Elliptic Operators in Subspaces

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Abstract

We construct elliptic theory in the subspaces, determined by pseudodifferential projections. The finiteness theorem as well as index formula are obtained for elliptic operators acting in the subspaces. Topological (K-theoretic) aspects of the theory are studied in detail.

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Contents

Introduction

In the present paper we construct the theory of elliptic operators acting in subspaces determined by pseudodifferential projections.

The classical elliptic theory deals with elliptic operators in Sobolev spaces. The main results are the Fredholm property of elliptic operators (on a closed manifold or a manifold with boundary) and an index formula for such problems. Unfortunately, for a number of applications the theory of elliptic operators in Sobolev spaces is not sufficient. Let us consider an example. In the theory of boundary value problems, an elliptic operator may not admit well-posed (Fredholm) boundary value problems in the Sobolev spaces. In particular, this is the case for some fundamental geometric operators (Dirac and Hirzebruch operators, etc.). Indeed, Atiyah and Bott [2] pointed out that an operator admits a Fredholm boundary value problem if and only if its principal symbol satisfies a certain condition. The above-mentioned geometric operators violate this condition. Thus, the following question arises: is it possible to construct an elliptic theory for operators violating the Atiyah–Bott condition? The answer is "yes". Namely, it turns out that an arbitrary elliptic operator admits a well-posed boundary value problem in Sobolev subspaces, rather than spaces. This is due to the absence of the Fredholm property in Sobolev spaces for non-Atiyah–Bott operators (the cokernel is infinite-dimensional). If we take the boundary values in a subspace of the Sobolev space, then the corresponding problem becomes well-posed [20], [16] (see also [6]). Boundary value problems for general elliptic pseudodifferential operators were constructed in a recent paper [19] in the framework of Boutet de Monvel–type algebra. This observation naturally leads to the construction of the theory of elliptic operators in *subspaces* of Sobolev spaces.

It turns out that such class of subspaces is related to *pseudodifferential projections*, and it is possible to prove the finiteness theorem on a compact manifold with or without boundary and carry out the index computation. We point out that to compute the index of an elliptic operator in subspaces, it is necessary to consider not only its principal symbol but also the subspaces where the operator acts. Earlier, the authors introduced a numerical functional defined on the set of subspaces subject to the so-called parity conditions for the sake of index computation. This functional takes dyadic values. We proved a simple and elegant index formula [17, 18] in terms of this functional. The functional is closely related to the η-invariant [1] of elliptic operators. Namely, under the parity conditions, the η-invariant of an operator A coincides with the value of the functional on the nonnegative spectral subspace of this operator. This equality and the above-mentioned index formula allowed solving an important problem, posed by P.Gilkey, of computing the fractional part of the η-invariant in topological terms [15]. The solution to this problem is of theoretical interest and also has important applications in different areas of algebraic topology, for example, in the theory of *pin* bordisms.

The paper is organized as follows. In the first section, we discuss subspaces defined by pseudodifferential projections, introduce symbols of subspaces and present examples. In the second section, we consider operators in such subspaces and introduce the notion of ellipticity. Here we prove the Fredholm property for elliptic operators and discuss properties of the index of elliptic operators in subspaces. *Even* and *odd* subspaces are introduced in Section 3. We show that these are related to spectral subspaces of *differential* self-adjoint operators of even or odd order, respectively. The dimension functional is defined for even and odd subspaces. The index theorem for elliptic operators in subspaces is obtained in Section 5. We conclude the paper with an appendix, where the relevant computations in K-theory are given.

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1 Subspaces, determined by pseudodifferential projections

Let M be a smooth manifold and E a vector bundle on M.

Definition 1 A linear subspace $\widehat{L} \subset C^\infty(M, E)$ is called *pseudodifferential* if there is a pseudodifferential projection

$$P : C^\infty(M, E) \to C^\infty(M, E)$$

of order zero that defines this subspace

$$\widehat{L} = \operatorname{Im} P.$$

Definition 2 The *symbol* of a pseudodifferential subspace is the subbundle $L \subset \pi^* E$, $\pi : S^* M \to M$, determined by the projection P according to the formula

$$L = \operatorname{Im} \sigma(P) \subset \pi^* E \in \operatorname{Vect}(S^* M).$$

The symbol of a subspace is a smooth vector bundle independent of the choice of projection P on $\widehat{L}$.

Proposition 1 *An arbitrary smooth subbundle $L \subset \pi^* E$ is the symbol for some subspace $\widehat{L} \subset C^\infty(M, E)$.*

Proof. We prove the proposition by constructing a pseudodifferential projection with the principal symbol projecting on the bundle L.

Denote by $\sigma(P)$ the orthogonal projection onto the subbundle L with respect to a metric on E. Let P_0 be a self-adjoint pseudodifferential operator of order zero with the principal symbol $\sigma(P)$. The desired projection P can be defined via the Cauchy-type integral

$$P = \frac{i}{2\pi} \int\limits_{|\lambda - 1| = \varepsilon} (P_0 - \lambda I)^{-1} \, d\lambda. \tag{1}$$

It is assumed that the number ε in (1) is chosen such that the circle $|\lambda - 1| = \varepsilon$, $0 < \varepsilon < 1$ contains no eigenvalues of the operator P_0. This proves the proposition.

Example 1 Consider the Hardy space

$$\widehat{L} \subset C^\infty(S^1)$$

of boundary values of functions holomorphic in the unit disk

$$D \subset C, \quad \partial D = S^1.$$

A Fourier series calculation shows that the Hardy space coincides with the spectral subspace of the elliptic self-adjoint operator $A = -id/d\varphi$ on the circle S^1, corresponding to nonnegative eigenvalues. The projection on this subspace is given by the formula

$$Pf(z) = \frac{1}{2\pi i} \lim_{r \to 1-0} \int \frac{f(\zeta)\, d\zeta}{\zeta - rz}.$$

Its principal symbol equals (e.g., see [14], Chapter 16)

$$\sigma(P)(\varphi, \xi) = \begin{cases} 1, & \xi = 1, \\ 0, & \xi = -1. \end{cases}$$

Hence, the symbol of the Hardy space is equal to

$$L_{\varphi,\xi} = \operatorname{Im} \sigma(P)(\varphi, \xi) = \begin{cases} C, & \xi = 1, \\ 0, & \xi = -1. \end{cases}$$

The Hardy space is the spectral subspace of a self-adjoint operator. We show in the next proposition that in the general case the spectral subspaces are pseudodifferential.

Proposition 2 *Let A be an elliptic self-adjoint operator of nonnegative order on the manifold M. Then the subspace $\widehat{L}_+(A)$, generated by the eigenvectors of A, corresponding to nonnegative eigenvalues, is pseudodifferential and its symbol $L_+(A)$ is equal to the nonnegative spectral subbundle, corresponding to the principal symbol $\sigma(A)$:*

$$L_+(A) = L_+(\sigma(A)).$$

Proof. It can be assumed that A is invertible (otherwise we replace it by $A+\varepsilon$; for ε sufficiently small, the spectral subspace remains the same). In this case the spectral projection P_+ onto $\widehat{L}_+(A)$ has the form

$$P_+ = \frac{|A| + A}{2|A|}, \quad |A| = \sqrt{A^2}.$$

By virtue of the work of Seeley [21], the operator $|A|$ is pseudodifferential. Consequently, the projection P_+ is pseudodifferential with the principal symbol equal to

$$\sigma(P_+) = \frac{|\sigma(A)| + \sigma(A)}{2|\sigma(A)|},$$

i.e. $\sigma(P_+)$ projects on the nonnegative spectral subbundle of the principal symbol of A, as desired.

Proposition 3 *The orthogonal complement of a pseudodifferential subspace is also pseudodifferential.*

Proof. Denote by P an arbitrary pseudodifferential projection on $\widehat{L}$. Then the following equality holds

$$\widehat{L} = \widehat{L}_+(2PP^* - Id).$$

To conclude the proof, it remains to apply Proposition 4 to the self-adjoint operator $2PP^* - Id$.

We refer the reader to the paper [5][1] for other methods of defining pseudodifferential subspaces.

2 Elliptic operators in subspaces

Consider two pseudodifferential subspaces $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$ and a pseudodifferential operator

$$D : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_2),$$

of order m, acting in the ambient spaces. Suppose also that this operator respects the subspaces:

$$D\widehat{L}_1 \subset \widehat{L}_2. \qquad (2)$$

[1] In this paper the authors call pseudodifferential subspaces "admissible".

Definition 3 The restriction of D

$$D : \widehat{L}_1 \longrightarrow \widehat{L}_2 \tag{3}$$

is called *operator of order m, acting in subspaces.*

The restriction of the principal symbol $\sigma(D)$ to the subbundle L_1 defines the homomorphism

$$\sigma(D) : L_1 \longrightarrow L_2 \tag{4}$$

of bundles over S^*M. Indeed, in terms of the projections $P_{1,2}$ on $\widehat{L}_{1,2}$ condition (2) takes the form $P_2 D P_1 = D P_1$. A similar equality for the principal symbols yields (4).

Definition 4 Homomorphism (4) is called the *principal symbol of operator in subspaces.*

Proposition 4 *An arbitrary bundle homomorphism* (4) *is the principal symbol for some operator in subspaces.*

Proof. Consider an arbitrary pseudodifferential operator

$$D_0 : C^\infty(M, E_1) \to C^\infty(M, E_2)$$

with the principal symbol $\sigma(D_0)$ equal to $\sigma(D)$ on $L \subset \pi^* E_1$ and arbitrary on the complementary subbundle. The operator

$$D = P_2 D_0, \quad \text{for} \quad \widehat{L}_2 = \operatorname{Im} P_2$$

is the desired pseudodifferential operator in subspaces $\widehat{L}_1, \widehat{L}_2$ with symbol (4).

Definition 5 An operator in subspaces is called *elliptic*, if its symbol (4) is a vector bundle isomorphism.

Theorem 1 *Elliptic operator D of order m in subspaces*

$$D : H^s(M, E_1) \supset \widehat{L}_1 \longrightarrow \widehat{L}_2 \subset H^{s-m}(M, E_2) \tag{5}$$

has the Fredholm property as an operator acting in the closures of the subspaces $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$ with respect to the Sobolev norm.

Proof (cf. [5, 9]). Let us construct the inverse (up to compact operators) of the initial (5). To this end, consider the symbol

$$\sigma(D)^{-1} : L_2 \longrightarrow L_1.$$

By virtue of Proposition 4, we construct an operator

$$D' : \widehat{L}_2 \longrightarrow \widehat{L}_1$$

with this symbol $\sigma(D') = \sigma(D)^{-1}$. Thus, we obtain

$$D'D = 1 + K_1 : \widehat{L}_1 \longrightarrow \widehat{L}_1, \quad DD' = 1 + K_2 : \widehat{L}_2 \longrightarrow \widehat{L}_2,$$

where $K_{1,2}$ denote compact operators. Therefore, D' is a regularizer of D. Hence, the operator D has the Fredholm property. This completes the proof of the theorem.

The homotopy invariance of the index of elliptic operators in subspaces is valid for the homotopies of the operator D as well as the homotopies of the subspaces $\widehat{L}_{1,2}$, where the operator acts.

Proposition 5 *Let*

$$D_t : \operatorname{Im} P_t \to \operatorname{Im} Q_t$$

be a smooth family of elliptic operators in subspaces (i.e. a family of operators $\{D_t\}$ and families of projections $\{P_t\}, \{Q_t\}$ which are smooth in the operator norm). Then the index of operators D_t remains constant

$$\operatorname{ind}(D_t, \operatorname{Im} P_t, \operatorname{Im} Q_t) = \text{Const}.$$

Proof. The proof is based on a reduction of the family D_t with varying subspaces to a family in fixed subspaces. We start with the following Lemma.

Lemma 1 *The projections P_t in a smooth one-parameter family of projections $\{P_t\}$ are equivalent. More precisely, there is a family of invertible operators $\{U_t\}$ such that*

$$P_t = U_t P_0 U_t^{-1}.$$

Proof of Lemma 1. The family U_t is defined as the solution of the Cauchy problem

$$\dot{U}_t = \left[\dot{P}_t, P_t\right] U_t, \quad U_0 = Id.$$

The desired property of the family is verified directly.

Proof of Proposition 5. Denote by U_t and V_t the families of invertibles corresponding to the families of projection P_t and Q_t. In this way

$$D_t : \operatorname{Im} P_t \to \operatorname{Im} Q_t$$

is equivalent to the operator

$$V_t^{-1} D_t U_t : \operatorname{Im} P_0 \to \operatorname{Im} Q_0$$

in the subspaces independent of the parameter t. The usual homotopy invariance of the index shows that the index of operators $V_t^{-1} D_t U_t$ remains the same. Consequently, the index of D_t is also constant.

Remark 1 The index of elliptic operators in subspaces *is not determined* by the principal symbol.

Indeed, consider a pair of finite-dimensional vector spaces $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$ and an arbitrary operator $D : \widehat{L}_1 \to \widehat{L}_2$. The principal symbol of D is zero

$$\pi^* E_1 \supset 0 \to 0 \subset \pi^* E_2,$$

while the index is

$$\operatorname{ind}\left(D, \widehat{L}_1, \widehat{L}_2\right) = \dim \widehat{L}_1 - \dim \widehat{L}_2$$

and it can take arbitrary values.

3 Subspaces with parity conditions

1. On the cotangent bundle T^*M of the manifold M, consider the involution

$$\alpha : T^*M \longrightarrow T^*M, \quad \alpha(x, \xi) = (x, -\xi).$$

Definition 6 A subspace $\widehat{L} \subset C^\infty(M, E)$ is called *even (odd)* with respect to the involution α, if its principal symbol L is invariant (antiinvariant) under the involution:

$$L = \alpha^* L, \quad (L \oplus \alpha^* L = \pi^* E). \tag{6}$$

Here both equalities are understood as coincidences of subbundles in the ambient $\pi^* E$. Any bundle L satisfying (6) is called *even (odd) bundle.*

Proposition 6 *Let A be a differential operator satisfying conditions of Proposition 2. Then the spectral subspace $\widehat{L}_+(A)$ is even or odd according to the parity of order of A.*

Proof. The principal symbol of a differential operator satisfies the equality

$$\alpha^* \sigma(A) = (-1)^{\operatorname{ord} A} \sigma(A).$$

Hence, by Proposition 2 we obtain the desired:

$$L_+(A) = \alpha^* L_+(A), \quad \text{or} \quad L_+(A) \oplus \alpha^* L_+(A) = \pi^* E.$$

Example 2 The Hardy space from Example 1 is odd.

Example 3 The space of closed differential forms

$$\widehat{L} = \ker d, \quad d : \Lambda^k(M) \to \Lambda^{k+1}(M)$$

as well as the space of exact forms

$$\widehat{L}' = \operatorname{Im} d, \quad d : \Lambda^{k-1}(M) \to \Lambda^k(M)$$

on a compact closed manifold are even subspaces with coinciding principal symbols. Indeed, $\widehat{L}$ is the spectral subspace of an elliptic self-adjoint operator of the second order

$$d\delta - \delta d, \quad \widehat{L} = \widehat{L}_+(d\delta - \delta d),$$

(where δ is the adjoint of the exterior differential d), while the subspace $\widehat{L}'$ differs from $\widehat{L}$ by a finite-dimensional subspace

$$\widehat{L} = \widehat{L}' \oplus (\ker d \cap \ker \delta)$$

by de Rham's theorem.

Ranks of even bundles can take arbitrary values. However, the rank of an odd bundle can not be arbitrary.

Remark 2 The dimension of an odd bundle $L \subset \pi^* E$ on a manifold M of dimension n satisfies the relation:

$$\left.\begin{array}{l} n-1 = 2k \\ n-1 = 2k+1 \end{array}\right\} \Rightarrow \dim L \ \text{ is a multiple of } 2^{k-1}. \tag{7}$$

Indeed, denote by

$$p_L : \pi^* E \to \pi^* E$$

the projection on L along the complementary subbundle $\alpha^* L$. The matrix-valued function

$$2p_L - 1$$

is an odd function on the sphere S^{n-1}

$$2p_L(-\xi) - 1 = 1 - 2p_L(\xi).$$

The conclusion (7) follows from the paper [10], where it is shown that an odd function on the sphere with values in invertible matrices satisfies (7).

Denote by $\widehat{\mathrm{Even}}(M)$ ($\widehat{\mathrm{Odd}}(M)$) abelian semigroups of even (odd) subspaces with respect to direct sums.

Definition 7 Two subspaces $\widehat{L}_1$ and $\widehat{L}_2$ are called *stably homotopic*, if direct sums $\widehat{L}_1 \oplus \widehat{L}_3$ and $\widehat{L}_2 \oplus \widehat{L}_3$ are homotopic for some pseudodifferential subspace $\widehat{L}_3$.

Consider the semigroups of classes of stably homotopic even or odd subspaces. The corresponding Grothendieck groups are denoted by $K\left(\widehat{\mathrm{Even}}(M)\right)$ and $K\left(\widehat{\mathrm{Odd}}(M)\right)$.

2. For even and odd subspaces, we define a functional, which is a generalization of dimension for finite-dimensional vector spaces. To this end, we define homomorphisms

$$Z \xrightarrow{i} K\left(\widehat{\mathrm{Even}}(M)\right), \quad Z \xrightarrow{i} K\left(\widehat{\mathrm{Odd}}(M)\right). \tag{8}$$

The former maps a natural number n to a finite-dimensional vector space of the same dimension (this subspace is even). For n negative, we put $i(n) = -i(-n)$.

We define i in the case of odd subspaces by the formula

$$i(n) = \left[\widehat{L} + n\right] - \left[\widehat{L}\right], \tag{9}$$

where $\widehat{L} \subset C^\infty(M, E)$ is an arbitrary odd subspace, while $\widehat{L} + n \subset C^\infty(M, E)$ is the sum of $\widehat{L}$ and an n-dimensional subspace in its complement. The difference (9) is independent of the choice of $\widehat{L}$.

Definition 8 Any additive functional

$$d : K\left(\widehat{\mathrm{Even}}\,(M)\right) \text{ or } K\left(\widehat{\mathrm{Odd}}\,(M)\right) \longrightarrow R,$$

of pseudodifferential subspaces is called *dimension functional,* if it satisfies the condition

$$d\,(i\,(n)) = n. \tag{10}$$

For even subspaces $\widehat{\mathrm{Even}}\left(M^{odd}\right)$ on odd-dimensional manifolds and odd subspaces $\widehat{\mathrm{Odd}}\,(M^{ev})$ on even-dimensional manifolds the following theorem was proved in [17, 18].

Theorem 2 *There is a unique dimension functional*

$$\begin{aligned} d &: K\left(\widehat{\mathrm{Even}}\left(M^{odd}\right)\right) \to \mathbf{R}, \\ d &: K\left(\widehat{\mathrm{Odd}}\,(M^{ev})\right) \longrightarrow R, \end{aligned}$$

which satisfies the complement property

$$d\left(\widehat{L}\right) + d\left(\widehat{L}^{\perp}\right) = 0. \tag{11}$$

Here $\widehat{L}^{\perp}$ denotes the orthogonal complement of $\widehat{L}$.

Proof.

A. Let us first prove the theorem for even subspaces.

The symbol of an even subspace $\widehat{L}$ is a vector bundle on the cosphere space S^*M. This bundle is invariant under the involution α. Hence, the following mapping is defined

$$\begin{array}{ccc} K\left(\widehat{\mathrm{Even}}\left(M^{odd}\right)\right) & \stackrel{j}{\longrightarrow} & K\left(P^*M\right), \\ \left[\widehat{L}\right] & \mapsto & [L]\,, \end{array} \tag{12}$$

here P^*M denotes the projectivization S^*M/α of the cosphere bundle with respect to the involution α. The following proposition is the main ingredient of the proof.

Proposition 7 *There is an exact sequence of groups*

$$0 \longrightarrow Z \stackrel{i}{\longrightarrow} K\left(\widehat{\mathrm{Even}}\left(M^{odd}\right)\right) \stackrel{j}{\longrightarrow} K\left(P^*M\right) \longrightarrow 0. \tag{13}$$

Proof of Proposition 7.

1) The mapping j is epimorphic, since an arbitrary vector bundle $\gamma \in \mathrm{Vect}\,(P^*M)$ can be realized as an even subbundle

$$\pi^*\gamma \subset C^N \in \mathrm{Vect}\,(S^*M)\,, \quad \pi : S^*M \to P^*M.$$

Therefore, there is an even subspace $\widehat{L}$ with symbol L. Consequently,

$$j\left[\widehat{L}\right] = [\gamma]\,.$$

2) Now we prove the exactness of the sequence in the second term: $\ker j \subset \operatorname{Im} i$. Suppose that

$$j\left(\left[\widehat{L}\right] - \left[C^\infty(M,F)\right]\right) = 0 \tag{14}$$

for an even subspace $\widehat{L} \subset C^\infty(M,E)$. Let us show that the subspace $\widehat{L}$ is homotopic to the space of sections $C^\infty(M,F)$, up to a finite-dimensional defect.

Condition (14) yields a vector bundle isomorphism

$$L \xrightarrow{\sigma} \pi^* F, \quad \pi : S^*M \to M,$$

which is even with respect to the cotangent variables $\sigma(x,-\xi) = \sigma(x,\xi)$. Consider an elliptic operator in (sub)spaces

$$\widehat{L} \xrightarrow{\widehat{\sigma}} C^\infty(M,F).$$

We may assume the invertibility of $\widehat{\sigma}$. Otherwise, we can modify $\widehat{L}$ by a finite-dimensional subspace. Let us define the homotopy connecting

$$\widehat{L} = \operatorname{Im} P \subset C^\infty(M, E\oplus F) \text{ to } C^\infty(M,F) = \operatorname{Im} P_F \subset C^\infty(M, E\oplus F),$$

by virtue of the homotopy of projections

$$\begin{aligned} P_\varphi &: C^\infty(M,E\oplus F) \to C^\infty(M,E\oplus F), \\ P_\varphi &= P\cos^2\varphi + P_F \sin^2\varphi + \cos\varphi\sin\varphi\left(P\widehat{\sigma}^{-1}P_F + P_F\widehat{\sigma}P\right). \end{aligned}$$

The subspaces $\widehat{L}_\varphi = \operatorname{Im} P_\varphi$ rotate during the homotopy from $\widehat{L}_0 = \widehat{L}$ towards $\widehat{L}_{\pi/2} = C^\infty(M,F)$ with the help of the isomorphism $\widehat{\sigma}$. This establishes the exactness of the sequence in the second term.

3) We verify, finally, that the mapping

$$i : Z \longrightarrow K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right)$$

on an odd-dimensional manifold is injective. Consider the converse $i(n) = 0$ for an $n > 0$. By the definition of the Grothendieck group $K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right)$, it follows that for some even subspace $\widehat{L}$ there is a homotopy of even subspaces

$$\widehat{L} \quad \text{and} \quad \widehat{L} + n,$$

where n denotes an n-dimensional subspace in the complement of $\widehat{L}$. Consider the corresponding homotopy of orthogonal projections P_t

$$\operatorname{Im} P_0 = \widehat{L}, \quad \operatorname{Im} P_1 = \widehat{L} + n.$$

By virtue of Lemma 1, the family of projections defines the family of invertibles U_t

$$P_t = U_t P_0 U_t^{-1}. \tag{15}$$

The principal symbols of U_t are even, since this property is valid for the projections P_t. Consider the following elliptic operator in subspaces

$$P_1 : \operatorname{Im} P_0 \longrightarrow \operatorname{Im} P_1.$$

Its index is, obviously, $-n$. On the other hand, substituting (15) in the last formula, we obtain an equivalent elliptic endomorphism

$$P_0 U_1^{-1} : \operatorname{Im} P_0 \longrightarrow \operatorname{Im} P_0.$$

This operator is analogous to generalized Toeplitz operators (e.g. see [4]). The index computation for endomorphisms in subspaces reduces to the index computation of the usual elliptic operators by the algebraic formula

$$\operatorname{ind}\left(P_0 U_1^{-1}, \operatorname{Im} P_0, \operatorname{Im} P_0\right) = \operatorname{ind}\left(P_0 U_1^{-1} P_0 + (1 - P_0)\right).$$

However, the operator $P_0 U_1^{-1} P_0 + (1 - P_0)$ has index zero, since it is defined on an odd-dimensional manifold and its principal symbol is even. Thus, we obtain the desired

$$n = \operatorname{ind}(P_1 : \operatorname{Im} P_0 \longrightarrow \operatorname{Im} P_1) = \operatorname{ind}\left(P_0 U_1^{-1} P_0 + (1 - P_0)\right) = 0.$$

This completes the proof of Proposition 7.

The complement property (11) implies that the dimension functional is zero on the space of vector bundle sections

$$d\left(C^\infty(M, E)\right) = 0.$$

The groups $K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right), K\left(P^*M\right),$ contain the subgroups generated by vector bundles:

$$\begin{array}{ccccccc} K(M) & \xrightarrow{p^*} & K(P^*M), & \qquad & K(M) & \xrightarrow{\alpha} & K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right), \\ & p : P^*M \to M & & & [E] & \mapsto & \left[C^\infty(M, E)\right]. \end{array}$$

These mappings are injective: the inverse of p^* can be defined by means of a nonsingular vector field on odd-dimensional M. Therefore, the following exact sequence is valid

$$0 \to Z \overset{i}{\to} K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right)/K(M) \overset{j}{\to} K(P^*M)/K(M) \to 0. \qquad (16)$$

The quotient $K(P^*M)/K(M)$ is a torsion group (see Appendix, where we show that the orders of its elements are powers of two). The tensor product of the sequence (16) with the ring of dyadic numbers $Z\left[\frac{1}{2}\right]$ gives an isomorphism

$$Z\left[\frac{1}{2}\right] \overset{i\otimes 1}{\to} \left\{K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right)/K(M)\right\} \otimes Z\left[\frac{1}{2}\right]. \qquad (17)$$

Hence, the dimension functional

$$d : K\left(\widehat{\operatorname{Even}}\left(M^{odd}\right)\right) \longrightarrow R,$$

satisfying the complement property, is unique and equal to the inverse mapping

$$d = (i \otimes 1)^{-1} : K\left(\widehat{\mathrm{Even}}\left(M^{odd}\right)\right) \to Z\left[\frac{1}{2}\right]$$

of the embedding i.

B. In the odd case we define the mapping [2]

$$\begin{array}{ccc} K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) & \xrightarrow{j} & K(M), \\ \left[\widehat{L}\right], \widehat{L} \subset C^{\infty}(M, E) & \mapsto & [E]. \end{array} \tag{18}$$

Proposition 8 *The sequence*

$$0 \longrightarrow Z \xrightarrow{i} K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \xrightarrow{j} K(M) \longrightarrow 0 \tag{19}$$

is exact modulo 2*-torsion. More precisely,* $\ker i = 0$, *while the groups* $\ker j / \operatorname{Im} i$, $\operatorname{coker} j = K(M) / \operatorname{Im} j$ *consist of elements of orders equal to powers of two.*

Proof of Proposition 8.

1) Let us verify that the cokernel of the mapping

$$j: K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \longrightarrow K(M)$$

is a 2-torsion group. To this end, it suffices to prove that an arbitrary bundle $E \in \mathrm{Vect}(M)$, for a sufficiently large number N, has a decomposition into the sum of two odd subspaces

$$C^{\infty}\left(M, 2^N E\right) = \widehat{L} \oplus \widehat{L}^{\perp}.$$

We start with the case of a trivial bundle.

We embed M into Euclidean space of some dimension N. The cotangent bundle embeds into the trivial bundle

$$T^*M \oplus \nu = R^N, \tag{20}$$

where ν denotes the normal bundle to M in R^N. Let $\mathrm{Cl}\left(C^N\right)$ be the Clifford algebra of the space C^N (e.g., see [13]). Then C^N as well as R^N act on the space $\mathrm{Cl}\left(C^N\right)$, $\dim \mathrm{Cl}\left(C^N\right) = 2^N$ by Clifford multiplication. Moreover, embedding (20) defines the vector bundle homomorphism

$$T^*M \xrightarrow{cl} \mathrm{End}\left(\mathrm{Cl}\left(C^N\right)\right),$$

induced by the Clifford multiplication. Consider the covariant differential $\nabla = d \otimes 1_{2^N}$ in the trivial bundle with fibre $\mathrm{Cl}\left(C^N\right)$. Let us define the self-adjoint Dirac-type operator

$$A = C^{\infty}\left(M, \mathrm{Cl}\left(C^N\right)\right) \xrightarrow{\nabla} C^{\infty}\left(M, T^*M \otimes \mathrm{Cl}\left(C^N\right)\right) \xrightarrow{cl} C^{\infty}\left(M, \mathrm{Cl}\left(C^N\right)\right).$$

[2] There is no direct analogy of the projectivization construction (12) for odd symbols. For example, the rank of an odd subbundle can not take arbitrary values, see Remark 2.

Its nonnegative spectral subspace is odd and the desired decomposition holds

$$2^N C^\infty(M) \simeq C^\infty\left(M, \mathrm{Cl}\left(C^N\right)\right) = \widehat{L}_+(A) \oplus \widehat{L}_+^\perp(A).$$

A decomposition for an arbitrary bundle $E \in \mathrm{Vect}(M)$ is obtained by means of a tensor product with the decomposition just described.

2) We verify the exactness of the sequence in the second term: $\ker j \subset \mathrm{Im}\, i$.

Suppose that a pair of subspaces satisfies the equality

$$j\left(\left[\widehat{L}_1\right] - \left[\widehat{L}_2\right]\right) = 0,$$

$\widehat{L}_1 \subset C^\infty(M, E), \widehat{L}_2 \subset C^\infty(M, F)$. By definition of the mapping j this implies that E and F are (stably) isomorphic

$$E \cong F.$$

We must prove that for some N odd subspaces $2^N \widehat{L}_{1,2} \subset C^\infty(M, E)$ are homotopic, modulo a finite-dimensional subspace. We construct the desired homotopy of the principal symbols of the subspaces in the following two lemmata.

Lemma 2 *The symbol $L \subset \pi^* E$ of an odd subspace on an even-dimensional manifold admits a vector bundle isomorphism*

$$2^N L \sim 2^N \alpha^* L$$

on S^*M for some natural number N. Furthermore, there is an even isomorphism

$$\sigma : 2^N \pi^* E \to 2^N \pi^* E,$$

transforming one bundle into another; this isomorphism can be chosen such that

$$[\sigma] = 0 \in K^1(P^*M).$$

Corollary 1 *The symbols $2^N L$, $2^N \alpha^* L$ of odd subspaces are homotopic.*

Indeed, the equality $[\sigma] = 0$ yields a homotopy σ_t of the homomorphism σ to unity. Hence, the bundles are deformed according to the formula $\sigma_t 2^N L$ (they remain odd during the homotopy).

Proof of Lemma 2. We show in the Appendix that the natural projection $S^*M \longrightarrow P^*M$ of the cosphere bundle onto its projectivization induces an isomorphism in K-theory (modulo 2-torsion)

$$K^*(P^*M) \to K^*(S^*M).$$

Consequently, for some N we obtain $\left[2^N L\right] = \left[2^N \alpha^* L\right]$. This proves the first part of the lemma.

Consider an arbitrary isomorphism

$$\sigma : 2^N L \to 2^N \alpha^* L.$$

We extend it to the whole space $2^N\pi^*E \supset 2^N L$ in accordance with the decomposition

$$\widetilde{\sigma} : 2^N\pi^*E = 2^N L \oplus 2^N\alpha^* L \to 2^N\alpha^* L \oplus 2^N L = 2^N\pi^*E$$

by the formula

$$\widetilde{\sigma}(\xi) = \sigma(\xi) \oplus \sigma(-\xi).$$

The desired even isomorphism is given by the formula

$$\widetilde{\sigma} \oplus \widetilde{\sigma}^{-1} : 2^{N+1}\pi^*E \to 2^{N+1}\pi^*E.$$

It is homotopic to identity and sends $2^{N+1}L$ to $2^{N+1}\alpha^*L$. Lemma 2 is proved.

Lemma 3 *The subbundles*

$$2^N(L_1 \oplus \alpha^* L_1) \quad \textit{and} \quad 2^N(L_2 \oplus \alpha^* L_2) \subset \pi^*\left(2^{N+1}E\right) \tag{21}$$

are homotopic.

Proof. Denote by p, q the projections on subbundles $L_{1,2}$. Suppose that the projections act along complementary subbundles $\alpha^* L_{1,2}$. Consider the following vector bundle endomorphism

$$\sigma : \begin{array}{c}\pi^*E\\ \oplus\\ \pi^*E\end{array} \longrightarrow \begin{array}{c}\pi^*E\\ \oplus\\ \pi^*E\end{array},$$

$$\sigma = \begin{pmatrix} qp + (1-q)(1-p) & (1-q)p + q(1-p) \\ q(1-p) + (1-q)p & qp + (1-q)(1-p) \end{pmatrix} = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

The former representation shows that σ is even, while the latter (as a composition of operators transposing the subspaces) shows that

$$[\sigma] = 0 \in K^1(S^*M),$$

i.e. this operator is homotopic to identity. Meanwhile in the Appendix we show that the kernel of the mapping

$$K^1(P^*M) \longrightarrow K^1(S^*M)$$

induced by the natural projection, consists of elements of orders 2^N. Thus, the subbundles (21) are homotopic. This completes the proof of Lemma 3.

By virtue of Lemma 2 any pair of odd bundles $L_{1,2} \subset \pi^*E$ defines homotopies

$$2^{N+1}L_{1,2} = 2^N L_{1,2} \oplus 2^N L_{1,2} \sim 2^N L_{1,2} \oplus 2^N\alpha^* L_{1,2} = 2^N(L_{1,2} \oplus \alpha^* L_{1,2}),$$

while Lemma 3 yields a homotopy of subbundles (21). Therefore the bundles $2^N L_{1,2}$ are homotopic. The following proposition completes the proof of point 2).

Lemma 4 *Consider a homotopy L_t of symbols of subspaces and a pair of subspaces $\widehat{L}_0$ and $\widehat{L}_1$ corresponding to L_0 and L_1. Then these subspaces define equal elements of the group $K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right)$ modulo the subgroup $\mathrm{Im}\, i$, generated by finite-dimensional vector spaces.*

Proof. Indeed, consider a family of pseudodifferential projections P_t with principal symbols projecting on L_t such that $\widehat{L}_{0,1} = \mathrm{Im}\, P_{0,1}$. The desired family can be constructed by virtue of Proposition 1. In the general case this family is piecewise smooth, see (1). In addition, the projection P_t changes by a finite-dimensional projection at the discontinuity points. In other words, the subspaces $\widehat{L}_{0,1}$ are homotopic modulo a finite-dimensional vector space. This completes the proof of the lemma.

Now we show that in the case of odd subspaces the mapping (8)

$$i : Z \longrightarrow K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right)$$

is injective on even-dimensional manifolds. The proof is similar to the corresponding proof of Lemma 7. The family of invertible operators U_t has even principal symbols as before

$$\sigma\left(U_t\right)(x, -\xi) = \sigma\left(U_t\right)(x, \xi).$$

Let us show that in the case of odd subspaces the index of a Toeplitz type operator

$$P_0 U_1^{-1} : \mathrm{Im}\, P_0 \longrightarrow \mathrm{Im}\, P_0, \quad \mathrm{Im}\, P_0 = \widehat{L}$$

is zero. The index is again computed by the formula

$$\mathrm{ind}\left(P_0 U_1^{-1}, \mathrm{Im}\, P_0, \mathrm{Im}\, P_0\right) = \mathrm{ind}\left(P_0 U_1^{-1} P_0 + (1 - P_0)\right).$$

Operator $P_0 U_1^{-1} P_0 + (1 - P_0)$ has the following principal symbol

$$\sigma\left(U_1^{-1}\right)\Big|_L \oplus 1_{\alpha^* L}.$$

Consider also the symbol $\alpha^*\left(\sigma\left(U_1^{-1}\right)\Big|_L \oplus 1_{\alpha^* L}\right)$. The indices of operators corresponding to these symbols are equal, since the involution α^* on an even-dimensional manifold induces the identity map modulo torsion in K-theory (see Appendix). On the other hand,

$$\alpha^*\left(\sigma\left(U_1^{-1}\right)\Big|_L \oplus 1_{\alpha^* L}\right) = \sigma\left(U_1^{-1}\right)\Big|_{\alpha^* L} \oplus 1_L.$$

Hence, the sum of the two symbols is equal to

$$\sigma\left(U_1^{-1}\right)\Big|_L \oplus 1_{\alpha^* L} \bigoplus \alpha^*\left(\sigma\left(U_1^{-1}\right)\Big|_L \oplus 1_{\alpha^* L}\right) \simeq \sigma\left(U_1^{-1}\right) \bigoplus 1.$$

We obtain, in particular, the desired equality of the index

$$2\mathrm{ind}\left(P_0 U_1^{-1} P_0 + (1 - P_0)\right) = \mathrm{ind} U_1^{-1} = 0.$$

This completes the proof of Proposition 8.

Proposition 9 *The sequence* (19) *admits the following splitting*

$$j' : K(M) \to K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \otimes Z\left[\frac{1}{2}\right]$$

(i.e. $jj' = Id$). The mapping sends a vector bundle $E \in \mathrm{Vect}\,(M)$ to a decomposition of the space of sections $C^\infty\left(M, 2^N E\right)$ into a direct sum of odd subspaces

$$j'\,[E] = \left[\widehat{L} \oplus \widehat{L}^\perp\right] \otimes \frac{1}{2^{N+1}}, \quad C^\infty\left(M, 2^N E\right) = \widehat{L} \oplus \widehat{L}^\perp. \tag{22}$$

Proof. The element

$$j'\,[E] = \left[\widehat{L} \oplus \widehat{L}^\perp\right] \otimes \frac{1}{2^{N+1}} \in K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \otimes Z\left[\frac{1}{2}\right], \quad \text{for } C^\infty\left(M, 2^N E\right) = \widehat{L} \oplus \widehat{L}^\perp$$

is independent of the choice of an odd subspace in $C^\infty\left(M, 2^N E\right)$, since another odd subspace $\widehat{L}_1 \subset C^\infty\left(M, 2^N E\right)$, by virtue of Lemma 3, satisfies the equality

$$2^{N'}\left[\widehat{L} \oplus \widehat{L}^\perp\right] = 2^{N'}\left[\widehat{L}_1 \oplus \widehat{L}_1^\perp\right].$$

Thus, the mapping j' is well defined. The desired property $jj' = Id$ follows immediately from the definition.

Proposition 9 is proved.

It follows from Propositions 8 and 9 that, modulo 2-torsion, the mapping i defines an isomorphism

$$Z\left[\frac{1}{2}\right] \overset{i\otimes 1}{\to} \left\{K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \otimes Z\left[\frac{1}{2}\right]\right\} \Big/ \left\{j' K\left(M\right) \otimes Z\left[\frac{1}{2}\right]\right\}. \tag{23}$$

As a consequence, the dimension functional

$$d : K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \longrightarrow R,$$

satisfying the complement property is unique and is equal to the inverse mapping of the injection i

$$d = (i \otimes 1)^{-1} : K\left(\widehat{\mathrm{Odd}}\left(M^{ev}\right)\right) \to Z\left[\frac{1}{2}\right].$$

This completes the proof of the theorem.

Remark 3 Property (10) can be equivalently expressed in terms of the *relative index* [8], [22]: for two subspaces $\widehat{L}_{1,2}$ with coinciding principal symbols, the following formula is valid

$$d\left(\widehat{L}_1\right) - d\left(\widehat{L}_2\right) = \mathrm{ind}\left(P_2 : \widehat{L}_1 \to \widehat{L}_2\right)$$

(where the subspace $\widehat{L}_2$ is determined by projection P_2, while the operator $P_2 : \widehat{L}_1 \to \widehat{L}_2$ has the Fredholm property).

Remark 4 The dimension functional of subspaces can be expressed in terms of the Atiyah–Patodi–Singer η-invariant [1]. More precisely, the following theorem is valid.

Theorem 3 (*see* [17, 18]) *Let A be an elliptic self-adjoint differential operator. For its nonnegative spectral subspace $\widehat{L}_+ (A)$ the following formula is valid*

$$d\left(\widehat{L}_+ (A)\right) = \eta (A), \tag{24}$$

provided the order of the operator and the dimension of the manifold have opposite parities.

4 The homotopy classification of elliptic operators in subspaces

Throughout the remaining sections we consider elliptic operators

$$D : \widehat{L}_1 \longrightarrow \widehat{L}_2$$

acting in subspaces with parity conditions (Definition 6)

$$\widehat{L}_{1,2} \in \widehat{\mathrm{Even}} (M) \quad \text{or } \widehat{\mathrm{Odd}} (M) .$$

Operators of the form

$$\widehat{L}_1 \xrightarrow{Id} \widehat{L}_1$$

are called trivial.

Definition 9 Elliptic operators D_1 and D_2 are called *stably homotopic* , if for some trivial operators D_3,D_3' the direct sums $D_1 \oplus D_3$ and $D_2 \oplus D_3'$ are homotopic.

Denote by $\mathrm{Ell}^{ev/odd} (M)$ an abelian group of stable homotopy classes of elliptic operators in subspaces with parity conditions. A similar group for the usual elliptic operators is denoted by $\mathrm{Ell} (M)$.

It turns out that for an elliptic operator in subspaces with parity conditions one can define an elliptic operator

$$\widetilde{D} : C^\infty (M, F_1) \longrightarrow C^\infty (M, F_2)$$

in spaces of sections of vector bundles. More precisely, in the case of odd subspaces $\widehat{L}_{1,2} \in \widehat{\mathrm{Odd}} (M)$, this operator has principal symbol

$$\sigma(\widetilde{D}) : \ L_1 \oplus \alpha^* L_1 \longrightarrow L_2 \oplus \alpha^* L_2$$

equal to

$$\sigma\left(\widetilde{D}\right) = \sigma (D) \oplus \alpha^* \sigma (D) .$$

In the even case $\widehat{L}_{1,2} \in \widehat{\mathrm{Even}}\,(M)$, this operator is an endomorphism

$$\widetilde{D} : C^{\infty}(M, E_1) \longrightarrow C^{\infty}(M, E_1), \tag{25}$$

with the principal symbol $\sigma(\widetilde{D})$

$$\sigma(\widetilde{D}) : \; L_1 \oplus L_1^{\perp} \longrightarrow L_1 \oplus L_1^{\perp}$$

equal to

$$\sigma\left(\widetilde{D}\right) = \left[\alpha^{*}\sigma(D)\right]^{-1}\sigma(D) \oplus 1.$$

When the parity of the subspaces is opposite to the parity of dimension of the manifold, the operator $D : \widehat{L}_1 \longrightarrow \widehat{L}_2$ is determined up to homotopy by the operator $\widetilde{D}$ and the value of the dimension functional d of the subspaces $\widehat{L}_{1,2}$.

Theorem 4 (homotopy classification of operators in subspaces). *For elliptic operators in pseudodifferential subspaces the following isomorphism of groups is valid*

$$\begin{array}{ccc} \mathrm{Ell}^{ev/odd}\left(M^{odd/ev}\right) \otimes Z\left[\frac{1}{2}\right] & \stackrel{\chi}{\longrightarrow} & \mathrm{Ell}\left(M^{odd/ev}\right) \otimes Z\left[\frac{1}{2}\right] \oplus Z\left[\frac{1}{2}\right], \\ \left[D : \widehat{L}_1 \longrightarrow \widehat{L}_2\right] & \mapsto & \left[\widetilde{D}\right] \otimes \frac{1}{2} \oplus \left(d\left(\widehat{L}_1\right) - d\left(\widehat{L}_2\right)\right). \end{array} \tag{26}$$

Remark 5 It follows from (26) that the groups

$$\mathrm{Ell}^{ev/odd}(M) \;\text{ and }\; \mathrm{Ell}(M) \oplus Z$$

are isomorphic modulo 2-torsion.

Proof. We construct the inverse mapping

$$\chi' : \mathrm{Ell}(M) \otimes Z\left[\frac{1}{2}\right] \oplus Z\left[\frac{1}{2}\right] \to \mathrm{Ell}^{ev/odd}(M) \otimes Z\left[\frac{1}{2}\right]. \tag{27}$$

A. Let us consider even subspaces. In this case the mapping χ' is induced on the first term by the inclusion of the usual elliptic operators in the set of elliptic operators in even subspaces:

$$\chi'([D] \oplus 0) = [D], \qquad D : C^{\infty}(M, E) \to C^{\infty}(M, F). \tag{28}$$

The mapping on the second term is defined by the formula

$$\chi'(0 \oplus k) = [k \to 0], \tag{29}$$

where $k \to 0$ denotes the operator equal to zero, it is defined on a subspace of dimension k. This zero operator acts in the zero-dimensional space. Let us verify that χ' is the inverse of χ.

First, consider the composition $\chi \circ \chi'$. Its second component

$$Z\left[\frac{1}{2}\right] \to Z\left[\frac{1}{2}\right]$$

is the identity map by the definition of the dimension functional d (see (10)). By virtue of (25), the first component has the form

$$\frac{1-\alpha^*}{2} : \mathrm{Ell}\,(M) \otimes Z\left[\frac{1}{2}\right] \longrightarrow \mathrm{Ell}\,(M) \otimes Z\left[\frac{1}{2}\right].$$

The homotopy classification of usual elliptic operators [3]

$$\mathrm{Ell}\,(M) \simeq K\left(T^*M\right)$$

together with Proposition 10 (see Appendix) imply that the mapping $(1-\alpha^*)/2$ on an odd-dimensional manifold is equal to identity. Thus, we obtain $\chi \circ \chi' = Id$.

Now, we prove the surjectivity of χ'. Let

$$D : \widehat{L}_1 \longrightarrow \widehat{L}_2$$

be an elliptic operator in even subspaces. We claim that this operator can be reduced to some usual elliptic operator, modulo an operator, acting in finite-dimensional spaces.

The isomorphism (17) yields for some N a homotopy of the direct sums of subspaces $2^N\widehat{L}_{1,2} \subset C^\infty\left(M, 2^N E_{1,2}\right)$ to spaces of sections of vector bundles, modulo finite-dimensional subspaces. By Lemma 1 we have the families of invertible operators

$$U_t, V_t : C^\infty\left(M, 2^N E_{1,2}\right) \longrightarrow C^\infty\left(M, 2^N E_{1,2}\right),$$

realizing the homotopies of the subspaces

$$\begin{aligned} U_0 &= 1, \qquad\qquad\quad V_0 = 1 \\ U_1\left(2^N\widehat{L}_1\right) &= C^\infty\left(M, E_1'\right), V_1\left(2^N\widehat{L}_2\right) = C^\infty\left(M, E_2'\right). \end{aligned}$$

Consider the following homotopy of elliptic operators in subspaces

$$V_t D U_t^{-1} : U_t\left(2^N\widehat{L}_1\right) \longrightarrow V_t\left(2^N\widehat{L}_2\right).$$

It connects $2^N D$ at $t=0$ to the usual elliptic operator at $t=1$

$$V_1 D U_1^{-1} : C^\infty\left(M, E_1'\right) \longrightarrow C^\infty\left(M, E_2'\right).$$

Thus χ' is surjective. The homotopy classification in the case of even subspaces is obtained.

B. The major steps of the proof of the homotopy classification for odd subspaces are similar to the even case just described.

The inverse mapping χ' for odd subspaces is defined on the second component (see (27)) as follows

$$\chi'(0 \oplus k) = \left[\widehat{L} + k \to \widehat{L}\right].$$

Here $\widehat{L}$ is an odd subspace, while $\widehat{L}+k\to\widehat{L}$ is the projection on $\widehat{L}$. Let us define χ' on the first term according to the formula

$$\begin{array}{ccc} \chi':\mathrm{Ell}\,(M)\otimes Z\left[\frac{1}{2}\right] & \longrightarrow & \mathrm{Ell}^{odd}\,(M)\otimes Z\left[\frac{1}{2}\right] \\ \left[D:C^\infty\left(M,E_1\right)\to C^\infty\left(M,E_2\right)\right] & \mapsto & \left[2^N D:\widehat{L}_1\oplus\widehat{L}_1^\perp\to\widehat{L}_2\oplus\widehat{L}_2^\perp\right]\otimes\frac{1}{2^N}, \end{array} \tag{30}$$

in terms of a decomposition into a sum of odd subspaces (see (22))

$$C^\infty\left(M,2^N E_{1,2}\right)=\widehat{L}_{1,2}\oplus\widehat{L}_{1,2}^\perp.$$

The map is well defined, i.e., it is independent of the choice of a decomposition.

A straightforward computation shows that $\chi\chi'$ is equal to

$$\frac{1+\alpha^*}{2}\oplus 1:\mathrm{Ell}\,(M)\otimes Z\left[\frac{1}{2}\right]\oplus Z\left[\frac{1}{2}\right]\to\mathrm{Ell}\,(M)\otimes Z\left[\frac{1}{2}\right]\oplus Z\left[\frac{1}{2}\right].$$

Taking into account that the manifold is even-dimensional, we obtain by the results of Appendix that this is the identity mapping.

We shall show that χ' is surjective. Consider an elliptic operator in odd subspaces

$$D:\widehat{L}_1\longrightarrow\widehat{L}_2.$$

We must prove that it can be reduced to an operator of the form (30), modulo an operator acting in finite-dimensional spaces.

For some natural N isomorphism (23) yields the homotopy of the subspaces $2^N\widehat{L}_{1,2}\subset C^\infty\left(M,2^N E_{1,2}\right)$ (modulo finite-dimensional vector spaces) to the direct sums of odd subspaces $\widehat{L}'_{1,2}\oplus\widehat{L}'^{\perp}_{1,2}$. Similar to the previous proof, the homotopies of subspaces can be lifted to homotopies of elliptic operators acting in the subspaces. Therefore, the direct sum $2^N D$ of the initial operator is homotopic to an operator of the form (30).

Thus, χ' is surjective. This establishes the homotopy classification of elliptic operators in odd subspaces.

The theorem is proved.

5 Index theorem

Let us define the index homomorphism on the group $\mathrm{Ell}\,(M)\oplus Z$

$$\begin{array}{ccc} \mathrm{ind}':\mathrm{Ell}\,(M)\oplus Z & \longrightarrow & Z \\ [D]\oplus k & \mapsto & \mathrm{ind}D+k. \end{array}$$

Theorem 5 (index theorem) *Let D be an elliptic operator in subspaces with parity conditions*

$$[D]\in\mathrm{Ell}^{ev/odd}\left(M^{odd/ev}\right).$$

The following index formula is valid

$$\mathrm{ind}D=\mathrm{ind}'\chi\,[D]\,.$$

Remark 6 In other words, the index of an elliptic operator in subspaces

$$D : \widehat{L}_1 \to \widehat{L}_2$$

is expressed by the formula

$$\operatorname{ind}\left(D, \widehat{L}_1, \widehat{L}_2\right) = \frac{1}{2}\operatorname{ind}\widetilde{D} + d\left(\widehat{L}_1\right) - d\left(\widehat{L}_2\right). \qquad (31)$$

Proof. The homotopy classification of elliptic operators in subspaces with parity conditions (Theorem 4) yields the equality

$$2^N [D] = 2^N \chi' \chi [D]$$

for some N. Thus

$$\operatorname{ind} D = \operatorname{ind} \chi' \chi [D].$$

It is not hard to check (see (28),(29)) that χ' preserves the index:

$$\operatorname{ind} \circ \chi' = \operatorname{ind}' : \operatorname{Ell}(M) \oplus Z \longrightarrow Z.$$

As a consequence, we obtain the index formula

$$\operatorname{ind} D = \operatorname{ind}' \chi [D].$$

Theorem is proved.

Example 4 Consider a self-adjoint elliptic operator from Example 3

$$A = d\delta - \delta d : \Lambda^1(M) \longrightarrow \Lambda^1(M)$$

on an odd-dimensional manifold M. The principal symbol of A is

$$\sigma(A) = \xi \wedge \xi \rfloor - \xi \rfloor \xi \wedge,$$

where $\xi \rfloor$ denotes the interior product with respect to Riemannian metric by the covector ξ. In particular, the symbol of the spectral subspace $\widehat{L}_+(A)$ at a point $\xi \neq 0$ is the line generated by the covector ξ itself. Therefore, even bundle $L_+(A)$ defines the so-called tautological line bundle on the projectivization P^*M. The tautological bundle is well known to be nontrivial. However, $L_+(A)$ has a natural trivialization on the (co)sphere bundle

$$\kappa : L_+(A) \longrightarrow C,$$
$$\kappa(x, \xi, \eta) = (\xi, \eta),$$

where (ξ, η) denotes the scalar product of two proportional vectors. The η-invariant of A was calculated in the paper [11]

$$\eta(A) = \dim\ker (d\delta + \delta d)|_{\Lambda^1(M)} - \dim\ker (d\delta + \delta d)|_{\Lambda^0(M)} = \dim H^1(M) - \dim H^0(M).$$

Thus, the index formula for any elliptic operator

$$D : \widehat{L}_{+}(A) \longrightarrow C^{\infty}(M)$$

has the form

$$\operatorname{ind}\left(D, \widehat{L}_{+}(A), C^{\infty}(M)\right) = \frac{1}{2}\operatorname{ind}\widetilde{D} + \dim H^{1}(M) - \dim H^{0}(M),$$

where the elliptic operator $\widetilde{D}$ has the principal symbol defined by the following formula

$$\sigma\left(\widetilde{D}\right)(x,\xi) = \sigma(D)(x,\xi)\left[\sigma(D)(x,-\xi)\right]^{-1} : \pi^{*}C \to \pi^{*}C.$$

The index of scalar operators on a manifold of dimension $\dim M \geq 3$ is trivial. To obtain a nontrivial index one can consider matrix operators.

Appendix. Action of antipodal involution in K-theory

On an even-dimensional manifold M consider the following involution of the cotangent bundle

$$\alpha : T^{*}M \longrightarrow T^{*}M, \quad \alpha(x,\xi) = (x,-\xi).$$

Denote by $P^{*}M$ the quotient space of the spheres with respect to the action of the involution α. It is a fibre bundle with real projective spaces as fibres. There are the following natural projections connecting M, $P^{*}M$, $S^{*}M$

$$S^{*}M \xrightarrow{\pi} P^{*}M \xrightarrow{p} M.$$

Proposition 10 .

1. *The mapping α induces involution in K-theory, this involution modulo 2-torsion is equal to $(-1)^{\dim M}$:*

   $$\alpha^{*} : K^{*}(T^{*}M) \otimes Z\left[\frac{1}{2}\right] \longrightarrow K^{*}(T^{*}M) \otimes Z\left[\frac{1}{2}\right], \quad \alpha^{*} = (-1)^{\dim M}.$$

   If M is a manifold with boundary, then the involution α^{*} has this property also on the group $K^{*}(T^{*}(M \backslash \partial M))$;

2. *On an even-dimensional M the projection $S^{*}M \xrightarrow{\pi} P^{*}M$ induces an isomorphism modulo 2-torsion*

   $$K^{*}(P^{*}M) \otimes Z\left[\frac{1}{2}\right] \to K^{*}(S^{*}M) \otimes Z\left[\frac{1}{2}\right];$$

3. *On an odd-dimensional manifold the projection $P^{*}M \xrightarrow{p} M$ induces an isomorphism modulo 2-torsion*

   $$K^{*}(M) \otimes Z\left[\frac{1}{2}\right] \to K^{*}(P^{*}M) \otimes Z\left[\frac{1}{2}\right].$$

Proof of these statements is an application of the Mayer-Vietoris principle [7]. The Mayer-Vietoris principle makes it possible to reduce the verification of the above properties on the entire manifold M to a similar check over a point.

1) Let us check properties 1-3 for the restriction of the mappings on the fibre over an arbitrary point $x \in M$ of the base:

$$K^*(T_x^*M) \xrightarrow{\alpha^*} K^*(T_x^*M),$$
$$K^*(P_x^*M) \xrightarrow{\pi^*} K^*(S_x^*M),$$
$$K^*(\{x\}) \xrightarrow{p^*} K^*(P_x^*M).$$

In the first case we obtain

$$T_x^*M = R^{\dim M}, \qquad K^*\left(R^{\dim M}\right) = Z,$$

the involution α preserves (reverses) the orientation of $R^{\dim M}$ together with the parity of dimension of M. Hence, we obtain the desired: $\alpha^* = (-1)^{\dim M}$.

In the second case we consider an even-dimensional M and the projection $\pi : S^{2n+1} \to RP^{2n+1}$. K-groups of spheres and projective spaces are well known (e.g., see [12])

$$\begin{aligned} K^0\left(RP^{2n+1}\right) &= Z \oplus Z_{2^n}, \quad K^0\left(S^{2n+1}\right) = Z, \\ K^1\left(RP^{2n+1}\right) &= Z, \qquad\quad K^1\left(S^{2n+1}\right) = Z. \end{aligned}$$

The first summand in K^0 corresponds to the dimension of vector bundles, while the projection π in K^1 groups acts as the multiplication by two

$$\begin{array}{ccc} \pi^* : K^1\left(RP^{2n+1}\right) = Z & \longrightarrow & K^1\left(S^{2n+1}\right) = Z \\ \\ n & \longmapsto & 2n. \end{array}$$

In the last (third) case of an odd-dimensional M, we consider the projection $p : RP^{2n} \to pt$. We have

$$\begin{aligned} K^0\left(RP^{2n}\right) &= Z \oplus Z_{2^n}, \quad K^0(pt) = Z, \\ K^1\left(RP^{2n}\right) &= 0, \qquad\quad K^1(pt) = 0. \end{aligned}$$

Both components Z denote the dimensions of vector bundles. Therefore, property 3 is satisfied over a point.

2) We claim that the following assertion is valid: suppose that the properties 1-3 are satisfied for two open subsets $U, V \subset M$ and for their intersection $U \cap V$. Then these properties are valid for the union $U \cup V$.

In the first case we write out a part of the Mayer-Vietoris exact sequence, corresponding to the inclusions $U \bigcap V \overset{i}{\subset} U \sqcup V \overset{j}{\subset} U \bigcup V$.

$$\begin{array}{ccccc} K^{*+1}(T^*(U \bigcap V)) & \xrightarrow{\delta} & K^*(T^*(U \bigcup V)) & \xrightarrow{j^*} & K^*(T^*U) \oplus K^*(T^*V) \\ \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* \oplus \alpha^* \\ K^{*+1}(T^*(U \bigcap V)) & \xrightarrow{\delta} & K^*(T^*(U \bigcup V)) & \xrightarrow{j^*} & K^*(T^*U) \oplus K^*(T^*V). \end{array}$$

Suppose that the left and the right involutions in the diagram are equal to $(-1)^{\dim M}$ (modulo 2-torsion). By a diagram chasing argument one deduces that the mapping α^* in the center also satisfies property 1. For example, on an even-dimensional manifold for $x \in K^*(T^*(U \bigcup V))$ we get

$$j^*(\alpha^* x - x) = 0 \Rightarrow \alpha^* x - x = \delta\alpha^* y, \alpha^* y = y \Rightarrow 2(\alpha^* x - x) = 0$$

(in this computation factors 2^N are omitted for brevity).

The last two cases are treated similarly. For example, the projection $\pi : S^*M \to P^*M$, corresponding to odd-dimensional M, acts on the Mayer-Vietoris exact sequence

$$\begin{array}{ccccccc} \ldots \to & K^*(P^*(U \bigcup V)) \to & K^*(P^*(U \sqcup V)) \to & K^*(P^*(U \bigcap V)) \to \ldots \\ & \downarrow \pi^* & \downarrow \pi^* & \downarrow \pi^* \\ \ldots \to & K^*(S^*(U \bigcup V)) \to & K^*(S^*(U \sqcup V)) \to & K^*(S^*(U \bigcap V)) \to \ldots \end{array}$$

By 5-lemma the mapping π^* on the left is an isomorphism modulo 2-torsion.

The property concerning the group $K^*(T^*(M \backslash \partial M))$ follows from the exact sequence of the pair $T^*M|_{\partial M} \subset T^*M$

$$\begin{array}{ccccccc} \to & K^{*+1}(T^*M|_{\partial M}) \otimes Z\left[\frac{1}{2}\right] & \to & K^*(T^*(M \backslash \partial M)) \otimes Z\left[\frac{1}{2}\right] & \to & K^*(T^*M) \otimes Z\left[\frac{1}{2}\right] & \to \\ & \downarrow (-1)^{\dim M} & & \downarrow \alpha & & \downarrow (-1)^{\dim M} & \\ \to & K^{*+1}(T^*M|_{\partial M}) \otimes Z\left[\frac{1}{2}\right] & \to & K^*(T^*(M \backslash \partial M)) \otimes Z\left[\frac{1}{2}\right] & \to & K^*(T^*M) \otimes Z\left[\frac{1}{2}\right] & \to \end{array}$$

which carries the action of the involution α.

3) Consider a good (see [7]) finite covering $\{U_\beta\}$ of the manifold M by contractible open sets. Over any U_β the properties 1-3 are valid by the first section of the proof. Let us consider all subsets in $\{U_\beta\}$.

Passing from the coverings consisting of a single element to the covering of the whole manifold M and applying the assertion from the second part of the proof, we obtain the desired properties for the entire manifold M.

This completes the proof of Proposition 10.

Corollary 2 *On an odd-dimensional manifold the index of an elliptic operator with even or odd principal symbol is zero.*

Indeed, the symbol σ of such operator satisfies

$$[\alpha^*\sigma] = [\sigma] \in K(T^*M).$$

On the other hand, modulo 2-torsion we have

$$\alpha^*[\sigma] = -[\sigma]$$

by Proposition 10. Consequently, $[\sigma]$ is a torsion element and its index is zero.

Example 5 On the manifold $M = RP^{2n}$, $n > 1$ one can show that

$$K^1\left(T^*RP^{2n}\right) \simeq \mathbf{Z}_{2^n},$$

while the involution α^* is nontrivial

$$\alpha^* = -1 : K^1\left(T^*RP^{2n}\right) \to K^1\left(T^*RP^{2n}\right).$$

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