Analytic Representation of CR Functions on Hypersurfaces with Singularities

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Abstract

We prove a theorem on analytic representation of integrable CR functions on hypersurfaces with singular points. Moreover, the behaviour of representing analytic functions near singular points is investigated. We are aimed at explaining the new effect caused by the presence of a singularity rather than at treating the problem in full generality.

Contents

Introduction

We begin by recalling a well-known theorem on analytic representation of *CR* functions on a smooth hypersurface, cf. [AH72, Chi75].

Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$, $n > 1$, whose Dolbeault cohomology with coefficients in the sheaf of germs of holomorphic functions vanishes at step 1, i.e., $H^1(\mathcal{D}, \mathcal{O}) = 0$. Such is the case, in particular, if $\mathcal{D}$ is a domain of holomorphy in $\mathbb{C}^n$.

Suppose that $\mathcal{S}$ is a smooth (of class C^1) closed orientable hypersurface in $\mathcal{D}$, dividing $\mathcal{D}$ into two open sets $\mathcal{D}^+$ and $\mathcal{D}^-$. As known, there exists a real-valued function $\rho \in C^1(\mathcal{D})$ such that $\mathcal{S} = \{z \in \mathcal{D}\colon \rho(z) = 0\}$ and $\nabla\rho\,|_{\mathcal{S}} \neq 0$. We set

$$\mathcal{D}^{\pm} = \{z \in \mathcal{D}\colon \pm\rho(z) > 0\}$$

and orient $\mathcal{S}$ as the boundary of $\mathcal{D}^-$. Thus, $\mathcal{D}^- \cup \mathcal{S}$ is an oriented manifold with boundary.

As usual, a function $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ is said to be a *CR function* on $\mathcal{S}$ if it satisfies

$$\int_{\mathcal{S}} f\,\bar{\partial} v = 0$$

for all differential forms v of bidegree $(n, n-2)$ with coefficients of class $C^\infty(\mathcal{D})$ and a compact support in $\mathcal{D}$.

Theorem 0.1 ([AH72, Chi75]) *For any CR function $f \in L^1_{\mathrm{loc}}(\mathcal{S})$, there are functions $h^\pm$ holomorphic in $\mathcal{D}^\pm$, respectively, such that*

$$f = h^+ - h^- \quad \text{on} \quad \mathcal{S}. \tag{0.1}$$

We write $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$. More precisely, the equality (0.1) is interpreted as follows:

1) if $\mathcal{S} \in C^{k+1}$, $k \in \mathbb{Z}_+$, and $f \in C^{k,\lambda}_{\mathrm{loc}}(\mathcal{S})$, $0 < \lambda < 1$, then $h^\pm \in C^{k,\lambda}_{\mathrm{loc}}(\mathcal{S} \cup \mathcal{D}^\pm)$ and (0.1) is fulfilled at each point of $\mathcal{S}$;

2) if $\mathcal{S} \in C^1$ and $f \in L^p_{\mathrm{loc}}(\mathcal{S})$, $p \geq 1$, then for each point $z^0 \in \mathcal{S}$ there is a neighbourhood U such that

$$\lim_{\varepsilon 0+} \int_{\mathcal{S} \cap U} \left| \left(h^+(\zeta + \varepsilon\nu(\zeta)) - h^-(\zeta - \varepsilon\nu(\zeta))\right) - f(\zeta)\right|^p d\sigma = 0,$$

where $d\sigma$ is the Lebesgue measure on Γ, and $\nu(\zeta)$ the unit outward normal vector to $\mathcal{S}$ at a point $\zeta \in \mathcal{S}$.

It is worth pointing out, cf. [Kyt95, Ch. 2], that the boundary behaviour of $h^{\pm}$ near the hypersurface $\mathcal{S}$ coincides with that of the Bochner-Martinelli integral $M(z)$ of f,

$$-M(z) = \int_{\mathcal{S}} f(\zeta) U(\zeta, z), \quad z \in \mathcal{D}^{\pm},$$

where

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} \, d\bar{\zeta}[j] \wedge d\zeta$$

and

$$\begin{aligned} d\zeta &= d\zeta_1 \wedge \ldots \wedge d\zeta_n, \\ d\bar{\zeta}[j] &= d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{j-1} \wedge d\bar{\zeta}_{j+1} \wedge \ldots \wedge d\bar{\zeta}_n. \end{aligned}$$

The theorem on analytic representation is of crucial importance in the theory of *CR* functions, cf. [Khe85, Sto93].

If the hypersurface $\mathcal{S}$ bears singularities, we may define the tangential Cauchy-Riemann conditions only at smooth points of $\mathcal{S}$. Theorem 0.1 is no longer true for such hypersurfaces even in the case of point singularities, cf. Section 1.

The purpose of this paper is to bring together two areas in which the problem of analytic representation can be studied. The first of the two is complex analysis with its explicit integral formulas which enable one to treat also problems of piecewise smooth "real" geometry. The important point to note here is the nature of singularities which are purely "real," namely conical points, power-like cusps, etc. The second area is the analysis of pseudodifferential operators on manifolds with singular points, cf. [RST97]. It introduces rather specific tools of real analysis in the complex problem, such as special weighted Sobolev spaces, asymptotics, "regularisation" of operators near singularities, etc.

Using this approach we describe those locally integrable functions f on a hypersurface with singular points, which are still representable in the form (0.1), cf. Section 2. Moreover, we specify the asymptotic behaviour of $h^{\pm}(z)$ close to every singular point, cf. Section 4.

No attempt has been made here to develop the theory for hypersurfaces with higher order singularities, such as "real" edges, corners, etc., and their "complex" analogues. Rather than do it we are interested in explaining the new effect caused by the presence of a singularity.

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1 Example of a non-representable *CR* function

Let $\mathcal{D}$ be the unit bidisk in $\mathbb{C}^2$,

$$\begin{array}{rcl} \mathcal{D} & = & \{z \in \mathbb{C}^2 : \ |z_1| < 1, \ |z_2| < 1\}, \\ \mathcal{S} & = & \{z \in \mathcal{D} : \ |z_1| = |z_2|\}. \end{array}$$

The origin $O(0,0)$ is a singular point of $\mathcal{S}$. Indeed, $\mathcal{S} = \{z \in \mathcal{D} : \ \rho(z) = 0\}$ where

$$\rho(z) = z_1\bar{z}_1 - z_2\bar{z}_2,$$

and $\nabla\rho(z)$ vanishes at the only point $z = O$ on $\mathcal{S}$. Obviously, this is a conical point.

Consider the open sets $\mathcal{D}^{\pm} = \{z \in \mathcal{D} : \ \pm\rho(z) > 0\}$ and the holomorphic function

$$f(z) = \frac{1}{z_1 z_2}$$

away from the planes $z_j = 0$, for $j = 1, 2$. The restriction of f is a smooth *CR* function on $\mathcal{S} \setminus \{O\}$.

Furthermore, we have $f \in L^1(\mathcal{S})$. To prove this, write

$$\begin{array}{rcl} z_1 & = & x_1 + iy_1, \\ z_2 & = & x_2 + iy_2 \end{array}$$

and parametrise $\mathcal{S}$ by

$$\begin{array}{rclcrcl} x_1 & = & r\,\cos\varphi_1, & & x_2 & = & r\,\cos\varphi_2, \\ y_1 & = & r\,\sin\varphi_1; & & y_2 & = & r\,\sin\varphi_2, \end{array}$$

where $0 < r \le 1$ and $0 \le \varphi_1, \varphi_2 < 2\pi$. Then the Gramian has the form

$$G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

whence

$$\begin{array}{rcl} d\sigma & = & \sqrt{\det G}\, dr d\varphi_1 d\varphi_2 \\ & = & \sqrt{2} r^2\, dr d\varphi_1 d\varphi_2. \end{array}$$

It follows that

$$\begin{array}{rcl} \displaystyle\int_{\mathcal{S}} |f|\, d\sigma & = & \displaystyle\int_{\mathcal{S}} \frac{1}{|z_1 z_2|}\, d\sigma \\ & = & \displaystyle\sqrt{2} \int_0^1 dr \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \\ & = & \sqrt{2}\, (2\pi)^2 \end{array}$$

is finite, as desired.

Suppose that f meets the conclusion of Theorem 0.1, i.e., $f = h^+ - h^-$ on $\mathcal{S} \setminus \{O\}$, where $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$ are continuous up to $\mathcal{S} \setminus \{O\}$. By the Cauchy theorem in one dimension it follows that

$$\int_{\substack{|z_1|=1/2\\|z_2|=1/2}} h^\pm(z)\, dz_1 dz_2 = 0$$

while

$$\int_{\substack{|z_1|=1/2\\|z_2|=1/2}} \frac{1}{z_1 z_2}\, dz_1 dz_2 = (2\pi i)^2$$

is different from zero. The contradiction shows that f can not be represented as the difference of holomorphic functions on $\mathcal{D}^\pm$.

The main obstruction to such a representation lies in the fact that the cohomology $H^1(\mathcal{D} \setminus \{O\}, \mathcal{O})$ is non-trivial.

Yet another interpretation of this example consists in the following. The point O is not *removable* for the integrable *CR* functions on $\mathcal{S}$, i.e., for a *CR* function f on $\mathcal{S} \setminus \{O\}$, the inclusion $f \in L^1(\mathcal{S})$ does not guarantee in general that f is a *CR* function on all of $\mathcal{S}$. In the case of smooth hypersurfaces $\mathcal{S}$, removable singularities of integrable *CR* functions are studied in [Jör88], [Kyt89], [Sto93], [KR95]. As a rule, isolated points are removable. The above example shows that if z^0 is a singular point of $\mathcal{S}$ then it can be unremovable for integrable *CR* functions.

However, we show that Theorem 0.1 remains still true for some kind singular points and *CR* functions having certain growth close to these points.

2 Analytic representation

Consider a hypersurface $\mathcal{S}$ in $\mathcal{D}$ with one singular point z^0, i.e., $\mathcal{S} \setminus \{z^0\}$ is smooth. Assume moreover that there exist a neighbourhood U of z^0, a neighbourhood V of $0 \in \mathbb{C}^n$ and a diffeomorphism $h : U \to V$ with the property that

$$h(\mathcal{S} \cap U) = \{z \in V :\ \varphi(x_n) = \sqrt{|x'|^2 + |y|^2},\ 0 \le x_n \le \varepsilon^0\}$$

where $\varphi \in C^1[0, \varepsilon^0]$ satisfies $\varphi(0) = 0$ and $\varphi(x_n) > 0$ if $x_n > 0$, and

$$\begin{array}{rcl} z_j &=& x_j + i y_j, \quad j = 1, \dots, n; \\ x' &=& (x_1, \dots, x_{n-1}). \end{array}$$

Denote $\Gamma = h(\mathcal{S} \cap U)$ and

$$\begin{array}{rcl} \Gamma_\varepsilon &=& \{z \in \Gamma :\ 0 \le x_n \le \varepsilon\}, \\ \mathcal{S}_\varepsilon &=& h^{-1}(\Gamma_\varepsilon), \end{array}$$

for $0 < \varepsilon \leq \varepsilon^0$.

If $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ then the function

$$S_f(\varepsilon) = \frac{1}{\mathrm{vol}(\partial \mathcal{S}_\varepsilon)} \int_{\partial \mathcal{S}_\varepsilon} |f| \, d\omega_\varepsilon$$

is, by the Fubini theorem, finite for almost all $0 < \varepsilon \leq \varepsilon_0$, where $d\omega_\varepsilon$ is the Lebesgue measure on $\partial \mathcal{S}_\varepsilon$, and $\mathrm{vol}(\partial \mathcal{S}_\varepsilon)$ the area of $\partial \mathcal{S}_\varepsilon$.

Theorem 2.1 *Assume that $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ is a CR function on $\mathcal{S} \setminus \{z^0\}$, satisfying $S_f(\varepsilon) = o\,(1/\varphi^{2n-2}(\varepsilon))$ as $\varepsilon 0$. Then Theorem 0.1 holds for f, more precisely, we have $f = h^+ - h^-$ on $\mathcal{S} \setminus \{z^0\}$, where $h^\pm \in \mathcal{O}(\mathcal{D}^\pm)$ and the boundary behaviour of $h^\pm$ near $\mathcal{S} \setminus \{z^0\}$ is actually the same as that in Theorem 0.1.*

The important point to note here is the condition $S_f(\varepsilon) = o\,(1/\varphi^{2n-2}(\varepsilon))$ as $\varepsilon 0$, on f. The "worse" the singularity of $\mathcal{S}$ at z^0, the faster $\varphi(\varepsilon)$ tends to 0 as $\varepsilon 0$, and so the wider the class of functions f meeting the additional condition. This is a general observation concerning the analysis on manifolds with cusps, cf. [RST97].

3 The proof

To prove Theorem 2.1 we may assume without loss of generality that $\mathcal{D} = U$ and h is the identity diffeomorphism. Thus, we have $z^0 = 0$, $\mathcal{S} = \Gamma$ and $\mathcal{S}_\varepsilon = \Gamma_\varepsilon$, for $0 < \varepsilon \leq \varepsilon^0$. In this case $\partial \mathcal{S}_\varepsilon$ is a $(2n-2)$-dimensional sphere in $\mathbb{R}^{2n-1}$, namely

$$\partial \mathcal{S}_\varepsilon = \{(x', y) \in \mathbb{R}^{2n-1} : \ |x'|^2 + |y|^2 = \varphi^2(\varepsilon)\}.$$

Consider the current $f[\mathcal{S}]^{0,1}$ in $\mathcal{D} \setminus \{0\}$. Since f is a *CR* function on $\mathcal{S} \setminus \{0\}$, this current $f[\mathcal{S}]^{0,1}$ is $\bar{\partial}$-closed, i.e., $\bar{\partial}\,(f[\mathcal{S}]^{0,1}) = 0$ in $\mathcal{D} \setminus \{0\}$. We next show that under the conditions of Theorem 2.1 $f[\mathcal{S}]^{0,1}$ extends to a $\bar{\partial}$-closed current on all of $\mathcal{D}$.

Lemma 3.1 *Let $\mathcal{D} \subset \mathbb{R}^N$, $N > 2$, and let f be a locally integrable function on $\mathcal{S} \setminus \{0\}$. If*

$$\int_0^{\varepsilon^0} S_f(\varepsilon)\, \varphi^{N-2}(\varepsilon)\, d\varepsilon < \infty,$$

then $f[\mathcal{S}]$ defines a current on all of $\mathcal{D}$ in a canonical way .

Proof. Let us parametrise $\mathcal{S}$ by

$$\begin{cases} x_1 & = & \varphi(t) & \sin\theta_1 & \sin\theta_2 & \dots & \sin\theta_{N-3} & \sin\theta_{N-2}, \\ x_2 & = & \varphi(t) & \sin\theta_1 & \sin\theta_2 & \dots & \sin\theta_{N-3} & \cos\theta_{N-2}, \\ \dots & \dots & \dots & \dots & \dots & \dots & & \\ x_{N-2} & = & \varphi(t) & \sin\theta_1 & \cos\theta_2, & & & \\ x_{N-1} & = & \varphi(t) & \cos\theta_1, & & & & \\ x_N & = & t, & & & & & \end{cases}$$

where $t \in [0, \varepsilon^0]$, and $\theta_1, \dots, \theta_{N-2}$ are polar coordinates on the unit sphere $\mathbb{S}^{N-2}$ in $\mathbb{R}^{N-1}$,

$$\begin{array}{ccccc} 0 & \le & \theta_1 & < & 2\pi, \\ 0 & \le & \theta_j & \le & \pi, \quad \text{for} \quad j = 2, \dots, N-2. \end{array}$$

If G is the Gramian of this parametrisation then it a simple matter to check that

$$\det G = \left(1 + (\varphi'(t))^2\right) (\varphi(t))^{2(N-2)} \sin\theta_2 \dots \sin^{N-3}\theta_{N-2}.$$

Hence the Lebesgue measure $d\sigma$ on $\mathcal{S}$ is

$$d\sigma = (\varphi(t))^{N-2} \sqrt{1 + (\varphi'(t))^2}\, dt \wedge d\omega, \tag{3.1}$$

$d\omega$ being the standard area form on the unit sphere $\mathbb{S}^{N-2}$.

For each differential form g of degree $N-1$ with smooth coefficients and compact support in $\mathcal{D}$, we get

$$\begin{aligned} \left|\int_{\mathcal{S}} fg\right| & \le c \int_{\mathcal{S}} |f|\, d\sigma \\ & = c \int_0^{\varepsilon^0} (\varphi(t))^{N-2} \sqrt{1 + (\varphi'(t))^2}\, dt \int_{\mathbb{S}^{N-2}} |f|\, d\omega \\ & \le C \int_0^{\varepsilon^0} (\varphi(t))^{N-2} S_f(t)\, dt \end{aligned}$$

where c and C are positive constants depending on $\mathcal{S}$ and g. Hence the lemma follows.

□

Lemma 3.1 shows that under assumptions of Theorem 2.1, $f[\mathcal{S}]^{0,1}$ is a current on all of $\mathcal{D}$ in a canonical way. In fact, it is of order 0, i.e., extends to $(n, n-1)$-forms with continuous coefficients.

Lemma 3.2 *Suppose f is a locally integrable CR function on $\mathcal{S} \setminus \{0\}$. Then, for almost all $\varepsilon \in (0, \varepsilon^0]$, we have*

$$\int_{\mathcal{S} \setminus \mathcal{S}_\varepsilon} f\, \bar{\partial} v = - \int_{\partial \mathcal{S}_\varepsilon} f v$$

whenever v is a differential form of bidegree $(n, n-2)$ with smooth coefficients and compact support in $\mathcal{D}$.

Proof. See [Kyt95, Ch. 2].

□

Having disposed of these preliminary steps, we can now return to the proof of Theorem 2.1.

By Lemma 3.2,

$$\begin{aligned}\langle \bar\partial \left(f[\mathcal{S}]^{0,1}\right), v\rangle &= \lim_{\varepsilon\to 0+} \int_{\mathcal{S}\setminus\mathcal{S}_\varepsilon} f\,\bar\partial v \\ &= -\lim_{\varepsilon\to 0+} \int_{\partial\mathcal{S}_\varepsilon} f v \\ &= 0\end{aligned}$$

for all $(n, n-2)$-forms v with smooth coefficients and compact support in $\mathcal{D}$. Indeed,

$$\begin{aligned}|\int_{\partial\mathcal{S}_\varepsilon} f v| &\le c \int_{\partial\mathcal{S}_\varepsilon} |f|\, d\omega_\varepsilon \\ &= c\ \sigma_{2n-1} \left(\varphi(\varepsilon)\right)^{2n-2}\ S_f(\varepsilon)\end{aligned}$$

where c is a positive constant independent of ε, and σ_{2n-1} the area of the unit sphere in $\mathbb{R}^{2n-1}$. We have used the fact that

$$\mathrm{vol}(\partial\mathcal{S}_\varepsilon) = \sigma_{2n-1} \left(\varphi(\varepsilon)\right)^{2n-2}.$$

As $S_f(\varepsilon) = o\left(1/\varphi^{2n-2}(\varepsilon)\right)$ when $\varepsilon \to 0$, the desired equality follows. Thus, the current $f[\mathcal{S}]^{0,1}$ is $\bar\partial$-closed in $\mathcal{D}$.

The rest of the proof of Theorem 2.1 runs as the proof of Theorem 0.1. We first recall a $\bar\partial$-homotopy formula of [HL75]. Namely, let

$$F = \sum_{j=1}^{n} F_j \left(\partial/\partial\bar z_j\right)$$

be a vector field in $\mathbb{C}^n$ whose coefficients are distributions on all of $\mathbb{C}^n$, satisfying

$$\sum_{j=1}^{n} \frac{\partial F_j}{\partial \bar z_j} = \delta,$$

δ being the Dirac delta-function. Then, given any current T of bidegree (p,q) with compact support, we have

$$T = F\#\bar\partial T + \bar\partial\left(F\#T\right)$$

where $F\#T$ stands for the contraction of T by F.

Consider

$$\begin{aligned} F_j &= c_n\, \bar{z}_j/|z|^{2n}, \quad j = 1, \ldots, n; \\ T &= \chi\, f[\mathcal{S}]^{0,1}, \end{aligned}$$

where c_n is a suitable constant and $\chi \in C^\infty_{\mathrm{comp}}(\mathcal{D})$ a function equal to 1 in a polycylindrical neighbourhood U of a point $z^0 \in \mathcal{D}$. This yields

$$-\,(F\#T)(z) = \int_{\mathcal{S}} \chi(\zeta) f(\zeta)\, U(\zeta, z), \quad z \notin \mathcal{S}. \tag{3.2}$$

Since $\bar\partial\,(\chi f[\mathcal{S}]^{0,1}) = 0$ in U, the current $F\#\bar\partial T$ has smooth coefficients in U and is $\bar\partial$-closed there. By Grothendieck's lemma, $F\#\bar\partial T$ is $\bar\partial$-exact in U, i.e., $F\#\bar\partial T = \bar\partial u$ for some smooth function u. Hence it follows that

$$T = \bar\partial\,(F\#T + u)$$

in U. In particular, the function $h_U = F\#T + u$ is holomorphic in $U \setminus \mathcal{S}$. We write $h_U^\pm$ for its restriction to $\mathcal{D}^\pm \cap U$.

Approximating $\mathcal{D}$ from within by domains U where an analytic representation $h_U^\pm$ has already been constructed, and using a familiar process of improving the convergence of a series, we arrive at an analytic representation $h^\pm$ in the whole domain $\mathcal{D}$.

Formula (3.2) shows that the difference of boundary values of the functions $h_U^\pm$ on $\mathcal{S} \cap U$ coincides with the jump of the Bochner-Martinelli integral on $\mathcal{S} \cap U$. Hence, it is equal to u, which completes the proof.

Remark 3.3 *As is shown in the proof of Theorem 6.1 of* [Kyt95, Ch. 2]*, the difference $h^\pm - M$ is a smooth function in $\mathcal{D}$. Therefore, the boundary behaviour of the functions $h^\pm$ near the singular point is completely defined by that of the Bochner-Martinelli integral $M(z)$ of f.*

4 Estimates of representing functions

The rest part of the paper is devoted to the study of boundary behaviour of the Bochner-Martinelli integral near singular points.

Recall that $\rho(z)$ stands for the defining function of the hypersurface $\mathcal{S}$ close to the singular point $O(0,0)$, i.e.,

$$\begin{aligned} \rho(z) &= \varphi^2(x_n) - |x'|^2 - |y|^2 \\ &= \varphi^2\,(\Re z_n) - |z'|^2 - (\Im z_n)^2 \end{aligned}$$

where $z' = (z_1, \ldots, z_{n-1})$. We assume that $\varphi'(x_n) > 0$ for $x_n \in (0, \varepsilon^0]$, and $\varphi(x_n) = x_n^p \psi(x_n)$ near $x_n = 0$, where p is an integer ≥ 1 and $\psi(x_n)$ a continuous function with $\psi(0) \neq 0$.

Set $\mathcal{S}_{\varepsilon^0} = \{\zeta \in \mathcal{D} : \rho(\zeta) = 0,\ 0 \leq \Re\zeta_n \leq \varepsilon^0\}$. Given a function $f \in L^1(\mathcal{S}_{\varepsilon^0})$, we consider the integral

$$\mathcal{P}_m(z) = \int_{\mathcal{S}_{\varepsilon^0}} f(\zeta)\, \frac{d\sigma(\zeta)}{|\zeta - z|^m}, \quad z \in \mathcal{D} \setminus \mathcal{S},$$

where $m > 0$.

In the sequel we restrict our attention to $z \in \mathcal{D}^+$. If $\tilde{z}$ is the projection of z to the axis Ox_n, then

$$|\tilde{z}|^2 \leq |z|^2 \leq |\tilde{z}|^2 + \varphi^2(|\tilde{z}|).$$

Hence

$$1 \leq \frac{|z|}{|\tilde{z}|} \leq \sqrt{1 + \left(\frac{\varphi(|\tilde{z}|)}{|\tilde{z}|}\right)^2}. \tag{4.1}$$

The right-hand side of (4.1) is dominated by a constant $C > 0$ independent of z, for the limit

$$\lim_{|\tilde{z}| \to 0} \frac{\varphi(|\tilde{z}|)}{|\tilde{z}|} = \varphi'(0)$$

exists and $\varphi'(x_n) \in C[0, \varepsilon^0]$. Therefore, (4.1) implies that z and $\tilde{z}$ are actually equivalent when $z \to 0$.

Since $\varphi(|z|) = |z|^p \psi(|z|)$, the functions $\varphi(|z|)$ and $\varphi(|\tilde{z}|)$ are also equivalent as $z \to 0$. From this we deduce that the estimate of $\mathcal{P}_m(\tilde{z})$ in terms of the function $\varphi(|\tilde{z}|)$ to be next obtained, will actually be valid for every point $z \in \mathcal{D}^+$.

Let $z = \tilde{z}$ tend to 0 along Ox_n. In this case we have $z = (0, \ldots, 0, x_n)$, with $x_n > 0$. Denote $\zeta_j = \tau_j + iv_j$, for $j = 1, \ldots, n$. Using equality (3.1) for the measure $d\sigma$ we get

$$\begin{aligned} \mathcal{P}_m(z) &= \int_0^{\varepsilon^0} (\varphi(\tau_n))^{2n-2} \sqrt{1 + (\varphi'(\tau_n))^2}\, d\tau_n \int_{\mathbb{S}^{2n-2}} \frac{f(\zeta) d\omega}{((\tau_n - x_n)^2 + |\tau'|^2 + |v|^2)^{m/2}} \\ &= \int_0^{\varepsilon^0} \frac{(\varphi(\tau_n))^{2n-2}}{((\tau_n - x_n)^2 + (\varphi(\tau_n))^2)^{m/2}} \sqrt{1 + (\varphi'(\tau_n))^2}\, d\tau_n \int_{\mathbb{S}^{2n-2}} f(\zeta) d\omega. \end{aligned}$$

As

$$\int_{\mathbb{S}^{2n-2}} f(\zeta) d\omega = \sigma_{2n-1}\, S_f(\tau_n),$$

it follows that

$$|\mathcal{P}_m(z)| \leq c \int_0^{\varepsilon^0} \frac{(\varphi(\tau_n))^{2n-2}}{((\tau_n - x_n)^2 + (\varphi(\tau_n))^2)^{m/2}}\, S_f(\tau_n)\, d\tau_n \tag{4.2}$$

where c is equal to σ_{2n-1} times the supremum of $\sqrt{1+(\varphi'(\tau_n))^2}$ in τ_n over $[0,\varepsilon^0]$.

In order to estimate the integral on the right-hand side of (4.2), we need auxiliary material. Let $\tau_n^* = \tau_n^*(x_n)$ stand for the minimum point of the function $(\tau_n - x_n)^2 + (\varphi(\tau_n))^2$ over $\tau_n \in [0,\varepsilon^0]$.

Lemma 4.1 *The following formulas hold:*

$$\begin{aligned}\lim_{x_n\to 0+} \frac{x_n}{\tau_n^*(x_n)} &= 1+(\varphi'(0))^2,\\ \lim_{x_n\to 0+} \frac{(\tau_n^*-x_n)^2+(\varphi(\tau_n^*))^2}{(\varphi(x_n))^2} &= \frac{1}{1+(\varphi'(0))^2}.\end{aligned}$$

Proof. At the point of minimum we clearly have $x_n - \tau_n^* = \varphi(\tau_n^*)\,\varphi'(\tau_n^*)$, whence

$$\frac{x_n}{\tau_n^*} - 1 = \frac{\varphi(\tau_n^*)}{\tau_n^*}\,\varphi'(\tau_n^*).$$

Since $x_n \to 0+$ and $\varphi(0) = 0$, it follows that $\tau_n^*(x_n) \to 0+$. Hence we conclude that

$$\begin{aligned}\lim_{x_n\to 0+} \frac{x_n}{\tau_n^*(x_n)} &= \lim_{x_n\to 0+} \left(1+\frac{\varphi(\tau_n^*)}{\tau_n^*}\,\varphi'(\tau_n^*)\right)\\ &= 1+(\varphi'(0))^2,\end{aligned}$$

as desired.

To prove the second formula, we consider separately two cases, namely $\varphi'(0) = 0$ and $\varphi'(0) > 0$.

If $\varphi'(0) = 0$, then

$$\begin{aligned}\lim_{x_n\to 0+} \frac{x_n}{\tau_n^*} &= 1,\\ \lim_{x_n\to 0+} \frac{\varphi(x_n)}{\varphi(\tau_n^*)} &= 1\end{aligned}$$

whence

$$\begin{aligned}\lim_{x_n\to 0+} \frac{(\tau_n^*-x_n)^2+(\varphi(\tau_n^*))^2}{(\varphi(x_n))^2} &= \lim_{x_n\to 0+} \frac{(\varphi(\tau_n^*))^2((\varphi'(\tau_n^*))^2+1)}{(\varphi(x_n))^2}\\ &= 1.\end{aligned}$$

On the other hand, if $\varphi'(0) > 0$ then

$$\begin{aligned}\lim_{x_n\to 0+} \frac{\varphi(\tau_n^*)}{\varphi(x_n)} &= \lim_{x_n\to 0+} \frac{\varphi\left(\frac{x_n}{1+(\varphi'(0))^2}\right)}{\varphi(x_n)}\\ &= \frac{1}{1+(\varphi'(0))^2},\end{aligned}$$

which is due to L'Hospital's rule. Hence

$$\begin{aligned}\lim_{x_n\to 0+}\frac{(\tau_n^*-x_n)^2+(\varphi(\tau_n^*))^2}{(\varphi(x_n))^2} &= \lim_{x_n\to 0+}\frac{(\varphi(\tau_n^*))^2((\varphi'(\tau_n^*))^2+1)}{(\varphi(x_n))^2}\\ &= \left(\lim_{x_n\to 0+}\frac{\varphi(\tau_n^*)}{\varphi(x_n)}\right)^2\left(1+(\varphi'(0))^2\right)\\ &= \frac{1}{1+(\varphi'(0))^2},\end{aligned}$$

which establishes the formula. □

Our next objective is to estimate the integral

$$\mathcal{I}_s(x)=\int_0^{\varepsilon^0}\frac{d\tau}{((\tau-x)^2+(\varphi(\tau))^2)^{s/2}}, \tag{4.3}$$

for $s\in\mathbb{R}$.

Lemma 4.2 *As defined above, the integral $\mathcal{I}_s(x)$ meets the following estimates:*

1) *If $2\le s$, then*

$$\mathcal{I}_s(x)=O\left(\frac{1}{(\varphi(x))^{s-1}}\right)\quad as\quad x\to 0+.$$

2) *If $1\le s<2$, then*

$$\mathcal{I}_s(x)=O\left(\frac{|\log\varphi(x)|}{(\varphi(x))^{s-1}}\right)\quad as\quad x\to 0+.$$

3) *If $s<1$, then*

$$\mathcal{I}_s(x)=O(1)\quad as\quad x\to 0+.$$

Proof.

1) Let $s\ge 2$. Then

$$\mathcal{I}_s(x)=\frac{1}{(\varphi(x))^{s-2}}\int_0^{\varepsilon^0}\frac{(\varphi(x))^{s-2}\,d\tau}{((\tau-x)^2+(\varphi(\tau))^2)^{s/2}}.$$

Using Lemma 4.1, we obtain

$$\begin{aligned}|\mathcal{I}_s(x)| &\le \frac{c}{(\varphi(x))^{s-2}}\int_0^{\varepsilon^0}\frac{(\varphi(x))^{s-2}\,d\tau}{((\tau^*-x)^2+(\varphi(\tau^*))^2)^{\frac{s-2}{2}}((\tau-x)^2+(\varphi(\tau))^2)}\\ &\le \frac{C}{(\varphi(x))^{s-2}}\int_0^{\varepsilon^0}\frac{d\tau}{(\tau-x)^2+(\varphi(\tau))^2},\end{aligned} \tag{4.4}$$

the constants c and C being independent of $x \in (0, \varepsilon^0]$.

Let us prove that

$$a \le \frac{(\tau - x)^2 + (\varphi(\tau))^2}{(\tau - x)^2 + (\varphi(x))^2} \le A \tag{4.5}$$

for all $x, \tau \in (0, \varepsilon^0]$, where a and A are positive constants independent of τ and x. To this end, set $y = \varphi(x)$. Since $\varphi'(x) > 0$ for all $x \in (0, \varepsilon^0]$, and $\varphi(x) = x^p\, \psi(x)$ where $\psi(0) \neq 0$ and $\psi'(x) > 0$, the inverse function $x = \varphi^{-1}(y)$ has the form

$$\varphi^{-1}(y) = y^{\frac{1}{p}}\, \eta(y)$$

where $\eta(0) \neq 0$.

Set

$$\begin{cases} \tau - x &= w, \\ x &= x \end{cases}$$

and

$$\begin{cases} w &= \vartheta \tilde{w}, \\ y &= \vartheta \tilde{y} \end{cases}$$

where $\tilde{w}^2 + \tilde{y}^2 = 1$. Then

$$\begin{aligned} \frac{(\tau - x)^2 + (\varphi(\tau))^2}{(\tau - x)^2 + (\varphi(x))^2} &= \frac{w^2 + (\varphi(w + \varphi^{-1}(y)))^2}{w^2 + y^2} \\ &= \frac{\vartheta^2 \tilde{w}^2 + (\varphi(\vartheta \tilde{w} + \varphi^{-1}(\vartheta \tilde{y})))^2}{\vartheta^2} \\ &= \tilde{w}^2 + \left(\frac{\varphi(\vartheta \tilde{w} + \varphi^{-1}(\vartheta \tilde{y}))}{\vartheta} \right)^2 \end{aligned}$$

which gives

$$\begin{aligned} \frac{(\tau - x)^2 + (\varphi(\tau))^2}{(\tau - x)^2 + (\varphi(x))^2} &= \tilde{w}^2 + \left(\left(\vartheta^{\frac{p-1}{p}} \tilde{w} + \tilde{y}^{\frac{1}{p}} \eta(\vartheta \tilde{y}) \right)^p \psi \left(\vartheta \tilde{w} + (\vartheta \tilde{y})^{\frac{1}{p}} \eta(\vartheta \tilde{y}) \right) \right)^2 \\ &\le A, \end{aligned}$$

the constant A does not depend on τ and x.

Considering the reverse fraction, we obtain an estimate

$$\frac{(\tau - x)^2 + (\varphi(x))^2}{(\tau - x)^2 + (\varphi(\tau))^2} \le \frac{1}{a},$$

with a a constant independent of τ and x. This yields (4.5). Finally, combining (4.4) and (4.5) we get

$$|\mathcal{I}_s(x)| \;\le\; \frac{C}{a} \frac{1}{(\varphi(x))^{s-2}} \int_0^{\varepsilon^0} \frac{d\tau}{(\tau - x)^2 + (\varphi(x))^2}$$

$$\begin{aligned}
&\leq \frac{C}{a}\,\frac{1}{(\varphi(x))^{s-1}}\,\arctan\frac{\tau-x}{\varphi(x)}\,\Big|_0^{\varepsilon^0}\\
&\leq \frac{C}{a}\,\pi\,\frac{1}{(\varphi(x))^{s-1}},
\end{aligned}$$

as desired.

2) Suppose $1 \leq s < 2$. Once again we make use of the estimate (4.5) to obtain

$$\begin{aligned}
|\mathcal{I}_s(x)| &= \frac{1}{(\varphi(x))^{s-1}}\int_0^{\varepsilon^0}\frac{(\varphi(x))^{s-1}\,d\tau}{((\tau-x)^2+(\varphi(\tau))^2)^{s/2}}\\
&\leq \frac{1}{a^{\frac{s-1}{2}}}\,\frac{1}{(\varphi(x))^{s-1}}\int_0^{\varepsilon^0}\frac{d\tau}{\sqrt{(\tau-x)^2+(\varphi(x))^2}}\\
&= \frac{1}{a^{\frac{s-1}{2}}}\,\frac{1}{(\varphi(x))^{s-1}}\,\log\left|(\tau-x)+\sqrt{(\tau-x)^2+(\varphi(x))^2}\right|\,\Big|_0^{\varepsilon^0}
\end{aligned}$$

whence

$$\mathcal{I}_s(x) = O\left(\frac{|\log\left(\sqrt{x^2+(\varphi(x))^2}-x\right)|}{(\varphi(x))^{s-1}}\right)$$

when $x \to 0+$. This establishes the formula.

3) If $s < 1$ then

$$\begin{aligned}
|\mathcal{I}_s(x)| &\leq \max\left(\frac{1}{a^{\frac{s}{2}}},\frac{1}{A^{\frac{s}{2}}}\right)\int_0^{\varepsilon^0}\frac{d\tau}{((\tau-x)^2+(\varphi(x))^2)^{s/2}}\\
&\leq \max\left(\frac{1}{a^{\frac{s}{2}}},\frac{1}{A^{\frac{s}{2}}}\right)\int_0^{\varepsilon^0}\frac{d\tau}{|\tau-x|^s},
\end{aligned}$$

and the proof is complete.

□

We are now in a position to prove the main result of this section giving sharp estimates of the potential $\mathcal{P}_m(z)$.

Theorem 4.3 *Suppose that* $f \in L^1(\mathcal{S}_{\varepsilon^0})$ *and* $S_f(\tau_n) = O\left(1/\varphi^N(\tau_n)\right)$ *as* $\tau_n \to 0+$, *for some* $N \leq 2n-2$.

1) *If* $2n-m \leq N$, *then*

$$|\mathcal{P}_m(z)| = O\left(\frac{1}{(\varphi(|z|))^{N+m-2n+1}}\right) \quad as \quad |z| \to 0.$$

2) *If* $2n - m - 1 \le N < 2n - m$, *then*

$$|\mathcal{P}_m(z)| = O\left(\frac{|\log \varphi(|z|)|}{(\varphi(|z|))^{N+m-2n+1}}\right) \quad as \quad |z| \to 0.$$

3) *If* $N < 2n - m - 1$, *then*

$$|\mathcal{P}_m(z)| = O(1) \quad as \quad |z| \to 0.$$

Proof. Indeed, let us continue the inequality (4.2), thus obtaining

$$\begin{aligned} |\mathcal{P}_m(z)| &\le C \int_0^{\varepsilon^0} \frac{d\tau_n}{((\tau_n - x_n)^2 + (\varphi(\tau_n))^2)^{\frac{N-2n+2+m}{2}}} \\ &= C\, \mathcal{I}_{N-2n+2+m}(|z|), \end{aligned}$$

the last integral being introduced by (4.3). To complete the proof it remains to apply Lemma 4.2.

□

Remark 4.4 *It is easy to see that Theorem 4.3 actually remains valid for all* $z \in \mathcal{D}^-$.

Consider the Bochner-Martinelli integral $M(z)$ of the function f. Since $|M(z)| \le c\,|\mathcal{P}_{2n-1}(z)|$, with c a constant independent of z, Theorem 4.3 for $m = 2n - 1$ implies the following statements.

Corollary 4.5 *Under the assumptions of Theorem 4.3, the following estimates hold for* $z \in \mathcal{D} \setminus \mathcal{S}$:

1) *If* $1 \le N$, *then*

$$|M(z)| = O\left(\frac{1}{(\varphi(|z|))^N}\right) \quad as \quad |z| \to 0.$$

2) *If* $0 \le N < 1$, *then*

$$|M(z)| = O\left(\frac{|\log \varphi(|z|)|}{(\varphi(|z|))^N}\right) \quad as \quad |z| \to 0.$$

3) *If* $N < 0$, *then*

$$|M(z)| = O(1) \quad as \quad |z| \to 0.$$

Remark 3.3 enables us to apply Corollary 4.5 to highlight the boundary behaviour of the representing analytic functions $h^{\pm}(z)$ of Theorem 2.1.

Corollary 4.6 *Let $f \in L^1_{\mathrm{loc}}(\mathcal{S})$ be a CR function on $\mathcal{S} \setminus \{z^0\}$ satisfying $S_f(\varepsilon) = O\left(1/\varphi^N(\varepsilon)\right)$ as $\varepsilon \to 0$, for some $N < 2n-2$. Then,*

1) *For $1 \le N$, we have*

$$|h^{\pm}(z)| = O\left(\frac{1}{(\varphi(|z-z^0|))^N}\right) \quad \text{as} \quad z \to z^0.$$

2) *For $0 \le N < 1$, we have*

$$|h^{\pm}(z)| = O\left(\frac{|\log \varphi(|z-z^0|)|}{(\varphi(|z-z^0|))^N}\right) \quad \text{as} \quad z \to z^0.$$

3) *For $N < 0$, we have*

$$|h^{\pm}(z)| = O\left(1\right) \quad \text{as} \quad z \to z^0.$$

In particular, the functions $h^{\pm}(z)$ are of finite order of growth when $z \to z^0$, provided that so is f.

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