Asymptotics of Solutions to Elliptic Equations on Manifolds with Corners [*†]

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Abstract

We show an explicit link between the nature of a singular point and behaviour of the coefficients of the equation, under which formal asymptotic expansions are still available.

Contents

1 Introduction

The paper studies asymptotics of solutions to elliptic differential equations on a closed manifold M with singular points. Near any singular point such an equation reduces to an ordinary differential equation on the half-axis $r \in \bar{\mathbb{R}}_+$, whose coefficients take their values in a pseudodifferential algebra on a compact closed manifold X. They have special degenerations at $r = 0$ which are determined by the geometry of the singular point.

While the most interesting case is when X itself bears singularities, many specific features can be observed by ordinary differential equations with scalar-valued coefficients on $\bar{\mathbb{R}}_+$. The best general reference here is the encyclopaedic book [Fed93] describing the developments of the last years in the area of asymptotic methods for linear ordinary differential equations.

The asymptotics are well understood for solutions of the Fuchs-type equations near $r = 0$. These correspond to conical singularities, an equivalent designation being *regular singular points*. Topologically each singular point of the underlying manifold is equivalent to a conical point. However, the corresponding homeomorphism does not preserve the C^∞ structure in general, which results in irregular singular points.

By the "formal asymptotic solution" is understood a function which satisfies the equation to some degree of accuracy. The algorithm for the construction of such solutions is extremely complicated for the case of irregular singular points. The series obtained this way terminate only in exceptional cases and usually diverge. Moreover, the existence of formal asymptotic solutions does not always imply the existence of real solutions having such asymptotic behaviour, cf. [Fed93, p. 16].

The recent book [MKR97] summarises the progress in asymptotic expansions of solutions to elliptic boundary value problems in domains with point singularities. Roughly speaking, such expansions have been obtained only for the equations whose coefficients are sufficiently "flat" in a neighbourhood of the singular point to survive under a singular change of variables blowing up the singular point to a conical one. In the case of irregular singular points full asymptotic expansions are known only in a restricted number of cases. Determining these expansions is essentially equivalent to solving the differential equation.

The aim of this paper is to show an explicit link between the nature of a singular point and behaviour of the coefficients of the equation near this point, which still ensures transparent formal asymptotic solutions. More precisely, in a "punctured" neighbourhood of a singular point we choose generalised polar coordinates $(r, x) \in (0, \varepsilon) \times X$ where r is the distance to the singular point and X a compact closed manifold called the *link*. In these coordinates a typical differential operator is of the form

$$A = (\delta'(r))^m \sum_{j=0}^{m} A_j(r) \left(\frac{1}{\delta'(r)} D_r \right)^j$$

where $\delta(r)$ is a C^∞ function on $(0, \varepsilon)$, such that $\delta(r) \searrow -\infty$ as $r \to 0$, and the coefficients A_j are C^∞ functions on $(0, \varepsilon)$ with values in $\Psi^{m-j}(X; w - j)$, a pseudodifferential algebra on X, cf. [RST97]. In order to get asymptotic results, it is necessary to require that A_j, $j = 0, 1, \ldots, m$, are continuous up to $r = 0$.

In case $\delta'(r)$ is equal to r^{-p-1} near $r = 0$, with $p = 0, 1, \ldots$, the above equality gives typical differential operators on manifolds with power-like cusps. For the particular value $p = 0$, we get the general form of linear operators in a neighbourhood of a conical singularity (so-called Fuchs-type operators).

We restrict ourselves to those operators A which are elliptic up to the singular points. This implies, in particular, that $A_m(r)$ is an elliptic operator of order 0 on X, for each $r \in [0, \varepsilon)$. Hence we may assume without loss of generality that $A_m(r) \equiv 1$ modulo operators of order -1 on X. We neglect these compact remainders and require $A_m(r)$ to be the identity operator. On the other hand, the multiplicative factor $(\delta'(r))^m$ can be traced back to weighted Sobolev spaces. Using a change of variables $t = \delta(r)$ suggested by the geometry of the singular point, we push forward the equation from a neighbourhood of $r = 0$ to that of $t = -\infty$. We thus arrive at

$$D^m u(t) + \sum_{j=0}^{m-1} C_j(t)\, D^j u(t) = f(t), \qquad t < T, \tag{1.1}$$

where $C_j(t) = A_j(\delta^{-1}(t))$ is a continuous function on $[-\infty, T)$ with values in $\Psi^{m-j}(X; w - j)$.

By a special change of variables equation (1.1) reduces to a first order system

$$DU(t) - C(t)\, U(t) = F(t), \qquad t < T, \tag{1.2}$$

$C(t)$ being an $(m \times m)$-matrix with entries in $\Psi^m(X; w)$. This operator equation is a central theme of many research papers and books, cf. Daletskii and Krein [DK70]. In most cases $i\, C(t)$ is required to be a generator of a semigroup or a bounded operator in a Banach space, an assumption which is violated for the differential operators.

For general $C(t)$ independent of t, equation (1.2) was studied in detail by Agmon and Nirenberg [AN63]. In particular, they derived asymptotic formulas for solutions of exponential growth under the condition that the spectrum of the operator C consists of normal eigenvalues located (except possibly for a finite number) in some double angular sector containing the imaginary axis, the angle being less than π. The asymptotic behaviour is described by

$$U(t) \sim \sum_{\Im \lambda_\nu \leq -\gamma} e^{i\lambda_\nu t} \sum_{k=0}^{\varrho_\nu - 1} \mathcal{P}^{(\nu)}_{\varrho_\nu - 1 - k}(t)\, \Phi^{(\nu)}_k \quad \text{as} \quad t \to -\infty, \tag{1.3}$$

where γ is some number characterising the growth of the solution, ϱ_ν is a partial multiplicity of the eigenvalue λ_ν of C, $\mathcal{P}^{(\nu)}_k(t)$ are polynomials of degree k, and $\Phi^{(\nu)}_k$ are eigen- and associated functions corresponding to λ_ν. Returning to (1.1) we obtain an asymptotic formula (1.3) for solutions in the case of constant coefficients $C_j(t)$, $\Phi^{(\nu)}_k$ being eigen- and associated functions of the operator pencil

$$\sigma(\lambda) = \lambda^m + \sum_{j=0}^{m-1} C_j \lambda^j.$$

These results were extended by Pazy [Paz67] to equations whose coefficients differ from constants by exponentially decreasing terms.

For solutions of parabolic boundary problems in domains with a smooth boundary which is characteristic at isolated points, similar asymptotic formulas were obtained by Kondrat'ev [Kon66]. He extended these results also to elliptic boundary value problems in domains with conical points on the boundary, cf. [Kon67].

Evgrafov [Evg61] treated the asymptotic behaviour for $t \to -\infty$ of the solution of (1.2) in the case when the operator $C(t)$ tends to an operator C in some weak sense as $t \to -\infty$. His formula reads

$$U(t) = e^{i \int_T^t \lambda(s) ds} \left(c\, \Phi(t) + o(1) \right)$$

where $\lambda(t)$ is an eigenvalue of the operator pencil $C(t)$ tending to a simple eigenvalue of the operator C as $t \to -\infty$, $\Phi(t)$ is the corresponding eigenfunction of $C(t)$, and c is a constant. In [Evg61] it is assumed that the resolvent $(\lambda - C)^{-1}$ of the operator C is completely continuous and has at most a finite number of poles outside of a double angular sector containing the imaginary axis; away from this sector $\|(\lambda - C)^{-1}\| \leq c\,(1 + |\lambda|)^{-1}$ provided that $|\lambda|$ is large enough.

Maz'ya and Plamenevskii [MP72] extended the theorem of [Evg61] to equations of an arbitrary order. Their results can also be interpreted as a generalisation of the asymptotic formula of [AN63] to the case of equation (1.1) with variable coefficients. For a further progress, we refer the reader to [Pla73]; applications in elliptic theory on manifolds with singularities are elaborated by Grisvard [Gri85], Schulze [Sch88], Nazarov and Plamenevskii [NP91], Rebahi [Reba00], and others.

Let us comment on the main results of this paper. The operators of $\Psi^m(X;w)$ act in weighted Sobolev spaces $H^{s,w}(X)$ on X, $s \in \mathbb{R}$ being a smoothness and $w \in \mathbb{R}^Q$ a tuple of weights. If $s' \geq s$ and $w' \geq w$, the latter inequality being understood component-wise, then $H^{s',w'}(X) \hookrightarrow H^{s,w}(X)$; moreover, this embedding is compact provided $s' > s$ and $w' > w$. An operator $\Psi \in \Psi^m(X;w)$ extends to a continuous mapping $H^{s,w}(X) \to H^{s-m,w-m}(X)$, for each $s \in \mathbb{R}$, where $w - m = (w_1 - m, \ldots, w_Q - m)$.

Consider the operator pencil $\sigma(\lambda)$ defined on a space $H^{s,w}(X)$. By the above, it can thought of as a family in $\mathcal{L}(H^{s,w}(X), H^{s-m,w-m}(X))$ parametrised by $\lambda \in \mathbb{C}$, for any $s \in \mathbb{R}$. We assume that, for some integer $s \geq p$, the operator $\sigma(\lambda)$ has a bounded inverse $\sigma^{-1}(\lambda)$ everywhere in the complex plane, with the exception of a discrete set, and this inverse is a meromorphic function of the parameter λ. All the poles of $\sigma^{-1}(\lambda)$, except possibly for a finite number, are required to lie in some double angular sector containing the imaginary axis, the angle being less than π. If moreover outside of this angular region we have an estimate

$$\sum_{j=0}^{s} |\lambda|^j \, \|u\|_{H^{s-j,w-j}(X)} \leq c \sum_{j=0}^{s-m} |\lambda|^j \, \|\sigma(\lambda)u\|_{H^{s-j-m,w-j-m}(X)}$$

for all $u \in H^{s,w}(X)$ and $|\lambda|$ large enough, the operator pencil $\sigma(\lambda)$ is said to be *elliptic*.

Our standing assumption on $\sigma(\lambda)$ is that $\sigma(0) = C_0$ is an invertible operator $H^{s,w}(X) \to H^{s-m,w-m}(X)$. Were the $\mathbb{Z}$-graded algebra $\Psi^{\cdot}(X;w)$ spectral invariant, it would follow that the inverse of $\sigma(0)$ actually belongs to $\Psi^{-m}(X; w-m)$, and thus $\sigma(0)$ is invertible for all s.

Recall that a number $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $\sigma(\lambda)$ if there exists a non-zero function $\varphi_0 \in H^{s,w}(X)$, such that $\sigma(\lambda_0)\varphi_0 = 0$. The function φ_0 is called an *eigenfunction* of $\sigma(\lambda)$ at λ_0. If $\varphi_1, \ldots, \varphi_{\varrho-1} \in H^{s,w}(X)$ satisfy the equations

$$\sum_{k=0}^{K} \frac{1}{(K-k)!} \sigma^{(K-k)}(\lambda_0)\varphi_k = 0 \tag{1.4}$$

for $K = 1, \ldots, \varrho - 1$, then the system $(\varphi_0, \varphi_1, \ldots, \varphi_{\varrho-1})$ is said to be a *Jordan chain* of length ϱ corresponding to the eigenvalue λ_0. The elements $\varphi_1, \ldots, \varphi_{\varrho-1}$ are called *associated functions*. The maximal length of the Jordan chains corresponding to an eigenfunction φ_0 is called the *rank* of φ_0. If moreover the chains

$$\left(\varphi_0^{(\imath)}, \varphi_1^{(\imath)}, \ldots, \varphi_{\varrho_\imath - 1}^{(\imath)}\right)_{\imath=1,\ldots,I}$$

form a complete set of Jordan chains corresponding to λ_0, then the integer $\mathfrak{n} = \varrho_1 + \ldots + \varrho_I$ is called the *algebraic multiplicity* of λ_0.

Theorem 1.1 *Let $C_j(t) \to C_j$ in $\mathcal{L}(H^{s-j,w-j}(X), H^{s-m,w-m}(X))$ when $t \to -\infty$, for each $j = 0, \ldots, m-1$. Assume that in the strip $-\mu < \Im\lambda < -\gamma$ there lie N eigenvalues of $\sigma(\lambda)$ (counting the multiplicities), and that there are no eigenvalues of this pencil on the lines $\Im\lambda = -\mu$ and $\Im\lambda = -\gamma$. Then the solution $u \in H^{s,\gamma}(-\infty, T)$ of equation (1.1) with $f \in H^{s-m,\mu}(-\infty, T)$ has the form*

$$u(t) = c_1\, s_1(t) + \ldots + c_N\, s_N(t) + R(t)$$

where $s_1, \ldots, s_N$ are solutions of the homogeneous equation which do depend on u; $c_1, \ldots, c_N$ constants; and $R \in H^{s,\mu}(-\infty, T)$.

Thus, any solution $u \in H^{s,\gamma}(-\infty, T)$ of (1.1) with a "good" right-hand side f can be written as the sum of several singular functions and a "remainder" which behaves better at infinity. Of course, this meets our definition of asymptotics but unfortunately the singular functions are in general not explicit.

Theorem 1.1 goes back at least as far as [MP72]. As but one consequence of this result we mention a so-called *Relative Index Theorem* which reads as follows.

Corollary 1.2 *Let A be an elliptic differential operator of order m on a manifold M with a corner v, acting as $H^{s,w,\gamma}(M) \to H^{s-m,w-m,\gamma-m}(M)$ where the weight $\gamma \in \mathbb{R}$ is related to v. Suppose in the strip $-\mu < \Im\lambda < -\gamma$ there lie N eigenvalues of the conormal symbol of A at v (counting the multiplicities), and there are no eigenvalues of this symbol on the lines $\Im\lambda = -\mu$ and $\Im\lambda = -\gamma$. Then the difference of the indices of A evaluated on $H^{s,w,\gamma}(M)$ and $H^{s,w,\mu}(M)$ is equal to N.*

Consider the equation (1.1) for $t < T$ and associate with it the operator pencil $\sigma(t, \lambda) = \lambda^m + \sum_{j=0}^{m-1} C_j(t)\lambda^j$. We say that $\sigma(t, \lambda)$ *stabilises* to the pencil $\sigma(\lambda)$ as $t \to -\infty$ if the following conditions are satisfied:

1) $C_j(t) \to C_j$ in the norm of $\mathcal{L}(H^{s-j,w-j}(X), H^{s-m,w-m}(X))$ as $t \to -\infty$, for each $j = 0, \ldots, m-1$.
2) $D^k C_j(t) \to 0$ in the norm of $\mathcal{L}(H^{s-j,w-j}(X), H^{s-m,w-m}(X))$ as $t \to -\infty$, for each $k = 1, \ldots, s$ and $j = 0, \ldots, m-1$.

For equations with abstract operator-valued coefficients, it is required that also formal adjoints $C_j^*(t)$ and C_j^* meet conditions 1) and 2), cf. [MP72]. In our case the formal adjoints inherit the properties of $C_j(t)$ and C_j, for these latter are bounded.

Note that condition 2) just amounts to saying that the coefficients of (1.1) are *slowly varying* as $t \to -\infty$ (cf. [RST97]). Under this condition, a Fredholm theory for the equation (1.1) is available. However, this theory falls short of providing explicit asymptotic solutions in full generality.

One may ask whether the continuity condition 1) makes it possible to show useful asymptotic expansions of solutions to (1.1). If the derivatives $D^k C_j(t)$, $k = 1, \ldots, s+1$, are bounded in a neighbourhood of $t = -\infty$, then 1) implies 2). Theorem 1.1 shows that if 1) is fulfilled then the singularity at $t = -\infty$ gives rise to a finite number of singular solutions. However, an irregular singular point is a complicated conglomeration of singularities. For example, the equation

$$D^2 u + i\left(\frac{p(p+t-1)}{t(p+t)} + 1\right) Du + \frac{p(p+t-1)}{t(p+t)} u = 0$$

has solutions $u_1 = (-t)^p$ and $u_2 = e^{-t}$. The first of these bears a singularity of the same type as in the case of a regular singular point, for the second $t = -\infty$ is an essential singular point (see [Fed93]).

Asymptotic formulas are first of all necessarily conjectured and up to this point it is difficult to formulate general principles. After a formula has been guessed its proof breaks into two stages. By using a suitable change of variables and unknown functions the equation is reduced to the form $(T+P)u = f$, where the equation $Tu = f$ is solvable in an explicit form, and the operator P can be regarded as a small perturbation. The equation $Tu = f - Pu$ is then solved as an inhomogeneous equation with right-hand side $f - Pu$, and one studies the resulting integral equation. As T one might take the operator with coefficients frozen at the singular point, here $t = -\infty$. Indeed, equations with constant coefficients meet the Euler theory, as is described by (1.3). By the above, the stabilisation given by 1)-3) falls outside the limits of "small perturbations." To meet this heuristic concept, we need some restrictions on the speed, at which $D^k C_j(t)$ tend to zero when $t \to -\infty$. For this purpose, we set

$$\epsilon_k(t) = \max_{j=0,\ldots,m-1} \left\| D^k \left(C_j(t) - C_j \right) \right\|_{\mathcal{L}(H^{s-j,w-j}(X), H^{s-m,w-m}(X))},$$

for $k = 0, \ldots, s$.

A *solution* of (1.1) is defined to be a function $u(t)$ with values in $H^{s,w}(X)$ which possesses strong derivatives $D^j u(t)$ in $H^{s-j,w-j}(X)$, $j = 1, \ldots, m$, for almost all $t < T$, and which satisfies the equation (1.1).

For any $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$, we introduce $H^{s,\gamma}(-\infty, T)$ to be the space of all functions on the interval $(-\infty, T)$ with values in $H^{s,w}(X)$, such that the norm

$$\|u\|_{H^{s,\gamma}(-\infty,T)} = \left(\int_{-\infty}^{T} e^{-2\gamma t} \sum_{j=0}^{s} \|D^j u(t)\|^2_{H^{s-j,w-j}(X)} dt \right)^{1/2}$$

is finite. In particular, $H^{0,\gamma}(-\infty, T)$ consists of all square integrable functions on $(-\infty, T)$ with values in $H^{0,w}(X)$ with respect to the measure $e^{-2\gamma t} dt$.

Let $\Im\lambda = -\gamma_i$, $i = 0, \pm 1, \ldots$, be all horizontal lines every of which contains at least one eigenvalue of the operator pencil $\sigma(\lambda)$, and $\gamma_{i-1} < \gamma_i$. Assume that on a line $\Im\lambda = -\gamma_\nu$ there lies only one eigenvalue λ_0 of the limit pencil $\sigma(\lambda)$. Let moreover $\sigma(t,\lambda)$ stabilise to $\sigma(\lambda)$ as $t \to -\infty$, and let only one eigenvalue $\lambda(t)$ of the pencil $\sigma(t,\lambda)$ tend to λ_0 as $t \to -\infty$. We write

$$\left(\varphi_0^{(\imath)}(t), \varphi_1^{(\imath)}(t), \ldots, \varphi_{\varrho_\imath - 1}^{(\imath)}(t)\right)_{\imath = 1, \ldots, I}$$

for the corresponding eigenchains, where the numbers $\varrho_\imath$ and I do not depend on $t \in (-\infty, T)$.

Theorem 1.3 *Suppose*

$$\int_{-\infty}^{t_0} t^{2(2\varrho-1)} \left(\epsilon_k(t)\right)^2 dt < \infty,$$

for $k = 1, \ldots, s$, *where* ϱ *is the maximum of* $\varrho_\imath$. *Let* $u(t)$ *be a solution of the homogeneous equation* (1.1) *for* $t < T$, *such that* $u \in H^{s,\gamma}(-\infty, T)$ *with* $\gamma_{\nu-1} < \gamma < \gamma_\nu$. *Then,*

$$u(t) = e^{i\int_T^t \lambda(s)ds} \left(\sum_{\imath=1}^{I} \sum_{k=0}^{\varrho_\imath - 1} \mathcal{P}_{\varrho_\imath - 1 - k}^{(\imath)}(t)\, \varphi_k^{(\imath)}(t) + R(t) \right) \qquad (1.5)$$

where $\mathcal{P}_k^{(\imath)}(t)$ *are polynomials of degree* k *and* $R \in H^{s,0}(-\infty, T)$.

Were $\lambda(s)$ independent of s, we would deduce under the assumptions of Theorem 1.3 that

$$\begin{aligned} e^{i\int_T^t \lambda(s)ds} R(t) &= e^{(\gamma_\nu + i\Re\lambda)(t-T)} R(t) \\ &\in H^{s,\gamma_\nu}(-\infty, T) \end{aligned}$$

which belongs to $H^{s,\gamma}(-\infty, T)$. Hence, the remainder in formula (1.5) behaves better than $u(t)$ itself, as $t \to -\infty$, showing the asymptotic character of this formula.

If the coefficients $C_j(t)$ bear a transparent structure in a neighbourhood of $t = -\infty$, then asymptotic behaviour of solutions can be described more precisely. Suppose

$$C_j(t) = \sum_{\imath=0}^{J} C_{j\imath} \frac{1}{t^\imath} + C_{j,J+1}(t)\, \frac{1}{t^{J+1}}, \qquad j = 0, \ldots, m-1, \qquad (1.6)$$

on the interval $(-\infty, T)$, where $C_{j\iota} \in \Psi^{m-j}(X; w-j)$, for $\iota \le J$, and $C_{j,J+1}(t)$ is a C^∞ function on $(-\infty, T)$ with values in $\Psi^{m-j}(X; w-j)$. Moreover, we require the derivatives $D^k C_{j,J+1}(t)$ to be bounded on $(-\infty, T)$ in the norm of $\mathcal{L}(H^{s-j,w-j}(X), H^{s-m,w-m}(X))$, for $k = 0, \ldots, s$.

Were $C_j(z)$ holomorphic functions in a punctured neighbourhood of the point at infinity, the structure (1.6) would correspond to a *removable singularity* at $z = \infty$. Let n_j be the largest number with the property that $C_{j\iota} = 0$ for $\iota < n_j$, so that the sum in (1.6) in fact starts with $\iota = n_j$. In the notation of [Fed93, p. 16] the quantity $\varrho = \max(-n_j/(m-j)) + 1$ is called the *rank* of the singular point $z = \infty$. Then, $z = \infty$ is a regular or irregular singular point according to whether $\varrho = 0$ or $\varrho \ge 1$. The analytic theory of differential equations says that if ϱ is an integer, then the homogeneous equation (1.1) admits formal solutions of the form $u(z) = z^p e^{\mathcal{P}(z^{1/N})} \Psi(z^{-1/Q})$ where $\mathcal{P}(\zeta)$ is a polynomial, $\Psi(\zeta)$ a formal power series, and N, Q natural numbers. As mentioned, such series are usually divergent. Moreover, there are other solutions to (1.1) besides the series of the above form (cf. *ibid.*, p. 18).

Suppose on the line $\Im\lambda = -\gamma_\nu$ there is a single eigenvalue λ_0 of the limit pencil $\sigma(\lambda)$, to which there corresponds only one eigenchain $(\varphi_0, \varphi_1, \ldots, \varphi_{\varrho-1})$. This condition does not arise in any essential way and is introduced only to simplify the description.

Theorem 1.4 *Under certain algebraic assumptions, any solution $u(t)$ of the homogeneous equation* (1.1)*, which belongs to the space $H^{s,\gamma}(-\infty, T)$ with $\gamma_{\nu-1} < \gamma < \gamma_\nu$, has the form*

$$u(t) = t^p \, e^{i \sum\limits_{k=0}^{\varrho-1} \lambda_k t^{\frac{\varrho-k}{\varrho}}} \left(c \left(\sum_{\iota=0}^{\varrho-1} \frac{1}{t^{\frac{\iota}{\varrho}}} \sum_{k=0}^{\iota} c_{\iota k} \varphi_k + \sum_{\iota=\varrho}^{\varrho J-1} \frac{1}{t^{\frac{\iota}{\varrho}}} \psi_\iota \right) + \frac{1}{t^J} R(t) \right) \qquad (1.7)$$

where c is a constant depending on the solution $u(t)$; the constants p, λ_k and $c_{\iota k}$ and the functions ψ_ι do not depend on the solution; and $R \in H^{s,0}(-\infty, T)$.

The constants p, λ_k and $c_{\iota k}$ and the functions ψ_ι are computed by means of a finite number of algebraic operations. It is worth emphasising that the additional conditions are of purely algebraic character because the algorithm for the construction of formal solutions is extremely cumbersome for the general case.

Changing the coordinate along the corner axis by $t = \delta(r)$, for $r \in (0, \varepsilon)$, we may trace back Theorems 1.3 and 1.4 to solutions of the homogeneous equation $Au = 0$ near the singular point. As

$$(D^k C_j)(\delta(r)) = \mathbf{D}^k A_j(r)$$

for any $k \in \mathbb{Z}_+$, where

$$\mathbf{D}_r = \frac{1}{\delta'(r)} D_r,$$

the stabilisation of the coefficients at $r = 0$ turns out to be specified by the quantities

$$\tilde{\epsilon}_k(r) = \max_{j=0,\dots,m-1} \left\| \mathbf{D}^k \left(A_j(r) - A_j(0) \right) \right\|_{\mathcal{L}(H^{s-j,w-j}(X),H^{s-m,w-m}(X))},$$

for $k = 0, \dots, s$.

For any $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$, we introduce $H^{s,\gamma}(0,\varepsilon)$ to be the space of all functions on the interval $(0,\varepsilon)$ with values in $H^{s,w}(X)$, such that the norm

$$\|u\|_{H^{s,\gamma}(0,\varepsilon)} = \left(\int_0^\varepsilon e^{-2\gamma\delta(r)} \sum_{j=0}^{s} \|\mathbf{D}^j u(r)\|^2_{H^{s-j,w-j}(X)} d\delta(r) \right)^{1/2}$$

is finite.

Let $\Im z = -\gamma_i$, $i = 0, \pm 1, \dots$, be all horizontal lines every of which contains at least one eigenvalue of the operator pencil $\sigma(A)(z) = \sigma(A)(0,z)$, where

$$\sigma(A)(r,z) = \sum_{j=0}^{m} A_j(r)\, z^j,$$

and let $\gamma_{i-1} < \gamma_i$. Suppose on a line $\Im z = -\gamma_\nu$ there is only one eigenvalue z_0 of the limit pencil $\sigma(A)(z)$. Let moreover $\sigma(A)(r,z)$ stabilise to $\sigma(A)(z)$ as $r \to 0$, in the sense that $\tilde{\epsilon}_k(r)$ tends to 0 as $r \to 0$, for $k = 0, \dots, s$, and let only one eigenvalue $z(r)$ of the pencil $\sigma(A)(r,z)$ tend to z_0 as $r \to 0$. We write

$$\left(\varphi_0^{(\imath)}(r), \varphi_1^{(\imath)}(r), \dots, \varphi_{\varrho_\imath - 1}^{(\imath)}(r) \right)_{\imath = 1,\dots,I}$$

for the corresponding eigenchains, where the numbers $\varrho_\imath$ and I do not depend on $r \in (0,\varepsilon)$.

Theorem 1.5 *Suppose*

$$\int_0^{r_0} (\delta(r))^{2(2\varrho - 1)} \left(\tilde{\epsilon}_k(r) \right)^2 d\delta(r) < \infty,$$

for $k = 1, \dots, s$, where ϱ is the maximum of $\varrho_\imath$. Let $u(r)$ be a solution of the homogeneous equation $Au = 0$ for $r \in (0,\varepsilon)$, such that $u \in H^{s,\gamma}(0,\varepsilon)$ with $\gamma_{\nu-1} < \gamma < \gamma_\nu$. Then,

$$u(r) = e^{i \int_\varepsilon^r z(\vartheta) d\delta(\vartheta)} \left(\sum_{\imath=1}^{I} \sum_{k=0}^{\varrho_\imath - 1} \mathcal{P}^{(\imath)}_{\varrho_\imath - 1 - k}(r)\, \varphi_k^{(\imath)}(r) + R(r) \right) \qquad (1.8)$$

where $\mathcal{P}_k^{(\imath)}(r)$ are polynomials of degree k and $R \in H^{s,0}(0,\varepsilon)$.

Assuming the coefficients $A_j(r)$ to be of class C^s up to $r=0$, we look for a restriction on the geometry of the singular point at $r=0$ under which Theorem 1.5 is applicable. To this end, let $\delta'(r) = r^{-p-1}$ close to $r=0$, where $p \geq 0$. Then, the stabilisation condition of Theorem 1.5 is fulfilled provided that

$$\int_0 (\delta(r))^{2(2\varrho-1)} \frac{dr}{\delta'(r)} < \infty,$$

which is equivalent to $p < 1/(2\varrho - 3/2)$. In the case of simple eigenvalues this becomes $p < 2$.

If the coefficients $A_j(r)$ have a transparent structure close to $r=0$, then the asymptotic behaviour of solutions can be described more precisely. Namely, suppose

$$A_j(r) = \sum_{\iota=0}^{J} A_{j\iota} \frac{1}{\delta(r)^{\iota}} + A_{j,J+1}(r) \frac{1}{\delta(r)^{J+1}}, \qquad j = 0, \ldots, m-1, \tag{1.9}$$

on the interval $(0,\varepsilon)$, where $A_{j\iota} \in \Psi^{m-j}(X; w-j)$, for $\iota \leq J$, and $A_{j,J+1}(r)$ is a C^∞ function on $(0,\varepsilon)$ with values in $\Psi^{m-j}(X; w-j)$. We also require the derivatives $\mathbf{D}^k A_{j,J+1}(r)$ to be bounded on $(0,\varepsilon)$ in $\mathcal{L}(H^{s-j,w-j}(X), H^{s-m,w-m}(X))$, for $k = 0, \ldots, s$.

Suppose on the line $\Im z = -\gamma_\nu$ there is a single eigenvalue z_0 of the pencil $\sigma(A)(z)$, to which there corresponds only one eigenchain $(\varphi_0, \varphi_1, \ldots, \varphi_{\varrho-1})$.

Theorem 1.6 *Under certain algebraic assumptions, any solution $u(r)$ of the equation $Au = 0$, which belongs to the space $H^{s,\gamma}(0,\varepsilon)$ with $\gamma_{\nu-1} < \gamma < \gamma_\nu$, has the form*

$$u(r) = \delta(r)^p \, e^{i \sum_{k=0}^{\varrho-1} z_k \delta(r)^{\frac{\varrho-k}{\varrho}}} \left(c \left(\sum_{\iota=0}^{\varrho-1} \frac{1}{\delta(r)^{\frac{\iota}{\varrho}}} \sum_{k=0}^{\iota} c_{\iota k} \varphi_k + \sum_{\iota=\varrho}^{\varrho J-1} \frac{1}{\delta(r)^{\frac{\iota}{\varrho}}} \psi_\iota \right) + \frac{1}{\delta(r)^J} R(r) \right) \tag{1.10}$$

where c is a constant depending on the solution $u(r)$; the constants p, z_k and $c_{\iota k}$ and the functions ψ_ι do not depend on the solution; and $R \in H^{s,0}(0,\varepsilon)$.

Note that Schulze [Sch94] considers algebras of pseudodifferential operators acting in weighted Sobolev spaces with asymptotics. As the operators of order zero include those of multiplication by functions, it follows that the coefficients should meet asymptotic expansions at $r=0$ like asymptotics of functions under consideration. Thus, it is to be expected that the expansions (1.10) in turn survive under more general conditions on the coefficients than (1.9) (cf. [ST99b]).

2 Operator pencils

Let X be a compact closed manifold with singularities. On X there live $\mathbb{Z}$-graded algebras $\Psi^{m-j}(X; w-j)$ of pseudodifferential operators acting in weighted Sobolev spaces $H^{s,w-j}(X)$. These latter are parametrised by $s \in \mathbb{R}$, a smoothness, and $w \in \mathbb{R}^Q$, a tuple of weights. By the very definition, $H^{s,w-j}(X)$ are the completions of C^∞ functions with a compact support on the smooth part of X under certain weight norms.

For fixed $s \in \mathbb{R}$ and $w \in \mathbb{R}^Q$, we apply the approach of [MP72] in the context of Hilbert spaces

$$H_j = H^{s-m+j,w-m+j}(X),$$

where $j = 0, 1, \ldots, m$. We give the proofs only for $s = m$. For the general case, the reader may consult [Pla73].

Suppose the coefficients $C_j = C_j(t)$ in (1.1) are independent of t. For every $j = 0, 1, \ldots, m-1$, we have $C_j \in \Psi^{m-j}(X; w-j)$, hence C_j induces a continuous mapping $H_{m-j} \to H_0$. We assume that $C_0 \colon H_m \to H_0$ is invertible. Thus, the inverse C_0^{-1} is defined on all of H_0 and maps it continuously onto H_m.

If $C_j \in \Psi^{m-j}(X; w-m)$, then C_j is a closable operator in H_0. Moreover, the domain of the minimal operator associated with C_j contains H_{m-j}. The operator C_0 with the domain H_m in H_0 is closed anyway, for it is continuously invertible.

We introduce an operator pencil $\sigma(\lambda) = \sum_{j=0}^{m} C_j \lambda^j$, with $C_m = \mathrm{Id}$, which is also known as the *conormal symbol* of (1.1). It is regarded as a mapping from H_m to H_0.

Lemma 2.1 *For each $\lambda \in \mathbb{C}$ with the possible exception of a discrete set Σ, there exists an inverse $\sigma^{-1}(\lambda)$ mapping continuously H_0 onto H_m.*

Proof. Indeed, we have

$$\begin{aligned} \|C_0^{-1} C_j u\|_{H_m} &\leq c\, \|C_j u\|_{H_0} \\ &\leq C\, \|u\|_{H_{m-j}} \end{aligned}$$

for all $u \in H_m$, with c and C constants independent of u. Since H_m is compactly imbedded into H_{m-j}, it follows that $C_0^{-1} C_j$ is a compact operator in H_m, for $j = 1, \ldots, m$. By a theorem on holomorphic operator-valued functions, cf. [GK69], the operator

$$C_0^{-1}\, \sigma(\lambda) = \mathrm{Id} + \sum_{j=1}^{m} C_0^{-1} C_j \lambda^j$$

has a bounded inverse in H_m for all complex λ except for a set of isolated points. This is equivalent to the desired assertion.

$\square$

Assume that the operator pencil $\sigma(\lambda)$ satisfies the following additional conditions:

1) There is an angular sector $S = \{\lambda \in \mathbb{C} : \ |\arg(\pm\lambda)| \le \vartheta, \ |\lambda| \ge R\}$ around the real axis, with $\vartheta \in (0, \pi/2)$ and $R > 0$, which does not meet Σ.
2) For all $\lambda \in S$, an estimate

$$\sum_{j=0}^{m} |\lambda|^j \|u\|_{H_{m-j}} \le c \ \|\sigma(\lambda)u\|_{H_0} \tag{2.1}$$

holds whenever $u \in H_m$, with c a constant independent of λ and u.

Both 1) and 2) are mere parts of the concept of ellipticity on a manifold with corners.

Consider the equation

$$\sigma(\lambda)u = f, \tag{2.2}$$

for a given $f \in H_0$. Applying a trick used by Calderón in studying a Cauchy problem [Cal58], we reduce (2.2) to a first order system linear in λ. Namely, set

$$\begin{array}{rcl} u_1 & = & \lambda u_2 + C_1 u_m, \\ \dots & \dots & \dots \\ u_{m-1} & = & \lambda u_m + C_{m-1} u_m, \\ u_m & = & u, \end{array}$$

then the equation (2.2) reduces to $\lambda u_1 + C_0 u_m = f$.

Denote by $\mathcal{H}$ the direct sum of m copies of H_0, i.e., $\mathcal{H} = \oplus_1^m H_0$. In $\mathcal{H}$ define the operator

$$C = \begin{pmatrix} 0 & \dots & 0 & -C_0 \\ 1 & \dots & 0 & -C_1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -C_{m-1} \end{pmatrix} \tag{2.3}$$

with the domain $\operatorname{Dom} C = (\oplus_1^{m-1} H_0) \oplus H_m$. Given any $F = (f_1, \dots, f_m)$ in $\mathcal{H}$, consider the system

$$\lambda U - CU = F \tag{2.4}$$

for an unknown vector-valued function $U = (u_1, ..., u_m)$ of $\operatorname{Dom} C$. Obviously, equation (2.2) is equivalent to the system (2.4) for the particular right-hand side $F = (f, 0, \dots, 0)$.

The connection between the operators $\sigma(\lambda)$ and $\lambda \, \mathrm{Id} - C$ is explained in the following lemma. We write the latter operator $\lambda - C$ for short.

Lemma 2.2

1) *The operators* $\sigma^{-1}(\lambda)$ *and* $(\lambda - C)^{-1}$ *exist for the same values of* λ, $\sigma^{-1}(\lambda)$ *being defined on* H_0 *and* $(\lambda - C)^{-1}$ *on* $\mathcal{H}$.

2) *A number* $\lambda_0 \in \mathbb{C}$ *is an eigenvalue of the pencil* $\sigma(\lambda)$ *if and only if it is an eigenvalue of* C. *Moreover, the multiplicities of* λ_0 *with respect to* $\sigma^{-1}(\lambda)$ *and* C *are identical.*

3) *For any solution* $U = (u_1, \ldots, u_m)$ *of the equation* (2.4), *an estimate holds*

$$\sum_{j=0}^{m} |\lambda|^j \|u_m\|_{H_{m-j}} + \sum_{j=1}^{m-1} |\lambda|^j \|u_j\|_{H_0} \leq c \sum_{j=1}^{m} |\lambda|^{j-1} \|f_j\|_{H_0} \tag{2.5}$$

whenever $\lambda \in S$, *the constant* c *being independent of* λ *and* U.

Proof. It suffices to express the components $u_1, \ldots, u_{m-1}$ by means of u_m and the components of F. The estimate (2.5) is equivalent to (2.1).

□

From this lemma it follows that $\lambda = 0$ does not belong to the spectrum of the operator C, and therefore C is closed. Moreover, the resolvent $(\lambda - C)^{-1}$ is a meromorphic operator-valued function whose poles are located outside of the set S. Let $(\gamma_i)_{i \in \mathbb{Z}}$ be the increasing sequence of real numbers such that every line

$$\Gamma_{-\gamma_i} = \{\lambda \in \mathbb{C} : \ \Im \lambda = -\gamma_i\}$$

contains at least one pole of the resolvent $(\lambda - C)^{-1}$, and the latter is holomorphic away from these lines.

An immediate verification shows that the inverse operator C^{-1} has the form

$$C^{-1} = \begin{pmatrix} -C_1 C_0^{-1} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ -C_{m-1} C_0^{-1} & 0 & \ldots & 1 \\ -C_0^{-1} & 0 & \ldots & 0 \end{pmatrix},$$

all the operators $C_j C_0^{-1}$ being bounded in H_0. Indeed, as $H_m \hookrightarrow H_{m-j}$, we obtain

$$\begin{aligned} \|C_j C_0^{-1} f\|_{H_0} &\leq c \, \|C_0^{-1} f\|_{H_{m-j}} \\ &\leq C \, \|f\|_{H_0} \end{aligned}$$

for all $f \in H_0$, with c and C constants independent of f.

Before identifying the adjoint of C in the sense of Hilbert spaces, we make some comments on the algebras $\Psi^m(X; w)$. For any $A \in \Psi^m(X; w)$, the formal adjoint A^* is available in the space $\Psi^m(X; -w + m)$. While A operates as $H^{s,w}(X) \to H^{s-m,w-m}(X)$, for any $s \in \mathbb{R}$, the formal adjoint A^* does as $H^{-s+m,-w+m}(X) \to H^{-s,-w}(X)$. On the other hand, we may think of A as an unbounded operator in the Hilbert space $H^{s-m,w-m}(X)$, the domain of A being dense. Then we define A^{adj}, the adjoint of A in the sense of Hilbert spaces, as usual, so that A^{adj} is an unbounded operator in $H^{s-m,w-m}(X)$. Hence, A^* differs from A^{adj}. However, these operators are linked through an isomorphism of Banach spaces $\star\colon H^{s-m,w-m}(X) \to H^{-s+m,-w+m}(X)$, given by $(u, \star v)_{H^{0,0}(X)} = (u,v)_{H^{s-m,w-m}(X)}$ for all $u \in H^{s-m,w-m}(X)$. Under this isomorphism, we have $A^* = \star A^{\mathrm{adj}} \star^{-1}$, and the domain of A^{adj} consists precisely of those $g \in H^{s-m,w-m}(X)$ which satisfy $A^* \star g \in H^{-s+m,-w+m}(X)$. This allows one to substitute A^* for A^{adj} in many contexts. Our basic assumption here is that the coefficient $C_0 \in \Psi^m(X;w)$ extends to an invertible mapping $H^{s,w}(X) \to H^{s-m,w-m}(X)$. It would then be a fine property of the algebra $\Psi^m(X;w)$, called the *spectral invariance*, guaranteeing that the inverse C_0^{-1} be still induced by an operator in $\Psi^{-m}(X;w-m)$. While many known pseudodifferential algebras on singular spaces are spectral invariant, this property has nothing to do with asymptotics. We avoid this additional assumption, thus choosing to work with adjoints in the sense of Hilbert spaces. However, since the definition of the formal adjoint relies only on the scalar product in $H^{0,0}(X)$, we shall occasionally substitute C_j^* for C_j^{adj}, keeping in mind the relation $C_j^* = \star C_j^{\mathrm{adj}} \star^{-1}$.

From what has already been proved, it follows that the bounded operator $(C^{-1})^{\mathrm{adj}} = (C^{\mathrm{adj}})^{-1}$ defined on $\mathcal{H}$ has the form

$$(C^{\mathrm{adj}})^{-1} = \begin{pmatrix} -(C_1C_0^{-1})^{\mathrm{adj}} & \dots & -(C_{m-1}C_0^{-1})^{\mathrm{adj}} & -(C_0^{-1})^{\mathrm{adj}} \\ 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \end{pmatrix}. \tag{2.6}$$

Our next goal is to specify the inverse of operator (2.6). Let

$$\begin{array}{rcl} G & = & (g_1, \dots, g_m), \\ V & = & (v_1, \dots, v_m) \end{array}$$

be arbitrary elements of $\mathcal{H}$. The equality $(C^{\mathrm{adj}})^{-1}V = G$ means that

$$\begin{array}{rcl} -\sum\limits_{j=1}^{m-1} (C_jC_0^{-1})^{\mathrm{adj}} v_j - (C_0^{-1})^{\mathrm{adj}} v_m & = & g_1, \\ v_1 & = & g_2, \\ \dots & \dots & \dots \\ v_{m-1} & = & g_m \end{array}$$

whence

$$\begin{array}{rcl} -g_1 - \sum\limits_{j=1}^{m-1} (C_jC_0^{-1})^{\mathrm{adj}} g_{j+1} & = & (C_0^{-1})^{\mathrm{adj}} v_m \\ & \in & \mathrm{Dom}\, C_0^{\mathrm{adj}} \end{array}$$

and

$$C_0^{\mathrm{adj}}\left(-g_1-\sum_{j=1}^{m-1}(C_jC_0^{-1})^{\mathrm{adj}}g_{j+1}\right)=v_m.$$

Thus, the domain of C^{adj} is

$$\mathrm{Dom}\, C^{\mathrm{adj}}=\{G\in\mathcal{H}:\ g_1+\sum_{j=1}^{m-1}(C_jC_0^{-1})^{\mathrm{adj}}g_{j+1}\in\mathrm{Dom}\, C_0^{\mathrm{adj}}\}$$

where C_0 is regarded as acting in H_0 with domain H_m, and the adjoint itself has the form

$$C^{\mathrm{adj}}=\begin{pmatrix}1&\dots&0&0\\ \dots&\dots&\dots&\dots\\ 0&\dots&1&0\\ 0&\dots&0&-C_0^{\mathrm{adj}}\end{pmatrix}\begin{pmatrix}0&1&\dots&0\\ \dots&\dots&\dots&\dots\\ 0&0&\dots&1\\ 1&(C_1C_0^{-1})^{\mathrm{adj}}&\dots&(C_{m-1}C_0^{-1})^{\mathrm{adj}}\end{pmatrix}.$$

The equation for an eigenfunction $\Psi=(\psi_1,\dots,\psi_m)$ of C^{adj} can be written as

$$\begin{array}{rcl}\psi_2&=&\bar\lambda_0\psi_1,\\ \dots&\dots&\dots\\ \psi_m&=&\bar\lambda_0\psi_{m-1},\\ -C_0^{\mathrm{adj}}\left(1+\sum\limits_{j=1}^{m-1}(C_jC_0^{-1})^{\mathrm{adj}}\bar\lambda_0^j\right)\psi_1&=&\bar\lambda_0^m\psi_1,\end{array}\qquad(2.7)$$

where $\lambda_0\neq 0$ and we require

$$\left(1+\sum_{j=1}^{m-1}(C_jC_0^{-1})^{\mathrm{adj}}\bar\lambda_0^j\right)\psi_1\in\mathrm{Dom}\, C_0^{\mathrm{adj}}.$$

Let us consider the adjoint operator pencil $\sigma^{\mathrm{adj}}(\lambda)$, $\lambda\in\mathbb{C}$, defined by the equality

$$(\sigma(\lambda)u,g)_{H_0}=\left(u,\sigma^{\mathrm{adj}}(\lambda)g\right)_{H_0}$$

for all $u\in H_m$ and $g\in\mathrm{Dom}\,\sigma^{\mathrm{adj}}(\lambda)$.

It is a property of Hilbert structures in the Sobolev spaces under study that the operator $\star$ maps $H^{s-j,w-j}(X)$ onto $H^{-s+2m-j,-w+2m-j}(X)$ continuously, for each $j=0,1,\dots,m$. Hence we deduce that the space $H_{m-j}=H^{s-j,w-j}(X)$ belongs to the domain of C_j^{adj}. Moreover, from the equality $C_j^{\mathrm{adj}}=\star^{-1}C_j^*\star$ we get

$$\begin{array}{rcl}\|C_j^{\mathrm{adj}}g\|_{H_0}&=&\|C_j^*\star g\|_{H^{-s+m,-w+m}(X)}\\ &\le& c\,\|\star g\|_{H^{-s+j,-w+j}(X)}\\ &\le& c\,\|\star g\|_{H^{-s+2m-j,-w+2m-j}(X)}\\ &\le& C\,\|g\|_{H^{s-j,w-j}(X)}\end{array}$$

whenever $g \in H_{m-j}$, the constants c and C do not depend on g.

For $j = 0$ we even claim that $\operatorname{Dom} C_0^{\mathrm{adj}} = H_m$. Indeed, since C_0 is an invertible operator $H^{s,w}(X) \to H^{s-m,w-m}(X)$, it is elliptic. Therefore, the formal adjoint C_0^* is an elliptic operator, too, thus inducing a Fredholm mapping $H^{-s+m,-w+m}(X) \to H^{-s,-w}(X)$. By duality, this mapping is actually an isomorphism. We make a purely technical additional assumption on C_0 that C_0^* restricts to an isomorphism $H^{-s+2m,-w+2m}(X) \to H^{-s+m,-w+m}(X)$. It follows that $C_0^{\mathrm{adj}} = \star^{-1} C_0^* \star$ is an invertible operator $H_m \to H_0$, which gives our assertion.

Lemma 2.3 *The domain* $\operatorname{Dom} \sigma^{\mathrm{adj}}(\lambda)$ *of* $\sigma^{\mathrm{adj}}(\lambda)$ *is equal to* H_m*, and for* $g \in H_m$

$$\sigma^{\mathrm{adj}}(\lambda) g = \sum_{j=0}^{m} C_j^{\mathrm{adj}} \bar{\lambda}^j\, g. \tag{2.8}$$

Proof. Since $(C_j^{\mathrm{adj}} (C_0^{\mathrm{adj}})^{-1})^{\mathrm{adj}}$ is a closure of $C_0^{-1} C_j$, this closure is bounded in H_0, for $0 \le j \le m$. Thus, also the closure of $C_0^{-1} \sigma(\lambda)$ is bounded in H_0 whence

$$\sigma^{\mathrm{adj}}(\lambda) = \left(C_0 \left(C_0^{-1} \sigma(\lambda) \right) \right)^{\mathrm{adj}}, \tag{2.9}$$

the right-hand side being the closure of $(C_0^{-1}\sigma(\lambda))^{\mathrm{adj}} C_0^{\mathrm{adj}}$.

Further,

$$\begin{aligned} \left(C_0^{-1} \sigma(\lambda) \right)^{\mathrm{adj}} &= \sum_{j=0}^{m} \left(C_0^{-1} C_j \right)^{\mathrm{adj}} \bar{\lambda}^j \\ &= \sum_{j=0}^{m} C_j^{\mathrm{adj}} \left(C_0^{\mathrm{adj}} \right)^{-1} \bar{\lambda}^j, \end{aligned}$$

and so substituting this expression to (2.9) enables us to conclude that $\sigma^{\mathrm{adj}}(\lambda)$ is the closure of $\sum_{j=0}^m C_j^{\mathrm{adj}} \bar{\lambda}^j$.

Let $u_\nu \in H_m$, $\nu \in \mathbb{N}$, and $f_\nu = \sum_{j=0}^m C_j^{\mathrm{adj}} \bar{\lambda}^j\, u_\nu$, where

$$\begin{aligned} u_\nu &\to u, \\ f_\nu &\to f \end{aligned}$$

in H_0. By the above,

$$\| C_0^{\mathrm{adj}} (u_\mu - u_\nu) \|_{H_0} \le \| f_\mu - f_\nu \|_{H_0} + C \sum_{j=1}^{m} \| u_\mu - u_\nu \|_{H_{m-j}} |\lambda|^j$$

for all $\mu, \nu \in \mathbb{N}$.

Since the space H_m is compactly embedded in H_{m-j}, for $j > 0$, to any $\varepsilon > 0$ there corresponds a constant $c(\varepsilon)$ such that

$$\|u\|_{H_{m-j}} \leq \varepsilon \, \|u\|_{H_m} + c(\varepsilon) \, \|u\|_{H_0}$$

whenever $u \in H_m$ Thus

$$\|u_\mu - u_\nu\|_{H_m} \leq c \, (\|f_\mu - f_\nu\|_{H_0} + \|u_\mu - u_\nu\|_{H_0}) \, ,$$

with c a constant independent of μ and ν, whence $u \in H_m$. Therefore, the equality (2.8) holds.

□

In order to dispense with the assumption on the behaviour of C_0^* on the space $H^{-s+2m,-w+2m}(X)$ it suffices to argue as above, with C_j^{adj} replaced by C_j^*.

Let λ_0 be a simple eigenvalue of $\sigma(\lambda)$. Then $\bar{\lambda}_0$ is a simple eigenvalue of the pencil $\sigma^{\mathrm{adj}}(\lambda)$. Denote by ψ_0 the corresponding eigenfunction in H_m, i.e., $\sigma^{\mathrm{adj}}(\bar{\lambda}_0)\psi_0 = 0$. Set

$$\begin{array}{lcl} \psi_1 & = & \psi_0, \\ \psi_2 & = & \bar{\lambda}_0 \, \psi_1, \\ \dots & \dots & \dots \\ \psi_m & = & \bar{\lambda}_0 \, \psi_{m-1}, \end{array} \tag{2.10}$$

then the equation $\sigma^{\mathrm{adj}}(\bar{\lambda}_0)\psi_0 = 0$ reduces to

$$- \sum_{j=0}^{m-1} C_j^{\mathrm{adj}} \psi_{j+1} = \bar{\lambda}_0 \, \psi_m. \tag{2.11}$$

Since $\psi_m \in H_m$, we get

$$\begin{array}{rcl} -C_0^{\mathrm{adj}} \left(1 + \displaystyle\sum_{j=1}^{m-1} (C_j C_0^{-1})^{\mathrm{adj}} \bar{\lambda}_0^j \right) \psi_1 & = & -C_0^{\mathrm{adj}} \left(1 + \displaystyle\sum_{j=1}^{m-1} (C_0^{\mathrm{adj}})^{-1} C_j^{\mathrm{adj}} \bar{\lambda}_0^j \right) \psi_1 \\ & = & - \displaystyle\sum_{j=0}^{m-1} C_j^{\mathrm{adj}} \psi_{j+1}, \end{array}$$

and so comparing (2.7) and (2.11) shows that $\Psi = (\psi_1, \dots, \psi_m)$ is an eigenfunction of the operator C^{adj}. By (2.10), all the components of Ψ belong to H_m.

3 Equations with constant coefficients

Consider the equation

$$D^m u(t) + \sum_{j=0}^{m-1} C_j \, D^j u(t) = f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $D = -i\,\partial/\partial t$ and the coefficients C_j satisfy the conditions of the preceding section.

For $\gamma \in \mathbb{R}$, let us introduce a space $H^{m,\gamma}(\mathbb{R})$ to be the completion of C^∞ functions with a compact support in $\mathbb{R}$ and values in $H^{m,w}(X)$ with respect to the norm

$$\|u\|_{H^{m,\gamma}(\mathbb{R})} = \left(\int_{\mathbb{R}} e^{-2\gamma t} \sum_{j=0}^{m} \|D^j u(t)\|^2_{H^{m-j,w-j}(X)} dt \right)^{1/2}.$$

As mentioned in Section 1, by $D^j u(t)$ are meant the *strong* derivatives in the corresponding norms.

Given any $f \in H^{0,\gamma}(\mathbb{R})$, we look for a solution of (3.1) in the space $H^{m,\gamma}(\mathbb{R})$. By means of the substitution

$$\begin{array}{rcl} u_1 & = & Du_2 + C_1 u_m, \\ \dots & \dots & \dots \\ u_{m-1} & = & Du_m + C_{m-1} u_m, \\ u_m & = & u \end{array}$$

the equation (3.1) is reduced to the system

$$DU(t) - CU(t) = F(t), \quad t \in \mathbb{R}, \tag{3.2}$$

where $U = (u_1, \dots, u_m)$, $F = (f, 0, \dots, 0)$ and C is the operator defined by (2.3).

We now introduce the spaces $\mathcal{H}_U^{m,\gamma}$ and $\mathcal{H}_F^{m-1,\gamma}$ of vector-valued functions with the norms

$$\begin{aligned} \|U\|_{\mathcal{H}_U^{m,\gamma}} &= \left(\int_{\mathbb{R}} e^{-2\gamma t} \left(\sum_{j=0}^{m} \|D^j u_m(t)\|^2_{H_{m-j}} + \sum_{i=1}^{m-1} \sum_{j=0}^{i} \|D^j u_i(t)\|^2_{H_0} \right) dt \right)^{\frac{1}{2}}, \\ \|F\|_{\mathcal{H}_F^{m-1,\gamma}} &= \left(\int_{\mathbb{R}} e^{-2\gamma t} \left(\sum_{i=1}^{m} \sum_{j=0}^{i-1} \|D^j f_i(t)\|^2_{H_0} \right) dt \right)^{\frac{1}{2}}. \end{aligned}$$

It is easily verified that the operator $D - C$ induces a continuous mapping of $\mathcal{H}_U^{m,\gamma}$ into $\mathcal{H}_F^{m-1,\gamma}$.

Lemma 3.1 *Let $\gamma \neq \gamma_i$, for $i \in \mathbb{Z}$, (γ_i) being defined after Lemma 2.2. Then, the equation (3.2) has a unique solution $U \in \mathcal{H}_U^{m,\gamma}$ for each right-hand side $F \in \mathcal{H}_F^{m-1,\gamma}$. Moreover, there exists a constant c independent of F, such that*

$$\|U\|_{\mathcal{H}_U^{m,\gamma}} \leq c\, \|F\|_{\mathcal{H}_F^{m-1,\gamma}}.$$

Proof. It suffices to apply the Fourier transform to both sides of (3.2), thus reducing this equation to the equation $\lambda\hat{U} - C\hat{U} = \hat{F}$ for the Fourier images,

$$\hat{F}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\lambda t} F(t) dt.$$

By assumption, the line $\Im\lambda = -\gamma$ is free from the poles of the resolvent $(\lambda - C)^{-1}$. Moreover, the inequality (2.5) is fulfilled for all λ on the line $\Gamma_{-\gamma}$ whence

$$U(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty-i\gamma}^{+\infty-i\gamma} e^{i\lambda t} (\lambda - C)^{-1} \hat{F}(\lambda)\, d\lambda, \tag{3.3}$$

and the lemma follows.

□

This lemma implies that equation (3.1) with a right-hand side f in $H^{0,\gamma}(\mathbb{R})$ has a unique solution $u \in H^{m,\gamma}(\mathbb{R})$, provided that there are no eigenvalues of the pencil $\sigma(\lambda)$ on the line $\Gamma_{-\gamma}$.

Lemma 3.2 *Assume the line $\Gamma_{-\gamma}$ is free from the poles of the resolvent $(\lambda - C)^{-1}$. Let moreover $\theta(t)$ be a real-valued function of class C^{m-1} on $\mathbb{R}$, such that*

$$\|((D - i\gamma) - C)^{-1}(\theta(t) - \gamma)\|_{\mathcal{L}(\mathcal{H}_U^{m,0})} < 1. \tag{3.4}$$

Then, for any right-hand side $F(t)$ satisfying $\exp(-\int_0^t \theta(s)ds)F(t) \in \mathcal{H}_F^{m-1,0}$, there is a unique solution $U(t)$ of (3.2) satisfying $\exp(-\int_0^t \theta(s)ds)U(t) \in \mathcal{H}_U^{m,0}$.

Note that (3.4) is fulfilled if the derivatives of $\theta(t) - \gamma$ up to order $m-1$ are small enough.

Proof. In equation (3.2) put

$$\begin{aligned} U(t) &= \exp\left(\int_0^t \theta(s)ds\right) \tilde{U}(t), \\ F(t) &= \exp\left(\int_0^t \theta(s)ds\right) \tilde{F}(t), \end{aligned}$$

thus reducing this equation to

$$(D - i\gamma)\,\tilde{U} - C\tilde{U} - i\,(\theta(t) - \gamma)\,\tilde{U} = \tilde{F}.$$

It suffices to prove the existence of a unique solution $\tilde{U} \in \mathcal{H}_U^{m,0}$ to this latter equation, for an arbitrary right-hand side $\tilde{F} \in \mathcal{H}_F^{m-1,0}$. By Lemma 3.1 we have

$$\tilde{U} - i\,((D - i\gamma) - C)^{-1}\,(\theta(t) - \gamma)\,\tilde{U} = ((D - i\gamma) - C)^{-1}\,\tilde{F},$$

the right-hand side $((D-i\gamma)-C)^{-1}\tilde{F}$ belonging to $\mathcal{H}_U^{m,0}$. This equation can be solved in a unique way in view of (3.4), as desired.

$\square$

For a real number T, let $\chi_T(t)$ be a non-negative C^∞ function on $\mathbb{R}$, such that $\chi_T(t)=1$, for $t<T-1$, and $\chi_T(t)=0$, for $t\geq T$.

Lemma 3.3 *Let $U(t)$ satisfy*

$$DU(t)-CU(t)=F(t), \quad t<T,$$

and $\chi_T U\in\mathcal{H}_U^{m,\gamma}$ where $\gamma_{\nu-1}<\gamma<\gamma_\nu$. If $\chi_T F\in\mathcal{H}_F^{m-1,\mu}$, with $\gamma<\mu<\gamma_\nu$, then actually $\chi_T U\in\mathcal{H}_U^{m,\mu}$.

Proof. Set $\tilde{U}=\chi_T U$, then $\tilde{U}\in\mathcal{H}_U^{m,\gamma}$, and the equation for $U(t)$ reduces to

$$\begin{aligned} D\tilde{U}-C\tilde{U} &= \chi_T\,(DU-CU)+(D\chi_T)\,U \\ &= \chi_T F+(D\chi_T)\,U \\ &:= \tilde{F}, \end{aligned}$$

$\tilde{F}$ being supported on the semiaxis $(-\infty,T]$.

The Fourier transform $\mathcal{F}\tilde{F}(\lambda)$ of the function $\tilde{F}(t)$ is an analytic function in the half-plane $\Im\lambda>-\mu$. According to (3.3),

$$\tilde{U}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty-i\gamma}^{+\infty-i\gamma} e^{i\lambda t}\,(\lambda-C)^{-1}\,\mathcal{F}\tilde{F}(\lambda)\,d\lambda$$

for $t\in\mathbb{R}$.

In the strip $-\mu\leq\Im\lambda\leq-\gamma$ the integrand is analytic. Taking into account the estimate (2.5) for the resolvent $(\lambda-C)^{-1}$ and the fact that $\mathcal{F}\tilde{F}(\lambda)$ is rapidly decreasing at $\Re\lambda=\pm\infty$, we readily deduce that the integration path $\Gamma_{-\gamma}$ can be changed to $\Gamma_{-\mu}$, i.e.,

$$\tilde{U}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty-i\mu}^{+\infty-i\mu} e^{i\lambda t}\,(\lambda-C)^{-1}\,\mathcal{F}\tilde{F}(\lambda)\,d\lambda.$$

Using once again Lemma 3.1 we conclude that the latter formula defines a unique solution $\tilde{U}\in\mathcal{H}_U^{m,\mu}$ to the equation $D\tilde{U}-C\tilde{U}=\tilde{F}$, which proves the lemma.

$\square$

4 Strong perturbations

Consider the equation with variable coefficients

$$D^m u(t) + \sum_{j=0}^{m-1} C_j(t)\, D^j u(t) = f(t), \qquad t \in \mathbb{R}, \tag{4.1}$$

where $C_j(t)$ is a continuous function on $\mathbb{R}$ with values in $\Psi^{m-j}(X; w-j)$, for $j = 0, 1, \ldots, m-1$.

For every fixed $t \in \mathbb{R}$, the operator $C_j(t)$ induces a continuous mapping $H_{m-j} \to H_0$. We think of $C_j(t)$ as an unbounded operator in H_0 whose domain does contain H_{m-j}. The minimal closure of $C_j(t)$, if exists, coincides with $C_j(t)$ on H_{m-j}. Moreover, we assume that $C_0(t)$ is a closed operator in H_0 with domain H_m.

Our basic assumption is the following. For each $j = 0, 1, \ldots, m-1$, the function $C_j(t)$ has bounded derivatives up to order j in the strong topology of $\mathcal{L}(H_{m-j}, H_0)$. This amounts to saying that $\|D^k C_j(t) u\|_{H_0} \le c\, \|u\|_{H_{m-j}}$ for all $u \in H_{m-j}$ and $k \le j$, with c a constant independent of u and t [1].

We shall reduce equation (4.1) to a first order system. To this end, write this equation in the form

$$D^m u(t) + \sum_{j=0}^{m-1} D^j \left(B_j(t) u(t) \right) = f(t), \qquad t \in \mathbb{R}, \tag{4.2}$$

where

$$B_j(t) = C_j(t) + c_{j,j+1}\, DC_{j+1}(t) + \ldots + c_{j,m-1}\, D^{m-1-j} C_{m-1}(t) \tag{4.3}$$

and c_{jk} are some combinatorial integers. From the conditions on $C_j(t)$ and (4.3) we see that

$$\|B_j(t) u\|_{H_0} \le c\, \|u\|_{H_{m-j}} \tag{4.4}$$

for all $u \in H_{m-j}$ and $j = 0, 1, \ldots, m-1$, the constant c being independent of u and t.

Put $u_m = u$ and introduce new functions by

$$\begin{array}{rcl} u_1 & = & Du_2 + B_1(t)\, u_m, \\ \ldots & \ldots & \ldots \\ u_{m-1} & = & Du_m + B_{m-1}(t)\, u_m, \end{array}$$

then (4.2) is equivalent to this system enlarged by the additional equation $Du_1 + B_0(t)\, u_m = f(t)$.

[1] In fact, a weaker assumption is sufficient, namely that the norm of $D^k C_{j+k}(t)$ in $\mathcal{L}(H_{m-j}, H_0)$ is uniformly bounded in $t \in \mathbb{R}$, for each $k \le m-1-j$.

Consider the operator

$$C(t) = \begin{pmatrix} 0 & \dots & 0 & -B_0(t) \\ 1 & \dots & 0 & -B_1(t) \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -B_{m-1}(t) \end{pmatrix} \tag{4.5}$$

acting in $\mathcal{H}$. The domain of this operator is $\operatorname{Dom} C(t) = (\oplus_1^{m-1} H_0) \oplus H_m$. By the above, the equation (4.1) is equivalent to the system

$$DU(t) - C(t)U(t) = F(t), \qquad t \in \mathbb{R}, \tag{4.6}$$

if $F = (f, 0, \dots, 0)$.

Moreover, a solution $U = (u_1, \dots, u_m)$ of (4.6) belongs to $\mathcal{H}_U^{m,\gamma}$ if and only if the solution $u = u_m$ of (4.1) belongs to $H^{m,\gamma}(\mathbb{R})$.

Let C be the matrix-valued operator defined by formula (2.3). As usual, we write

$$\|U\|_C = \|U\|_{\mathcal{H}} + \|CU\|_{\mathcal{H}}, \quad U \in \operatorname{Dom} C, \tag{4.7}$$

for the graph norm of C. Completeness of the space $\operatorname{Dom} C$ under this norm is equivalent to the closedness of the operator C. Since $C_0 : H_m \to H_0$ is invertible, we get

$$\|U\|_C \sim \left(\sum_{j=1}^{m-1} \|u_j\|_{H_0} \right) + \|u_m\|_{H_m}$$

for $U = (u_1, \dots, u_m)$.

Following Gokhberg and Krein [GK57], we call an operator T acting in $\mathcal{H}$ *C-bounded* if $\operatorname{Dom} C \subset \operatorname{Dom} T$ and $\|TU\|_{\mathcal{H}} \le c\,\|U\|_C$ for all $U \in \operatorname{Dom} C$. The smallest constant c for which the last inequality holds is said to be the *C-norm* of the operator T.

In view of (4.4) it is clear that the operator $C(t)$ is C-bounded uniformly in $t \in \mathbb{R}$.

Suppose

$$\|B_j(t) - C_j\|_{\mathcal{L}(H_{m-j}, H_0)} \to 0$$

as $t \to -\infty$, for $j = 0, 1, \dots, m-1$. Then the C-norm of the difference $C(t) - C$ tends to zero when $t \to -\infty$. If λ_0 is a simple eigenvalue of the operator C then for t small enough, there exists a simple eigenvalue $\lambda_0(t)$ of the operator $C(t)$ (cf. [GK57]). Let $\Phi(t)$ be the eigenfunction corresponding to this eigenvalue, and let Φ be the eigenfunction of C corresponding to λ_0. Then

$$\begin{aligned} \lambda_0(t) &\to \lambda_0, \\ \Phi(t) &\to \Phi \end{aligned}$$

as $t \to -\infty$, cf. *ibid.*

Lemma 4.1 *Let*

$$\begin{array}{rcll} \|B_j(t)-C_j\|_{\mathcal{L}(H_{m-j},H_0)} & < & \epsilon, & \\ \|D^k B_j(t)\|_{\mathcal{L}(H_{m-j},H_0)} & < & \epsilon, & k=1,\ldots,j, \end{array} \tag{4.8}$$

where ϵ is a sufficiently small positive constant. Then, given any $F \in \mathcal{H}_F^{m-1,\gamma}$ with $\gamma_{\nu-1} < \gamma < \gamma_\nu$, the equation (4.6) has a unique solution $U \in \mathcal{H}_U^{m,\gamma}$.

Proof. The equation (4.6) can be written in the form

$$U-(D-C)^{-1}(C(t)-C)U = (D-C)^{-1}F, \tag{4.9}$$

where $(D-C)^{-1}$ is defined by (3.3). By choosing $\epsilon > 0$ small enough we may guarantee that

$$\begin{array}{l} \|(D-C)^{-1}(C(t)-C)\|_{\mathcal{L}(\mathcal{H}_U^{m,\gamma})} \\ \quad \le \|(D-C)^{-1}\|_{\mathcal{L}(\mathcal{H}_F^{m-1,\gamma},\mathcal{H}_U^{m,\gamma})}\|C(t)-C\|_{\mathcal{L}(\mathcal{H}_U^{m,\gamma},\mathcal{H}_F^{m-1,\gamma})} \\ \quad < 1, \end{array}$$

hence (4.9) has a unique solution in $\mathcal{H}_U^{m,\gamma}$, as desired.

□

It is worth pointing out that the norm of $C(t)-C$ in $\mathcal{L}(\mathcal{H}_U^{m,\gamma}, \mathcal{H}_F^{m-1,\gamma})$ is majorised by the norms $\|D^k(B_j(t)-C_j)\|_{\mathcal{L}(H_{m-j},H_0)}$, for $j=0,1,\ldots,m-1$ and $k=0,1,\ldots,j$.

Lemma 4.2 *Let*

$$\begin{array}{rcll} \|B_j(t)-C_j\|_{\mathcal{L}(H_{m-j},H_0)} & \to & 0 & \text{as} \quad t\to-\infty, \\ \|D^k B_j(t)\|_{\mathcal{L}(H_{m-j},H_0)} & \to & 0 & \text{as} \quad t\to-\infty, \end{array} \tag{4.10}$$

for $k=1,\ldots,j$. Suppose $U(t)$ satisfies

$$DU(t)-C(t)U(t) = F(t), \quad t<T,$$

and $\chi_T U \in \mathcal{H}_U^{m,\gamma}$ where $\gamma_{\nu-1} < \gamma < \gamma_\nu$. If $\chi_T F \in \mathcal{H}_F^{m-1,\mu}$, with $\gamma < \mu < \gamma_\nu$, then $\chi_T U \in \mathcal{H}_U^{m,\mu}$.

Proof. Fix an arbitrary $\Theta < T-1$ and set $\tilde{U} = \chi_\Theta U$. It is easy to verify that

$$D\tilde{U}(t)-C(t)\tilde{U}(t) = \tilde{F}(t), \tag{4.11}$$

where $\tilde{F} = \chi_\Theta F + (D\chi_\Theta)\,U$. We define $\tilde{F}$ to be equal to zero on the interval (Θ,∞) and rewrite (4.11) in the form

$$D\tilde{U}(t)-C\tilde{U}(t) = \chi_T(t)\,(C(t)-C)+\tilde{F}(t)$$

on the entire real axis. Since $\tilde{U} \in \mathcal{H}_U^{m,\gamma}$, the right-hand side of the latter equation belongs to $\mathcal{H}_F^{m-1,\gamma}$. Thus this equation is equivalent to

$$\tilde{U} - K_\gamma \tilde{U} = \mathcal{R}_\gamma \tilde{F}, \tag{4.12}$$

where $\mathcal{R}_\gamma$ denotes the inverse of $D - C$ acting from $\mathcal{H}_F^{m-1,\gamma}$ to $\mathcal{H}_U^{m,\gamma}$, and $K_\gamma = \mathcal{R}_\gamma \, \chi_T \, (C(t) - C)$.

As $\tilde{F} \in \mathcal{H}_F^{m-1,\mu}$, we may look for a solution of (4.12) in the space $\mathcal{H}_U^{m,\mu}$. To this end, we rewrite (4.12) in the form

$$\tilde{\tilde{U}} - K_\mu \tilde{\tilde{U}} = \mathcal{R}_\mu \tilde{F}, \tag{4.13}$$

where $\mathcal{R}_\mu$ and K_μ are operators defined in the same way as $\mathcal{R}_\gamma$ and K_γ, the only difference being in replacing γ by μ. By the above, conditions (4.10) ensure that

$$\begin{array}{rcl} \|K_\gamma\|_{\mathcal{L}(\mathcal{H}_U^{m,\gamma})} & < & 1, \\ \|K_\mu\|_{\mathcal{L}(\mathcal{H}_U^{m,\mu})} & < & 1, \end{array}$$

for a sufficiently small T depending on γ and μ. Therefore, both equations (4.12) and (4.13) can be solved by the method of successive approximations, namely

$$\begin{array}{rcl} \tilde{U} & = & \sum\limits_{\nu=0}^{\infty} K_\gamma^\nu \, \mathcal{R}_\gamma \tilde{F}, \\ \tilde{\tilde{U}} & = & \sum\limits_{\nu=0}^{\infty} K_\mu^\nu \, \mathcal{R}_\mu \tilde{F}. \end{array}$$

Since $\tilde{F} \in \mathcal{H}_F^{m,\mu}$ and $\tilde{F}$ vanishes for $t \geq T$, the corresponding terms of these series are identical by Lemma 3.3. This finishes the proof.

$\square$

5 Structure of solutions

The following *Structure Theorem* for solutions of the inhomogeneous equation can also be regarded as a *Regularity Theorem.*

Theorem 5.1 *Let* (4.10) *be fulfilled. Suppose N eigenvalues of C (counting the multiplicities) are located in the strip $-\mu < \Im\lambda < -\gamma$, and there are no eigenvalues of C on the lines $\Im\lambda = -\mu$ and $\Im\lambda = -\gamma$. Then any solution to $DU(t) - C(t)U(t) = F(t)$, $t < T$, such that $\chi_T U \in \mathcal{H}_U^{m,\gamma}$ and $\chi_T F \in \mathcal{H}_F^{m-1,\mu}$, has the form*

$$U(t) = \left(\sum_{\nu=1}^{N} c_\nu S_\nu(t) \right) + R(t),$$

where S_ν are linearly independent solutions of the homogeneous equation which do not depend on U; c_ν constants; and $\chi_T R \in \mathcal{H}_U^{m,\mu}$.

Proof. We prove this theorem under the assumption that there is only one simple eigenvalue of C in the strip $-\mu < \Im\lambda < -\gamma$. The proof in the general case is analogous.

Consider the equation with a constant operator (3.2) where $F(t)$ is a function with a support on the semiaxis $t < 0$, belonging to $\mathcal{H}_F^{m-1,\mu}$ (and therefore to the space $\mathcal{H}_F^{m-1,\gamma}$). This equation has a unique solution in each of the spaces $\mathcal{H}_U^{m,\gamma}$ and $\mathcal{H}_U^{m,\mu}$, cf. Lemma 3.1. As above, we denote $\mathcal{R}_\gamma$ the inverse of $D - C$ acting from $\mathcal{H}_F^{m-1,\gamma}$ to $\mathcal{H}_U^{m,\gamma}$, and $\mathcal{R}_\mu$ the inverse of $D - C$ acting from $\mathcal{H}_F^{m-1,\mu}$ to $\mathcal{H}_U^{m,\mu}$.

The Fourier transform $\hat{F}(\lambda)$ of $F(t)$ is analytic in the strip $-\mu < \Im\lambda < -\gamma$. Therefore,

$$\begin{aligned}\mathcal{R}_\gamma F(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty-i\gamma}^{+\infty-i\gamma} e^{i\lambda t} (\lambda - C)^{-1} \hat{F}(\lambda)\, d\lambda \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty-i\mu}^{+\infty-i\mu} e^{i\lambda t} (\lambda - C)^{-1} \hat{F}(\lambda)\, d\lambda + e^{i\lambda_0 t} \left(\hat{F}(\lambda_0), \Psi\right)_{\mathcal{H}} \Phi\end{aligned} \tag{5.1}$$

where Φ is the eigenvector of the operator C corresponding to the eigenvalue λ_0, and Ψ is the eigenvector of the operator C^{adj} corresponding to the eigenvalue $\bar{\lambda}_0$.

In evaluating the residue of the integrand in (5.1) we made use of the representation

$$(C - \lambda)^{-1} = \frac{(\cdot, \Psi)_{\mathcal{H}}}{\lambda_0 - \lambda} \Phi + H(\lambda)$$

in a neighbourhood of λ_0, $H(\lambda)$ being an analytic operator-valued function (cf. Keldysh [Kel51]). Furthermore,

$$\begin{aligned}\left| \left(\hat{F}(\lambda_0), \Psi\right)_{\mathcal{H}} \right| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\lambda_0 t} (F(t), \Psi)_{\mathcal{H}}\, dt \right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\lambda_0 t + \mu t} \left(e^{-\mu t} F(t), \Psi\right)_{\mathcal{H}}\, dt \right| \\ &\le c \int_{-\infty}^{0} e^{(\Im\lambda_0 + \mu)t} \|e^{-\mu t} F(t)\|_{\mathcal{H}} dt \\ &\le C\, \|F\|_{\mathcal{H}_F^{m-1,\mu}},\end{aligned}$$

the constants c and C being independent of F. Thus, (5.1) implies

$$\mathcal{R}_\gamma F = \mathcal{R}_\mu F + \phi(F)\, e^{i\lambda_0 t} \Phi \tag{5.2}$$

where the functional $\phi(F)$ satisfies the estimate

$$|\phi(F)| \le C\, \|F\|_{\mathcal{H}_F^{m-1,\mu}}.$$

Now consider the equation $DU(t) - C(t)U(t) = F(t)$, for $t < T$. Introduce the function $\tilde{U} = \chi_T U$. It will cause no confusion if we use the same letter U to designate $\tilde{U}$, for the previous function U no longer appears. Substituting this new function into the equation, we obtain $DU - C(t)U = \chi_T F + (D\chi_T)\, U$, the right-hand side $\tilde{F} = \chi_T F + (D\chi_T)\, U$ being supported on the interval $(-\infty, T]$. Furthermore, let us denote $\Delta = C(t) - C$ and rewrite the latter equation in the form

$$DU - CU = \Delta U + \tilde{F}.$$

Suppose that $U \in \mathcal{H}_U^{m,\gamma}$. The equation yields

$$U = \mathcal{R}_\gamma \, \Delta U + \mathcal{R}_\gamma \tilde{F}$$

on all of $\mathbb{R}$. Taking into account that the support of U belongs to $(-\infty, T]$, we get

$$U = \chi\, \mathcal{R}_\gamma \, \Delta U + \chi\, \mathcal{R}_\gamma \tilde{F} \tag{5.3}$$

where $\chi = \chi_{T+1}$. To solve this equation by the method of successive approximations, we put

$$\begin{array}{rcl} U_0 & = & \chi\, \mathcal{R}_\gamma \tilde{F}, \\ U_{\nu+1} & = & \chi\, \mathcal{R}_\gamma \, \Delta U_\nu + \chi\, \mathcal{R}_\gamma \tilde{F}, \qquad \nu = 0, 1, \ldots . \end{array}$$

Along with (5.3) we shall consider the equation

$$\Upsilon = \chi\, \mathcal{R}_\mu \, \Delta \Upsilon + \chi\, \mathcal{R}_\mu \tilde{F} \tag{5.4}$$

and, accordingly,

$$\begin{array}{rcl} \Upsilon_0 & = & \chi\, \mathcal{R}_\mu \tilde{F}, \\ \Upsilon_{\nu+1} & = & \chi\, \mathcal{R}_\mu \, \Delta \Upsilon_\nu + \chi\, \mathcal{R}_\mu \tilde{F}, \qquad \nu = 0, 1, \ldots . \end{array}$$

In view of (4.10) we can assume, by decreasing T if necessary, that the norms of the operators $\mathcal{R}_\gamma \Delta$ and $\mathcal{R}_\mu \Delta$ acting in $\mathcal{H}_U^{m,\gamma}$ and $\mathcal{H}_U^{m,\mu}$, respectively, are less than 1. Therefore, both equations (5.3) and (5.4) can be solved by the method of successive approximations. Moreover, we invoke (5.2) to conclude that

$$\begin{array}{rcl} U_0 & = & \chi\, \mathcal{R}_\mu \, \tilde{F} + \phi(\tilde{F})\, \chi\, e^{i\lambda_0 t} \Phi \\ & = & \Upsilon_0 + \phi(\tilde{F})\, \mathcal{A} \end{array}$$

where $\mathcal{A} = \chi\, e^{i\lambda_0 t} \Phi$.

From now on we write $\mathcal{R}_\gamma$ instead of $\chi\, \mathcal{R}_\gamma$ and $\mathcal{R}_\mu$ instead of $\chi\, \mathcal{R}_\mu$. We claim that

$$\begin{array}{rcl} U_\nu & = & \Upsilon_\nu + \phi(\Delta \Upsilon_{\nu-1} + \tilde{F})\, \mathcal{A} + \phi(\Delta \Upsilon_{\nu-2} + \tilde{F})\, \mathcal{R}_\gamma \Delta\, \mathcal{A} \\ & & + \ldots + \phi(\Delta \Upsilon_0 + \tilde{F})\, (\mathcal{R}_\gamma \Delta)^{\nu-1} \mathcal{A} + \phi(\tilde{F})\, (\mathcal{R}_\gamma \Delta)^\nu \mathcal{A} \end{array}$$

for all $\nu = 0, 1, \ldots$. To prove this by induction, we note that the equality is true for $\nu = 0$. We assume that it holds for all $\nu \leq N$, and show it for $\nu = N + 1$. We have

$$\begin{aligned} U_{N+1} &= \mathcal{R}_\gamma \, \Delta U_N + \mathcal{R}_\gamma \tilde{F} \\ &= \mathcal{R}_\gamma \, \Delta U_N + \mathcal{R}_\mu \tilde{F} + \phi(\tilde{F}) \, \mathcal{A} \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_\gamma \, \Delta U_N &= \mathcal{R}_\gamma \, \Delta \Upsilon_N + \phi(\Delta \Upsilon_{N-1} + \tilde{F}) \, \mathcal{R}_\gamma \Delta \, \mathcal{A} + \phi(\Delta \Upsilon_{N-2} + \tilde{F}) \, (\mathcal{R}_\gamma \Delta)^2 \mathcal{A} \\ & \quad + \ldots + \phi(\Delta \Upsilon_0 + \tilde{F}) \, (\mathcal{R}_\gamma \Delta)^N \mathcal{A} + \phi(\tilde{F}) \, (\mathcal{R}_\gamma \Delta)^{N+1} \mathcal{A}. \end{aligned}$$

As

$$\mathcal{R}_\gamma \, \Delta \Upsilon_N = \mathcal{R}_\mu \, \Delta \Upsilon_N + \phi(\Delta \Upsilon_N) \, \mathcal{A},$$

we arrive at the equality

$$\begin{aligned} U_{N+1} &= \Upsilon_{N+1} + \phi(\Delta \Upsilon_N + \tilde{F}) \, \mathcal{A} + \phi(\Delta \Upsilon_{N-1} + \tilde{F}) \, \mathcal{R}_\gamma \Delta \, \mathcal{A} \\ & \quad + \ldots + \phi(\Delta \Upsilon_0 + \tilde{F}) \, (\mathcal{R}_\gamma \Delta)^N \mathcal{A} + \phi(\tilde{F}) \, (\mathcal{R}_\gamma \Delta)^{N+1} \mathcal{A}, \end{aligned} \tag{5.5}$$

as required.

Further,

$$\begin{aligned} \Delta \Upsilon_0 &= \Delta \mathcal{R}_\mu \, \tilde{F}, \\ \Delta \Upsilon_1 &= (\Delta \mathcal{R}_\mu)^2 \, \tilde{F} + \Delta \mathcal{R}_\mu \, \tilde{F}, \\ \ldots & \ldots \ldots \\ \Delta \Upsilon_N &= \sum_{\nu=1}^{N+1} (\Delta \mathcal{R}_\mu)^\nu \, \tilde{F}, \end{aligned}$$

and so the formula (5.5) can be written as

$$U_{N+1} = \Upsilon_{N+1} + \sum_{n=0}^{N+1} \sum_{\nu=0}^{N+1-n} \phi \left((\Delta \mathcal{R}_\mu)^\nu \, \tilde{F} \right) (\mathcal{R}_\gamma \Delta)^n \, \mathcal{A}. \tag{5.6}$$

The double series

$$\sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \phi \left((\Delta \mathcal{R}_\mu)^\nu \, \tilde{F} \right) (\mathcal{R}_\gamma \Delta)^n \, \mathcal{A}$$

is easily verified to converge absolutely. Therefore, we may take the limit in (5.6), as $N \to \infty$, thus obtaining

$$\begin{aligned} U(t) &= \Upsilon(t) + \sum_{n=0}^{\infty} \phi \left(\sum_{\nu=0}^{\infty} (\Delta \mathcal{R}_\mu)^\nu \, \tilde{F} \right) (\mathcal{R}_\gamma \Delta)^n \, \mathcal{A} \\ &= \Upsilon(t) + c \, S(t) \end{aligned}$$

where $\Upsilon \in \mathcal{H}_U^{m,\mu}$ and

$$\begin{aligned} S(t) &= \sum_{n=0}^{\infty} (\mathcal{R}_\gamma \Delta)^n \, \mathcal{A}, \\ c &= \phi \left(\sum_{\nu=0}^{\infty} (\Delta \mathcal{R}_\mu)^\nu \, \tilde{F} \right). \end{aligned}$$

Note that $S(t)$ is independent of $U(t)$. Only the constant c depends on the solution.

□

As is seen from the proof, we tacitly assume that the number T is small enough. However, it is completely controlled by the norms (4.10) and may be chosen independently of U and F.

Corollary 5.2 *Let the assumptions of Theorem 5.1 be fulfilled. Then any solution* $u \in H^{m,\gamma}(-\infty, T)$ *of equation* (1.1) *with* $f \in H^{0,\mu}(-\infty, T)$ *has the form*

$$u(t) = c_1 \, s_1(t) + \ldots + c_N \, s_N(t) + R(t)$$

where $s_1, \ldots, s_N$ *are solutions of the homogeneous equation which do depend on* u*;* $c_1, \ldots, c_N$ *constants; and* $R \in H^{m,\mu}(-\infty, T)$*.*

In the proof of Theorem 5.1 the conditions (4.10) are needed only to guarantee the smallness of the norms of operators $\mathcal{R}_\gamma \Delta$ and $\mathcal{R}_\mu \Delta$, for T small enough. We did not really have to use the special form of the matrix $C(t)$. Therefore, to prove Corollary 5.2 we could have applied the usual way of reducing the equation (1.1) to a first order system, namely taking derivatives of $u(t)$ as new unknown functions. In order to make the norms of the resulting operators $\mathcal{R}_\gamma \Delta$ and $\mathcal{R}_\mu \Delta$ small, the following conditions are sufficient instead of (4.10):

$$\| C_j(t) - C_j \|_{\mathcal{L}(H_{m-j}, H_0)} \to 0 \quad \text{as} \quad t \to -\infty, \tag{5.7}$$

for $j = 0, 1, \ldots, m-1$. Thus, we arrive at Theorem 1.1 stated in the Introduction.

6 Smoothness of the spectrum

In what follows it will be assumed that the eigenvalues and eigenfunctions of operators are differentiable a sufficient number of times. In this section we show conditions on the operators under which their eigenvalues and eigenfunctions are as smooth as needed.

Let C be a closed operator acting in $\mathcal{H}$. We shall denote $\mathcal{H}_C$ the complete space obtained by introducing the graph norm $\|\cdot\|_C$ in the domain of C, cf. (4.7).

Suppose $C(t)$, $t < T$, is a family of operators with domains $\operatorname{Dom} C(t)$ containing $\operatorname{Dom} C$, such that

$$\|C(t) - C\|_{\mathcal{L}(\mathcal{H}_C,\mathcal{H})} \to 0 \quad \text{as} \quad t \to -\infty. \tag{6.1}$$

Note that if λ is a regular point of the limiting operator C, then λ is a regular point for $C(t)$, too, if t is sufficiently small (see, e.g., [GK57]).

Lemma 6.1 *Let* (6.1) *be fulfilled and $C(t)$ have a strong derivative in $\mathcal{L}(\mathcal{H}_C, \mathcal{H})$, for any $t < T$. Then the resolvent $R_{C(t)}(\lambda) = (C(t) - \lambda)^{-1}$ also has a derivative in $\mathcal{L}(\mathcal{H}, \mathcal{H}_C)$, and*

$$D R_{C(t)}(\lambda) = -R_{C(t)}(\lambda)\,(DC(t))\,R_{C(t)}(\lambda). \tag{6.2}$$

Proof. We have

$$\frac{1}{\Delta t}\left(R_{C(t+\Delta t)}(\lambda) - R_{C(t)}(\lambda)\right) = \frac{1}{\Delta t}\,R_{C(t)}(\lambda)\,(C(t) - C(t+\Delta t))\,R_{C(t+\Delta t)}(\lambda) \tag{6.3}$$

provided Δt is small enough. By choosing Δt sufficiently small we may actually assume that $\|(C(t) - C(t+\Delta t))\,R_{C(t)}(\lambda)\|_{\mathcal{L}(\mathcal{H})} < 1$. Then, we invoke the equality

$$\begin{aligned} R_{C(t+\Delta t)}(\lambda) &= \left(\left(1 - (C(t) - C(t+\Delta t))\,R_{C(t)}(\lambda)\right)(C(t) - \lambda)\right)^{-1} \\ &= R_{C(t)}(\lambda)\left(1 + \sum_{\nu=1}^{\infty}\left((C(t) - C(t+\Delta t))\,R_{C(t)}(\lambda)\right)^{\nu}\right) \end{aligned}$$

to see that $R_{C(t+\Delta t)}(\lambda)$ is continuous in Δt in the norm of $\mathcal{L}(\mathcal{H}, \mathcal{H}_C)$. Taking into account the differentiability of $C(t)$, we can pass to the limit in (6.3), as $\Delta t \to 0$. This gives (6.2).

$\square$

By (6.2), we get

$$\|D R_{C(t)}(\lambda)\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_C)} \le c\,\|DC(t)\|_{\mathcal{L}(\mathcal{H}_C,\mathcal{H})}$$

where $c = \|R_{C(t)}(\lambda)\|^2_{\mathcal{L}(\mathcal{H},\mathcal{H}_C)}$. Note that this constant is majorised uniformly in t, for t small enough.

Corollary 6.2 *Suppose* (6.1) *is fulfilled and $C(t)$ has strong derivatives up to order K in $\mathcal{L}(\mathcal{H}_C, \mathcal{H})$, for any $t < T$. Then the resolvent $R_{C(t)}(\lambda)$ also has the derivatives up to order K in $\mathcal{L}(\mathcal{H}, \mathcal{H}_C)$, and*

$$\|D^k R_{C(t)}(\lambda)\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_C)} \le c \max_{p_1+\ldots+kp_k=k} \|DC(t)\|^{p_1}_{\mathcal{L}(\mathcal{H}_C,\mathcal{H})} \cdots \|D^k C(t)\|^{p_k}_{\mathcal{L}(\mathcal{H}_C,\mathcal{H})} \tag{6.4}$$

for all $k = 1, \ldots, K$.

Proof. Indeed, we apply (6.2) k times to conclude that the derivative $D^k R_{C(t)}(\lambda)$ is a linear combination of operators of the form

$$R_{C(t)}(\lambda) \left(D^{k_1} C(t)\right) R_{C(t)}(\lambda) \dots R_{C(t)}(\lambda) \left(D^{k_N} C(t)\right) R_{C(t)}(\lambda)$$

where $k_1, \dots, k_N > 0$ and $k_1 + \dots + k_N = k$. Hence the estimate (6.4) follows immediately. $\square$

Let λ be a simple eigenvalue of C, and Φ the corresponding eigenvector. If (6.1) is fulfilled, then there is a simple eigenvalue $\lambda(t)$ of $C(t)$, for t small enough, and $\lambda(t) \to \lambda$ as $t \to -\infty$ (cf. [GK57]). We write $\Phi(t)$ for the corresponding eigenvector of $C(t)$.

Lemma 6.3 *Suppose the condition* (6.1) *is satisfied and, moreover,*

$$\begin{array}{rcll} \int_{-\infty}^{t_0} \|DC(t)\|_{\mathcal{L}(\mathcal{H}_C, \mathcal{H})}\, dt & < & \infty, & \\ \|D^k C(t)\|_{\mathcal{L}(\mathcal{H}_C, \mathcal{H})} & \leq & \epsilon(t), & k = 1, \dots, m, \end{array} \tag{6.5}$$

$\epsilon(t)$ being bounded in $t < T$. Then, both $\lambda(t)$ and $\Phi(t)$ possess the derivatives up to order m, and the following estimates hold:

$$\begin{array}{rclrcl} \int_{-\infty}^{t_0} |D\lambda(t)|\, dt & < & \infty, & \max\limits_{k \leq m} |D^k \lambda(t)| & \leq & c\, \epsilon(t); \\ \int_{-\infty}^{t_0} \|D\Phi(t)\|_C\, dt & < & \infty, & \max\limits_{k \leq m} \|D^k \Phi(t)\|_C & \leq & c\, \epsilon(t). \end{array} \tag{6.6}$$

Proof. Introduce the operator

$$P(t) = \frac{1}{2\pi i} \int_{\mathfrak{c}} (C(t) - \lambda)^{-1}\, d\lambda,$$

where the path $\mathfrak{c}$ is chosen so that all its points are regular for $C(t)$ and a unique eigenvalue $\lambda(t)$ of $C(t)$ is located inside $\mathfrak{c}$ for all $t < T$. It is known, cf. [GK57], that $P(t)$ is a projection onto the eigenspace of $C(t)$ corresponding to $\lambda(t)$. Since the resolvent $R_{C(t)}(\lambda)$ is differentiable in t, so is the operator $P(t)$, and

$$\begin{array}{rcl} \|P(t)U\|_C & \leq & c\, \|U\|_{\mathcal{H}}, \\ \|DP(t)U\|_C & \leq & c\, \|DC(t)\|_{\mathcal{L}(\mathcal{H}_C, \mathcal{H})}\, \|U\|_{\mathcal{H}}. \end{array} \tag{6.7}$$

Denote by Ψ the eigenvector of C^{adj} corresponding to the eigenvalue $\bar{\lambda}$. The norm of Ψ can be determined by $(\Phi, \Psi)_{\mathcal{H}} = 1$.

We now observe that $(P(t)\Phi, \Psi)_{\mathcal{H}} \to 1$ when $t \to -\infty$. Hence it follows that $(P(t)\Phi, \Psi)_{\mathcal{H}} \neq 0$ for all $t < T$ small enough. As $C(t)P(t)\Phi = \lambda(t)P(t)\Phi$, we get

$$\lambda(t) = \frac{(C(t)P(t)\Phi, \Psi)_{\mathcal{H}}}{(P(t)\Phi, \Psi)_{\mathcal{H}}}.$$

Since both $C(t)$ and $P(t)$ are differentiable, we deduce that $\lambda(t)$ has a first derivative. Moreover, applying (6.7) yields

$$|D\lambda(t)| \leq c\, \|DC(t)\|_{\mathcal{L}(\mathcal{H}_C,\mathcal{H})}, \tag{6.8}$$

the constant c being independent of $t < T$.

Further, the normalised eigenfunction $\Phi(t)$ can be expressed as

$$\Phi(t) = \frac{P(t)\Phi}{\|P(t)\Phi\|_{\mathcal{H}}}$$

whence

$$\|D\Phi(t)\|_C \leq c\, \|DC(t)\|_{\mathcal{L}(\mathcal{H}_C,\mathcal{H})}, \tag{6.9}$$

with c a constant independent of $t < T$.

From the inequalities (6.8) and (6.8) the estimates (6.6) for the derivatives of order $k \leq 1$ follow. The higher order derivatives can be estimated in a similar way.

□

Now we clarify under what conditions on the coefficients of the differential equation (4.1) the operator $C(t)$ arising by reduction of this equation to a first order system has properties (6.5). Let us recall that the reduction leads to the operator (4.5), the entries $B_j(t)$ being given by (4.3). Obviously, conditions (6.5) are fulfilled if

$$\begin{array}{rcll} \int_{-\infty}^{t_0} \|DB_j(t)\|_{\mathcal{L}(H_{m-j},H_0)}\, dt & < & \infty, & \\ \|D^k B_j(t)\|_{\mathcal{L}(H_{m-j},H_0)} & \leq & \epsilon(t), & k = 1,\ldots,m, \end{array}$$

for each $j = 0,1,\ldots,m-1$. These latter are in turn consequences of the conditions

$$\begin{array}{rcll} \int_{-\infty}^{t_0} \|D^k C_j(t)\|_{\mathcal{L}(H_{m-j},H_0)}\, dt & < & \infty, & k = 1,\ldots,j+1; \\ \|D^k C_j(t)\|_{\mathcal{L}(H_{m-j},H_0)} & \leq & \epsilon(t), & k = 1,\ldots,j+m, \end{array} \tag{6.10}$$

$\epsilon(t)$ being bounded in $t < T$.

Lemma 6.4 *Assume that the coefficients $C_j(t)$ satisfy conditions* (6.10). *Then* (6.6) *holds.*

As mentioned after (4.7), the norm in the space $\mathcal{H}_C$ is equivalent to the norm $U \mapsto \left(\sum_{j=1}^{m-1} \|u_j\|_{H_0}\right) + \|u_m\|_{H_m}$, for $U = (u_1,\ldots,u_m)$. Therefore, (6.6) implies, in particular, that

$$\max_{k\leq m} \left(\left(\sum_{j=1}^{m-1} \|D^k\varphi_j(t)\|_{H_0} \right) + \|D^k\varphi_m(t)\|_{H_m} \right) \leq c\, \epsilon(t),$$

where $\varphi_j(t)$, $j = 1, \ldots, m$, are the components of $\Phi(t)$.

An easy computation shows that $\varphi_m(t)$ is actually an eigenfunction of the operator pencil

$$\tilde{\sigma}(t,\lambda) = \sum_{j=0}^{m} B_j(t)\lambda^j,$$

where $B_m = \mathrm{Id}$.

Assume that the adjoints $C_j^{\mathrm{adj}}(t)$ of $C_j(t)$ in H_0 satisfy conditions similar to (6.10). The arguments of Section 2 still apply to the adjoint operator $C^{\mathrm{adj}}(t)$ and the adjoint bundle $\tilde{\sigma}^{\mathrm{adj}}(t,\lambda)$. As a result we obtain that $\tilde{\sigma}^{\mathrm{adj}}(t,\lambda)$ has the form

$$\tilde{\sigma}^{\mathrm{adj}}(t,\lambda)g = \sum_{j=0}^{m} B_j^{\mathrm{adj}}(t)\bar{\lambda}^j\, g,$$

with the domain $\mathrm{Dom}\,\tilde{\sigma}^{\mathrm{adj}}(t,\lambda) = H_m$. Moreover, the components of the eigenvector $\Psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ of $C^{\mathrm{adj}}(t)$ corresponding to the eigenvalue $\bar{\lambda}(t)$ are related by

$$\begin{array}{rcl} \psi_1(t) & = & \psi(t), \\ \psi_2(t) & = & \bar{\lambda}(t)\,\psi_1(t), \\ \ldots & \ldots & \ldots \\ \psi_m(t) & = & \bar{\lambda}(t)\,\psi_{m-1}(t), \end{array} \tag{6.11}$$

where $\psi(t)$ is the eigenfunction of the operator pencil $\tilde{\sigma}^{\mathrm{adj}}(t,\lambda)$ corresponding to $\bar{\lambda}(t)$.

Since the coefficients of $\tilde{\sigma}^{\mathrm{adj}}(t,\lambda)$ are assumed to behave similarly to those of $\tilde{\sigma}(t,\lambda)$, we get

$$\begin{array}{rcl} \int_{-\infty}^{t_0} \|D\psi_1(t)\|_{H_m} dt & < & \infty, \\ \max\limits_{k\le m} \|D^k\psi_1(t)\|_{H_m} & \le & c\,\epsilon(t), \end{array}$$

the constant c being independent of t small enough. Thus, combining the equalities (6.11) with the estimates for the derivatives of $\lambda(t)$ given in (6.6), we obtain

$$\begin{array}{rcl} \int_{-\infty}^{t_0} \|D\psi_j(t)\|_{H_m} dt & < & \infty, \\ \max\limits_{k\le m} \|D^k\psi_j(t)\|_{H_m} & \le & c\,\epsilon(t), \end{array} \tag{6.12}$$

for every $j = 1, \ldots, m$. This proves

Lemma 6.5 *Let* (6.10) *be fulfilled, with $C_j(t)$ replaced by $C_j^{\mathrm{adj}}(t)$. Then the estimates* (6.12) *hold for the components $\psi_j(t)$, $j = 1, \ldots, m$, of the eigenvector $\Psi(t)$ of $C^{\mathrm{adj}}(t)$.*

7 Splitting of a first order system

Lemma 7.1 *Let λ_0 be a simple pole of the resolvent $R_{\mathfrak{C}}(\lambda) = (\mathfrak{C} - \lambda)^{-1}$ of a closed operator $\mathfrak{C}$ with a domain dense in a Hilbert space $\mathcal{H}$, so that*

$$R_{\mathfrak{C}}(\lambda)F = \frac{(F, \Psi_0)_{\mathcal{H}}}{\lambda_0 - \lambda}\, \Phi_0 + H(\lambda)F$$

for all $F \in \mathcal{H}$, where Φ_0 (resp. Ψ_0) is the eigenvector of $\mathfrak{C}$ (resp. $\mathfrak{C}^{\mathrm{adj}}$) corresponding to the eigenvalue λ_0 (resp. $\bar{\lambda}_0$), with $(\Phi_0, \Psi_0)_{\mathcal{H}} = 1$, and $H(\lambda)$ is holomorphic near λ_0. Then the resolvent of the operator $\mathfrak{C}_\kappa = \mathfrak{C} - \kappa\, (\cdot, \Psi_0)_{\mathcal{H}} \Phi_0$ has the form

$$R_{\mathfrak{C}_\kappa}(\lambda)F = \frac{(F, \Psi_0)_{\mathcal{H}}}{\lambda_0 - \lambda - \kappa}\, \Phi_0 + H(\lambda)F. \tag{7.1}$$

Proof. Consider the equation $(\mathfrak{C}_\kappa - \lambda)U = F$ or, which is the same,

$$(\mathfrak{C} - \lambda)U - \kappa\, (U, \Psi_0)_{\mathcal{H}}\, \Phi_0 = F, \tag{7.2}$$

the latter being equivalent to

$$U = R_{\mathfrak{C}}(\lambda)F + \kappa\, (U, \Psi_0)_{\mathcal{H}}\, R_{\mathfrak{C}}(\lambda)\Phi_0,$$

for $\lambda \neq \lambda_0$. As $R_{\mathfrak{C}}(\lambda)\Phi_0 = (\lambda_0 - \lambda)^{-1}\Phi_0$, we get

$$U = \frac{(F, \Psi_0)_{\mathcal{H}}}{\lambda_0 - \lambda}\, \Phi_0 + H(\lambda)F + \kappa\, \frac{(U, \Psi_0)_{\mathcal{H}}}{\lambda_0 - \lambda}\, \Phi_0. \tag{7.3}$$

Multiplying (7.2) by Ψ_0 yields

$$(\lambda_0 - \lambda)\, (U, \Psi_0)_{\mathcal{H}} - \kappa\, (U, \Psi_0)_{\mathcal{H}} = (F, \Psi_0)_{\mathcal{H}}$$

whence

$$(U, \Psi_0)_{\mathcal{H}} = \frac{(F, \Psi_0)_{\mathcal{H}}}{\lambda_0 - \lambda - \kappa}.$$

Substituting this into (7.3) gives (7.1), as desired.

□

Suppose $\|B_j(t) - C_j\|_{\mathcal{L}(H_{m-j}, H_0)} \to 0$ as $t \to -\infty$. Let only one simple eigenvalue λ_0 of C be located on a line $\Im\lambda = -\gamma_\nu$. As the C-norm of $C(t) - C$ tends to zero as $t \to -\infty$, there exists a simple eigenvalue $\lambda(t)$ of $C(t)$, for t large enough. Denote by $\Phi(t)$ the eigenvector of $C(t)$ corresponding to $\lambda(t)$, and by $\Psi(t)$ the eigenvector of $C^{\mathrm{adj}}(t)$ corresponding to $\bar{\lambda}(t)$. We normalise them by requiring

$$\begin{aligned} (\Phi(t), \Psi(t))_{\mathcal{H}} &= 1, \\ \|\Phi(t)\|_{\mathcal{H}} &= 1. \end{aligned} \tag{7.4}$$

As is shown in Section 6, if the derivatives $DB_j(t)$ and $DB_j^{\mathrm{adj}}(t)$ exist, so do the derivatives $D\Phi(t)$ and $D\Psi(t)$ in $\mathcal{H}$. Put

$$\begin{array}{rcl} \Upsilon(t) & = & U(t) - c(t)\,\Phi(t), \\ c(t) & = & (U(t), \Psi(t))_{\mathcal{H}}, \end{array} \tag{7.5}$$

then the normalisation conditions on $\Phi(t)$ and $\Psi(t)$ imply

$$(\Upsilon(t), \Psi(t))_{\mathcal{H}} = 0. \tag{7.6}$$

If $U(t)$ is a solution of the equation $DU - C(t)U = 0$ on the semiaxis $t < T$, then

$$D\Upsilon(t) + Dc(t)\,\Phi(t) + c(t)\,D\Phi(t) - C(t)\Upsilon(t) - c(t)\lambda(t)\Phi(t) = 0. \tag{7.7}$$

Let us multiply both sides of this equality by $\Psi(t)$. In view of (7.4) we obtain

$$(D\Upsilon(t), \Psi(t))_{\mathcal{H}} + Dc(t) + c(t)\,(D\Phi(t), \Psi(t))_{\mathcal{H}} - (C(t)\Upsilon(t), \Psi(t))_{\mathcal{H}} - c(t)\lambda(t) = 0.$$

Obviously,

$$\begin{array}{rcl} (C(t)\Upsilon(t), \Psi(t))_{\mathcal{H}} & = & (\Upsilon(t), C^{\mathrm{adj}}(t)\Psi(t))_{\mathcal{H}} \\ & = & \lambda(t)\,(\Upsilon(t), \Psi(t))_{\mathcal{H}} \\ & = & 0 \end{array}$$

whence

$$(D\Upsilon(t), \Psi(t))_{\mathcal{H}} + Dc(t) + c(t)\,(D\Phi(t), \Psi(t))_{\mathcal{H}} - c(t)\lambda(t) = 0. \tag{7.8}$$

Multiplying (7.8) by $\Phi(t)$ and subtracting the result from (7.7) leads to the equality

$$D\Upsilon(t) + c(t)\,(D\Phi(t) - (D\Phi(t), \Psi(t))_{\mathcal{H}}\Phi(t)) - C(t)\Upsilon(t) - (D\Upsilon(t), \Psi(t))_{\mathcal{H}}\Phi(t) = 0. \tag{7.9}$$

By (7.6), we have

$$(D\Upsilon(t), \Psi(t))_{\mathcal{H}} = (\Upsilon(t), D\Psi(t))_{\mathcal{H}},$$

hence the equations (7.8) and (7.9) can be written in the form

$$\begin{array}{rcl} D\Upsilon(t) & = & C(t)\Upsilon(t) + c_{11}(t)\,\Upsilon(t) + c_{12}(t)\,c(t), \\ Dc(t) & = & \lambda(t)c(t) + c_{21}(t)\,\Upsilon(t) + c_{22}(t)\,c(t), \end{array} \tag{7.10}$$

where

$$\begin{array}{rcl} c_{11}(t)\,\Upsilon(t) & = & (\Upsilon(t), D\Psi(t))_{\mathcal{H}}\Phi(t), \\ -c_{12}(t) & = & D\Phi(t) - (D\Phi(t), \Psi(t))_{\mathcal{H}}\Phi(t), \\ -c_{21}(t)\,\Upsilon(t) & = & (\Upsilon(t), D\Psi(t))_{\mathcal{H}}, \\ -c_{22}(t) & = & (D\Phi(t), \Psi(t))_{\mathcal{H}}. \end{array}$$

Let us extend the functions $\Phi(t)$ and $\Psi(t)$ to the whole real axis so that the differences $\Phi(t)-\Phi_0$ and $\Psi(t)-\Psi_0$ and the derivatives $D^k\Phi(t)$ and $D^k\Psi(t)$, $k \leq m$, be sufficiently small in the norm of $\mathcal{H}_C$ for all $t \in \mathbb{R}$. Introduce the operators

$$\begin{array}{rcl} C_\kappa \Upsilon(t) & = & C\Upsilon(t) - \kappa\,(\Upsilon(t), \Psi_0)_{\mathcal{H}}\,\Phi_0, \\ C_\kappa(t)\Upsilon(t) & = & C(t)\Upsilon(t) - \kappa\,(\Upsilon(t), \Psi(t))_{\mathcal{H}}\,\Phi(t) \end{array}$$

where κ is a complex number which we choose so that the line Γ_0 do not contain eigenvalues of the operator $C_\kappa - \lambda_0$.

By (7.6), we get $C_\kappa(t)\Upsilon(t) = C(t)\Upsilon(t)$. Therefore, the system (7.10) finally becomes

$$\begin{array}{rcl} D\Upsilon(t) & = & C_\kappa(t)\Upsilon(t) + c_{11}(t)\,\Upsilon(t) + c_{12}(t)\,c(t), \\ Dc(t) & = & \lambda(t)c(t) + c_{21}(t)\,\Upsilon(t) + c_{22}(t)\,c(t). \end{array} \tag{7.11}$$

Thus we have proved

Lemma 7.2 *Assume that $\|B_j(t)-C_j\|_{\mathcal{L}(H_{m-j},H_0)} \to 0$ as $t \to -\infty$. Let the derivatives $DB_j(t)$ and $DB_j^{\mathrm{adj}}(t)$ exist and the norms*

$$\begin{array}{l} \|DB_j(t)\|_{\mathcal{L}(H_{m-j},H_0)}, \\ \|DB_j^{\mathrm{adj}}(t)\|_{\mathcal{L}(H_{m-j},H_0)} \end{array}$$

be bounded uniformly in $t \ll 0$, for $j = 0, 1, \ldots, m-1$. If $U(t)$ is a solution to $DU - C(t)U = 0$, $t < T$, such that $\chi_T U \in \mathcal{H}_U^{m,\gamma}$, then the functions $\Upsilon(t)$ and $c(t)$ satisfy (7.11) for small t.

8 Asymptotics under weak stabilisation

In this section we derive an asymptotic representation of a solution to the homogeneous equation (4.1) under the condition that the coefficients tend to constant operators as $t \to -\infty$. This result is an immediate consequence of an asymptotic formula for solutions $U(t)$ of the system $DU - C(t)U = 0$, for $t < T$.

Lemma 8.1 *Let the following conditions be fulfilled:*

1) *The operators C_j and C_j^{adj} meet the assumptions of Section 2.*
2) *The domain of $C_0(t)$ coincides with H_m, for each $t < T$.*
3) *$\|B_j(t) - C_j\|_{\mathcal{L}(H_{m-j},H_0)} \to 0$ as $t \to -\infty$.*

4) $\|D^k C_j(t)\|_{\mathcal{L}(H_{m-j},H_0)} \le \epsilon(t)$ *for all* $k = 1, \ldots, j+m$, *where* $\epsilon(t) \le c$.
5) *The adjoints* $C_j^{\mathrm{adj}}(t)$ *possess properties* 2) *and* 4).

Then, given any solution $U(t)$ *to* $DU - C(t)U = 0$ *on the semiaxis* $t < T$, *satisfying* $\chi_T U \in \mathcal{H}_U^{m,\gamma}$, *and any eigenvector* $\Psi(t)$ *of the operator* $C^{\mathrm{adj}}(t)$, *we have*

$$|D^k\,(U(t), D\Psi(t))_{\mathcal{H}}| \le C\,\epsilon(t) \left(\sum_{s=0}^{k-1} \|D^s u_m(t)\|_{H_0} + \sum_{j=1}^{m-1} \|u_j(t)\|_{H_0} \right)$$

for all $k = 1, \ldots, m-1$.

Proof. We have

$$D^k\,(U(t), D\Psi(t))_{\mathcal{H}} = D^{k-1}\left((DU(t), D\Psi(t))_{\mathcal{H}} - (U(t), D^2\Psi(t))_{\mathcal{H}}\right)$$

for all $t < T$. Since $DU = C(t)U$, it follows that

$$(DU(t), D\Psi(t))_{\mathcal{H}} = \sum_{j=1}^{m-1}(u_j(t), D\psi_{j+1}(t))_{\mathcal{H}} - \sum_{j=1}^{m}(B_{j-1}(t)u_m(t), D\psi_j(t))_{\mathcal{H}}.$$

We now observe that every component $D\psi_j(t)$ belongs to H_m, by Lemma 6.5. Hence $D\psi_j(t) \in \mathrm{Dom}\, B_{j-1}^{\mathrm{adj}}(t)$, showing

$$(DU(t), D\Psi(t))_{\mathcal{H}} = \sum_{j=1}^{m-1}(u_j(t), D\psi_{j+1}(t))_{\mathcal{H}} - \sum_{j=1}^{m}(u_m(t), B_{j-1}^{\mathrm{adj}}(t)D\psi_j(t))_{\mathcal{H}}.$$

Applying the estimates of the derivatives $D^s\psi_j(t)$ given by (6.12), we easily obtain

$$|D^{k-1}\sum_{j=1}^{m}(u_m(t), B_{j-1}^{\mathrm{adj}}(t)D\psi_j(t))_{\mathcal{H}}| \le c\,\epsilon(t)\sum_{s=0}^{k-1}\|D^s u_m(t)\|_{H_0} \tag{8.1}$$

provided that $k \le m$.

We now turn to the derivative

$$D^{k-1}\sum_{j=1}^{m-1}(u_j(t), D\psi_{j+1}(t))_{\mathcal{H}}.$$

From

$$\begin{array}{rcll} Du_1 & = & -B_0(t)u_m, & \\ Du_j & = & u_{j-1} - B_{j-1}(t)u_m, & \quad j = 2, \ldots, m, \end{array} \tag{8.2}$$

it follows that

$$
\begin{aligned}
& D^{k-1} \sum_{j=1}^{m-1} (u_j, D\psi_{j+1})_{\mathcal{H}} \\
& = D^{k-2} \left(\sum_{j=1}^{m-1} (Du_j, D\psi_{j+1})_{\mathcal{H}} - \sum_{j=1}^{m-1} (u_j, D^2\psi_{j+1})_{\mathcal{H}} \right) \\
& = D^{k-2} \left(\sum_{j=2}^{m-1} (u_{j-1}, D\psi_{j+1})_{\mathcal{H}} - \sum_{j=1}^{m-1} (B_{j-1}(t)u_m, D\psi_{j+1})_{\mathcal{H}} - \sum_{j=1}^{m-1} (u_j, D^2\psi_{j+1})_{\mathcal{H}} \right).
\end{aligned}
$$

Moving the operators $B_{j-1}(t)$ from u_m to $D\psi_{j+1}$ enables us to conclude, as in the proof of (8.1), that

$$
|D^{k-2} \sum_{j=1}^{m-1} (B_{j-1}(t)u_m, D\psi_{j+1})_{\mathcal{H}}| \le c\,\epsilon(t) \sum_{s=0}^{k-2} \|D^s u_m(t)\|_{H_0}.
$$

Further, the remaining terms can be rewritten as

$$
\begin{aligned}
& D^{k-2} \left(\sum_{j=2}^{m-1} (u_{j-1}, D\psi_{j+1})_{\mathcal{H}} - \sum_{j=1}^{m-1} (u_j, D^2\psi_{j+1})_{\mathcal{H}} \right) \\
& \quad = D^{k-3} \left(\sum_{j=2}^{m-1} (Du_{j-1}, D\psi_{j+1})_{\mathcal{H}} - \sum_{j=2}^{m-1} (u_{j-1}, D^2\psi_{j+1})_{\mathcal{H}} \right. \\
& \qquad \left. - \sum_{j=1}^{m-1} (Du_j, D^2\psi_{j+1})_{\mathcal{H}} + \sum_{j=1}^{m-1} (u_j, D^3\psi_{j+1})_{\mathcal{H}} \right),
\end{aligned}
$$

and so using formulas (8.2) again transforms the right-hand side of the last equality to

$$
D^{k-3} \left(\sum_{j=3}^{m-1} (u_{j-2}, D\psi_{j+1})_{\mathcal{H}} - 2 \sum_{j=2}^{m-1} (u_{j-1}, D^2\psi_{j+1})_{\mathcal{H}} + \sum_{j=1}^{m-1} (u_j, D^3\psi_{j+1})_{\mathcal{H}} + \ldots \right),
$$

the dots meaning terms containing operators $B_{j-1}(t)$. These terms are estimated in the same manner as inequality (8.1).

We continue in this fashion to arrive at the desired estimate.

$\square$

Before formulating our next lemma, we recall that the function $c(t)$ is defined by (7.5).

Lemma 8.2 *Under the assumptions of Lemma 8.1, let moreover $U(t)$ be a solution to $DU - C(t)U = 0$ on the semiaxis $t < T$, such that $\chi_T U \in \mathcal{H}_U^{m,\gamma}$,* *for some $\gamma_{\nu-1} < \gamma < \gamma_\nu$. Then, for every $\mu < \gamma_\nu$, the following estimate is fulfilled:*

$$\int_{-\infty}^{T} e^{-2\mu t} \sum_{j=0}^{m} |D^j c(t)|^2 dt < \infty.$$

Proof. Indeed, Lemma 4.2 implies that $\chi_T U \in \mathcal{H}_U^{m,\mu}$. As

$$\begin{aligned} |c(t)| &\le \|U(t)\|_{\mathcal{H}} \|\Psi(t)\|_{\mathcal{H}}, \\ \|\Psi(t)\|_{\mathcal{H}} &\le c\, \epsilon(t), \end{aligned}$$

$\epsilon(t)$ being uniformly bounded in $t < T$, we get

$$\begin{aligned} \int_{-\infty}^{T} e^{-2\mu t} |c(t)|^2 dt &\le C \int_{-\infty}^{T} e^{-2\mu t} \|U(t)\|_{\mathcal{H}}^2 dt \\ &< \infty. \end{aligned}$$

We now proceed by induction. Assume that

$$\int_{-\infty}^{T} e^{-2\mu t} \sum_{j=0}^{J} |D^j c(t)|^2 dt < \infty,$$

for some $J \le m-1$, and prove that this inequality remains valid with J replaced by $J+1$. We have

$$\begin{aligned} D^{J+1} c(t) &= D^J \left((DU(t), \Psi(t))_{\mathcal{H}} - (U(t), D\Psi(t))_{\mathcal{H}} \right) \\ &= D^J \left(C(t) U(t), \Psi(t) \right)_{\mathcal{H}} - D^J \left(U(t), D\Psi(t) \right)_{\mathcal{H}}. \end{aligned}$$

Further,

$$\begin{aligned} (C(t)U(t), \Psi(t))_{\mathcal{H}} &= \lambda(t)\ (U(t), \Psi(t))_{\mathcal{H}} \\ &= \lambda(t)\ c(t), \end{aligned}$$

and so in view of the boundedness of the derivatives $D^k \lambda(t)$, for $k \le m$, we get

$$\int_{-\infty}^{T} e^{-2\mu t} |D^J \left(C(t) U(t), \Psi(t) \right)_{\mathcal{H}}|^2 dt < \infty.$$

On the other hand, the expression $D^J \left(U(t), D\Psi(t) \right)_{\mathcal{H}}$ can be estimated by Lemma 8.1, as desired.

□

Corollary 8.3 *Let the hypotheses of Lemma 8.2 be satisfied. Then the inequalities*

$$\begin{aligned} \int_{\mathbb{R}} e^{-2\mu t} \left(\sum_{j=0}^{m} \|D^j (\chi_T(t) v_m(t))\|_{H_{m-j}}^2 + \sum_{i=1}^{m-1} \sum_{j=0}^{i} \|D^j (\chi_T(t) v_i(t))\|_{H_0}^2 \right) dt &< \infty, \\ \int_{\mathbb{R}} e^{-2\mu t} \sum_{j=0}^{m} |D^j (\chi_T(t) c(t))|^2 dt &< \infty \end{aligned}$$

hold, where $v_i(t)$*,* $i = 1, \dots, m$*, are the components of the vector-valued function* $\Upsilon(t)$ *defined in* (7.5)*.*

Proof. The proof follows immediately from Lemma 8.2, the boundedness of the norms of $D^k\Phi(t)$ (cf. Lemma 6.3), and the fact that $\chi_T U \in \mathcal{H}_U^{m,\mu}$.

□

Let us go back to system (7.11). Suppose $T \ll 0$ is a sufficiently small number. Put

$$\begin{array}{rcl} \tilde{U}(t) & = & \chi_T(t)U(t) \\ & = & \tilde{\Upsilon}(t) + \tilde{c}(t)\,\Phi(t) \end{array}$$

where

$$\begin{array}{rcl} \tilde{\Upsilon}(t) & = & \chi_T(t)\Upsilon(t), \\ \tilde{c}(t) & = & \chi_T(t)c(t). \end{array}$$

The functions $\tilde{\Upsilon}(t)$ and $\tilde{c}(t)$ satisfy the following system on the whole real axis

$$\begin{array}{rcl} D\tilde{\Upsilon}(t) & = & C_\kappa(t)\tilde{\Upsilon}(t) + c_{11}(t)\,\tilde{\Upsilon}(t) + c_{12}(t)\,\tilde{c}(t) + R(t), \\ D\tilde{c}(t) & = & \lambda(t)\tilde{c}(t) + c_{21}(t)\,\tilde{\Upsilon}(t) + c_{22}(t)\,\tilde{c}(t) + r(t), \end{array}$$

where

$$\begin{array}{rcl} R(t) & = & (D\chi_T(t))\,\Upsilon(t), \\ r(t) & = & (D\chi_T(t))\,c(t) \end{array}$$

are functions with compact supports. By abuse of notation, we write U, Υ and c instead of $\tilde{U}$, $\tilde{\Upsilon}$ and $\tilde{c}$, respectively. This will not lead to any misunderstanding, for the old functions U, Υ and c no longer appear. Thus, the system becomes

$$\begin{array}{rcl} D\Upsilon(t) & = & C_\kappa(t)\Upsilon(t) + c_{11}(t)\,\Upsilon(t) + c_{12}(t)\,c(t) + R(t), \\ Dc(t) & = & \lambda(t)c(t) + c_{21}(t)\,\Upsilon(t) + c_{22}(t)\,c(t) + r(t), \end{array} \tag{8.3}$$

the supports of $\Upsilon(t)$ and $c(t)$ being located on the half-axis $t < T$.

Let $\lambda(t)$ be an eigenvalue of the operator $C(t)$, such that $\lambda(t) \to \lambda = \tau - i\gamma_\nu$ as $t \to -\infty$. Let moreover $\mu < \gamma_\nu$.

Denote $\tilde{\chi}_N(t)$ a smooth function mit values in $[0, 1]$, such that $\tilde{\chi}_N(t) = 1$, for $t \le N' < N$, and $\tilde{\chi}_N(t) = 0$, for $t > N$. Choosing $N' \ll N$ sufficiently small, we may arrange that the derivatives of $\tilde{\chi}_N(t)$ be small. Introduce the function

$$\lambda_N(t) = (1 - \tilde{\chi}_N(t))\,\lambda(t) + \tilde{\chi}_N(t)\,(\tau - i\mu)\,.$$

Obviously,

$$\begin{array}{rcl} \Im\,(\lambda_N(t) - \lambda(t)) & \ge & 0, \\ \lambda_N(t) & = & \lambda(t), \quad \text{for} \quad t > N, \\ \Im\lambda_N(t) & = & -\mu, \quad \text{for} \quad t < N', \end{array}$$

the first inequality being fulfilled if N is small enough. Put

$$\begin{aligned} Y(t) &= \Upsilon(t)\exp\left(i\int\limits_t^T \lambda(s)ds\right), \\ y(t) &= c(t)\exp\left(i\int\limits_t^T \lambda(s)ds\right). \end{aligned}$$

In the same way, using $\lambda_N(t)$ instead of $\lambda(t)$, we introduce functions $Y_N(t)$ and $y_N(t)$. In view of (8.3) they satisfy

$$\begin{aligned} DY_N(t) &= (C_\kappa(t)-\lambda_N(t))\,Y_N(t)+c_{11}(t)\,Y_N(t)+c_{12}(t)\,y_N(t)+R_N(t), \\ Dy_N(t) &= (\lambda(t)-\lambda_N(t))\,y_N(t)+c_{21}(t)\,Y_N(t)+c_{22}(t)\,y_N(t)+r_N(t), \end{aligned} \tag{8.4}$$

where $R_N(t)$ and $r_N(t)$ are the functions obtained from $R(t)$ and $r(t)$ by multiplication by $\exp(i\int_t^T \lambda_N(s)ds)$.

From the definitions of $\lambda_N(t)$, $Y_N(t)$ and $y_N(t)$ one obtains, by Corollary 8.3, that

$$\begin{aligned} \int_{\mathbb{R}}\left(\sum_{j=0}^{m}\|D^jY_{N,m}(t)\|^2_{H_{m-j}}+\sum_{i=1}^{m-1}\sum_{j=0}^{i}\|D^jY_{N,i}(t)\|^2_{H_0}\right)dt &\le \mathrm{const}(N), \\ \int_{\mathbb{R}}\sum_{j=0}^{m}|D^jy_N(t)|^2dt &\le \mathrm{const}(N), \end{aligned} \tag{8.5}$$

where $Y_{N,i}(t)$ are the components of $Y_N(t)$. The task is now to find an estimate for $Y_N(t)$ and $y_N(t)$ with a constant independent of N.

Lemma 8.4 *Let the assumptions of Lemma 8.1 hold. Suppose $Y_N(t)$, $y_N(t)$ is a solution to* (8.4)*, satisfying* (8.5)*. Then*

$$|D^k(Y_N,D\Psi)_{\mathcal{H}}|\le C\,\epsilon(t)\left(\sum_{s=0}^{k-1}\|D^sY_{N,m}\|_{H_0}+\sum_{j=1}^{m-1}\|Y_{N,j}\|_{H_0}+\sum_{s=0}^{k}|D^sy_N|\right).$$

for all $k=1,\ldots,m-1$.

Proof. Put $U_N(t)=Y_N(t)+y_N(t)\Phi(t)$. Then

$$(Y_N(t),D\Psi(t))_{\mathcal{H}}=(U_N(t),D\Psi(t))_{\mathcal{H}}-y_N(t)\,(\Phi(t),D\Psi(t))_{\mathcal{H}}.$$

We claim that

$$DU_N(t)=(C(t)-\lambda_N(t))\,U_N(t)$$

for all $t<T-1$. Indeed, since $Y_N(t)$ and $y_N(t)$ satisfy (8.4) and both $R_N(t)$ and $r_N(t)$ vanish for all $t<T-1$, a straightforward verification shows that

the above equality reduces to $C_\kappa(t)\Upsilon_N(t) = C(t)\Upsilon_N(t)$, which is obviously fulfilled.

Thus, in much the same way as in the proof of Lemma 8.1 we arrive at an estimate

$$|D^k\,(U_N(t), D\Psi(t))_{\mathcal{H}}| \le c\,\epsilon(t) \left(\sum_{s=0}^{k-1} \|D^s U_{N,m}(t)\|_{H_0} + \sum_{j=1}^{m-1} \|U_{N,j}(t)\|_{H_0} \right).$$

On the other hand,

$$|D^k\,(y_N(t)\,(\Phi(t), D\Psi(t))_{\mathcal{H}})\,| \le c\,\epsilon(t) \sum_{s=0}^{k} |D^s y_N(t)|,$$

as is easy to check by Lemmas 6.3 and 6.5. Since

$$\begin{aligned} &\sum_{s=0}^{k-1} \|D^s U_{N,m}(t)\|_{H_0} + \sum_{j=1}^{m-1} \|U_{N,j}(t)\|_{H_0} \\ &\qquad \le C \left(\sum_{s=0}^{k-1} \|D^s Y_{N,m}(t)\|_{H_0} + \sum_{j=1}^{m-1} \|Y_{N,j}(t)\|_{H_0} + \sum_{s=0}^{k-1} |D^s y_N(t)| \right), \end{aligned}$$

the lemma follows.

□

Our next objective will be to estimate the function $y_N(t)$ and its derivatives.

Lemma 8.5 *Let the hypotheses of Lemma 8.1 be fulfilled, the function $\epsilon(t)$ of 4) tending to zero as $t \to -\infty$. Suppose moreover that*

$$\begin{array}{rcl} \int_{-\infty}^{t_0} \|D^k C_j(t)\|_{\mathcal{L}(H_{m-j}, H_0)} dt & < & \infty, \\ \int_{-\infty}^{t_0} \|D^k C_j^{\mathrm{adj}}(t)\|_{\mathcal{L}(H_{m-j}, H_0)} dt & < & \infty, \end{array}$$

for every $j = 0, 1, \ldots, m-1$ and $k = 1, \ldots, j+1$, and

$$\Delta = \sup_{\substack{t<T \\ k\le m}} |D^k\,(\lambda(t) - \lambda_N(t))\,|$$

is sufficiently small. Then, for T small enough, the estimates

$$\begin{array}{rcl} \displaystyle\sum_{k=1}^{m} |D^k y_N(t)| & \le & C\,\Delta\,|y_N(t)| + \epsilon(t) \left(\displaystyle\sum_{k=0}^{m-1} \|D^k Y_{N,m}(t)\|_{H_0} + \sum_{j=1}^{m-1} \|Y_{N,j}(t)\|_{H_0} \right), \\ \displaystyle\sup_{t<T} |y_N(t)| & \le & C + \varepsilon(T) \displaystyle\sup_{t<T} \sum_{j=1}^{m} \|Y_{N,j}(t)\|_{H_0} \end{array} \tag{8.6}$$

are true, with C and $\varepsilon(T)$ independent of N, $\varepsilon(T) \to 0$ as $T \to -\infty$.

Proof. From the second equation of (8.4) it follows that

$$y_N(t) = -\int_t^T e^{-i\int_t^\vartheta (\lambda(s)-\lambda_N(s))ds}\,(c_{21}(\vartheta)Y_N(\vartheta) + c_{22}(\vartheta)y_N(\vartheta) + r_N(\vartheta))\,d\vartheta,$$

for $y_N(t)$ vanishes for $t \geq T$. As $\Im(\lambda(s) - \lambda_N(s)) \leq 0$, we get

$$\sup_{t<T} |y_N(t)| \leq \int_{-\infty}^T \left(\|c_{21}(\vartheta)\| \sum_{j=1}^m \|Y_{N,j}(t)\|_{H_0} + |c_{22}(\vartheta)|\,|y_N(\vartheta)| + |r_N(\vartheta)| \right) d\vartheta \tag{8.7}$$

where $\|c_{21}(\vartheta)\|$ and $|c_{22}(\vartheta)|$ are integrable functions, cf. Lemmas 6.4 and 6.5, and $r_N(\vartheta)$ vanishes for $\vartheta \leq T-1$. For $T \ll 0$, the estimate (8.7) is easily seen to imply (8.6).

Further, differentiating the second equality (8.4) k times, $0 \leq k \leq m-1$, results in

$$\begin{aligned} D^{k+1}y_N(t) &= \sum_{s=0}^k \binom{k}{s} D^{k-s}\left(\lambda(t) - \lambda_N(t)\right) D^s y_N(t) \\ &\quad + D^k\left(c_{21}(t)Y_N(t)\right) + D^k\left(c_{22}(t)y_N(t)\right) + D^k r_N(t), \end{aligned}$$

for any $t \in \mathbb{R}$. The derivatives of $c_{21}(t)Y_N(t) = -(Y_N(t), D\Psi(t))_{\mathcal{H}}$ are estimated by Lemma 8.4. Estimating the remaining terms is not difficult. Thus, (8.7) implies the estimates

$$\begin{aligned} |D^{k+1}y_N(t)| &\leq C\,\Delta \sum_{s=0}^k |D^s y_N(t)| \\ &\quad + c\,\epsilon(t) \left(\sum_{s=0}^{k-1} \|D^s Y_{N,m}(t)\|_{H_0} + \sum_{j=1}^{m-1} \|Y_{N,j}(t)\|_{H_0} + \sum_{s=0}^k |D^s y_N(t)| \right), \end{aligned}$$

for $k = 1, \ldots, m-1$.

Summarising these inequalities and taking into account that $\epsilon(t) \to 0$ when $t \to -\infty$, we obtain the first estimate (8.6) for sufficiently small T.

$\square$

Lemma 8.6 *Let the eigenvalue $\lambda(t)$ of $C(t)$ have a limit $\lambda = \tau - i\gamma_\nu$ as $t \to -\infty$, and no other eigenvalues of the limiting operator C be located on the line $\Gamma_{-\gamma_\nu}$. Assume moreover that $U \in \mathcal{H}_U^{m,0}$ is supported in $(-\infty, T]$ and satisfies*

$$DU(t) - (C_\kappa(t) - \lambda_N(t))\,U(t) = F(t), \tag{8.8}$$

$F \in \mathcal{H}_F^{m-1,0}$ being a function of compact support. Then, for T small enough, we have

$$\|U\|_{\mathcal{H}_U^{m,0}} \leq c\,\|F\|_{\mathcal{H}_F^{m-1,0}},$$

with c a constant independent of N (and F).

Proof. The functions $\lambda(t)$ and $\lambda_N(t)$ are defined for $t < T$ only, where T is sufficiently small. Let us define them on the whole real axis in such a way that $\lambda(t) \to \lambda = \tau - i\gamma_\nu$ as $t \to \mp\infty$, and

$$\lambda_N(t) = (1 - \tilde{\chi}_N(t))\,\lambda(t) + \tilde{\chi}_N(t)\,(\tau - i\mu),$$

the derivatives $D^k(\lambda(t) - \lambda_N(t))$, $k \le m-1$, being sufficiently small on all of $\mathbb{R}$. The operator $C_\kappa(t)$ can be thought of as being defined on the whole axis and such that the norm $\|C_\kappa(t) - C_\kappa\|_{\mathcal{L}(\mathcal{H}_U^{m,0}, \mathcal{H}_F^{m-1,0})}$ is small. Write the equality (8.8) as

$$DU(t) - (C_\kappa - \lambda)U(t) = (C_\kappa(t) - C_\kappa)U(t) + (\lambda - \lambda_N(t))U(t) + F(t),$$

for $t \in \mathbb{R}$. By Lemma 7.1 there are no poles of the resolvent of $C_\kappa - \lambda$ on the line Γ_0. Hence the desired estimate follows.

□

Lemma 8.7 *Let the assumptions of Lemma 8.5 be satisfied, and, moreover,* $\int_{-\infty}^{t_0} (\epsilon(t))^2 dt < \infty$. *Suppose the eigenvalue $\lambda(t)$ of $C(t)$ tends to a simple eigenvalue $\tau - i\gamma_\nu$ of C, when $t \to -\infty$, and no other eigenvalues of C are located on $\Gamma_{-\gamma_\nu}$. Then*

$$\|Y_N\|_{\mathcal{H}_U^{m,0}} \le c \tag{8.9}$$

provided that $T \ll 0$ is sufficiently small, the constant c being independent of N.

Proof. Put $F(t) = c_{11}(t)Y_N(t) + c_{12}(t)y_N(t) + R_N(t)$, then the first equation of (8.4) becomes

$$DY_N(t) = (C_\kappa(t) - \lambda_N(t))\,Y_N(t) + F(t).$$

By Lemma 8.6, we get

$$\begin{aligned} \|Y_N\|_{\mathcal{H}_U^{m,0}} &\le c\,\|F\|_{\mathcal{H}_F^{m-1,0}} \\ &\le c\left(\|c_{11}Y_N\|_{\mathcal{H}_F^{m-1,0}} + \|c_{12}y_N\|_{\mathcal{H}_F^{m-1,0}} + \|R_N\|_{\mathcal{H}_F^{m-1,0}}\right), \end{aligned}$$

where c is independent of N. The first term on the right is estimated by Lemma 8.4, for $c_{11}(t)Y_N(t) = (Y_N(t), D\Psi(t))_{\mathcal{H}}\Phi(t)$. We thus obtain

$$\begin{aligned} &\|c_{11}Y_N\|^2_{\mathcal{H}_F^{m-1,0}} \\ &\le c\int_{-\infty}^{T} (\epsilon(t))^2 \left(\sum_{k=0}^{m-1} \|D^k Y_{N,m}(t)\|^2_{H_0} + \sum_{j=1}^{m-1} \|Y_{N,j}(t)\|^2_{H_0} + \sum_{k=0}^{m-1} |D^k y_N(t)|^2\right) dt \end{aligned}$$

whence, by (8.6),

$$\|c_{11}Y_N\|^2_{\mathcal{H}_F^{m-1,0}} \le C' + (\varepsilon(T))^2 \sup_{t<T} \sum_{j=1}^{m} \|Y_{N,j}(t)\|^2_{H_0}$$
$$+ C'' \int_{-\infty}^{T} (\epsilon(t))^2 \left(\sum_{k=0}^{m-1} \|D^k Y_{N,m}(t)\|^2_{H_0} + \sum_{j=1}^{m-1} \|Y_{N,j}(t)\|^2_{H_0} \right) dt,$$

where C', C'' and $\varepsilon(T)$ do not depend on N, $\varepsilon(T) \to 0$ as $T \to -\infty$. The remaining terms $\|c_{12}y_N\|_{\mathcal{H}_F^{m-1,0}}$ and $\|R_N\|_{\mathcal{H}_F^{m-1,0}}$ can be estimated in an analogous manner. Therefore,

$$\|Y_N\|_{\mathcal{H}_U^{m,0}} \le c' + c'' q(T)\, \|Y_N\|_{\mathcal{H}_U^{m,0}}$$

where $q(T) \to 0$ when $T \to -\infty$. Hence the estimate (8.9) for small $T \ll 0$ follows.

□

Theorem 8.8 *Let the assumptions of Lemma 8.1 be fulfilled, and let moreover*

$$\int_{-\infty}^{t_0} t^2 (\epsilon(t))^2 dt < \infty.$$

Suppose $\lambda(t)$ is an eigenvalue of $C(t)$ tending to a simple eigenvalue $\tau - i\gamma_\nu$ of C, as $t \to -\infty$, and there are no other eigenvalues of C on $\Gamma_{-\gamma_\nu}$. Then any solution to $DU - C(t)U = 0$, $t < T$, such that $\chi_T U \in \mathcal{H}_U^{m,\gamma}$, $\gamma_{\nu-1} < \gamma < \gamma_\nu$, has the form

$$U(t) = e^{i\int_T^t \lambda(s)ds} (c\,\Phi(t) + R(t)) \tag{8.10}$$

where c is a constant depending on the solution, and $\chi_T R \in \mathcal{H}_U^{m,0}$.

Proof. Decomposing the system $DU - C(t)U = 0$ as above, cf. (7.5), one obtains

$$U(t) = e^{i\int_T^t \lambda(s)ds} (y(t)\Phi(t) + Y(t))\,.$$

The limit of Y_N as $N \to -\infty$ is equal to $\chi_T Y$. Passing to the limit in (8.9), we get $\chi_T Y \in \mathcal{H}_U^{m,0}$.

The function $y(t)$ satisfies $Dy(t) = c_{21}(t)Y(t) + c_{22}(t)y(t) + r(t)$, for $t < T$, whence

$$y(t) = c_0 - \int_t^T (c_{21}(s)Y(s) + c_{22}(s)y(s) + r(s))\, ds,$$

c_0 being a constant. It is easy to see that the latter integral converges as $t \to -\infty$. Therefore,

$$y(t) = c + \int_{-\infty}^{t} (c_{21}(s)Y(s) + c_{22}(s)y(s) + r(s))\, ds.$$

The conditions of the theorem guarantee that the last integral belongs to the space $H^{m,0}(-\infty, T)$. This establishes the desired formula.

□

We are now in a position to *prove* Theorem 1.3 stated in the Introduction. To this end, we introduce the operator

$$\tilde{C}(t) = \begin{pmatrix} 0 & \dots & 0 & -C_0(t) \\ 1 & \dots & 0 & -C_1(t) \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -C_{m-1}(t) \end{pmatrix}.$$

Denote by $\Lambda(t)$ the eigenvalue of $\tilde{C}(t)$ tending to the eigenvalue λ of C, as $t \to -\infty$, with $\Im\lambda = -\gamma_\nu$. Our next goal is to show that the difference $\Delta(t) = \lambda(t) - \Lambda(t)$ meets the estimate

$$\int_{-\infty}^{t_0} t^2 |\Delta(t)|^2 dt < \infty. \tag{8.11}$$

Put

$$\tilde{P}(t) = \frac{1}{2\pi i} \int_{\mathfrak{c}} \left(\tilde{C}(t) - \lambda\right)^{-1} d\lambda,$$

where $\mathfrak{c}$ is a path containing in its interior only one eigenvalue $\Lambda(t)$ of $\tilde{C}(t)$, for $t < T$. We have

$$\begin{aligned} \left(\tilde{C}(t) - \lambda\right)^{-1} &= \left(\left(\mathrm{Id} - (C(t) - \tilde{C}(t)) R_{C(t)}(\lambda)\right)(C(t) - \lambda)\right)^{-1} \\ &= R_{C(t)}(\lambda) \left(\mathrm{Id} + \sum_{\nu=1}^{\infty} ((C(t) - \tilde{C}(t)) R_{C(t)}(\lambda))^\nu\right). \end{aligned} \tag{8.12}$$

Let Φ be an eigenvector of the limiting operator C. Then $\tilde{P}(t)\Phi \to \Phi$ in $\mathcal{H}$, as $t \to -\infty$. The vector

$$\tilde{\Phi}(t) = \frac{\tilde{P}(t)\Phi}{\|\tilde{P}(t)\Phi\|_{\mathcal{H}}}$$

is a normalised eigenvector of $\tilde{C}(t)$. By (8.12) it is easy to check that the difference $\Phi(t) - \tilde{\Phi}(t)$ belongs to $\mathcal{H}_U^{m,0}$. Further,

$$\begin{aligned} \lambda(t) &= \frac{(C(t)P(t)\Phi, \Psi)_{\mathcal{H}}}{(P(t)\Phi, \Psi)_{\mathcal{H}}}, \\ \Lambda(t) &= \frac{(\tilde{C}(t)\tilde{P}(t)\Phi, \Psi)_{\mathcal{H}}}{(\tilde{P}(t)\Phi, \Psi)_{\mathcal{H}}} \end{aligned}$$

where $P(t)$ is a projection operator into an eigenspace of $C(t)$ and Ψ is an eigenvector of $C^{\mathrm{adj}}(t)$. Using (8.12) and the properties of $C_j(t)$, we arrive at (8.11).

If $U(t)$ is a solution to the system $DU - C(t)U = 0$, $t < T$, satisfying $\chi_T U \in \mathcal{H}_U^{m,\gamma}$, $\gamma_{\nu-1} < \gamma < \gamma_\nu$, then by Theorem 8.8

$$\begin{aligned} U(t) &= e^{i\int_T^t \lambda(s)ds} \left(c\,\Phi(t) + R(t)\right) \\ &= e^{i\int_T^t \Lambda(s)ds} \left(c\,\tilde{\Phi}(t) + \tilde{R}(t)\right), \end{aligned}$$

where $\chi_T \tilde{R} \in \mathcal{H}_U^{m,0}$.

To complete the proof of Theorem 1.3, it suffices to observe that $\Lambda(t)$ is an eigenvalue of the operator pencil $\sigma(t,\lambda) = \sum_{j=0}^m C_j(t)\lambda^j$ and that $\tilde{\varphi}_m(t)$, the last component of $\tilde{\Phi}(t)$, is an eigenvector of this pencil.

$\square$

9 Applications

The Fuchs-type operators of the standard cone theory (cf. [Sch94]) are assumed to have C^∞ coefficients up to $r = 0$. This just amounts to saying that the coefficients bear Taylor asymptotics near $r = 0$. However, the solutions of homogeneous Fuchs-type equations bear asymptotics spanned by $r^{i\lambda_\nu} (\log r)^k c_{\nu k}(x)$, where $\lambda_\nu \in \mathbb{C}$, $k \in \mathbb{Z}_+$, and $c_{\nu k}(x)$ are C^∞ functions on the link. Thus, to organise a cone calculus in spaces with asymptotics, we may actually allow more general asymptotics of the coefficients, provided that these latter remain slowly varying at $r = 0$. More precisely, we demand $A_j(r)$ to satisfy

$$A_j(r) \sim A_j(0) + \sum_{\nu=1}^{\infty} \sum_{k=0}^{m_\nu} r^{iz_\nu} (\log r)^k c_{\nu k}(x) \tag{9.1}$$

as $r \to 0$, where $(z_\nu)_{\nu\in\mathbb{N}}$ is a sequence of points in the complex plane with the property that

$$\begin{aligned} 0 > \Im z_1 \geq \Im z_2 \geq \dots, \\ \Im z_\nu \to -\infty \quad \text{as} \quad \nu \to \infty, \end{aligned}$$

and $m_\nu \in \mathbb{Z}_+$. This property allows one to give a precise meaning to the asymptotic sum on the right side of (9.1). Assume that (9.1) can be differentiated in r sufficiently many times. Since

$$(rD_r)\left(r^{iz} (\log r)^k\right) = (z - ik/\log r)\left(r^{iz} (\log r)^k\right),$$

the condition of Theorem 1.5 will be established for all $k \geq 1$ once we prove it for $k = 1$. In this particular case it becomes

$$\begin{aligned} \int_0^{r_0} (\log r)^{2(2\varrho-1)}\, |r^{iz_1}\, (\log r)^{m_1}\,|^2\, \frac{dr}{r} &= \int_0^{r_0} (\log r)^{2(2\varrho-1+m_1)}\, r^{-2\Im z_1}\, \frac{dr}{r} \\ &< \infty, \end{aligned}$$

which is obviously fulfilled because of the condition $\Im z_1 < 0$. Thus, Theorem 1.5 applies to Fuchs-type equations with coefficients as non-smooth, as are the asymptotics.

10 A refinement of asymptotic formulas

The asymptotic formula for a solution of $DU - C(t)U = 0$, $t < T$, established in Theorem 8.8 can be refined if the operator $C(t)$ has a special behaviour near $t = -\infty$.

Suppose $C(t)$ can be written as

$$C(t) = \sum_{\iota=0}^{J} O_\iota \frac{1}{t^\iota} + O_{J+1}(t),$$

where

$$O_0 = \begin{pmatrix} 0 & \dots & 0 & -C_0 \\ 1 & \dots & 0 & -C_1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -C_{m-1} \end{pmatrix}$$

and

$$O_\iota = \begin{pmatrix} 0 & \dots & 0 & -B_{0,\iota} \\ 0 & \dots & 0 & -B_{1,\iota} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -B_{m-1,\iota} \end{pmatrix},$$

for $\iota = 1, \dots, J+1$, Here, $B_{j,\iota}$, $\iota \leq J$, are constant operators, and $B_{j,J+1}$ depends on t, the domains of all entries containing H_m. Moreover, we require the norms of $D^k B_{j,J+1}(t)$ in $\mathcal{L}(H_{m-j}, H_0)$ to be bounded by c/t^{J+1+k}, for each $k \leq m$.

Let $\Gamma_{-\gamma_\nu}$ contain only one eigenvalue λ_0 of O_0, and λ_0 be simple. We look for a solution to $DU - C(t)U = 0$, $t < T$, satisfying $\chi_T U \in \mathcal{H}_U^{m,\gamma}$, $\gamma_{\nu-1} < \gamma < \gamma_\nu$, of the form

$$U(t) = t^p\, e^{i\lambda_0 t} \left(c_0 \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota} + U_{J+1}(t) \right),$$

the vectors $U_0, U_1, \ldots, U_J$ and the number p being determined from the conditions

$$\begin{aligned}(O_0 - \lambda_0)U_0 &= 0, \\ (O_0 - \lambda_0)U_\iota &= \sqrt{-1}\,(\iota - 1 - p)\,U_{\iota-1} - \sum_{k=0}^{\iota-1} O_{\iota-k}U_k,\end{aligned} \tag{10.1}$$

for $\iota = 1, \ldots, J$.

From the first equation it follows that U_0 is an eigenvector of O_0. We denote Ψ_0 a corresponding eigenvector of O_0^{adj} and normalise the vectors U_0 and Ψ_0 so that $\|U_0\|_{\mathcal{H}} = 1$ and $(U_0, \Psi_0)_{\mathcal{H}} = 1$.

Consider the second equation of (10.1), namely

$$(O_0 - \lambda_0)U_1 = -\sqrt{-1}\,p\,U_0 - O_1U_0.$$

It has a solution if and only if

$$-\sqrt{-1}\,p\,(U_0, \Psi_0)_{\mathcal{H}} = (O_1U_0, \Psi_0)_{\mathcal{H}}\,,$$

i.e., $p = \sqrt{-1}\,(O_1U_0, \Psi_0)_{\mathcal{H}}$. If p is chosen in this way, then the equation has a solution

$$U_1 = Y_1 + c\,U_0,$$

where Y_1 is a particular solution and c an arbitrary constant.

The equation for U_2 has the form

$$(O_0 - \lambda_0)U_2 = \sqrt{-1}\,(1-p)\,U_1 - (O_2U_0 + O_1U_1)\,,$$

the solvability condition being

$$\sqrt{-1}\,(1-p)\,(U_1, \Psi_0)_{\mathcal{H}} = (O_2U_0 + O_1U_1, \Psi_0)_{\mathcal{H}}\,.$$

This condition can in turn be satisfied by a proper choice of the constant c, namely

$$c = -\sqrt{-1}\,(O_2U_0 + O_1Y_1, \Psi_0)_{\mathcal{H}} - (1-p)\,(Y_1, \Psi_0)_{\mathcal{H}}\,.$$

We continue in this fashion to show that the system (10.1) is solvable. The vectors $U_0, U_1, \ldots, U_{J-1}$ are determined uniquely, and U_J can be written in the form

$$U_J = Y_J + c(t)\,U_0,$$

where Y_J is a particular solution of the corresponding inhomogeneous equation and $c(t)$ an *arbitrary* function of t. Therefore, the solution $U(t)$ can be written as

$$U(t) = t^p\,e^{i\lambda_0 t}\left(c_0 \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota} + c(t)\,U_0 + Y_{J+1}(t)\right),$$

with $c(t)$ an arbitrary function and $Y_{J+1}(t)$ an unknown vector-valued function, the meanings of U_J and $c(t)$ having been changed. Let us define the function $c(t)$ by

$$c(t) = \left(t^{-p}\, e^{-i\lambda_0 t} U(t) - c_0 \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota}, \Psi_0 \right)_{\mathcal{H}},$$

then $(Y_{J+1}(t), \Psi_0)_{\mathcal{H}} = 0$ for all $t < T$. Choose the constant c_0 in such a way that $c(t) \to 0$ when $t \to -\infty$. An easy computation shows that $c(t)$ and $Y_{J+1}(t)$ meet the system

$$\begin{array}{rcl} Dc(t) & = & \left(\left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) \right) U_0, \Psi_0 \right)_{\mathcal{H}} c(t) \\ & + & c_0 \left(\left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) - D \right) \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota}, \Psi_0 \right)_{\mathcal{H}} \\ & + & \left(\left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) \right) Y_{J+1}(t), \Psi_0 \right)_{\mathcal{H}}, \\ DY_{J+1}(t) & = & \left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) \right) Y_{J+1}(t) \\ & + & c_0 \left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) - D \right) \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota} \\ & + & \left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) \right) c(t) U_0 - Dc(t)\, U_0. \end{array}$$

Moreover,

$$\begin{array}{l} \left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) - D \right) \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota} \\ = \sqrt{-1}\, (p - J)\, U_J \frac{1}{t^{J+1}} + \sum_{\iota=J+1}^{2J} \frac{1}{t^\iota} \sum_{k=\iota-J}^{J} O_{\iota-k} U_k + O_{J+1}(t) \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota} \\ = O\left(\frac{1}{t^{J+1}} \right), \end{array}$$

which is clear from the choice of $U_0, U_1, \ldots, U_J$. Thus, the above system reduces to

$$\begin{array}{rcl} Dc(t) & = & \left(\left(\sum_{\iota=2}^{J} O_\iota \frac{1}{t^\iota} + O_{J+1}(t) \right) U_0, \Psi_0 \right)_{\mathcal{H}} c(t) \\ & + & c_0 \left(O\left(\frac{1}{t^{J+1}} \right), \Psi_0 \right)_{\mathcal{H}} \end{array}$$

$$
\begin{aligned}
& + \left(\left(\sum_{\iota=0}^{J} O_\iota \frac{1}{t^\iota} + O_{J+1}(t) \right) Y_{J+1}(t), \Psi_0 \right)_{\mathcal{H}}, \\
DY_{J+1}(t) &= \left(\sqrt{-1}\, p \frac{1}{t} - \lambda_0 + C(t) \right) Y_{J+1}(t) \\
&+ c_0 \left(O\left(\frac{1}{t^{J+1}} \right) - \left(O\left(\frac{1}{t^{J+1}} \right), \Psi_0 \right)_{\mathcal{H}} U_0 \right) \\
&- \left(\left(\sum_{\iota=0}^{J} O_\iota \frac{1}{t^\iota} + O_{J+1}(t) \right) Y_{J+1}(t), \Psi_0 \right)_{\mathcal{H}} U_0 + R(t),
\end{aligned} \tag{10.2}
$$

where $R(t)$ can be expressed linearly in terms of $c(t)U_0$ with coefficients decreasing at least as rapidly as $1/t$.

We solve the first equation of (10.2) with respect to $c(t)$ and substitute the resulting expression into the second, thus obtaining

$$
DY_{J+1}(t) = (C_\kappa - \lambda_0 + Q(t))\, Y_{J+1}(t) + F(t),
$$

where the norm $\|\chi_T Q\|_{\mathcal{L}(\mathcal{H}_U^{m,0})}$ is sufficiently small, for $T \ll 0$, and $F(t)$ is a vector-valued function satisfying

$$
\exp\left(\int_t^T \theta(s) ds \right) \chi_T F \in \mathcal{H}_F^{m-1,0},
$$

with $\theta(t) = \chi_T(t)(J + \Delta)/(-t)$ and $\Delta \in (0, 1/2)$. Applying Lemma 3.2 we deduce that

$$
\exp\left(\int_t^T \theta(s) ds \right) \chi_T Y_{J+1} \in \mathcal{H}_U^{m,0}.
$$

A similar estimate for $c(t)$ follows from the first equation (10.2). Thus, we have proved

Theorem 10.1 *Suppose the line $\Im\lambda = -\gamma_\nu$ contains only one eigenvalue λ_0 of C, and this eigenvalue is simple. Then any solution to the equation $DU - C(t)U = 0$, $t < T$, satisfying $\chi_T U \in \mathcal{H}_U^{m,\gamma}$, with $\gamma_{\nu-1} < \gamma < \gamma_\nu$, has the form*

$$
U(t) = t^p\, e^{i\lambda_0 t} \left(c_0 \sum_{\iota=0}^{J} U_\iota \frac{1}{t^\iota} + U_{J+1}(t) \right),
$$

where U_0 is an eigenvector of O_0 corresponding to the eigenvalue λ_0, Ψ_0 an eigenvector of O_0^{adj} corresponding to $\bar{\lambda}_0$, p is given by $\sqrt{-1}(O_1 U_0, \Psi_0)_{\mathcal{H}}$, and $\exp(\int_t^T \theta(s) ds)\, \chi_T U_{J+1} \in \mathcal{H}_U^{m,0}$.

11 Relative Index Theorem

Suppose A is an elliptic differential operator of order m on a manifold M with a corner v, acting as $H^{s,w,\gamma}(M) \to H^{s-m,w-m,\gamma-m}(M)$ where the weight $\gamma \in \mathbb{R}$ is related to v. From a Regularity Theorem it follows that the index of the operator A does not depend on s. In this section we study the dependence of the index on γ.

Choose a neighbourhood O of v on M such that $O \setminus \{v\}$ is diffeomorphic to $(0,\varepsilon) \times X$. Using a change of coordinates $t = \delta(r)$, we reduce the equation $Au = f$ near v to (1.1). The operator pencil $\sigma(\lambda)$ associated with the equation (1.1) is known to be just the conormal symbol of A at v. We write $\sigma_{\mathbf{F}}(A)(v,z)$ for it.

Our basic assumption is that the coefficients of A *stabilise* at the corner v, i.e., the coefficients $C_j(t)$ meet the conditions 1)-3) after Corollary 1.2. This just amounts to saying that the coefficients of A are *continuous* and *slowly varying* at v.

Theorem 11.1 *Assume that in the strip $-\mu < \Im z < -\gamma$ there lie N eigenvalues of the symbol $\sigma_{\mathbf{F}}(A)(v,z)$ (counting the multiplicities), and that there are no eigenvalues of this pencil on the lines $\Im\lambda = -\mu$ and $\Im\lambda = -\gamma$. Then any solution $u \in H^{s,w,\gamma}(M)$ of the equation $Au = f$ with $f \in H^{s-m,w-m,\mu-m}(M)$ has the form*

$$u = c_1\, u_1 + \ldots + c_N\, u_N + R$$

in a neighbourhood of v, where $u_1, \ldots, u_N$ are elements of the space $H^{s,w,\gamma}(M)$ with support near v which satisfy $Au = 0$ in a smaller neighbourhood of v and are linearly independent modulo $H^{s,w,\mu}(M)$; $c_1, \ldots, c_N$ are constants; and $R \in H^{s,w,\mu}(M)$.

Proof. Let ω be a C^∞ function on M with a support in a sufficiently small neighbourhood of v, such that $\omega \equiv 1$ near v. Then ωu satisfies the equation

$$A\,(\omega u) = F,$$

with $F = \omega f + [A,\omega]u$ and $[A,\omega] = A\omega - \omega A$ denoting the commutator of A and ω. By our assumption on ω, the function F is an element of the space $H^{s-m,w-m,\mu-m}(M)$ with a support in a neighbourhood of v. Passing to the coordinates $(t,x) \in (-\infty,T) \times X$ near v, where $t = \delta(r)$, we can apply Theorem 1.1. Thus, we get

$$\omega u = c_1\, u_1 + \ldots + c_N\, u_N + R \tag{11.1}$$

in a neighbourhood of v, where $u_1, \ldots, u_N$ are solutions of the homogeneous equation $Au_\nu = 0$, such that $\delta_* u_\nu$ are linearly independent modulo the space

$H^{s,\mu}(-\infty,T)$, and $\delta_* R \in H^{s,\mu}(-\infty,T)$. Without loss of generality we may assume that the supports of u_ν and R are contained in a small neighbourhood of v. Indeed, otherwise we multiply (11.1) by a function $\tilde{\omega} \in C^\infty_{\mathrm{comp}}[0,\varepsilon)$ equal to 1 on the support of ω, to obtain (11.1) with $\tilde{\omega} u_\nu$ and $\tilde{\omega} R$ instead of u_ν and R. Since $\delta_* (1-\tilde{\omega}) u_\nu$ lies in $H^{s,\mu}(-\infty,T)$, for each $\nu = 1,\ldots,N$, the functions $\delta_* \tilde{\omega} u_1, \ldots, \delta_* \tilde{\omega} u_N$ are also linearly independent modulo the space $H^{s,\mu}(-\infty,T)$. This completes the proof.

□

We emphasize that N is the sum of the algebraic multiplicities of all eigenvalues of $\sigma_{\mathbf{F}}(A)(v,z)$ lying in the strip $-\mu < \Im z < -\gamma$.

Corollary 11.2 *Suppose in the strip $-\mu < \Im\lambda < -\gamma$ there lie N eigenvalues of the conormal symbol of A at v (counting the multiplicities), and there are no eigenvalues of this symbol on the lines $\Im\lambda = -\mu$ and $\Im\lambda = -\gamma$. Then the equation $Au = 0$ has at most N solutions in $H^{s,w,\gamma}(M)$ which are linearly independent modulo $H^{s,w,\mu}(M)$.*

Proof. Pick a C^∞ function ω on M with a support in a small neighbourhood of v, such that $\omega \equiv 1$ near v. By Theorem 11.1, every solution $u \in H^{s,w,\gamma}(M)$ of $Au = 0$ admits a decomposition (11.1), where $u_1,\ldots,u_N$ satisfy $Au_\nu = 0$ near v and are linearly independent modulo the space $H^{s,w,\mu}(M)$, while $R \in H^{s,w,\mu}(M)$. Hence we obtain

$$u = c_1\, u_1 + \ldots + c_N\, u_N + \tilde{R},$$

where $\tilde{R} = R + (1-\omega)u$ is clearly an element of $H^{s,w,\mu}(M)$. This proves our assertion.

□

Let A_γ denote A mapping as $H^{s,w,\gamma}(M) \to H^{s-m,w-m,\gamma-m}(M)$. Furthermore, let $U_1,\ldots,U_Q \in H^{s,w,\gamma}(M)$ be a maximal set of solutions of the homogeneous equation $Au = 0$ which are linearly independent modulo the space $H^{s,w,\mu}(M)$. From Corollary 11.2 we deduce that $Q \le N$ and every solution $u \in H^{s,w,\gamma}(M)$ of $Au = 0$ fulfills

$$u = \sum_{q=1}^{Q} c_q\, U_q$$

modulo $H^{s,w,\mu}(M)$. The set $(U_1,\ldots,U_Q)$ is called a *basis* in $\ker A_\gamma$ modulo the space $H^{s,w,\mu}(M)$.

As was shown in the proof of Corollary 11.2, each element U_q of this basis meets

$$U_q = \sum_{\nu=1}^{N} c_{q\nu}\, u_\nu$$

modulo $H^{s,w,\mu}(M)$, where $u_1, \ldots, u_N \in H^{s,w,\gamma}(M)$ satisfy the homogeneous equation near v, and $(c_{q\nu})$ is a $(Q \times N)$-matrix of the rank Q. There is no loss of generality in assuming that $(c_{q\nu})$ has the form (T_1, T_2), with T_1 a nondegenerate $(Q \times Q)$-matrix. Hence, we get

$$T_1^{-1} U = \left(\mathrm{Id}_Q, T_1^{-1} T_2\right) u$$

modulo $H^{s,w,\mu}(M)$, where Id_Q is the identity $(Q \times Q)$-matrix, and U, u denote the columns with entries U_q and u_ν, respectively. Thus, we can assume without loss of generality that

$$U_q = u_q + \sum_{\nu=Q+1}^{N} c_{q\nu}\, u_\nu \tag{11.2}$$

modulo $H^{s,w,\mu}(M)$.

Any basis $(U_1, \ldots, U_Q)$ in $\ker A_\gamma$ modulo the space $H^{s,w,\mu}(M)$, which has the form (11.2) is called *canonical.* From now on we assume that some canonical basis is given.

Lemma 11.3 *Assume that in the strip $-\mu < \Im \lambda < -\gamma$ there lie N eigenvalues of $\sigma_{\mathbf{F}}(A)(v, z)$ (counting the multiplicities), and there are no eigenvalues of this symbol on the lines $\Im \lambda = -\mu$ and $\Im \lambda = -\gamma$. Let Q be the maximal number of solutions of $Au = 0$ in $H^{s,w,\gamma}(M)$ which are linearly independent modulo the space $H^{s,w,\mu}(M)$. Then the equation $A^* g = 0$ has exactly $N - Q$ solutions in $H^{m-s,m-w,m-\mu}(M)$ which are linearly independent modulo the space $H^{m-s,m-w,m-\gamma}(M)$.*

Proof. Let $G_1, \ldots, G_J \in H^{m-s,m-w,m-\mu}(M)$ be a basis in $\ker A_\mu^*$ modulo the space $H^{m-s,m-w,m-\gamma}(M)$.

We first show that $J + Q \le N$. Obviously, there exists a system $f_1, \ldots, f_J$ in the space $H^{s-m,w-m,\mu-m}(M)$, such that

$$\begin{array}{rcll} (f_i, G_j) & = & \delta_{i,j} & \text{for all} \quad i, j = 1, \ldots, J; \\ (f_i, G) & = & 0 & \text{for all} \quad G \in \ker A_\gamma^*, \end{array}$$

where $(\cdot, \cdot)$ denotes the scalar product in $H^{0,0,0}(M)$, and $\delta_{i,j}$ the Kronecker delta. Since A_γ is a Fredholm operator, the latter condition implies the existence of solutions $\tilde{u}_i \in H^{s,w,\gamma}(M)$ of the equation $A \tilde{u}_i = f_i$, for $i = 1, \ldots, J$. Suppose there is a linear combination

$$u = \sum_{i=1}^{J} \tilde{c}_i\, \tilde{u}_i + \sum_{q=1}^{Q} c_q\, U_q$$

of $\tilde{u}_1, \ldots, \tilde{u}_J$ and $U_1, \ldots, U_Q$, which belongs to the space $H^{s,w,\mu}(M)$. Then we obtain

$$\begin{aligned} 0 &= (u, A^* G_j) \\ &= (Au, G_j) \\ &= \sum_{i=1}^{J} \tilde{c}_i \, (f_i, G_j) + \sum_{q=1}^{Q} c_q \, (AU_q, G_j) \\ &= \tilde{c}_j \end{aligned}$$

for $j = 1, \ldots, J$. Moreover, we have $c_1 = \ldots = c_Q = 0$, for the elements $G_1, \ldots, G_Q$ are linearly independent modulo the space $H^{s,w,\mu}(M)$. On the other hand, Theorem 11.1 shows that any element $\tilde{u}_1, \ldots, \tilde{u}_J$ and $U_1, \ldots, U_Q$ is a linear combination of $u_1, \ldots, u_N$ modulo $H^{s,w,\mu}(M)$. Hence it follows that $J + Q \leq N$, as desired.

We now suppose that $J < N - Q$. Since $Au_\nu = 0$ in a neighbourhood of the corner v, we can define the moments

$$m_{j\nu} = (Au_\nu, G_j)\,,$$

for $j = 1, \ldots, J$ and $\nu = Q+1, \ldots, N$. Pick a non-zero solution $(c_{Q+1}, \ldots, c_N)$ in $\mathbb{C}^{N-Q}$ of the linear system

$$\sum_{\nu=Q+1}^{N} m_{j\nu}\, c_\nu = 0, \quad j = 1, \ldots, J.$$

Then, $u = c_{Q+1}\, u_{Q+1} + \ldots + c_N\, u_N$ satisfies

$$(Au, G_j) = 0, \quad \text{for each} \quad j = 1, \ldots, J,$$

whence $(Au, G) = 0$ for all $G \in \ker A_\mu^*$. Since A_μ is a Fredholm operator, we conclude that Au belongs to the range of A_μ. This means that there is an element $\tilde{u} \in H^{s,w,\mu}(M)$ such that $U_{Q+1} = u - \tilde{u}$ is a solution of the equation $AU_{Q+1} = 0$. However, then $U_1, \ldots, U_{Q+1}$ form a system of $Q+1$ solutions of the homogeneous equation on M which are, by (11.2), linearly independent modulo $H^{s,w,\mu}(M)$. This contradicts the maximality of $U_1, \ldots, U_Q$, which completes the proof.

$\square$

We have used only the fact that the *transpose* A' of A acting from $H^{s,w,\gamma}(M)$ to $H^{s-m,w-m,\gamma-m}(M)$ can be specified as the *formal adjoint* A^* with respect to the non-degenerate sesquilinear pairing $H^{s,w,\gamma}(M) \times H^{-s,-w,-\gamma}(M) \to \mathbb{C}$ induced by the scalar product in $H^{0,0,0}(M)$. To prove this it suffices to observe that the spaces $H^{s,w,\gamma}(M)$ are defined as completions of C^∞ functions with compact supports on the smooth part of M with respect to appropriate weighted norms.

Lemma 11.3 leads to the following interesting consequence for the index of the operator A.

Theorem 11.4 *Suppose in the strip $-\mu < \Im\lambda < -\gamma$ there lie N eigenvalues of the conormal symbol of A at v (counting the multiplicities), and there are no eigenvalues of this symbol on the lines $\Im\lambda = -\mu$ and $\Im\lambda = -\gamma$. Then the difference of the indices of A evaluated on $H^{s,w,\gamma}(M)$ and $H^{s,w,\mu}(M)$ is equal to N.*

Proof. Indeed, denote A_γ the operator $H^{s,w,\gamma}(M) \to H^{s-m,w-m,\gamma-m}(M)$ induced by A. Then

$$\begin{aligned} \text{ind } A_\gamma &= \dim\ker A_\gamma - \dim\ker A_\gamma^*, \\ \text{ind } A_\mu &= \dim\ker A_\mu - \dim\ker A_\mu^*. \end{aligned}$$

Set

$$Q = \dim\ker A_\gamma - \dim\ker A_\mu,$$

then from Lemma 11.3 it follows that

$$\dim\ker A_\mu^* = \dim\ker A_\gamma^* + (N - Q).$$

Thus,

$$\begin{aligned} \text{ind } A_\gamma &= (\dim\ker A_\mu + Q) - \big(\dim\ker A_\mu^* - (N-Q)\big) \\ &= \text{ind } A_\mu + N \end{aligned}$$

showing the theorem.

□

Theorem 11.4 is actually a direct consequence of the *Structure Theorem* 11.1 and the following abstract result of functional analysis. Let $A\colon H \to \tilde{H}$ be a Fredholm mapping of Hilbert spaces which restricts to a Fredholm mapping $A_\Sigma\colon \Sigma \to \tilde{\Sigma}$, both embeddings $\Sigma \hookrightarrow H$ and $\tilde{\Sigma} \hookrightarrow \tilde{H}$ being continuous. Suppose there are $u_1, \dots, u_N \in H$ linearly independent modulo Σ, such that $Au_\nu \in \tilde{\Sigma}$, for $\nu = 1, \dots, N$, and every $u \in H$ with $Au \in \tilde{\Sigma}$ can be written in the form $u = \sum_{\nu=1}^N c_\nu u_\nu + R$, where $c_1, \dots, c_N$ are constants and $R \in \Sigma$. Then, $\text{ind } A = \text{ind } A_\Sigma + N$.

Theorem 11.4 seems to be new even for manifolds with cusps, i.e., when the link X has no singularities. However, in this case it can be easily derived from the Relative Index Theorem for manifolds with conical singularities. Indeed, every elliptic operator on a manifold with cusps can be continuously deformed through elliptic operators to an operator whose "coefficients" are independent of r close to every cuspidal point. They are in fact equal to the "coefficients" of the original operator, frozen at the singular point. Such operators survive under arbitrary homeomorphisms of a small neighbourhood of a cusp, which are C^∞ away from the cusp. Hence changing the variables by $s = \exp(\delta(r))$ reduces the operators to those of the Fuchs type, while the index remains the same.

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