BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH EXITS TO INFINITY

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Abstract. We construct a new calculus of boundary value problems with the transmission property on a non-compact smooth manifold with boundary and conical exits to infinity. The symbols are classical both in covariables and variables. The operators are determined by principal symbol tuples modulo operators of lower orders and weights (such remainders are compact in weighted Sobolev spaces). We develop the concept of ellipticity, construct parametrices within the algebra and obtain the Fredholm property. For the existence of Shapiro-Lopatinskij elliptic boundary conditions to a given elliptic operator we prove an analogue of the Atiyah-Bott condition.

Key words: pseudo-differential boundary value problems, elliptic operators on non-compact manifolds, Atiyah-Bott condition

2000 Mathematics Subject Classification: 35S15, 58J40.

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Contents

Introduction

Elliptic differential (and pseudo-differential) boundary value problems are particularly simple on either a compact smooth manifold with smooth boundary or on a non-compact manifold under local aspects, e.g., elliptic regularity or parametrix constructions. This concerns pseudo-differential operators with the transmission property, cf. Boutet de Monvel [3], or Rempel and Schulze [16], with ellipticity of the boundary data in the sense of a pseudo-differential analogue of the Shapiro-Lopatinskij condition. An essential achievement consists of the algebra structure of boundary value problems and of the fact that parametrices of elliptic operators can be expressed within the algebra. There is an associated boundary symbol algebra that can be viewed as a parameter-dependent calculus of pseudo-differential operators on the half-axis, the inner normal to the boundary (with respect to a chosen Riemannian metric).

The global calculus of pseudo-differential boundary value problems on non-compact or non-smooth manifolds is more complicated. In fact, it is only known in a number of special situations, for instance, on non-compact smooth manifolds with exits to infinity, modelled near the boundary by an infinite half-space, cf. the references below, and then globally generated by charts with a specific behaviour of transition maps. Pseudo-differential boundary value problems are also studied on manifolds with singularities, e.g., conical singularities by Schrohe and Schulze [20],[21] or edge and corner singularities by Rabinovich, Schulze and Tarkhanov [14],[15]. An anisotropic theory of boundary value problems on an infinite cylinder and parabolicity are studied in Krainer [12]. Moreover, essential steps for an algebra of operator-valued symbols for manifolds with edges may be found in Schrohe and Schulze [22],[23],[24]. The latter theory belongs to the concept of operator algebras with operator-valued symbols with a specific twisting in the involved parameter spaces, expressed by strongly continuous groups of isomorphisms in those spaces. The calculus of pseudo-differential operators based on symbols and Sobolev spaces with such twistings was introduced in Schulze [25] in connection with pseudo-differential operators on manifolds with edges, cf. also the monograph [26]. This is, in fact, also a concept to establish algebras of boundary value problems; the corresponding theory is elaborated in [27] for symbols that have not necessarily the transmission property. The case with the transmission property is automatically included, except for the aspect of types in Green and trace operators in Boutet de Monvel's operators; a characterisation in twisted operator-valued symbol terms is contained in Schulze [28] and also in [27]. An application of the edge pseudo-differential calculus for boundary value problems is the crack theory that is treated in a new monograph of Kapanadze and Schulze [10], cf. also the article [29]. An essential tool for this theory are pseudo-differential boundary value problems on manifolds with exits to infinity.

The main purpose of the present paper is to single out a convenient subalgebra of a global version of Boutet de Monvel's algebra on a smooth manifold with exits to infinity. Such a calculus with general non-classical symbols (without the edge-operator machinery) has been studied by Schrohe [18],[19]. We are interested in classical symbols both in covariables and variables. This is useful in applications (e.g., in edge boundary value problems, or crack theory), where explicit criteria for the global ellipticity of boundary conditions are desirable. Our approach reduces all symbol information of the elliptic theory to a compact subset in the space of covariables and variables, though the underlying manifold is not compact. Moreover, we derive a new topological criterion for the existence of global boundary conditions satisfying the Shapiro-Lopatinskij condition, when an elliptic interior symbol with the transmission condition is given. This is an analogue of the Atiyah-Bott condition, well-known for the case of compact smooth manifolds with smooth boundary, cf. Atiyah and Bott [1] and Boutet de Monvel [3]. Note that our algebra can also be regarded as a special calculus of pseudo-differential operators on a manifold with edges that have exits to infinity. The edge here is the boundary and the model cone (of the wedge) the inner normal. Some ideas of our theory seem to generalise to the case of edges in general, though there are also essential differences. The main new point in general is that the transmission property is to be dismissed completely. A theory analogous to the present one without the transmission property would be of independent interest. New elements that appear in this context are smoothing Mellin and Green operators with non-trivial asymptotics near the boundary. Continuity properties of such operators "up to infinity" are studied in Seiler [32], cf. also [31].

Acknowledgement: The authors are grateful to T. Krainer of the University of Potsdam for useful remarks to the manuscript.

1 Pseudo-differential operators with exit symbols

1.1 Standard material on pseudo-differential operators

First we recall basic elements of the standard pseudo-differential calculus as they are needed for the more specific structures in boundary value problems below.

Let $S^\mu(U\times\mathbb{R}^n)$ for $\mu\in\mathbb{R}$ and $U\subseteq\mathbb{R}^m$ open denote the space of all $a(x,\xi)\in C^\infty(U\times\mathbb{R}^n)$ that satisfy the symbol estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)|\le c\langle\xi\rangle^{\mu-|\beta|} \tag{1}$$

for all $\alpha\in\mathbb{N}^m,\beta\in\mathbb{N}^n$ and all $x\in K$ for arbitrary $K\Subset U,\xi\in\mathbb{R}^n$, with constants $c=c(\alpha,\beta,K)>0;\langle\xi\rangle=(1+|\xi|^2)^{\frac{1}{2}}$.

Moreover, let $S^{(\mu)}(U\times(\mathbb{R}^n\setminus 0))$ be the space of all $f\in C^\infty(U\times(\mathbb{R}^n\setminus 0))$ with the property $f(x,\lambda\xi)=\lambda^\mu f(x,\xi)$ for all $\lambda\in\mathbb{R}_+,(x,\xi)\in U\times(\mathbb{R}^n\setminus 0)$. Then we have $\chi(\xi)S^{(\mu)}(U\times(\mathbb{R}^n\setminus 0))\subset S^\mu(U\times\mathbb{R}^n)$ for any excision function $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ (i.e., $\chi(\xi)=0$ for $|\xi|<c_0$, $\chi(\xi)=1$ for $|\xi|>c_1$ for certain $0<c_0<c_1$). We then define $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n)$ to be the subspace of all $a(x,\xi)\in S^\mu(U\times\mathbb{R}^n)$ such that there are elements $a_{(\mu-j)}(x,\xi)\in S^{(\mu-j)}(U\times(\mathbb{R}^n\setminus 0))$, $j\in\mathbb{N}$, with $a(x,\xi)-\sum_{j=0}^N\chi(\xi)a_{(\mu-j)}(x,\xi)\in S^{\mu-(N+1)}(U\times\mathbb{R}^n)$ for all $N\in\mathbb{N}$. Symbols in $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n)$ are called classical. The functions $a_{(\mu-j)}$ (uniquely determined by a) are called the homogeneous components of a of order $\mu-j$, and we call

$$\sigma_\psi(a)(x,\xi):=a_{(\mu)}(x,\xi)$$

the homogeneous principal symbol of order μ (if the order μ is known by the context, otherwise we also write σ_ψ^μ instead of σ_ψ). We do not repeat here all known properties of symbol spaces, such as the relevant Fréchet topologies, asymptotic sums, etc., but tacitly use them. For details we refer to standard expositions on pseudo-differential analysis, e.g., Hörmander [9] or Treves [33], or to the more general scenario with operator-valued sumbols below, where scalar symbols are a special case.

Often we have $m=2n$, $U=\Omega\times\Omega$ for open $\Omega\subseteq\mathbb{R}^n$. In that case symbols are also denoted by $a(x,x',\xi)$, $(x,x')\in\Omega\times\Omega$. The Leibniz product between symbols $a(x,\xi)\in S^\mu(\Omega\times\mathbb{R}^n)$, $b(x,\xi)\in S^\nu(\Omega\times\mathbb{R}^n)$ is denoted by $\#$, i.e.,

$$a(x,\xi)\#b(x,\xi)\sim\sum_\alpha\frac{1}{\alpha!}(D_\xi^\alpha a(x,\xi))\partial_x^\alpha b(x,\xi)$$

($D_x = (\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}), \partial_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$). If notation or relations refer to both classical or non-classical elements, we write (cl) as subscript. In this sense we define the spaces of classical or non-classical pseudo-differential operators to be

$$L^\mu_{(\mathrm{cl})}(\Omega) = \{\mathrm{Op}(a) : a(x, x', \xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)\}. \tag{2}$$

Here, Op is the pseudo-differential action, based on the Fourier transform $F = F_{x\to\xi}$ in $\mathbb{R}^n$, i.e., $\mathrm{Op}(a)u(x) = \iint e^{i(x-x')\xi} a(x, x', \xi) u(x') \mathrm{d}x' \text{đ}\xi$, $\text{đ}\xi = (2\pi)^{-n}\mathrm{d}\xi$. As usual, this is interpreted in the sense of oscillatory integrals, first for $u \in C_0^\infty(\Omega)$, and then extended to more general distribution spaces. Recall that $L^{-\infty}(\Omega) = \cap_{\mu\in\mathbb{Z}} L^\mu(\Omega)$ is the space of all operators with kernel in $C^\infty(\Omega \times \Omega)$.

It will be also be important to employ parameter-dependent variants of pseudo-differential operators, with parameters $\lambda \in \mathbb{R}^l$, treated as additional covariables. We set

$$L^\mu_{\mathrm{cl}}(\Omega; \mathbb{R}^l) = \{\mathrm{Op}(a)(\lambda) : a(x, x', \xi, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{n+l}_{\xi,\lambda})\},$$

using the fact that $a(x, x', \xi, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{n+l})$ implies $a(x, x', \xi, \lambda_0) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)$ for every fixed $\lambda_0 \in \mathbb{R}^l$. In particular, we have $L^{-\infty}(\Omega; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(\Omega))$ with the identification $L^{-\infty}(\Omega) \cong C^\infty(\Omega \times \Omega)$, and $\mathcal{S}(\mathbb{R}^l, E)$ being the Schwartz space of E-valued functions.

Concerning distribution spaces, especially Sobolev spaces, we employ here the usual notation. $L^2(\mathbb{R}^n)$ is the space of square integrable functions in $\mathbb{R}^n$ with the standard scalar product. Then $H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n_\xi)\}, s \in \mathbb{R}$, is the Sobolev space of smoothness $s \in \mathbb{R}$, $\hat{u}(\xi) = (F_{x\to\xi}u)(\xi)$. Analogous spaces make sense on a C^∞ manifold X. Let us assume in this section that X is closed and compact. Let $\mathrm{Vect}(X)$ denote the set of all complex C^∞ vector bundles on X and $H^s(X, E)$, $E \in \mathrm{Vect}(X)$, the space of all distributional sections in E of Sobolev smoothness $s \in \mathbb{R}$. Furthermore, define $L^\mu_{(\mathrm{cl})}(X; E, F; \mathbb{R}^l)$ for $\mu \in \mathbb{R}$, $E, F \in \mathrm{Vect}(X)$, to be the set of all parameter-dependent pseudo-differential operators $A(\lambda)$ (with local classical or non-classical symbols) on X, acting between spaces of distributional sections, i.e.,

$$A(\lambda) : H^s(X, E) \longrightarrow H^{s-\mu}(X, F),\ \lambda \in \mathbb{R}^l.$$

For $l = 0$ we simply write $L^\mu_{(\mathrm{cl})}(X; E, F)$. The homogeneous principal symbol of order μ of an operator $A \in L^\mu_{(\mathrm{cl})}(X; E, F)$ will be denoted by $\sigma_\psi(A)$ (or $\sigma_\psi(A)(x, \xi)$ for $(x, \xi) \in T^*X \setminus 0$) which is a bundle homomorphism

$$\sigma_\psi(A) : \pi^*E \longrightarrow \pi^*F \quad \text{for} \quad \pi : T^*X \setminus 0 \longrightarrow X.$$

Similarly, for $A(\lambda) \in L^\mu_{\mathrm{cl}}(X; E, F; \mathbb{R}^l)$ we have a corresponding parameter-dependent homogeneous principal symbol of order μ that is a bundle homomorphism $\pi^*E \to \pi^*F$ for $\pi : T^*X \times \mathbb{R}^l \setminus 0 \longrightarrow X$ (here, 0 indicates $(\xi, \lambda) = 0$).

1.2 Operators with the transmission property

Boundary value problems on a smooth manifold with smooth boundary will be formulated for operators with the transmission property with respect to the boundary. We will employ the transmission property in its simplest version for classical symbols.

Let $S^\mu_{(\mathrm{cl})}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n) = \{a = \tilde{a}\,|_{\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^n} : \tilde{a}(x, \xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R} \times \mathbb{R}^n)\}$, where $\Omega \subseteq \mathbb{R}^{n-1}$ is an open set, $x = (y, t) \in \Omega \times \mathbb{R}, \xi = (\eta, \tau)$. Moreover, define $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\mathrm{tr}}$ to be the subspace of all $a(x, \xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ such that

$$D_t^k D_\eta^\alpha \{a_{(\mu-j)}(y, t, \eta, \tau) - (-1)^{\mu-j} a_{(\mu-j)}(y, t, -\eta, -\tau)\} = 0 \tag{3}$$

on the set $\{(y,t,\eta,\tau) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : y \in \Omega, t = 0, \eta = 0, \tau \in \mathbb{R}\backslash 0\}$, for all $k \in \mathbb{N}, \alpha \in \mathbb{N}^{n-1}$ and all $j \in \mathbb{N}$. Set $S^{\mu}_{\rm cl}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\rm tr} = \{a = \tilde{a}\mid_{\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n} : \tilde{a}(x,\xi) \in S^{\mu}_{\rm cl}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\rm tr}\}$. Symbols in $S^{\mu}_{\rm cl}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\rm tr}$ or in $S^{\mu}_{\rm cl}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\rm tr}$ are said to have the transmission property with respect to $t = 0$.

Pseudo-differential operators with symbols $a \in S^{\mu}_{\rm cl}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\rm tr}$ are defined by the rule

$$\mathrm{Op}^+(a)u(x) = \mathrm{r}^+\mathrm{Op}(\tilde{a})\mathrm{e}^+u(x), \tag{4}$$

where $\tilde{a} \in S^{\mu}_{\rm cl}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\rm tr}$ is any extension of a to $\Omega \times \mathbb{R}$ and e^+ is the operator of extension by zero from $\Omega \times \overline{\mathbb{R}}_+$ to $\Omega \times \mathbb{R}$, while r^+ is the operator of restriction from $\Omega \times \mathbb{R}$ to $\Omega \times \mathbb{R}_+$. As is well-known, $\mathrm{Op}^+(a)$ for $S^{\mu}_{\rm cl}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\rm tr}$ induces a continuous operator

$$\mathrm{Op}^+(a) : C_0^\infty(\Omega \times \overline{\mathbb{R}}_+) \longrightarrow C^\infty(\Omega \times \overline{\mathbb{R}}_+) \tag{5}$$

(that is independent of the choice of the extension $\tilde{a}$) and extends to a continuous operator

$$\mathrm{Op}^+(a) : [\varphi]H^s(\Omega \times \mathbb{R}_+) \longrightarrow H^{s-\mu}(\Omega \times \mathbb{R}_+) \tag{6}$$

for arbitrary $\varphi \in C_0^\infty(\Omega \times \overline{\mathbb{R}}_+)$ and $s \in \mathbb{R}, s > -\frac{1}{2}$. Here, for simplicity, we assume $\Omega \subset \mathbb{R}^{n-1}$ to be a domain with smooth boundary; then $H^s(\Omega \times \mathbb{R}_+) = H^s(\mathbb{R}^n)|_{\Omega \times \mathbb{R}_+}$. Moreover, if E is a Fréchet space that is a (left) module over an algebra A, $[\varphi]E$ for $\varphi \in A$ denotes the closure of $\{\varphi e : e \in E\}$ in E.

1.3 Calculus on a closed manifold with exits to infinity

A further important ingredient in our theory is the calculus of pseudo-differential operators on a non-compact smooth manifold with conical exits to infinity. The simplest example is the Euclidian space $\mathbb{R}^n$. It can be viewed as a local model for the general case.

The global pseudo-differential calculus in $\mathbb{R}^n$ with weighted symbols and weighted Sobolev spaces has been introduced by Parenti [13] and further developed by Cordes [4]. The case of manifolds with exits to infinity has been investigated by Schrohe [17]. The substructure with classical (in covariables and variables) symbols is elaborated in Hirschmann [8], see also Schulze [27], Section 1.2.3. In Section 1.4 below we shall develop the corresponding operator valued calculus with classical symbols.

Let $S^{\mu;\delta}(\mathbb{R}^n \times \mathbb{R}^n) =: S^{\mu;\delta}$ for $\mu, \delta \in \mathbb{R}$ denote the set of all $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ that satisfy the symbol estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le c\langle\xi\rangle^{\mu-|\beta|}\langle x\rangle^{\delta-|\alpha|} \tag{7}$$

for all $\alpha, \beta \in \mathbb{N}^n, (x,\xi) \in \mathbb{R}^{2n}$, with constants $c = c(\alpha,\beta) > 0$.

This space is Fréchet in a canonical way. Like for standard symbol spaces we have natural embeddings of spaces for different μ, δ. Moreover, asymptotic sums can be carried out in these spaces when the orders in one group of variables x and ξ, or in both variables tend to $-\infty$. Basic notions and results in this context may be found in [30], Section 1.4. Recall that

$$\bigcap_{\mu,\delta \in \mathbb{R}} S^{\mu;\delta}(\mathbb{R}^n \times \mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) =: S^{-\infty;-\infty}(\mathbb{R}^n \times \mathbb{R}^n).$$

We are interested in symbols that are classical both in ξ and in x. To this end we introduce some further notation. Set

$$S_\xi^{(\mu)} = \{a(x,\xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \backslash 0)) : a(x, \lambda\xi) = \lambda^\mu a(x,\xi) \text{ for all } \lambda > 0,\ (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash 0)\}$$

and define analogously the space $S_x^{(\delta)}$ by interchanging the role of x and ξ. Moreover, we set

$$S_{\xi;x}^{(\mu;\delta)} = \{a(x,\xi) \in C^\infty((\mathbb{R}^n \backslash 0) \times (\mathbb{R}^n \backslash 0)) : a(\lambda x, \tau\xi) = \lambda^\delta \tau^\mu a(x,\xi) \text{ for all } \lambda > 0, \tau > 0,\ (x,\xi) \in (\mathbb{R}^n \backslash 0) \times (\mathbb{R}^n \backslash 0)\}.$$

It is also useful to have $S_{\xi;\mathrm{cl}_x}^{(\mu);\delta}$ defined to be the subspace of all $a(x,\xi) \in S_\xi^{(\mu)}$ such that $a(x,\xi)|_{|\xi|=1} \in C^\infty(S^{n-1}, S_{\mathrm{cl}_x}^\delta(\mathbb{R}^n))$ where $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ (clearly, cl_x means that symbols are classical in x with x being treated as a covariable), and $S_{\mathrm{cl}_\xi;x}^{\mu;(\delta)}$ is defined in an analogous manner, by interchanging the role of x and ξ.

Let $S_\xi^{[\mu]}$ defined to be the subspace of all $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that there is a $c = c(a)$ with

$$a(x, \lambda\xi) = \lambda^\mu a(x,\xi) \text{ for all } \lambda \geq 1, x \in \mathbb{R}^n, |\xi| \geq c.$$

In an analogous manner we define $S_x^{[\delta]}$ by interchanging the role of x and ξ. Clearly, for every $a(x,\xi) \in S_\xi^{[\mu]}$ there is a unique element $\sigma_\psi^\mu(a) \in S_\xi^{(\mu)}$ with $a(x,\xi) = \sigma_\psi^\mu(a)(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi| \geq c$ for a constant $c = c(a) > 0$. Analogously, for every $b(x,\xi) \in S_x^{[\delta]}$ there is a unique $\sigma_\mathrm{e}^\delta(b) \in S_x^{(\delta)}$ with $b(x,\xi) = \sigma_\mathrm{e}^\delta(b)(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|x| \geq c$ for some $c = c(b) > 0$.

Set $S^{\mu;[\delta]} = S^{\mu;\delta} \cap S_x^{[\delta]}$, $S^{[\mu];\delta} = S^{\mu;\delta} \cap S_\xi^{[\mu]}$. Let $S_{\mathrm{cl}_\xi}^{\mu;[\delta]}$ be the subspace of all $a(x,\xi) \in S^{\mu;[\delta]}$ such that there are elements $a_k(x,\xi) \in S_\xi^{[\mu-k]} \cap S_x^{[\delta]}$, $k \in \mathbb{N}$, with

$$a(x,\xi) - \sum_{k=0}^N a_k(x,\xi) \in S^{\mu-(N+1);\delta}$$

for all $N \in \mathbb{N}$. Clearly, the remainders automatically belong to $S^{\mu-(N+1);[\delta]}$. Moreover, define $S_{\mathrm{cl}_\xi}^{\mu;\delta}$ to be the subspace of all $a(x,\xi) \in S^{\mu;\delta}$ such that there are elements $a_k(x,\xi) \in S^{[\mu-k];\delta}$, $k \in \mathbb{N}$, with

$$a(x,\xi) - \sum_{k=0}^N a_k(x,\xi) \in S^{\mu-(N+1);\delta}$$

for all $N \in \mathbb{N}$. By interchanging the role of x and ξ we obtain analogously the spaces $S_{\mathrm{cl}_x}^{[\mu];\delta}$ and $S_{\mathrm{cl}_x}^{\mu;\delta}$.

Definition 1.1 *The space $S_{\mathrm{cl}_{\xi;x}}^{\mu;\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ of classical (in ξ and x) symbols of order $(\mu;\delta)$ is defined to be the set of all $a(x,\xi) \in S^{\mu;\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ such that there are sequences*

$$a_k(x,\xi) \in S_{\mathrm{cl}_x}^{[\mu-k];\delta}, k \in \mathbb{N} \quad \text{and} \quad b_l(x,\xi) \in S_{\mathrm{cl}_\xi}^{\mu;[\delta-l]}, l \in \mathbb{N},$$

such that

$$a(x,\xi) - \sum_{k=0}^N a_k(x,\xi) \in S_{\mathrm{cl}_x}^{\mu-(N+1);\delta} \text{ and } a(x,\xi) - \sum_{l=0}^N b_l(x,\xi) \in S_{\mathrm{cl}_\xi}^{\mu;\delta-(N+1)}$$

for all $N \in \mathbb{N}$.

Remark 1.2 *It can easily be proved that* $S^{[\mu];\delta}_{\mathrm{cl}_x} \subset S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}$, $S^{\mu;[\delta]}_{\mathrm{cl}_\xi} \subset S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}$, *where* $S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}} = S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$.

The definition of $S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}$ gives rise to well-defined maps

$$\sigma^{\mu-k}_\psi : S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}} \longrightarrow S^{(\mu-k);\delta}_{\xi;\mathrm{cl}_x},\ k \in \mathbb{N} \ \text{ and } \ \sigma^{\delta-l}_{\mathrm{e}} : S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}} \longrightarrow S^{\mu;(\delta-l)}_{\mathrm{cl}_\xi;x},\ l \in \mathbb{N},$$

namely $\sigma^{\mu-k}_\psi(a) = \sigma^{\mu-k}_\psi(a_k), \sigma^{\delta-l}_{\mathrm{e}}(a) = \sigma^{\delta-l}_{\mathrm{e}}(b_l)$, with the notation of Definition 1.1. From the definition we also see that $\sigma^{\mu-k}_\psi(a)$ is classical in x of order δ and $\sigma^{\delta-l}_{\mathrm{e}}(a)$ is classical in ξ of order μ. So we can form the corresponding homogeneous components $\sigma^{\delta-l}_{\mathrm{e}}(\sigma^{\mu-k}_\psi(a))$ and $\sigma^{\mu-k}_\psi(\sigma^{\delta-l}_{\mathrm{e}}(a))$ in x and ξ, respectively. Then we have $\sigma^{\delta-l}_{\mathrm{e}}(\sigma^{\mu-k}_\psi(a)) = \sigma^{\mu-k}_\psi(\sigma^{\delta-l}_{\mathrm{e}}(a)) =: \sigma^{\mu-k;\delta-l}_{\psi,\mathrm{e}}(a)$ for all $k, l \in \mathbb{N}$.

For $a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$ we set

$$\sigma_\psi(a) := \sigma^\mu_\psi(a), \sigma_{\mathrm{e}}(a) := \sigma^\delta_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a) := \sigma^{\mu;\delta}_{\psi,\mathrm{e}}(a)$$

and define

$$\sigma(a) = (\sigma_\psi(a), \sigma_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a)).$$

Remark 1.3 $a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $\sigma(a) = 0$ *implies* $a(x,\xi) \in S^{\mu-1;\delta-1}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$. *Moreover, from* $\sigma(a)$ *we can recover* $a(x,\xi)$ mod $S^{\mu-1;\delta-1}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$ *by setting*

$$a(x,\xi) = \chi(\xi)\sigma_\psi(a)(x,\xi) + \chi(x)\{\sigma_{\mathrm{e}}(a)(x,\xi) - \chi(\xi)\sigma_{\psi,\mathrm{e}}(a)(x,\xi)\},$$

where χ *is any excision function in* $\mathbb{R}^n$. *More generally, let* $p_\psi(x,\xi) \in S^{(\mu);\delta}_{\xi;\mathrm{cl}_x}$, $p_{\mathrm{e}}(x,\xi) \in S^{\mu;(\delta)}_{\mathrm{cl}_\xi;x}$ *and* $p_{\psi,\mathrm{e}}(x,\xi) \in S^{(\mu;\delta)}_{\xi;x}$ *be arbitrary elements with* $\sigma_{\mathrm{e}}(p_\psi) = \sigma_\psi(p_{\mathrm{e}}) = p_{\psi,\mathrm{e}}$. *Then* $a(x,\xi) = \chi(\xi)p_\psi(x,\xi) + \chi(x)\{p_{\mathrm{e}}(x,\xi) - \chi(\xi)p_{\psi,\mathrm{e}}(x,\xi)\} \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$, *and we have* $\sigma_\psi(a) = p_\psi,\ \sigma_{\mathrm{e}}(a) = p_{\mathrm{e}},\ \sigma_{\psi,\mathrm{e}}(a) = p_{\psi,\mathrm{e}}$.

Example 1.4 *Let us concider a symbol of the form*

$$a(x,\xi) = \omega(x)b(x,\xi) + (1-\omega(x))x^{-m}\sum_{|\alpha|\le m} x^\alpha a_\alpha(\xi)$$

with a cut-off function ω *in* $\mathbb{R}^n$ *(i.e.,* $\omega \in C_0^\infty(\mathbb{R}^n)$, $\omega = 1$ *in a neighbourhood of the origin) and symbols* $b(x,\xi) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$, $a_\alpha(\xi) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n)$, $|\alpha| \le m$ *(in the notation of Section* 1.1*). Then we have* $a(x,\xi) \in S^{\mu;0}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$, *where*

$$\sigma_\psi(a)(x,\xi) = \omega(x)\sigma_\psi(b)(x,\xi) + (1-\omega(x))x^{-m}\sum_{|\alpha|\le m} x^\alpha \sigma_\psi(a)_\alpha(\xi),$$

$$\sigma_{\mathrm{e}}(a)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(\xi), \quad \sigma_{\psi,\mathrm{e}}(x,\xi) = \sum_{|\alpha|=m} \sigma_\psi(a_\alpha)(\xi).$$

Let us now pass to spaces of global pseudo-differential operators in $\mathbb{R}^n$. We formulate some relations both for the classical and non-classical case and indicate it by subscript $(\mathrm{cl}_{\xi;x})$ at the spaces of symbols and (cl) at the spaces of operators. Set

$$L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n) = \{\mathrm{Op}(a) : a(x,\xi) \in S^{\mu;\delta}_{(\mathrm{cl}_{\xi;x})}(\mathbb{R}^n \times \mathbb{R}^n)\},$$

cf. (1.1). As is well-known Op induces isomorphisms

$$\mathrm{Op} : S^{\mu;\delta}_{(\mathrm{cl}_{\xi;x})}(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n) \tag{8}$$

for all $\mu, \delta \in \mathbb{R}$. Recall that $L^{-\infty;-\infty}(\mathbb{R}^n) = \bigcap_{\mu,\delta\in\mathbb{R}} L^{\mu;\delta}(\mathbb{R}^n)$ equals the space of all integral operators with kernels in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Let us form the weighted Sobolev spaces

$$H^{s;\varrho}(\mathbb{R}^n) = \langle x\rangle^{-\varrho} H^s(\mathbb{R}^n)$$

for $s, \varrho \in \mathbb{R}$. Then every $A \in L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n)$ induces continuous operators

$$A : H^{s;\varrho}(\mathbb{R}^n) \longrightarrow H^{s-\mu;\varrho-\delta}(\mathbb{R}^n) \tag{9}$$

for all $s, \varrho \in \mathbb{R}$. Moreover, A restricts to a continuous operator

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{10}$$

For $A \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)$ we set

$$\sigma_\psi(A) = \sigma_\psi(a),\ \sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(a),\ \sigma_{\psi,\mathrm{e}}(A) = \sigma_{\psi,\mathrm{e}}(a),$$

where $a = \mathrm{Op}^{-1}(A)$, according to relation (8).

Remark 1.5 *The pseudo-differential operator calculus globally in $\mathbb{R}^n$ with weighted symbols and weighted Sobolev spaces can be generalised to the case of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \ni (x, \tilde{x})$ with different weights for large $|x|$ or $|\tilde{x}|$. Instead of (7) the symbol estimates are $|D_x^\alpha D_{\tilde{x}}^{\tilde{\alpha}} D_\xi^\beta a(x, \tilde{x}, \xi)| \leq c\langle\xi\rangle^{\mu-|\beta|}\langle x\rangle^{\delta-|\alpha|}\langle\tilde{x}\rangle^{\tilde{\delta}-|\tilde{\alpha}|}$ for all $\alpha, \tilde{\alpha}, \beta$ and $(x, \tilde{x}) \in R^{n+\tilde{n}}, \xi \in \mathbb{R}^{n+\tilde{n}}$, with constants $c = (\alpha, \tilde{\alpha}, \beta)$. Such a theory is elaborated in Gerisch [6].*

We now formulate the basic elements of the pseudo-differential calculus on a smooth manifold M with conical exits to infinity, as it is necessary for boundary value problems below. For simplicity we restrict ourselves to the case of charts that are conical "near infinity". This is a special case of a more general framework of Schrohe [17]. Our manifolds M are defined as unions

$$M = K \cup \bigcup_{j=1}^{k} [1 - \varepsilon, \infty) \times X_j$$

for some $0 < \varepsilon < 1$, where $X_j, j = 1, \ldots, k$, are closed compact C^∞ manifolds, K is a compact smooth manifold with smooth boundary ∂K that is diffeomorphic to the disjoint union $\bigcup_{j=1}^k X_j$, identified with $\{1 - \varepsilon\} \times \bigcup_{j=1}^k X_j$ by a gluing map. On the conical exits to infinity $[1 - \varepsilon, \infty) \times X_j$ we fix Riemannian metrics of the form $\mathrm{d}r^2 + r^2 g_j,\ r \in [1 - \varepsilon, \infty)$, with Riemannian metrics g_j on $X_j, j = 1, \ldots, k$. Moreover, we choose a Riemannian metric on M that restricts to these metrics on the conical exits. Since X_j may have different connected components we may (and will) assume $k = 1$ and set $X = X_1$.

Let $\mathrm{Vect}(M)$ denote the set of all smooth complex vector bundles on M that we represent over $[1, \infty) \times X$ as pull-backs of bundles on X with respect to the canonical projection $[1, \infty) \times X \to X$. Hermitian metrics in the bundles are assumed to be homogeneous of order 0 with respect to homotheties along $[1, \infty)$. On M we fix an open covering by neighbourhoods

$$\{U_1, \ldots, U_L, U_{L+1}, \ldots, U_N\} \tag{11}$$

with $(U_1 \cup \ldots U_L) \cap ([1,\infty) \times X) = \emptyset$ and $U_j \cong (1-\varepsilon,\infty) \times U_j^1$, where $\{U_j^1\}_{j=L+1,\ldots N}$ is an open covering of X. Concerning charts $\chi_j : U_j \to V_j$ to open sets $V_j, j = L+1,\ldots,N$, we choose them of the form $V_j = \{x \in \mathbb{R}^n : |x| > 1-\varepsilon, \frac{x}{|x|} \in V_j^1\}$ for certain open sets $V_j^1 \subset S^{n-1}$ (the unit sphere in $\mathbb{R}^n$). Transition diffeomorphisms are assumed to be homogeneous of order 1 in $r = |x|$ for $r \geq 1$.

Let us now define weighted Sobolev spaces $H^{s;\varrho}(M,E)$ of distributional sections in $E \in \mathrm{Vect}(M)$ of smoothness $s \in \mathbb{R}$ and weight $\varrho \in \mathbb{R}$ (at infinity). To this end, let $\varphi_j \in C^\infty(U_j), j = L+1,\ldots,N$, be a system of functions that are pull-backs $\chi_j^* \tilde{\varphi}_j$ under the chosen charts $\chi_j : U_j \to V_j$, where $\tilde{\varphi}_j \in C^\infty(\mathbb{R}^n), \tilde{\varphi}_j = 0$ for $|x| < 1 - \frac{\varepsilon}{2}, \tilde{\varphi}_j = 0$ in a neighbourhood of $\{x : |x| > 1-\varepsilon, \frac{x}{|x|} \in \partial U_j^1\}$, and $\tilde{\varphi}_j(\lambda x) = \tilde{\varphi}_j(x)$ for all $|x| \geq 1, \lambda \geq 1$. In addition we prescribe the values of $\tilde{\varphi}_j = (\chi_j^*)^{-1}\varphi_j$ on U_j^1 in such a way that $\sum_{j=L+1}^N \varphi_j \equiv 1$ for all points in M that correspond to $|x| \geq 1$ in local coordinates. Given an $E \in \mathrm{Vect}(M)$ of fibre dimension k we choose trivialisations that are compatible with $\chi_j : U_j \to V_j, j = L+1,\ldots,N$, $\tau_j : E|_{U_j} \to V_j \times \mathbb{C}^k$, and homogeneous of order 0 with respect to homotheties in $r \in [1,\infty)$. Then we can easily define $H^{s;\varrho}(M,E)$ as a subspace of $H^s_{\mathrm{loc}}(M,E)$ in an invariant way by requiring $(\tau_j)_*(\varphi_j u) \in H^{s;\varrho}(\mathbb{R}^n,\mathbb{C}^k) = H^{s;\varrho}(\mathbb{R}^n) \otimes \mathbb{C}^k$ for every $L+1 \leq j \leq N$, where $(\tau_j)_*$ denotes the push-forward of sections under τ_j. Setting

$$\mathcal{S}(M,E) = \mathrm{proj}\lim\{H^{l;l}(M,E) : l \in \mathbb{N}\} \tag{12}$$

we get a definition of the Schwartz space of sections in E. By means of the chosen Riemannian metric on M and the Hermitian metric in E we get $L^2(M,E) \cong H^{0;0}(M,E)$ with a corresponding scalar product.

Moreover, observe that the operator spaces $L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n)$ have evident $m \times k$-matrix valued variants $L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n;\mathbb{C}^k,\mathbb{C}^m) = L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n) \otimes \mathbb{C}^m \otimes \mathbb{C}^k$. They can be localised to open sets $V \subset \mathbb{R}^n$ that are conical in the large (i.e., $x \in V, |x| \geq R$ implies $\lambda x \in V$ for all $\lambda \geq 1$, for some $R = R(V) > 0$). Then, given bundles E and $F \in \mathrm{Vect}(M)$ of fibre dimensions k and m, respectively, we can invariantly define the spaces of pseudo-differential operators

$$L^{\mu;\delta}_{(\mathrm{cl})}(M;E,F)$$

on M as subspaces of all standard pseudo-differential operators A of order $\mu \in \mathbb{R}$, acting between distributional sections in E and F, such that

(i) the push-forwards of $\varphi_j A \tilde{\varphi}_j$ with respect to the trivialisations of $E|_{U_j}, F|_{U_j}$ belong to $L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n;\mathbb{C}^k,\mathbb{C}^m)$ for all $j = L+1,\ldots,N$ and arbitrary functions $\varphi_j, \tilde{\varphi}_j$ of the above kind (recall that "cl" means classical in ξ and x),

(ii) $\psi A \tilde{\psi} \in L^{-\infty;-\infty}(M;E,F)$ for arbitrary $\psi, \tilde{\psi} \in C^\infty(M)$ with $\mathrm{supp}\psi \cap \mathrm{supp}\tilde{\psi} = \emptyset$ and $\psi, \tilde{\psi}$ homogeneous of order zero for large r (on the conical exits of M).

Here, $L^{-\infty;-\infty}(M;E,F)$ is the space of all integral operators on M with kernels in $\mathcal{S}(M,F)\hat{\otimes}_\pi \mathcal{S}(M,E^*)$ (integration on M refers to the measure associated with the chosen Riemannian metric; E^* is the dual bundle to E).

Note that the operators $A \in L^{\mu;\delta}(M;E,F)$ induce continuous maps

$$A : H^{s;\varrho}(M,E) \longrightarrow H^{s-\mu;\varrho-\delta}(M,F)$$

for all $s, \varrho \in \mathbb{R}$, and A restricts to a continuous map $\mathcal{S}(M,E) \to \mathcal{S}(M,F)$.

To define the symbol structure we restrict ourselves to classical operators. First, to $A \in L^{\mu;\delta}_{\mathrm{cl}}(M;E,F)$ we have the homogeneous principal symbol of order μ

$$\sigma_\psi(A) : \pi_\psi^* E \longrightarrow \pi_\psi^* F, \quad \pi_\psi : T^*M \setminus 0 \longrightarrow M. \tag{13}$$

The exit symbol components of order δ and (μ,δ) are defined near $r = \infty$ on the conical exit $(R,\infty) \times X$ for any $R \geq 1 - \varepsilon$. Given trivialisations

$$\tau_j : E|_{U_j} \longrightarrow V_j \times \mathbb{C}^k, \quad \vartheta_j : F|_{U_j} \longrightarrow V_j \times \mathbb{C}^l, \tag{14}$$

of E, F on U_j we have the symbols

$$\sigma_{\mathrm{e}}(A_j)(x,\xi) \text{ for } (x,\xi) \in V_j \times \mathbb{R}^n, \qquad \sigma_{\psi,\mathrm{e}}(A_j)(x,\xi) \text{ for } (x,\xi) \in V_j \times (\mathbb{R}^n \setminus 0),$$

where A_j is the push-forward of $A|_{U_j}$ with respect to (14). They behave invariant with respect to the transition maps and define globally bundle homomorphisms

$$\sigma_{\mathrm{e}}(A) : \pi_{\mathrm{e}}^* E \longrightarrow \pi_{\mathrm{e}}^* F, \quad \pi_{\mathrm{e}} : T^*M|_{X^\wedge_\infty} \longrightarrow X^\wedge_\infty, \tag{15}$$

$$\sigma_{\psi,\mathrm{e}}(A) : \pi_{\psi,\mathrm{e}}^* E \longrightarrow \pi_{\psi,\mathrm{e}}^* F, \quad \pi_{\psi,\mathrm{e}} : (T^*M \setminus 0)|_{X^\wedge_\infty} \longrightarrow X^\wedge_\infty. \tag{16}$$

In this notation X_∞ means the base of $[R,\infty) \times X$ "at infinity" with an obvious geometric meaning (for instance, for $M = \mathbb{R}^n$ we have $X_\infty \cong S^{n-1}$, interpreted as the manifold that completes $\mathbb{R}^n$ to a compact space at infinity), and $X^\wedge_\infty = \mathbb{R}_+ \times X_\infty$.

An operator $A \in L^{\mu,\delta}_{\mathrm{cl}}(M;E,F)$ is called elliptic if (13), (15) and (16) are isomorphisms.

An operator $P \in L^{-\mu;-\delta}_{\mathrm{cl}}(M;F;E)$ is called a parametrix of A if $PA - I \in L^{-\infty;-\infty}(M;E,E)$, $AP - I \in L^{-\infty;-\infty}(M;F,F)$.

Theorem 1.6 *Let $A \in L^{\mu;\delta}_{\mathrm{cl}}(M;E,F)$ be elliptic. Then the operator*

$$A : H^{s;\varrho}(M,E) \longrightarrow H^{s-\mu;\varrho-\delta}(M,F)$$

is Fredholm for every $s, \varrho \in \mathbb{R}$, and there is a parametrix $P \in L^{-\mu;-\delta}_{\mathrm{cl}}(M;F,E)$.

1.4 Calculus with operator-valued symbols

As noted in the beginning the theory of boundary value problems can be formulated in a convenient way in terms of pseudo-differential operators with operator-valued symbols. Given a Hilbert space $\boldsymbol{E}$ with a strongly continuous group $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms, acting on $\boldsymbol{E}$, we define the Sobolev space $\mathcal{W}^s(\mathbb{R}^q, \boldsymbol{E})$ of $\boldsymbol{E}$-valued distributions of smoothness $s \in \mathbb{R}$ to be the completion of $\mathcal{S}(\mathbb{R}^q, \boldsymbol{E})$ with respect to the norm $\{\int \langle\eta\rangle^{2s} ||\kappa^{-1}(\eta)\hat{u}(\eta)||^2_{\boldsymbol{E}} d\eta\}^{\frac{1}{2}}$. Here, $\kappa(\eta) := \kappa_{\langle\eta\rangle}$, and $\hat{u}(\eta) = (F_{y\to\eta} u)(\eta)$ is the Fourier transform in $\mathbb{R}^q$. Given an open set $\Omega \subseteq \mathbb{R}^q$ there is an evident notion of spaces $\mathcal{W}^s_{\mathrm{comp}}(\Omega, \boldsymbol{E})$ and $\mathcal{W}^s_{\mathrm{loc}}(\Omega, \boldsymbol{E})$. Moreover, if $\boldsymbol{E}$ and $\tilde{\boldsymbol{E}}$ are Hilbert spaces with strongly continuous groups of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, we define the symbol space $S^\mu(U \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$, $\mu \in \mathbb{R}$, $U \subseteq \mathbb{R}^p$ open, to be the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}}))$ (with $\mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}})$ being equipped with the norm topology) such that

$$||\tilde{\kappa}^{-1}(\eta)\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa(\eta)||_{\mathcal{L}(\boldsymbol{E},\tilde{\boldsymbol{E}})} \leq c\langle\eta\rangle^{\mu-|\beta|}$$

for all $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$ and all $y \in K$ for arbitrary $K \Subset U, \eta \in \mathbb{R}^q$, with constants $c = c(\alpha,\beta,K) > 0$.

Let $S^{(\mu)}(U \times (\mathbb{R}^q \setminus 0); \boldsymbol{E}, \tilde{\boldsymbol{E}})$ denote the set of all $f(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus 0), \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}}))$ such that $f(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda f(y,\eta)\kappa_\lambda^{-1}$ for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in U \times (\mathbb{R}^q \setminus 0)$. Furthermore, let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ (the space of classical operator-valued symbols of order μ) defined to be the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}}))$ such that there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus 0); \boldsymbol{E}, \tilde{\boldsymbol{E}})$, $j \in \mathbb{N}$, with

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q)$$

for all $N \in \mathbb{N}$ (with χ being any excision function in η). Set $\sigma_\partial(a)(y,\eta) := a_{(\mu)}(y,\eta)$ for the homogeneous principal symbol of $a(y,\eta)$ of order μ.

In the case $U = \Omega \times \Omega$, $\Omega \subseteq \mathbb{R}^q$ open, the variables in U will also be denoted by (y, y'). Similarly to (2) we set

$$L^\mu_{(\mathrm{cl})}(\Omega; \boldsymbol{E}, \tilde{\boldsymbol{E}}) = \{\mathrm{Op}(a) : a(y,y',\eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})\}, \tag{17}$$

where Op refers to the action in the y-variables on Ω, while the values of amplitude functions are operators in $\mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}})$. For $A \in L^\mu_{\mathrm{cl}}(\Omega; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ we set $\sigma_\partial(A)(y,\eta) = a_{(\mu)}(y,y',\eta)|_{y'=y}$, called the homogeneous principal symbol of A of order μ. Every $A \in L^\mu(\Omega; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ induces continuous operators

$$A : \mathcal{W}^s_{\mathrm{comp}}(\Omega, \boldsymbol{E}) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{\boldsymbol{E}})$$

for each $s \in \mathbb{R}$. More details of this kind on the pseudo-differential calculus with operator-valued symbols may be found in [26], [30]. In particular, all elements of the theory have a reasonable generalisation to Fréchet spaces $\boldsymbol{E}$ and $\tilde{\boldsymbol{E}}$, written as projective limits of corresponding scales of Hilbert spaces, where the strong continuous actions are defined by extensions or restrictions to the Hilbert spaces of the respective scales [30], Section 1.3.1. This will tacitly be used below. Let us now pass to an analogue of the global pseudo-differential calculus of Section 1.3 with operator-valued symbols. Let $S^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ for $\mu, \delta \in \mathbb{R}$ denote the space of all $a(y,\eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}}))$ that satisfy the symbol estimates

$$||\tilde{\kappa}^{-1}(\eta)\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa(\eta)||_{\mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}})} \leq c\langle\eta\rangle^{\mu-|\beta|}\langle y\rangle^{\delta-|\alpha|}$$

for all $\alpha, \beta \in \mathbb{N}^q$, $(y,\eta) \in \mathbb{R}^{2q}$, with constants $c = c(\alpha,\beta) > 0$. This space is Fréchet, and again, like for standard symbols, we have generalisations of the structures from the local spaces to the global ones. Further details are given in [30], [5], see also [31].

We now define operator-valued symbols that are classical both in η and y, where the group actions on $\boldsymbol{E}, \tilde{\boldsymbol{E}}$ are taken as the identities for all $\lambda \in \mathbb{R}_+$ when y is treated as as a covariable. Similarly to the scalar case we set

$$S^{(\mu)}_\eta = \{a(y,\eta) \in C^\infty(\mathbb{R}^q \times (\mathbb{R}^q \setminus 0), \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}})) : a(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a(y,\eta)\kappa_\lambda^{-1} \text{ for all } \lambda > 0,\ (y,\eta) \in \mathbb{R}^q \times (\mathbb{R}^q \setminus 0)\},$$

$$S^{(\delta)}_y = \{a(y,\eta) \in C^\infty((\mathbb{R}^q \setminus 0) \times \mathbb{R}^q, \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}})) : a(\lambda y, \eta) = \lambda^\delta a(y,\eta) \text{ for all } \lambda > 0,\ (y,\eta) \in (\mathbb{R}^q \setminus 0) \times \mathbb{R}^q\},$$

and

$$S^{(\mu;\delta)}_{\eta;y} = \{a(y,\eta) \in C^\infty((\mathbb{R}^q \setminus 0) \times (\mathbb{R}^q \setminus 0), \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}})) : a(\lambda y, \tau\eta) = \lambda^\delta \tau^\mu \tilde{\kappa}_\tau a(y,\eta)\kappa_\tau^{-1} \text{ for all } \lambda > 0, \tau > 0,\ (y,\eta) \in (\mathbb{R}^q \setminus 0) \times (\mathbb{R}^q \setminus 0)\}.$$

Moreover, let $S^{(\mu);\delta}_{\eta;\mathrm{cl}_y}$ defined to be the subspace of all $a(y,\eta) \in S^{(\mu)}_\eta$ such that $a(y,\eta)|_{|\eta|=1} \in C^\infty(S^{q-1}, S^\delta_{\mathrm{cl}_y}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}))$ (where in $S^\delta_{\mathrm{cl}_y}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}))$ the spaces $\boldsymbol{E}$ and $\tilde{\boldsymbol{E}}$ are endowed with the identities for all $\lambda \in \mathbb{R}_+$ as the corresponding group actions), and $S^{\mu;(\delta)}_{\mathrm{cl}_\eta;y}$ the subspace of all $a(y,\eta) \in S^{(\delta)}_y$ such that $a(y,\eta)|_{|y|=1} \in C^\infty(S^{q-1}, S^\mu_{\mathrm{cl}_\eta}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}))$.

Let $S^{[\mu]}_\eta$ defined to be the set of all $a(y,\eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}}))$ such that there is a $c = c(a)$ with

$$a(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a(y,\eta)\kappa_\lambda^{-1} \quad \text{for all } \lambda \geq 1, y \in \mathbb{R}^q, |\eta| \geq c.$$

Similarly, the space $S^{[\delta]}_y$ is defined to be the set of all $a(y,\eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(\boldsymbol{E}, \tilde{\boldsymbol{E}}))$ such that there is a $c = c(a)$ with

$$a(\lambda y, \eta) = \lambda^\delta a(y,\eta) \quad \text{for all } \lambda \geq 1, |y| \geq c, \eta \in \mathbb{R}^q.$$

Clearly, for every $a(y,\eta) \in S^{[\mu]}_\eta$ there is a unique element $\sigma^\mu_\partial(a) \in S^{(\mu)}_\eta$ with $a(y,\eta) = \sigma^\mu_\partial(a)(y,\eta)$ for all $(y,\eta) \in \mathbb{R}^q \times \mathbb{R}^q$ with $|\eta| \geq c$ for a constant $c = c(a) > 0$. Analogously, for every $b(y,\eta) \in S^{[\delta]}_y$ there is a unique $\sigma^\delta_{\mathrm{e}'}(b) \in S^{(\delta)}_y$ with $b(y,\eta) = \sigma^\delta_{\mathrm{e}'}(b)(y,\eta)$ for all $(y,\eta) \in \mathbb{R}^q \times \mathbb{R}^q$ with $|y| \geq c$ for some $c = c(b) > 0$. Set $S^{\mu;[\delta]} = S^{\mu;\delta} \cap S^{[\delta]}_y$, $S^{[\mu];\delta} = S^{\mu;\delta} \cap S^{[\mu]}_\eta$. Moreover, let $S^{\mu;[\delta]}_{\mathrm{cl}_\eta}$ denote the subspace of all $a(y,\eta) \in S^{\mu;[\delta]}$ such that there are elements $a_k(y,\eta) \in S^{[\mu-k]}_\eta \cap S^{[\delta]}_y$, $k \in \mathbb{N}$, with $a(y,\eta) - \sum_{k=0}^N a_k(y,\eta) \in S^{\mu-(N+1);\delta}$ for all $N \in \mathbb{N}$. Similarly, we define $S^{[\mu];\delta}_{\mathrm{cl}_y}$ to be the subspace of all $a(y,\eta) \in S^{[\mu];\delta}$ such that there are elements $b_l(y,\eta) \in S^{[\mu]}_\eta \cap S^{[\delta-l]}_y$, $l \in \mathbb{N}$, with $a(y,\eta) - \sum_{l=0}^N a_l(y,\eta) \in S^{\mu;\delta-(N+1)}$ for all $N \in \mathbb{N}$.

Let $S^{\mu;\delta}_{\mathrm{cl}_\eta}$ defined to be the set of all $a(y,\eta) \in S^{\mu;\delta}$ such that there are elements $a_k(y,\eta) \in S^{[\mu-k];\delta}$, $k \in \mathbb{N}$, satisfying the relation $a(y,\eta) - \sum_{k=0}^N a_k(y,\eta) \in S^{\mu-(N+1);\delta}$ for all $N \in \mathbb{N}$.

Analogously, define $S^{\mu;\delta}_{\mathrm{cl}_y}$ to be the set of all $a(y,\eta) \in S^{\mu;\delta}$ such that there are elements $a_l(y,\eta) \in S^{\mu;[\delta-l]}$, $l \in \mathbb{N}$, satisfying the relation $a(y,\eta) - \sum_{l=0}^N a_l(y,\eta) \in S^{\mu;\delta-(N+1)}$ for all $N \in \mathbb{N}$.

Note that $S^{[\mu]}_\eta \cap S^{[\delta]}_y \subset S^{[\mu];\delta} \cap S^{\mu;[\delta]}$.

Definition 1.7 *The space $S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ of classical (in y and η) symbols of order $(\mu;\delta)$ is defined to be the set of all $a(y,\eta) \in S^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ such that there are sequences $a_k(y,\eta) \in S^{[\mu-k];\delta}_{\mathrm{cl}_y}$, $k \in \mathbb{N}$, and $b_l(y,\eta) \in S^{\mu;[\delta-l]}_{\mathrm{cl}_\eta}$, $l \in \mathbb{N}$, with*

$$a(y,\eta) - \sum_{k=0}^N a_k(y,\eta) \in S^{\mu-(N+1);\delta}_{\mathrm{cl}_y} \text{ and } a(y,\eta) - \sum_{l=0}^N b_l(y,\eta) \in S^{\mu;\delta-(N+1)}_{\mathrm{cl}_\eta}$$

for all $N \in \mathbb{N}$.

Remark 1.8 *We have $S^{[\mu];\delta}_{\mathrm{cl}_y} \subset S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}$, $S^{\mu;[\delta]}_{\mathrm{cl}_\eta} \subset S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}$, where $S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}} = S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$.*

Given $a \in S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}$ we set $\sigma^{\mu-k}_\partial(a) = \sigma^{\mu-k}_\partial(a_k)$, $\sigma^{\delta-l}_{\mathrm{e}'}(a) = \sigma^{\delta-l}_{\mathrm{e}'}(b_l)$, with the notation of Definition 1.7. This gives us well-defined maps

$$\sigma^{\mu-k}_\partial : S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}} \longrightarrow S^{(\mu-k);\delta}_{\eta;\mathrm{cl}_y}, \; k \in \mathbb{N}, \qquad \sigma^{\delta-l}_{\mathrm{e}'} : S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}} \longrightarrow S^{\mu;(\delta-l)}_{\mathrm{cl}_\eta;y}, \; l \in \mathbb{N}.$$

The corresponding homogeneous components $\sigma_{\mathrm{e}'}^{\delta-l}(\sigma_\partial^{\mu-k}(a))$ and $\sigma_\partial^{\mu-k}(\sigma_{\mathrm{e}'}^{\delta-l}(a))$ in y and η, respectively, are compatible in the sense

$$\sigma_{\mathrm{e}'}^{\delta-l}(\sigma_\partial^{\mu-k}(a)) = \sigma_\partial^{\mu-k}(\sigma_{\mathrm{e}'}^{\delta-l}(a)) =: \sigma_{\partial,\mathrm{e}'}^{\mu-k;\delta-l}(a)$$

for all $k, l \in \mathbb{N}$. For $a(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ we set

$$\sigma_\partial(a) = \sigma_\partial^\mu(a),\ \sigma_{\mathrm{e}'}(a) = \sigma_{\mathrm{e}'}^\delta(a),\ \sigma_{\partial,\mathrm{e}'}(a) = \sigma_{\partial,\mathrm{e}'}^{\mu;\delta}(a)$$

and define

$$\sigma(a) = (\sigma_\partial(a), \sigma_{\mathrm{e}'}(a), \sigma_{\partial,\mathrm{e}'}(a)). \tag{18}$$

Remark 1.9 *$a(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ and $\sigma(a) = 0$ implies $a(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu-1;\delta-1}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$. Moreover, if χ is any excision function in $\mathbb{R}^q$, we have $\chi(\eta)\sigma_\partial(a)(y,\eta) + \chi(y)\{\sigma_{\mathrm{e}'}(a)(y,\eta) - \chi(\eta)\sigma_{\partial,\mathrm{e}'}(a)(y,\eta)\} = a(y,\eta) \bmod S_{\mathrm{cl}_{\eta;y}}^{\mu-1;\delta-1}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$. More generally, let $p_\partial(y,\eta) \in S_{\eta;\mathrm{cl}_y}^{(\mu);\delta}, p_{\mathrm{e}'}(y,\eta) \in S_{\mathrm{cl}_\eta;y}^{\mu;(\delta)}$ and $p_{\partial,\mathrm{e}'}(y,\eta) \in S_{\eta;y}^{(\mu;\delta)}$ be arbitrary elements with $\sigma_{\mathrm{e}'}(p_\partial) = \sigma_\partial(p_{\mathrm{e}'}) = p_{\partial,\mathrm{e}'}$. Then $a(y,\eta) = \chi(\eta)p_\partial(y,\eta)$ $+ \chi(y)\{p_{\mathrm{e}'}(y,\eta) - \chi(\eta)p_{\partial,\mathrm{e}'}(y,\eta)\} \in S_{\mathrm{cl}_{\eta;y}}^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$, and we have $\sigma_\partial(a) = p_\partial,\ \sigma_{\mathrm{e}'}(a) = p_{\mathrm{e}'},\ \sigma_{\partial,\mathrm{e}'}(a) = p_{\partial,\mathrm{e}'}$.*

Remark 1.10 *An element $a(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ is uniquely determined by the sequence*

$$\{\sigma_\partial^{\mu-j}(a)(y,\eta), \sigma_{\mathrm{e}'}^{\delta-j}(a)(y,\eta)\}_{j\in\mathbb{N}}, \tag{19}$$

mod $S^{-\infty;-\infty}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$.

In fact, by the construction of Remark 1.9 we can form

$$a_0(y,\eta) = \chi(\eta)\sigma_\partial^\mu(a)(y,\eta) + \chi(y)\{\sigma_{\mathrm{e}'}^\delta(a)(y,\eta) - \chi(\eta)\sigma_{\partial,\mathrm{e}'}^{\mu;\delta}(a)(y,\eta)\}$$

for any fixed excision function χ, where $b_1(y,\eta) = a(y,\eta) - a_0(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu-1;\delta-1}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$. Then $\sigma_\partial^{\nu-1}(b_1)(y,\eta), \sigma_{\mathrm{e}'}^{\delta-1}(b_1)(y,\eta)$ is completely determined by (19), and we can form

$$a_1(y,\eta) = \chi(\eta)\sigma_\partial^{\mu-1}(b_1)(y,\eta) + \chi(y)\{\sigma_{\mathrm{e}'}^{\delta-1}(b_1)(y,\eta) - \chi(\eta)\sigma_{\partial,\mathrm{e}'}^{\mu-1;\delta-1}(b_1)(y,\eta)\}.$$

Then $b_2 := b_1 - a_1 \in S_{\mathrm{cl}_{\eta;y}}^{\mu-2;\delta-2}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ or $b_2 = a - a_0 - a_1 \in S_{\mathrm{cl}_{\eta;y}}^{\mu-2;\delta-2}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$. Continuing this procedure successively we get a sequence of symbols $a_k(y,\eta)$ $\in S_{\mathrm{cl}_{\eta;y}}^{\mu-k;\delta-k}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}), k \in \mathbb{N}$, with $a(y,\eta) - \sum_{k=0}^N a_k(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu-(N+1);\delta-(N+1)}$ $(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ for all N. Thus we can recover $a(y,\eta)$ as an asymptotic sum $a(y,\eta) \sim \sum_{k=0}^\infty a_k(y,\eta)$.

According to the generalities about symbols with exit behaviour, cf. [5], Proposition 1.5, we can produce $a(y,\eta)$ as a convergent sum $a(y,\eta) = \sum_{k=0}^\infty \boldsymbol{\chi}(\frac{y,\eta}{c_k})a_k(y,\eta)$, where $\boldsymbol{\chi}$ is an excision function in $\mathbb{R}^q \times \mathbb{R}^q$ and c_k a sequence of non-negative reals, tending to infinity sufficiently fast, as $k \to \infty$.

Remark 1.11 *$a(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}_0, \tilde{\boldsymbol{E}})$, $b(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\nu;\kappa}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \boldsymbol{E}_0)$ implies $(ab)(y,\eta) \in S_{\mathrm{cl}_{\eta;y}}^{\mu+\nu;\delta+\kappa}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$, and we have*

$$\sigma(ab) = \sigma(a)\sigma(b) \tag{20}$$

with componentwise multiplication.

Remark 1.12 *Operator-valued symbols of the classes* $S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ *can also be multiplied by scalar ones, namely* $b(y,\eta) \in S^{\nu;\kappa}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q)$*. We then obtain* $(ab)(y,\eta) \in S^{\mu+\nu;\delta+\kappa}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ *including the symbol relation* (20)*. In particular, we have*

$$S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}) = \langle\eta\rangle^{\mu}\langle y\rangle^{\delta} S^{0;0}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}).$$

Set

$$L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}) = \{\mathrm{Op}(a) : a(y,\eta) \in S^{\mu}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})\}. \tag{21}$$

Notice that the subscript "cl" on the left hand side means "classical" both in η and y (in contrast to the corresponding notation in (17)). Similarly to (8) we have isomorphisms

$$\mathrm{Op} : S^{\mu;\delta}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}) \longrightarrow L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}) \tag{22}$$

for all $\mu, \delta \in \mathbb{R}$. This is a consequence of the same kind of oscillatory integral arguments as in the scalar case, cf. [5], Proposition 1.11.

Set

$$\mathcal{W}^{s;\varrho}(\mathbb{R}^q, \boldsymbol{E}) = \langle y\rangle^{-\varrho}\mathcal{W}^{s}(\mathbb{R}^q, \boldsymbol{E}),$$

endowed with the norm $||u||_{\mathcal{W}^{s;\varrho}(\mathbb{R}^q,\boldsymbol{E})} = ||\langle y\rangle^{\varrho}u||_{\mathcal{W}^{s}((\mathbb{R}^q,\boldsymbol{E})}$. Then every $A \in L^{\mu;\delta}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ induces continuous operators

$$A : \mathcal{W}^{s;\varrho}(\mathbb{R}^q, \boldsymbol{E}) \longrightarrow \mathcal{W}^{s-\mu;\varrho-\delta}(\mathbb{R}^q, \tilde{\boldsymbol{E}}) \tag{23}$$

for all $s, \varrho \in \mathbb{R}$, cf. [5], Proposition 1.21. In addition, A is continuous in the sense

$$A : \mathcal{S}(\mathbb{R}^q, \boldsymbol{E}) \longrightarrow \mathcal{S}(\mathbb{R}^q, \tilde{\boldsymbol{E}}), \tag{24}$$

cf. [5], Proposition 1.8.

Remark 1.13 *If we have an operator* $\mathrm{Op}(a)$ *for* $a(y,\eta) \in S^{\mu;\delta}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$*, choose any* $\alpha \in \mathbb{R}$ *and form the operator* $\langle y\rangle^{\alpha}\mathrm{Op}(a)\langle y\rangle^{-\alpha} : \mathcal{S}(\mathbb{R}^q, \boldsymbol{E}) \to \mathcal{S}(\mathbb{R}^q, \tilde{\boldsymbol{E}})$*. Then, there is a unique symbol* $a_{\alpha}(y,\eta) \in S^{\mu;\delta}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ *such that*

$$\langle y\rangle^{\alpha}\mathrm{Op}(a)\langle y\rangle^{-\alpha} = \mathrm{Op}(a_{\alpha}). \tag{25}$$

Thus, (25) *has an extension by continuity to a continuous operator* (23) *for all* $s, \varrho \in \mathbb{R}$*.*

This is a direct consequence of $\langle y\rangle^{\alpha}\mathrm{Op}(a)\langle y\rangle^{-\alpha} \in L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}})$ and of the isomorphism (22).

2 Boundary value problems in the half-space

2.1 Operators on the half-axis

The operator-valued symbols in the present set-up will take their values in a certain algebra of operators on the half-axis. The essential features of this algebra may be described in terms of block-matrices

$$\boldsymbol{a} = \begin{pmatrix} \mathrm{op}^{+}(a) & 0 \\ 0 & 0 \end{pmatrix} + g : \begin{array}{c} H^{s}(\mathbb{R}_{+}) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_{+}) \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \tag{26}$$

for $\mu \in \mathbb{Z}$ and $N_-, N_+ \in \mathbb{N}$ ($s \in \mathbb{R}$ will be specified below). The operator $\mathrm{op}^+(a) = \mathrm{r}^+\mathrm{op}(a)\mathrm{e}^+$ is defined for symbols $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})_{\mathrm{tr}}$, i.e., symbols of order μ with the transmission property, where r^+ and e^+ are restriction and extension operators as in (4), while op(.) denotes the pseudo-differential action on $\mathbb{R}$, i.e.,

$$\mathrm{op}(a)u(t) = \iint e^{i(t-t')\tau} a(\tau)u(t')\mathrm{d}t'\mathrm{d}\tau.$$

(Here, for the moment, we consider symbols with constant coefficients). Moreover, g is a Green operator of type $d \in \mathbb{N}$ on the half-axis, defined as a sum

$$g = g_0 + \sum_{j=1}^{d} g_j \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{27}$$

for continuous operators

$$g_j : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \quad \text{with} \quad g_j^* : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix}$$

(here, $*$ denotes the adjoint with respect to the corresponding scalar products in $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_\pm}$), and we set $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$. Let $\Gamma^d(\overline{\mathbb{R}}_+; N_-, N_+)$ denote the space of all such operators (27), and let $D^{\mu,d}(\overline{\mathbb{R}}_+; N_-, N_+)$ denote the space of all operators (26), $s > d - \frac{1}{2}$, for arbitrary $a \in S^\mu_{\mathrm{cl}}(\mathbb{R})_{\mathrm{tr}}$ and $g \in \Gamma^d(\overline{\mathbb{R}}_+; N_-, N_+)$ (for $N_- = N_+ = 0$ we write $\Gamma^d(\overline{\mathbb{R}}_+)$ and $D^{\mu,d}(\overline{\mathbb{R}}_+)$, respectively). The following properties are part of the boundary symbol calculus for boundary value problems, cf. Boutet de Monvel [3], or Rempel and Schulze [16].

Theorem 2.1 *$\boldsymbol{a} \in D^{\mu,d}(\overline{\mathbb{R}}_+; N_0, N_+)$ and $\boldsymbol{b} \in D^{\nu,e}(\overline{\mathbb{R}}_+; N_-, N_0)$ implies $\boldsymbol{ab} \in D^{\mu+\nu,h}(\overline{\mathbb{R}}_+; N_-, N_+)$ for $h = \max(\nu + d, e)$.*

Theorem 2.2 *Let $\boldsymbol{a} \in D^{\mu,d}(\overline{\mathbb{R}}_+; N_-, N_+)$ where $a(\tau) \neq 0$ for all $\tau \in \mathbb{R}$, and assume that $\boldsymbol{a}$ defines an invertible operator* (26) *for some $s_0 \in \mathbb{R}, s_0 > \max(\mu, d) - \frac{1}{2}$. Then* (26) *is invertible for all $s \in \mathbb{R}, s > \max(\mu, d) - \frac{1}{2}$. In addition*

$$\boldsymbol{a} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

is invertible, and we have $\boldsymbol{a}^{-1} \in D^{-\mu,(d-\mu)^+}(\overline{\mathbb{R}}_+; N_+, N_-)$; here $\nu^+ = \max(\nu, 0)$.

2.2 Boundary symbols associated with interior symbols

In this section we introduce a special symbol class on $\mathbb{R}^n$ that gives rise to operator-valued symbols in the sense of Section 1.4.

Let $S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)_{\asymp}$ defined to be the subspace of all $a(x,\xi) \in S^\mu_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)$ vanishing on the set

$$T_R := \{x = (y,t) \in \mathbb{R}^n : |x| \geq R,\ |t| > R|y|\} \tag{28}$$

for some constant $R = R(a)$; In an analogous manner we define the more general space $S^{\mu;\delta}(\mathbb{R}^n \times \mathbb{R}^n)_{\asymp}$. Set $S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{tr},\asymp} = S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)_{\asymp} \cap S^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{tr}}$ and

$S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr},\asymp} = \{a = \tilde{a}|_{\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n} : \tilde{a}(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{tr},\asymp}\}, \overline{\mathbb{R}}^n_+ = \mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+$. Similarly, we can define the spaces $S^{\mu;\delta}_{(\mathrm{cl}_\xi)}(\mathbb{R}^n \times \mathbb{R}^n)_\asymp$, $S^{\mu;\delta}_{\mathrm{cl}_\xi}(\mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{tr},\asymp}$, $S^{\mu;\delta}_{\mathrm{cl}_\xi}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr},\asymp}$, where cl_ξ means symbols that are only classical in ξ. For $a \in S^{\mu;\delta}_{\mathrm{cl}_\xi}(\mathbb{R}^n \times \mathbb{R}^n)_\asymp$ we form $\mathrm{op}(a)(y,\eta)u(t) = \iint e^{i(t-t')\tau} a(y,t,\eta,\tau)u(t')dt'đ\tau$ and set $\mathrm{op}^+(a)(y,\eta) = \mathrm{r}^+\mathrm{op}(a)(y,\eta)\mathrm{e}^+$, where r^+ and e^+ are of analogous meaning on $\mathbb{R}$ as the corresponding operators r^+ and e^+ in Section 1.2

We also form $\mathrm{op}^+(a)(y,\eta)$ for symbols $a(y,t,\eta,\tau) \in S^{\mu;\delta}_{\mathrm{cl}_\xi}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr},\asymp}$; the extension e^+ includes an extension of a to a corresponding $\tilde{a}$, though $\mathrm{op}^+(a)(y,\eta)$ does not depend on the choice of $\tilde{a}$.

Proposition 2.3 *$a(x,\xi) \in S^{\mu;\delta}(\mathbb{R}^n \times \mathbb{R}^n)_\asymp$ implies*

$$\mathrm{op}(a)(y,\eta) \in S^{\mu;\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$$

for every $s \in \mathbb{R}$.

The simple proof is left to the reader.

Proposition 2.4 *$a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_\xi}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr},\asymp}$ implies*

$$\mathrm{op}^+(a)(y,\eta) \in S^{\mu;\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+))$$

for every $s > -\frac{1}{2}$ and

$$\mathrm{op}^+(a)(y,\eta) \in S^{\mu;\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)).$$

The proof of this result can be given similarly to Theorem 2.2.11 in [20].

Given a symbol $a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_\xi}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$ we call the operator family

$$\mathrm{op}^+(a|_{t=0})(y,\eta) : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+),$$

$s > -\frac{1}{2}$ (or $\mathrm{op}^+(a|_{t=0})(y,\eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$) the boundary symbol associated with $a(x,\xi)$.

Remark 2.5 *For $a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$ we have*

$$\mathrm{op}^+(a|_{t=0})(y,\eta) \in S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)),$$

$s > -\frac{1}{2}$, and $\mathrm{op}^+(a|_{t=0})(y,\eta) \in S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$.

For $a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ we form

$$\sigma(a) = (\sigma_\psi(a), \sigma_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a); \sigma_\partial(a), \sigma_{\mathrm{e}'}(a), \sigma_{\partial,\mathrm{e}'}(a)),$$

for $\sigma_\psi(a) = \sigma_\psi(\tilde{a})|_{\overline{\mathbb{R}}^n_+ \times (\mathbb{R}^n \setminus 0)}$, $\sigma_{\mathrm{e}}(a) = \sigma_{\mathrm{e}}(\tilde{a})|_{(\overline{\mathbb{R}}^n_+ \setminus 0) \times \mathbb{R}^n}$, $\sigma_{\psi,\mathrm{e}}(a) = \sigma_{\psi,\mathrm{e}}(\tilde{a})|_{(\overline{\mathbb{R}}^n_+ \setminus 0) \times (\mathbb{R}^n \setminus 0)}$ with an $\tilde{a} \in S^{\mu;\delta}_{\mathrm{cl}_\xi}(\mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{tr}}$ where $a = \tilde{a}|_{\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n}$, and

$$\begin{aligned} \sigma_\partial(a)(y,\eta) &:= \sigma_\partial(\mathrm{op}^+(a|_{t=0}))(y,\eta), \quad (y,\eta) \in \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus 0), \\ \sigma_{\mathrm{e}'}(a)(y,\eta) &:= \sigma_{\mathrm{e}'}(\mathrm{op}^+(a|_{t=0}))(y,\eta), \quad (y,\eta) \in (\mathbb{R}^{n-1} \setminus 0) \times \mathbb{R}^{n-1}, \\ \sigma_{\partial,\mathrm{e}'}(a)(y,\eta) &:= \sigma_{\partial,\mathrm{e}'}(\mathrm{op}^+(a|_{t=0}))(y,\eta), \quad (y,\eta) \in (\mathbb{R}^{n-1} \setminus 0) \times (\mathbb{R}^{n-1} \setminus 0), \end{aligned}$$

where the right hand sides are understood in the sense of (18). (Here, e$'$ is used for the exit symbol components along $y \in \mathbb{R}^{n-1}$, while e indicates exit symbol components of interior symbols with respect to $x \in \mathbb{R}^n$).

It is useful to decompose symbols in $S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ into a $\asymp$-part and an interior part by a suitable partition of unity.

Definition 2.6 *A function $\chi_\asymp \in C^\infty(\overline{\mathbb{R}}^n_+)$ is called a global admissible cut-off function in $\overline{\mathbb{R}}^n_+$ if*

(i) $0 \le \chi_\asymp(x) \le 1$ *for all* $x \in \overline{\mathbb{R}}^n_+$,

(ii) there is an $R > 0$ *such that* $\chi_\asymp(\lambda x) = \chi_\asymp(x)$ *for all* $\lambda \ge 1, |x| > R$,

(iii)$_g$ $\chi_\asymp(x) = 1$ *for* $0 \le t < \varepsilon$ *for some* $\varepsilon > 0, \chi_\asymp(x) = 0$ *for* $|x| \ge R, t > \tilde{R}|y|$ *and* $\chi_\asymp(x) = 0$ *for* $|x| \le R, t > \tilde{\varepsilon}$ *for some* $\tilde{\varepsilon} > \varepsilon$ *and non-negative reals* R *and* $\tilde{R}$.

A function $\chi_\asymp \in C^\infty(\overline{\mathbb{R}}^n_+)$ is called a local admissible cut-off function in $\overline{\mathbb{R}}^n_+$ if it has the properties $(i), (ii)$ and

(iii)$_l$ $\chi_\asymp(x) = \nu(x)(1-\omega(x))$ *for* $\omega = \tilde{\omega}|_{\overline{\mathbb{R}}^n_+}$ *for some* $\tilde{\omega} \in C_0^\infty(\mathbb{R}^n), 0 \le \tilde{\omega}(x) \le 1$ *for all* $x \in \mathbb{R}^n$ *and* $\tilde{\omega}(x) = 1$ *in a neighbourhood of* $x = 0$ *and* $\nu = \varkappa|_{\overline{\mathbb{R}}^n_+}$ *for some* $\varkappa \in C^\infty(\mathbb{R}^n \setminus 0)$ *with* $\varkappa(\lambda x) = \varkappa(x)$ *for all* $\lambda \in \mathbb{R}_+, x \in \mathbb{R}^n \setminus 0$, *such that for some* $y \in \mathbb{R}^{n-1}$ *with* $|y| = 1$, *and certain* $0 < \varepsilon < \tilde{\varepsilon} < \frac{1}{2}$ *we have* $\varkappa(x) = 1$ *for all* $x \in S^{n-1} \cap \overline{\mathbb{R}}^n_+$ *with* $|x-y| < \varepsilon$ *and* $\varkappa(x) = 0$ *for all* $x \in S^{n-1} \cap \overline{\mathbb{R}}^n_+$ *with* $|x-y| > \tilde{\varepsilon}$.

For $a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ and any (local or global) admissible cut-off function $\chi_\asymp$ we then get a decomposition

$$a(x,\xi) = \chi_\asymp(x)a(x,\xi) + (1-\chi_\asymp(x))a(x,\xi)$$

where $a_\asymp(x,\xi) := \chi_\asymp(x)a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_\asymp$ and $(1-\chi_\asymp(x))a(x,\xi) \in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$.

Remark 2.7 *The operator of multiplication $M_{\chi_\asymp}$ by any $\chi_\asymp \in C^\infty(\mathbb{R}^n)$ with $\chi_\asymp(\lambda x) = \chi_\asymp(x)$ for all $|x| > R$ for some $R > 0$ and $\lambda \ge 1$, can be regarded as an element in $L^{0;0}_{\mathrm{cl}}(\mathbb{R}^n)$. In other words, we have $M_{\chi_\asymp}A, AM_{\chi_\asymp} \in L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n)$ for every $A \in L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n)$. If $\chi_\asymp$ and $\tilde{\chi}_\asymp$ are two such functions with* $\mathrm{supp}\chi_\asymp \cap \mathrm{supp}\tilde{\chi}_\asymp = \emptyset$ *we have $\chi_\asymp A \tilde{\chi}_\asymp \in L^{-\infty;-\infty}(\mathbb{R}^n)$ for arbitrary $A \in L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^n)$. A similar observation is true in the operator-valued case.*

2.3 Green symbols

Pseudo-differential boundary value problems are described by a symbol structure that reflects an analogue of Green's function and generates boundary (and potential) conditions of elliptic boundary value problems. This is summarised by the following definition.

Definition 2.8 *The space $\mathcal{R}^{\mu,0;\delta}_{\mathrm{G}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; N_-, N_+)$ of Green symbols of order $\mu \in \mathbb{R}$, type 0 and weight $\delta \in \mathbb{R}$ is defined to be space of all operator-valued symbols*

$$g(y,\eta) \in S^{\mu;\delta}_{\mathrm{cl}_\eta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+})$$

such that

$$g^*(y,\eta) \in S^{\mu;\delta}_{\mathrm{cl}_\eta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; L^2(\mathbb{R}_+)\oplus\mathbb{C}^{N_+}, \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{N_-}).$$

Moreover, the space $\mathcal{R}^{\mu,d;\delta}_{\mathrm{G}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$ *of Green symbols of order* $\mu\in\mathbb{R}$, *type* $d\in\mathbb{N}$ *and weight* $\delta\in\mathbb{R}$ *is defined to be the space of all operator families of the form*

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta)\begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{29}$$

for arbitrary $g_j \in \mathcal{R}^{\mu-j,0;\delta}_{\mathrm{G}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$, $j=0,\dots,d$.

Proposition 2.9 *Every* $g(y,\eta)\in\mathcal{R}^{\mu,d;\delta}_{\mathrm{G}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$ *belongs to* $S^{\mu;\delta}_{\mathrm{cl}_\eta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^s(\mathbb{R}_+)\oplus\mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{N_+})$ *for every real* $s > d-\frac{1}{2}$.

The specific aspect in our symbol calculus near exits to infinity consists of classical elements, here with respect to $y\in\mathbb{R}^{n-1}$. Let $\mathcal{R}^{\mu,d;\delta}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$ denote the subspace of all $g(y,\eta)\in\mathcal{R}^{\mu,d;\delta}_{\mathrm{G}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$ of the form (29) for $g_j(y,\eta)\in\mathcal{R}^{\mu-j,0;\delta}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$, where $\mathcal{R}^{\mu,0;\delta}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$ is defined to be the space of all

$$g(y,\eta)\in S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; L^2(\mathbb{R}_+)\oplus\mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{N_+})$$

with

$$g^*(y,\eta)\in S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; L^2(\mathbb{R}_+)\oplus\mathbb{C}^{N_+}, \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{N_-}).$$

Similarly to Proposition 2.9 we have

$$\mathcal{R}^{\mu,d;\delta}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)\subset S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^s(\mathbb{R}_+)\oplus\mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{N_+}) \tag{30}$$

for all $s>d-\frac{1}{2}$. Applying (18) we then get the triple of principal symbols

$$\sigma(g) = (\sigma_\partial(g), \sigma_{\mathrm{e}'}(g), \sigma_{\partial,\mathrm{e}'}(g)). \tag{31}$$

Remark 2.10 *There is a direct analogue of Remark* 1.9 *in the framework of Green symbols that we do not repeat in this version in detail. Let us only observe that we can recover* $g(y,\eta)$ *from* (31) $\mathrm{mod}\,\mathcal{R}^{\mu-1,d;\delta-1}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$ *by* $\chi(\eta)\sigma_\partial(g)(y,\eta) + \chi(y)\{\sigma_{\mathrm{e}'}(g)(y,\eta) - \chi(\eta)\sigma_{\partial,\mathrm{e}'}(g)(y,\eta)\}\in\mathcal{R}^{\mu,d;\delta}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; N_-, N_+)$.

Lemma 2.11 *Let* $\varphi(y,t)\in C^\infty(\mathbb{R}^q\times\overline{\mathbb{R}}_+)$ *and assume that for some* $\delta\in\mathbb{R}$ *the following estimates hold:* $\sup_{t\in\overline{\mathbb{R}}_+}|D_y^\alpha\varphi(y,t)|\le c\langle y\rangle^{\delta-|\alpha|}$ *for all* $y\in\mathbb{R}^q$ *and all* $\alpha\in\mathbb{N}^q$, *with constants* $c=c(a)>0$. *Then the operator* $M_{\varphi(y,t)}$ *of multiplication by* $\varphi(y,t)$ *fulfils the relation*

$$M_{\varphi(y,t)}\in S^{0;\delta}(\mathbb{R}^q\times\mathbb{R}^q; L^2(\mathbb{R}_+), L^2(\mathbb{R}_+)). \tag{32}$$

Proof. We have to check the symbol estimates

$$||\kappa^{-1}(\eta)\{D_y^\alpha D_\eta^\beta M_{\varphi(y,t)}\}\kappa(\eta)||_{\mathcal{L}(L^2(\mathbb{R}_+))}\le c\langle y\rangle^{\delta-|\alpha|}$$

for all $\alpha, \beta \in \mathbb{N}^q$ and all $(y, \eta) \in \mathbb{R}^q \times \mathbb{R}^q$ with suitable $c = c(\alpha, \beta) > 0$. Because $M_{\varphi(y,t)}$ is independent of η it suffices to consider $\beta = 0$. Using $\kappa^{-1}(\eta) D_y^\alpha M_{\varphi(y,t)} \kappa(\eta) = D_y^\alpha M_{\varphi(y,t\langle y \rangle^{-1})}$ we get for $u \in L^2(\mathbb{R}_+)$

$$||\kappa^{-1}(\eta)\{D_y^\alpha D_\eta^\beta M_{\varphi(y,t)}\}\kappa(\eta) u(t)||_{L^2(\mathbb{R}_+)} = ||D_y^\alpha \varphi(y, t\langle y \rangle^{-1}) u(t)||_{L^2(\mathbb{R}_+)} \leq$$

$$\sup_{t \in \overline{\mathbb{R}}_+} |D_y^\alpha \varphi(y, t\langle y \rangle^{-1})| ||u||_{L^2(\mathbb{R}_+)} \leq c \langle y \rangle^{\delta - |\alpha|} ||u||_{L^2(\mathbb{R}_+)}. \quad \square$$

Lemma 2.12 *Let $\varphi(y,t) \in C^\infty(\mathbb{R}^q \times \overline{\mathbb{R}}_+)$ be a function such that there are constants $m, \delta \in \mathbb{R}$ such that $\sup_{t \in \overline{\mathbb{R}}_+} |D_y^\alpha D_t^M \varphi(y,t)| \leq c \langle y \rangle^{\delta - |\alpha|} \langle t \rangle^{m-M}$ for all $y \in \mathbb{R}^q, t \in \overline{\mathbb{R}}_+$ and all $\alpha \in \mathbb{N}^q, M \in \mathbb{N}$, with constants $c = c(\alpha, M) > 0$. Then we have*

$$M_{\varphi(y,t)} \in S^{0,\delta}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{33}$$

Proof. Let us express the Schwartz space as a projective limit

$$\mathcal{S}(\overline{\mathbb{R}}_+) = \operatorname{proj\,lim}\{\langle t \rangle^{-k} H^k(\mathbb{R}_+) : k \in \mathbb{N}\}.$$

An operator b is continuous in $\mathcal{S}(\overline{\mathbb{R}}_+)$ if for every $k \in \mathbb{N}$ there is an $l = l(k) \in \mathbb{N}$ such that

$$||b||_{\mathcal{L}(\langle t \rangle^{-l} H^l(\mathbb{R}_+), \langle t \rangle^{-k} H^k(\mathbb{R}_+))} \leq c$$

for certain $c = c(k, l) > 0$. The symbol estimates for (33) require for every $k \in \mathbb{N}$ an $l = l(k) \in \mathbb{N}$ such that

$$||\kappa^{-1}(\eta)\{D_y^\alpha M_{\varphi(y,t)}\}\kappa(\eta) u||_{\langle t \rangle^{-k} H^k(\mathbb{R}_+)} \leq c \langle y \rangle^{\delta - |\alpha|} ||u||_{\langle t \rangle^{-l} H^l(\mathbb{R}_+)}, \tag{34}$$

for constants $c > 0$ depending on k, l, α, for all α and k. Similarly to the proof of Lemma 2.11 the η-derivatives may be ignored. Estimate (34) is equivalent to

$$||\langle t \rangle^k D_y^\alpha \varphi(y, t\langle \eta \rangle^{-1}) \langle t \rangle^{-l} v||_{H^k(\mathbb{R}_+)} \leq c \langle y \rangle^{\delta - |\alpha|} ||v||_{H^l(\mathbb{R}_+)} \tag{35}$$

for all $v \in H^l(\mathbb{R}_+)$. Setting $l = k + m^+$ for $m^+ = \max(m, 0)$ we get (35) from the system of simpler estimates

$$||D_t^j\{\langle t \rangle^{-m^+} D_y^\alpha \varphi(y, t\langle \eta \rangle^{-1}) v(t)\}||_{L^2(\mathbb{R}_+)} \leq c \langle y \rangle^{\delta - |\alpha|} ||v||_{H^l(\mathbb{R}_+)}$$

for all $0 \leq j \leq k$. The function $D_t^j\{\langle t \rangle^{-m^+} D_y^\alpha \varphi(y, t\langle \eta \rangle^{-1}) v(t)\}$ is a sum of expressions of the form

$$v_{j_1 j_2 j_3}(t) = c \langle t \rangle^{-m^+ - j_1} \langle \eta \rangle^{-j_2} (D_t^{j_2} D_y^\alpha \varphi)(y, t\langle \eta \rangle^{-1}) D_t^{j_3} v(t)$$

for $j_1 + j_2 + j_3 = j$ and constants $c = c(j_1, j_2, j_3)$. We now employ the assumption on φ, namely $\sup_{t \in \overline{\mathbb{R}}_+} |(D_y^\alpha D_t^{j_2} \varphi)(y, t\langle \eta \rangle^{-1})| \leq \langle y \rangle^{\delta - |\alpha|} \langle t\langle \eta \rangle^{-1} \rangle^{m - j_2}$. Using $\langle t\langle \eta \rangle^{-1} \rangle^{m-j_2} \leq \langle t \rangle^{m - j_2}$ for $m - j_2 \geq 0$ and $\langle t\langle \eta \rangle^{-1} \rangle^{m-j_2} \leq 1$ for $m - j_2 < 0$ we immediately get

$$||v_{j_1 j_2 j_3}(t)||_{L^2(\mathbb{R}_+)} \leq c \langle \eta \rangle^{\delta - |\alpha|} ||D_t^{j_3} v||_{L^2(\mathbb{R}_+)}$$

for all $y \in \mathbb{R}^q$, with different constants $c > 0$. This gives us finally the estimates (35). $\square$

2.4 The algebra of boundary value problems

Definition 2.13 *$\mathcal{R}^{\mu,d;\delta}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_-,N_+)$ for $(\mu,d)\in\mathbb{Z}\times\mathbb{N},\delta\in\mathbb{R}$, is defined to be the set of all operator families*

$$\boldsymbol{a}(y,\eta)=\begin{pmatrix}\mathrm{op}^+(a)(y,\eta) & 0\\ 0 & 0\end{pmatrix}+g(y,\eta)$$

for arbitrary $a(x,\xi)\in S^{\mu;\delta}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)_{\mathrm{tr},\asymp}$ and $g(y,\eta)\in\mathcal{R}^{\mu,d;\delta}_{\mathrm{G,cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_-,N_+)$.

Observe that the components of

$$\sigma(\boldsymbol{a}):=(\sigma_\psi(\boldsymbol{a}),\sigma_{\mathrm{e}}(\boldsymbol{a}),\sigma_{\psi,\mathrm{e}}(\boldsymbol{a});\sigma_\partial(\boldsymbol{a}),\sigma_{\mathrm{e}'}(\boldsymbol{a}),\sigma_{\partial,\mathrm{e}'}(\boldsymbol{a})) \tag{36}$$

for $\boldsymbol{a}(y,\eta)\in\mathcal{R}^{\mu,d;\delta}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_-,N_+)$, given by

$$\sigma_\psi(\boldsymbol{a}):=\sigma_\psi(a),\ \ \sigma_{\mathrm{e}}(\boldsymbol{a}):=\sigma_{\mathrm{e}}(a),\ \ \sigma_{\psi,\mathrm{e}}(\boldsymbol{a}):=\sigma_{\psi,\mathrm{e}}(a),$$

and

$$\begin{aligned}\sigma_\partial(\boldsymbol{a})&=\begin{pmatrix}\sigma_\partial(\mathrm{op}^+(a|_{t=0})) & 0\\ 0 & 0\end{pmatrix}+\sigma_\partial(g),\\ \sigma_{\mathrm{e}'}(\boldsymbol{a})&=\begin{pmatrix}\sigma_{\mathrm{e}'}(\mathrm{op}^+(a|_{t=0})) & 0\\ 0 & 0\end{pmatrix}+\sigma_{\mathrm{e}'}(g),\\ \sigma_{\partial,\mathrm{e}'}(\boldsymbol{a})&=\begin{pmatrix}\sigma_{\partial,\mathrm{e}'}(\mathrm{op}^+(a|_{t=0})) & 0\\ 0 & 0\end{pmatrix}+\sigma_{\partial,\mathrm{e}'}(g)\end{aligned}$$

are uniquely determined by $\boldsymbol{a}(y,\eta)$, and that $\sigma(\boldsymbol{a})=0$ implies $\boldsymbol{a}(y,\eta)\in\mathcal{R}^{\mu-1,d;\delta-1}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_-,N_+)$.

Moreover, $\boldsymbol{a}(y,\eta)\in\mathcal{R}^{\mu,d;\delta}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_0,N_+)$ and $\boldsymbol{b}(y,\eta)\in\mathcal{R}^{\nu,e;\varrho}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_-,N_0)$ implies $(\boldsymbol{ab})(y,\eta)\in\mathcal{R}^{\mu+\nu,h;\delta+\varrho}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};N_-,N_+)$ for $h=\max(\nu+d,e)$ where $\sigma(\boldsymbol{ab})=\sigma(\boldsymbol{a})\sigma(\boldsymbol{b})$ (with componentwise multiplication).

Next we define spaces of smoothing operators in the half-space. The space $\mathcal{B}^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+;N_-,N_+)$ is defined to be the set of all block matrix operators

$$\mathcal{A}=\begin{pmatrix}A & K\\ T & C\end{pmatrix}:\begin{matrix}\mathcal{S}(\overline{\mathbb{R}}_+)\\ \oplus\\ \mathcal{S}(\mathbb{R}^{n-1},\mathbb{C}^{N_-})\end{matrix}\longrightarrow\begin{matrix}\mathcal{S}(\overline{\mathbb{R}}_+)\\ \oplus\\ \mathcal{S}(\mathbb{R}^{n-1},\mathbb{C}^{N_+})\end{matrix},$$

where

(i) $Au(y,t)=\iint_{\mathbb{R}^n_+}a(y,t,y',t')u(y',t')dy'dt'$ for certain $a(y,t,y',t')\in\mathcal{S}(\overline{\mathbb{R}}^n_+\times\overline{\mathbb{R}}^n_+)(=\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)|_{\overline{\mathbb{R}}^n_+\times\overline{\mathbb{R}}^n_+})$, $u\in\mathcal{S}(\overline{\mathbb{R}}^n_+)$,

(ii) $Kv(y,t)=\sum_{l=1}^{N_-}K_lv_l(y,t)$ for $K_lv_l(y,t)=\int_{\mathbb{R}^{n-1}}k_l(y,t,y')v_l(y')dy'$ for certain $k_l(y,t,y')\in\mathcal{S}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1})(=\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^{n-1})|_{\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1}})$, for $v=(v_l)_{l=1,\dots,N_-}\in\mathcal{S}(\mathbb{R}^{n-1},\mathbb{C}^{N_-})$,

(iii) $Tu(y)=(T_mu(y))_{m=1,\dots,N_+}$ for $T_mu(y)=\iint_{\overline{\mathbb{R}}^n_+}b_m(y,y',t')u(y',t')dy'dt'$ for certain $b_m(y,y',t')\in\mathcal{S}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+)(=\mathcal{S}(\mathbb{R}^{n-1}\times\mathbb{R}^n)|_{\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+})$, $m=1,\dots,N_+$ for $u\in\mathcal{S}(\overline{\mathbb{R}}^n_+)$,

(iv) $Cv(y) = (\sum_{l=1}^{N_-} \int c_{lm}(y,y')v_l(y')\mathrm{d}y')_{m=1,\ldots,N_+}$ for certain $c_{lm}(y,y') \in \mathcal{S}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $l = 1,\ldots,N_-$, $m = 1,\ldots,N_+$.

$\mathcal{B}^{-\infty,d;-\infty}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ for $d \in \mathbb{N}$ is the space of all operators

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{d} \mathcal{C}_j \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix}$$

for arbitrary $\mathcal{C}_j \in \mathcal{B}^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+; N_-, N_+)$, $j = 0,\ldots,d$.

Let $L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)_\smile$ denote the subspace of all $P \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)$ such that there is an $R > 0$ with $\varphi P \psi = 0$ for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}\varphi, \mathrm{supp}\psi \subseteq \mathbb{R}^n \setminus T_R$, cf. (28). Moreover, we set $L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n_+)_\smile = \{P = \tilde{P}|_{\mathbb{R}^n_+} : \tilde{P} \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)_\smile\}$. For $P = \tilde{P}|_{\mathbb{R}^n_+}, \tilde{P} \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)_\smile$ we define

$$\sigma(P) = (\sigma_\psi(\tilde{P})|_{\overline{\mathbb{R}}^n_+ \times (\mathbb{R}^n \setminus 0)}, \sigma_{\mathrm{e}}(\tilde{P})|_{(\overline{\mathbb{R}}^n_+ \setminus 0) \times \mathbb{R}^n}, \sigma_{\psi,\mathrm{e}}(\tilde{P})|_{(\overline{\mathbb{R}}^n_+ \setminus 0) \times (\mathbb{R}^n \setminus 0)}) \tag{37}$$

Definition 2.14 *The space $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ for $(\mu, d) \in \mathbb{Z} \times \mathbb{N}, \delta \in \mathbb{R}$, is defined to be the set of all operators*

$$\mathcal{A} = \mathrm{Op}(\boldsymbol{a}) + \mathcal{P} + \mathcal{C} \tag{38}$$

for arbitrary $\boldsymbol{a}(y,\eta) \in \mathcal{R}^{\mu,d;\delta}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; N_-, N_+), \mathcal{P} \in \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$ with $P \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)_\smile$ and $\mathcal{C} \in \mathcal{B}^{-\infty,d;-\infty}(\overline{\mathbb{R}}^n_+; N_-, N_+)$. Moreover, we set

$$B^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+) = \mathrm{u.l.c}\, \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+). \tag{39}$$

Similarly, we get the subspaces of so-called Green operators (of order μ, type d, and weight δ) $\mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ and $B^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ when we require amplitude functions to belong to $\mathcal{R}^{\mu,d;\delta}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; N_-, N_+)$ and $R^{\mu,d;\delta}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, respectively. For $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ we write $\mathrm{ord}\mathcal{A} = (\mu; \delta)$.

Note that particularly simple elements in $B^{\mu,0;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ are differential operators

$$A = \sum_{|\alpha| \le \mu} a_\alpha(x) D_x^\alpha \tag{40}$$

with coefficients $a_\alpha = \tilde{a}_\alpha|_{\overline{\mathbb{R}}^n_+}$ where $\tilde{a}_\alpha(x) \in S^\delta_{\mathrm{cl}}(\mathbb{R}^n_x)$.

Theorem 2.15 *For every $\mu \in \mathbb{Z}$ the space $B^{\mu,0;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ (cf. the notation (39)) contains an element R^μ that induces isomorphisms*

$$R^\mu : H^{s;\varrho}(\mathbb{R}^n_+) \longrightarrow H^{s-\mu;\varrho}(\mathbb{R}^n_+) \tag{41}$$

for all $s, \varrho \in \mathbb{R}$ as well as isomorphisms

$$R^\mu : \mathcal{S}(\overline{\mathbb{R}}^n_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}^n_+), \tag{42}$$

where $R^{-\mu} := (R^\mu)^{-1} \in B^{-\mu,0;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$.

This is a well-known result for $\varrho = 0$, proved in this form for all $s \in \mathbb{R}$ in Grubb [7]; note that for $s \le -\frac{1}{2}$ we have to compose the pseudo-differential operators from the right by an extension operator $l : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$, while for $s > -\frac{1}{2}$ we can take e^+. Let us mention for completeness that order reductions for $s > \mu^+ - \frac{1}{2}$ have been constructed before by Boutet de Monvel [3]. The symbols from [7] have the form

$$r^\mu_-(\xi) = \left(\chi(\frac{\tau}{a\langle\eta\rangle})\langle\eta\rangle - i\tau\right)^\mu \tag{43}$$

$\xi = (\eta, \tau) \in \mathbb{R}^n$ for a sufficiently large constant $a > 0$ and a function $\chi \in \mathcal{S}(\mathbb{R})$ with $F^{-1}\chi$ supported in $\mathbb{R}_-$ and $\chi(0) = 1$. It was proved in [20], Section 5.3, that (43) is a classical symbol in ξ. In other words, we have $r^\mu_-(\xi) \in S^{\mu;0}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$, and we can set $R^\mu u = \mathrm{r}^+ \mathrm{Op}(r^\mu_-) l$ where $l = \mathrm{e}^+$ for $s > -\frac{1}{2}$. It is now trivial that R^μ induces isomorphisms for all s, ϱ, because the operators with symbols (43) in $\mathbb{R}^n$ belong to $L^{\mu;0}_{\mathrm{cl}}(\mathbb{R}^n)$, cf. (9).

Note that the operator R^μ is elliptic of order $(\mu; 0)$ in the sense of Definition 2.21 below.

Writing $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ in the form (38) we set

$$(\sigma_\psi(\mathcal{A}), \sigma_\mathrm{e}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A})) = (\sigma_\psi(\boldsymbol{a}) + \sigma_\psi(P), \sigma_\mathrm{e}(\boldsymbol{a}) + \sigma_\mathrm{e}(P), \sigma_{\psi,\mathrm{e}}(\boldsymbol{a}) + \sigma_{\psi,\mathrm{e}}(P)),$$

where we use notation from (36) and (37). Moreover, we define

$$(\sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A})) = (\sigma_\partial(\boldsymbol{a}), \sigma_{\mathrm{e}'}(\boldsymbol{a}), \sigma_{\partial,\mathrm{e}'}(\boldsymbol{a})), \tag{44}$$

cf. the notation in (36). Finally, we set

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\mathrm{e}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}); \sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A})), \tag{45}$$

called the principal symbol of the operator $\mathcal{A}$.

Let us set

$$\mathrm{symb}\, \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+) = \{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)\}.$$

Remark 2.16 *The space* $\mathrm{symb}\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ *can easily be defined intrinsically, i.e., as a space of symbol tuples* $(p_\psi, p_\mathrm{e}, p_{\psi,\mathrm{e}}; p_\partial, p_{\mathrm{e}'}, p_{\partial,\mathrm{e}'})$ *with natural compatibility conditions between the components. Then* $\sigma : \mathcal{A} \to \sigma(\mathcal{A})$ *is a surjective map*

$$\sigma : \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+) \longrightarrow \mathrm{symb}\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+), \tag{46}$$

and there is a linear right inverse

$$\mathrm{op} : \mathrm{symb}\, \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+) \longrightarrow \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+) \tag{47}$$

of σ. *Moreover, we have* $\ker \sigma = \mathcal{B}^{\mu-1,d;\delta-1}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$. *Any choice of a map* (47) *with* (46) *is called an operator convention.*

Remark 2.17 *An operator* $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ *induces continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}^n_+) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}^n_+) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, \mathbb{C}^{N_+}) \end{array}. \tag{48}$$

This is an immediate consequence of the fact that $\mathcal{A}$ can be written in the form

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ \tilde{A} \mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{Op}(g) + \mathcal{C} \tag{49}$$

for an $\tilde{A} \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n)_{\mathrm{tr}}$ $(:= \{\tilde{A} \in L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n) : \tilde{A}$ has the transmission property with respect to $t = 0\})$, and $\mathcal{C} \in \mathcal{B}^{-\infty,d;-\infty}(\overline{\mathbb{R}}^n_+; N_-, N_+)$, where (48) is clear for $\mathcal{C}$, while the continuity for the other ingredients, immediately follow from (10) and (24).

Theorem 2.18 *$\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_0, N_+)$ and $\mathcal{B} \in \mathcal{B}^{\nu,e;\varrho}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_0)$ implies $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h;\delta+\varrho}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ for $h = \max(\nu + d, e)$, and we have*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

(with componentwise multiplication). If $\mathcal{A}$ or $\mathcal{B}$ is a Green operator then so is $\mathcal{AB}$.

The proof of this theorem is very close to the corresponding proof in Boutet de Monvel's calculus in local terms. Therefore, we only sketch the typical novelty in the framework with weights. Compositions of the form $(\mathcal{G}+\mathcal{C})\mathcal{A}$ or $\mathcal{B}(\tilde{\mathcal{G}}+\tilde{\mathcal{C}})$ for smoothing operators $\mathcal{C}, \tilde{\mathcal{C}}$ and Green operators $\mathcal{G} = \mathrm{Op}(g), \tilde{\mathcal{G}} = \mathrm{Op}(\tilde{g})$ in the corresponding spaces are again of the type Green plus smoothing operator (this can easily be verified, if we represent $\mathcal{A}$ or $\mathcal{B}$ like (38)). It remains to consider compositions $(\mathrm{r}^+\tilde{A}\mathrm{e}^+)(\mathrm{r}^+\tilde{B}\mathrm{e}^+)$ that equal $\mathrm{r}^+\tilde{A}\tilde{B}\mathrm{e}^+ + \mathrm{r}^+\tilde{A}(1-\Theta^+)\tilde{B}\mathrm{e}^+$, where Θ^+ denotes the characteristic function of $\mathbb{R}^n_+$. The first summand is as desired, while $\mathrm{r}^+\tilde{A}(1-\Theta^+)\tilde{B}\mathrm{e}^+$ has to be recognised as an element $\mathrm{Op}(g)$ (modulo a smoothing remainder) for some Green symbol $g(y,\eta)$ of weight $\delta+\varrho$ for $|y| \to \infty$. Here, we can write $\tilde{A} = \chi_{\asymp}\tilde{A} + (1-\chi_{\asymp})\tilde{A}$ and $\tilde{B} = \chi_{\asymp}\tilde{B} + (1-\chi_{\asymp})\tilde{B}$ for a certain global admissible cut-off function $\chi_{\asymp}$ in $\mathbb{R}^n$, cf. Definition 2.6. Then, $\mathrm{r}^+\chi_{\asymp}\tilde{A}(1-\Theta^+)\chi_{\asymp}\tilde{B}\mathrm{e}^+$ is obviously of the asserted form because the weight contributions for $t \to \infty$ are cutted out, while $\mathrm{r}^+(1-\chi_{\asymp})\tilde{A}(1-\Theta^+)\chi_{\asymp}\tilde{B}\mathrm{e}^+$, $\mathrm{r}^+\chi_{\asymp}\tilde{A}(1-\Theta^+)(1-\chi_{\asymp})\tilde{B}\mathrm{e}^+$ and $\mathrm{r}^+(1-\chi_{\asymp})\tilde{A}(1-\Theta^+)(1-\chi_{\asymp})\tilde{B}\mathrm{e}^+$ are smoothing, cf. Remark 2.7.

Remark 2.19 *The operator $\mathcal{S}^\varrho$ of multiplication by $\mathrm{diag}(\langle x\rangle^\varrho, \langle y\rangle^\varrho \otimes \mathrm{id}_{\mathbb{C}^N})$ belongs to $\mathcal{B}^{0,0;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N, N)$ and induces isomorphisms*

$$\mathcal{S}^\varrho : \begin{array}{c} H^{s;\varrho}(\mathbb{R}^n_+) \\ \oplus \\ H^{s;\varrho}(\mathbb{R}^{n-1}, \mathbb{C}^N) \end{array} \longrightarrow \begin{array}{c} H^{s}(\mathbb{R}^n_+) \\ \oplus \\ H^{s}(\mathbb{R}^{n-1}, \mathbb{C}^N) \end{array}$$

for all $s \in \mathbb{R}$. Moreover, we have

$$\mathcal{S}^{\varrho-\delta}\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)\mathcal{S}^{-\varrho} = \mathcal{B}^{\mu,d;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$$

(clearly, the dimensions in the factors of the latter relation are assumed to be N_+ and N_-, respectively).

Theorem 2.20 *$\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} H^{s;\varrho}(\mathbb{R}^n_+) \\ \oplus \\ H^{s;\varrho}(\mathbb{R}^{n-1}, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(\mathbb{R}^n_+) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\mathbb{R}^{n-1}, \mathbb{C}^{N_+}) \end{array} \tag{50}$$

for all $s, \varrho \in \mathbb{R}, s > d - \frac{1}{2}$. If $\mathrm{ord}\mathcal{A} < (\mu;\delta)$ (i.e., the relation $<$ holds for both components) the operator (50) is compact.

Proof. Write $\mathcal{A}$ in the form (49). The assertion for $\mathcal{C}$ is obvious. Concerning $\mathrm{r}^+\tilde{A}\mathrm{e}^+$ it suffices to apply (9) and (10), combined with the properties (6) and (5). It remains $\mathrm{Op}(g)$. For simplicity we assume g to be of the form of an upper left corner, i.e., $g(y,\eta) \in R^{\mu,d;\delta}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, cf. the notation in Definition 2.8. The other entries in the general block matrix case can be treated in a similar manner which is left to the reader. So we show the continuity

$$\mathrm{Op}(g) : H^{s;\varrho}(\mathbb{R}^n_+) \longrightarrow H^{s-\mu;\varrho-\delta}(\mathbb{R}^n_+),$$

$s > d - \frac{1}{2}$. Applying Theorem 2.15 it suffices to prove the continuity of

$$R^{s-\mu}\mathrm{Op}(g)R^{-s} : H^{0;\varrho}(\mathbb{R}^n_+) \longrightarrow H^{0;\varrho-\delta}(\mathbb{R}^n_+).$$

The symbol $g(y,\eta)$ may be written in the form $g(y,\eta) = \langle y\rangle^{\delta} a(y,\eta)$ for $a(y,\eta) \in R^{\mu,d;0}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Now, using Theorem 2.18 we get $\mathrm{op}(a)R^{-s} = \mathrm{Op}(b) + C$ for some $b(y,\eta) \in R^{\mu-s,0;0}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $C \in B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$. In the following C may be ignored; we then have to show the continuity of

$$R^{s-\mu}\langle y\rangle^{\delta}\mathrm{Op}(b) : H^{0;\varrho}(\mathbb{R}^n_+) \longrightarrow H^{0;\varrho-\delta}(\mathbb{R}^n_+). \tag{51}$$

From the general pseudo-differential calculus with operator-valued symbols we know that

$$\langle y\rangle^{\delta}\mathrm{Op}(b)\langle y\rangle^{-\delta} = \mathrm{Op}(c)$$

with some symbol $c(y,\eta)$ that in this case again belongs to $R^{s-\mu,0;0}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Thus, the operator in (51) gets the form

$$R^{s-\mu}\mathrm{Op}(c)\langle y\rangle^{\delta} = \mathrm{Op}(d)\langle y\rangle^{\delta} + C$$

for another $C \in B^{-\infty,0;-\infty}$ and $d(y,\eta) \in R^{0,0;0}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. We may concentrate on the proof of the continuity

$$\mathrm{Op}(d)\langle y\rangle^{\delta} : H^{0;\varrho}(\mathbb{R}^n_+) \longrightarrow H^{0;\varrho-\delta}(\mathbb{R}^n_+)$$

for arbitrary $\varrho \in \mathbb{R}$. This is equivalent to the continuity of

$$\langle x\rangle^{\varrho-\delta}\mathrm{Op}(d)\langle y\rangle^{\delta}\langle x\rangle^{-\varrho} : L^2(\mathbb{R}^n_+) \longrightarrow L^2(\mathbb{R}^n_+),$$

$x = (y,t)$. By the argument used above in connection with (51) we find an $f(y,\eta) \in R^{0,0;0}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ such that

$$\langle x\rangle^{\varrho-\delta}\mathrm{Op}(d)\langle y\rangle^{\delta}\langle x\rangle^{-\varrho} = \langle x\rangle^{\varrho-\delta}\langle y\rangle^{\delta-\varrho}\mathrm{Op}(f)\langle y\rangle^{\varrho}\langle x\rangle^{-\varrho}.$$

Now $\varphi(y,t) := \langle y\rangle^{\varrho}\langle x\rangle^{-\varrho}$ for $\varrho \geq 0$ satisfies the assumptions of Lemma 2.11 for $\delta = 0$. For $\varrho - \delta \leq 0$ we can apply Lemma 2.11 once again for $\psi(y,t) = \langle x\rangle^{\varrho-\delta}\langle y\rangle^{\delta-\varrho}$ and get

$$\langle x\rangle^{\varrho-\delta}\mathrm{Op}(d)\langle y\rangle^{\delta}\langle x\rangle^{-\varrho} = \mathrm{Op}(M_{\psi})\mathrm{Op}(f)\mathrm{Op}(M_{\varphi})$$

as a continuous operator $L^2(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+)$. Let us now examine the case $\varrho \geq 0$ but $\varrho - \delta \geq 0$. We then write $\langle x\rangle^{\varrho-\delta}\langle y\rangle^{\delta-\varrho} = \psi(y,t)\langle t\rangle^{\varrho-\delta}$ for $\psi(y,t) = \langle x\rangle^{\varrho-\delta}\langle y\rangle^{\delta-\varrho}\langle t\rangle^{\delta-\varrho}$. The function ψ satisfies the assumptions of Lemma 2.11, while $M_{\langle t\rangle^{\varrho-\delta}}$ belongs to $S^{0,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$ by Lemma 2.12. This gives us

$$M_{\langle t\rangle^{\varrho-\delta}} f(y,\eta) \in S^{0,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+), L^2(\mathbb{R}_+))$$

Hence, setting again $\varphi(y,t) = \langle y\rangle^{\varrho}\langle x\rangle^{-\varrho}$, we see that

$$\langle x\rangle^{\varrho-\delta}\langle y\rangle^{\delta-\varrho}\mathrm{Op}(f)\langle y\rangle^{\varrho}\langle x\rangle^{-\varrho} = \mathrm{Op}(M_\psi)\mathrm{Op}(M_{\langle t\rangle^{\varrho-\delta}}f)\mathrm{Op}(M_\varphi)$$

is a continuous operator $L^2(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+)$. In an analogous manner we can proceed in the remaining cases concerning the sign of ϱ and $\varrho - \delta$. The compactness of $\mathcal{A}$ for $\mathrm{ord}\mathcal{A} < (\mu;\delta)$ then follows from the continuity of $\mathcal{A}$ to spaces of better smoothness and weight and from corresponding compact embeddings of Sobolev spaces. $\square$

2.5 Ellipticity

The principal symbol structure of the preceding section gives rise to an adequate notion of ellipticity of pseudo-differential boundary value problems globally on the half-space.

Definition 2.21 *An operator $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ is called elliptic (of order $(\mu;\delta)$) if*

(i) $\sigma_\psi(\mathcal{A})(x,\xi)$ *for all* $(x,\xi) \in \overline{\mathbb{R}}^n_+ \times (\mathbb{R}^n \setminus 0)$,
$\sigma_{\mathrm{e}}(\mathcal{A})(x,\xi)$ *for all* $(x,\xi) \in (\overline{\mathbb{R}}^n_+ \setminus 0) \times \mathbb{R}^n$,
$\sigma_{\psi,\mathrm{e}}(\mathcal{A})(x,\xi)$ *for all* $(x,\xi) \in (\overline{\mathbb{R}}^n_+ \setminus 0) \times (\mathbb{R}^n \setminus 0)$
are non-zero,

(ii) $\sigma_\partial(\mathcal{A})(y,\eta)$ *for all* $(y,\eta) \in \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus 0)$,
$\sigma_{\mathrm{e}'}(\mathcal{A})(y,\eta)$ *for all* $(y,\eta) \in (\mathbb{R}^{n-1} \setminus 0) \times \mathbb{R}^{n-1}$,
$\sigma_{\partial,\mathrm{e}'}(\mathcal{A})(y,\eta)$ *for all* $(y,\eta) \in (\mathbb{R}^{n-1} \setminus 0) \times (\mathbb{R}^{n-1} \setminus 0)$
are isomorphisms

$$\begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}.$$

Remark 2.22 *Condition (ii) in the latter definition can equivalently be replaced by bijectivities in the sense*

$$\begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

for any $s > \max(\mu, d) - \frac{1}{2}$.

Definition 2.23 *Given $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$, an operator $\mathcal{P} \in \mathcal{B}^{-\mu,e;-\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_+, N_-)$ for some $e \in \mathbb{N}$ is called a parametrix of $\mathcal{A}$ if $\mathcal{P}\mathcal{A} - \mathcal{I} \in \mathcal{B}^{-\infty,d_l;-\infty}(\overline{\mathbb{R}}^n_+; N_-, N_-)$ and $\mathcal{A}\mathcal{P} - \mathcal{I} \in \mathcal{B}^{-\infty,d_r;-\infty}(\overline{\mathbb{R}}^n_+; N_+, N_+)$ for certain $d_l, d_r \in \mathbb{N}$.*

We shall see below that the ellipticity of an operator $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_-, N_+)$ entails the existence of a parametrix. First we want to construct further examples of elliptic boundary value problems.

The Dirichlet problem for $c - \Delta$, with the Laplace operator $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, and a constant $c > 0$ is represented by the operator

$$\mathcal{A}_1 = \begin{pmatrix} c - \Delta \\ \mathrm{r}' \end{pmatrix} : H^s(\mathbb{R}^n_+) \longrightarrow \begin{matrix} H^{s-2}(\mathbb{R}^n_+) \\ \oplus \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}). \end{matrix} \tag{52}$$

For convenience we pass to

$$\mathcal{A}_2 = \begin{pmatrix} c-\Delta \\ Q\mathrm{r}' \end{pmatrix} : H^s(\mathbb{R}^n_+) \longrightarrow \begin{matrix} H^{s-2}(\mathbb{R}^n_+) \\ \oplus \\ H^{s-2}(\mathbb{R}^{n-1}) \end{matrix} \tag{53}$$

where Q is an order reduction on the boundary that we take of the form $Q = \mathrm{Op}_y(\langle\eta\rangle^{\frac{3}{2}})$, such that $Q : H^s(\mathbb{R}^{n-1}) \longrightarrow H^{s-\frac{3}{2}}(\mathbb{R}^{n-1})$ is an isomorphism for all $s \in \mathbb{R}$. Then we have $\mathcal{A}_2 \in \mathcal{B}^{2,1;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; 0, 1)$. We want to show that (53) is an isomorphism for all $s > \frac{3}{2}$ and construct the inverse. We have for $x = (y,t), \xi = (\eta, \tau)$

$$\sigma_\psi(\mathcal{A}_2) = |\xi|^2, \ \sigma_{\mathrm{e}}(\mathcal{A}_2) = c + |\xi|^2, \ \sigma_{\psi,\mathrm{e}}(\mathcal{A}_2) = |\xi|^2,$$

$$\sigma_\partial(\mathcal{A}_2) = \begin{pmatrix} |\eta|^2 - \partial_t^2 \\ |\eta|^{\frac{3}{2}}\mathrm{r}' \end{pmatrix}, \sigma_{\mathrm{e}'}(\mathcal{A}_2) = \begin{pmatrix} c + |\eta|^2 - \partial_t^2 \\ \langle\eta\rangle^{\frac{3}{2}}\mathrm{r}' \end{pmatrix}, \sigma_{\partial,\mathrm{e}'}(\mathcal{A}_2) = \begin{pmatrix} |\eta|^2 - \partial_t^2 \\ |\eta|^{\frac{3}{2}}\mathrm{r}' \end{pmatrix}.$$

Hence $\mathcal{A}_2$ is elliptic in the sense of Definition 2.21. First we invert the operator family

$$\begin{pmatrix} \alpha - \partial_t^2 \\ \beta\mathrm{r}' \end{pmatrix} : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \tag{54}$$

where $\alpha := (c+|\eta|^2)^{\frac{1}{2}}, \beta := \langle\eta\rangle^{\frac{3}{2}}$. Let us write $l_\pm(\tau) = \alpha \pm i\tau$; then $l_-(\tau)l_+(\tau) = \alpha^2+\tau^2$ and $\alpha^2 - \partial_t^2 = \mathrm{op}^+(l_-l_+) = \mathrm{op}^+(l_-)\mathrm{op}^+(l_+)$ (the latter identity is true because l_- is a minus function; $\mathrm{op}^+(l_-) : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$ is an isomorphism). Thus, to invert (54), it suffices to consider

$$\begin{pmatrix} \mathrm{op}^+(l_+) \\ \beta\mathrm{r}' \end{pmatrix} : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C} \end{matrix}$$

which is an isomorphism, because $\mathrm{op}^+(l_+) : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$ is surjective and $\beta\mathrm{r}'$ induces an isomorphism of $\ker \mathrm{op}^+(l_+) = \{\gamma e^{-\alpha t} : \gamma \in \mathbb{C}\}$ to $\mathbb{C}$. Let us form the potential $k = k(\alpha) : \mathbb{C} \to \mathcal{S}(\overline{\mathbb{R}}_+)$, defined by $k\gamma = \gamma\beta^{-1}e^{-\alpha t}, \gamma \in \mathbb{C}$. Then

$$\begin{pmatrix} \mathrm{op}^+(l_+) \\ \beta\mathrm{r}' \end{pmatrix} (\mathrm{op}^+(l_+^{-1}) \quad k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

because $\mathrm{r}'\mathrm{op}^+(l_+^{-1}) = 0$, and hence

$$\begin{pmatrix} \mathrm{op}^+(l_+) \\ \beta\mathrm{r}' \end{pmatrix}^{-1} = (\mathrm{op}^+(l_+^{-1}) \quad k).$$

Consider now $a(\tau) = \alpha^2 + \tau^2$. The operator in (54) can be written

$$\begin{pmatrix} \mathrm{op}^+(a) \\ \beta\mathrm{r}' \end{pmatrix} = \begin{pmatrix} \mathrm{op}^+(l_-) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{op}^+(l_+) \\ \beta\mathrm{r}' \end{pmatrix}$$

and hence

$$\begin{pmatrix} \mathrm{op}^+(a) \\ \beta\mathrm{r}' \end{pmatrix}^{-1} = (\mathrm{op}^+(l_+^{-1}) \quad k) \begin{pmatrix} \mathrm{op}^+(l_-^{-1}) & 0 \\ 0 & 1 \end{pmatrix} =$$

$$(\mathrm{op}^+(l_+^{-1})\mathrm{op}^+(l_-^{-1}) \quad k).$$

Here $\mathrm{op}^+(l_+^{-1})\mathrm{op}^+(l_-^{-1}) = \mathrm{op}^+(a^{-1}) + g$ for a certain $g \in \Gamma^0(\overline{\mathbb{R}}_+)$. It follows altogether

$$\begin{pmatrix} \mathrm{op}^+(a) \\ \beta \mathrm{r}' \end{pmatrix}^{-1} = (\mathrm{op}^+(a^{-1}) + g \qquad k),$$

i.e., we calculated the inverse of (54). Inserting now the expression for $\alpha = \alpha(\eta), \beta = \beta(\eta)$, we easily see that the ingredients of

$$\sigma_{\mathrm{e}'}(\mathcal{A}_2)^{-1} = (\mathrm{op}^+(a^{-1})(\eta) + g(\eta) \qquad k(\eta)) \tag{55}$$

belong to $\mathcal{R}_{\mathrm{cl}}^{-2,0;0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; 1, 0)$ (they are, of course, independent of y), and it is clear that $\mathcal{A}_2^{-1} = \mathrm{Op}_y(\sigma_{\mathrm{e}'}(\mathcal{A}_2)^{-1})$ which belongs to $\mathcal{B}_{\mathrm{cl}}^{-2,0;0}(\overline{\mathbb{R}}_+^n; 1, 0)$. The method of calculating (55) gives us analogously $\sigma_\partial(\mathcal{A}_2)^{-1}$ and $\sigma_{\partial,\mathrm{e}'}(\mathcal{A}_2)^{-1}$, and

$$\sigma(\mathcal{A}_2^{-1}) = (|\xi|^{-2}, (c + |\xi|^2)^{-1}, |\xi|^{-2}; \sigma_\partial(\mathcal{A}_2)^{-1}, \sigma_{\mathrm{e}'}(\mathcal{A}_2)^{-1}, \sigma_{\partial,\mathrm{e}'}(\mathcal{A}_2)^{-1}).$$

It is then obvious how to express $\mathcal{A}_1^{-1}$, namely

$$\mathcal{A}_1^{-1} = \mathcal{A}_2^{-1} \begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix}.$$

Remark 2.24 *Similar arguments apply to the Neumann problem for $c - \Delta$ in the half-space, with $\mathrm{r}'\partial_t$ in place of r'. To get an element with unified orders we can pass to the boundary operator $R\mathrm{r}'\partial_t$ for $R = \mathrm{Op}_y(\langle\eta\rangle^{\frac{1}{2}})$. We see that*

$$\begin{pmatrix} c - \Delta \\ R\mathrm{r}'\partial_t \end{pmatrix} \in \mathcal{B}_{\mathrm{cl}}^{2,2;0}(\overline{\mathbb{R}}_+^n; 0, 1)$$

is also elliptic in the sense of Definition 2.21 and even invertible as an operator $H^s(\mathbb{R}_+^n) \to H^{s-2}(\mathbb{R}_+^n) \oplus H^{s-2}(\mathbb{R}^{n-1})$ for $s > \frac{3}{2}$. The inverse belongs to $\mathcal{B}_{\mathrm{cl}}^{-2,0;0}(\overline{\mathbb{R}}_+^n; 1, 0)$. We shall construct in Section 4.3 below a general class of further examples of this kind.

Theorem 2.25 *For every $N \in \mathbb{N}$ there exist elliptic elements $\mathcal{A}_N^+ \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(\overline{\mathbb{R}}_+^n; 0, N)$ and $\mathcal{A}_N^- \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(\overline{\mathbb{R}}_+^n; N, 0)$ that induce isomorphisms*

$$\mathcal{A}_N^+ : H^s(\mathbb{R}_+^n) \longrightarrow \begin{matrix} H^s(\mathbb{R}_+^n) \\ \oplus \\ H^s(\mathbb{R}^{n-1}, \mathbb{C}^N) \end{matrix},$$

$$\mathcal{A}_N^- : \begin{matrix} H^s(\mathbb{R}_+^n) \\ \oplus \\ H^s(\mathbb{R}^{n-1}, \mathbb{C}^N) \end{matrix} \longrightarrow H^s(\mathbb{R}_+^n)$$

for all $s > -\frac{1}{2}$, where $\mathcal{A}_N^- = (\mathcal{A}_N^+)^{-1}$.

Proof. Let us start from the above operator $\mathcal{A}_2$ and form

$$\mathcal{A}_0 = \mathcal{R}^{s_0-2}\mathcal{A}_2\tilde{\mathcal{R}}^{-s_0} \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(\overline{\mathbb{R}}_+^n; 0, 1) \tag{56}$$

for any fixed $s_0 > 2$, where $\tilde{\mathcal{R}} := R^1 \in B_{\mathrm{cl}}^{1,0;0}(\overline{\mathbb{R}}_+^n)$ is the order reducing element from Theorem 2.15 and $\mathcal{R} := \mathrm{diag}(R^1, R')$ for $R' = \mathrm{Op}_y(\langle\eta\rangle \otimes \mathrm{id}_{\mathbb{C}^N})$. Then, setting $\mathcal{A}_1^+ = \mathcal{A}_0$, we can form $\mathcal{A}_N^+$ inductively by

$$\mathcal{A}_N^+ = \begin{pmatrix} A_N^+ \\ T_N^+ \end{pmatrix} := \begin{pmatrix} A_{N-1}^+ & 0 \\ T_{N-1}^+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1^+ \\ T_1^+ \end{pmatrix} = \begin{pmatrix} A_{N-1}^+ A_1^+ \\ T_{N-1}^+ A_1^+ \\ T_1^+ \end{pmatrix}.$$

Here, $\mathcal{A}_1^+ = \begin{pmatrix} A_1^+ \\ T_1^+ \end{pmatrix}$. Moreover, from the above construction of $\mathcal{A}_2^{-1}$ and Theorem 2.15 it follows that we may set $\mathcal{A}_N^- := (\mathcal{A}_N^+)^{-1}$. □

2.6 Parametrices and Fredholm property

Theorem 2.26 *Let $\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(\overline{\mathbb{R}}_+^n; N_-, N_+)$ be elliptic. Then*

$$\mathcal{A} : \begin{array}{c} H^{s;\varrho}(\mathbb{R}_+^n) \\ \oplus \\ H^{s;\varrho}(\mathbb{R}^{n-1}, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(\mathbb{R}_+^n) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\mathbb{R}^{n-1}, \mathbb{C}^{N_+}) \end{array} \tag{57}$$

is a Fredholm operator for every $s > \max(\mu, d) - \frac{1}{2}$ and every $\varrho \in \mathbb{R}$, and $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{B}_{\mathrm{cl}}^{-\mu,(d-\mu)^+;-\delta}(\overline{\mathbb{R}}_+^n; N_+, N_-)$ where $d_l = \max(\mu, d)$ and $d_r = (d-\mu)^+$ (cf. the notation in Definition 2.23).

The proof of this theorem will be given below after some preparations.

Remark 2.27 *Applying Remark 2.19 we can reduce the proof of Theorem 2.26 to the case $\delta = 0$. In other words, it suffices to consider the operator $\mathcal{S}^{\varrho-\delta}\mathcal{A}\mathcal{S}^{-\varrho} \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;0}(\overline{\mathbb{R}}_+^n; N_-, N_+)$. Furthermore, we can reduce orders and pass to*

$$\mathcal{A}_0 := \mathcal{R}_{(+)}^{s_0-\mu}(\mathcal{S}^{\varrho-\delta}\mathcal{A}\mathcal{S}^{-\varrho})\mathcal{R}_{(-)}^{-s_0} \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(\overline{\mathbb{R}}_+^n; N_-, N_+)$$

for any choice of $s_0 > \max(\mu, d)$, where $\mathcal{R}_{(\pm)} = \mathrm{diag}(R^1, R'_{N_\pm})$ for $R'_{N_\pm} := \mathrm{Op}(\langle\eta\rangle \otimes \mathrm{id}_{\mathbb{C}^{N_\pm}})$, cf. similarly (56). Clearly, the ellipticity of $\mathcal{A}$ is equivalent to that of $\mathcal{A}_0$, and the construction of a parametrix $\mathcal{P}_0$ for $\mathcal{A}_0$ gives us immediately a parametrix $\mathcal{P}$ of $\mathcal{A}$, namely

$$\mathcal{P} = \mathcal{S}^{-\varrho}\mathcal{R}_{(-)}^{-s_0}\mathcal{P}_0\mathcal{R}_{(+)}^{s_0-\mu}\mathcal{S}^{\varrho-\delta}. \tag{58}$$

So we mainly concentrate on the case $\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(\overline{\mathbb{R}}_+^n; N_-, N_+)$.

Let $p(x,\xi) \in S_{\mathrm{cl}}^{0;0}(\overline{\mathbb{R}}_+^n \times \mathbb{R}^n)_{\mathrm{tr},\asymp}$ be a symbol with

$$\sigma_\psi(p) \neq 0 \qquad \text{for all} \qquad (x,\xi) \in \overline{\mathbb{R}}_+^n \times (\mathbb{R}^n \backslash 0), \tag{59}$$

$$\sigma_{\mathrm{e}}(p) \neq 0 \qquad \text{for all} \qquad (x,\xi) \in (\overline{\mathbb{R}}_+^n \backslash 0) \times \mathbb{R}^n, \tag{60}$$

$$\sigma_{\psi,\mathrm{e}}(p) \neq 0 \qquad \text{for all} \qquad (x,\xi) \in (\overline{\mathbb{R}}_+^n \backslash 0) \times (\mathbb{R}^n \backslash 0). \tag{61}$$

Set

$$b'_{11}(y,\eta) := \mathrm{op}^+(p|_{t=0})(y,\eta), \tag{62}$$

and consider the operator families

$$\sigma_\partial(b'_{11})(y,\eta), \sigma_{\mathrm{e}'}(b'_{11})(y,\eta), \sigma_{\partial,\mathrm{e}'}(b'_{11})(y,\eta) : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+), \tag{63}$$

$\sigma_\partial(b'_{11})$ for $(y,\eta) \in \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1}\backslash 0)$, $\sigma_{\mathrm{e}'}(b'_{11})$ for $(y,\eta) \in (\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R}^{n-1}$, $\sigma_{\partial,\mathrm{e}'}(b'_{11})$ for $(y,\eta) \in (\mathbb{R}^{n-1}\backslash 0) \times (\mathbb{R}^{n-1}\backslash 0)$.

These are families of Fredholm operators parametrised by the corresponding sets of (y,η)-variables.

Proposition 2.28 *For every $\varepsilon > 0$ there exists an $R = R_\varepsilon > 0$ such that*

$$||\sigma_\partial(b'_{11})(y,\eta) - \sigma_{\partial,\mathrm{e}'}(b'_{11})(y,\eta)||_{\mathcal{L}(L^2(\mathbb{R}_+))} < \varepsilon \tag{64}$$

for all $|y| > R$ and $\eta \in \mathbb{R}^{n-1}\backslash 0$,

$$||\sigma_\partial(b'_{11})(y,\eta) - \sigma_{\mathrm{e}'}(b'_{11})(y,\eta)||_{\mathcal{L}(L^2(\mathbb{R}_+))} < \varepsilon \tag{65}$$

for all $|y| > R$ and $|\eta| > R$,

$$||\sigma_{\mathrm{e}'}(b'_{11})(y,\eta) - \sigma_{\partial,\mathrm{e}'}(b'_{11})(y,\eta)||_{\mathcal{L}(L^2(\mathbb{R}_+))} < \varepsilon \tag{66}$$

for all $|y| \in \mathbb{R}^{n-1}\backslash 0$ and $|\eta| > R$.

Proof. Let us first verify (64). Both $\mathrm{op}(\sigma_\psi(p)|_{t=0})(y,\eta)$ and $\mathrm{op}(\sigma_{\psi,\mathrm{e}}(p)|_{t=0})(y,\eta)$ can be regarded as parameter-dependent families of pseudo-differential operators $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with parameter $y \in \mathbb{R}^{n-1}$, smoothly dependent on η with $|\eta| = 1$.

But

$$\mathrm{op}(\sigma_\psi(p|_{t=0}) - \sigma_{\psi,\mathrm{e}}(p|_{t=0}))(y,\eta) \tag{67}$$

is of order -1 in the parameter. A well-known result on operator norms of parameter-dependent pseudo-differential operators, cf., e.g., [30], Section 1.2.2, tells us that the $\mathcal{L}(L^2(\mathbb{R}_+))$-norm of (67) tends to zero for $|y| \to \infty$, in this case uniformly for $|y| = 1$. Thus, composing (67) from the right with e^+ and from the left with r^+ we get relation (64) for all $|y| \geq R$, $R = R_\varepsilon$, and $|\eta| = 1$.

In a similar way we can argue for (66), now with $\eta \in \mathbb{R}^{n-1}\backslash 0$ as parameter and smooth dependence on y with $|y| = 1$. This gives us relation (66). Estimate (65) is then an obvious consequence of (64) and (66).

Corollary 2.29 *Under the conditions of Proposition* 2.28 *there exists an $R = R_\varepsilon > 0$ such that the Fredholm families*

$$\sigma_\partial(b'_{11})(y,\eta) : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+) \qquad \textit{for} \qquad 0 \leq |y| \leq R,\ |\eta| = R$$

and

$$\sigma_{\mathrm{e}'}(b'_{11})(y,\eta) : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+) \qquad \textit{for} \qquad |y| = R,\ 0 \leq |\eta| \leq R,$$

satisfy $||\sigma_\partial(b'_{11})(y,\eta) - \sigma_{\mathrm{e}'}(b'_{11})(y,\eta)||_{\mathcal{L}(L^2(\mathbb{R}_+))} < \varepsilon$ for all $|y| = |\eta| = R$.

Let $\varepsilon > 0$, and set

$$T_\varepsilon = \{(y,\eta) \in \mathbb{R}^{2(n-1)} : |y| = |\eta| = R_\varepsilon\},\ D_\varepsilon = T_\varepsilon \times [0,1],$$

and form

$$Z^j_\varepsilon = \{(y,\eta) \in \mathbb{R}^{2(n-1)} :\ |y| \leq R_\varepsilon + j,\ |\eta| = R_\varepsilon\},$$
$$H^j_\varepsilon = \{(y,\eta) \in \mathbb{R}^{2(n-1)} :\ |y| = R_\varepsilon,\ |\eta| \leq R_\varepsilon + j\}$$

for $j = 0, 1, \infty$. Define the spaces $\mathbb{L}^j_\varepsilon = (Z^j_\varepsilon \cup_d H^j_\varepsilon) \cup_b D_\varepsilon / \sim$, where $\cup_d$ is the disjoint union, while $\cup_b$ is the disjoint union combined with the projection to the quotient space, given by natural identifications $T_\varepsilon \cap Z^j_\varepsilon \cong T_\varepsilon \times \{0\}$, $T_\varepsilon \cap H^j_\varepsilon \cong T_\varepsilon \times \{1\}$.

Write for abbreviation $Z_\varepsilon = Z^0_\varepsilon$, $H_\varepsilon = H^0_\varepsilon$, $\mathbb{L}_\varepsilon = \mathbb{L}^0_\varepsilon$. Moreover, let $D_{\varepsilon,\tau} := T_\varepsilon \times [0,\tau]$ and $\mathbb{L}_{\varepsilon,\tau} := Z_\varepsilon \cup_d H_\varepsilon \cup_b D_{\varepsilon,\tau}$, $0 \leq \tau \leq 1$, where $\cup_b$ is defined by means the identifications $T_\varepsilon \cap Z_\varepsilon \cong T_\varepsilon \times \{0\}$, $T_\varepsilon \cap H_\varepsilon \cong T_\varepsilon \times \{\tau\}$. Thus $\mathbb{L}_\varepsilon = \mathbb{L}_{\varepsilon,1}$, and we set $\mathbb{B}_\varepsilon = \mathbb{L}_{\varepsilon,0}$.

Define an operator function $F(m), m \in \mathbb{L}_\varepsilon$, by the following relations:

$$F(y,\eta) = \sigma_\partial(b'_{11})(y,\eta) \qquad \text{for} \qquad m=(y,\eta)\in Z_\varepsilon,$$

$$F(y,\eta) = \sigma_{\mathrm{e}'}(b'_{11})(y,\eta) \qquad \text{for} \qquad m=(y,\eta)\in H_\varepsilon,$$

$$F(y,\eta,\delta) = \delta\sigma_\partial(b'_{11})(y,\eta) + (1-\delta)\sigma_{\mathrm{e}'}(b'_{11})(y,\eta) \qquad \text{for} \qquad m=(y,\eta,\delta)\in D_\varepsilon.$$

From Corollary 2.29 we get

$$||F(y,\eta,\delta) - F(y,\eta,\delta')||_{\mathcal{L}(L^2(\mathbb{R}_+))} \le |\delta-\delta'|\varepsilon \tag{68}$$

for all $(y,\eta,\delta),(y,\eta,\delta')\in D_\varepsilon, 0\le\delta,\delta'\le 1$. We have

$$F\in C(\mathbb{L}_\varepsilon, \mathcal{L}(L^2(\mathbb{R}_+))), \tag{69}$$

and $F|_{Z_\varepsilon}, F|_{H_\varepsilon}$ are continuous families of Fredholm operators. Relation (68) shows that (69) is a family of Fredholm operators for all $m\in\mathbb{L}_\varepsilon$, provided $\varepsilon>0$ is sufficiently small. We then get an index element $\mathrm{ind}_{\mathbb{L}_\varepsilon} F \in K(\mathbb{L}_\varepsilon)$. Because of $K(\mathbb{L}_{\varepsilon,\tau})\cong K(\mathbb{L}_\varepsilon)$ for all $0\le\tau\le 1$, $\mathrm{ind}_{\mathbb{L}_\varepsilon} F$ represents, in fact, an element in $K(\mathbb{B}_\varepsilon)$ that we denote by

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(b'_{11})(y,\eta), \sigma_{\mathrm{e}'}(b'_{11})(y,\eta)\}\in K(\mathbb{B}_\varepsilon). \tag{70}$$

Our next objective is to check, whether the operator family $b'_{11}(y,\eta)$ for an elliptic symbol $p(x,\xi)\in S^{0;0}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)_{\mathrm{tr}}$ can be completed to a block matrix valued symbol

$$\boldsymbol{b}'(y,\eta) = \begin{pmatrix} b'_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}(y,\eta)\in S^{0;0}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\boldsymbol{E},\tilde{\boldsymbol{E}}), \tag{71}$$

$$\boldsymbol{E} = \begin{matrix} L^2(\mathbb{R}_+)\\ \oplus \\ \mathbb{C}^{N_-}\end{matrix}, \quad \tilde{\boldsymbol{E}} = \begin{matrix} L^2(\mathbb{R}_+)\\ \oplus \\ \mathbb{C}^{N_+}\end{matrix} \tag{72}$$

with suitable N_-, N_+, such that the homogeneous symbols

$$\sigma_\partial(\boldsymbol{b}')(y,\eta)\in S^{(0)}_\eta, \quad (y,\eta)\in\mathbb{R}^{n-1}\times(\mathbb{R}^{n-1}\backslash 0), \tag{73}$$

$$\sigma_{\mathrm{e}'}(\boldsymbol{b}')(y,\eta)\in S^{(0)}_y, \quad (y,\eta)\in(\mathbb{R}^{n-1}\backslash 0)\times\mathbb{R}^{n-1}, \tag{74}$$

$$\sigma_{\partial,\mathrm{e}'}(\boldsymbol{b}')(y,\eta)\in S^{(0;0)}_{\eta;y}, \quad (y,\eta)\in(\mathbb{R}^{n-1}\backslash 0)\times(\mathbb{R}^{n-1}\backslash 0), \tag{75}$$

are isomorphisms.

Theorem 2.30 *Let $p(x,\xi)\in S^{0;0}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)_{\mathrm{tr}}$ be $(\sigma_\psi,\sigma_{\mathrm{e}},\sigma_{\psi,\mathrm{e}})$-elliptic, i.e., relations* (59), (60) *and* (61) *are fulfiled. Then the following conditions are equivalent:*

(*i*) *The families of Fredholm operators $L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$*

$$\sigma_\partial(b'_{11})(y,\eta) \quad \text{for} \quad (y,\eta)\in\mathbb{R}^{n-1}\times(\mathbb{R}^{n-1}\backslash 0), \tag{76}$$

$$\sigma_{\mathrm{e}'}(b'_{11})(y,\eta) \quad \text{for} \quad (y,\eta)\in(\mathbb{R}^{n-1}\backslash 0)\times\mathbb{R}^{n-1}, \tag{77}$$

$$\sigma_{\partial,\mathrm{e}'}(b'_{11})(y,\eta) \quad \text{for} \quad (y,\eta)\in(\mathbb{R}^{n-1}\backslash 0)\times(\mathbb{R}^{n-1}\backslash 0) \tag{78}$$

can be completed to $D^{0,0}(\overline{\mathbb{R}}_+;N_-,N_+)$-valued families of isomorphisms (73), (74) *and* (75), *respectively.*

(ii)

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(b'_{11}), \sigma_{e'}(b'_{11})\} \in \pi_+^* K(\{+\}), \tag{79}$$

where $\pi_+ : \mathbb{B}_\varepsilon \to \{+\}$ *is the projection of* $\mathbb{B}_\varepsilon$ *to a single point* $\{+\}$, $(K(\{+\}) = \mathbb{Z})$.

Proof. $(i) \Rightarrow (ii)$: In the construction of the proof we choose $\varepsilon > 0$ sufficiently small. Assume that we have isomorphism-valued symbols (73), (74) and (75), associated with the given upper left corners (76), (77) and (78). Then the above Fredholm family $F(m)$ on $\mathbb{L}_\varepsilon$, associated with $\{\sigma_\partial(b'_{11}), \sigma_{e'}(b'_{11})\}$ has the property $\mathrm{ind}_{\mathbb{L}_\varepsilon} F = [\mathbb{C}^{N_+}] - [\mathbb{C}^{N_-}]$, i.e., $\mathrm{ind}_{\mathbb{L}_\varepsilon} F \in \mathbb{Z}$ which implies $\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(b'_{11}), \sigma_{e'}(b'_{11})\} \in \mathbb{Z} \cong \pi_+^* K(\{+\})$.

$(ii) \Rightarrow (i)$: Condition $\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(b'_{11}), \sigma_{e'}(b'_{11})\} \in \pi_+^* K(\{+\})$ implies the existence of numbers $N_\pm \in \mathbb{N}$ with $\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(b'_{11}), \sigma_{e'}(b'_{11})\} = [\mathbb{C}^{N_+}] - [\mathbb{C}^{N_-}]$. Replacing $N_\pm$ by $N_\pm + M$ for sufficiently large M and denoting the enlarged numbers again by $N_\pm$ we find operator families

$$k(m) : \mathbb{C}^{N_-} \longrightarrow L^2(\mathbb{R}_+),\ t(m) : L^2(\mathbb{R}_+) \longrightarrow \mathbb{C}^{N_+},\ q(m) : \mathbb{C}^{N_-} \longrightarrow \mathbb{C}^{N_+},$$

such that

$$f(m) := \begin{pmatrix} F(m) & k(m) \\ t(m) & q(m) \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \tag{80}$$

is a family of isomorphisms, continuously parametrised by $\mathbb{L}_\varepsilon$. It is evident that they can be chosen as $D^{0;0}(\overline{\mathbb{R}}_+; N_-, N_+)$-valued functions, similarly to the construction of bijective boundary symbols in the local algebra of boundary value problems with the transmission property. In addition it is clear that the functions $k(m)$, $t(m)$ and $q(m)$ can be chosen to be smooth in (y, η).

Let us now define a Fredholm family $F^1(m)$ for $m \in \mathbb{L}_\varepsilon^1$ by

$$F^1(m) = F(m) \quad \text{for} \quad m \in \mathbb{L}_\varepsilon,$$

$$F^1(m) = (1 - \lambda)\sigma_\partial(b'_{11})(y, \eta) + \lambda\sigma_{\partial,e'}(b'_{11})(y, \eta)$$

$$\text{for} \quad R_\varepsilon \le |y| \le R_\varepsilon + 1, |\eta| = R_\varepsilon, \quad \text{where} \quad \lambda = R_\varepsilon - |y|,$$

$$F^1(m) = (1 - \lambda)\sigma_{e'}(b'_{11})(y, \eta) + \lambda\sigma_{\partial,e'}(b'_{11})(y, \eta)$$

$$\text{for} \quad |y| = R_\varepsilon,\ R_\varepsilon \le |\eta| \le R_\varepsilon + 1, \quad \text{where} \quad \lambda = R_\varepsilon - |\eta|.$$

Estimates (64) and (66) show that F^1 is a family of Fredholm operators on $\mathbb{L}_\varepsilon^1$, provided $\varepsilon > 0$ is sufficiently small. We can construct a family of isomorphisms

$$f^1(m) := \begin{pmatrix} F^1(m) & k^1(m) \\ t^1(m) & q^1(m) \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}, \tag{81}$$

$m \in \mathbb{L}_\varepsilon^1$, similarly as $f(m)$ (if necessary, we take N_-, N_+ larger than before), where $f^1|_{\mathbb{L}_\varepsilon} = f$. Since $F(m)$ is a-priori given on $\mathbb{L}_\varepsilon^\infty$, we can also form

$$\tilde{f}^1(m) := \begin{pmatrix} F(m) & k^1(m) \\ t^1(m) & q^1(m) \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix},$$

$m \in \mathbb{L}_\varepsilon^1$. Due to (64) and (66) this is a family of Fredholm operators. Clearly, we may choose $\tilde{f}^1(m)$ in such a way that $\tilde{f}^1|_{Z_\varepsilon^1}$ and $\tilde{f}^1|_{H_\varepsilon^1}$ are smooth in (y, η). Let us finally look at $\mathbb{L}_\varepsilon^\infty$. The operator function f^1, first given on $\mathbb{L}_\varepsilon^1$, canonically extends to $\mathbb{L}_\varepsilon^\infty$ by homogeneity of order zero to $\mathbb{L}_\varepsilon^\infty \backslash \mathbb{L}_\varepsilon^1$ in y and η. Let f^∞ denote this extension,

$$f^\infty(m) := \begin{pmatrix} F^\infty(m) & k^\infty(m) \\ t^\infty(m) & q^\infty(m) \end{pmatrix} \tag{82}$$

i.e., $f^\infty|_{\mathbb{L}_\varepsilon^1} = f^1$. Since f^∞ is obtained by homogeneous extension of a family of isomorphisms, it is again isomorphism-valued. Moreover, we can also form

$$\tilde{f}^\infty(m) := \begin{pmatrix} F(m) & k^\infty(m) \\ t^\infty(m) & q^\infty(m) \end{pmatrix}$$

which is a family of isomorphisms because of the corresponding property of (82) and relations (64) and (66).

Then, to get (73), (74) and (75), it is suffices to define $\sigma_\partial(\boldsymbol{b}')(y,\eta)$ as the extension by homogeneity 0 in η of $\tilde{f}^\infty|_{Z_\varepsilon^\infty}$ to $\mathbb{R}^{n-1} \times (\mathbb{R}^{n-1}\backslash 0)$, $\sigma_{\mathrm{e}'}(\boldsymbol{b}')(y,\eta)$ as the extension by homogeneity 0 in y of $\tilde{f}^\infty|_{H_\varepsilon^\infty}$ to $(\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R}^{n-1}$ and $\sigma_{\partial,\mathrm{e}'}(\boldsymbol{b}')(y,\eta)$ as the extension by homogeneity 0 in y and η of $\tilde{f}^\infty|_{\{|y|=R_\varepsilon+1, |\eta|=R_\varepsilon+1\}}$ to $(\mathbb{R}^{n-1}\backslash 0) \times (\mathbb{R}^{n-1}\backslash 0)$. To justify the notation in (73), (74) and (75) (i.e., to generate the latter homogeneous functions in terms of a symbol (71)) we can first form $\boldsymbol{b}''(y,\eta) = \chi(\eta)\sigma_\partial(\boldsymbol{b}')(y,\eta) + \chi(y)\{\sigma_{\mathrm{e}'}(\boldsymbol{b}')(y,\eta) - \chi(\eta)\sigma_{\partial,\mathrm{e}'}(\boldsymbol{b}')(y,\eta)\} \in S^{0;0}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \boldsymbol{E}, \tilde{\boldsymbol{E}})$, cf. the second part of Remark 1.9, and then define $\boldsymbol{b}'(y,\eta)$ by replacing the upper left entry of $\boldsymbol{b}''(y,\eta)$ by $b'_{11}(y,\eta)$. $\square$

Remark 2.31 *Notice that Theorem* 2.30 *is an analogue of the Atiyah-Bott condition for the existence of elliptic boundary conditions to an elliptic operator A, cf. also Section* 3.4 *below.*

The canonical projection $T^*\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ restricted to the subset $\mathbb{B}_\varepsilon \subset T^*\mathbb{R}^{n-1}$ gives us a projection $\pi_\varepsilon : \mathbb{B}_\varepsilon \to B_\varepsilon := \{y \in \mathbb{R}^{n-1} : |y| \leq R_\varepsilon\}$. Condition (79) *can equivalently be written*

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(b'_{11}), \sigma_{\mathrm{e}'}(b'_{11})\} \in \pi_\varepsilon^* K(B_\varepsilon),$$

since B_ε is contractible to a point $\{+\}$.

Corollary 2.32 *Given a symbol $p(x,\xi) \in S^{0;0}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$ that is $(\sigma_\psi, \sigma_{\mathrm{e}}, \sigma_{\psi,\mathrm{e}})$-elliptic, under the condition* (79) *for $b'_{11}(y,\eta) = \mathrm{op}^+(p|_{t=0})(y,\eta)$ we find a*

$$\boldsymbol{b}(y,\eta) \in \mathcal{R}^{0,0;0}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; N_-, N_+)$$

for suitable $N_-, N_+ \in \mathbb{N}$, such that (76), (77) *and* (78) *are isomorphims $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-} \to L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}$, cf. Definition* 2.21. *To construct $\boldsymbol{b}(y,\eta)$ it suffices to define $\boldsymbol{b}(y,\eta)$ by replacing the upper left entry of $\boldsymbol{b}''(y,\eta)$ by $\mathrm{op}^+(p_\asymp)(y,\eta)$ for $p_\asymp(x,\xi) = \chi_\asymp(x)p(x,\xi)$ with some global admissible cut-off function $\chi_\asymp$, cf. Definition* 2.6.

Proposition 2.33 *Let $G \in B^{0,0;0}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ be an operator such that $A := 1 + G$ is elliptic in the sense of Definition* 2.21. *Then there is a $\tilde{G} \in B^{0,0;0}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ such that $\tilde{A} := 1 + \tilde{G}$ is a parametrix of A, i.e., $A\tilde{A} - 1, \tilde{A}A - 1 \in B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$.*

Proof. Let us first observe that for every $g \in \Gamma^0(\overline{\mathbb{R}}_+)$ (i.e., $g \in \mathcal{L}(L^2(\mathbb{R}_+))$) with $g, g^* : L^2(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+)$ being continuous, cf. Section 2.1, we have $ag, ga \in \Gamma^0(\overline{\mathbb{R}}_+)$

for every $a \in \mathcal{L}(L^2(\mathbb{R}_+))$. Then, if $1+g : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ for a $g \in \mathcal{L}(L^2(\mathbb{R}_+))$ is invertible, we have $a = (1+g)^{-1} \in \mathcal{L}(L^2(\mathbb{R}_+))$ and $a(1+g) = 1 = a + ag$, i.e., $a = 1 + \tilde{g}$ for $\tilde{g} = -ag \in \Gamma^0(\overline{\mathbb{R}}_+)$. Analogous conclusions are valid for the symbols $\sigma_\partial(1+G), \sigma_{\mathrm{e}'}(1+G)$ and $\sigma_{\partial,\mathrm{e}'}(1+G)$. Then, setting

$$\tilde{g}_\partial(y,\eta) := \sigma_\partial(1+G)^{-1}(y,\eta) - 1, \tilde{g}_{\mathrm{e}'}(y,\eta) := \sigma_{\mathrm{e}'}(1+G)^{-1}(y,\eta) - 1,$$

$$\tilde{g}_{\partial,\mathrm{e}'}(y,\eta) := \sigma_{\partial,\mathrm{e}'}(1+G)^{-1}(y,\eta) - 1,$$

we can form $\tilde{g}(y,\eta) := \chi(\eta)\tilde{g}_\partial(y,\eta) + \chi(y)(\tilde{g}_{\mathrm{e}'}(y,\eta) - \chi(\eta)\tilde{g}_{\partial,\mathrm{e}'}(y,\eta)) \in R^{0,0;0}_{G,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ cf. Remark 1.9. For $\tilde{G}_1 = \mathrm{Op}_y(\tilde{g})$ we then have $(1+G)(1+\tilde{G}_1) = 1 + \tilde{G}_2$ where $\tilde{G}_2 \in B^{-1,0;-1}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$. Then $\tilde{G}_2^j \in B^{-j,0;-j}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ for all j, and we can carry out the asymptotic sum $\sum_{j=0}^\infty (-1)^j \tilde{G}_2^j$ in the class of operators $1 + B^{-1,0;-1}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ (which is just a version of the formal Neumann series argument in our operator class). In other words, we can find a $\tilde{G}_3 \in B^{-1,0;-1}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ such that $(1+\tilde{G}_2)(1+\tilde{G}_3) = 1 + C$ for $C \in B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$. Because $(1+\tilde{G}_1)(1+\tilde{G}_3) = 1 + \tilde{G}$ for some $\tilde{G} \in B^{0,0;0}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ we get $(1+G)(1+\tilde{G}) = 1 + C$. Similar arguments from the left yield a $\tilde{\tilde{G}} \in B^{0,0;0}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ with $(1+\tilde{\tilde{G}})(1+G) - 1 \in B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$. Then a standard algebraic argument gives us $\tilde{G} = \tilde{\tilde{G}} \bmod B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$. In other words $\tilde{A} = 1 + \tilde{G}$ is as desired. □

Proof of Theorem 2.26. As noted in Remark 2.27 we may content ourselves with the case $\mu = d = \delta = 0$. The ellipticity of $\mathcal{A}$ with respect to $(\sigma_\psi(\mathcal{A}), \sigma_{\mathrm{e}}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}))$ allows us to form a symbol $p(x,\xi) = \chi(\xi)\sigma_\psi(\mathcal{A})^{-1}(x,\xi) + \chi(x)\{\sigma_{\mathrm{e}}(\mathcal{A})^{-1}(x,\xi) - \chi(\xi)\sigma_{\psi,\mathrm{e}}(\mathcal{A})^{-1}(x,\xi)\} \in S^{0;0}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$, where $(\sigma_\psi(p), \sigma_{\mathrm{e}}(p), \sigma_{\psi,\mathrm{e}}(p)) = (\sigma_\psi(\mathcal{A})^{-1}, \sigma_{\mathrm{e}}(\mathcal{A})^{-1}, \sigma_{\psi,\mathrm{e}}(\mathcal{A})^{-1})$. We now observe that $p(x,\xi)$ meets the assumption of Theorem 2.30. In fact, the original symbol $a(x,\xi)$ belonging to $\mathcal{A}$ satisfies these conditions because the assumed bijectivities just correspond to the ellipticity of $\mathcal{A}$ with respect to $(\sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A}))$. Hence relation (80) with respect to $a(x,\xi)$ is fulfiled. This implies the corresponding relation with respect to $p(x,\xi)$ because the index element in $\pi_+^* K(\{+\})$ is just the inverse of that for $a(x,\xi)$. By construction we have $p(x,\xi)a(x,\xi) = 1 + r(x,\xi)$ for an $r(x,\xi) \in S^{-1;-1}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$. This yields

$$p(x,\xi)\#a(x,\xi) = 1 + \tilde{r}(x,\xi) \tag{83}$$

for an $\tilde{r}(x,\xi) \in S^{-1;-1}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$. A formal Neumann series argument, applied to $1 + \tilde{r}(x,\xi)$ in terms of the Leibniz multiplication $\#$ gives us a symbol $\tilde{q}(x,\xi) \in S^{-1;-1}_{\mathrm{cl}_{\xi;x}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)_{\mathrm{tr}}$ such that $(1+\tilde{q}(x,\xi))\#(1+\tilde{r}(x,\xi)) = 1 \bmod S^{-\infty;-\infty}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$. Setting $\tilde{p}(x,\xi) = (1+\tilde{q}(x,\xi))\#p(x,\xi)$ from relation (83) we get $\tilde{p}(x,\xi)\#a(x,\xi) = 1 \bmod S^{-\infty;-\infty}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$. Applying Corollary 2.32 to $\tilde{p}(x,\xi)$ we can generate a $\boldsymbol{b}(y,\eta)$ of the asserted kind, more precisely $\boldsymbol{b}(y,\eta) \in \mathcal{R}^{0,0;0}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Then the operator

$$\tilde{\mathcal{P}} := \mathrm{Op}(\boldsymbol{b}) + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$$

for $R := (1 - \chi_{\asymp}(x))\mathrm{Op}_x(\tilde{p})$, cf. Definition 2.6, has the property $\mathcal{A}\tilde{\mathcal{P}} = \mathcal{I} + \mathcal{G}$ for some $\mathcal{G} \in \mathcal{B}^{0,0;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_+, N_+)$. Since $\mathcal{A}$ and $\tilde{\mathcal{P}}$ are both elliptic also $\mathcal{I} + \mathcal{G}$ is elliptic. Applying Theorem 2.25 to $N = N_+$ we can pass to the elliptic operator $\mathcal{A}_{N_+}(\mathcal{I}+\mathcal{G})\mathcal{A}^{-1}_{N_+}$ that has the form $1 + G$ for a $G \in B^{0,0;0}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$. Proposition 2.33 gives us a $\tilde{G} \in$

$B^{0,0;0}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ such that $(1+G)(1+\tilde{G}) = 1+C$ for a $C \in B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$. It follows that $\mathcal{A}_{N_+}\mathcal{A}\tilde{\mathcal{P}}\mathcal{A}^{-1}_{N_+}(1+\tilde{G}) = 1+C$ for an element $C \in B^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+)$. This yields

$$\mathcal{A}\tilde{\mathcal{P}}\mathcal{A}^{-1}_{N_+}(1+\tilde{G})\mathcal{A}_{N_+} = \mathcal{A}^{-1}_{N_+}(1+C)\mathcal{A}_{N_+} = 1+\mathcal{C}$$

for a remainder $\mathcal{C} \in \mathcal{B}^{-\infty,0;-\infty}(\overline{\mathbb{R}}^n_+; N_+, N_+)$. Hence,

$$\mathcal{P}_0 := \tilde{\mathcal{P}}\mathcal{A}^{-1}_{N_+}(1+\tilde{G})\mathcal{A}_{N_+} \in \mathcal{B}^{0,0;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; N_+, N_-)$$

is a right parametrix of $\mathcal{A}$. In an analogous manner we can construct a parametrix from the left; then a standard argument shows that $\mathcal{P}_0$ is also a left parametrix. In other words, when we go back to the original orders of Theorem 2.26, we get a parametrix $\mathcal{P}$ by formula (58), where its type is $(d-\mu)^+$ and the types d_l and d_r of remainders are an immediate consequence of Theorem 2.18. The Fredholm property of (57) follows from the fact that the remainders are compact operators in the respective spaces, since they improve smoothness and weight. This completes the proof of Theorem 2.26. □

3 The global theory

3.1 Boundary value problems on smooth manifolds

The calculus of boundary value problems that we intend to develop in Section 3.2 below on a manifold with exits to infinity will be a substructure of a corresponding calculus on a general (not necessarily compact) smooth manifold with smooth boundary. This is, in fact, Boutet de Monvel's algebra [3] that we employ as the corresponding background. Concerning details, cf. the monograph of Rempel and Schulze [16] or Schulze [30], Chapter 4. For future references we want to give a brief description.

Let M be a smooth manifold with smooth boundary ∂M, choose vector bundles $E, F \in \mathrm{Vect}(M)$, $J^-, J^+ \in \mathrm{Vect}(\partial M)$, and set $\boldsymbol{v} = (E, F; J^-, J^+)$. We then have the space $\mathcal{B}^{-\infty,0}(M;\boldsymbol{v})$ of all smoothing operators $C_0^\infty(M,E) \oplus C_0^\infty(\partial M, J^-) \to C^\infty(M,F) \oplus C^\infty(\partial M, J^+)$ of type 0 that are given by corresponding C^∞ kernels, smooth up to boundary (in the corresponding variables on M). Integrations refer to Riemannian metrics on M and ∂M that we keep fixed in the sequel, further to Hermitian metrics in the occurring vector bundles. Assume that the Riemannian metric on M induces the product metric of $(\partial M) \times [0,1)$ in a collar neighbourhood of ∂M. Incidentally we employ $2M$, the double of M, obtained by gluing together two copies of M along ∂M by an identification diffeomorphism. On M we have the space $\mathrm{Diff}^j(M; E, F)$ of all differential operators of order j acting between sections in the bundles E and F. Then $\mathcal{B}^{-\infty,d}(M;\boldsymbol{v})$, the space of all smoothing operators on M of type $d \in \mathbb{N}$, is defined to be the set of all

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=1}^{d} \mathcal{G}_j \begin{pmatrix} D^j & 0 \\ 0 & 0 \end{pmatrix}$$

for arbitrary $\mathcal{G}_0, \ldots, \mathcal{G}_d \in \mathcal{B}^{-\infty,0}(M;\boldsymbol{v})$ and $D^j \in \mathrm{Diff}^j(M; E, F)$. To introduce the space of Green operators on M we first consider an open set $\Omega \subseteq \mathbb{R}^{n-1}$, $n = \mathrm{dim} M$, and define $\mathcal{R}^{\mu,d}_G(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$, the space of all Green symbols of order μ and type d, to be the set of all

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix}$$

for $g_j(y,\eta) \in \mathcal{R}_G^{\mu-j,0}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$. Here $\mathcal{R}_G^{\nu,0}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+) \ni g(y,\eta)$ is given by the conditions

$$g(y,\eta) \in S_{\mathrm{cl}}^{\nu}(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{N_+}),$$

$$g^*(y,\eta) \in S_{\mathrm{cl}}^{\nu}(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{N_+}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{N_-}),$$

$x = (y,t)$ is the splitting of variables in local coordinates near ∂M (cf. analogously Definition 2.8), and k, m, N_- and N_+ are the fibre dimensions of E, F, J^- and J^+, respectively. Now $\mathcal{B}_G^{\mu,d}(M; \boldsymbol{v})$ is defined to be the set of all operators of the form $\mathcal{G}_0 + \mathcal{C}$ for arbitrary $\mathcal{C} \in \mathcal{B}^{-\infty,d}(M; \boldsymbol{v})$ and operators $\mathcal{G}_0$ that are concentrated in a collar neighbourhood of ∂M and are locally finite sums of operators of the form $\mathrm{Op}(g)$ for certain $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$. The pull-backs refer to charts $U \to \Omega \times \overline{\mathbb{R}}_+$ for coordinate patches U near ∂M and trivialisations of the involved bundles; "$\mathcal{G}_0$ concentrated near ∂M" means that for certain functions $\varphi, \psi \in C^\infty(M)$ that equal 1 in a collar neighbourhood of ∂M and 0 outside another collar neighbourhood of ∂M we have $\mathcal{G}_0 = \mathcal{M}_\varphi \mathcal{G}_0 \mathcal{M}_\psi$, cf. similar notation in (92) below.

Finally, let $L_{\mathrm{cl}}^{\mu}(2M; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ for $\tilde{E}, \tilde{F} \in \mathrm{Vect}(2M)$ denote the subspace of all $\tilde{A} \in L_{\mathrm{cl}}^{\mu}(2M; \tilde{E}, \tilde{F})$ (classical "in ξ-variables") pseudo-differential operators on $2M$ of order μ acting between sections of the bundles $\tilde{E}, \tilde{F}$ that have the transmission property with respect to ∂M. We employ the standard Sobolev spaces $H_{\mathrm{comp}}^s(M, E), H_{\mathrm{loc}}^s(M, E)$ of smoothness $s \in \mathbb{R}$ for bundles $E \in \mathrm{Vect}(M)$. "comp" and "loc" are understood in the sense $H_{\mathrm{comp}}^s(M, E) = H_{\mathrm{comp}}^s(2M, \tilde{E})|_M$, $H_{\mathrm{loc}}^s(M, E) = H_{\mathrm{loc}}^s(2M, \tilde{E})|_M$ for any $\tilde{E} \in \mathrm{Vect}(2M)$ with $E = \tilde{E}|_M$. For every $\tilde{A} \in L_{\mathrm{cl}}^{\mu}(2M; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ and $E = \tilde{E}|_M, F = \tilde{F}|_M$ we can form $\mathrm{r}^+ \tilde{A} \mathrm{e}^+$, where e^+ is the extension by zero from $\mathrm{int}M$ to $2M$ and r^+ the restriction from $2M$ to $\mathrm{int}M$; this gives us continuous operators

$$\mathrm{r}^+ \tilde{A} \mathrm{e}^+ : H_{\mathrm{comp}}^s(M, E) \longrightarrow H_{\mathrm{loc}}^{s-\mu}(M, F)$$

for all $s > -\frac{1}{2}$.

Definition 3.1 *The space $\mathcal{B}^{\mu,d}(M; \boldsymbol{v})$ for $\mu \in \mathbb{Z}, d \in \mathbb{N}, \boldsymbol{v} = (E, F; J^-, J^+)$, is defined to be the set of all operators*

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ \tilde{A} \mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} \tag{84}$$

for arbitrary $\tilde{A} \in L_{\mathrm{cl}}^{\mu}(2M; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ and $\mathcal{G} \in \mathcal{B}_G^{\mu,d}(M; \boldsymbol{v})$.

An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v})$ induces continuous operators

$$\mathcal{A} : \begin{array}{c} H_{\mathrm{comp}}^s(M, E) \\ \oplus \\ H_{\mathrm{comp}}^s(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} H_{\mathrm{loc}}^{s-\mu}(M, F) \\ \oplus \\ H_{\mathrm{loc}}^{s-\mu}(\partial M, J^+) \end{array}$$

for all $s > d - \frac{1}{2}$ (which entails continuity between C^∞ sections. In particular, if M is compact, "comp" and "loc" are superfluous, and we get continuous operators

$$\mathcal{A} : \begin{array}{c} H^s(M, E) \\ \oplus \\ H^s(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(M, F) \\ \oplus \\ H^{s-\mu}(\partial M, J^+). \end{array} \tag{85}$$

The principal symbol structure of $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ consists of a pair

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})),$$

where $\sigma_\psi(\mathcal{A})$, the homogeneous principal interior symbol of order μ, is a bundle homomorphism

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(\tilde{A})|_{T^*M\setminus 0} : \pi_\psi^* E \longrightarrow \pi_\psi^* F, \tag{86}$$

for $\pi_\psi : T^*M\setminus 0 \longrightarrow M$, and $\sigma_\partial(\mathcal{A})$, the homogeneous principal boundary symbol of order μ, a bundle homomorphism

$$\sigma_\partial(\mathcal{A}) : \pi_\partial^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_\partial^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{87}$$

for $\pi_\partial : T^*(\partial M)\setminus 0 \to \partial M$. Alternatively, $\sigma_\partial(\mathcal{A})$ may be regarded as a homomorphism

$$\sigma_\partial(\mathcal{A}) : \pi_\partial^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_\partial^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{88}$$

for all $s > d - \frac{1}{2}$, cf. Remark 2.22. Setting $\mathrm{symb}\mathcal{B}^{\mu,d}(M;\boldsymbol{v}) = \{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})\}$ there is a map

$$\mathrm{op} : \mathrm{symb}\mathcal{B}^{\mu,d}(M;\boldsymbol{v}) \longrightarrow \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$$

with $\sigma \circ \mathrm{op} = \mathrm{id}$ on the symbol space. We have $\sigma(\mathcal{A}) = 0 \Rightarrow \mathcal{A} \in \mathcal{B}^{\mu-1,d}(M;\boldsymbol{v})$; if M is compact, the operator (85) is compact when its symbol vanishes.

Theorem 3.2 *Let M be compact; then $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v}), \boldsymbol{v} = (E_0, F; J_0, J^+)$, and $\mathcal{B} \in \mathcal{B}^{\nu,e}(M;\boldsymbol{w}), \boldsymbol{w} = (E, E_0; J^-, J_0)$, implies $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h}(M;\boldsymbol{v}\circ\boldsymbol{w})$, for $h = \max(\nu + d, e)$, $\boldsymbol{v}\circ\boldsymbol{w} = (E, F; J^-, J^+)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ (with componentwise multiplication). An analogous result holds for general M when we replace the composition by $\mathcal{A}\mathcal{M}_\varphi\mathcal{B}$ for a compactly supported $\varphi \in C^\infty(M)$ where $\sigma(\mathcal{A}\mathcal{M}_\varphi\mathcal{B}) = \sigma(\mathcal{A}\mathcal{M}_\varphi)\sigma(\mathcal{B})$.*

An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ is called elliptic, if both (86), and (87) are isomorphisms (the second condition is equivalent to the bijectivity of (88) for all $s > \max(\mu, d) - \frac{1}{2}$).

Theorem 3.3 *Let M be compact. Then the following conditions are equivalent:*

(i) $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ *is elliptic,*

(ii) the operator (85) *is Fredholm for some $s = s_0 > \max(\mu, d) - \frac{1}{2}$.*

If $\mathcal{A}$ is elliptic, then (85) *is a Fredholm operator for all $s > \max(\mu, d) - \frac{1}{2}$, and there is a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+}(M;\boldsymbol{v}^{-1})$ of $\mathcal{A}$ in the sense*

$$\mathcal{PA} - \mathcal{I} \in \mathcal{B}^{-\infty,d_l}(M;\boldsymbol{v}_l), \;\; \mathcal{AP} - \mathcal{I} \in \mathcal{B}^{-\infty,d_r}(M;\boldsymbol{v}_r) \tag{89}$$

for $d_l = \max(\mu, d)$, $\boldsymbol{v}_l = (E, E; J^-, J^-)$, $d_r = (d-\mu)^+$, $\boldsymbol{v}_r = (F, F; J^+, J^+)$.

Remark 3.4 *Ellipticity of $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ for non-compact M entails the existence of a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+}(M;\boldsymbol{v}^{-1})$, where* (89) *is to be replaced by*

$$\mathcal{M}_\psi\mathcal{P}\mathcal{M}_\varphi\mathcal{A} - \mathcal{M}_\varphi \in \mathcal{B}^{-\infty,d_l}(M;\boldsymbol{v}_l), \;\; \mathcal{M}_\varphi\mathcal{A}\mathcal{M}_\psi\mathcal{P} - \mathcal{M}_\varphi \in \mathcal{B}^{-\infty,d_r}(M;\boldsymbol{v}_r)$$

for arbitrary $\varphi, \psi \in C_0^\infty(M)$ with $\varphi\psi = \varphi$ (and $\mathcal{M}_\varphi, \mathcal{M}_\psi$ being the multiplication operators, containing evident tensor products with the identity maps in the respective vector bundles).

3.2 Calculus on manifolds with exits to infinity

In this paper a manifold M with boundary and conical exits to infinity is defined to be a smooth manifold with smooth boundary containing a submanifold C that is diffeomorphic (in the sense of manifolds with boundary) to $(1-\varepsilon,\infty)\times X$ with a smooth compact manifold X with smooth boundary Y, where $M\backslash C$ is compact. Concerning the local descriptions we proceed similarly to Section 1.3 above. To simplify the considerations we assume (without loss of generality) that there is a smooth manifold $2M$ without boundary (the double of M) where $2M$ has conical exits to infinity, cf. Section 1.3, with $2X$ being the base of the infinite part of $2M$ that is diffeomorphic to $(1-\varepsilon,\infty)\times(2X)$. Here $2X$, the double of X, is obtained from two copies of X, glued together along the common boundary Y by an identification diffeomorphism to a smooth closed compact manifold.

To describe the pseudo-differential calculus of boundary value problems on M we mainly concentrate on C; the calculus on the "bounded" part of M has been explained in Section 3.1. If $\{\tilde{U}_j\}_{j=1,\dots,N}$ denotes an open covering of $2M$ of analogous meaning as (11), we have the subsystem $\{\tilde{U}_j\}_{j=L+1,\dots,N}$ of "infinite" neighbourhoods. Without loss of generality we can choose the numeration in such a way that $\tilde{U}_j\cap\partial M=\emptyset$ for $j=L+1,\dots,B$, $\tilde{U}_j\cap\partial M\neq\emptyset$ for $j=B+1,\dots,N$, for a certain $L+1\leq B\leq N$. Similarly to Section 1.3 we have charts

$$\tilde{\chi}_j:\tilde{U}_j\longrightarrow\tilde{V}_j,\quad j=B+1,\dots,N$$

where $\tilde{V}_j=\{x\in\mathbb{R}^n:|x|>1-\varepsilon,\frac{x}{|x|}\in V_j^1\}$ for certain open sets $\tilde{V}_j^1\subset S^{n-1}$, $n=\dim(2M)$. We may (and will) assume that $\tilde{U}_j$ has the form $2U_j$ for an infinite neighbourhood U_j on $M, U_j\cap\partial M\neq\emptyset$, that is glued together with its counterpart to $\tilde{U}_j=2U_j$ along $\tilde{U}_j\cap\partial M$, where $\tilde{\chi}_j:\tilde{U}_j\cap\partial M\to\tilde{V}_j\cap\mathbb{R}^{n-1}$ and

$$\chi_j:=\tilde{\chi}_j|_{U_j}:U_j\longrightarrow\tilde{V}_j\cap\overline{\mathbb{R}}^n_+=:V_j,\quad j=B+1,\dots,N. \tag{90}$$

Let $U\subset M$ be a neighbourhood of M that equals U_j for some $B+1\leq j\leq N$, and let $\chi:U\to V\subset\overline{\mathbb{R}}^n_+$ be the chart corresponding to (90). We call U a local admissible neighbourhood and any $\varphi\in C^\infty(U)$ a local admissible cut-off function on M if $\varphi=\chi^*\varkappa$ for some local admissible cut-off function $\varkappa$ in $\overline{\mathbb{R}}^n_+$ (that is supported in V), cf. Definition 2.6. Moreover, the above-mentioned infinite part $C\cong(1-\varepsilon,\infty)\times X$ of M allows us to define global admissible neighbourhoods on M, namely sets of the form $(1-\varepsilon,\infty)\times Y\times[0,\beta)$ for some (small) $\beta>0$, where $Y\times[0,\beta)$ denotes a corresponding collar neighbourhood of Y in X. Then a $\varphi_\asymp\in C^\infty(M)$ is called a global admissible cut-off function on M if $0\leq\varphi_\asymp\leq 1, \operatorname{supp}\varphi_\asymp\subset(1-\frac{\varepsilon}{2},\infty)\times Y\times[0,\beta)$, $\varphi_\asymp=1$ for $m\in(1,\infty)\times Y\times[0,\frac{\beta}{2})$, and $\varphi(\lambda m)=\varphi(m)$ for all $\lambda\geq 1, m\in(R,\infty)\times Y\times[0,\frac{\beta}{2})$ for some $R>1$.

Given a vector bundle $E\in\operatorname{Vect}(M)$ we fix an $\tilde{E}\in\operatorname{Vect}(2M)$ such that $E=\tilde{E}|_M$. In Section 1.3 we have defined weighted Sobolev spaces $H^{s;\varrho}(2M,\tilde{E})$ for $s,\varrho\in\mathbb{R}$. Let $H_0^{s;\varrho}(M,E)$ denote the subspace of all $u\in H^{s;\varrho}(2M,\tilde{E})$ with $\operatorname{supp}u\subseteq M$. Similarly, denoting by M_- the negative counterpart of M in $2M$, we have $H_0^{s;\varrho}(M_-,E_-)$ for $E_-=\tilde{E}|_{M_-}$. Let r^+ be the operator of restriction to $\operatorname{int}M=M\backslash\partial M$, and set

$$H^{s;\varrho}(M,E)=\{\mathrm{r}^+u:u\in H^{s;\varrho}(2M,\tilde{E})\}. \tag{91}$$

There is then an isomorphism of (91) to the space $H^{s;\varrho}(2M,\tilde{E})/H_0^{s;\varrho}(M_-,E_-)$ which gives us a Banach space structure on (91) (in fact, a Hilbert space structure) via the quotient topology. Similarly to (12) we introduce the Schwartz space $\mathcal{S}(M,E)$ of sections in E.

Let $L^{\mu;\delta}_{\mathrm{cl}}(2M;\tilde{E},\tilde{F})_{\mathrm{tr}}$ for $\tilde{E},\tilde{F}\in \mathrm{Vect}(2M)$ denote the subspace of all $\tilde{A}\in L^{\mu;\delta}_{\mathrm{cl}}(2M;\tilde{E},\tilde{F})$ that have the transmission property with respect to ∂M. Then, if e^+ is the operator of extension by zero from M to $2M$, analogously to (4) we form $\mathrm{r}^+\tilde{A}\mathrm{e}^+$ for arbitrary $\tilde{A}\in L^{\mu;\delta}_{\mathrm{cl}}(2M;\tilde{E},\tilde{F})_{\mathrm{tr}}$ and get continuous operators

$$\mathrm{r}^+\tilde{A}\mathrm{e}^+ : H^{s;\varrho}(M,E)\longrightarrow H^{s-\mu;\varrho-\delta}(M,F)$$

for all $s>-\frac{1}{2}$ and $\varrho\in\mathbb{R}$.

In order to introduce the global space of pseudo-differential boundary value problems on M we first introduce the smoothing elements of type 0. Let $E,F\in\mathrm{Vect}(M)$, $J^-,J^+\in\mathrm{Vect}(\partial M)$. Recall that all bundles are equipped with Hermitian metrics (homogeneous of order zero in the axial variable of the conical exits). Moreover, on M and ∂M we have fixed Riemannian metrics such that the metric on ∂M is induced by that on M. There are then associated measures $\mathrm{d}m$ on M and $\mathrm{d}n$ on ∂M. Now $\mathcal{B}^{-\infty,0;-\infty}(M;\boldsymbol{v})$ for $\boldsymbol{v}=(E,F;J^-,J^+)$ is defined to be the space of all operators

$$\mathcal{C}=\begin{pmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix} : \begin{matrix} H^{s;\varrho}(M,E)\\ \oplus \\ H^{s;\varrho}(\partial M,J^-)\end{matrix} \longrightarrow \begin{matrix}\mathcal{S}(M,F)\\ \oplus\\ \mathcal{S}(\partial M,J^+)\end{matrix}$$

$s,\varrho\in\mathbb{R}$ such that C_{ij} are integral operators with kernels c_{ij}, where $c_{11}(m,m')\in\mathcal{S}(M,F)\hat{\otimes}_\pi\mathcal{S}(M,E^*)$, $c_{12}(m,n')\in\mathcal{S}(M,F)\hat{\otimes}_\pi\mathcal{S}(\partial M,(J^-)^*)$, $c_{21}(n,m')\in\mathcal{S}(\partial M,J^+)\hat{\otimes}_\pi\mathcal{S}(M,E^*)$, $c_{22}(n,n')\in\mathcal{S}(\partial M,J^+)\hat{\otimes}_\pi\mathcal{S}(\partial M,(J^-)^*)$ and

$$(C_{11}u)(m)=\int_M (c_{11}(m,m'),u(m'))_E\,\mathrm{d}m'$$

with $(.,.)_E$ denoting the pointwise pairing in the fibers of E, etc. Let $\mathrm{Diff}^{j;\delta}_{\mathrm{cl}}(M;E,E)$ be the space of all differential operators of order j on M (acting on sections of the bundles E) that belong to $L^{j;\delta}_{\mathrm{cl}}(M;E,E)$ (cf., in particular, formula (40)). Then the space $\mathcal{B}^{-\infty,d;-\infty}(M;\boldsymbol{v})$ of all smoothing operators of type $d\in\mathbb{N}$ is defined to be the set of all

$$\mathcal{C}=\mathcal{C}_0+\sum_{j=1}^{d}\mathcal{C}_j\begin{pmatrix} D^j & 0\\ 0 & 0\end{pmatrix}$$

for arbitrary $\mathcal{C}_j\in\mathcal{B}^{-\infty,0;-\infty}(M;\boldsymbol{v})$ and $D^j\in\mathrm{Diff}^{j;0}_{\mathrm{cl}}(M;E,E)$.

Next we introduce the space of classical Green operators on M, that is an analogue of $\mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+;N_-,N_+)$, cf. Definition 2.14. First, for arbitrary $k,m\in\mathbb{N}$ there is an evident block-matrix version $\mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+;k,m;N_-,N_+)$. Every operator $\mathcal{Q}$ in this space is continuous in the sense

$$\mathcal{Q}: \begin{matrix} H^{s;\varrho}(\mathbb{R}^n_+,\mathbb{C}^k)\\ \oplus\\ H^{s;\varrho}(\mathbb{R}^{n-1},\mathbb{C}^{N_-})\end{matrix} \longrightarrow \begin{matrix} H^{s-\mu;\varrho-\delta}(\mathbb{R}^n_+,\mathbb{C}^m)\\ \oplus \\ H^{s-\mu;\varrho-\delta}(\mathbb{R}^{n-1},\mathbb{C}^{N_+})\end{matrix}$$

for $s>d-\frac{1}{2}$. If $\varkappa$ and ϑ are local admissible cut-off functions in $\overline{\mathbb{R}}^n_+$, we have

$$\mathcal{M}_\varkappa\mathcal{Q}\mathcal{M}_\vartheta\in\mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+;k,m;N_-,N_+), \tag{92}$$

for every $\mathcal{Q} \in \mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+; k, m; N_-, N_+)$, where $\mathcal{M}_\varkappa$ is the operator of multiplication by $\mathrm{diag}(\varkappa \otimes \mathrm{id}_{\mathbb{C}^m}, \varkappa|_{\mathbb{R}^{n-1}} \otimes \mathrm{id}_{\mathbb{C}^{N_+}})$ and similarly $\mathcal{M}_\vartheta$. Given bundles $E, F \in \mathrm{Vect}(M)$, $J^-, J^+ \in \mathrm{Vect}(\partial M)$, an operator

$$\mathcal{G}: \begin{array}{c} H^{s;\varrho}(M,E) \\ \oplus \\ H^{s;\varrho}(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(M,F) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\partial M, J^+) \end{array} \tag{93}$$

is said to be supported in a global admissible neighbourhood of ∂M if there are global admissible cut-off functions $\varphi_\asymp, \psi_\asymp$ on M such that $\mathcal{G} = \mathcal{M}_{\varphi_\asymp}\mathcal{G}\mathcal{M}_{\psi_\asymp}$. Similarly, we say that a $\mathcal{Q} \in \mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+; k, m; N_-, N_+)$ is supported in a local admissible set in $\overline{\mathbb{R}}^n_+$ if $\mathcal{Q}$ satisfies a relation $\mathcal{Q} = \mathcal{M}_\varkappa \mathcal{Q} \mathcal{M}_\vartheta$ for certain local admissible cut-off functions $\varkappa$ and ϑ. If $\chi : U \to V$ is one of the charts (90), we have an associated chart $\chi' : U \cap \partial M \to V \cap \mathbb{R}^{n-1}$, and there are corresponding trivialisations of the bundles E, F and J^-, J^+, respectively. χ gives rise to a push-forward of operators

$$\boldsymbol{\chi}_* \mathcal{M}_\varphi \mathcal{G} \mathcal{M}_\psi : \begin{array}{c} H^{s;\varrho}(\mathbb{R}^n_+, \mathbb{C}^k) \\ \oplus \\ H^{s;\varrho}(\mathbb{R}^{n-1}, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(\mathbb{R}^n_+, \mathbb{C}^m) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\mathbb{R}^{n-1}, \mathbb{C}^{N_+}) \end{array},$$

where k, m and N_-, N_+ are the fibre dimensions of the bundles E, F and J^-, J^+, respectively, and φ, ψ local admissible cut-off functions supported by U.

Now $\mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(M; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J^-, J^+)$ is defined to be the set of all operators $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{C}$, where $\mathcal{C} \in \mathcal{B}^{-\infty,d;-\infty}(M; \boldsymbol{v})$ and

(i) $\mathcal{G}_0$ is supported in $\cup_{B+1 \le j \le N} U_j$, cf. (90), where

$$\boldsymbol{\chi}_{j*} \mathcal{M}_{\varphi_j} \mathcal{G}_0 \mathcal{M}_{\psi_j} \in \mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+; k, m; N_-, N_+)$$

for arbitrary local admissible cut-off functions φ_j and ψ_j on M supported in U_j, $B + 1 \le j \le N$.

(ii) $\mathcal{G}_1$ is an operator (93) that is supported in a collar neighbourhood of the boundary of the finite part M, i.e., $\partial(M \backslash \overline{C})$, and it is a Green operator of order μ and type d in Boutet de Monvel's algebra on $M \backslash \overline{C}$.

It can be easily be proved that this is a correct definition; in fact, the operators in the space $\mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(\overline{\mathbb{R}}^n_+; k, m; N_-, N_+)$, supported in an admissible set in $\overline{\mathbb{R}}^n_+$, are invariant under the transition maps generated by the charts and corresponding trivialisations of the involved bundles.

Definition 3.5 *The space $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M; \boldsymbol{v})$ for $\mu \in \mathbb{Z}, d \in \mathbb{N}, \delta \in \mathbb{R}$ and $\boldsymbol{v} = (E, F; J^-, J^+)$, $E, F \in \mathrm{Vect}(M)$, $J^-, J^+ \in \mathrm{Vect}(\partial M)$, is defined to be the set of all operators*

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ \tilde{A} \mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} \tag{94}$$

for arbitrary $\tilde{A} \in L^{\mu;\delta}_{\mathrm{cl}}(2M; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ (with $\tilde{E}|_M = E, \tilde{F}|_M = F$) and $\mathcal{G} \in \mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(M; \boldsymbol{v})$.

Theorem 3.6 *Every operator $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J^-, J^+)$, induces continuous operators*

$$\mathcal{A}: \begin{array}{c} H^{s;\varrho}(M,E) \\ \oplus \\ H^{s;\varrho}(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(M,F) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\partial M, J^+) \end{array}$$

for all real $s > d - \frac{1}{2}$ and all $\varrho \in \mathbb{R}$. In particular, $\mathcal{A}$ is also continuous in the sense

$$\mathcal{A} : \begin{array}{c} \mathcal{S}(M, E) \\ \oplus \\ \mathcal{S}(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} \mathcal{S}(M, F) \\ \oplus \\ \mathcal{S}(\partial M, J^+) \end{array}.$$

This result is an easy consequence of Theorem 2.20 and Remark 2.17.

Similarly to the global principal symbol structure of operators on a closed manifold with exit to infinity, cf. Section 1.3, we now introduce global principal symbols for an operator $\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J^-, J^+)$ for $E, F \in \mathrm{Vect}(M), J^-, J^+ \in \mathrm{Vect}(\partial M)$. The principal interior symbols only depend on $\tilde{A}$ in (94). According to formulas (13), (15), (16), we have $(\sigma_\psi(\tilde{A}), \sigma_{\mathrm{e}}(\tilde{A}), \sigma_{\psi,\mathrm{e}}(\tilde{A}))$ for any $\tilde{A} \in L_{\mathrm{cl}}^{\mu;\delta}(2M; \tilde{E}, \tilde{F})$, where

$$\sigma_\psi(\tilde{A}) : \pi_\psi^* \tilde{E} \longrightarrow \pi_\psi^* \tilde{F}, \quad \pi_\psi : T^*(2M)\backslash 0 \longrightarrow 2M,$$

$$\sigma_{\mathrm{e}}(\tilde{A}) : \pi_{\mathrm{e}}^* \tilde{E} \longrightarrow \pi_{\mathrm{e}}^* \tilde{F}, \quad \pi_{\mathrm{e}} : T^*(2M)|_{(2X)^\wedge_\infty} \longrightarrow (2X)^\wedge_\infty,$$

$$\sigma_{\psi,\mathrm{e}}(\tilde{A}) : \pi_{\psi,\mathrm{e}}^* \tilde{E} \longrightarrow \pi_{\psi,\mathrm{e}}^* \tilde{F}, \quad \pi_{\psi,\mathrm{e}} : (T^*(2M)\backslash 0)|_{(2X)^\wedge_\infty} \longrightarrow (2X)^\wedge_\infty.$$

Restricting this to M (and taking for the projections the same notation) we get

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(\tilde{A})|_{T^*M\backslash 0} : \pi_\psi^* E \longrightarrow \pi_\psi^* F, \quad \pi_\psi : T^*M\backslash 0 \longrightarrow M, \tag{95}$$

$$\sigma_{\mathrm{e}}(\mathcal{A}) := \sigma_{\mathrm{e}}(\tilde{A})|_{T^*M|_{X^\wedge_\infty}} : \pi_{\mathrm{e}}^* E \longrightarrow \pi_{\mathrm{e}}^* F, \quad \pi_{\mathrm{e}} : T^*M|_{X^\wedge_\infty} \longrightarrow X^\wedge_\infty, \tag{96}$$

$$\sigma_{\psi,\mathrm{e}}(\mathcal{A}) := \sigma_{\psi,\mathrm{e}}(\tilde{A})|_{(T^*M\backslash 0)|_{X^\wedge_\infty}} : \pi_{\psi,\mathrm{e}}^* E \longrightarrow \pi_{\psi,\mathrm{e}}^* F, \quad \pi_{\psi,\mathrm{e}} : (T^*M\backslash 0)|_{X^\wedge_\infty} \longrightarrow X^\wedge_\infty. \tag{97}$$

Concerning the principal boundary symbol components we first have

$$\sigma_\partial(\mathcal{A}) : \pi_\partial^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_\partial^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{98}$$

for $\pi_\partial : T^*(\partial M)\backslash 0 \to \partial M$, according to the inclusion $\mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M; \boldsymbol{v}) \subset \mathcal{B}^{\mu,d}(M; \boldsymbol{v})$, $E' = E|_{\partial M}, F' = F|_{\partial M}$, cf. Section 3.1. Moreover, the $\mathrm{e}'-$ and (∂, e')-components of (44) (in the corresponding $(m \times k)$ block matrix-valued version) have a simple invariant meaning with respect to the transition maps from the local representations of $\mathcal{A}$ on the infinite part of M. The system of the local boundary ($\mathrm{e}'-$ and $(\partial, \mathrm{e}')-$) symbols in the sense of (44) gives us bundle homomorphisms

$$\sigma_{\mathrm{e}'}(\mathcal{A}) : \pi_{\mathrm{e}'}^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_{\mathrm{e}'}^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{99}$$

for $\pi_{\mathrm{e}'} : T^*(\partial M)|_{Y^\wedge_\infty} \to Y^\wedge_\infty$ and

$$\sigma_{\partial,\mathrm{e}'}(\mathcal{A}) : \pi_{\partial,\mathrm{e}'}^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_{\partial,\mathrm{e}'}^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{100}$$

for $\pi_{\partial,\mathrm{e}'} : (T^*(\partial M)\backslash 0)|_{Y^\wedge_\infty} \to Y^\wedge_\infty$. Note that $\mathcal{S}(\overline{\mathbb{R}}_+)$ may be replaced by Sobolev spaces on the half-axis for $s > d - \frac{1}{2}$, cf. analogously Section 3.1. Let

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_{\mathrm{e}}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}); \sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A})) \tag{101}$$

for $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})$, and set $\mathrm{symb}\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v}) = \{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})\}$. We then have a direct generalisation of Remark 2.16; the obvious details are left to the reader.

Note that there are natural compatibility properties between the components of $\sigma(\mathcal{A})$.

Theorem 3.7 *$\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})$, $\boldsymbol{v} = (E_0, F; J_0, J^+)$, and $\mathcal{B} \in \mathcal{B}^{\nu,e;\varrho}_{\mathrm{cl}}(M;\boldsymbol{w})$, $\boldsymbol{w} = (E, E_0; J^-, J_0)$, implies $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h;\delta+\varrho}_{\mathrm{cl}}(M;\boldsymbol{v}\circ\boldsymbol{w})$ for $h = \max(\nu+d, e)$ and $\boldsymbol{v}\circ\boldsymbol{w} = (E, F; J^-, J^+)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ (with componentwise multiplication).*

Theorem 3.7 is the global version of Theorem 2.18 and, in fact, a direct consequence of this local composition result.

3.3 Ellipticity, parametrices and Fredholm property

Definition 3.8 *An operator $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J^-, J^+)$ is called elliptic of order (μ, δ) if all bundle homomorphisms* (95), (96), (97), (98), (99), (100) *are isomorphisms.*

Similarly to Remark 2.22, in the conditions for (98), (99), (100) we may replace $\mathcal{S}(\overline{\mathbb{R}}_+)$ by $H^s(\mathbb{R}_+)$ and $H^{s-\mu}(\mathbb{R}_+)$, respectivly, for $s > \max(\mu, d) - \frac{1}{2}$.

Definition 3.9 *Given $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J^-, J^+)$ an operator $\mathcal{P} \in \mathcal{B}^{-\mu,e;-\delta}_{\mathrm{cl}}(M;\boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} = (F, E; J^+, J^-)$ and some $e \in \mathbb{N}$ is called a parametrix of $\mathcal{A}$ if*

$$\mathcal{PA} - \mathcal{I} \in \mathcal{B}^{-\infty,d_l;-\infty}(M;\boldsymbol{v}_l), \quad \mathcal{AP} - \mathcal{I} \in \mathcal{B}^{-\infty,d_r;-\infty}(M;\boldsymbol{v}_r)$$

for certain $d_l, d_r \in \mathbb{N}$, and $\boldsymbol{v}_l = (E, E; J^-, J^-)$, $\boldsymbol{v}_r = (F, F; J^+, J^+)$.

Note that the Theorem 3.7 entails $\sigma(\mathcal{A})^{-1} = \sigma(\mathcal{P})$ (with componentwise inversion) where $\mathcal{P}$ is a parametrix of $\mathcal{A}$.

Theorem 3.10 *Let $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})$ be elliptic. Then*

$$\mathcal{A} : \begin{array}{c} H^{s;\varrho}(M, E) \\ \oplus \\ H^{s;\varrho}(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(M, F) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\partial M, J^+) \end{array} \tag{102}$$

is a Fredholm operator for every $s > \max(\mu, d) - \frac{1}{2}$ and every $\varrho \in \mathbb{R}$, and $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+;-\delta}_{\mathrm{cl}}(M;\boldsymbol{v}^{-1})$, where $d_l = \max(\mu, d)$ and $d_r = (d-\mu)^+$ (cf. the notation in Definition 3.9*).*

The proof of this result can be given similarly to Theorem 2.26. Alternatively, the methods of Section 2.6 can also be used to first construct $\sigma(\mathcal{A})^{-1}$ and to form $\tilde{\mathcal{P}} := \mathrm{op}(\sigma(\mathcal{A})^{-1}) \in \mathcal{B}^{-\mu,(d-\mu)^+;-\delta}(M;\boldsymbol{v}^{-1})$. Then we get $\tilde{\mathcal{P}}\mathcal{A} - \mathcal{I} \in \mathcal{B}^{-1,e;-1}(M;\boldsymbol{v}_l)$ for some e, and we get $\mathcal{P}$ itself by a formal Neumann series argument.

Remark 3.11 *Let $\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v})$ be elliptic. Then we have elliptic regularity of solutions in the following sense. $\mathcal{A}u = f \in H^{s-\mu;\varrho-\delta}(M, F) \oplus H^{s-\mu;\varrho-\delta}(\partial M, J^+)$ for any $s > \max(\mu, d) - \frac{1}{2}$ and $\varrho \in \mathbb{R}$ and $u \in H^{r;-\infty}(M, E) \oplus H^{r;-\infty}(\partial M, J^-)$, $r > \max(\mu, d) - \frac{1}{2}$, implies $u \in H^{s;\varrho}(M, E) \oplus H^{s;\varrho}(\partial M, J^-)$.*

In fact, we can argue in a standard manner. Composing $\mathcal{A}u = f$ from the left by $\mathcal{P}$ we get $\mathcal{P}\mathcal{A}u = (1+\mathcal{G})u \in H^{s;\varrho}(M,E) \oplus H^{s;\varrho}(\partial M, J^-)$ and $\mathcal{G}u \in \mathcal{S}(M,E) \oplus \mathcal{S}(\partial M, J^-)$ which yields the assertion.

Remark 3.12 *From Remark* 3.11 *we easily obtain that the kernel of $\mathcal{A}$ is a finite-dimensional subspace of $\mathcal{S}(M,E) \oplus \mathcal{S}(\partial M, J^-)$ (and as such independent of s and ϱ). Moreover, it can easily be shown that there is a finite-dimensional subspace $\mathcal{N}_- \subset \mathcal{S}(M,F) \oplus \mathcal{S}(\partial M, J^+)$ such that* $\mathrm{im}\mathcal{A} + \mathcal{N}_- = H^{s-\mu;\varrho-\delta}(M,F) \oplus H^{s-\mu;\varrho-\delta}(\partial M, J^+)$ *for all s, where* $\mathrm{im}\mathcal{A}$ *means the image in the sense of* (102)*. Thus* $\mathrm{ind}\mathcal{A}$ *(the index of* (102)*) is independent of* $s > \max(\mu, d) - \frac{1}{2}$ *and of* $\varrho \in \mathbb{R}$.

Remark 3.13 *Let* $\mathcal{A}_i \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M; \boldsymbol{v}_i)$, $\boldsymbol{v}_i = (E, F; J_i^-, J_i^+), i = 1,2,$ *be elliptic operators where $\mathcal{A}_1$ has the same upper left corner as $\mathcal{A}_2$; then there is an analogue of Agranovich-Dynin formula for the indices* $\mathrm{ind}\mathcal{A}_i$, $i = 1,2:$ *There exists an elliptic operator* $\mathcal{B} \in L^{0;0}_{\mathrm{cl}}(\partial M; J_2^+ \oplus J_1^-, J_1^+ \oplus J_2^-)$ *such that*

$$\mathrm{ind}\mathcal{A}_1 - \mathrm{ind}\mathcal{A}_2 = \mathrm{ind}\mathcal{B}.$$

The idea of the proof is completely analogous to the corresponding result for a compact, smooth manifold with boundary, cf. Rempel and Schulze [16], Section 3.2.1.3. The operator $\mathcal{B}$ can be evaluated explicitely by applying reductions of orders and weights (cf., also Theorem 4.13 below) and using a parametrix of $\mathcal{A}_2$.

3.4 Construction of global elliptic boundary conditions

An essential point in the analysis of elliptic boundary value problems is the question whether an element

$$A \in B^{\mu,d;\delta}_{\mathrm{cl}}(M; E, F) \tag{103}$$

that is elliptic with respect to the interior symbol tuple $(\sigma_\psi(A), \sigma_{\mathrm{e}}(A), \sigma_{\psi,\mathrm{e}}(A))$ can be regarded as the upper left corner of an operator

$$\mathcal{A} \in \mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M; \boldsymbol{v}) \text{ for } \boldsymbol{v} = (E, F; J^-, J^+) \tag{104}$$

for a suitable choice of bundles $J^-, J^+ \in \mathrm{Vect}(\partial M)$ and additional entries of the block matrix, such that $\mathcal{A}$ is elliptic in the sense of Definition 3.8. We want to give the general answer and by this extend the well-known Atiyah-Bott condition from [1]. Atiyah and Bott formulated a topological obstruction for the existence of Shapiro-Lopatinskij elliptic boundary conditions for elliptic differential operators on a compact smooth manifold (concerning the corresponding conditions for pseudo-differential boundary value problems cf. Boutet de Monvel [3]). To formulate the result in our situation, without loss of generality we consider the case $\mu = d = \delta = 0$. The general case is then a consequence of a simple reduction of orders, types and weights, applying Theorem 4.13 and Remark 4.14 below. The constructions for Theorem 2.30 above can be generalised to a given $(\sigma_\psi, \sigma_{\mathrm{e}}, \sigma_{\psi,\mathrm{e}})$-elliptic operator $A \in B^{0,0;0}_{\mathrm{cl}}(M; E, F)$ as follows. Starting point are the boundary symbols

$$\sigma_\partial(A)(y,\eta) \text{ for } (y,\eta) \in T^*(\partial M)\setminus 0,$$

$$\sigma_{\mathrm{e}'}(A)(y,\eta) \text{ for } (y,\eta) \in T^*(\partial M)|_{Y^\wedge_\infty},$$

$$\sigma_{\partial,\mathrm{e}'}(A)(y,\eta) \text{ for } (y,\eta) \in (T^*(\partial M)\setminus 0)|_{Y^\wedge_\infty},$$

as operator families

$$E'_y \otimes L^2(\mathbb{R}_+) \longrightarrow F'_y \otimes L^2(\mathbb{R}_+),$$

(in contrast to (98)-(100) we now prefer $L^2(\mathbb{R}_+)$ instead of $\mathcal{S}(\overline{\mathbb{R}}_+)$, according to the considerations in Section 2.6). For points $y \in \partial M$ belonging to the infinite exit to infinity $\cong (1-\varepsilon, \infty) \times Y_\infty$ it makes sense to talk about $|y| > R$ (this simply means that the associated axial variable is larger than R). First there is an obvious analogue of Proposition 2.28 that refers to points $(y,\eta) \in T^*(\partial M)$ for $y \in (1-\varepsilon,\infty) \times Y_\infty$.

Proposition 3.14 *For every $\varepsilon > 0$ there exists an $R = R_\varepsilon > 0$ such that*

$$||\sigma_\partial(A)(y,\eta) - \sigma_{\partial,\mathrm{e}'}(A)(y,\eta)||_{\mathcal{L}(E'_y\otimes L^2(\mathbb{R}_+),F'_y\otimes L^2(\mathbb{R}_+))} < \varepsilon \tag{105}$$

for all $|y| > R$ and $\eta \neq 0$,

$$||\sigma_\partial(A)(y,\eta) - \sigma_{\mathrm{e}'}(A)(y,\eta)||_{\mathcal{L}(E'_y\otimes L^2(\mathbb{R}_+),F'_y\otimes L^2(\mathbb{R}_+))} < \varepsilon \tag{106}$$

for all $|y| > R$ and $|\eta| > R$,

$$||\sigma_{\mathrm{e}'}(A)(y,\eta) - \sigma_{\partial,\mathrm{e}'}(A)(y,\eta)||_{\mathcal{L}(E'_y\otimes L^2(\mathbb{R}_+),F'_y\otimes L^2(\mathbb{R}_+))} < \varepsilon \tag{107}$$

for all $|y| \in (1-\varepsilon,\infty) \times Y_\infty$ and $|\eta| > R$.

Corollary 3.15 *There is an $R = R_\varepsilon > 0$ such that*

$$||\sigma_\partial(A)(y,\eta) - \sigma_{\mathrm{e}'}(A)(y,\eta)||_{\mathcal{L}(E'_y\otimes L^2(\mathbb{R}_+),F'_y\otimes L^2(\mathbb{R}_+))} < \varepsilon$$

for all $|y| = |\eta| = R$.

For $\varepsilon > 0$ we set

$$T_\varepsilon = \{(y,\eta) \in T^*(\partial M) : |y| = |\eta| = R_\varepsilon\}, \quad D_\varepsilon = T_\varepsilon \times [0,1]$$

and

$$Z^j_\varepsilon = \{(y,\eta) \in T^*(\partial M) : y \in \partial M \backslash \{|y| > R_\varepsilon + j\}, |\eta| = R_\varepsilon\},$$
$$H^j_\varepsilon = \{(y,\eta) \in T^*(\partial M) : |y| = R_\varepsilon, |\eta| \leq R_\varepsilon + j\}$$

for $j = 0, 1, \infty$. Moreover, let $\mathbb{L}^j_\varepsilon = (Z^j_\varepsilon \cup_d H^j_\varepsilon) \cup_b D_\varepsilon / \sim$, with $\cup_d$ being the disjoint union and $\cup_b$ the disjoint union combined with the projection to the quotient space that is given by natural identifications $T_\varepsilon \cap Z^j_\varepsilon \cong T_\varepsilon \times \{0\}$, $T_\varepsilon \cap H^j_\varepsilon \cong T_\varepsilon \times \{1\}$. Write $Z_\varepsilon = Z^0_\varepsilon$, $H_\varepsilon = H^0_\varepsilon$, $\mathbb{L}_\varepsilon = \mathbb{L}^0_\varepsilon$. Furthermore, for $0 \leq \tau \leq 1$ we set $D_{\varepsilon,\tau} := T_\varepsilon \times [0,\tau]$ and form $\mathbb{L}_{\varepsilon,\tau} := Z_\varepsilon \cup_d H_\varepsilon \cup_b D_{\varepsilon,\tau}$, $0 \leq \tau \leq 1$, where $\cup_b$ is the disjoint union combined with the projection from the identification $T_\varepsilon \cap Z_\varepsilon \cong T_\varepsilon \times \{0\}$, $T_\varepsilon \cap H_\varepsilon \cong T_\varepsilon \times \{\tau\}$. We now introduce an operator function $F(m), m \in \mathbb{L}_\varepsilon$, as follows:

$$F(y,\eta) = \sigma_\partial(A)(y,\eta) \qquad \text{for} \qquad m = (y,\eta) \in Z_\varepsilon, \tag{108}$$

$$F(y,\eta) = \sigma_{\mathrm{e}'}(A)(y,\eta) \qquad \text{for} \qquad m = (y,\eta) \in H_\varepsilon, \tag{109}$$

$$F(y,\eta,\delta) = \delta\sigma_\partial(A)(y,\eta) + (1-\delta)\sigma_{\mathrm{e}'}(A)(y,\eta) \qquad \text{for} \qquad m = (y,\eta,\delta) \in D_\varepsilon. \tag{110}$$

We then have an operator familiy

$$F(m) : E'_y \otimes L^2(\mathbb{R}_+) \longrightarrow F'_y \otimes L^2(\mathbb{R}_+)$$

continuously depending on $m \in \mathbb{L}_\varepsilon$, and F is Fredholm operator-valued, provided $\varepsilon > 0$ is sufficiently small. This gives us an index element $\mathrm{ind}_{\mathbb{L}_\varepsilon} F \in K(\mathbb{L}_\varepsilon)$. For analogous reasons as above in connection with (70) we form

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(A)(y,\eta), \sigma_{\mathrm{e}'}(A)(y,\eta)\} \in K(\mathbb{B}_\varepsilon), \tag{111}$$

$\mathbb{B}_\varepsilon := \mathbb{L}_{\varepsilon,0} \subset T^*(\partial M)$. The canonical projection $T^*(\partial M) \to \partial M$ induces a projection $\pi_\varepsilon : \mathbb{B}_\varepsilon \to B_\varepsilon$ where

$$B_\varepsilon := \partial M \backslash \{y \in \partial M : |y| > R_\varepsilon\}.$$

Given an arbitrary $(\sigma_\psi, \sigma_\mathrm{e}, \sigma_{\psi,\mathrm{e}})$-elliptic operator (103) we set

$$A_0 = R_F^{s_0-\mu} \mathcal{S}^{-\delta} A R_E^{-s_0} \tag{112}$$

for any $s_0 > \max(\mu, d) - \frac{1}{2}$, where $R_F^{s_0-\mu} \in B_{\mathrm{cl}}^{s_0-\mu,0;0}(M; F, F)$ and $R_E^{-s_0} \in B_{\mathrm{cl}}^{-s_0,0;0}(M; E, E)$ are order reducing operators in the sence of Remark 4.14, and $\mathcal{S}^{-\delta}$ a weight reducing factor on M of a similar meaning as that in Remark 2.19. Then we have $A_0 \in B_{\mathrm{cl}}^{0,0;0}(M; E, F)$, and A_0 is also $(\sigma_\psi, \sigma_\mathrm{e}, \sigma_{\psi,\mathrm{e}})$-elliptic. In the sequel the choice of the specific order and weight reducing factors is unessential.

The following theorem is an analogue of the Atiyah-Bott condition, formulated in [1] for the case of differential operators on a smooth compact manifold with boundary, and established by Boutet de Monvel [3] for pseudo-differential boundary value problems with the transmission property.

Theorem 3.16 *Let M be a smooth manifold with boundary and conical exits to infinity, $E, F \in \mathrm{Vect}(M)$, and let $A \in B_{\mathrm{cl}}^{\mu,d;\delta}(M; E, F)$ be a $(\sigma_\psi, \sigma_\mathrm{e}, \sigma_{\psi,\mathrm{e}})$-elliptic operator. Then there exists an elliptic operator* (104) *having A as the upper left corner if and only if the operator* (112) *satisfies the condition*

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(A_0), \sigma_{\mathrm{e}'}(A_0)\} \in \pi_\varepsilon^* K(B_\varepsilon), \tag{113}$$

for a (sufficiently small) $\varepsilon > 0$, $\pi_\varepsilon : \mathbb{B}_\varepsilon \to B_\varepsilon$.

If (113) *holds, for any choice of the additional bundles $J^-, J^+ \in \mathrm{Vect}(\partial M)$ in the sense of* (104) *we have*

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(A_0), \sigma_{\mathrm{e}'}(A_0)\} = \pi_\varepsilon^* \left([J^+|_{B_\varepsilon}] - [J^-|_{B_\varepsilon}]\right). \tag{114}$$

Proof. First note that the criterion of Theorem 3.16 does not depend on the choice of order reductions. Moreover, such reductions allow us to pass from $A_0 \in B_{\mathrm{cl}}^{0,0;0}(M; E, F)$ and an associated $\mathcal{A}_0 \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(M; \boldsymbol{v})$ with A_0 as upper left corner to the corresponding operators $A \in B_{\mathrm{cl}}^{\mu,d;\delta}(M; E, F)$ and $\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M; \boldsymbol{v})$. Thus, without loss of generality we assume $\mu = d = \delta = 0$ and talk about A and $\mathcal{A}$, respectively. Clearly, the existence of an elliptic $\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{0,0;0}(M; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J^-, J^+)$, to a given $((\sigma_\psi, \sigma_e, \sigma_{\psi,e})$-) elliptic $A \in B_{\mathrm{cl}}^{0,0;0}(M; E, F)$ implies

$$\mathrm{ind}_{\mathbb{B}_\varepsilon}\{\sigma_\partial(A), \sigma_{\mathrm{e}'}(A)\} = \pi_\varepsilon^*\{[J^+|_{B_\varepsilon}] - [J^-|_{B_\varepsilon}]\}, \tag{115}$$

because the role of the bundles J^-, J^+ in the components of $(\sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A}))$ is just that they fill up the Fredholm families $(\sigma_\partial(A), \sigma_{\mathrm{e}'}(A), \sigma_{\partial,\mathrm{e}'}(A))$ to block matrices of isomorphisms; combining this with Corollary 3.15 we get the desired index relation. Conversely assume that (115) holds. Then the construction of an elliptic operator $\mathcal{A}$ in terms of A takes place on the level of boundary symbols. In other words, the

Fredholm families have to be first completed to block matrices of isomorphisms. This can be done when we also include (110) into the construction, in order to deal with continuous Fredholm families, and then drop the "superfluous" part on D_ε. Thus the first step to find $\mathcal{A}$ is to fill up $F(m), m \in \mathbb{L}_\varepsilon$, to a family of isomorphisms

$$\mathcal{F}(m) = \begin{pmatrix} F(m) & K(m) \\ t(m) & Q(m) \end{pmatrix} : \begin{matrix} E'_y \otimes L^2(\mathbb{R}_+) \\ \oplus \\ J^-_y \end{matrix} \longrightarrow \begin{matrix} F'_y \otimes L^2(\mathbb{R}_+) \\ \oplus \\ J^+_y \end{matrix},$$

$m \in \mathbb{L}_\varepsilon$. Here we employ the fact that the additional finite-dimensional vector spaces corresponding to the entries $\mathcal{F}(m)_{ij}$ for $i + j > 1$ are fibres in some bundles J^- and J^+ on B_ε, using the hypothesis on $F(m)$, further local representations with respect to $y \in B_\varepsilon$ and the invariance under the transition maps. Similarly to the local theory we find $\mathcal{F}(m)$ (locally) in form of $D^{0,0}(\overline{\mathbb{R}}_+; k, k; N_-, N_+)$-valued families (here, k is the fibre dimension both of E and F, and $N_\pm$ are the fibre dimensions $J^\pm$, and we employ a corresponding generalisation of the notation of Section 2.1 to $k \times k$-matrices in the upper left corners), smoothly dependent on (y, η) on Z_ε or H_ε. In this construction $\varepsilon > 0$ is chosen sufficiently small, i.e., $R = R_\varepsilon$ large enough. The construction so far gives us $\sigma_\partial(\mathcal{A})|_{Z_\varepsilon}$ and $\sigma_{\mathrm{e}'}(\mathcal{A})|_{H_\varepsilon}$. Extending $\sigma_\partial(\mathcal{A})|_{Z_\varepsilon}$ (by κ_λ-homogeneity) for all $\eta \neq 0$ and $\sigma_{\mathrm{e}'}(\mathcal{A})|_{H_\varepsilon}$ (by usual homogeneity) for all $|y| \geq R_\varepsilon$ we get $\sigma_\partial(\mathcal{A})$ and $\sigma_{\mathrm{e}'}(\mathcal{A})$ everywhere. Next we form $\sigma_{\partial,\mathrm{e}'}(\mathcal{A}) = \sigma_{\mathrm{e}'}(\sigma_\partial(\mathcal{A})) = \sigma_\partial(\sigma_{\mathrm{e}'}(\mathcal{A}))$. Thus we have an elliptic symbol tuple $\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\mathrm{e}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}); \sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A}))$, where the first three components equal the given ones, namely $(\sigma_\psi(\mathcal{A}), \sigma_\mathrm{e}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}))$. By virtue of $\sigma(\mathcal{A}) \in \mathrm{symb}\mathcal{B}^{0,0;0}_{\mathrm{cl}}(M; \boldsymbol{v})$ we can apply an operator convention

$$\mathrm{op} : \mathrm{symb}\mathcal{B}^{0,0;0}_{\mathrm{cl}}(M; \boldsymbol{v}) \longrightarrow \mathcal{B}^{0,0;0}_{\mathrm{cl}}(M; \boldsymbol{v})$$

to get $\mathcal{A}$ itself. □

Remark 3.17 *As is well-known for compact smooth manifolds with boundary there are in general elliptic differential operators that violate the Atiyah-Bott condition. An example is the Cauchy-Riemann operator $\overline{\partial}_z$ in a disk in the complex plane. One may ask what happens for $\overline{\partial}_z$, say, in a half-plane $\{z \in \mathbb{C} : \mathrm{Im} z \geq 0\}$. In this case the Atiyah-Bott condition is, of course, violated, too, but the operator $\overline{\partial}_z$ is worse. In fact, there is no constant $c \in \mathbb{C}$ such that $c + \overline{\partial}_z$ is $(\sigma_\psi, \sigma_\mathrm{e}, \sigma_{\psi,\mathrm{e}})$-elliptic, such that also for that reason there are no global elliptic operators $\mathcal{A}$ in the half-plane with $\sigma_\psi(\mathcal{A}) = \sigma_\psi(\overline{\partial}_z)$.*

4 Parameter-dependent operators and applications

4.1 Basic observations

As noted in the beginning the theory of pseudo-differential boundary value problems on a manifold with exits is motivated by a number of interesting applications. In this connection boundary value problems appear as parameter-dependent operator families, where parameters $\lambda \in \mathbb{R}$ are involved like additional covariables in the symbols. All essential notions and results have reasonable analogues in the parameter-dependent case, though there are some specific new aspects. The parameter-dependent ellipticity that we formulate below in also of interest for the (non-parameter-dependent) algebras themselves, insofar, as we shall see, they provide a tool to construct order reducing elements within the algebras in a transparent way.

First we have a direct analogue of the symbol classes with the transmission property $S^{\mu}_{\mathrm{cl}}(\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^{n+l}_{\xi,\lambda})_{\mathrm{tr}}$, cf. Section 1.2, where ξ is to be replaced by (ξ,λ). Concerning symbol estimates, the parameter-dependent case is not a new situation; in estimate (1) we admitted independent dimensions of $x-$ and $\xi-$variables, anyway. Similarly, we can talk about weighted symbol classes $S^{\mu;\delta}(\mathbb{R}^n_x\times\mathbb{R}^{n+l}_{\xi,\lambda})$, where ξ in the estimates (7) is replaced by (ξ,λ). The material of Section 1.3 on weighted symbols that are classical in x and ξ has an evident parameter-dependent analogue, in other words, we have the symbol classes

$$S^{\mu;\delta}_{\mathrm{cl}_{\xi,\lambda;x}}(\mathbb{R}^n_x\times\mathbb{R}^{n+l}_{\xi,\lambda}) \tag{116}$$

including the (λ-dependent) principal symbols

$$\sigma(a)=(\sigma_\psi(a)(x,\xi,\lambda),\sigma_{\mathrm{e}}(a)(x,\xi,\lambda),\sigma_{\psi,\mathrm{e}}(a)(x,\xi,\lambda))$$

for all $a(x,\xi,\lambda)$ belonging to (116), with $\sigma_\psi(a)(x,\xi,\lambda)$ being given on $\mathbb{R}^n\times(\mathbb{R}^{n+l}\setminus 0)$, $\sigma_{\mathrm{e}}(a)(x,\xi,\lambda)$ on $(\mathbb{R}^n\setminus 0)\times\mathbb{R}^{n+l}$ and $\sigma_{\psi,\mathrm{e}}(a)(x,\xi,\lambda)$ on $(\mathbb{R}^n\setminus 0)\times(\mathbb{R}^{n+l}\setminus 0)$. We set $L^{\mu;\delta}_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{R}^l)=\{\mathrm{Op}_x(a)(\lambda):a(x,\xi,\lambda)\in S^{\mu;\delta}_{\mathrm{cl}_{\xi,\lambda;x}}(\mathbb{R}^n\times\mathbb{R}^{n+l})\}$, where Op_x is a bijection between the parameter-dependent symbol and operator spaces for all $\mu,\delta\in\mathbb{R}$. Then, in particular,

$$L^{-\infty;-\infty}(\mathbb{R}^n;\mathbb{R}^l)=\mathcal{S}(\mathbb{R}^l,L^{-\infty;-\infty}(\mathbb{R}^n)),$$

where $L^{-\infty;-\infty}(\mathbb{R}^n)$ is identified with $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$.

If M is a manifold with exits to infinity in the sense of Section 1.3, we also have the global spaces of (classical) parameter-dependent operators $L^{\mu;\delta}_{\mathrm{cl}}(M;E,F;\mathbb{R}^l)$ for $E,F\in\mathrm{Vect}(M)$. (Clearly, there is also the non-classical context, but we want to employ homogeneous principal symbols; thus we content ourselves with the classical case). The parameter-dependent homogeneous principal symbols for $A\in L^{\mu;\delta}_{\mathrm{cl}}(M;E,F;\mathbb{R}^l)$ are bundle homomorphisms

$$\sigma_\psi(A):\pi^*_\psi E\longrightarrow\pi^*_\psi F,\quad \pi_\psi:(T^*M\times\mathbb{R}^l)\setminus 0\longrightarrow M, \tag{117}$$

$$\sigma_{\mathrm{e}}(A):\pi^*_{\mathrm{e}}E\longrightarrow\pi^*_{\mathrm{e}}F,\quad \pi_{\mathrm{e}}:T^*M|_{X_\infty}\times\mathbb{R}^l\longrightarrow X_\infty, \tag{118}$$

$$\sigma_{\psi,\mathrm{e}}(A):\pi^*_{\psi,\mathrm{e}}E\longrightarrow\pi^*_{\psi,\mathrm{e}}F,\quad \pi_{\psi,\mathrm{e}}:((T^*M\times\mathbb{R}^l)\setminus 0)|_{X_\infty}\longrightarrow X_\infty, \tag{119}$$

here, 0 means $(\xi,\lambda)=0$.

Notice that $A\in L^{\mu;\delta}_{\mathrm{cl}}(M;E,F;\mathbb{R}^l)$ implies $A(\lambda_0)\in L^{\mu;\delta}_{\mathrm{cl}}(M;E,F)$ for every fixed $\lambda_0\in\mathbb{R}^l$. Clearly, the associated principal symbols $\sigma_\psi(A(\lambda_0)),\sigma_{\psi,\mathrm{e}}(A(\lambda_0))$ do not depend on λ_0. In this connection we also call (117), (118) and (119) the parameter-dependent principal symbols of $A(\lambda)$. Every $A(\lambda)\in L^{\mu;\delta}_{\mathrm{cl}}(M;E,F;\mathbb{R}^l)$ gives rise to families of continuous operators

$$A(\lambda):H^{s;\varrho}(M,E)\longrightarrow H^{s-\mu;\varrho-\delta}(M,F) \tag{120}$$

for all $s,\varrho\in\mathbb{R}$. Let $\nu\geq\mu$ and set

$$b_{\mu,\nu}(\lambda)=\begin{cases}\langle\lambda\rangle^{\mu} & \text{for } \nu\geq 0,\\ \langle\lambda\rangle^{\mu-\nu} & \text{for } \nu\leq 0.\end{cases} \tag{121}$$

We then have the following result:

Theorem 4.1 *Let $A(\lambda) \in L^{\mu;\delta}_{\mathrm{cl}}(M; E, F; \mathbb{R}^l)$ be regarded as a family of continuous operators*

$$A(\lambda) : H^{s;\varrho}(M, E) \longrightarrow H^{s-\nu;\varrho-\delta}(M, F)$$

for every $\nu \geq \mu$. Then there is a constant $m > 0$ such that the operator norm fulfils the estimate

$$||A(\lambda)||_{\mathcal{L}(H^{s;\varrho}(M,E),H^{s-\nu;\varrho-\delta}(M,F))} \leq m b_{\mu,\nu}(\lambda) \tag{122}$$

for all $\lambda \in \mathbb{R}^l$.

We have no explicit reference for this result, though the proof is not really difficult; so the details are left to the reader.

An operator $A(\lambda) \in L^{\mu;\delta}_{\mathrm{cl}}(M; E, F; \mathbb{R}^l)$ is called parameter-dependent elliptic if (117), (118) and (119) are isomorphisms.

Theorem 4.2 *Let $A(\lambda) \in L^{\mu;\delta}_{\mathrm{cl}}(M; E, F; \mathbb{R}^l)$ be parameter-dependent elliptic. Then there is a parameter-dependent parametrix $P(\lambda) \in L^{-\mu;-\delta}_{\mathrm{cl}}(M; F, E; \mathbb{R}^l)$, i.e.,*

$$P(\lambda)A(\lambda) - I \in L^{-\infty;-\infty}(M; E, E; \mathbb{R}^l), \quad A(\lambda)P(\lambda) - I \in L^{-\infty;-\infty}(M; F, F; \mathbb{R}^l).$$

Moreover, there is a $C > 0$ such that (120) *are isomorphisms for all $|\lambda| \geq C$ and all $s, \varrho \in \mathbb{R}$.*

The proof of the first part of the theorem is straightforward, the second assertion is a direct consequence.

Next let M be a smooth manifold with smooth boundary, not necessarily compact. There is then a direct parameter-dependent analogue of the class of pseudo-differential boundary value problems $\mathcal{B}^{\mu,d}(M; \boldsymbol{v})$, cf. Definition 3.1, namely

$$\mathcal{B}^{\mu,d}(M; \boldsymbol{v}; \mathbb{R}^l). \tag{123}$$

To define (123) we simply have to replace the ingredients of (84) by the corresponding parameter-dependent versions $\mathrm{r}^+\tilde{A}(\lambda)\mathrm{e}^+$ and $\mathcal{G}(\lambda)$, respectively. Here, $\tilde{A}(\lambda) \in L^{\mu}_{\mathrm{cl}}(2M; \tilde{E}, \tilde{F}; \mathbb{R}^l)_{\mathrm{tr}}$ with obvious meaning of notation (recall that "cl" here only means "classical" in the covariables, though M may be non-compact) and $\mathcal{G}(\lambda) \in \mathcal{B}^{\mu,d}_G(M; \boldsymbol{v}; \mathbb{R}^l)$, also being defined along the lines of the class without parameters (all symbols simply contain λ as extra covariable, i.e., (ξ, λ) instead of ξ in the interior and (η, λ) instead of η near the boundary), and the parameter-dependent smoothing operators are given by

$$\mathcal{B}^{-\infty,d}(M; \boldsymbol{v}; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, \mathcal{B}^{-\infty,d}(M; \boldsymbol{v})), \tag{124}$$

where $\mathcal{B}^{-\infty,d}(M; \boldsymbol{v})$ is equipped with its standard Fréchet topology.

For $\mathcal{A} \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v}; \mathbb{R}^l)$ we have parameter-dependent homogeneous principal symbols, namely

$$\sigma_\psi(\mathcal{A}) : \pi^*_\psi E \longrightarrow \pi^*_\psi F, \quad \pi_\psi : (T^*M \times \mathbb{R}^l)\backslash 0 \longrightarrow M, \tag{125}$$

$$\sigma_\partial(\mathcal{A}) : \pi^*_\partial E \longrightarrow \pi^*_\partial F, \quad \pi_\partial : (T^*(\partial M) \times \mathbb{R}^l)\backslash 0 \longrightarrow \partial M. \tag{126}$$

$\mathcal{A} \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v}; \mathbb{R}^l)$ implies $\mathcal{A}(\lambda_0) \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v})$ for every fixed $\lambda_0 \in \mathbb{R}^l$, and we call (125), (126) the parameter-dependent principal symbols of $\mathcal{A}(\lambda)$ if we want to distinguish them from the usual ones of $\mathcal{A}(\lambda_0)$ that are independent of λ_0.

An element $\mathcal{A} \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v}; \mathbb{R}^l)$ is called parameter-dependent elliptic if (125), (126) are isomorphisms.

Theorem 4.3 *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v}; \mathbb{R}^l)$ be parameter-dependent elliptic. Then there is a parameter-dependent parametrix $\mathcal{P} \in \mathcal{B}^{-\infty,(d-\mu)^+}(M; \boldsymbol{v}^{-1}; \mathbb{R}^l)$ in a similar sense as in Remark* 3.4*; here, the remainders are smoothing in the sense of* (124).

The proof is similar to that of Theorem 3.10 above.

Theorem 4.4 *Let M be a compact smooth manifold with boundary, and let $\mathcal{A} \in \mathcal{B}^{\mu,d}(M; \boldsymbol{v}; \mathbb{R}^l)$ be parameter-dependent elliptic. Then there is a $C > 0$ such that*

$$\mathcal{A}(\lambda) : \begin{array}{c} H^s(M, E) \\ \oplus \\ H^s(\partial M, J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(M, F) \\ \oplus \\ H^{s-\mu}(\partial M, J^+) \end{array}$$

are isomorphisms for all $|\lambda| \geq C$ and all $s > \max(\mu, d) - \frac{1}{2}$.

Theorem 4.4 is a direct corollary of Theorem 4.3.

Remark 4.5 *In the cases that we discussed so far in the parameter-dependent set-up (i.e., "closed" manifolds with exits to infinity or smooth compact manifolds with boundary), where elliptic operators induce isomorphisms between the Sobolev spaces for large $|\lambda|$, we can easily conclude that the inverse maps belong λ-wise to the corresponding algebras in the non-parameter-dependent sense (as such they are reductions of orders in the algebras). It suffices to observe that when $1 + \{$smoothing operator$\}$ in one of our algebras is invertible, the inverse is of analogous structure and can be composed with the parametrix. This can even be done in the parameter-dependent framework for large $|\lambda|$, such that, in fact, the inverses for large $|\lambda|$ are also in the corresponding parameter-dependendent class.*

4.2 Boundary value problems for the case with exits to infinity

In the preceding section we extended some "standard" pseudo-differential algebras to the parameter-dependent variant, namely the algebra on a "closed" smooth manifold M with (conical) exits to infinity and the algebra of boundary value problems with the transmission property on a smooth manifold M with boundary (compact or non-compact). Now we formulate the calculus on a smooth manifold M with boundary and (conical) exits to infinity. In other words, we extend the material of Sections 2.2, 2.3, 2.4, 2.5, 2.6, 3.2 and 3.3 to the parameter-dependent case. This is to a large extent straightforward; so we content ourselves with the basic definitions and crucial points. Let us consider the parameter-dependent variant of symbol and operator spaces of Section 1.4 that is an operator-valued generalisation of the corresponding scalar symbols and operator spaces, respectively, as they are studied in Section 1.3. Similarly to the remarks in the beginning of the preceding section the essential constructions for the operator-valued symbols with parameters are practically the same as those without parameters. In particular, we have the parameter-dependent spaces of symbols $S^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{R}^{q+l}; \boldsymbol{E}, \tilde{\boldsymbol{E}}) =: S^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}; \mathbb{R}^l)$ based on strongly continuous groups of isomorphisms $\{\kappa_\tau\}_{\tau \in \mathbb{R}_+}$ on $\boldsymbol{E}$, $\{\tilde{\kappa}_\tau\}_{\tau \in \mathbb{R}_+}$ on $\tilde{\boldsymbol{E}}$, and the parameter-dependent spaces of pseudo-differential operators $L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q; \boldsymbol{E}, \tilde{\boldsymbol{E}}; \mathbb{R}^l)$ or $L^{\mu;\delta}_{(\mathrm{cl})}(M; \boldsymbol{E}, \tilde{\boldsymbol{E}}; \mathbb{R}^l)$. In the latter operator space M is, of course, a "closed" manifold with conical exits to infinity. Let us examine the behaviour of the operator norm with respect to the parameter λ. In the present situation the corresponding analogues of estimates (122) refer to global weighted Sobolev spaces

$$\mathcal{W}^{s;\varrho}(M, \boldsymbol{E}) \qquad (127)$$

(on a closed manifold M with conical exits to infinity) that are defined as subspaces of $\mathcal{W}^s_{\mathrm{loc}}(M, \boldsymbol{E})$ locally modelled by $\langle y\rangle^{-\varrho}\mathcal{W}^s(\mathbb{R}^q, \boldsymbol{E})|_\Gamma$ for $q = \dim M$ and suitable open subsets $\Gamma \subset \mathbb{R}^q$ that are conical in the large (recall that the global weighted spaces $H^{s;\varrho}(M)$ in Section 1.3 have been introduced by a similar scheme). Recall that for strongly continuous groups of isomorphisms $\{\kappa_\tau\}_{\tau\in\mathbb{R}_+}$ on $\boldsymbol{E}$ and $\{\tilde\kappa_\tau\}_{\tau\in\mathbb{R}_+}$ on $\tilde{\boldsymbol{E}}$ there are constants K and $\tilde K$, respectively, such that

$$||\kappa_\tau||_{\mathcal{L}(\boldsymbol{E})} \le c\langle\tau\rangle^K, \quad ||\tilde\kappa_\tau||_{\mathcal{L}(\tilde{\boldsymbol{E}})} \le \tilde c\langle\tau\rangle^{\tilde K}$$

for all $\tau \in \mathbb{R}_+$ and certain constants $c, \tilde c > 0$; $\langle\tau\rangle = (1+\tau^2)^{\frac{1}{2}}$.

Theorem 4.6 *Let $A(\lambda) \in L^{\mu;\delta}_{(\mathrm{cl})}(M; \boldsymbol{E}, \tilde{\boldsymbol{E}}; \mathbb{R}^l)$ be regarded as a family of continuous operators*

$$A(\lambda) : \mathcal{W}^{s;\varrho}(M, \boldsymbol{E}) \longrightarrow \mathcal{W}^{s-\nu;\varrho-\delta}(M, \tilde{\boldsymbol{E}})$$

for some $\nu \ge \mu$. Then there is a constant $m > 0$ such that the operator norm fulfils the estimate

$$||A(\lambda)||_{\mathcal{L}(\mathcal{W}^{s;\varrho}(M,\boldsymbol{E}),\mathcal{W}^{s-\nu;\varrho-\delta}(M,\tilde{\boldsymbol{E}}))} \le m b_{\mu+K+\tilde K,\nu+K+\tilde K}(\lambda)$$

for all $\lambda \in \mathbb{R}^l$, cf. (121).

For the case of compact M (and spaces $\mathcal{W}^s(M, \boldsymbol{E}) = \mathcal{W}^{s;0}(M, \boldsymbol{E})$) a similar theorem is proved in Behm [2]. This extends to the non-compact case with conical exits and weighted spaces in a similar manner as in the scalar situation; for the corresponding technique, cf. Dorschfeldt, Grieme, and Schulze [5] and Seiler [32].

Let us now return to the case of a smooth manifold M with boundary and conical exits to infinity. First, there is the space

$$\mathcal{B}^{-\infty,d;-\infty}(M; \boldsymbol{v}; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l; \mathcal{B}^{-\infty,d;-\infty}(M; \boldsymbol{v})) \tag{128}$$

for $\boldsymbol{v} = (E, F; J^-, J^+)$ of parameter-dependent smoothing operators on M, using $\mathcal{B}^{-\infty,d;-\infty}(M; \boldsymbol{v})$ in its canonical Fréchet topology.

Another simple ingredient of the class

$$\mathcal{B}^{\mu,d;\delta}(M; \boldsymbol{v}; \mathbb{R}^l) \tag{129}$$

that will be defined below is the space of all operator families $\mathrm{r}^+\tilde A(\lambda)\mathrm{e}^+$, where

$$\tilde A(\lambda) \in L^{\mu;\delta}_{\mathrm{cl}}(2M; \tilde E, \tilde F; \mathbb{R}^l)_{\mathrm{tr}}. \tag{130}$$

Here, "cl" means classical in covariables and variables in the local representations on conical subsets of M; the transmission property including parameters with respect to ∂M has been defined in Section 4.1; the interpretation of the weight δ at infinity is the same as in Section 1.3. Furthermore, we have a direct analogue of $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; k, m; N_-, N_+)$, cf. the $m \times k$ block matrix-version of Definition 2.14 in the parameter-dependent case, namely

$$\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; k, m; N_-, N_+; \mathbb{R}^l).$$

An inspection of all ingredients shows that (except for $\mathcal{P}$ and $\mathcal{C}$ in equality (38) that we already defined above) the only new point is to replace the amplitude function $\boldsymbol{a}(y, \eta)$ in (38) by $\boldsymbol{a}(y, \eta, \lambda)$ from the space $\mathcal{R}^{\mu,d;\delta}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1+l}; k, m; N_-, N_+)$, the parameter-dependent $m \times k$-block matrix version of the corresponding space in Definition 2.13. Finally, we get (129) by a straightforward generalisation of the constructions for Definition 3.5. In fact, only the above-mentioned ingredients are involved, except for evident invariance properties (under transition maps) of corresponding subspaces of parameter-dependent Green operators and localisation by admissible cut-off functions (those are the same as for the case without parameters). Summing up we have introduced all data of the following definition.

Definition 4.7 *The space* $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v};\mathbb{R}^l)$ *for* $\mu \in \mathbb{Z}, d \in \mathbb{N}, \delta \in \mathbb{R}$ *and* $\boldsymbol{v} = (E,F;J^-,J^+)$ *is defined to be the set of all operator families*

$$\mathcal{A}(\lambda) = \begin{pmatrix} \mathrm{r}^+\tilde{A}(\lambda)\mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}(\lambda), \tag{131}$$

$\lambda \in \mathbb{R}^l$, *for arbitrary* $\tilde{A}(\lambda) \in L^{\mu;\delta}_{\mathrm{cl}}(2M;\tilde{E},\tilde{F};\mathbb{R}^l)_{\mathrm{tr}}$ *(with* $\tilde{E}|_M = E, \tilde{F}|_M = F$*) and* $\mathcal{G}(\lambda) \in \mathcal{B}^{\mu,d;\delta}_{G,\mathrm{cl}}(M;\boldsymbol{v};\mathbb{R}^l)$.

Remark 4.8 *By definition we have* $\mathcal{B}^{\mu,d;\delta}_{\mathrm{cl}}(M;\boldsymbol{v};\mathbb{R}^l) \subset \mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ *where the right hand side is understood in the sense of* (123).

Applying the definition of global parameter-dependent symbols (117), (118), (119) to $\tilde{A}$ on $2M$ and restricting them to M (similarly to (95),(96),(97)) we get the parameter-dependent principal interior symbols

$$\sigma_\psi(A) : \pi^*_\psi E \longrightarrow \pi^*_\psi F, \quad \pi_\psi : (T^*M \times \mathbb{R}^l)\backslash 0 \longrightarrow M, \tag{132}$$

$$\sigma_{\mathrm{e}}(A) : \pi^*_{\mathrm{e}} E \longrightarrow \pi^*_{\mathrm{e}} F, \quad \pi_{\mathrm{e}} : T^*M|_{X^\wedge_\infty} \times \mathbb{R}^l \longrightarrow X^\wedge_\infty, \tag{133}$$

$$\sigma_{\psi,\mathrm{e}}(A) : \pi^*_{\psi,\mathrm{e}} E \longrightarrow \pi^*_{\psi,\mathrm{e}} F, \quad \pi_{\psi,\mathrm{e}} : ((T^*M \times \mathbb{R}^l)\backslash 0)|_{X^\wedge_\infty} \longrightarrow X^\wedge_\infty. \tag{134}$$

A direct generalisation of (98), (99), (100) to the parameter-dependent case gives us the parameter-dependent principal boundary symbols

$$\sigma_\partial(\mathcal{A}) : \pi^*_\partial \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi^*_\partial \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{135}$$

for $\pi_\partial : (T^*(\partial M) \times \mathbb{R}^l)\backslash 0 \to \partial M$,

$$\sigma_{\mathrm{e}'}(\mathcal{A}) : \pi^*_{\mathrm{e}'} \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi^*_{\mathrm{e}'} \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{136}$$

for $\pi_{\mathrm{e}'} : T^*(\partial M)|_{Y^\wedge_\infty} \times \mathbb{R}^l \to Y^\wedge_\infty$ and

$$\sigma_{\partial,\mathrm{e}'}(\mathcal{A}) : \pi^*_{\partial,\mathrm{e}'} \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi^*_{\partial,\mathrm{e}'} \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{137}$$

for $\pi_{\partial,\mathrm{e}'} : ((T^*(\partial M) \times \mathbb{R}^l)\backslash 0)|_{Y^\wedge_\infty} \to Y^\wedge_\infty$. Further explanation to the latter bundle homomorphisms is unnecessery, because the only novelty are the additional covariables $\lambda \in \mathbb{R}^l$.

Remark 4.9 *$\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M;\boldsymbol{v};\mathbb{R}^l)$ implies $\mathcal{A}(\lambda_0) \in \mathcal{B}_{\mathrm{cl}}^{\mu,d}(M;\boldsymbol{v})$ for every fixed $\lambda_0 \in \mathbb{R}^l$, and the symbols $\sigma_\psi(\mathcal{A}(\lambda_0)), \sigma_{\psi,\mathrm{e}}(\mathcal{A}(\lambda_0)), \sigma_\partial(\mathcal{A}(\lambda_0)), \sigma_{\partial,\mathrm{e}'}(\mathcal{A}(\lambda_0))$ do not depend on λ_0. If necessary we point out that* (132), (133), (134), (135), (136), (137) *are the parameter-dependent principal symbols of $\mathcal{A}(\lambda)$.*

Remark 4.10 *There is an obvious analogue of the composition result of Theorem* 3.7 *for the parameter-dependent case, including the symbol rule, where in the present case $\sigma(\mathcal{A})$ is the tuple of parameter-dependent principal symbols* (132)-(137)*, similarly to* (101)*.*

Definition 4.11 *An operator $\mathcal{A} \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M;\boldsymbol{v};\mathbb{R}^l)$ is called parameter-dependent elliptic if all principal symbol homomorphisms* (132)-(137) *are isomorphisms. An operator $\mathcal{P}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{-\mu,e;-\delta}(M;\boldsymbol{v}^{-1};\mathbb{R}^l)$ for some $e \in \mathbb{N}$ is called a parameter-dependent parametrix if*

$$\mathcal{P}(\lambda)\mathcal{A}(\lambda) - \mathcal{I} \in \mathcal{B}^{-\infty,d_l;-\infty}(M;\boldsymbol{v}_l;\mathbb{R}^l), \quad \mathcal{A}(\lambda)\mathcal{P}(\lambda) - \mathcal{I} \in \mathcal{B}^{-\infty,d_r;-\infty}(M;\boldsymbol{v}_r;\mathbb{R}^l)$$

for certain $d_l, d_r \in \mathbb{N}$, and $\boldsymbol{v}_l = (E,E;J^-,J^-), \boldsymbol{v}_r = (F,F;J^+,J^+)$.

Theorem 4.12 *Let M be a smooth manifold with boundary and (conical) exits to infinity, and $\mathcal{A}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M;\boldsymbol{v};\mathbb{R}^l)$, $\boldsymbol{v} = (E,F;J^-,J^+)$, be parameter-dependent elliptic. Then there exists a parameter-dependent parametrix $\mathcal{P}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{-\mu,(d-\mu)^+;-\delta}(M;\boldsymbol{v}^{-1};\mathbb{R}^l)$, where the types in the remainders are $d_l = \max(\mu,d)$, $d_r = (d-\mu)^+$. Moreover,*

$$\mathcal{A}(\lambda) : \begin{array}{c} H^{s;\varrho}(M,E) \\ \oplus \\ H^{s;\varrho}(\partial M,J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu;\varrho-\delta}(M,F) \\ \oplus \\ H^{s-\mu;\varrho-\delta}(\partial M,J^+) \end{array} \tag{138}$$

is a family of Fredholm operators of index 0 for every $s > \max(\mu,d) - \frac{1}{2}$, and there is a constant $C > 0$ such that (138) *are isomorphisms for all $|\lambda| \geq C$.*

The basic idea of proving results of this type has been briefly discussed in Remark 4.5 above. Also in the present situation of Theorem 4.12 we first construct a parameter-dependent parametrix $\mathcal{P}(\lambda)$ by inverting the parameter-dependent principal symbol of $\mathcal{A}(\lambda)$ and get $\mathcal{P}(\lambda)\mathcal{A}(\lambda) - \mathcal{I} = \mathcal{C}_l(\lambda), \mathcal{A}(\lambda)\mathcal{P}(\lambda) - \mathcal{I} = \mathcal{C}_r(\lambda)$, with smoothing operators in Boutet de Monvel's algebra. By virtue of (128) it is fairly obvious that $\mathcal{I} + \{\text{smoothing operator}\}$ is invertible in the same class, such that it can be composed with $\mathcal{P}(\lambda)$.

Theorem 4.13 *Let M be a smooth manifold with boundary and (conical) exits to infinity, and let $E \in \mathrm{Vect}(M), \mu \in \mathbb{Z}, \delta \in \mathbb{R}$. Then there exists a parameter-dependent elliptic element $R_E^{\mu;\delta}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{\mu,0;\delta}(M;E,E;\mathbb{R}^l)$ that induces isomorphisms*

$$R_E^{\mu;\delta}(\lambda) : H^{s;\varrho}(M,E) \longrightarrow H^{s-\mu;\varrho-\delta}(M,E) \tag{139}$$

for all $s, \varrho \in \mathbb{R}$ and all $\lambda \in \mathbb{R}^l$, and we have $R_E^{\mu;\delta}(\lambda)^{-1} \in \mathcal{B}_{\mathrm{cl}}^{-\mu,0;-\delta}(M;E,E;\mathbb{R}^l)$. Similarly, for every $J \in \mathrm{Vect}(\partial M)$ and $\nu, \delta \in \mathbb{R}$ there exists a parameter-dependent elliptic element $R_J'^{\nu;\delta}(\lambda) \in L_{\mathrm{cl}}^{\nu;\delta}(\partial M;J,J;\mathbb{R}^l)$ that induces isomorphisms

$$R_J'^{\nu;\delta}(\lambda) : H^{s;\varrho}(\partial M,J) \longrightarrow H^{s-\nu;\varrho-\delta}(\partial M,J) \tag{140}$$

for all $s, \varrho \in \mathbb{R}$ and all $\lambda \in \mathbb{R}^l$, and we have $R_J'^{\nu;\delta}(\lambda)^{-1} \in L_{\mathrm{cl}}^{-\nu;-\delta}(\partial M;J,J;\mathbb{R}^l)$.

Remark 4.14 *Combining the latter theorem with Remark 4.9, by inserting any fixed $\lambda_0 \in \mathbb{R}^l$ into (139) and (140) we get order reducing operators in the operator spaces $\mathcal{B}_{\mathrm{cl}}^{\mu,0;\delta}(M;E,E)$ and $L^{\nu;\delta}(\partial M;J,J)$, respectively.*

Remark 4.15 *The concept of algebras of parameter-dependent operators can also be formulated for more general parameter sets $\Lambda \subseteq \mathbb{R}^q$ that have the property $\lambda \in \Lambda \Rightarrow c\lambda \in \Lambda$ for all $c \geq 1$. Examples are*

$$\Lambda = \mathbb{R}^l \backslash 0, \quad \Lambda = \mathbb{R}^l \backslash \{|\lambda| < C\}$$

or single rays Λ in $\mathbb{R}^q$. For such Λ all our operator classes have corresponding variants, e.g., $L_{\mathrm{cl}}^{\mu;\delta}(M;E,F;\Lambda)$, cf. Theorem 4.1, $\mathcal{B}_{\mathrm{cl}}^{\mu,d}(M;\boldsymbol{v};\Lambda)$, cf. formula (123), etc. The behaviour of operators in these spaces for small $\lambda \in \Lambda$ remains unspecified; we assume, for instance, smoothness in λ. The parameter-dependent symbols now refer to $\lambda \in \Lambda$, and we have evident generalisations of the corresponding parameter-dependent ellipticities and parametrices.

Let us explicitly formulate a corresponding extension of Theorem 4.12 in the version with Λ:

Theorem 4.16 *Let $\mathcal{A}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M;\boldsymbol{v};\Lambda)$, $\boldsymbol{v} = (E,F;J^-,J^+)$, be parameter-dependent elliptic. Then there exists a parameter-dependent parametrix $\mathcal{P}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{-\mu,(d-\mu)^+;-\delta}(M;\boldsymbol{v}^{-1};\Lambda)$, with the above-mentioned types d_l and d_r of remainders. Furthermore, the operators (138) are Fredholm and of index 0 for all $s > \max(\mu,d) - \frac{1}{2}$, there is a constant $C > 0$ such that (138) are isomorphisms for all $|\lambda| \geq C$, and we have $\mathcal{A}^{-1} \in \mathcal{B}_{\mathrm{cl}}^{-\mu,(d-\mu)^+;-\delta}(M;\boldsymbol{v}^{-1};\Lambda_C)$ for $\Lambda_C = \{\lambda \in \Lambda : |\lambda| \geq C\}$.*

Remark 4.17 *The spaces $\mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M;\boldsymbol{v};\Lambda)$, $\boldsymbol{v} = (E,F;J^-,J^+)$, (as well as the other operator spaces, e.g., from Sections 3.1 or 4.1) can easily be generalised to the case of Douglis-Nirenberg orders (DN-orders) with a corresponding ellipticity; the results carry over to the variant with DN-orders.*

The Douglis-Nirenberg generalisation refers to representations of the bundles as direct sums $E = \oplus_{m=1}^k E_m$, $F = \oplus_{n=1}^l F_n$, $J^- = \oplus_{i=1}^b J_i^-$, $J^+ = \oplus_{j=1}^c J_j^+$. Operators are then represented as block matrices, composed with diagonal matrices of order reductions on M and ∂M, respectively. The constructions are straightforward, so we do not really discuss the details, but in some cases below we need notation. This concerns DN-orders for the boundary operators, where

$$\mathcal{A}(\lambda) = \mathcal{R}(\lambda)\tilde{\mathcal{A}}(\lambda)\mathcal{Q}^{-1}(\lambda)$$

with $\tilde{\mathcal{A}}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M;\boldsymbol{v};\Lambda)$, $\boldsymbol{v} = (E,F;\oplus_{i=1}^b J_i^-, \oplus_{j=1}^c J_j^+)$, and

$$\mathcal{R}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{diag}(R_j'^{\gamma_j}(\lambda)) \end{pmatrix}, \quad \mathcal{Q}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{diag}(Q_i'^{\beta_i}(\lambda)) \end{pmatrix}$$

with 1 denoting identity operators referring to F and E and order reducing operator families

$$Q_i'^{\beta_i}(\lambda) \in L_{\mathrm{cl}}^{\beta_i;0}(\partial M;J_i^-;J_i^-;\Lambda),\ i = 1,\ldots,b,$$

$$R_j'^{\gamma_j}(\lambda) \in L_{\mathrm{cl}}^{\gamma_j;0}(\partial M;J_j^+;J_j^+;\Lambda),\ j = 1,\ldots,c,$$

$\beta_i, \gamma_j \in \mathbb{R}$. Such a situation is customary in elliptic boundary value problems for differential operators with differential boundary conditions. In this particular case all bundles J_i^- are of fibre dimension 0, while parametrices refer to the case that the bundles J_j^+ are of fibre dimension 0. The parameter-dependent ellipticity of $\mathcal{A}(\lambda)$ is defined to be the parameter-dependent ellipticity of $\tilde{\mathcal{A}}(\lambda)$, and we get a parameter-dependent parametrix of $\mathcal{A}(\lambda)$ by $\mathcal{P}(\lambda) = \mathcal{Q}(\lambda)\tilde{\mathcal{P}}(\lambda)\mathcal{R}^{-1}(\lambda)$ when $\tilde{\mathcal{P}}(\lambda)$ denotes a parameter-dependent parametrix of $\tilde{\mathcal{A}}(\lambda)$.

In any case the involved orders are known and fixed. Therefore, given $(\beta_1, \ldots, \beta_b)$ and $(\gamma_1, \ldots, \gamma_c)$, we set

$$\boldsymbol{\mathcal{B}}_{\mathrm{cl}}^{\mu,d;\delta}(M; \boldsymbol{v}; \Lambda) = \{\mathcal{R}(\lambda)\tilde{\mathcal{A}}(\lambda)\mathcal{Q}^{-1}(\lambda) : \tilde{\mathcal{A}}(\lambda) \in \mathcal{B}_{\mathrm{cl}}^{\mu,d;\delta}(M; \boldsymbol{v}; \Lambda)\}. \tag{141}$$

Parametrices then belong (by notation) to $\boldsymbol{\mathcal{B}}_{\mathrm{cl}}^{-\mu,(d-\mu)^+;-\delta}(M; \boldsymbol{v}^{-1}; \Lambda)$.

4.3 Relations to the edge pseudo-differential calculus

In this section we want to discuss relations between our calculus of boundary value problems on non-compact manifolds with exits and the theory of boundary value problems in domains with edges. Particularly simple edge configurations occur in models of the crack theory. In local terms the situation can be described by $(\mathbb{R}^2 \backslash \mathbb{R}_+) \times \Omega$, where $\Omega \subseteq \mathbb{R}^q$ plays the role of a crack boundary (for crack problems in $\mathbb{R}^3$ we have $q = 1$), $\mathbb{R}^2$ is the normal plane to the crack boundary, and $\mathbb{R}_+ \subset \mathbb{R}^2$ is a coordinate half-axis corresponding to the intersection of the crack with $\mathbb{R}^2$. This situation is studied in detail in Kapanadze and Schulze [11]. A special aspect of this approach is that the crack boundary is regarded as an edge and $\mathbb{R}^2 \backslash \mathbb{R}_+$ as an infinite model cone with the origin of $\mathbb{R}^2$ as the tip of the cone. More precisely, the cone consists of a configuration, where two copies of $\mathbb{R}_+$ constitute the slit in $\mathbb{R}^2$ with separate elliptic boundary conditions on the $\pm$-sides. The edge symbol calculus for this situation may be regarded as a parameter-dependent "infinite" cone theory, consisting of the calculus on a bounded part of the cone near the tip and that in the exit sense elsewhere. The latter calculus treats both sides of the slit separately, and its contribution can be formulated in terms of parameter-dependent boundary value problems in the half-space, together with a localisation. We now formulate a result that is typical for this theory. Let

$$A = \sum_{|\alpha|+|\beta| \le m} a_{\alpha\beta}(x, y) D_x^\alpha D_y^\beta$$

be a differential operator in $U \times \Omega \ni (x, y)$, $U \subseteq \mathbb{R}^n$ open, containing the origin, and $\Omega \subseteq \mathbb{R}^q$ open (for simplicity we assume A to be scalar; the considerations for systems are analogous), with coefficients $a_{\alpha\beta} \in C^\infty(U \times \Omega)$. For the edge symbol calculus the coefficients are to be frozen at $x = 0$. This gives us an operator family

$$\sigma_\wedge(A)(y, \eta) := \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta}(0, y) D_x^\alpha \eta^\beta : H^s(\mathbb{R}_+^n) \longrightarrow H^{s-m}(\mathbb{R}_+^n).$$

Set $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$. Then we have

$$\sigma_\wedge(A)(y, \lambda\eta) = \lambda^m \kappa_\lambda \sigma_\wedge(A)(y, \eta) \kappa_\lambda^{-1} \tag{142}$$

for all $\lambda \in \mathbb{R}_+$ and all y, η.

Let A be elliptic, and let $T = (\mathrm{r}'B_1, \ldots, \mathrm{r}'B_N)$ be a vector of trace operators, $\mathrm{r}'u = u|_{\mathbb{R}^{n-1}}$, for differential operators

$$B_j = \sum_{|\alpha|+|\beta|\le m_j} b^j_{\alpha\beta}(x,y) D_x^\alpha D_y^\beta$$

with cofficients $b^j_{\alpha\beta} \in C^\infty(U \times \Omega)$ and $m_j < m$ for all j. Assume that the boundary value problem

$$Au = f \;\text{ in }\; U \cap \mathbb{R}^n_+, \qquad Tu = g \;\text{ on }\; U \cap \mathbb{R}^{n-1} \tag{143}$$

is elliptic in the sense that the trace operators satisfy the Shapiro-Lopatinskij condition with respect to A (clearly, N is known by the problem, e.g., if m is even and $n+q \ge 3$, we have $N = \frac{m}{2}$). Let us form

$$\sigma_\wedge(T)(y,\eta) = (\sigma_\wedge(T_1)(y,\eta), \ldots, \sigma_\wedge(T_N)(y,\eta))$$

where $\sigma_\wedge(T_j)(y,\eta) = \mathrm{r}' \sum_{|\alpha|+|\beta|=m_j} b^j_{j\alpha}(0,y) D_x^\alpha \eta^\beta \;:\; H^s(\mathbb{R}^n_+) \longrightarrow H^{s-m_j-\frac{1}{2}}(\mathbb{R}^{n-1})$, $s > m_j + \frac{1}{2}$. Then

$$\sigma_\wedge(T_j)(y, \lambda\eta) = \lambda^{m_j+\frac{1}{2}} \sigma_\wedge(T_j)(y,\eta)\kappa_\lambda^{-1} \tag{144}$$

for all $\lambda \in \mathbb{R}_+$ and all $(y,\eta) \in \Omega \times (\mathbb{R}^q \backslash 0)$. We have

$$\sigma_\wedge(\mathcal{A})(y,\eta) := \begin{pmatrix} \sigma_\wedge(A)(y,\eta) \\ \sigma_\wedge(T)(y,\eta) \end{pmatrix} \in C^\infty(\Omega, \boldsymbol{\mathcal{B}}^{m,d;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; \boldsymbol{v}; \mathbb{R}^q \backslash 0)) \tag{145}$$

with the above-mentioned interpretation of the order superscript (and $\gamma_j = -m + m_j + \frac{1}{2}$, $j = 1, \ldots, N$, while the numbers β_j disappear in this case), $d = \max_j(m_j+1)$, $\boldsymbol{v} = (1,1;0,N)$.

Theorem 4.18 *Let* $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ *be elliptic in* $U \times \Omega$. *Then*

$$\sigma_\wedge(\mathcal{A})(y,\eta) : H^s(\mathbb{R}^n_+) \longrightarrow \begin{pmatrix} H^{s-m}(\mathbb{R}^n_+) \\ \oplus \\ \oplus_{j=1}^N H^{s-m_j-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{pmatrix} \tag{146}$$

is a family of invertible operators for all $(y,\eta) \in \Omega \times (\mathbb{R}^q \backslash 0)$ *and all* $s > \max(m,d) - \frac{1}{2}$, *and we have*

$$\sigma_\wedge(\mathcal{A})^{-1}(y,\eta) \in C^\infty(\Omega, \boldsymbol{\mathcal{B}}^{-m,0;0}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+; \boldsymbol{v}^{-1}; \mathbb{R}^q \backslash 0)),$$

cf. the notation in Remark 4.15 *and formula* (141).

Proof. By assumption A is elliptic in $U \times \Omega$, i.e., $\sigma_\psi(A)(x,y,\xi,\eta) \ne 0$ for all $(x,y) \in U \times \Omega$ and $(\xi,\eta) \ne 0$. Thus the η-dependent family $a(y,\eta) = \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta}(0,y) D_x^\alpha \eta^\beta$ of differential operators with respect to x (smoothly dependent on y) is parameter-dependent elliptic with parameters $\eta \in \mathbb{R}^{n-1} \backslash 0$ with respect to $(\sigma_\psi, \sigma_\mathrm{e}, \sigma_{\psi,\mathrm{e}})$, uniformly on compact subsets with respect to y). For similar reasons, the Shapiro-Lopatinskij condition of the original boundary value problem (143), i.e., the invertibility of

$$\sigma_\partial \begin{pmatrix} A \\ T \end{pmatrix} (x', y, \xi', \eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^N \end{matrix}$$

for all $(x', y) \in (U \cap \mathbb{R}^{n-1}) \times \Omega, (\xi', \eta) \neq 0$, gives us parameter-dependent ellipticity with parameter $\eta \in \mathbb{R}^n \backslash 0$ with respect to $(\sigma_\partial, \sigma_{e'}, \sigma_{\partial,e'})$ (uniformly on compact subsets with respect to y). Applying Theorem 4.12 we find for every $y \in \Omega$ a constant $C > 0$ (which can obviously be chosen uniformly on compact subsets of Ω) such that (146) is invertible for $|\eta| > C$. Because of the κ_λ-homogeneity of $\sigma_\wedge(\mathcal{A})(y, \eta)$ (i.e., relations (142) and (144)) we get the invertibility of (146) for all $\eta \neq 0$. Concerning the asserted nature of the inverse we can apply Theorem 4.12. □

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