{2}{B} \frac{\delta f}{\delta x}(\overset{3}{A}, \overset{3}{A}) + \overset{2}{[} B, A] \frac{\delta^2 f}{\delta x^2}(\overset{1}{A}, \overset{3}{A}, \overset{3}{A}) \\ &= \overset{2}{B} \frac{\partial f}{\partial x}(\overset{3}{A}) + \overset{2}{[} B, A] \frac{\delta^2 f}{\delta x^2}(\overset{1}{A}, \overset{3}{A}, \overset{3}{A}), \end{aligned} \tag{1.18}$$

and, with regard for (1.17), we find that

$$f(A+\varepsilon B)-f(A)=\varepsilon\frac{\partial f}{\partial x}(A)B+O(h)+O(\varepsilon^2).$$

Now if $[[B,A],A]$ is also small (of the order of h), then we can apply the same transformation to the second term on the right-hand side in (1.18), thus obtaining

$$f(A+\varepsilon B)=f(A)+\varepsilon\frac{\partial f}{\partial x}(A)B+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(A)[B,A]+O(\varepsilon^2)+O(\varepsilon h)$$

(for the special case in which $\varepsilon=h$, the remainder is $O(h^2)$). In the semiclassical setting, the commutators usually satisfy

$$K(A,B)=O(h^{\operatorname{length}(K)}),$$

where length (K) is the *length* of the commutator K, defined inductively as follows:

$$\operatorname{length}(A)=\operatorname{length}(B)=0,$$
$$\operatorname{length}([K_1,K_2])=1+\operatorname{length}(K_1)+\operatorname{length}(K_2).$$

In this case one can obtain very nice expansions of $f(A+hB)$ for functions $f(x)$ that are independent of h or depend on h regularly. For details on computations of this sort, we refer the reader to [4].

We only note a different way for obtaining such expansions. If, instead of $f(A+\varepsilon B)$, we deal with $f(\overset{1}{A}+\varepsilon\overset{2}{B})$, then obtaining expansions in powers of ε is fairly easy:

$$f(\overset{1}{A}+\varepsilon\overset{2}{B})=\sum_{k=0}^{\infty}\frac{1}{k!}(\varepsilon)^k f^{(k)}(A).$$

Thus, the problem is to reduce $f(A+\varepsilon B)$ to $f(\overset{1}{A}+\varepsilon\overset{2}{B})$. This can be accomplished as follows:

$$f(A+\varepsilon B)-f(\overset{1}{A}+\varepsilon\overset{2}{B})=f(\overset{2}{[\![}\,A+\varepsilon B]\!])-f(\overset{1}{A}+\varepsilon\overset{3}{B})$$

$$= (\overset{2}{[\![} A + \varepsilon B)]\!] - \overset{1}{A} - \varepsilon \overset{2}{B}) \frac{\delta f}{\delta x} (\overset{2}{[\![} A + \varepsilon B]\!], \overset{1}{A} + \varepsilon \overset{2}{B})$$

$$= (\overset{2}{[\![} A + \varepsilon B]\!] - \overset{1}{A} - \varepsilon \overset{3}{B}) \frac{\delta f}{\delta x} (\overset{2}{[\![} A + \varepsilon B]\!], \overset{0}{A} + \varepsilon \overset{5}{B})$$

$$= (\overset{2}{[\![} A + \varepsilon B]\!] - \overset{1}{A} - \varepsilon \overset{3}{B}) \frac{\delta f}{\delta x} (\overset{4}{[\![} A + \varepsilon B]\!], \overset{0}{A} + \varepsilon \overset{5}{B})$$

$$- \varepsilon \overset{3}{[} B, A] \frac{\delta^2 f}{\delta x^2} (\overset{2}{[\![} A + \varepsilon B]\!], \overset{4}{[\![} A + \varepsilon B]\!], \overset{0}{A} + \varepsilon \overset{5}{B}).$$

Now by Rule 3 the first term on the right-hand side is zero, and we obtain

$$f(A+\varepsilon B) = f(\overset{1}{A} + \varepsilon \overset{2}{B}) + \varepsilon \overset{3}{[} A + B] \frac{\delta^2 f}{\delta x^2} (\overset{2}{[\![} A+\varepsilon B]\!], \overset{4}{[\![} A+\varepsilon B]\!], \overset{1}{A} + \varepsilon \overset{5}{B}).$$

If $\varepsilon = h$ and the commutators satisfy the above-cited smallness condition, then

$$f(A + hB) = f(\overset{1}{A} + h \overset{2}{B}) + \frac{h}{2} \overset{2}{[} A, B] \frac{\partial^2 f}{\partial x^2} (\overset{1}{A}) + O(h^2).$$

The subsequent terms in this expansion can be computed in terms of a relatively complicated diagram technique, for which we refer the reader to the book [2] and the literature cited there.

Generally, the computation tools of noncommutative analysis involve quite a lot of formulas, many of which can be found in the above-mentioned books and papers.

1.4 Main Tools of Noncommutative Analysis

In the preceding section we gave a number of formulas that prove useful in various applications of noncommutative analysis. Even more important are some general principals, tricks. and techniques specific to noncommutative analysis and forming the basis of numerous applications of it.

It is convenient to consider a slightly more general case than in the preceding sections. We assume that our "operators" are elements of some algebra $\mathcal{A}$, not necessarily represented by operators acting on some linear space; this algebra is assumed to be equipped with some notion of convergence. Next, we consider some space $\mathcal{F}$ of admissible symbols $f(x)$, which are functions of one variable x (depending on specific applications. x ranges over the real line or the complex plane). $\mathcal{F}$ is also assumed to be equipped with some notion of convergence, and we assume that $\mathcal{F}$ is proper in the following sense.

Definition 9 We say that the symbol space $\mathcal{F}$ is proper if

(1) $\mathcal{F}$ contains the space $\mathcal{P}$ of polynomials;

(2) $\mathcal{F}$ is an algebra with respect to (pointwise) multiplication of functions, and the multiplication is continuous;

(3) the difference derivative $\frac{\delta f}{\delta x}(x, y)$ of any symbol $f \in \mathcal{F}$ belongs to $\mathcal{F}\hat{\otimes}\mathcal{F}$. Moreover, the mapping

$$f \mapsto \frac{\delta f}{\delta x}$$

is continuous from $\mathcal{F}$ to $\mathcal{F}\hat{\otimes}\mathcal{F}$.

We recall that $\mathcal{F}\hat{\otimes}\mathcal{F}$ is the projective tensor product of $\mathcal{F}$ by itself, that is, the unique linear space such that any continuous bilinear map $\chi : \mathcal{F} \times \mathcal{F} \to E$ into a space E with convergence can be factored through $\mathcal{F}\hat{\otimes}\mathcal{F}$ in a unique way:

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{F} & \stackrel{\chi}{\to} & E \\ i \searrow & & \nearrow \widetilde{\chi} \\ & \mathcal{F}\hat{\otimes}\mathcal{F} & \end{array} .$$

In this diagram, $i : \mathcal{F} \times \mathcal{F} \to \mathcal{F}\hat{\otimes}\mathcal{F}$ is the canonical bilinear mapping, and $\widetilde{\chi}$ is the unique continuous mapping making this diagram commutative.

We do not touch the problem of the *existence* of $\mathcal{F}\hat{\otimes}\mathcal{F}$; cf. [5]. We only mention that $\mathcal{F}\hat{\otimes}\mathcal{F}$ can be obtained as the completion of the usual (algebraic) tensor product $\mathcal{F} \otimes \mathcal{F}$ in the strongest convergence on $\mathcal{F} \otimes \mathcal{F}$ in which the induced map $\tilde{\chi}$ is continuous. The elements of $\mathcal{F} \otimes \mathcal{F}$ can be thought of as functions of two variables of the form

$$f(x, y) = \sum f_j(x) g_j(y)$$

(the sum is finite). Hence we obtain the interpretation of $\mathcal{F} \hat{\otimes} \mathcal{F}$ (after the completion) as functions of two variables.

(Of course, one needs to prove that, say, the embedding $\mathcal{F} \otimes \mathcal{F} \subset C^\infty(\mathbf{R}^2)$ extends by continuity to an embedding $\mathcal{F} \hat{\otimes} \mathcal{F} \subset C^\infty(\mathbf{R}^2)$.)

Now we define an $\mathcal{F}$-generator in $\mathcal{A}$ as an element $A \in \mathcal{A}$ such that there exists a homomorphism

$$\mu = \mu_A : \mathcal{F} \to \mathcal{A}$$

of algebras such that

$$\mu_A(p) = p(A) \tag{1.19}$$

for each polynomial $p \in \mathcal{P}$ (needless to say, it suffices to require this property for linear functions $p(x)$).

In this situation, all theorems of the preceding sections still apply, including the uniqueness Theorem 2.

By $\mathcal{A}_{\mathcal{F}} \subset \mathcal{A}$ we denote the set of $\mathcal{F}$-generators in $\mathcal{A}$.

Our immediate aim is to study how $\mathcal{F}$-generators in $\mathcal{A}$ are related to $\mathcal{F}$-generators in some algebras associated with $\mathcal{A}$.

For an algebra $\mathcal{A}$ with convergence, we consider the following algebras with convergence, associated with $\mathcal{A}$:

(i) The algebra End $(\mathcal{A})$ of all continuous linear operators acting on the linear space $\mathcal{A}$ (the algebra structure of $\mathcal{A}$ is ignored).

(ii) The algebra $\mathrm{Mat}_n(\mathcal{A})$ of $n \times n$ matrices whose entries are elements of $\mathcal{A}$.

We shall study some specific classes of $\mathcal{F}$-generators in these algebras. Let us start from End $(\mathcal{A})$. We define the *left regular representation*

$$\begin{aligned} L : \mathcal{A} &\to \mathrm{End}\,(\mathcal{A}), \\ A &\mapsto L_A, \end{aligned}$$

where L_A is given by the formula

$$L_A(B) = AB, \quad B \in \mathcal{A},$$

and the *right regular representation*, which is in fact an antirepresentation,

$$\begin{aligned} R : \mathcal{A} &\to \operatorname{End}(\mathcal{A}), \\ A &\mapsto R_A, \\ R_A(B) &= BA, \quad B \in \mathcal{A}. \end{aligned}$$

The difference

$$ad = L - R$$

is called the *adjoint representation* and acts by taking commutators:

$$ad_A(B) = [A, B].$$

By virtue of the *Jacobi identity*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$$

for commutators, the adjoint representation of any element of $\mathcal{A}$ proves to be a derivation of $\mathcal{A}$:

$$ad_A(BC) = ad_A(B)C + B ad_A(C)$$

for any $B, C \in \mathcal{A}$.

Thus, we take some element $A \in \mathcal{A}_\mathcal{F}$, and the problem is, what can we say about L_A, R_A and ad_A?

Theorem 10 (i) *The mapping L is a one-to-one correspondence between $\mathcal{A}_\mathcal{F}$ and $\operatorname{End}(\mathcal{F})_\mathcal{F} \cap \mathcal{R}$, where $\mathcal{R} \subset \operatorname{End}(\mathcal{A})$ is the set of right-invariant endomorphisms:*

$$\begin{aligned} \varphi \in \mathcal{R} \quad &\Leftrightarrow \quad \varphi(AB) = \varphi(A)B \quad \forall A, B \in \mathcal{A} \\ &\Leftrightarrow \quad \varphi = L_A \text{ for some } A \in \mathcal{A}. \end{aligned}$$

(ii) *The mapping R is a one-to-one correspondence between $\mathcal{A}_{\mathcal{F}}$ and* $\operatorname{End}(\mathcal{A})_{\mathcal{F}} \cap \mathcal{L}$, *where $\mathcal{L} \subset \operatorname{End}(\mathcal{A})$ is the set of left-invariant endomorphisms:*

$$\begin{aligned} \varphi \in \mathcal{L} \quad &\Leftrightarrow \quad \varphi(BA) = B\varphi(A) \quad \forall A, B \in \mathcal{A} \\ &\Leftrightarrow \quad \varphi = R_A \text{ for some } A \in \mathcal{A}. \end{aligned}$$

(iii) *$ad(\mathcal{A}_{\mathcal{F}}) \subset \operatorname{End}(\mathcal{A})_{\mathcal{F}}$ provided that $\mathcal{F}$ has the following* invariance property: *for any $\lambda, \mu \in \mathbf{R}$ (resp., $\mathbf{C}$), the mapping*

$$f(x) \mapsto f(\lambda x + \mu y)$$

is a continuous mapping of $\mathcal{F}$ into $\mathcal{F}\hat{\otimes}\mathcal{F}$. Moreover, this mapping continuously depends on λ and μ.

Proof. Items (i) and (ii) are completely similar, and we shall only prove (i). Since $\mathcal{A}$ is an algebra with unit, the mapping L has a trivial kernel, and we only need to prove the implications

$$A \in \mathcal{A}_{\mathcal{F}} \Leftrightarrow L_A \in \operatorname{End}(\mathcal{A})_{\mathcal{F}}.$$

Let us first prove ($\Rightarrow$). Let $A \in \mathcal{A}_{\mathcal{F}}$. We set

$$f(L_A)B \stackrel{\text{def}}{=} f(A)B$$

for every $B \in \mathcal{A}$. If $p \in \mathcal{P}$ is a polynomial, then

$$f(L_A)B = f(A)B = p(A)B = L_{p(A)}B = p(L_A)B,$$

since L is a homomorphism of algebras. Next,

$$f(L_A)(g(L_A)(B)) = f(A)g(A)B = [fg](A)B = [fg](L_A)(B),$$

and we see that $f \mapsto f(L_A)$ is a homomorphism. Next, let us prove ($\Leftarrow$). Suppose that $L_A \in \operatorname{End}(\mathcal{A})_{\mathcal{F}}$. For each $f \in \mathcal{F}$, consider the endomorphism

$$f(L_A) : \mathcal{A} \to \mathcal{A}.$$

We claim that $f(L_A) \subset \mathcal{R}$. Indeed, for any $B \in A$ one has

$$[L_A, R_B] = 0.$$

By the commutation formula,

$$[f(L_A), R_B] = \overset{2}{[}\, L_A, R_B] \frac{\delta f}{\delta x}(\overset{1}{L_A}, \overset{3}{L_A}) = 0,$$

which just means that $f(L_A) \in \mathcal{R}$. It follows that there exists a unique element $C \in \mathcal{A}$ such that $f(L_A) = L_C$. We set, by definition,

$$f(A) \stackrel{\text{def}}{=} C.$$

Since L is monomorphic, we readily find that

$$f(A)\delta(A) = (fg)(A).$$

Next, it can be proved that

$$L^{-1} : \mathcal{R} \to \mathcal{A}$$

is continuous, and hence we see that the mapping $f \mapsto f(A)$, being the composition of μ_{L_A} with L^{-1}, is continuous as well.

Now let us prove (ii). Let $A \in \mathcal{A}_{\mathcal{F}}$. For any $f \in \mathcal{F}$, $f(x-y) \in \mathcal{F}\hat{\otimes}\mathcal{F}$ by virtue of the invariance condition, and we set

$$f(ad_A)B \stackrel{\text{def}}{=} f(\overset{3}{A} - \overset{1}{A})\ \overset{2}{B}$$

for any $B \in \mathcal{A}$. Then

$$\begin{aligned} f(ad_A)g(ad_A)B &= f(\overset{3}{A} - \overset{1}{A}) \overset{2}{[\![}\, g(\overset{3}{A} - \overset{1}{A})\ \overset{2}{B}]\!] \\ &= f(\overset{5}{A} - \overset{0}{A}) g(\overset{3}{A} - \overset{1}{A})\ \overset{2}{B} \\ &= f(\overset{3}{A} - \overset{1}{A}) g(\overset{3}{A} - \overset{1}{A})\ \overset{2}{B} = [fg](ad_A)(B), \end{aligned}$$

and remains to verify that

$$p(\overset{3}{A} - \overset{1}{A})\ \overset{2}{B} = p(ad_A)\ \overset{2}{B}$$

for any *polynomial* $p(x)$. In view of the above, it suffices to prove this for $p(x) = x$, which is tautological:

$$(\overset{3}{A} - \overset{1}{A})\ \overset{2}{B} = AB - BA = [A, B] = ad_A(B).$$

The proof of the theorem is complete.

Remark 3 In a similar way, one can define functions of every linear combination $\lambda L_A + \mu R_A$; however, this is particularly useful for the case $\lambda = -\mu = 1$, considered above, and the case $\lambda = \mu = 1$, where we deal with the anticommutation operator

$$an_A = L_A + R_A.$$

Functions of an_A are defined by the similar formula

$$f(an_A)(B) = f(\overset{1}{A} + \overset{3}{A})\,\overset{2}{B}, \quad B \in A.$$

This operator proves useful if one deals with operators satisfying anti-commutation relations, which is quite common in quantum field theory.

Before proceeding to the matrix algebras Mat $_n(\mathcal{A})$, let us prove the following important theorem.

Theorem 11 *The sets $\mathcal{A}_{\mathcal{F}}$ are compatible with algebra homomorphisms. More precisely, if $\mathcal{A}$ and $\mathcal{B}$ are two algebras with convergence and*

$$\kappa : \mathcal{A} \to \mathcal{B}$$

is a continuous homomorphism of algebras, then

$$\kappa(\mathcal{A}_{\mathcal{F}}) \subset \mathcal{B}_{\mathcal{F}}, \tag{1.20}$$

and moreover,

$$\kappa(f(A)) = f(\kappa(A)) \tag{1.21}$$

for any $f \in \mathcal{F}$ and $A \in \mathcal{A}_{\mathcal{F}}$.

The *proof* is trivial; one just takes (1.21) as the *definition* of $f(\kappa(A))$. (A special case of this argument has already been used in the proof of Theorem 10.)

In particular, the assertion of Theorem 11 remains valid for the special case of inner automorphisms of the algebra $\mathcal{A}$ itself, and we obtain the following important corollary.

Corollary 12 *Let $A \in \mathcal{A}_{\mathcal{F}}$, and let $U \in \mathcal{A}$ be invertible. Then $U^{-1}AU \in \mathcal{A}_{\mathcal{F}}$ and*

$$f(U^{-1}AU) = U^{-1}f(A)U$$

for every $f \in \mathcal{F}$.

Now we focus our attention on $\mathcal{F}$-generators in matrix algebras. We shall consider only the case $n = 2$, that is, the algebra $\mathrm{Mat}_2(\mathcal{A})$ of 2×2 block matrices with entries elements of $\mathcal{A}$:

$$a = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \mathcal{A}.$$

Theorem 13 *Let $A \in \mathcal{A}_{\mathcal{F}}$, $B \in \mathcal{A}$, and $[A, B] = 0$. Then*

$$a = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{A})_{\mathcal{F}}.$$

Proof. We need to construct a homomorphism

$$\mu_a : \mathcal{F} \to \mathrm{Mat}_2(\mathcal{A})$$

such that

$$\mu_a(p) = p(a)$$

for any polynomial $p(x)$. To this end, we set

$$\mu_a(f) = \begin{pmatrix} f(A) & f'(A)B \\ 0 & f(A) \end{pmatrix}. \tag{1.22}$$

Let us prove that (1.22) is a homomorphism:

$$\begin{pmatrix} f(A) & f'(A)B \\ 0 & f(A) \end{pmatrix} \begin{pmatrix} g(A) & g'(A)B \\ 0 & g(A) \end{pmatrix}$$
$$= \begin{pmatrix} f(A)g(A) & f(A)g'(A)B + f'(A)Bg(A) \\ 0 & f(A)g(A) \end{pmatrix}$$
$$= \begin{pmatrix} (f_g)(A) & (fg)'(A)B \\ 0 & (fg)(A) \end{pmatrix}.$$

(Here we have essentially used the fact that B commutes with A and hence with $g(A)$.)

Next, if $f(x) = x$, then (1.22) gives

$$\mu_a(f) = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = a,$$

which completes the proof of the theorem.

Theorem 14 *If $A_1, A_2 \in \mathcal{A}_\mathcal{F}$, then*

$$A_1 \oplus A_2 \equiv \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{A})_\mathcal{F}.$$

Proof. $f(A_1 \oplus A_2) = f(A_1) \oplus f(A_2)$.

Strange as it may seem, we can combine Theorems 13 and 14 and obtain a result involving no commutativity requirement like that in Theorem 13.

Theorem 15 *Suppose that $A, B \in \mathcal{A}_\mathcal{F}$ and $C \in \mathcal{A}$. Then*

$$a = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{A})_\mathcal{F}.$$

Proof. For every $f \in \mathcal{F}$, we set

$$f(a) = \begin{pmatrix} f(A) & \frac{\delta f}{\delta x}(\overset{3}{A}, \overset{1}{B})\,\overset{2}{C} \\ 0 & f(B) \end{pmatrix}. \tag{1.23}$$

Then $f(a) = a$ for $f(x) \equiv x$, and moreover,

$$\begin{aligned} f(a)g(a) &= \begin{pmatrix} f(A) & \dfrac{\delta f}{\delta x}(\overset{3}{A}, \overset{1}{B})\,\overset{2}{C} \\ 0 & f(B) \end{pmatrix} \begin{pmatrix} g(A) & \dfrac{\delta g}{\delta x}(\overset{3}{A}, \overset{1}{B})\,\overset{2}{C} \\ 0 & g(B) \end{pmatrix} \\ &= \begin{pmatrix} f(A)g(A) & \dfrac{\delta f}{\delta x}(\overset{3}{A}, \overset{1}{B})g(\overset{1}{B})\,\overset{2}{C} + f(\overset{3}{A})\dfrac{\delta g}{\delta x}(\overset{3}{A}, \overset{1}{B})\,\overset{2}{C} \\ 0 & f(B)g(B) \end{pmatrix}. \end{aligned}$$

Now

$$f(A)g(A) = [fg](A), \quad f(B)g(B) = [fg](B),$$

and, using the identity

$$\frac{\delta}{\delta x}[fg](x,y) = \frac{\delta f}{\delta x}(x,y)g(y) + f(x)\frac{\delta g}{\delta x}(x,y),$$

we find that the off-diagonal entry in the last matrix is equal to

$$\frac{\delta(fg)}{\delta x}(\overset{2}{A},\overset{1}{B})\overset{2}{C}\,.$$

Consequently,

$$f(a)g(a) = (fg)(a),$$

and the proof of the theorem is complete.

Various identities that can be obtained readily in End $(\mathcal{A})$ and $\mathrm{Mat}_n(\mathcal{A})$ (examples are given in the above theorems) can often be used to prove important facts in the algebra $\mathcal{A}$ itself. For example, let us show how Theorem 15 can be used to prove the commutation formulas of Theorems 4 and 5. First, let us consider the assertion of Theorem 4. We consider the operator

$$a = \begin{pmatrix} B & [A,B] \\ 0 & B \end{pmatrix}.$$

Lemma 16

$$f(a) = \begin{pmatrix} f(B) & [A,f(B)] \\ 0 & f(B) \end{pmatrix}. \tag{1.24}$$

Indeed, for $f(x) \equiv x$ we get $f(a) = a$, and moreover,

$$\begin{aligned} f(a)g(a) &= \begin{pmatrix} f(B)g(B) & f(B)[A,g(B)] + [A,f(B)]g(B) \\ 0 & f(B)g(B) \end{pmatrix} \\ &= \begin{pmatrix} f(B)g(B) & [A,f(B)g(B)] \\ 0 & f(B)g(B) \end{pmatrix}; \end{aligned}$$

hence f defined in (1.24) is indeed a homomorphism and gives $f(a)$ by definition.

On the other hand, we can apply Theorem 15 to the element a. By comparing the off-diagonal entries in the two expressions for $f(A)$, we find that

$$[A, f(B)] = [\overset{2}{A}, B]\frac{\delta f}{\delta x}(\overset{1}{B}, \overset{3}{B}),$$

that is, the assertion of Theorem 4 holds.

To obtain the assertion of Theorem 5, consider the operator

$$a = \begin{pmatrix} C & D \\ 0 & B \end{pmatrix},$$

where

$$D = AB - CA.$$

Lemma 17

$$f(a) = \begin{pmatrix} f(C) & Af(B) - f(C)A \\ 0 & f(B) \end{pmatrix}. \tag{1.25}$$

Indeed, for $f(x) \equiv x$ we again have $f(a) = a$. Next,

$$\begin{aligned} & f(a)g(a) \\ &= \begin{pmatrix} f(C)g(C) & f(C)(Ag(B) - g(C)A) + (Af(B) - f(C)A)g(B) \\ 0 & f(B)g(B) \end{pmatrix} \\ &= \begin{pmatrix} [fg(C) & A[fg](B) - [fg](C)A \\ 0 & [fg](B) \end{pmatrix} = (fg)(a), \end{aligned}$$

as desired. Applying Theorem 15 to $f(a)$ and comparing off-diagonal entries, we obtain

$$Af(B) - f(C)A = \frac{\delta f}{\delta x}(\overset{3}{C}, \overset{1}{B})\, \overset{2}{D},$$

which coincides with the assertion of Theorem 5.

Let us now prove a more general version of theorem 15, which will prove useful in the next section. Suppose that $E = E_1 \oplus E_2$ is a linear space with convergence represented as the sum of two subspaces; then End (E) can be represented as the 2×2 block matrix

$$\operatorname{end}(E) = \begin{pmatrix} \operatorname{End}(E_1) & \operatorname{Hom}(E_2, E_1) \\ \operatorname{Hom}(E_1, E_2) & \operatorname{End}(E_2) \end{pmatrix};$$

this means that any element $a \in \operatorname{End}(E)$ can be represented by a block matrix of the form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $a_{11} \in \operatorname{End}(E_1)$, $a_{12} \in \operatorname{Hom}(E_2, E_1)$, etc. and the product of two elements can be computed by the usual matrix product rule.

Theorem 18 *Let*

$$a = \begin{pmatrix} C & D \\ 0 & B \end{pmatrix} \in \operatorname{End}(E),$$

and let $C \in \operatorname{End}(E_1)_{\mathcal{F}}$ *and* $B \in \operatorname{End}(E_2)_{\mathcal{F}}$. *Then* $a \in \operatorname{End}(E)_{\mathcal{F}}$, *and*

$$f(a) = \begin{pmatrix} f(C) & \overset{2}{D} \frac{\delta f}{\delta x}(\overset{3}{C}, \overset{1}{B}) \\ 0 & f(B) \end{pmatrix}.$$

The *proof* reproduces that of Theorem 15 word for word.

Corollary 19 *Suppose that* $C \in \operatorname{End}(E_1)_{\mathcal{F}}$, $B \in \operatorname{End}(E_2)_{\mathcal{F}}$, *and*

$$\gamma : E_2 \to E_1$$

is a linear operator such that

$$\gamma B = C\gamma. \tag{1.26}$$

(In this case, one says that γ *is intertwining operator for the pair* (B, C)*.) Then* γ *is an intertwining operator for the pair* $(f(B), f(C))$ *with an arbitrary function* $f \in \mathcal{F}$:

$$\gamma f(B) = f(C)\gamma. \tag{1.27}$$

Proof. Without assuming (1.26) momentarily, in the assertion of Theorem 18 we take

$$D = \gamma B - C\gamma.$$

We claim that then

$$f(D) = \begin{pmatrix} f(C) & \gamma f(B) - f(C)\gamma \\ 0 & f(B) \end{pmatrix} \tag{1.28}$$

(the argument is the same as in the above proof of Theorem 5). Comparing this with the assertion of Theorem 18, we obtain

$$\gamma f(B) - f(C)\gamma = \overset{2}{[\![} \gamma B - C\gamma]\!] \frac{\delta f}{\delta x}(\overset{3}{C}, \overset{1}{B}).$$

Now it remains to use (1.26), and we arrive at the desired relation (1.27).

So far, we have stated our assertions in this section for functions of a single operator. Appropriate analogs of these assertions hold for functions of several operators. Let us give a list of the most common formulas:

(i) $U^{-1} f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n) U = f(\overset{1}{[\![} U^{-1} A_1 U]\!], \ldots, \overset{n}{[\![} U^{-1} A_n U]\!]);$

(ii) if $\gamma B_j = C_j \gamma$ for $j = 1, \ldots, n$, then

$$\gamma \cdot f(\overset{1}{B}_1, \ldots, \overset{n}{B}_n) = f(\overset{1}{C}_1, \ldots, \overset{n}{C}_n) \cdot \gamma;$$

(iii) $f(\overset{1}{L}_{A_1}, \ldots, \overset{n}{L}_{A_n}) = L_{f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)};$

(iv) $f(\overset{1}{R}_{A_1}, \ldots, \overset{n}{R}_{A_n}) = R_{f(\overset{n}{A}_1, \ldots, \overset{1}{A}_n)}$ (note the reverse order of indices in the last formula, owing to the fact that R is an antihomomorphism);

(v) $f(\overset{1}{ad}_{A_1}, \ldots, \overset{n}{ad}_{A_n})(C) = f(\overset{1}{A}_1 - \overset{-1}{A}_1, \ldots, \overset{n}{A}_n - \overset{-n}{A}_n)\, \overset{0}{C};$

(vi) if $\chi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of algebras, then

$$\chi[f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)] = f(\chi(\overset{1}{A}_1), \ldots, \chi(\overset{n}{A}_n)).$$

The reader can readily reconstruct the assumptions (omitted here and quite similar to those for the case in which $f \in \mathcal{F}$) under which these assertions hold. We only mention that in our case $f \in \mathcal{F}_n = \underbrace{\mathcal{F}\hat{\otimes}\dots\hat{\otimes}\mathcal{F}}_{n \text{ copies}}$.

1.5 Composition Laws and Ordered Representations

The technique of noncommutative analysis, developed in the preceding chapter, readily permits us to quantize a classical system once we know the quantization of "basic" variables. In other words, if any dynamic variable in the classical system has the form $f(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ is the set of basic variables (in the simplest situation, $n = 2m$, where m is the number of degrees of freedom, and $(x_1, \dots, x_{2m}) = (q_1, \dots, q_m, p_1, \dots, p_m)$ is the set of coordinates and momenta), and if we know that the basic variables $x_1, \dots, x_n$ are represented in the quantum setting by some operators $A_1, \dots, A_n$, then to each dynamic variable $f = f(x_1, \dots, x_n)$ we assign the quantum operator

$$\hat{f} = f(\overset{1}{A_1}, \dots, \overset{n}{A_n}). \tag{1.29}$$

Needless to say, we must take care that the classical variable f belongs to $\mathcal{F}_n = \mathcal{F}\hat{\otimes}\dots\hat{\otimes}\mathcal{F}$, where $\mathcal{F}$ is such that the A_j are $\mathcal{F}$-generators.

The choice of a different ordering of operators in (1.29) makes little difference from the semiclassical point of view, since all commutators $[A_j, A_k]$ in quantum mechanics use to be $O(h)$, and so it follows from the index permutation theorem (theorem 3 in § 1.2) that the new definition will differ from (1.29) by $O(h)$. One of the most important properties of quantum systems is that their dynamic variables, like those of the corresponding classical systems, form algebras, i.e., the set of dynamic variables is closed with respect to the operators of addition, multiplication, and multiplication by scalars. While addition and multiplication by scalars are reflected by the very same operations at the classical level, multiplication is not:

$$\hat{f}\hat{g} \neq \widehat{fg}.$$

And so we face the problem of computing the symbol of the product $\hat{f}\hat{g}$, i.e., the classical dynamic variable h such that

$$\hat{h} = \hat{f}\hat{g},$$

in terms of f and g. This computation is of interest both in quantum mechanics (where it describes the composition law in the algebra of quantum observables) and in the theory of differential equations (where it reduces the task of computing the inverse A^{-1} of a differential operator $A = f\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right) \equiv \hat{f}$ to solving an equation for the symbol of the inverse).

In the next chapter we consider some situations in which the composition can be computed in a closed form explicitly. Chapter 3 deals with the cases in which the composition law can be computed only asymptotically (or even exists only in the asymptotic sense). However, first of all we have to establish some general conditions under which the composition law is possible at all and reveal the structure of the composition law. This is carried out in this section.

Thus we deal with the following situation. Let $\mathcal{A}$ be an algebra with convergence, and let some elements

$$A_1, \ldots, A_n \in \mathcal{A}_{\mathcal{F}}$$

be given, where $\mathcal{F}$ is a proper class of symbols in the sense of Definition 9 § 1.4. We consider the subspace (not necessarily closed)

$$\Lambda = \{B \in \mathcal{A} \mid B = f(\overset{1}{A_1}, \ldots, \overset{n}{A_n}), f \in \mathcal{F}_n\}$$

of $\mathcal{A}$ formed by operators B representable in the form

$$B = f(\overset{1}{A_1}, \ldots, \overset{n}{A_n})$$

with *given* ordering $\overset{1}{A_1}, \ldots, \overset{n}{A_n}$ of the operators $A_1, \ldots, A_n$ and with symbols $f \in \mathcal{F}\hat{\otimes} \ldots \hat{\otimes}\mathcal{F}$. The main questions we are dealing with in this section are as follows:

- Is $\Lambda \subset \mathcal{A}$ a subalgebra[2]?

[2]Not necessarily closed.

- If Λ is a subalgebra, how can one compute the product in Λ in terms of symbols?

In the following, for brevity we write $\hat{f}$ or $f(A)$ instead of $f(A_1, \dots, A_n)$, so that A is understood as the n-tuple $A = (\overset{1}{A}_1, \dots, \overset{n}{A}_n)$.

First, let us establish some *necessary* conditions for Λ to be a subalgebra. The following lemma is obvious from the definitions.

Lemma 20 *If Λ is a subalgebra, then there exist functions $f_{jk} \in \mathcal{F}_n$, $j, k = 1, \dots, n$, such that*

$$A_j A_k = f_{jk}(A). \tag{1.30}$$

(Needless to say, for $j \geq k$ one can take $f_{jk}(x) = x_j x_k$.)

Thus, the operators $A_1, \dots, A_n$ must satisfy a system of relations of the form (1.30). Such systems will be referred to as *general commutation relations*.

The condition given in Lemma 20 is not sufficient. We have given this lemma for the only reason that commutation relations are usually a starting point for constructing the composition law and proving that Λ is an algebra. In subsequent chapters we indicate special classes of relations which do imply that Λ is an algebra.

In fact, if Λ is an algebra, then a stronger condition holds. Consider the product $A_j \circ f(A)$, where $f \in \mathcal{F}_n$ is arbitrary.

If λ is an algebra, than $A_j \circ F(A) \in \Lambda$, that is.

$$A_j \circ f(A) = g(A) \tag{1.31}$$

for some $g \in \mathcal{F}_n$. For given j, formula (1.31) specifies a well-defined mapping

$$\widetilde{l}_j : \mathcal{F}_n \to \mathcal{F}_n / \operatorname{Ker} \mu, \tag{1.32}$$

where μ is the mapping $f \mapsto f(A)$. We suppose that the mapping (1.32) can be lifted to a continuous linear mapping

$$l_j : \mathcal{F}_n \to \mathcal{F}_n. \tag{1.33}$$

The n-tuple $l = (l_1, \dots, l_n)$ of operators l_j acting on $\mathcal{F}_n$ is called the *left ordered representation* of the n-tuple $A = (\overset{1}{A}_1, \dots, \overset{n}{A}_n)$. We note that the left ordered representation depends on the ordering, not only on the operators $A_1, \ldots, A_n$ themselves. (In [5] an example is given of two operators A, B such that the ordered representation exists for the pair $\overset{1}{A}, \overset{2}{B}$ and does not exist for the pair $\overset{2}{A}, \overset{1}{B}$).

It turns out that the existence of a left regular representation is almost sufficient for Λ to be an algebra. More precisely, the following theorem is valid

Theorem 21 *Suppose that the tuple* $A = (\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ *has a left ordered representation, i.e., a tuple* $l = (\overset{1}{l}_1, \ldots, \overset{n}{l}_n)$ *of operators* $l_j : \mathcal{F}_n \to \mathcal{F}_n$ *such that*

$$(l_j f)(A) = A_j \circ f(A), \quad j = 1, \ldots, n, \tag{1.34}$$

for each symbol $f \in \mathcal{F}_n$. *Then* Λ *is a left module over*

$$\mathcal{P} = \{B \in \mathcal{A} \,|\, B = p(A) \textit{ for some polynomial } p\}.$$

The *proof* is easy. We represent a polynomial $p(x) = \sum a_\alpha x^\alpha$ as a sum of monomials and prove the desired assertion for each of the monomials by induction on the order of the monomial, using the operators l_j.

However, this theorem, though sufficient in a number of applications, is as weak as it is easy. Fortunately, there is a stronger theorem, which tells us under what conditions Λ is not only a $\mathcal{P}$-module, but an algebra as well. This theorem involves an additional condition, a very natural one, imposed on the left ordered representation.

Theorem 22 *Let* $A = (\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ *be an* n*-tuple of* $\mathcal{F}$*-generators in* $\mathcal{A}$. *Suppose that* A *has a left ordered representation* $l = (l_1, \ldots, l_n)$ *such that each* l_j *is an* $\mathcal{F}$*-generator in* $\mathcal{F}_n$. *Then*

$$\Lambda = \{f(A) \,|\, f \in \mathcal{F}_n\}$$

is a subalgebra of $\mathcal{A}$ *(not necessarily closed). The composition law in* Λ *is given by the formula*

$$\begin{aligned} &f(A)g(A) = h(A), \\ &h(x) = [f(l)g](x), \quad f, g \in \mathcal{F}_n. \end{aligned} \tag{1.35}$$

Thus, to obtain the symbol of the product of two operators with symbols in $\mathcal{F}_n$, we must just substitute the left ordered representation into the first symbol and act with the operator thus obtained on the second symbol. This is possible since the l_j are $\mathcal{F}$-generators and so $f(l)$ is a well-defined operator in $\mathcal{F}_n$ for $f \in \mathcal{F}_n$. As a result, the symbol space $\mathcal{F}_n$ is equipped with a binary operation, which will be denoted by $*$:

$$f * g \stackrel{\text{def}}{=} f(l)g. \tag{1.36}$$

This operation is obviously bilinear (but not necessarily associative; associativity and related issues will be discussed later on in this section). In terms of this "twisted multiplication" (1.36), Theorem 22 can be stated as follows: *the mapping*

$$\begin{aligned} \mu : \mathcal{F}_n &\to \mathcal{A} \\ f &\mapsto f(A) \end{aligned} \tag{1.37}$$

is a "homomorphism", that is, μ is linear and

$$\mu(f * g) = \mu(f)\,\mu(g). \tag{1.38}$$

(On the right-hand side in (1.38), we have the usual multiplication in $\mathcal{A}$.)

Let us proceed to the proof of Theorem 22.

Proof of Theorem 22. Condition (1.34), which express the fact that $l = (l_1, \ldots, l_n)$ is a left regular representation of A, can be put in the following form: the diagrams

$$\begin{array}{ccc} \mathcal{F}_n & \xrightarrow{\ \mu\ } & \mathcal{A} \\ {\scriptstyle l_j}\downarrow & & \downarrow{\scriptstyle L_{A_j}} \\ \mathcal{F}_n & \xrightarrow{\ \mu\ } & \mathcal{A} \end{array}$$

$j = 1, \ldots, n$, commute. (Recall that L_{A_j} is the operator of left multiplication by A_j.) In other words, μ is an intertwining operator for each pair (l_j, L_{A_j}). Now we are in a position to use Corollary 19 and Theorem 10 (more precisely, their several-operator analogs (ii) and (iii) listed in the end of the preceding section). Using (ii), we find that the diagram

$$\begin{array}{ccc} \mathcal{F}_n & \xrightarrow{\mu} & \mathcal{A} \\ f(l)\downarrow & & \downarrow f(L_A) \\ \mathcal{F}_n & \xrightarrow{\mu} & \mathcal{A}. \end{array}$$

commutes for every $f \in \mathcal{F}_n$. In other words,

$$[f(l)g](A) = f(L_A)(g(A)).$$

It remains to notice that, by (iii),

$$f(L_A)(g(A)) = L_{f(A)}g(A) \equiv f(A)g(A).$$

The proof is complete.

Remark 4 Let us also state, without proof, the following assertion, intermediate between Theorems 21 and 22. Suppose that l is a left regular representation of a tuple A of $\mathcal{F}$-generators. Next, let $\mathcal{F}_{(0)} \subset \mathcal{F}$ be a subalgebra. We assume that each l_j, $j = 1, \ldots, n$, is an $\mathcal{F}_{(0)}$-generator in $\mathcal{F}_n$. Then functions $f(l) \equiv f(\overset{1}{l}_1, \ldots, \overset{n}{l}_n)$ are well defined for

$$f \in \mathcal{F}_{(0)n} = \underbrace{\mathcal{F}_{(0)}\hat{\otimes} \ldots \hat{\otimes}\mathcal{F}_{(0)}}_{n \text{ times}} \subset \mathcal{F}_n.$$

We set

$$\begin{aligned} \Lambda_{(0)} &= \{f(A) \,|\, f \in \mathcal{F}_{(0)n}\}, \\ \Lambda &= \{f(A) \,|\, f \in \mathcal{F}_n\}. \end{aligned}$$

These are linear subspaces of $\mathcal{A}$ (not necessarily closed). The assertion is as follows.

Under the above conditions,

$$\Lambda_{(0)}\Lambda \subset \Lambda,$$

and

$$f(A)g(A) = [f(l)g](A), \quad f \in \mathcal{F}_{(0)n}, \quad g \in \mathcal{F}_n.$$

For $\mathcal{F}_{(0)} = \mathcal{F}$ we obtain Theorem 22, and for $\mathcal{F}_{(0)} = \mathcal{F}$ (the set of polynomials) we return to Theorem 21.

Now we return to the binary operation (1.36) on the symbol space $\mathcal{F}_n$. We are interested in the following closely related problems:

(a) Is the operation $(*)$ associative? That is, is $(\mathcal{F}_n, *)$ an associative algebra?

(b) Is the left regular representation unique?

(c) Is the mapping $\mu : f \mapsto f(A)$ a monomorphism?

It turns out that problems (a)–(c) are, in a sense, very close; that is, the positive answer to one of them implies the positive answer to the others. This is however true only if we shift a bit to a different setting. The root of the problem can be explained as follows. By way of example, let us consider pairwise commuting operators $A_1, \ldots, A_n$:

$$[A_j, A_k] = 0, \quad j, k = 1, \ldots, n. \tag{1.39}$$

Then, obviously, one can take l_j to be the operator of multiplication by x_j:

$$l_j f(x_1, \ldots, x_n) = x_j f(x_1, \ldots, x_n). \tag{1.40}$$

Then $f * g = fg$ (the usual pointwise multiplication of functions), and $(\mathcal{F}_n, *)$ is an associative algebra. Suppose for simplicity that $\mathcal{A} = \mathcal{F}_n$. As the operators A_j satisfying relations (1.39), we can take the following tuples of operators:

(A) $A_j = x_j$ (the operator of multiplication by x_j, $j = 1, \ldots, n$.

(B) $A_j = 0$ (zero operator), $j = 1, \ldots, N$.

In case (A), the mapping μ is a monomorphism, and the left regular representation is unique. In case (B), μ is not a monomorphism ($\mu : f(x) \mapsto f(0)$), and the left regular representations is nonunique (say, we can take $l_j = 0$, $j = 1, \ldots, n$, instead of (1.40)).

The explanation is that in case (B) the operators A_j satisfy a lot of additional identities that are not corollaries of (1.39). To avoid complications, we slightly modify the notion of left ordered representation. Namely, we shall deal with left ordered representations of systems of relations rather than of individual tuples $(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$.

Definition 23 A *relation* is an identity of the form

$$\varphi(\overset{k_1}{B}_{j_1}, \dots, \overset{k_m}{B}_{j_m}) = 0, \tag{1.41}$$

where m is a given number, $\varphi \in \mathcal{F}_m$, $j_1, \dots, j_m \in \{1, \dots, n\}$, and $k_1, \dots, k_m$ are distinct Feynman indices. One says that an n-tuple $A = (\overset{1}{A}_1, \dots, \overset{n}{A}_n)$ of $\mathcal{F}$-generators satisfies relation (1.41) if

$$\varphi(\overset{k_1}{A}_{j_1}, \dots, \overset{k_m}{A}_{j_m}) = 0.$$

(Thus, the "native" Feynman indices of the tuple are disregarded, and the numbers prescribed by the relation are used instead.)

Let Σ be a system of relations (the symbols φ occuring in these relations will be numbered by a subscript α, $\varphi = \varphi_\alpha$, $\alpha = 1, \dots, N$).

Definition 24 A tuple $l = (\overset{1}{l}_1, \dots, \overset{n}{l}_n)$ of $\mathcal{F}$-generators $l_j : \mathcal{F}_n \to \mathcal{F}_n$ is called a *left ordered representation* of the system Σ if it is a left ordered representation of every n-tuple $A = (\overset{1}{A}_1, \dots, \overset{n}{A}_n)$ of $\mathcal{F}$-generators satisfying Σ and if the following *regularity condition* is satisfied.

$$f(l)1 = f(x), \tag{1.42}$$

where 1 is the symbol identically equal to unity.

Remark 5 The regularity condition (1.42) is quite natural, since, of course,

$$f(A)1 = f(A)$$

for any symbol f. In terms of the twisted multiplication (1.36), this means that 1 is the right unit: $f * 1 = 1$. (Note that, by the definition of the twisted multiplication, $1 * f = f$, so that in fact 1 is the two-sided unit in this case.)

Now we introduce a notion central to the problem that we deal with in this section.

Definition 25 Let $l = (\overset{1}{l}_1, \dots, \overset{n}{l}_n)$ be a left ordered representation of a system Σ of relations. We say that the *generalized Jacobi condition* holds for l and Σ if l itself satisfies all relations in Σ (i.e. l is a representation of Σ).

Theorem 26 *Let Σ be a system of relations admitting at least one left ordered representation l. Then the following conditions are equivalent.*

(i) *There exists a tuple A of $\mathcal{F}$-generators satisfying the system Σ such that $\mu : f \mapsto f(A)$ is a monomorphism;*

(ii) *The left ordered representation is unique;*

(iii) *The generalized Jacobi condition holds for l and Σ.*

If these conditions are satisfied, then the twisted multiplication $f * g = f(l)g$ is associative, and for every representation A of the system Σ, the mapping

$$\begin{aligned} \mu : \mathcal{F}_n &\rightarrow \mathcal{A}, \\ A &\mapsto f(A) \end{aligned}$$

is a homomorphism of associative algebras $(\mathcal{F}_n, *)$ and $\mathcal{A}$.

Since under the generalized Jacobi condition the left regular representation is unique, we see that this condition in fact expresses a property of a system of relations itself. In the next chapter, we shall see that for the classical Lie commutation relations the generalized Jacobi condition degenerates into the ordinary Jacobi identity for commutators.

Proof of Theorem 26. (i)$\Rightarrow$(ii). We argue by contradiction. If there are two left ordered representations, l and $\widetilde{l}$, such that, say, $l_k \neq \widetilde{l}_k$ for some k, then there is a symbol $f(x)$ such that $h = l_k f \widetilde{l}_k f \neq 0$. Then

$$h(A) = (l_k f)(A) - (\widetilde{l}_k f)(A) = A_k f(A) - A_k f(A) = 0$$

for any n-tuple A of $\mathcal{F}$-generators satisfying Σ. This, however, contradicts (i).

(ii)⇒(iii). We again argue by contradiction. Let the generalized Jacobi condition fail. Then, for some α,

$$\varphi_\alpha(\overset{k_1}{l}_{j_1}, \dots, \overset{k_m}{l}_{j_m}) \neq 0.$$

Hence there is a symbol $g(x)$ such that

$$h = \varphi_\alpha(\overset{k_1}{l}_{j_1}, \dots, \overset{k_m}{l}_{j_m})g \neq 0.$$

However,

$$h(A) = [\![\varphi_\alpha(\overset{k_1}{A}_{j_1}, \dots, \overset{k_m}{A}_{j_m})]\!]g(A) = 0 \tag{1.43}$$

for any n-tuple $A = (A_1, \dots, A_n)$ of $\mathcal{F}$-generators satisfying Σ. (The proof of the left equality is essentially the same as that of the composition formula (1.35).) Now we set

$$[\widetilde{l}_j f](x) = [l_j f](x) + f(0) \cdot h(x), \quad j = 1, \dots, n.$$

This is well defined, since the functional $f \mapsto f(0)$ is continuous on $\mathcal{F}_n$, With regard for (1.43), the new operators $\widetilde{l}_j$ also form a left ordered representation $\widetilde{l}$ of Σ, which does not however coincide with l.

(iii)⇒(i) Under condition (iii), the operators l_j themselves satisfy Σ. By virtue of the regularity condition (1.42), the mapping $f \mapsto f(l)$ is a monomorphism (a right inverse is given by $B \mapsto B(1)$. Thus we arrive at (i).

It remains to prove the associativity of the twisted multiplication under conditions (i)–(iii). (That μ is a homomorphism is obvious.) We have

$$\begin{aligned} (f * g) * h - f * (g * h) &= \{[(f * g) * h](l) - [f * (g * h)](l)\}1 \\ &= \{(f(l)g(l))h(l) - f(l)(g(l)h(l))\}(1) = 0 \end{aligned}$$

(we have used condition (iii)).

The proof of the theorem is complete.

Remark 6 Quite similar results hold for *right ordered representations* determined by the formula

$$f(A)A_j = (r_j f)(A), \quad j = 1, \dots, n.$$

We do not reproduce these results here, leaving the obvious restatement to the reader. We only note in passing that the only subtle point is the *reversed order* of Feynman indices in the definition of twisted multiplication and the composition formula:

$$f * g = g(\overset{n}{r}_1, \dots, \overset{1}{r}_n)f. \tag{1.44}$$

(The same pertains to the Jacobi condition: the operators $r_1, \dots, r_n$ must satisfy

$$\varphi_\alpha(\overset{-k_1}{r}_{j_1}, \dots, \overset{-k_n}{r}_{j_n}) = 0.) \tag{1.45}$$

Finally, note that if the Jacobi condition is satisfied, then the left and right regular representations can be reconstructed from each other by the formulas

$$r_j f = f(l)(x_j), \quad l_j f = f(\overleftarrow{r})(x_j), \tag{1.46}$$

where the backward arrow stands for the reverse ordering of operators. Numerous examples of computing ordered representations for specific systems of operators and/or commutation relations can be found in the subsequent chapters.

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