ON ITERATIONS OF DOUBLE LAYER POTENTIALS

Alexander Shlapunov [1]

Abstract

We prove the existence of $H^p(D)$-limit of iterations of double layer potentials constructed with the use of Hodge parametrix on a smooth compact manifold X, D being an open connected subset of X. This limit gives us an orthogonal projection from Sobolev space $H^p(D)$ to a closed subspace of $H^p(D)$-solutions of an elliptic operator P of order $p \geq 1$. Using this result we obtain formulae for Sobolev solutions to the equation $Pu = f$ in D whenever these solutions exist. This representation involves the sum of a series whose terms are iterations of double layer potentials. Similar regularization is constructed also for a P-Neumann problem in D.

1 Introduction

This paper is based on the following simple observation. Consider an operator equation $Au = f$ with a bounded linear operator $A : H_0 \to H_1$ in Hilbert spaces H_0, H_1 and assume that for every $u \in H_0$ the following formula holds true

$$u = \Pi_1 u + \Pi_2 A u$$

where Π_1 is a projection from H_0 to the kernel of A. Then one can hope that, under reasonable conditions, the element $\Pi_2 f$ defines a solution to the equation $Au = f$.

It is well known that Hodge theory for partial differential operator P with injective symbol on a compact manifold X gives an L^2-orthogonal projection to space of solutions to equation $Pu = 0$ on the whole X. In this paper using the Hodge theory we construct an orthogonal projection from Sobolev space $H^p(D)$, D being an open connected subset of X and p being the order of P, to a closed subspace of $H^p(D)$-solutions to equation $Pu = 0$ in D (see Sections 2, 3). Let us briefly sketch some motivations for this investigation.

First, local solvability for linear partial differential operators with injective symbol and smooth coefficients is a long standing problem of the theory of overdetermined systems (see, for instance, [8]). With the use of this result we succeed in proving a representation formula for H^p-solutions to the equation $Pu = f$ on open subsets of X for an operator P with injective

[1] This work was done under financial support of Institute für Mathematik, Universität Potsdam and RFFI, grant 96–15–9626 for leading scientific schools.

symbol whenever these solutions exist (see Section 4). This representation involves the sum of a series whose terms are iterations of double layer potentials, while solvability of $Pu = f$ is equivalent to the convergence of the series together with the orthogonality to $\ker P^\star$ (the last one is a trivial necessary condition).

The second motivation is that this method gives a possibility to construct a similar regularization for a P-Neumann problem (see Sections 5, 6) For the CAUCHY-RIEMANN system $P = \overline{\partial}$ in $\mathbb{C}^n$ $(n > 1)$, the double layer potentials involved in the regularization formulae are the MARTINELLI-BOCHNER integrals. In this case, results similar to ours were obtained by A.V. ROMANOV [6].

Theorem 1.1 *Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected boundary ∂D of class C^1, and Mu is the Martinelli–Bochner integral for $u \in H^1(D)$. Then in the strong operator topology of $H^1(D)$ there exists a limit $\lim_{N\to\infty} M^N = \Pi_1$ where Π_1 is a projection from $H^1(D)$ onto the closed subspace of holomorphic $H^1(D)$-functions.*

Using this theorem ROMANOV (see [6]) obtained an explicit formula for a solution $u \in H^1(D)$ to the equation $\overline{\partial} u = f$ where D is a pseudo convex domain with a smooth boundary, and f is a $\overline{\partial}$ -closed (0,1)-form with coefficients in $H^1(D)$.

2 Hodge theory on a compact manifold.

Let X be a C^∞-manifold with dim $X = n$, E and F be smooth $\mathbb{C}$-vector bundles over X, and $do_p(E \to F)$ be the vector space of smooth linear partial differential operators of order $\leq p$ between the vector bundles E and F. Let E^* be the dual bundle of E, and let $(.,.)_x$ be a HERMITIAN metric on E. Then $*_E : E \to E^*$ is defined by $< *_E v, u >_x = (u, v)_x$ (where u, v are sections of E and $< .,. >_x$ is the pairing of E and E^*). Let Λ^q be the bundle of complex valued exterior forms of degree q $(q = 1, 2, \ldots)$ over X, and dx be a volume form on X. We denote ${}^tP \in do_p(F^* \to E^*)$ the transposed operator, and by $P^* = *_E^{-1}({}^tP)*_F \in do_p(F \to E)$ the (formal) adjoint operator for $P \in do_p(E \to F)$ (see, for instance, [8]).

Let $S_P(U)$ stand for the space of weak solutions to the equation $Pu = 0$ on an open set U in X. For every domain (i.e. open connected set) D in X we will denote $L^2(E_{|D})$ the Hilbert space of all measurable functions defined on D, for which $(u, u)_{L^2(E_{|D})} = \int_D (u, u)_x dx < \infty$. We also denote by $H^m(E_{|D})$ the SOBOLEV space of distribution sections of E over D having weak derivatives in $L^2(E_{|D})$ up to order m and by $S^m_P(D)$ the closed linear subspace of $H^m(E_{|D})$ of weak solutions of the equation $Pu = 0$ in D.

Let $T^*(X)$ be the (real) cotangent bundle over the manifold X and $\sigma(P)$ be the principal symbol of the operator P. From now on we assume that the symbol $\sigma(x,\zeta)$ is injective for all $x \in X$ and $\zeta \in T^*(X) \setminus \{0\}$. Then the Laplacian $\Delta = P^*P$ is an elliptic operator of order $2p$ on X and there exists Hodge parametrix for it (see, for example, [8], §8):

Theorem 2.1 *There exist pseudodifferential operators $K : \mathcal{D}'(E) \to C^\infty(E)$ and $\Phi : \mathcal{D}'(E) \to \mathcal{D}'(E)$ such that:*

(1) K is the L^2-projection on the finite dimensional space $S_P(X)$ having the kernel $K(x,y) = \sum_i h_i(x) \otimes h_i(y)^\star$ where $\{h_i\}$ is an orthogonal basis for $S_P(X)$;

(2) $PK = KP^* = 0$ and $\Phi K = K\Phi = 0$;

(3) Φ is a pseudodifferential operator of order $(-2p)$ and for all $u \in \mathcal{D}'(E)$ we have $(P^*P)\Phi u = u - Ku$, $\Phi(P^*Pu) = u - Ku$.

Then integrating by parts we see that

$$\int_X < {}^tP^*\Phi(x,y), Pu(y) >_y dy \; + \int_X < K(x,y), u(y) >_y dy = u(x),\; x \in X \tag{2.1}$$

for every $u \in H^p(E)$. Consider, for $u, v \in H^p(E)$, the HERMITIAN form

$$h(u,v) = \int_X (Pu, Pv)_x dx + \int_X (Ku, Kv)_x dx.$$

Proposition 2.2 *The* HERMITIAN *form $h(\cdot,\cdot)$ is a scalar product in $H^p(E)$ defining a topology equivalent to the original one. K is the orthogonal projection with respect to $h(.,.)$ from $H^p(E)$ onto $S(X)$. Moreover denoting by*

$$Tf(x) = \int_X < {}^tP^*(y)\Phi(x,y), f(y) >_y dy$$

we obtain $h(Tf,u) = \int_X (f, Pu)_x dx$ for all $f \in L^2(F)$, $u \in H^p(E)$.

Proof. The coefficients of P are C^∞ - functions, therefore, $Pu \in L^2(F)$. It follows from (2.1) that $h(u,u) = 0$ implies $u \equiv 0$ in X. Since $(\cdot,\cdot)_x$ is a HERMITIAN metric, $h(.,.)$ is a scalar product on $H^p(E)$. Because K is a smoothing operator we conclude that $\|.\|_{H^p(E)}$ is not weaker than $\sqrt{h(.,.)}$.

Further, (2.1) and boundedness theorem for pseudodifferential operators (see [5], 1.2.3.5) imply that there exists a constant $c_1 > 0$ such that for every $u \in H^p(E)$ we have $\|u\|^2_{H^p(E)} \le c_1 \left(\|Pu\|^2_{L^2(F)} + \|Ku\|^2_{H^p(E)} \right)$. Since K is a projection on a finite dimensional space $S_P(X)$ we see that there exists a constant $c_2 > 0$ such that for every $u \in H^p(E)$ we have $\|Ku\|^2_{H^p(E)} \le c_2 \, \|Ku\|^2_{L^2(E)}$. This proves the equivalence of the topologies.

If $f \in C^\infty(F)$, $u \in H^p(E)$ then integrating by parts we deduce that $Tf = \Phi(P^* f)$. Therefore, due to Theorem 2.1, $KTf = 0$ and

$$h(Tf, u) = \int_X (P\Phi(P^* f), Pu)_x dx = \int_X (P^* P\Phi(P^* f), u)_x dx =$$

$$\int_X (P^* f, u)_x dx = \int_X (f, Pu)_x dx.$$

Now, because $C^\infty(F)$ is dense in $L^2(F)$ we obtain the required statement on the integral T.

Finally, for $u, v \in H^p(E)$ we have

$$h(Ku, v) = h(u, v) - h(TPu, v) = \int_X (Ku, Kv)_x dx,$$

i.e. K is a self-adjoint operator with respect to $h(.,.)$ in $H^p(E)$ with $K^2 = I$, which was to be proved. □

Proposition 2.3 *Let $f \in L^2(F)$. There exists $u \in H^p(E)$ satisfying $Pu = f$ on X if and only if*

$$\int_X (f, g)_x dx = 0 \textit{ for all } g \in \ker T. \tag{2.2}$$

Moreover, if (2.2) holds then $PTf = f$ on X.

Proof. The necessity of (2.2) follows from Proposition 2.2.

On the other hand, Theorem 2.1 and (2.1) imply that for every $f \in L^2(F)$ we have $TPTf = Tf$, i.e $(f - PTf) \in \ker T$. Since both f and PTf are orthogonal to $\ker T$ we conclude that $PTf = f$. □

Remark 2.4 *It follows from Proposition 2.2 that $g \in \ker T$ if and only if $P^* g = 0$ weakly on X, i.e. $\ker T = S^0_{P^*}(X)$.*

If P is included into some elliptic complex of differential operators on X, say, $\{E^i, P^i\}$, for vector bundles E^i and $P^i \in do_{p_i}(E^i \to E^{i+1})$ where $P = P^0$ then the cohomology of the complex is finite (see, for instance, [8]) and condition (2.2) may be replaced by

$$P^1 f = 0 \text{ on } X \text{ and } \int_X (f, g)_x dx = 0 \text{ for all } g \in S^0_{P^*}(X) \cap S^0_{P^1}(X). \tag{2.3}$$

Remark 2.5 *Since X is compact manifold and the complex (E^i, P^i) is elliptic we see that the "harmonic" space $\mathfrak{H}(X) = S^0_{P^*}(X) \cap S^0_{P^1}(X)$ is finite dimensional.*

Actually the Hodge parametrix may be useful on open subsets of X too. Denote by $G_P(.,.)$ a Green operator for $P \in do_p(E \to F)$ (see [8], p.82).

Let D be a relatively compact domain (i.e. open connected subset) in X with smooth boundary. Define the operators T_D K_D and M by

$$(K_D u)(x) = \int_D < K(x,y), u(y) >_y dy \quad (x \in X),$$

$$(Mu)(x) = -\int_{\partial D} G_P({}^tP^*(y,D)\Phi(x,y), u(y)) \quad (x \in X \backslash \partial D),$$

$$(T_D f)(x) = \int_D < {}^tP^*(y,D)\Phi(x,y), f(y) >_y dy \quad (x \in X) \tag{2.4}$$

for $u \in H^p(E_{|D})$, $f \in L^2(F_{|D})$

By boundedness theorem for pseudodifferential operators (see [5]) and Stokes' formula we have

$$(Mu)(x) + (K_D u)(x) + (T_D P u)(x) = \begin{cases} u(x), & x \in D, \\ 0, & x \in X \setminus \overline{D} \end{cases} \tag{2.5}$$

for every $u \in H^p(E_{|D})$. Then the boundedness theorem for pseudodifferential operators implies that the integrals M, K_D and T_D given above define linear bounded operators $M : H^p(E_{|D}) \to H^p(E_{|D})$, $K_D : H^p(E_{|D}) \to H^p(E_{|D})$ and $T_D : L^2(F_{|D}) \to H^p(E_{|D})$.

Example 2.6 Let Y be a relatively compact domain with smooth boundary ∂Y in an open set $\tilde{X} \subset \mathbb{R}^n$, $n > 1$, and P be an operator with injective symbol on $\tilde{X}$. Assume that P^*P has a bilateral fundamental solution on $\tilde{X}$ and let Φ is the Green function of the Dirichlet problem for the elliptic operator P^*P in Y. Nacinovich and Shlapunov [3] constructed a scalar product $h_D(.,.)$ on $H^p(E_{|D})$ defining an equivalent topology and such that the limit of iterations of double layer potentials $\lim_{N\to\infty} M^N$ gives the orthogonal projection to $S^p(D)$. They also proved that the integral $T_D f(x)$ satisfies $h_D(T_D f, v) = \int_D (f, Pv)_x dx$ for all $f \in L^2(F_{|D})$ and $v \in H^p(E_{|D})$. This case corresponds to the Hodge decomposition for the Dirichlet problem for P^*P in Y with $K = 0$ (cf. also Schulze, Shlapunov, Tarkhanov [7]).

In the sequel we will prove a similar result for the integrals T_D and M in our more general situation.

3 Construction of the scalar product $h_D(.,.)$.

For our purposes we need information on solvability of the DIRICHLET problem for the operator $\Delta = P^*P$ on a subdomain D of X.

Let U be a neighborhood of ∂D in X, and F_j $(0 \leq j \leq p-1)$ be vector bundles over U. Fix a Dirichlet system, say, $\{B_j\}_{j=0}^{p-1}$, of boundary differential operators $B_j \in do_j(E_{|U} \to F_j)$. This means that $\sigma(B_j)(x,\zeta)$ have the maximal ranks for all $x \in U$ and vectors ζ conormal to ∂D.

Problem 3.1 *Let* $\oplus_{j=0}^{p-1}\psi_j \in \oplus H^{p-j-1/2}(F_{j|\partial D})$ *and* $\phi \in L^2(E_{|D})$. *Find a section* $\psi \in H^p(E_{|D})$ *such that*

$$\begin{cases} P^*P\psi = \phi & in\, D; \\ (B_j\psi)_{|\partial D} = \psi_j & (0 \leq j \leq p-1). \end{cases}$$

We denote by $H_0^p(E_{|D})$ the space

$$H_0^p(E_{|D}) = \{u \in H^p(E_{|D}) : B_j u = 0 \ on\ \partial D\ for\ 0 \leq j \leq p-1\}.$$

Then $H_0^p(E_{|D})$ is the closure of $\mathcal{D}(E_{|D})$ in $H^p(E_{|D})$. In the following well-known statement $Z_0(D)$ stands for $S_P(D) \cap H_0^p(E_{|D})$ and $Z_0^\perp(D)$ consists of sections $\psi \in H^p(E_{|D})$ satisfying $\int_D (\psi(x), v(x))_x dx = 0$ for all $v \in Z_0(D)$.

Lemma 3.2 *Problem 3.1 is solvable if and only if*

$$\int_D (\phi, v)_x dx = 0 \quad for\ all\ v \in Z_0(D).$$

It has no more than a finite number of solutions; the difference between two solutions belongs to $Z_0(D)$. *Moreover, there exists a constant* $c > 0$ *such that*

$$\|\psi\|_{H^p(E_{|D})} \leq c \left(\|\phi\|_{L^2(E_{|D})} + \sum_{j=0}^{p-1} \|\psi_j\|_{H^{p-j-1/2}(F_{j|\partial D})} \right)$$

for every solution $\psi \in Z_0^\perp(D)$ *of Problem 3.1.*

Let $\tilde{S^p}_\Delta(X \backslash \overline{D})$ stand for $S^p_{P^*P}(X \setminus \overline{D}) \cap Z_0^\perp(X \setminus \overline{D})$. Using Lemma 3.2 we obtain a linear isomorphism

$$\tilde{S^p}_\Delta(X \backslash \overline{D}) \ni v \overset{\mathcal{R}^+}{\longrightarrow} \oplus_{j=0}^{p-1}(B_j v)_{|\partial D} \in \oplus_{j=0}^{p-1}(H^{p-j-1/2}(E_{|\partial D})).$$

Composing $(\mathcal{R}^+)^{-1}$ with the trace operator

$$H^p(E_{|D}) \ni u \overset{\mathcal{R}^-}{\longrightarrow} \oplus_{j=0}^{p-1}(B_j u)_{|\partial D} \in \oplus_{j=0}^{p-1}(H^{p-j-1/2}(E_{|\partial D})).$$

we obtain a continuous linear map $H^p(E_{|D}) \ni u \to S(u) \in \tilde{S^p}_\Delta(X \backslash \overline{D})$.

For $u \in H^p(E_{|D})$ we introduce now the following notation:

$$U(u)(x) = \begin{cases} u(x), & x \in D, \\ S(u)(x), & x \in X \setminus \overline{D}. \end{cases}$$

Because $(B_j S(u))_{|\partial D} = (B_j u)_{|\partial D}$ $(0 \le j \le p-1)$, we have $U(u) \in H^p(E)$.

Theorem 3.3 *The* HERMITIAN *form $h_D(u,v) = h(U(u), U(v))$ is a scalar product in $H^p(E_{|D})$ defining a topology equivalent to the original one. Moreover the integral $T_D f$ satisfies $h_D(T_D f, u) = \int_D (f, Pu)_x dx$ for all $f \in L^2(F_{|D})$, $u \in H^p(E_{|D})$.*

Proof. Proposition 2.2 implies that $\sqrt{(h_D(.,.))}$ is not weaker than the standard norm $\|.\|_{H^p(E_{|D})}$.

On the other hand,

$$h_D(u,u) = h(U(u), U(u)) \le c_1 \; \|U(u)\|^2_{H^p(E)} \le$$

$$2c_1 \; \left(\|u\|^2_{H^p(E_{|D})} + \|S(u)\|^2_{H^p(E_{|X\setminus D})} \right) \qquad \text{for all } u \in H^p(E_{|D}).$$

Now using Lemma 3.2 and continuity of the trace operator we see that there exist positive constants c_3, c_4 such that

$$\|S(u)\|^2_{H^p(E_{|X\setminus D})} \le c_3 \sum_{j=0}^{p-1} \|B_j u\|^2_{H^{p-j-1/2}(F_{j|\partial D})} \le c_4 \; \|u\|_{H^p(E_{|D})}.$$

for all $u \in H^p(E_{|D})$. This proves the equivalence of the topologies.

Proposition 3.4 *For every $u, v \in H^p(E_{|D})$, $f \in L^2(F_{|D})$*

$$h_D(T_D f, v) = \int_D (f, Pv)_x dx,$$

$$h_D((M + K_D)u, v) = \int_{X\setminus D} (PS(u), PS(v))_x dx + \int_X (KU(u), KU(v))_x dx.$$

Proof. Let $f \in \mathcal{D}(F_{|D})$. Then $T_D f \in H^p(E)$ (and even in $C^\infty(E)$). Let us show that $U(Tf_{|D}) = Tf$. For this we need to check that $(T_D f)_{|X\setminus\overline{D}} \in \tilde{S^p}_\Delta(X\backslash\overline{D})$. However, $T_D f = Tf = \Phi(P^* f)$ and therefore $P^*PT_D f = P^* f - KP^* f = P^* f$ on X. Because $f \in \mathcal{D}(F_{|D})$ we see that $P^*PT_D f = 0$ in $X \setminus \overline{D}$. Since $Z_0(X \setminus \overline{D}) \subset Z(X)$ and $\Phi(P^* f)$ is orthogonal to $Z(X)$ we conclude $T_D f_{|X\setminus\overline{D}} \in Z_0^\perp(X \setminus \overline{D})$, as desired.

Further, if $u \in H^p(E_{|D})$ then Proposition 2.2 implies that

$$h_D(T_D f, u) = h(Tf, U(u)) = \int_X (f, PU(u))_x dx = \int_D (f, Pu)_x dx.$$

Since $\mathcal{D}(F_{|D})$ is dense in $L^2(F_{|D})$ and the operator T_D is bounded, this formula holds for every $u \in H^p(E_{|D})$ and every $f \in L^2(F_{|D})$. Finally, (2.5) implies that

$$h_D((M + K_D)u, v) = h_D(u - T_D P u, v) =$$

$$\int_{X \setminus D} (PS(u), PS(v))_x dx + \int_X (KU(u), KU(v))_x dx.$$

□

This proves the theorem. □

Corollary 3.5 *The operators $T_D P : H^p(E_{|D}) \to H^p(E_{|D})$ and $(M + K_D) : H^p(E_{|D}) \to H^p(E_{|D})$ are bounded linear self-adjoint non-negative operators with $\|T_D P\| \leq 1$, $\|M + K_D\| \leq 1$.*

Now it is easy to see that $Z_0^\perp(D)$ is the orthogonal complement to $Z_0(D)$ in the space $H^p(E_{|D})$ with respect to $h_D(.,.)$. Indeed, $Z_0(D) \subset Z(X)$ because every element $u \in Z_0(D)$ may be extended by zero from D to X as a solution to $Pu = 0$ on X. Then $S(u) = 0$ for all $u \in Z_0(D)$ and $h_D(u, v) = \int_X (KU(u), KU(v))_x dx = \int_D (u, v)_x dx$.

Corollary 3.5 implies that it is possible to consider iterations $(M + K_D)^\nu$ and $(T_D P)^\nu$ of the integrals $(M + K_D)$ and $T_D P$ respectively in the Sobolev space $H^p(E_{|D})$. In the following statement $\Pi(\Sigma)$ stands for the orthogonal (with respect to $h_D(.,.)$) projection to a closed subspace Σ in $H^p(E_{|D})$.

Corollary 3.6 *In the strong operator topology in $H^p(E_{|D})$*

$$\lim_{\nu \to \infty} (M + K_D)^\nu = \Pi(S_P^p(D)), \ \lim_{\nu \to \infty} (T_D P)^\nu = \Pi(\ker(M + K_D)).$$

In the strong operator topology in $L^2(F_{|D})$

$$\lim_{\nu \to \infty} (I - PT_D)^\nu = \Pi(\ker(T_D)).$$

Proof. It follows from Corollary 3.5 that

$$\lim_{\nu \to \infty} (M + K_D)^\nu = \Pi(I - K_D - M), \qquad \lim_{\nu \to \infty} (I - PT_D)^\nu = \ker(PT_D),$$

$$\lim_{\nu \to \infty} (T_D P)^\nu = \ker(I - T_D P)$$

in the strong operator topology in $H^p(E_{|D})$ (see, for instance, [3], §2, or [4] for compact operators). Theorem 3.4 and (2.5) imply that $\ker(I - T_D P) = \ker(M + K_D)$, $\ker T_D P = S_P^p(D)$ and $\ker PT_D = \ker T_D$. □

4 Solvability conditions for equation $Pu = f$

In this section we will use Corollary 3.6 to investigate solvability of equation $Pu = f$ in D. In particular, when it is solvable we will obtain an expression of the solution by means of a series which can be computed from the data.

Corollary 4.1 *In the strong operator topology in $H^p(E_{|D})$*

$$I = \Pi(S^p(D)) + \sum_{\mu=0}^{\infty}(M + K_D)^\mu(T_D P), \tag{4.1}$$

$$I = \Pi(\ker(M + K_D)) + K_D + \sum_{\mu=0}^{\infty}(T_D P)^\mu M. \tag{4.2}$$

In the strong operator topology in $L^2(F_{|D})$

$$I = \Pi(\ker T_D) + \sum_{\mu=0}^{\infty} P(M + K_D)^\mu T_D. \tag{4.3}$$

Proof. Formula (2.5) implies that for every $\nu \in \mathbb{N}$

$$I = (I - PT_D)^N \ + \sum_{\mu=0}^{N-1}(I - PT_D)^\mu PT_D. \tag{4.4}$$

It is easy to see from (2.5) that

$$(I - PT_D)^\mu PT_D = P(I - T_D P)^\mu T_D = P(M + K_D)^\mu T_D.$$

Then using Theorem 3.6 we can pass to the limit for $N \to \infty$ in (4.4), obtaining (4.3). The proof for (4.1) and 4.2) is similar. □

Theorem 4.2 *Let $f \in L^2(F_{|D})$. There exists $u \in H^p(E)$ satisfying $Pu = f$ on D if and only if*

(1) *the series $Rf = \sum_{\mu=0}^{\infty}(M + K_D)^\mu T_D f$ converges in $H^p(E_{|D})$;*

(2) *$\int_D (g, f)_x dx = 0$ for all $g \in \ker T_D$.*

Moreover, if (1) and (2) hold then $PRf = f$ on D.

Proof. The necessity follows from Theorem 3.3 and Corollary 4.1.

Back, let (1) and (2) hold true. Then Corollary 4.1 implies that $f = \sum_{\mu=0}^{\infty} P(M + K_D)^\mu T_D f$. Because the series Rf converges in $H^p(E_{|D})$ we conclude that $f = PRf$. □

Nacinovich and Shlapunov [3] proved such a result for the case considered in Example 2.6.

Remark 4.3 *Corollary 3.6 imply that the solution $u = Rf$ to equation $Pu = f$ in D belongs to $(S^p_P(D))^\perp$ where $(S^p_P(D))^\perp$ is the orthogonal (with respect to $h_D(.,.)$) complement of $S^p_P(D)$ in $H^p(E_{|D})$; it is the unique solution belonging to this subspace. The partial sums $R_N f$ of the series Rf may be regarded as approximate solutions to equation $Pu = f$ in D. It easily follows from Corollaries 3.6 and 4.1 that $R_N f \in (S^p_P(D))^\perp$ for every $N \in \mathbb{N}$ and that $\lim_{N\to\infty} \|PR_N f - f - \Pi(\ker T_D) f\|_{L^2(F_{|D})} = 0$ for every $f \in L^2(F_{|D})$.*

Remark 4.4 *Let us denote by $\tilde{g}$ the extension of $g \in L^2(F_{|D})$ from D to X by zero. Then $0 = T_D g = T\tilde{g}$ and therefore $\ker T_D$ is the set of all functions g from $L^2(F_{|D})$ satisfying $P^*\tilde{g} = 0$ weakly on X (see Remark 2.4).*

If P can be included into an elliptic complex (E^i, P^i) then condition (2) in Theorem 4.2 may be replaced by

(2a) $P^1 f = 0$ on D and $\int_D (g, f)_x dx = 0$ for all $g \in \ker T_D \cap S^0_{P_1}(D)$.

Let, as before, $\{B_j\}_{j=0}^{p-1}$ be a Dirichlet system of order $(p-1)$ on ∂D, $\{C_j\}_{j=0}^{p-1}$ be the Dirichlet system dual to $\{B_j\}$ with respect to Green formula for the operator P (see, for instance, [9], Lemma 28.3), and let

$$\mathfrak{H}(D) = \{g \in L^2(F_{|D}) \text{ such that } P^*g = 0,\ P^1 g = 0 \text{ in } D,\ \text{weakly satisfying the boundary conditions } ({}^tC_j^* g)_{|\partial D} = 0,\ 0 \le j \le p-1\}.$$

We call $\mathfrak{H}(D)$ "harmonic" space for complex $\{E^i, P^i\}$ in D. By the ellipticity assumptions, $\mathfrak{H}(D) \subset C^\infty(F_{|D})$. It is not difficult to show that for the Dolbeault complex this definition of the harmonic space $\mathfrak{H}(D)$ is equivalent to the one given in [1]. It is easy to see that $\ker T_D \cap S^0_{P^1}(D) = \mathfrak{H}(D)$. However the space $\mathfrak{H}(D)$ fails to be finite-dimensional in general (cf. Remark 2.5).

5 Applications to P-Neumann problem

In this section we show how Theorem 3.3 can be used to study the P-Neumann problem associated to an elliptic operator $P \in do_p(E \to F)$.

Problem 5.1 *Let $\phi \in L^2(E_{|D})$ and $\psi_j \in H^{-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$ be given sections. Find $\psi \in H^p(E_{|D})$ such that*

$$\begin{cases} P^*P\psi = \phi & \text{in } D; \\ {}^tC_j^* P\psi = \psi_j & \text{on } \partial D \\ (0 \le j \le p-1). \end{cases}$$

The equation $P^*P\psi = \phi$ in D has to be understood in the sense of distributions, while the boundary values are interpreted in the variational sense :

$$\int_D (\phi, v)_x dx - \int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j}) B_j v, \psi_j >_y ds(y) =$$

$$\int_D (P\psi, Pv)_y dy \text{ for every } v \in C^\infty(E_{|\overline{D}}). \tag{5.1}$$

Proposition 5.2 *Let $\phi = 0$ and $\psi_j = 0$ for all $0 \le j \le p-1$. Then $\psi \in H^p(E_{|D})$ is a solution of* Problem 5.1 *if and only if $\psi \in S_P^p(D)$.*

Proof. Obviously, $\psi \in S_P^p(D)$ is a solution of Problem 5.1 with $\phi = 0$ and $\psi_j = 0$ $(0 \le j \le p-1)$. Conversely, if ψ is a solution of Problem 5.1 with $\phi = 0$ and $\psi_j = 0$ $(0 \le j \le p-1)$ then $T_D P\psi = 0$. Hence

$$\psi = (M + K_D)\psi = \lim_{\nu\to\infty} (M + K_D)^\nu \psi,$$

i.e. $\psi \in S_P^p(D)$ (see Corollary 3.6). □

The operator P^*P is elliptic with C^∞ coefficients, and the ranks of the symbols of the boundary operators $({}^tC_j^*)$ are maximal in a neighborhood of ∂D. Nevertheless, since, the space $S_P^p(D)$ is not finite dimensional in general, Proposition 5.2 shows that Problem 5.1 may be ill-posed.

In the following theorem we set

$$\widetilde{T_D}(\oplus\psi_j)(x) = \int_{\partial D} \sum_{j=0}^{p-1} < {}^tB_j^*(y)\Phi(x,y), \psi_j(y) >_y ds(y),$$

$$V(\phi)(x) = \int_D < \Phi(x,y), \phi(y) >_y dy.$$

Theorem 5.3 Problem 5.1 *is solvable if and only if*

(1) $\int_D (\phi, v)_x dx - \int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j}) B_j v, \psi_j >_y ds(y) = 0$
for every $v \in S_P^p(D)$;

(2) *the series*
$r(\phi, \oplus\psi_j) = \sum_{\mu=0}^\infty (M + K_D)^\mu (V(\phi) - \widetilde{T_D}(\oplus\psi_j))$
converges in the $H^p(E_{|D})$-norm.

If (1) and (2) hold then $r(\phi, \oplus\psi_j)$ is a solution to Problem 5.1.

Proof. Let Problem 5.1 be solvable and let $\psi \in H^p(E_{|D})$ be a solution. Then $V(\phi) - \widetilde{T_D}(\oplus\psi_j) = T_D P\psi$, and, due to Corollary 3.6, the series $RP\psi = r(\phi, \oplus\psi_j)$ converges in the $H^p(E_{|D})$-norm.

Conversely, assume that (1) and (2) hold true. Let us prove that the series $r(\phi, \oplus\psi_h)$ satisfies (5.1). First, we note that

$$T_D Pr(\phi, \oplus\psi_j) = (I - (M + K_D))r(\phi, \oplus\psi_j) = V(\phi) - \widetilde{T_D}(\oplus\psi_j). \quad (5.2)$$

Using (2.5) we see that for every $v \in C^\infty(E_{|D})$ and $x \in D$ we have

$$v(x) = \int_D < \Phi(x,y), P^*Pv(y) >_y dy + (K_D v)(x) - \int_{\partial D} G_{P^*P}(\Phi(x,y), v(y)).$$

Then Fubini theorem and (5.2) imply that

$$\int_D (Pr(\phi, \oplus\psi_j, Pv)_x dx = \int_D (T_D Pr(\phi, \oplus\psi_j), P^*Pv)_y dy -$$

$$\int_{\partial D} G_{P^*P}(T_D Pr(\phi, \oplus\psi_j), v(y)) = \int_D (V(\phi) - \widetilde{T_D}(\oplus\psi_j), P^*Pv)_y dy -$$

$$\int_{\partial D} G_{P^*P}(V(\phi) - \widetilde{T_D}(\oplus\psi_j), v(y)) =$$

$$\int_D (\phi, v - K_D v)_x dx - \int_{\partial D} \sum_{j=0}^{p-1} < *B_j(v - K_D v), \psi_j >_x dx.$$

Finally, since $K_D v \in S^p_P(D)$ condition (1) implies that (5.1) holds. □

Of course, if Shapiro-Lopatiskii condition holds true for Problem 5.1 then this problem is a Fredholm one and the series $r(\phi, \oplus\psi_j)$ converges for all data ϕ and ψ_j.

For $P = \overline{\partial}$ in $\mathbb{C}^n$ a similar formula for solutions to the $\overline{\partial}$-Neumann problem was obtained by Kytmanov (see [2], p.177). In the situation considered in Example 2.6 such a theorem was proved in [3].

6 Examples

Let P be a homogeneous $(l \times k)$-operator with constant coefficient in $\mathbb{R}^n$ $(n \geq 2)$ having injective symbol of order $p \geq 1$. In this case P^*P has the standard fundamental solution Φ of convolution type. Then for $n > 4p$ we have the Hodge decomposition in $L^2(\mathbb{R}^n)$ with $K = 0$ and with Φ "vanishing at infinity" (see [9], p. 74). In this case $S(u) \in [H^p(\mathbb{R}^n \setminus \overline{D})]^k$ is the solution "vanishing at infinity" to the exterior Dirichlet problem for P^*P and D and we are able to consider the scalar product $h_D(u,v) = \int_{\mathbb{R}^n} (PU(v))^*(x)(PU(u))(x)dx$ on $[H^p(D)]^k$. Thus this situation corresponds to compactification of $\mathbb{R}^n$ with one point at infinity.

Example 6.1 Let P be the gradient operator in $\mathbb{R}^n$. Then $(-P^*P)$ is the usual Laplace operator in $\mathbb{R}^n$, $\Phi = \phi_n$ is the standard fundamental solution to the Laplace operator in $\mathbb{R}^n$. The compatibility complex for P is de Rham complex and Problem 5.1 is the classical Neumann problem. It is well known to be an elliptic boundary value problem.

Example 6.2 Let P be the Cauchy-Riemann system in $\mathbb{C}^n$. In this case $(-4P^*P)$ is the usual Laplace operator in $\mathbb{R}^{2n}$. The compatibility complex for P is Dolbeault complex and Problem 5.1 is the $\overline{\partial}$-Neumann problem (see, for instance, [2]). It is well known to be an ill-posed boundary value problem.

ACKNOWLEDGMENTS

The author thanks prof. N.N. Tarkhanov and prof. B.-W. Schulze for many useful remarks.

References

[1] L. Hörmander. L^2-estimates and existence theorems for the $\overline{\partial}$ operator. Acta Math. **113(1–2)**, 89 –152, 1965.

[2] A.M. Kytmanov. The Bochner-Martinelli integral, and its applications. Nauka, Siberian branch, Novosibirsk, 1992 (Russian).

[3] M. Nacinovich, A. A. Shlapunov. On iterations of the Green integrals and their applications to elliptic differential complexes. *Math. Nachr.*, **180**, 1996, 243–286.

[4] M.M. Lavrent'ev, V.G. Romanov, S.P. Shishatskii. Ill-posed problems of mathematical physics and analysis. Nauka, Moscow, 1980, 286pp.

[5] S. Rempel, B.-W. Schulze. Index theory of elliptic boundary problems. Akademie-Verlag. Berlin, 1982.

[6] A.V. Romanov. Convergence of iterations of Bochner -Martinelli operator, and the Cauchy - Riemann system. Dokl. AN SSSR **242 (4)**, 780–783, 1978; English transl.: Soviet Math. Dokl **19 (5)** 1978.

[7] B.-W. Schulze, A. A. Shlapunov, N.N. Tarkhanov. Regularization of mixed boundary problem. Prepr. 99/9, Institute für Mathematik, Universität Potsdam, 1999, 32 pp.

[8] N.N. Tarkhanov. Parametrix method in theory of differential complexes. Nauka, Novosibirsk, 1990; English transl.: Complexes of differential operators. Kluwer Academic Publishers, Dordrecht, NL, 1995.

[9] N.N. Tarkhanov. Laurent series for solutions of elliptic systems. Novosibirsk, 1991; English transl.: The analysis of solutions of elliptic equations. Kluwer Academic Publishers, Dordrecht, NL, 1997.