On the Boundary Behavior of the Logarithmic Residue Integral

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Abstract

A formula of multidimensional logarithmic residue is proved for holomorphic maps with zeroes on the boundary of a bounded domain in $\mathbb{C}^n$.

Let D be a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary ∂D and let $w = \psi(z)$ be a holomorphic map from $\overline{D}$ to $\mathbb{C}^n$, having a finite set E_ψ of zeroes on $\overline{D}$. We recall the definition of the multiplicity of zero of a map ψ (see, for example, [1, §2]). Let $B(z, R) = \{\zeta : |\zeta - z| < R\}$ stand for the ball with center z and radius $R > 0$, and $S(z, R) = \partial B(z, R)$. We assume that a is a zero of ψ and $B(a, R)$ does not contain other zeroes of ψ. Then there is a ball $B(0, r)$ such that for almost all points $\zeta \in B(0, r)$ the map $w = \psi - \zeta$ has the same number of zeroes in $B(a, R)$. This number is referred to as the multiplicity of zero a and it is denoted μ_a.

For a point $z \in E_\psi \cap \partial D$ we consider a ball $B(z, R)$ which does not contain other zeroes of ψ, and we denote by $\tau_\psi(z)$ the expression

$$\tau_\psi(z) = \lim_{r \to +0} \frac{\mathcal{L}^{2n-1}[S(0, r) \cap \psi(B(z, R) \cap D)]}{\mathcal{L}^{2n-1}[S(0, r)]},$$

Here $\mathcal{L}^{2n-1}$ is the $(2n-1)$-Lebesgue measure.

In other words, we consider the solid angle of the tangent cone for the image $\psi(B(z, R) \cap D)$ at the point 0 rather than that for the domain D at the point z. (For a definition of the tangent cone we refer the reader to [6, §3.1.21]).

For $z \in E_\psi$ and for a sufficiently small neighborhood V_z of z, we have $B_\psi(z, r) = \{\zeta \in V_z : |\psi(\zeta)| < r\}$. Moreover, $S_\psi(z, r) = \{\zeta \in V_z : |\psi(\zeta)| = r\}$ is a relatively compact smooth $(2n-1)$-cycle in V_z (for almost all sufficiently small $r > 0$) by the Sard theorem.

We define the principal value v.p. $^\psi$ of the integral of any measurable function φ over a neighborhood $S \subset \partial D$ of the point $z \in E_\psi$ as follows:

$$\text{v.p.}^{\psi} \int\limits_S \varphi(\zeta)\, d\mathcal{L}^{2n-1}(\zeta) = \lim_{r \to +0} \int\limits_{S \setminus B_\psi(z, r)} \varphi(\zeta)\, d\mathcal{L}^{2n-1}(\zeta).$$

This definition is different from of usual definition of the Cauchy principal value v.p. , namely, we remove the "curved" ball $B_\psi(z, r)$ rather than the usual ball with center at z.

We introduce the kernel $U(\psi(\zeta))$ used in the multidimensional logarithmic residue formula (see, for example, [1, §3]). It is obtained from the Bochner-Martinelli kernel $U(w)$ by substitution $w = \psi(z)$. Recall that

$$U(w) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\overline{w}_k d\overline{w}[k] \wedge dw}{|w|^{2n}},$$

where $d\overline{w}[k] = d\overline{w}_1 \wedge \ldots \wedge d\overline{w}_{k-1} \wedge d\overline{w}_{k+1} \wedge \ldots \wedge d\overline{w}_n$, and $dw = dw_1 \wedge \ldots \wedge dw_n$.

The kernel $U(\psi(\zeta))$ is a closed differential form of type $(n, n-1)$ on $\overline{D}$ with singularities at the points $a \in E_\psi$. The explicit form of this kernel is

$$U(\psi(\zeta)) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} \frac{(-1)^{k-1} \overline{\psi_k(\zeta)} d\overline{\psi(\zeta)}[k] \wedge d\psi(\zeta)}{|\psi(\zeta)|^{2n}}.$$

We formulate our main result.

Theorem 1 *If a function F satisfies a Hölder condition of exponent $\gamma > 0$ in $\overline{D}$ (i.e., $F \in \mathcal{C}^\gamma(\overline{D})$) and F is holomorphic in D then*

$$\text{v.p.} \int\limits_{\partial D} F(\zeta)U(\psi(\zeta)) = \sum_{a\in E_\psi\cap D} \mu_a F(a) + \sum_{a\in E_\psi\cap \partial D} \tau_\psi(a)\mu_a F(a).$$

This formula presents the multidimensional logarithmic residue in the case of singularities on the boundary of D. If ψ does not have any zero on the boundary, it is the usual logarithmic residue formula of [1, §3]. For the case of simple zeroes $a \in \partial D$ it recovers the theorem of [5]. Moreover, the above theorem generalizes Theorem 20.7 from [2], which imposes additional conditions on the boundary ∂D and the map ψ.

We first prove Theorem 1 for the principal value v.p. $^\psi$, and then, using Proposition 1, we get it for usual Cauchy principal value. For the proof, we use Theorem 3.2.5 of [6]. We formulate it:

Let $\psi : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map and $m \leqslant n$. Then

$$\int\limits_A g(\psi(x))J_m\psi(x)\, d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} g(y)N(\psi|A,y)\ d\mathcal{H}^m(y), \tag{1}$$

where the set A is $\mathcal{L}^m$-measurable, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $N(\psi|A,y) < \infty$ for $\mathcal{H}^m$-almost all y.

Here $J_m\psi(x)$ is the m-dimensional Jacobian of the map ψ, $\mathcal{L}^m$ is the m-dimensional Lebesgue measure, $\mathcal{H}^m$ is the m-dimensional Hausdorff measure, and $N(\psi|A,y)$ is the multiplicity function of the map ψ, i.e. the number of preimages $\psi^{-1}(y)$ lying in A.

Proof. Consider the domain

$$D_r = D \setminus \bigcup_{a\in E_\psi\cap\partial D} B_\psi(a,r).$$

By the multidimensional logarithmic residue formula (see [1, §3]), we have

$$\int\limits_{\partial D_r} F(\zeta)U(\psi(\zeta)) = \sum_{a\in E_\psi\cap D} \mu_a F(a).$$

Moreover,

$$\text{v.p.} \int\limits_{\partial D} F(\zeta)U(\psi(\zeta)) = \lim_{r\to+0} \int\limits_{\partial D\setminus \bigcup\limits_{a\in E_\psi\cap\partial D} B_\psi(a,r)} F(\zeta)U(\psi(\zeta))$$

whence

$$\int\limits_{\partial D_r} F(\zeta)U(\psi(\zeta)) = \int\limits_{\partial D\setminus\cup_a B_\psi(a,r)} F(\zeta)U(\psi(\zeta)) - \sum_a \int\limits_{S_\psi(a,r)\cap D} F(\zeta)U(\psi(\zeta)).$$

Consider the integral

$$\begin{aligned}
&\int\limits_{S_\psi(a,r)\cap D} F(\zeta)U(\psi(\zeta)) \\
= &\int\limits_{S_\psi(a,r)\cap D} (F(\zeta)-F(a))U(\psi(\zeta)) + F(a) \int\limits_{S_\psi(a,r)\cap D} U(\psi(\zeta)). \qquad (2)
\end{aligned}$$

Further we use the Lojasiewicz inequality (see [4, Ch. 4]) which asserts that

$$|\zeta - a| \leqslant C|\psi(\zeta)|^\alpha \qquad (3)$$

for some positive numbers α and C and points ζ of a sufficiently small neighborhood of a.

Show that the first integral in the formula (2) tends to zero as $r \to +0$. The Hölder condition for the function F, equality (1) and inequality (3) imply

$$\begin{aligned}
&\int\limits_{S_\psi(a,r)\cap D} |F(\zeta)-F(a)|\frac{|\psi_k|}{|\psi(\zeta)|^{2n}}|d\overline{\psi}[k]\wedge d\psi| \\
\leqslant\ & C_1 \int\limits_{S_\psi(a,r)\cap D} |\psi(\zeta)|^{\gamma\alpha+1-2n}|d\overline{\psi}[k]\wedge d\psi| \\
\leqslant\ & C_1\mu_a \int\limits_{S(0,r)\cap\psi(D)} |w|^{\gamma\alpha+1-2n}\, d\mathcal{H}^{2n-1}(w) \\
\leqslant\ & C_2 \int\limits_{S(0,r)} |w|^{\gamma\alpha+1-2n}\, d\mathcal{L}^{2n-1}(w),
\end{aligned}$$

Since the map ψ is smooth, we have $\mathcal{H}^{2n-1}(\psi(S)) \leqslant C_3\mathcal{L}^{2n-1}(S)$, and so the last integral tends obviously to zero as $r \to +0$.

For the second integral of (2) we use the equality (1) to get

$$\begin{aligned}
\lim_{r\to+0} \int\limits_{S_\psi(a,r)\cap D} U(\psi(\zeta)) &= \lim_{r\to+0} \mu_a \int\limits_{S(0,r)\cap\psi(D)} U(w) \\
&= \mu_a\tau_\psi(a),
\end{aligned}$$

because

$$\int\limits_{S(0,r)\cap\psi(D)} U(w) = \frac{\mathcal{L}^{2n-1}[S(0,r)\cap\psi(D)]}{\mathcal{L}^{2n-1}[S(0,r)]}$$

by Lemma 2.1 of [2]. □

Let now $\psi = (\psi_1, \dots, \psi_n)$ be a holomorphic map of $\mathbb{C}^n$ with entire components, ψ having a unique zero at the origin, i.e. $\psi(0) = 0$ and $\psi(z) \neq 0$ for $z \neq 0$. The multiplicity of this zero of ψ we denote by μ.

We define integrals

$$\int_{\partial D_\zeta} f(\zeta)\, U(\psi(\zeta - z)) = \begin{cases} F^+(z), & z \in D, \\ F^-(z), & z \notin \overline{D}. \end{cases} \tag{4}$$

Corollary 1 *Suppose $f \in \mathcal{C}^\gamma(\partial D)$, $\gamma > 0$. Then the integrals $F^\pm$ extend continuously to ∂D and $F^+(z) - F^-(z) = \mu f(z)$ on ∂D.*

PROOF. Let us extend f to a function in a neighborhood V of the boundary of D, satisfying a Hölder condition of exponent γ. Prove that the function

$$\int_{\partial D_\zeta} (f(\zeta) - f(z)) U(\psi(\zeta - z))$$

is continuous in V.

To do this it is necessary to show that the integrals of the form

$$\int_{S_\zeta} (f(\zeta) - f(z)) \frac{\overline{\psi_k(\zeta - z)}}{|\psi(\zeta - z)|^{2n}} d\overline{\psi}[k] \wedge d\psi$$

converge absolutely in some neighborhood S of the point z on the surface ∂D. The inequality (3), if applied to $\psi(\zeta - z)$, and the Hölder continuity of f yield

$$|f(\zeta) - f(z)| \leqslant c|\zeta - z|^\gamma \leqslant c_1 |\psi(\zeta - z)|^{\gamma\alpha}$$

for the points ζ of a sufficiently small neighborhood of z.

The equality (1) implies in the same way as in the proof of Theorem 1 that

$$\begin{aligned} & \int_{S_\zeta} |f(\zeta) - f(z)| \frac{|\psi_k(\zeta - z)|}{|\psi(\zeta - z)|^{2n}} |d\overline{\psi}[k] \wedge d\psi| \\ \leqslant\ & c_1 \int_{S_\zeta} |\psi(\zeta - z)|^{\gamma\alpha + 1 - 2n} |d\overline{\psi}[k] \wedge d\psi| \\ \leqslant\ & c_1 \mu \int_{\psi(S)} |w|^{\gamma\alpha + 1 - 2n}\, d\mathcal{H}^{2n-1}(w) \\ \leqslant\ & c_2 \int_S |w|^{\gamma\alpha + 1 - 2n}\, d\mathcal{L}^{2n-1}(w), \end{aligned}$$

and the last integral is obviously convergent.

The equality

$$\int_{\partial D} U(\psi(\zeta - z)) = \begin{cases} \mu, & z \in D, \\ 0, & z \notin \overline{D}, \end{cases}$$

completes the proof. □

Proposition 1 *For any function $f \in \mathcal{C}^\gamma(\partial D)$, $\gamma > 0$, the equality*

$$v.p.\ {}^{\psi} \int_S f(\zeta) U(\psi(\zeta)) = \ v.p. \int_S f(\zeta) U(\psi(\zeta))$$

holds.

This proposition generalizes the assertion of [5] on the equality of principal values for the case of simple zeroes of ψ.

Proof. As it is shown in Corollary 1, the integral

$$\int_S (f(\zeta) - f(z)) U(\psi(\zeta))$$

absolutely converges, therefore the principal values are equal for the given integral.

It remains to prove that

$$\text{v.p. } {}^{\psi} \int_S U(\psi(\zeta)) = \ \text{v.p.} \int_S U(\psi(\zeta)).$$

We transform the integral on the left-hand side of this equality by the logarithmic residue formula. Namely,

$$\begin{aligned} &\int_{S \setminus B_\psi(z,r)} U(\psi(\zeta)) \\ &= \int_{\partial(D \cap B(z,R) \setminus B_\psi(z,r))} U(\psi(\zeta)) + \int_{D \cap S(z,R)} U(\psi(\zeta)) + \int_{D \cap S_\psi(z,r)} U(\psi(\zeta)) \\ &= -\int_{D \cap S(z,R)} U(\psi(\zeta)) + \int_{D \cap S_\psi(z,r)} U(\psi(\zeta)) \end{aligned}$$

for r small enough, where $S = \partial D \cap B(z,R)$. Therefore, we have to prove that

$$\lim_{r \to +0} \int_{D \cap S_\psi(z,r)} U(\psi(\zeta)) = \lim_{r \to +0} \int_{D \cap S(z,r)} U(\psi(\zeta)).$$

By Theorem 3.2.5 of [6] (equality (1)), we get

$$\begin{aligned} \int\limits_{D\cap S_\psi(z,r)} U(\psi(\zeta)) &= \mu_z \int\limits_{\psi(D)\cap S(0,r)} U(w), \\ \int\limits_{D\cap S(z,r)} U(\psi(\zeta)) &= \mu_z \int\limits_{\psi(D\cap S(z,r))} U(w). \end{aligned}$$

Hence, one needs to show that

$$\lim_{r\to+0} \int\limits_{\psi(D)\cap S(0,r)} U(w) = \lim_{r\to+0} \int\limits_{\psi(D\cap S(z,r))} U(w).$$

In the latter equality one can replace $\psi(D)$ by the tangent cone to $\psi(D)$ at the point 0. We denote it by Π. We show that

$$\int\limits_{\Pi\cap S(0,r_1)} U(w) = \int\limits_{\Pi\cap\psi(S(z,r_2))} U(w).$$

Consider the domain G bounded by the hypersurfaces $\Pi\cap S(0,r_1)$, $\Pi\cap\psi(S(z,r_2))$ and a part of the conic hypersurface $M\cap\partial\Pi$ (r_1 and r_2 are chosen so that the ball $B(0,r_1)$ contains the hypersurface $\psi(S(z,r_2))$). By the Bochner-Martinelli formula,

$$\int\limits_{\partial G} U(w) = 0,$$

whence

$$\int\limits_{\Pi\cap S(0,r_1)} U(w) - \int\limits_{\Pi\cap\psi(S(z,r_2))} U(w) = \int\limits_{M} U(w).$$

We show that

$$\int\limits_{M} U(w) = 0.$$

To this end, we pass from complex coordinates w to real coordinates $w_j = \xi_j + i\xi_{n+j}$, $j = 1,\dots,n$. Then (see [5] or [2, §20])

$$\begin{aligned} \operatorname{Re} U(w) &= \frac{(n-1)!}{2\pi^n}\sum_{k=1}^{2n}(-1)^{k-1}\frac{\xi_k}{|\xi|^{2n}}d\xi[k], \\ \operatorname{Im} U(w) &= -\frac{(n-2)!}{4\pi^n}d\left(\sum_{k=1}^{n}\frac{1}{|\xi|^{2n-2}}d\xi[k,n+k]\right), \quad n>1, \end{aligned}$$

and for $n = 1$

$$\operatorname{Im} U(w) = -\frac{d\ln|\xi|^2}{4\pi}.$$

The restriction of the differential form $\operatorname{Re} U(w)$ to the conic surface M (at the smooth points of M) is equal to 0. Indeed, let M be the zero set of the homogeneous real-valued function φ, i.e. $M = \{\xi : \varphi(\xi) = 0\}$. Then at the smooth points of M the restriction of the form $d\xi[k]$ to M is equal to $(-1)^{k-1}\gamma_k d\sigma$, where $\gamma_k = \dfrac{\partial\varphi}{\partial\xi_k}\cdot\dfrac{1}{|\operatorname{grad}\varphi|}$ are the components of the unit outward normal vector, and $d\sigma$ is the area element of M. Then

$$\begin{aligned}\sum_{k=1}^{2n}(-1)^{k-1}\frac{\xi_k}{|\xi|^{2n}}d\xi[k]\Bigg|_M &= \sum_{k=1}^{2n}\xi_k\frac{\partial\varphi}{\partial\xi_k}\cdot\frac{1}{|\operatorname{grad}\varphi||\xi|^{2n}}d\sigma\\ &= l\varphi\frac{1}{|\operatorname{grad}\varphi||\xi|^{2n}}d\sigma\\ &= 0\end{aligned}$$

by the Euler formula for homogeneous functions, l being the homogeneity degree of φ). Clearly, the $(2n-1)$-dimensional measure of the singular set is equal to 0.

The integration over M shall go as follows. We consider real lines on M of the form

$$L_b = \{\xi : \xi_j = b_j t, j = 1, \ldots, 2n,\, t \in \mathbb{R}\},$$

where $|b| = 1$. For fixed $b \in S(0,1)$, the variable t varies from some number $r_2(b)$ to r_1. The function $r_2(b)$ is measurable. Thus M is fibering over the cycle $\partial\Pi \cap S(0,1)$.

In this variable it is not difficult to show

$$\begin{aligned}\operatorname{Im} U(w) &= c_n d\left(\frac{dt}{t}\wedge\sum_{k,j}\pm b_k db[j,k,n+k]\right)\\ &= c_n\frac{dt}{t}\wedge\sum_{k=1}^{n} db[k,n+k],\end{aligned}$$

since the form containing the product of more than $(2n-2)$ differentials db_j vanishes on $S \cap \partial\Pi$. Then

$$\int_M \operatorname{Im} U(w) = c_n \int_{S(0,1)\cap\partial\Pi} \ln\frac{r_1}{r_2(b)}\sum_{k=1}^{n} db[k,n+k].$$

For almost all points $S \cap \partial\Pi$, the variables b_k, b_{n+k} are functions of other variables b_j, $j \neq k, n+k$. Therefore, the last integral takes the form

$$
\begin{aligned}
& \int\limits_{S(0,1)\cap\partial\Pi} \sum_{k=1}^{n} \ln \Phi_k(b_1, \dots, [k], \dots, [n+k], \dots, b_{2n}) db[k, n+k] \\
= & \int\limits_{S(0,1)\cap\Pi} d\left(\sum_{k=1}^{n} \ln \Phi_k(b_1, \dots, [k], \dots, [n+k], \dots, b_{2n}) db[k, n+k] \right) \\
= & \ 0
\end{aligned}
$$

by the Stokes formula. The proof of Theorem 1 is complete. □

Corollary 1 allows one to strengthen Theorem 1 of [3], which has been proved for smooth functions.

Corollary 2 *Let D be a bounded domain in $\mathbb{C}^n$ with connected smooth boundary. Given a function $f \in \mathcal{C}^{\gamma}(\partial D)$, if the integral $F^{-}(z)$ vanishes outside of $\overline{D}$ then f extends holomorphically to D.*

Proof completely repeats the proof of Theorem 1 of [3] with Corollary 1 thereof replaced by Corollary 1 of the present paper.

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References

[1] L.A. Aĭzenberg and A.P. Yuzhakov, *Integral Representations and Residues in Multidimensional Complex Analysis*, AMS, Providence, RI, 1983.

[2] A.M. Kytmanov, *The Bochner-Martinelli Integral and Its Applications,* Birkhäuser Verlag, Basel, Boston, Berlin, 1995.

[3] A.M. Kytmanov and S.G. Myslivets, *On holomorphy of functions representable by the logarithmic residue formula,* Sib. Math. J. 38 (1997), no. 2, 302–311.

[4] B. Malgrange, *Ideals of Differentiable Functions,* Oxford Univ. Press, 1966.

[5] B.B. Prenov and N.N. Tarkhanov, *On the singular Bochner-Martinelli integral,* Sib. Mat. Zh. 33 (1992), no. 2, 202–205 (Russian).

[6] H.Federer, *Geometric Measure Theory*, Springer Verlag, New-York, 1969.

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