HYPERGEOMETRIC SYSTEMS OF DIFFERENTIAL EQUATIONS AND AMOEBAS OF RATIONAL FUNCTIONS

T. Sadykov

Abstract

We study the approach to the theory of hypergeometric functions in several variables via a generalization of the Horn system of differential equations. A formula for the dimension of its solution space is given. Using this formula we construct an explicit basis in the space of holomorphic solutions to the generalized Horn system under some assumptions on its parameters. These results are applied to the problem of describing the complement of the amoeba of a rational function, which was posed in [12].

0. Introduction

There exist several approaches to the notion of a hypergeometric function depending on several complex variables. It can be defined as the sum of a power series of a certain form (such series are known as $\Gamma-$series) [10], as a solution to a system of partial differential equations [9], [11], [1], or as a Mellin-Barnes integral [14]. In the present paper we study the approach to the theory of hypergeometric functions via a generalization of the Horn system of differential equations. We consider the system of partial differential equations of hypergeometric type

$$x^{u_i} P_i \left(x \frac{\partial}{\partial x} \right) y(x) = Q_i \left(x \frac{\partial}{\partial x} \right) y(x), \ i = 1, \ldots, n. \tag{1}$$

Here the vectors $u_i = (u_{i1}, \ldots, u_{in}) \in \mathbb{Z}^n$ are assumed to be linearly independent, P_i, Q_i are nonzero polynomials in n complex variables and

$$x \frac{\partial}{\partial x} = \left(x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n} \right)^T.$$

We use the notation $x^{u_i} = x_1^{u_{i1}} \ldots x_n^{u_{in}}$. If $\{u_i\}_{i=1}^n$ form the standard basis of the lattice $\mathbb{Z}^n$ then the system (1) coincides with a classical system of partial differential equations which goes back to Horn and Mellin (see [13], § 1.2 of [10] and section 9.4 of [17]). It was originally introduced as a natural system of partial differential equations having a given series of hypergeometric type [10] as one of its solutions. In the present paper the system (1) is referred to as the generalized Horn system of hypergeometric differential equations. We call a function $y(x)$ hypergeometric if it satisfies (1) for some choice of the polynomials P_i, Q_i.

One of the reasons for studying the generalized Horn system is the fact that knowing the structure of solutions to (1) allows one to investigate the so-called amoeba of the inverse of a solution to (1). The notion of amoebas was introduced by Gelfand, Kapranov and Zelevinsky (see [12], Chapter 6, § 1). Given a mapping $f(x)$, its amoeba $\mathcal{A}_f$ is the image of the hypersurface $f^{-1}(0)$ under the map $(x_1, ..., x_n) \mapsto (\log|x_1|, ..., \log|x_n|)$. To describe the connected components of the complement of $\mathcal{A}_f$ for a Laurent polynomial $f(x)$ was pointed out in [12] (see Chapter 6, Remark 1.10) as a difficult and interesting problem. In section 4 we use the results on the structure of solutions to (1) for computing the number of connected components of the complement of amoebas of some rational functions. The problem of describing the class of rational hypergeometric functions was studied in a different setting in [6], [7]. The definition of a hypergeometric function used in these papers is based on the Gelfand-Kapranov-Zelevinsky system of differential equations [9], [10], [11].

Solutions to (1) are closely related to the notion of a generalized Horn series which is defined as a formal (Laurent) series

$$y(x) = x^{\gamma} \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s, \tag{2}$$

whose coefficients $\varphi(s)$ are characterized by the property that $\varphi(s + u_i) = \varphi(s) R_i(s)$. Here $R_i(s)$ are rational functions. We also use notations $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, $\operatorname{Re} \gamma_i \in [0, 1)$, $x^s = x_1^{s_1} \ldots x_n^{s_n}$. In the case when $\{u_i\}_{i=1}^n$ form the standard basis of $\mathbb{Z}^n$ we get the definition of the classical Horn series (see [10], § 1.2).

In the case of two or more variables the generalized Horn system (1) is in general not solvable in the class of series (2) without additional assumptions on the polynomials P_i, Q_i. In section 1 we investigate solvability of hypergeometric systems of equations and describe supports of solutions to the generalized Horn system. The necessary and sufficient conditions for a formal solution to the system (1) in the class (2) to exist are given in Theorem 1.

In section 2 we consider the $\mathcal{D}-$module associated with the generalized Horn system. We give a formula which allows one to compute the dimension of the space of holomorphic solutions to (1) at a generic point under some additional assumptions on the system under study (Theorem 8). We give also an estimate for the dimension of the solution space of (1) under less restrictive assumptions on the parameters of the system (Corollary 9). The author benefited greatly from reading paper [1] by A. Adolphson whose ideas are used in section 2.

In section 3 we consider the case when the polynomials P_i, Q_i can be factorized up to polynomials of degree 1 and construct an explicit basis in the space of holomorphic solutions to some systems of the Horn type. We show that in the case when

$R_i(s+u_j)R_j(s) = R_j(s+u_i)R_i(s)$, $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i(s) > \deg P_i(s)$, $i,j = 1,\ldots,n$, $i \neq j$, there exists a basis in the space of holomorphic solutions to (1) consisting of series (2) if the parameters of the system under study are sufficiently general (Theorem 10).

In section 4 we apply the results on the generalized Horn system to the problem of describing the complement of the amoeba of a rational function. We show how Theorem 1 can be used for studying Laurent series developments of a rational solution to (1). A class of rational hypergeometric functions with minimal number of connected components of the complement of the amoeba is described.

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1. Supports of solutions to the generalized Horn system

Suppose that the series (2) represents a solution to the system (1). Computing the action of the operator $x^{u_i} P_i\left(x\frac{\partial}{\partial x}\right) - Q_i\left(x\frac{\partial}{\partial x}\right)$ on this series we arrive at the following system of difference equations

$$\varphi(s+u_i)Q_i(s+\gamma+u_i) = \varphi(s)P_i(s+\gamma), \ i = 1,\ldots,n. \tag{3}$$

The system (3) is equivalent to (1) as long as we are concerned with those solutions to the generalized Horn system which admit a series expansion of the form (2). Let $\mathbb{Z}^n + \gamma$ denote the shift in $\mathbb{C}^n$ of the lattice $\mathbb{Z}^n$ with respect to the vector γ. Without loss of generality we assume that the polynomials $P_i(s), Q_i(s+u_i)$ are relatively prime for all $i = 1,\ldots,n$. In this section we shall describe nontrivial solutions to the system (3) (i.e. those ones which are not equal to zero identically). While looking for a solution to (3) which is different from zero on some subset S of $\mathbb{Z}^n$ we shall assume that the polynomials $P_i(s), Q_i(s)$, the set S and the vector γ satisfy the condition

$$|P_i(s+\gamma)| + |Q_i(s+\gamma+u_i)| \neq 0, \tag{4}$$

for any $s \in S$ and for all $i = 1,\ldots,n$. That is, for any $s \in S$ the equality $P_i(s+\gamma) = 0$ implies that $Q_i(s+\gamma+u_i) \neq 0$ and $Q_i(s+\gamma+u_i) = 0$ implies $P_i(s+\gamma) \neq 0$.

The system of difference equations (3) is in general not solvable without further restrictions on P_i, Q_i. Let $R_i(s)$ denote the rational function $P_i(s)/Q_i(s+u_i)$, $i = 1,...,n$. Increasing the argument s in the ith equation of (3) by u_j and multiplying the obtained equality by the jth equation of (3), we arrive at the relation $\varphi(s+u_i+u_j)/\varphi(s) = R_i(s+u_j)R_j(s)$. Analogously, increasing the argument in the jth equation of (3) by u_i and multiplying the result by the ith equation of (3), we arrive at the equality $\varphi(s+u_i+u_j)/\varphi(s) = R_j(s+u_i)R_i(s)$. Thus the conditions

$$R_i(s+u_j)R_j(s) = R_j(s+u_i)R_i(s), \ \ i,j = 1,...,n \tag{5}$$

are in general necessary for (3) to be solvable. The conditions (5) will be referred to as the compatibility conditions for the system (3). Throughout this paper we assume that the polynomials P_i, Q_i defining the generalized Horn system (1) satisfy (5).

Let U denote the matrix whose rows are the vectors $u_1, \ldots, u_n$. A set $S \subset \mathbb{Z}^n$ is said to be $U-$*connected* if any two points in S can be connected by a polygonal line with the vectors $u_1, \ldots, u_n$ as sides and vertices in S. Let $\varphi(s)$ be a solution to (3). We define the *support* of $\varphi(s)$ to be the subset of the lattice $\mathbb{Z}^n$ where $\varphi(s)$ is different from zero. A formal series $x^\gamma \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s$ is called a *formal solution* to the system (1) if the function $\varphi(s)$ satisfies the equations (3) at each point of the lattice $\mathbb{Z}^n$. The following Theorem gives necessary and sufficient conditions for a solution to the system (3) supported in some set $S \subset \mathbb{Z}^n$ to exist.

Theorem 1 *For $S \subset \mathbb{Z}^n$ define*

$$S_i^{'} = \{s \in S : s + u_i \notin S\}, \ S_i^{''} = \{s \notin S : s + u_i \in S\}, \ i = 1, \ldots, n.$$

Suppose that the conditions (4) are satisfied on S. Then there exists a solution to the system (3) supported in S if and only if the following conditions are fulfilled:

$$P_i(s+\gamma)|_{S_i^{'}} = 0, \ \ Q_i(s+\gamma+u_i)|_{S_i^{''}} = 0, \ i = 1, \ldots, n, \tag{6}$$

$$P_i(s+\gamma)|_{S \setminus S_i^{'}} \neq 0, \ \ Q_i(s+\gamma+u_i)|_{S} \neq 0, \ i = 1, \ldots, n. \tag{7}$$

Proof Necessity. Let $\varphi(s)$ be a solution to (3), $S = \operatorname{supp} \varphi$. Let $s^{(0)} \in S_i^{'}$. Since $\varphi(s^{(0)}) \neq 0$ and $\varphi(s^{(0)} + u_i) = 0$, it follows from the ith equation of (3) that $P_i(s^{(0)} + \gamma) = 0$. Analogously, if $s^{(0)} \in S_i^{''}$ then $\varphi(s^{(0)}) = 0$, $\varphi(s^{(0)} + u_i) \neq 0$, which yields $Q_i(s^{(0)} + \gamma + u_i) = 0$. This proves the necessity of the conditions (6) for the system (3) to be solvable. To show the necessity of (7) we assume that $P_i(s^{(0)} + \gamma) = 0$ for $s^{(0)} \in S$. Then the ith equation of (3) together with (4) gives $\varphi(s^{(0)} + u_i) = 0$, which means that $s^{(0)} \in S_i^{'}$. Next, if $Q_i(s^{(0)} + \gamma + u_i) = 0$ then it follows from (3) and (4) that $\varphi(s^{(0)}) = 0$, that is, $s^{(0)} \notin S$.

Sufficiency. We shall construct a function $\varphi_S(s)$ satisfying (3) and supported in S. Without loss of generality we may assume that the set S is $U-$connected. Choose an arbitrary point $s^{(0)} \in S$ and set $\varphi_S(s^{(0)}) = 1$. The equations (3) may be viewed as recurrent relations which allow one to compute the value of $\varphi_S(s^{(0)} \pm u_i)$ for any $i = 1, \ldots, n$ unless $P_i(s^{(0)} + \gamma) = 0$ or $Q_i(s^{(0)} + \gamma + u_i) = 0$. Repeating this argument, we can define $\varphi_S(s)$ for any $s \in S$ since by (7) the polynomial $Q_i(s + \gamma + u_i)$ does not vanish on S for any $i = 1, \ldots, n$ and since the polynomial $P_i(s+\gamma)$ vanishes on $S_i^{'}$ only. The function $\varphi_S(s)$ is well-defined since the compatibility conditions (5) are fulfilled. (These conditions imply that the value of $\varphi_S(s)$ at a point $s^{(1)} \in S$ obtained by iterating the equations (3) does not depend on the path connecting $s^{(0)}$ and $s^{(1)}$.)

Let us define $\varphi_S(s)$ to be zero outside S. The function $\varphi_S(s)$ satisfies the equations (3) on $\mathbb{Z}^n \setminus \left(\bigcup_{i=1}^n (S_i' \cup S_i'')\right)$ by the construction. This follows since $\varphi(s) = \varphi(s+u_i) = 0$ on $\mathbb{Z}^n \setminus \left(S \cup (\bigcup_{i=1}^n S_i'')\right)$, for any $i = 1,\ldots,n$, and since $\varphi(s)$ was defined through the equations (3) on $S \setminus \left(\bigcup_{i=1}^n S_i'\right)$. The conditions (6) yield that these equations are also satisfied on $\bigcup_{i=1}^n (S_i' \cup S_i'')$, which shows that the conditions (6),(7) are sufficient for a solution to (3) supported in S to exist. The proof is complete.

Theorem 1 will be used in section 3 for constructing an explicit basis in the space of holomorphic solutions to the generalized Horn system in the case when $\deg Q_i > \deg P_i$ and $Q_i(s+u_j) = Q_i(s)$, $i,j = 1,\ldots,n$, $i \neq j$. In the next section we compute the dimension of the space of holomorphic solutions to (1) at a generic point.

2. Holomorphic solutions to the generalized Horn system

Let G_i denote the differential operator $x^{u_i} P_i\left(x\frac{\partial}{\partial x}\right) - Q_i\left(x\frac{\partial}{\partial x}\right)$, $i = 1,\ldots,n$. Let $\mathcal{D}$ be the Weyl algebra in n variables [4], and define $\mathcal{M} = \mathcal{D}/\sum_{i=1}^n \mathcal{D}G_i$ to be the left $\mathcal{D}-$module associated with the system (1). Let $R = \mathbb{C}[z_1,\ldots,z_n]$ and $R[x] = R[x_1,\ldots,x_n] = \mathbb{C}[x_1,\ldots,x_n,z_1,\ldots,z_n]$. We make $R[x]$ into a left $\mathcal{D}-$module by defining the action of ∂_j on $R[x]$ by

$$\partial_j = \frac{\partial}{\partial x_j} + z_j. \tag{8}$$

Let $\Phi : \mathcal{D} \to R[x]$ be the $\mathcal{D}-$linear map defined by

$$\Phi(x_1^{a_1}\ldots x_n^{a_n}\partial_1^{b_1}\ldots\partial_n^{b_n}) = x_1^{a_1}\ldots x_n^{a_n} z_1^{b_1}\ldots z_n^{b_n}. \tag{9}$$

It is easily checked that Φ is an isomorphism of $\mathcal{D}-$modules. In this section we establish some properties of linear operators acting on $R[x]$. We aim to construct a commutative family of $\mathcal{D}-$linear operators $W_i : R[x] \to R[x]$, $i = 1,\ldots,n$ which satisfy the equality $\Phi(G_i) = W_i(1)$. The crucial point which requires additional assumptions on the parameters of the system (1) is the commutativity of the family $\{W_i\}_{i=1}^n$ which is needed for computing the dimension (as a $\mathbb{C}-$vector space) of the module $R[x]/\sum_{i=1}^n W_i R[x]$ at a fixed point $x^{(0)}$. We construct the operators W_i and show that they commute with one another under some additional assumptions on the polynomials $Q_i(s)$ (Lemma 4). However, no additional assumptions on the polynomials $P_i(s)$ are needed as long as the compatibility conditions (5) are fulfilled.

Following the spirit of Adolphson [1] we define operators $D_i : R[x] \to R[x]$ by setting

$$D_i = z_i\frac{\partial}{\partial z_i} + x_i z_i,\ i = 1,\ldots,n. \tag{10}$$

It was pointed out in [1] that the operators (10) form a commutative family of $\mathcal{D}-$linear operators. Let D denote the vector $(D_1, \ldots, D_n)$. For any $i = 1, \ldots, n$ we define operator $\nabla_i : R[x] \to R[x]$ by $\nabla_i = z_i^{-1} D_i$. This operator commutes with the operators ∂_j since both D_i and the multiplication by z_i^{-1} commute with ∂_j. Moreover, the operator ∇_i commutes with ∇_j for all $1 \leq i, j \leq n$ and with D_j for $i \neq j$. In the case $i = j$ we have $\nabla_i D_i = \nabla_i + D_i \nabla_i$.

Remark 1 A power of the operator $x_i \frac{\partial}{\partial x_i}$ admits the following expansion: $\left(x_i \frac{\partial}{\partial x_i}\right)^k = \sum_{j=1}^k S_{j,k} x_i^j \frac{\partial^j}{\partial x_i^j}$, where $S_{j,k}$ are the Stirling numbers of the second kind (see [2], page 89). This allows one to determine the constants $h_\alpha^{(i)}$ in the expansion of the operator $x_i P\left(x \frac{\partial}{\partial x}\right) - Q_i\left(x \frac{\partial}{\partial x}\right) = \sum_\alpha h_\alpha^{(i)} x^{\beta_\alpha^{(i)}} \frac{\partial^{|\alpha|}}{\partial x^\alpha}$.

The following lemma (which can be applied to a more general family of differential operators than $\{G_i\}_{i=1}^n$) gives $\mathcal{D}-$linear operators $W_i : R[x] \to R[x]$ which satisfy $\Phi(G_i) = W_i(1)$.

Lemma 2 *Let $m_1, \ldots, m_n \in \mathbb{N}_0$ and let $V_\alpha^{(i)}$, $|\alpha| \leq m_i$ be polynomials in n variables. Consider the family of generalized hypergeometric operators*

$$\tilde{G}_i = \sum_{|\alpha| \leq m_i} x^\alpha V_\alpha^{(i)} \left(x \frac{\partial}{\partial x}\right), \ i = 1, \ldots, n.$$

Let us define operators $\tilde{W}_i$, $i = 1, \ldots, n$ by

$$\tilde{W}_i = \sum_{|\alpha| \leq m_i} V_\alpha^{(i)}(D) \nabla^\alpha.$$

Then $\tilde{W}_i$ is a $\mathcal{D}-$linear operator on $R[x]$ satisfying $\Phi(\tilde{G}_i) = \tilde{W}_i(1)$.

Proof The $\mathcal{D}-$linearity of $\tilde{W}_i$ follows since the operators D_i and ∇_i are $\mathcal{D}-$linear, for all $i, j = 1, \ldots, n$. Thus we need to show that for any $\alpha, \beta \in \mathbb{N}_0^n$

$$(D_1^{\alpha_1} \ldots D_n^{\alpha_n} \nabla_1^{\beta_1} \ldots \nabla_n^{\beta_n})(1) = \Phi\left(x^\beta \left(x_1 \frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(x_n \frac{\partial}{\partial x_n}\right)^{\alpha_n}\right) \tag{11}$$

and that

$$(D_1^{\alpha_1} \ldots D_n^{\alpha_n})(1) = \Phi\left(\left(x_1 \frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(x_n \frac{\partial}{\partial x_n}\right)^{\alpha_n}\right). \tag{12}$$

The equation (11) follows from (12) since $D_i \nabla_j(1) = D_i(x_j) = x_j D_i(1)$ and $\Phi(x_j F) = x_j \Phi(F)$ for any differential operator $F \in \mathcal{D}$. For proving (12) we notice that $D_i^{\alpha_i}(1)$ can be written in the form $D_i^{\alpha_i}(1) = \sum_{k=1}^{\alpha_i} c_{k,\alpha_i} x_i^k z_i^k$. Since $D_i(1) = x_i z_i$ and

$$D_i^{k+1}(1) = c_{1,k} x_i z_i + \sum_{j=2}^k (j c_{j,k} + c_{j-1,k}) x_i^j z_i^j + x_i^{k+1} z_i^{k+1}$$

it follows that the constants $c_{j,k}$ are the Stirling numbers $S_{j,k}$ of the second kind as in Remark 1. Indeed, they are determined by the same recurrent relation with the same initial condition as $S_{j,k}$. Thus we get $D_i^{\alpha_i}(1) = \sum_{k=1}^{\alpha_i} S_{k,\alpha_i} x_i^k z_i^k$ and Remark 1 gives $D_i^{\alpha_i}(1) = \Phi\left(\left(x_i \frac{\partial}{\partial x_i}\right)^{\alpha_i}\right)$. The equality (12) follows now from the identities

$$D_1^{\alpha_1} \dots D_n^{\alpha_n}(1) = \prod_{i=1}^{n} D_i^{\alpha_i}(1),$$

$$\Phi\left(\left(x_1 \frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(x_n \frac{\partial}{\partial x_n}\right)^{\alpha_n}\right) = \prod_{i=1}^{n} \Phi\left(\left(x_i \frac{\partial}{\partial x_i}\right)^{\alpha_i}\right),$$

which hold for any $\alpha \in \mathbb{N}_0^n$. The proof is complete.

Thanks to Lemma 2 we may define operators $W_i = P_i(D)\nabla^{u_i} - Q_i(D)$ such that for any $i = 1, \dots, n$ W_i is a $\mathcal{D}-$linear operator satisfying the identity $\Phi(G_i) = W_i(1)$. It follows by the $\mathcal{D}-$linearity of W_i that $\sum_{i=1}^{n} W_i R[x]$ and $R[x]/\sum_{i=1}^{n} W_i R[x]$ can be considered as left $\mathcal{D}-$modules. The following argument is due to Adolphson (see [1], Theorem 4.4 and Lemma 4.12).

Theorem 3 *The following isomorphism holds true:*

$$\mathcal{M} \simeq R[x] \Big/ \left(\sum_{j=1}^{n} W_j R[x]\right). \tag{13}$$

Proof It follows by $\mathcal{D}-$linearity of the operators W_i that the sum $\sum_{j=1}^{n} \mathcal{D} G_j$ belongs to the kernel of the map

$$\mathcal{D} \to R[x] \Big/ \left(\sum_{j=1}^{n} W_j R[x]\right), \tag{14}$$

induced by the isomorphism of $\mathcal{D}-$modules $\Phi : \mathcal{D} \to R[x]$. To show that this sum is equal to the kernel it suffices to prove that for any $b = (b_1, \dots, b_n) \in \mathbb{N}_0^n$, and any $u_b(x) \in \mathbb{C}[x]$ there exists $\xi \in \mathcal{D} W_j$ such that $\Phi(\xi) = W_j\left(u_b(x) z^b\right)$. Let us define ξ by $\xi = u_b(x) \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{b_i} G_j$. Using $\mathcal{D}-$linearity of W_j and the equality $\Phi(G_i) = W_i(1)$ we obtain

$$\Phi\left(u_b(x) \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{b_i} G_j\right) = u_b(x) \prod_{i=1}^{n} \partial_i^{b_i} \Phi(G_j) =$$

$$u_b(x) \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + z_i\right)^{b_i} W_j(1) = W_j\left(u_b(x) \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + z_i\right)^{b_i}(1)\right) = W_j\left(u_b(x) z^b\right).$$

This shows that $\sum_{j=1}^{n} \mathcal{D}G_j$ coincides with the kernel of the map (14) and completes the proof of the Theorem.

In the general case the operators $W_i = P_i(D)\nabla^{u_i} - Q_i(D)$ do not commute since D_i does not commute with ∇_i. However, this family of operators may be shown to be commutative under some assumptions on the polynomials $Q_i(s)$ in the case when the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (5). The following Lemma holds.

Lemma 4 *The operators $W_i = P_i(D)\nabla^{u_i} - Q_i(D)$ commute with one another if and only if the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (5) and for any $i, j = 1, \dots, n$, $i \neq j$, $Q_i(s + u_j) = Q_i(s)$.*

Proof Since $\nabla_i = z_i^{-1} + D_i z_i^{-1}$ it follows that $\nabla_i D_i = \nabla_i + D_i \nabla_i$ and that ∇_i commutes with D_j for $i \neq j$. Hence for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$

$$\nabla_i D_1^{\alpha_1} \dots D_n^{\alpha_n} = D_1^{\alpha_1} \dots (D_i + 1)^{\alpha_i} \dots D_n^{\alpha_n} \nabla_i. \tag{15}$$

Let E_i^t denote the operator which increases the ith argument by t, that is, $E_i^t f(x) = f(x + te_i)$. Here $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{Z}^n$. It follows from (15) that

$$\nabla_i P_j(D) = (E_i^1 P_j)(D) \nabla_i. \tag{16}$$

For $\alpha \in \mathbb{Z}^n$ let E^α denote the composition $E_1^{\alpha_1} \circ \dots \circ E_n^{\alpha_n}$. Using (16) we compute the commutator of the operators W_i, W_j :

$$W_i W_j - W_j W_i = \Big(P_i(D)(E^{u_i} P_j)(D) - P_j(D)(E^{u_j} P_i)(D)\Big)\nabla^{u_i + u_j} +$$
$$\Big((E^{u_j} Q_i)(D) - Q_i(D)\Big) P_j(D) \nabla^{u_j} + \Big(Q_j(D) - (E^{u_i} Q_j)(D)\Big) P_i(D) \nabla^{u_i}. \tag{17}$$

Let us define the grade $g(x^\alpha z^\beta)$ of an element $x^\alpha z^\beta$ of the ring $R[x]$ to be $\alpha - \beta$. Notice that $g(D_i(x^\alpha z^\beta)) = \alpha - \beta$ and that $g(\nabla_i(x^\alpha z^\beta)) = \alpha - \beta + e_i$, for any $\alpha, \beta \in \mathbb{N}_0^n$. The result of the action of the operator in the right-hand side of (17) on $x^\alpha z^\beta$ consists of three terms whose grades are $\alpha - \beta + u_i + u_j$, $\alpha - \beta + u_j$ and $\alpha - \beta + u_i$. Thus the operators W_i, W_j commute if and only if

$$Q_i(D) = (E^{u_j} Q_i)(D), \; i, j = 1, \dots, n, \quad i \neq j, \tag{18}$$

and

$$P_i(D)(E^{u_i} P_j)(D) = P_j(D)(E^{u_j} P_i)(D), \; i, j = 1, \dots, n. \tag{19}$$

It follows from (18) that the condition $Q_i(s + u_j) = Q_i(s)$, $i, j = 1, \dots, n$, $i \neq j$ is necessary for the family $\{W_i\}_{i=1}^n$ to be commutative. Under this assumption on the polynomials $Q_i(s)$ the compatibility conditions (5) can be written in the form

$$P_i(s + u_j) P_j(s) = P_j(s + u_i) P_i(s), \; i, j = 1, \dots, n$$

and they are therefore equivalent to (19). The proof is complete.

For $x^{(0)} \in \mathbb{C}^n$ let $\hat{\mathcal{O}}_{x^{(0)}}$ be the $\mathcal{D}-$module of formal power series centered at $x^{(0)}$. Let $\mathbb{C}_{x^{(0)}}$ denote the set of complex numbers $\mathbb{C}$ considered as a $\mathbb{C}[x_1,\ldots,x_n]-$module via the isomorphism $\mathbb{C} \simeq \mathbb{C}[x_1,\ldots,x_n]/(x_1 - x_1^{(0)},\ldots,x_n - x_n^{(0)})$. We use the following isomorphism (see Proposition 2.5.26 in [5] or [1], § 4) between the space of formal solutions to $\mathcal{M}$ at $x^{(0)}$ and the dual space of $\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \mathcal{M}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \mathcal{M}, \mathbb{C}). \tag{20}$$

This isomorphism holds for any finitely generated $\mathcal{D}-$module. Using (13) and fixing the point $x = x^{(0)}$ we arrive at the isomorphism

$$\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \left(R[x] \Big/ \sum_{i=1}^{n} W_i R[x] \right) \simeq R \Big/ \sum_{i=1}^{n} W_{i,x^{(0)}} R, \tag{21}$$

where $W_{i,x^{(0)}}$ are obtained from the operators W_i by setting $x = x^{(0)}$. Combining (20) with (21) we see that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathbb{C}} \left(R \Big/ \sum_{i=1}^{n} W_{i,x^{(0)}} R,\ \mathbb{C} \right).$$

Thus the following Lemma holds true.

Lemma 5 *The number of linearly independent formal power series solutions to the system (1) at the point $x = x^{(0)}$ is equal to $\dim_{\mathbb{C}} R/\sum_{i=1}^{n} W_{i,x^{(0)}} R$.*

The following Lemma is motivated by the proof of Theorem 5.4 in [1].

Lemma 6 *Let $L_i : R \to R$, $i = 1,\ldots,n$ be a commutative family of linear operators such that there exists a regular sequence of homogeneous polynomials $f_1,\ldots,f_n$ in R with the property $L_i(h) = f_i h + \tilde{h}$, where $\deg \tilde{h} < \deg(f_i h)$. Then $R/\sum_{i=1}^{n} L_i R$ and $R/(f_1,\ldots,f_n)$ are isomorphic as $\mathbb{C}-$vector spaces. Here $(f_1,\ldots,f_n)$ is the ideal generated by $f_1,\ldots,f_n$.*

Proof Let $\{h_\alpha\}_{\alpha\in\Lambda}$ represent a $\mathbb{C}-$basis in $R/(f_1,\ldots,f_n)$ consisting of homogeneous polynomials. Let $h \in R$, $\deg h = k$, and let X denote the set of all linear combinations of $\{h_\alpha\}_{\alpha\in\Lambda}$. We use induction on k to show that $h \in X + \sum_{i=1}^{n} L_i R$. Since $h = \sum_{i=1}^{n} f_i v_i + \sum_\alpha c_\alpha h_\alpha$ for some $v_1,\ldots,v_n \in R$ and $c_\alpha \in \mathbb{C}$ it follows that

$$h - \sum_{i=1}^{n} L_i(v_i) = h - \sum_{i=1}^{n} f_i v_i - \sum_{i=1}^{n} \tilde{v}_i = \sum_\alpha c_\alpha h_\alpha - \sum_{i=1}^{n} \tilde{v}_i,$$

where $\deg\,(\sum_{i=1}^n \tilde{v}_i) < k$. By induction $\sum_{i=1}^n \tilde{v}_i \in X + \sum_{i=1}^n L_i R$ which shows that $h \in X + \sum_{i=1}^n L_i R$. Thus $\{h_\alpha\}_{\alpha\in\Lambda}$ represents a generating set for the quotient $R/\sum_{i=1}^n L_i R$.

Let us show that $\{h_\alpha\}_{\alpha\in\Lambda}$ represent a set of linearly independent elements in $R/\sum_{i=1}^n L_i R$. Let $w \in X$ and suppose that $w = \sum_{i=1}^n L_i v_i$ for some $v_i \in R$. We show by induction on $k = \max\limits_{i=1,\dots,n} \deg\,(f_i v_i)$ that $w = 0$. Let $v_i = v_i^{'} + v_i^{''}$, where $\deg\,(f_i v_i^{'}) = k$ and $\deg\,(f_i v_i^{''}) < k$. Let w_k be the homogeneous part of w of degree k. Since $\{h_\alpha\}_{\alpha\in\Lambda}$ are homogeneous it follows that $w_k \in X$. Using the equality $w = \sum_{i=1}^n L_i v_i$ we obtain $w_k = \sum_{i=1}^n f_i v_i^{'}$. Since $\{h_\alpha\}_{\alpha\in\Lambda}$ represent a basis in $R/(f_1,\dots,f_n)$ we have $X \cap \sum_{i=1}^n L_i R_i = \{0\}$. This shows that $\sum_{i=1}^n f_i v_i^{'} = 0$. By the regularity of the sequence $(f_1,\dots,f_n)$ there exists a skew-symmetric set $\{\eta_{ij}\}_{i,j=1}^n$ of homogeneous polynomials such that $v_i^{'} = \sum_{j=1}^n \eta_{ij} f_j$. Let us consider $\tilde{v}_i = v_i - \sum_{j=1}^n L_j(\eta_{ij})$. Since the family of the operators $\{L_i\}_{i=1}^n$ is commutative it follows that $\sum_{i,j=1}^n L_i L_j(\eta_{ij}) = 0$ and hence

$$w = \sum_{i=1}^n L_i(v_i) = \sum_{i=1}^n L_i \left(\tilde{v}_i + \sum_{j=1}^n L_j(\eta_{ij}) \right) = \sum_{i=1}^n L_i(\tilde{v}_i).$$

Since $f_i \sum_{j=1}^n L_j(\eta_{ij})$ and $f_i \sum_{j=1}^n \eta_{ij} f_j$ are equal up to the terms of degree less than k it follows that $\deg f_i \tilde{v}_i < k$. By induction on k we conclude that $w = 0$, which completes the proof of the Lemma.

Since $\{h_\alpha\}_{\alpha\in\Lambda}$ was shown to be a generating set for $R/\sum_{i=1}^n L_i R$ without using commutativity of the family $\{L_i\}_{i=1}^n$, we obtain the following corollary.

Corollary 7 *Let $L_i : R \to R$, $i = 1,\dots,n$ be a family of linear operators such that there exists a regular sequence of homogeneous polynomials $f_1,\dots,f_n$ in R with the property $L_i(h) = f_i h + \tilde{h}$, where $\deg \tilde{h} < \deg\,(f_i h)$. Then $\dim_{\mathbb{C}} R/\sum_{i=1}^n L_i R \le \dim_{\mathbb{C}} R/(f_1,\dots,f_n)$.*

For any differential operator $P \in \mathcal{D}$, $P = \sum_{|\alpha|\le m} c_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha$ its principal symbol $\sigma(P)(x,z) \in R[x]$ is defined by $\sigma(P)(x,z) = \sum_{|\alpha|=m} c_\alpha(x) z^\alpha$. Let $H_i(x,z) = \sigma(G_i)(x,z)$ be the principal symbols of the differential operators which define the generalized Horn system (1). Let $J \subset \mathcal{D}$ be the left ideal generated by $G_1,\dots,G_n$. By the definition (see [4], Chapter 5, § 2) the characteristic variety $\mathrm{char}(\mathcal{M})$ of the generalized Horn system is given by

$$\mathrm{char}(\mathcal{M}) = \{(x,z) \in \mathbb{C}^{2n} : \sigma(P)(x,z) = 0, \ \text{ for all } P \in J\}.$$

Let us define the set $U_{\mathcal{M}} \subset \mathbb{C}^n$ by $U_{\mathcal{M}} = \{x \in \mathbb{C}^n : \exists\, z \ne 0 \text{ such that } (x,z) \in \mathrm{Char}(\mathcal{M})\}$. Theorem 7.1 in [3, Chapter 5] yields that for $x^{(0)} \notin U_{\mathcal{M}}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_{x^{(0)}}).$$

It follows from [18, pages 148,146] that the $\mathbb{C}-$dimension of the factor of the ring R with respect to the ideal generated by the regular sequence of homogeneous polynomials $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ is equal to the product $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$. Since a sequence of n homogeneous polynomials in n variables is regular if and only if their common zero is the origin, it follows that $U_{\mathcal{M}} = \emptyset$ in our setting. Using Lemmas 5,6 and 7 we arrive at the following Theorem.

Theorem 8 *Suppose that the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (5) and that $Q_i(s + u_j) = Q_i(s)$ for any $i, j = 1, \ldots, n$, $i \neq j$. If the principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ of the differential operators $G_1, \ldots, G_n$ form a regular sequence at $x^{(0)}$ then the dimension of the space of holomorphic solutions to (1) at the point $x^{(0)}$ is equal to $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$.*

Using Corollary 7 we obtain the following result.

Corollary 9 *Suppose that the principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ of the differential operators $G_1, \ldots, G_n$ form a regular sequence at $x^{(0)}$. Then the dimension of the space of holomorphic solutions to (1) at the point $x^{(0)}$ is less than or equal to $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$.*

In the next section we, using Theorem 8, construct an explicit basis in the space of holomorphic solutions to the generalized Horn system under the assumption that P_i, Q_i can be represented as products of linear factors and that $\deg Q_i > \deg P_i$, $i = 1, \ldots, n$.

3. Explicit basis in the solution space of some hypergeometric systems of the Horn type

Throughout this section we assume that the polynomials $P_i(s), Q_i(s)$ defining the generalized Horn system (1) can be factorized up to polynomials of degree one. Suppose that $P_i(s), Q_i(s)$ satisfy the following conditions: $Q_i(s + u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$ for any $i, j = 1, \ldots, n$, $i \neq j$. In this section we will show how to construct an explicit basis in the solution space of such a system of partial differential equations under some additional assumptions which are always satisfied if the parameters of the system under study are sufficiently general.

Recall that U denotes the matrix whose rows are $u_1, \ldots, u_n$ and let U^T denote the transpose of U. Let $\Lambda = \left(U^T\right)^{-1}$, let $(\Lambda s)_i$ denote the ith component of the vector Λs and $d_i = \deg Q_i$. Under the above conditions the polynomials $Q_i(s)$ can be represented in the form

$$Q_i(s) = \prod_{j=1}^{d_i} ((\Lambda s)_i - \alpha_{ij}), \quad i = 1, \ldots, n, \quad \alpha_{ij} \in \mathbb{C}.$$

By the Ore–Sato theorem [16] (see also § 1.2 of [10] and Part 3 of [15]) the general solution to the system of difference equations (3) associated with (1) can be written in the form

$$\varphi(s) = t_1^{s_1} \dots t_n^{s_n} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s\rangle - c_i)}{\prod_{i=1}^{n} \prod_{j=1}^{d_i} \Gamma((\Lambda s)_i - \alpha_{ij} + 1)} \phi(s), \tag{22}$$

where $p \in \mathbb{N}_0$, $t_i, c_i \in \mathbb{C}$, $A_i \in \mathbb{Z}^n$ and $\phi(s)$ is an arbitrary function satisfying the periodicity conditions $\phi(s + u_i) \equiv \phi(s)$, $i = 1, \dots, n$. (Given polynomials P_i, Q_i satisfying the compatibility conditions (5), the parameters p, t_i, c_i, A_i of the solution $\varphi(s)$ can be computed explicitly. For a concrete construction of the function $\varphi(s)$ see Part 3 of [15].) The following Theorem holds true.

Theorem 10 *Suppose that the following conditions are fulfilled.*
1. For any $i, j = 1, \dots, n$, $i \neq j$ it holds $Q_i(s + u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$.
2. The difference $\alpha_{ij} - \alpha_{ik}$ is never equal to a real integer number, for any $i = 1, \dots, n$ and $j \neq k$.
3. For any multi-index $I = (i_1, \dots, i_n)$ with $i_k \in \{1, \dots, d_k\}$ the product $\prod_{i=1}^{p}(\langle A_i, s\rangle - c_i)$ never vanishes on the shifted lattice $\mathbb{Z}^n + \gamma_I$, where $\gamma_I = (\alpha_{1i_1}, \dots, \alpha_{ni_n})$.

Then the family consisting of $\prod_{i=1}^{n} d_i$ functions

$$y_I(x) = x^{\gamma_I} \sum_{s \in \mathbb{Z}^n \cap K_U} t^{s+\gamma_I} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s + \gamma_I\rangle - c_i)}{\prod_{k=1}^{n} \prod_{j=1}^{d_k} \Gamma((\Lambda s)_k + \alpha_{ki_k} - \alpha_{kj} + 1)} x^s \tag{23}$$

is a basis in the space of holomorphic solutions to the system (1) at any point $x \in (\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. Here K_U is the cone spanned by the vectors $u_1, \dots, u_n$.

Proof It follows from Theorem 1 and the assumptions 2,3 of Theorem 10 that the series (23) formally satisfies the generalized Horn system (1). Let χ_k denote the kth row of Λ. Since $\deg Q_i(s) > \deg P_i(s)$, $i = 1, \dots, n$ it follows by the construction of the function (22) (see [15], Part 3) that all the components of the vector $\triangle = \sum_{i=1}^{p} A_i - \sum_{i=1}^{n} d_i \chi_i$ are negative. Thus for any multi-index I the intersection of the half-space $\mathrm{Re}\langle \triangle, s\rangle \geq 0$ with the shifted octant $K_U + \gamma_I$ is a bounded set. Using the Stirling formula we conclude that the series (23) converges everywhere in $(\mathbb{C}^*)^n$ for any multi-index I (see also § 2.4 in [15]).

The series (23) corresponding to different multi-indices I, J are linearly independent since by the second assumption of Theorem 10 their initial monomials $x^{\gamma_I}, x^{\gamma_J}$ are different. Finally, the conditions of Theorem 8 are satisfied in our setting since the first assumption of Theorem 10 yields that the sequence of principal symbols $H_1(x^{(0)}, z), \dots, H_n(x^{(0)}, z) \in R$ of hypergeometric differential operators defining the generalized Horn system is regular for $x^{(0)} \in (\mathbb{C}^*)^n$. Hence by Theorem 8 the number of linearly independent holomorphic solutions to the system under study at a generic point equals $\prod_{i=1}^{n} d_i$. In this case $U_{\mathcal{M}} = \{x^{(0)} \in \mathbb{C}^n : x_1^{(0)} \dots x_n^{(0)} = 0\}$. Thus the series (23) span the space of holomorphic solutions to the system (1) at any point $x^{(0)} \in (\mathbb{C}^*)^n$. The proof is complete.

In the theory developed by Gelfand, Kapranov and Zelevinsky the conditions 2 and 3 of Theorem 10 correspond to the so-called nonresonant case (see [9], § 8.1). Thus the result on the structure of solutions to the generalized Horn system can be formulated as follows.

Corollary 11 *Let $x^{(0)} \in (\mathbb{C}^*)^n$ and suppose that $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$ for any $i, j = 1, \ldots, n$, $i \neq j$. If the parameters of the system (1) are nonresonant then there exists a basis in the space of holomorphic solutions to (1) near $x^{(0)}$ whose elements are given by series of the form (2).*

Let us now consider a simple example.

Example 1 Let $u_1 = (1,0), u_2 = (1,1)$ and consider the system of equations

$$\left\{ \begin{array}{ll} x^{u_1} y(x) & = \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right) y(x), \\ x^{u_2} y(x) & = \left(x_2 \frac{\partial}{\partial x_2} \right) y(x). \end{array} \right. \tag{24}$$

The principal symbols $H_1(x,z), H_2(x,z) \in R[x]$ of the differential operators defining the system (24) are given by $H_1(x,z) = -x_1 z_1 + x_2 z_2$, $H_2(x,z) = -x_2 z_2$. By Theorem 8 the dimension of the solution space of (24) at a generic point is equal to 1 since $\dim_{\mathbb{C}} R/(H_1(x,z), H_2(x,z)) = 1$ for $x_1 x_2 \neq 0$. For computing the solution to (24) explicitly we choose $\gamma = 0$ and consider the corresponding system of difference equations

$$\left\{ \begin{array}{ll} \varphi(s+u_1)(s_1 - s_2 + 1) & = \varphi(s), \\ \varphi(s+u_2)(s_2+1) & = \varphi(s). \end{array} \right. \tag{25}$$

The general solution to (25) is given by $\varphi(s) = (\Gamma(s_1 - s_2 + 1)\Gamma(s_2+1))^{-1} \phi(s)$, where $\phi(s)$ is an arbitrary function which is periodic with respect to the vectors u_1, u_2.

There exists only one subset of $\mathbb{Z}^2$ satisfying the conditions of Theorem 1, namely $S = \{(s_1, s_2) \in \mathbb{Z}^2 : s_1 - s_2 \geq 0,\ s_2 \geq 0\}$. Choosing $\phi(s) \equiv 1$ and using (23), we obtain the solution to (24):

$$y(x) = \sum_{\substack{s_1 - s_2 \geq 0, \\ s_2 \geq 0}} \frac{x_1^{s_1} x_2^{s_2}}{\Gamma(s_1 - s_2 + 1)\Gamma(s_2+1)} = \exp(x_1 x_2 + x_1). \tag{26}$$

It is straightforward to check that the solution space of (24) is indeed spanned by (26).

4. Amoebas of rational functions

In this section we use the results on the structure of solutions to the generalized Horn system for computing the number of Laurent expansions of some rational functions. This problem is closely related to the notion of the amoeba of a Laurent polynomial, which was introduced by Gelfand et al. in [12] (see Chapter 6, § 1). Given a Laurent polynomial f, its amoeba $\mathcal{A}_f$ is defined to be the image of the hypersurface $f^{-1}(0)$ under the map $(x_1, ..., x_n) \mapsto (\log|x_1|, ..., \log|x_n|)$. This name is motivated by the typical shape of $\mathcal{A}_f$ with tentacle-like asymptotes going off to infinity. The connected components of the complement of the amoeba are convex and each such component corresponds to a specific Laurent series development with the center at the origin of the rational function $1/f$ (see [12], Chapter 6, Corollary 1.6). The problem of finding all such Laurent series expansions of a given Laurent polynomial was posed in [12] (Chapter 6, Remark 1.10), where this problem is referred to as a difficult and interesting one.

Let $f(x_1, ..., x_n) = \sum_{\alpha \in S} a_\alpha x^\alpha$ be a Laurent polynomial. Here S is a finite subset of the integer lattice $\mathbb{Z}^n$ and each coefficient a_α is a non-zero complex number. The Newton polytope $\mathcal{N}_f$ of the polynomial f is defined to be the convex hull in $\mathbb{R}^n$ of the index set S. The following result was obtained in [8].

Theorem 12 *Let f be a Laurent polynomial. The number of Laurent series expansions with the center at the origin of the rational function $1/f$ is at least equal to the number of vertices of the Newton polytope $\mathcal{N}_f$ and at most equal to the number of integer points in $\mathcal{N}_f$.*

In the view of Corollary 1.6 in Chapter 6 of [12], Theorem 12 states that the number of connected components of the complement of the amoeba $\mathcal{A}_f$ is bounded from below by the number of vertices of $\mathcal{N}_f$ and from above by the number of integer points in $\mathcal{N}_f$. The lower bound has already been obtained in [12]. In this section we describe a class of rational functions for which the number of Laurent expansions attains the lower bound given by Theorem 12. Our main tool is Theorem 1 which allows one to describe supports of the Laurent series expansions of a rational function which can be treated as a solution to a generalized Horn system.

Proposition 13 *Let $u_1, ..., u_n \in \mathbb{Z}^n$ be linearly independent vectors, let $p \in \mathbb{N}$ and let $a_1, ..., a_n \in \mathbb{C}^*$ be nonzero complex numbers. For any of the rational functions*

$$\begin{aligned} y_1(x) &= (1 - a_1 x^{u_1} - \cdots - a_n x^{u_n})^{-1}, \\ y_2(x) &= ((1 - a_1 x^{u_1} - \cdots - a_{n-1} x^{u_{n-1}})^p - a_n x^{u_n})^{-1} \\ y_3(x) &= ((1 - a_1 x^{u_1})^p - a_2 x^{u_2} - \cdots - a_n x^{u_n})^{-1}, \end{aligned}$$

the number of its Laurent expansions with the center at the origin equals $n+1$. Thus the lower bound for the number of the connected components of the complement of the amoeba $\mathcal{A}_{y_i^{-1}}$ is attained for any $i = 1, 2, 3$.

Proof To make the idea of the proof more transparent we begin with the function $y_1(x)$ which is a special case of $y_2(x)$ and $y_3(x)$. Recall that U denotes the matrix whose rows are $u_1, ..., u_n$. Let $(\lambda_{ij}) = \Lambda = (U^T)^{-1}$ and $\nu_i = \lambda_{1i} + \cdots + \lambda_{ni}$.

1. The function $y_1(x)$ satisfies the following system of the Horn type

$$\begin{pmatrix} a_1 x^{u_1} \\ \cdots \\ a_n x^{u_n} \end{pmatrix} \left(\nu_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \nu_n x_n \frac{\partial}{\partial x_n} + 1 \right) y(x) = \Lambda \begin{pmatrix} x_1 \frac{\partial}{\partial x_1} \\ \cdots \\ x_n \frac{\partial}{\partial x_n} \end{pmatrix} y(x). \tag{27}$$

Indeed, after the change of variables $x_i(\xi_1, ..., \xi_n) = \xi_1^{\lambda_{1i}} ... \xi_n^{\lambda_{ni}}$ (whose inverse is $\xi_i = x^{u_i}$) the system (27) takes the form

$$a_i \xi_i \left(\xi_1 \frac{\partial}{\partial \xi_1} + \cdots + \xi_n \frac{\partial}{\partial \xi_n} + 1 \right) y(\xi) = \xi_i \frac{\partial}{\partial \xi_i} y(\xi), \quad i = 1, ..., n. \tag{28}$$

The function $(1 - a_1\xi_1 - \cdots - a_n\xi_n)^{-1}$ satisfies (28) and therefore the function $y_1(x)$ is a solution of (27). By Theorem 8 the space of holomorphic solutions to (27) has dimension one at a generic point and hence $y_1(x)$ is the only solution to this system. Thus the supports of the Laurent series expansions of $y_1(x)$ can be found by means of Theorem 1. There exist $n+1$ subsets of the lattice $\mathbb{Z}^n$ which satisfy the conditions in Theorem 1 and can give rise to a Laurent expansion of $y_1(x)$ with nonempty domain of convergence. These subsets are $S_0 = \{s \in \mathbb{Z}^n : (\Lambda s)_i \geq 0,\ i = 1, ..., n\}$ and $S_j = \{s \in \mathbb{Z}^n : \nu_1 s_1 + \cdots + \nu_n s_n + 1 \leq 0,\ (\Lambda s)_i \geq 0,\ i \neq j\}$, $j = 1, ..., n$. Besides $S_0, ..., S_n$ there can exist other subsets of $\mathbb{Z}^n$ satisfying the conditions in Theorem 1. (Such subsets "penetrate" some of the hyperplanes $(\Lambda s)_i = 0$, $\nu_1 s_1 + \cdots + \nu_n s_n + 1 = 0$ without intersecting them; subsets of this type can only appear if $|\det U| \geq 1$). However, none of these additional subsets gives rise to a convergent Laurent series and therefore does not define an expansion of $y_1(x)$. Indeed, in any series with the support in a "penetrating" subset at least one index of summation necessarily runs from $-\infty$ to ∞. Letting all the variables, except for that one which corresponds to this index, be equal to zero, we obtain a hypergeometric series in one variable. The classical result on convergence of one-dimensional hypergeometric series (see [10], § 1) shows that this series is necessarily divergent. Thus the number of Laurent series developments of $y_1(x)$ cannot exceed $n+1$. The Newton polytope of the polynomial $1/y_1(x)$ has $n+1$ vertices since the vectors $u_1, ..., u_n$ are linearly independent. Using Theorem 12 we conclude that the number of Laurent series expansions of $y_1(x)$ equals $n+1$. Thus the lower bound for the number of connected components of the amoeba complement is attained.

2. Recall that $x\frac{\partial}{\partial x}$ denotes the vector $\left(x_1\frac{\partial}{\partial x_1},...,x_n\frac{\partial}{\partial x_n}\right)^T$ and let $\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_i$ denote the ith component of the vector $\Lambda\left(x\frac{\partial}{\partial x}\right)$. Let $\mathcal{G}$ be the differential operator defined by

$$\mathcal{G} = \left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_1 + \cdots + \left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_{n-1} + p\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_n + p.$$

The function $y_2(x)$ is a solution to the following system of differential equations of hypergeometric type

$$\left\{\begin{array}{l} a_i x^{u_i}\mathcal{G}y(x) = \left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_i y(x), \quad i=1,...,n-1, \\ a_n x^{u_n}\left(\prod_{j=0}^{p-1}(\mathcal{G}+j)\right)y(x) = \left(\prod_{j=0}^{p-1}\left(p\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_n + j\right)\right)y(x). \end{array}\right. \tag{29}$$

Indeed, the same monomial change of variables as in the first part of the proof reduces (29) to the system

$$\left\{\begin{array}{l} a_i\xi_i\tilde{\mathcal{G}}y(x) = \xi_i\frac{\partial}{\partial\xi_i}y(x), \quad i=1,...,n-1, \\ a_n\xi_n\left(\prod_{j=0}^{p-1}(\tilde{\mathcal{G}}+j)\right)y(x) = \left(\prod_{j=0}^{p-1}\left(p\xi_n\frac{\partial}{\partial\xi_n}+j\right)\right)y(x), \end{array}\right. \tag{30}$$

where $\tilde{\mathcal{G}} = \xi_1\frac{\partial}{\partial\xi_1} + \cdots + \xi_{n-1}\frac{\partial}{\partial\xi_{n-1}} + p\xi_n\frac{\partial}{\partial\xi_n} + p$. The system (30) is satisfied by the function $\left((1-a_1\xi_1-\cdots-a_{n-1}\xi_{n-1})^p - a_n\xi_n\right)^{-1}$. This shows that $y_2(x)$ is indeed a solution to (29). Thus the support of a Laurent expansion of $y_2(x)$ must satisfy the conditions in Theorem 1. Notice that unlike (27), the system (29) can have solutions supported in subsets of the shifted lattice $\mathbb{Z}^n+\gamma$ for some $\gamma\in(0,1)^n$. Yet, such subsets are not of interest for us since we are looking for Laurent series developments of $y_2(x)$. The subsets $S_0 = \{s\in\mathbb{Z}^n : (\Lambda s)_i \geq 0,\, i=1,...,n\}$ and $S_j = \{s\in\mathbb{Z}^n : (\Lambda s)_1+\cdots+(\Lambda s)_{n-1}+p(\Lambda s)_n+p\leq 0,\, (\Lambda s)_i\geq 0,\, i\neq j\}$, $j=1,...,n$ satisfy the conditions in Theorem 1. The same arguments as in the first part of the proof show that no other subsets of $\mathbb{Z}^n$ satisfying the conditions in Theorem 1 can give rise to a convergent Laurent series which represents $y_2(x)$. This yields that the number of expansions of $y_2(x)$ is at most equal to $n+1$. The Newton polytope of the polynomial $1/y_2(x)$ has $n+1$ vertices since the vectors $u_1,...,u_n$ are assumed to be linearly independent. Using Theorem 12 we conclude that the number of Laurent series developments of $y_2(x)$ equals $n+1$.

3. Let $\mathcal{H}$ be the differential operator defined by

$$\mathcal{H} = p\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_2 + \cdots + p\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_n + p.$$

Using the same change of variables as in the first part of the proof, one checks that $y_3(x)$ solves the system

$$\left\{\begin{array}{l} a_1 x^{u_1}\left(\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_1+\mathcal{H}\right)y(x)=\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_1 y(x),\\ a_i x^{u_i}\frac{1}{p}\mathcal{H}\left(\prod\limits_{j=0}^{p-1}\left(\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_1+\mathcal{H}+j\right)\right)y(x)=\\ \qquad\qquad\left(\Lambda\left(x\frac{\partial}{\partial x}\right)\right)_i\left(\prod\limits_{j=0}^{p-1}(\mathcal{H}-p+j)\right)y(x),\quad i=2,...,n.\end{array}\right. \tag{31}$$

Analogously to the first part of the proof, we apply Theorem 1 to the system (31) and conclude that the number of Laurent expansions of $y_3(x)$ at most equals $n+1$. Thus it follows from Theorem 12 that the number of such expansions equals $n+1$. The proof is complete.

Remark 2 Suppose that the series (2) satisfies a generalized Horn system (this is equivalent to saying that the polynomials P_i, Q_i defined by the equalities $P_i(s)/Q_i(s+u_i)=\varphi(s+u_i)/\varphi(s)$ and the support S of (2) satisfy the conditions in Theorem 1). If (2) converges to a rational function $g(x)$ then the number of Laurent series developments of $g(x)$ can be found by means of Theorem 1. The problem of describing the class of rational hypergeometric functions was studied in [6] and [7]. Yet, the definition of a hypergeometric function used in these papers is based on the Gelfand-Kapranov-Zelevinsky system of differential equations [9], [10], [11], [1] rather than the Horn system.

Example 2 To show how Theorem 1 can be used for computing the number of Laurent series developments of a rational function satisfying a generalized Horn system, we consider an example which is not a special case of Proposition 13. Let $n=2$. The Szegö kernel of the domain $\{z\in\mathbb{C}^2 : |z_1|+|z_2|<1\}$ is given by the hypergeometric series

$$h(x_1,x_2)=\sum_{s_1,s_2\geq 0}\frac{\Gamma(2s_1+2s_2+2)}{\Gamma(2s_1+1)\Gamma(2s_2+1)}x_1^{s_1}x_2^{s_2}=$$

$$\frac{(1-x_1-x_2)(1+2x_1x_2-x_1^2-x_2^2)+8x_1x_2}{\left((1-x_1-x_2)^2-4x_1x_2\right)^2}. \tag{32}$$

(See [3], Chapter 3, § 14.) This series satisfies the system of equations

$$x_i\left(2x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+3\right)\left(2x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+2\right)y(x)=$$

$$\left(2x_i\frac{\partial}{\partial x_i}\right)\left(2x_i\frac{\partial}{\partial x_i}-1\right)y(x),\quad i=1,2.$$

There exist three subsets of the lattice $\mathbb{Z}^n$ which satisfy the conditions in Theorem 1, namely $\{s \in \mathbb{Z}^2 : s_1 \geq 0, s_2 \geq 0\}$, $\{s \in \mathbb{Z}^2 : s_1 \geq 0, s_1 + s_2 + 1 \leq 0\}$, $\{s \in \mathbb{Z}^2 : s_2 \geq 0, s_1 + s_2 + 1 \leq 0\}$. Using Theorem 1 we conclude that the number of Laurent expansions centered at the origin of the Szegö kernel (32) at most equals 3. The Newton polytope of the denominator of the rational function (32) is the simplex with the vertices $(0,0)$, $(4,0)$, $(0,4)$. By Theorem 12 the number of Laurent series developments of the Szegö kernel at least equals 3. Thus the number of Laurent expansions of (32) (or, equivalently, the number of connected components in the complement of the amoeba of its denominator) attains its lower bound.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF STOCKHOLM
S-10691 STOCKHOLM
SWEDEN
E-mail: timur@matematik.su.se