Ellipticity and Invertibility in the Cone Algebra on L_p-Sobolev Spaces

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Abstract: Given a manifold B with conical singularities, we consider the cone algebra with discrete asymptotics, introduced by Schulze, on a suitable scale of L_p-Sobolev spaces. Ellipticity is proven to be equivalent to the Fredholm property in these spaces; it turns out to be independent of the choice of p. We then show that the cone algebra is closed under inversion: whenever an operator is invertible between the associated Sobolev spaces, its inverse belongs to the calculus. We use these results to analyze the behaviour of these operators on $L_p(B)$.

Let B be a manifold with conical singularities. By definition, B is a smooth $(n+1)$-dimensional manifold outside a finite set of exceptional points. In a neighborhood of each point b in this collection, B has the structure of a cone whose cross-section, X_b, is a smooth compact manifold of dimension n. Following the standard procedure, we blow up at each b. We obtain locally the cylinder $[0,1)\times X_b$ and globally a manifold $\mathbb{B}$ with boundary which makes the analysis much more convenient. For simplicity, we assume that we only have one singularity.

Fixing a positive density on $\mathbb{B}$, we naturally have the notion of $L_p(\mathbb{B})$. Choosing a boundary defining function t, the space $L_p(B)$ consists of all measurable functions u on $\mathbb{B}$ such that

$$\int |u(y)|^p t^n(y)\,d\mu(y) < \infty.$$

We introduce a class of weighted Mellin L_p-Sobolev spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $s,\gamma\in\mathbb{R}$, $1<p<\infty$. For $s\in\mathbb{N}$ they are easily described as the set of all $u\in H^s_{p,loc}(\mathrm{int}\,\mathbb{B})$, for which, in local coordinates on $[0,1)\times X$,

$$t^{(n+1)/2-\gamma}(t\partial_t)^k\partial_x^\alpha u(t,x)\in L_p(\tfrac{dt}{t}dx),\qquad \forall\, k+|\alpha|\le s.$$

For $p=2$ we recover the notation used by Schulze, cf. [30]. Note that $L_p(B)$ coincides with $\mathcal{H}_p^{0,\gamma_p}(\mathbb{B})$ for $\gamma_p=(n+1)(1/2-1/p)$.

On $\mathbb{B}$ we consider the space $\bigcup_{\mu\in\mathbb{R}} C^\mu(\mathbb{B},\mathbf{g})$ of cone pseudodifferential operators as introduced by Schulze (the so-called 'weight-datum' $\mathbf{g}$ encodes information on the γ used). An operator $A\in C^\mu(\mathbb{B},\mathbf{g})$ induces a continuous mapping

$$A:\mathcal{H}_p^{s,\gamma}(\mathbb{B})\to\mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}) \tag{0.1}$$

for every choice of s and p. A basic question one is interested in is the Fredholm property of A.

The natural analog of the L_2-ellipticity condition is the requirement that (i) the interior principal symbol is elliptic and that (ii) the principal conormal symbol, namely the operator family

$$\{\sigma_M^\mu(A)(z) : \mathrm{Re}\, z = 1/2 - \gamma\} \subset \mathcal{L}(H_p^\mu(X), H_p^0(X))$$

is invertible by a parameter-dependent pseudodifferential operator of order μ; the parameter space here is the line $\{\mathrm{Re}\, w = 1/2 - \gamma\}$. It is clear from the standard theory and the boundedness of A that this condition is sufficient for the Fredholm property in (0.1) for all s and $1 < p < \infty$. We prove here that it also is necessary.

We conclude that the Fredholm property in (0.1) is independent of both s and p. On the other hand, it is well-known from the case $p = 2$ that the Fredholm property depends on γ. If A is elliptic for one choice of γ, then it is elliptic for all $\gamma \in \mathbb{R}$ except for a discrete set without accumulation point. In general, the index will jump in these points. The same is true for arbitrary p. In particular, the Fredholm spectrum of a zero order operator on $L_p(B)$ will in general depend on p, since γ_p varies with p.

We next deduce the spectral invariance of the algebra of zero order cone pseudodifferential operators in $\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}))$: Whenever A is invertible on $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ its inverse is an element of $C^0(\mathbb{B}, \mathbf{g}^{-1})$. As a consequence, the spectrum of $A \in C^0(\mathbb{B}, \mathbf{g})$ in $\mathcal{L}(H_p^{s,\gamma}(\mathbb{B}))$ is independent of s and p while it will in general depend on γ. Therefore also the spectrum of A, considered as an operator on $L_p(B)$, will depend on p. Using order reductions, one obtains analogous results for operators of arbitrary order.

The proof of the above statements on the independence of the Fredholm property and the invertibility on s and p relies on corresponding properties of the parameter-dependent pseudodifferential operators on a smooth manifold. As a by-product we see that the algebra of all parameter-dependent pseudodifferential operators of order zero is a Ψ-algebra in the sense of Gramsch [14].

The results in this paper will be applied to the L_p-theory of partial differential equations on singular spaces. As a first step they will be used in the analysis of resolvents to differential operators on manifolds with conical singularities [27]. Moreover, they play a role in the Fredholm theory for edge-degenerate operators and boundary value problems [26], where one needs to establish the ellipticity of the principal edge symbol as an operator on an infinite cone.

1 Notation and basics

We fix $1 < p < \infty$ and let p' be its dual number, i.e. $1/p + 1/p' = 1$.

For a Fréchet space E, we let $\mathcal{S}(\mathbb{R}^l, E)$ be the space of rapidly decreasing functions with values in E.

Throughout the text, $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and $\langle\xi, \lambda\rangle = (1 + |\xi|^2 + |\lambda|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$, where $\Lambda = \mathbb{R}^l$ for a certain integer $l \in \mathbb{N}_0$.

1.1 Manifolds with conical singularities

In this paper, B is a manifold with conical singularities of dimension $n+1$. By definition, B is a locally compact, second countable Hausdorff space, which is a smooth manifold outside a finite number of points. For each of these so-called *conical points*, b, there is a neighborhood U_b and a smooth compact n-dimensional manifold X_b such that U_b is homeomorphic to the cone $C_b = (\overline{\mathbb{R}}_+ \times X_b)/(\{0\} \times X_b)$, the point b being mapped to the tip of C_b. We then choose an atlas on B subject to the following conditions:

i) changes of coordinates outside the conical points are smooth,

ii) given two homeomorphisms φ_1, φ_2 mapping neighborhoods of a conical point b to open neighborhoods of the tip of C_b, the restriction of $\varphi_2 \circ \varphi_1^{-1}$ to $]0, \varepsilon[\, \times X_b$ (for sufficiently small $\varepsilon > 0$) extends to a smooth map on $[0, \varepsilon[\, \times X_b$.

For simplicity we shall assume that there is only one conical point b. We denote the corresponding cross-section by X. In view of the above extension property of the coordinate changes, we can identify $B \setminus \{b\}$ with the interior of a smooth manifold $\mathbb{B}$ with boundary X, the blow-up of B.

We fix a positive density on $\mathbb{B}$ and introduce the associated spaces $L_p(\mathbb{B})$, $1 < p < \infty$. Moreover, $L_p(B)$ is the space of all measurable functions f on $\mathbb{B}$ such that $|f|^p t^n$ is integrable for some boundary defining function t on $\mathbb{B}$. Note that this is the choice suggested by introducing polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\}$.

Near $\partial\mathbb{B}$ we shall often work with a collar neighborhood $[0, 1[\, \times X$ and use coordinates (t, x), $0 \le t < 1$, $x \in X$.

1.2 Pseudodifferential operators on compact manifolds

By $L^\mu(X; \Lambda)$ we denote the Fréchet space of parameter-dependent pseudodifferential operators of order $\mu \in \mathbb{R}$ on X, whose local symbols satisfy estimates

$$\sup \left\{ |D_\xi^\alpha D_\lambda^\gamma D_x^\beta a(x, \xi, \lambda)| \, \langle \xi, \lambda \rangle^{|\alpha|+|\gamma|-\mu} \mid (x, \xi, \lambda) \in \mathbb{R}^{2n} \times \Lambda \right\} < \infty$$

for all multi-indices α, β, γ. The residual class $L^{-\infty}(X; \Lambda)$ consists of integral operators with kernel in $\mathcal{S}(\Lambda, C^\infty(X \times X))$. For the subclass of classical operators, denoted by $L^\mu_{cl}(X; \Lambda)$, we require the symbols to allow asymptotic expansions $a \sim \sum a_{(\mu-j)}$ with $a_{(\mu-j)}$ positively homogeneous of degree $\mu - j$ in (ξ, λ). With $A \in L^\mu_{cl}(X; \Lambda)$ we associate its homogeneous principal symbol $\sigma^\mu_\psi(A) \in C^\infty((T^*X \times \Lambda) \setminus 0)$, a smooth function homogeneous of degree μ in the fibers over each $x \in X$.

If $H^s_p(X)$ are the standard Sobolev spaces on X, locally modelled over $H^s_p(\mathbb{R}^n) = \mathrm{op}(\langle \xi \rangle^{-s})(L_p(\mathbb{R}^n))$, each $A \in L^\mu(X)$ extends to continuous operators $A : H^s_p(X) \to H^{s-\mu}_p(X)$, and in that sense $L^\mu(X) \hookrightarrow \mathcal{L}(H^s_p(X), H^{s-\mu}_p(X))$ for each $s \in \mathbb{R}$.

We shall need the following result on spectral invariance of pseudodifferential operators. This can be proven with the help of a commutator characterization similarly as in [6] or [4]. The subscript (cl) indicates that it holds both for classical and non-classical operators.

1.1 Theorem. *For $A \in L^{\mu}_{(cl)}(X)$ the following properties are equivalent:*

a) $A : H^s_p(X) \to H^{s-\mu}_p(X)$ is invertible for some $s \in \mathbb{R}$.

b) $A : H^t_q(X) \to H^{t-\mu}_q(X)$ is invertible for all $t \in \mathbb{R}$ and all $1 < q < \infty$.

c) There exists a $B \in L^{-\mu}_{(cl)}(X)$ such that $AB = BA = 1$ on $C^\infty(X)$.

If the above conditions hold, we call A invertible in $L^{\mu}_{(cl)}(X)$.

1.3 Parameter-dependent pseudodifferential operators

We shall establish the L_p-spectral invariance property for parameter-dependent operators of order zero. The proof relies on a technique introduced by Gohberg [11] and Hörmander [17].

For $y, \eta \in \mathbb{R}^n$, $s > 0$, and fixed $0 < \tau < 1/2$ set

$$[S_s(y,\eta)u](x) = s^{\tau n/p} e^{isx\eta} u(s^\tau(x-y)), \qquad u \in L_p(\mathbb{R}^n). \tag{1.1}$$

1.2 Lemma. *$S_s(y,\eta)$ is an isometry with inverse given by*

$$[S_s^{-1}(y,\eta)u](x) = s^{-\tau n/p} e^{-is(y+s^{-\tau}x)\eta} u(y + s^{-\tau}x).$$

If $a \in S^0(\mathbb{R}^n; \mathbb{R}^n \times \Lambda)$ then

$$S_s^{-1}(y,\eta)\,\mathrm{op}(a)(s\lambda)\,S_s(y,\eta) = \mathrm{op}(a_s(y,\eta))(\lambda), \tag{1.2}$$

where $a_s(y,\eta) \in S^0(\mathbb{R}^n; \mathbb{R}^n \times \lambda)$ is defined by

$$a_s(y,\eta; x, \xi, \lambda) = a(y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda).$$

For $(\eta,\lambda) \neq 0$, $s \geq 1$, and all multiindices α, β, γ we have estimates

$$|D^\alpha_\xi D^\gamma_\lambda D^\beta_x a_s(y,\eta; x,\xi,\lambda)| \leq C_{\alpha\beta\gamma} \frac{\langle \xi \rangle^{|\alpha|+|\gamma|}}{|(\eta,\lambda)|^{|\alpha|+|\gamma|}} s^{(2\tau-1)(|\alpha|+|\gamma|)-\tau|\beta|}. \tag{1.3}$$

Proof: Note that, by Peetre's inequality,

$$\langle v + w \rangle^{-1} \leq \frac{c}{\langle v \rangle} \langle w \rangle \leq \frac{c}{|v|} \langle w \rangle$$

for $v \neq 0$, and that for $s \geq 1$ and $0 \leq \sigma \leq 1$

$$\langle s^\tau \sigma w \rangle \leq s^\tau \langle w \rangle .$$

This together with the usual symbol estimates of a shows the estimate for $a_s(y,\eta)$. The other statements are elementary. ∎

1.3 Lemma. *If $u \in L_p(\mathbb{R}^n)$, then $S_s(y,\eta)u \to 0$ weakly in $L_p(\mathbb{R}^n)$ for $s \to \infty$.*

PROOF: We have to show that $\langle S_s(y,\eta)u, v\rangle_{L_2(\mathbb{R}^n)} \to 0$ for all $v \in L_{p'}(\mathbb{R}^n)$. Since $\|S_s(y,\eta)\|_{\mathcal{L}(L_p(\mathbb{R}^n))} = 1$ we may assume that $u, v \in C_0^\infty(\mathbb{R}^n)$. Then

$$|\langle S_s(y,\eta)u, v\rangle| \leq \int s^{\tau n/p}|u(s^\tau(x-y))||v(x)|\, dx \leq s^{-\tau n/p'}\|v\|_{L_\infty(\mathbb{R}^n)}\|u\|_{L_1(\mathbb{R}^n)} \stackrel{s\to\infty}{\longrightarrow} 0.$$

∎

1.4 Lemma. *Let $a \in S^0(\mathbb{R}^n; \mathbb{R}^n \times \Lambda)$ and, for $y \in \mathbb{R}^n$ fixed, let*

$$\alpha = \alpha(y) = \liminf_{|(\eta,\lambda)|\to\infty} |a(y,\eta,\lambda)|.$$

Then there exist $0 \neq (\eta,\lambda) \in \mathbb{R}^n \times \Lambda$ and a sequence $s_k \to \infty$ such that

$$a_{s_k}(y,\eta;x,\xi,\lambda) = a(y + s_k^{-1}x, s_k\eta + s_k^\tau\xi, s_k\lambda) \stackrel{k\to\infty}{\longrightarrow} \beta = \beta(y) \tag{1.4}$$

for all (x,ξ), for some β with $|\beta| = \alpha$.

PROOF: We find a sequence $((\eta_k,\lambda_k))_{k\in\mathbb{N}}$ with $|(\eta_k,\lambda_k)| \to \infty$ such that $a(y,\eta_k,\lambda_k) \to \beta$ for some β with $|\beta| = \alpha$. By passing to a subsequence, we can assume without loss of generality that $\frac{(\eta_k,\lambda_k)}{|(\eta_k,\lambda_k)|} \to (\eta,\lambda)$ for some (η,λ). With $s_k := |(\eta_k,\lambda_k)|$ we write

$$\begin{aligned}|a_{s_k}(y,\eta;x,\xi,\lambda) - \beta(y)| \quad &\leq \quad |a_{s_k}(y,\eta;x,\xi,\lambda) - a(y,s_k\eta,s_k\lambda)| + \\ &\quad + |a(y,s_k\eta,s_k\lambda) - a(y,\eta_k,\lambda_k)| + |a(y,\eta_k,\lambda_k) - \beta(y)|.\end{aligned}$$

By the fundamental theorem of calculus and inequality (1.3), the first summand can be estimated from above by

$$\begin{aligned}&|x|\int_0^1 |(\nabla_x a_{s_k})(y,\eta,\sigma x,\sigma\xi,\lambda)|\, d\sigma \quad + \quad |\xi|\int_0^1 |(\nabla_\xi a_{s_k})(y,\eta,\sigma x,\sigma\xi,\lambda)|\, d\sigma \\ &\qquad \leq \; cs_k^{-\tau}|x| + cs_k^{2\tau-1}\frac{\langle\xi\rangle}{|(\eta,\lambda)|}|\xi| \stackrel{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

A similar argument applies to the second term, the third term converges to zero by assumption. ∎

1.5 Lemma. *Fix $y \in \mathbb{R}^n$ and let $(s_k)_{k\in\mathbb{N}}$ and (η,λ) be as in Lemma 1.4. Then*

$$\mathrm{op}(a_{s_k}(y,\eta))(s_k\lambda)u \stackrel{k\to\infty}{\longrightarrow} \beta(y)u$$

in $L_p(\mathbb{R}^n)$ for each $u \in L_p(\mathbb{R}^n)$.

PROOF: In view of (1.2) and since all $S_s(y,\eta)$ are isometric, we may assume that $u \in C_0^\infty(\mathbb{R}^n)$. Now,

$$[\mathrm{op}(a_{s_k}(y,\eta))(s_k\lambda)u](x) - \beta(y)u(x) = \int e^{ix\xi}(a_{s_k}(y,\eta;x,\xi,\lambda) - \beta(y))\hat{u}(\xi)\, đ\xi.$$

Since $a \in S^0(\mathbb{R}^n;\mathbb{R}^n \times \Lambda)$, the integrand is bounded from above by $c|\hat{u}|$ and, according to (1.4), converges pointwise to zero as $k \to \infty$. Lebesgue's theorem on dominated convergence therefore shows that the above expression tends to zero for each $x \in \mathbb{R}^n$. Integration by parts yields

$$\begin{aligned} x^\gamma & \int e^{ix\xi}(a_{s_k}(y,\eta;x,\xi,\lambda) - \beta(y))\hat{u}(\xi)\,đ\xi \\ &= \sum_{\gamma_1+\gamma_2=\gamma} c_{\gamma_1\gamma_2} \int e^{ix\xi}(D_\xi^{\gamma_1} a_{s_k}(y,\eta;x,\xi,\lambda) - \beta(y))D_\xi^{\gamma_2}\hat{u}(\xi)\,đ\xi \end{aligned}$$

for any $\gamma \in \mathbb{N}_0^n$. Therefore,

$$|[\mathrm{op}(a_{s_k}(y,\eta))(s_k\lambda)u](x) - \beta(y)u(x)|^p \leq c\,\langle x\rangle^{-(n+1)}$$

and a second application of Lebesgue's theorem shows the convergence in $L_p(\mathbb{R}^n)$. ∎

1.6 Theorem. *Let $A \in L^0(X;\Lambda)$ and assume the existence of $B(\lambda) \in \mathcal{L}(L_p(X))$, $\lambda \in \Lambda$, uniformly bounded in λ, such that*

$$B(\lambda)A(\lambda) = 1 - K(\lambda)$$

with $K(\lambda) \in \mathcal{L}(L_p(X))$ compact and $K(\lambda) \to 0$ as $|\lambda| \to \infty$. Then A is parameter-dependent elliptic.

Proof: Let $x \in X$, $\chi : U \to \mathbb{R}^n$ a chart around x, and $\varphi_0, \varphi_1 \in C_0^\infty(U)$ with $\varphi_0\varphi_1 = \varphi_1$ and $\varphi_j \equiv 1$ near x. Then $\chi^*(\varphi_0\,A(\lambda)\,\varphi_1) = \mathrm{op}(a)(\lambda)$ with a symbol $a \in S^0(\mathbb{R}^n;\mathbb{R}^n \times \Lambda)$. It clearly suffices to show the existence of constants $C, R > 0$ such that

$$|a(y,\eta,\lambda)| \geq C \qquad \forall\,|(\eta,\lambda)| \geq R, \tag{1}$$

where $y := \chi(x)$. By continuity, such an estimate then automatically holds in a neighborhood of y. If $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ near y and $(\varphi \circ \chi)\varphi_1 = (\varphi \circ \chi) =: \varphi_\chi$, then

$$\varphi_\chi B(\lambda)\varphi_0 A(\lambda)\varphi_1 - \varphi_\chi = \varphi_\chi K(\lambda)\varphi_1 - \varphi_\chi B(\lambda)(1-\varphi_0)A(\lambda)\varphi_1.$$

Due to the disjoint support of $(1-\varphi_0)$ and φ_1 we obtain operator families $\tilde{B}(\lambda), \tilde{K}(\lambda) \in \mathcal{L}(L_p(\mathbb{R}^n))$ having the same properties as $B(\lambda)$ and $K(\lambda)$, respectively, such that

$$\varphi[\tilde{B}(\lambda)\mathrm{op}(a)(\lambda) - 1] = \tilde{K}(\lambda).$$

Choose $0 \neq u \in C_0^\infty(\mathbb{R}^n)$ with $\varphi u(\cdot - y) = u(\cdot - y)$. Then

$$\begin{aligned} \|u\|_{L_p(\mathbb{R}^n)} &= \|S_s(y,\eta)u\|_{L_p(\mathbb{R}^n)} = \|\varphi S_s(y,\eta)u\|_{L_p(\mathbb{R}^n)} \\ &= \|\varphi\tilde{B}(\lambda)\mathrm{op}(a)(\lambda)S_s(y,\eta)u - \tilde{K}(\lambda)S_s(y,\eta)u\|_{L_p(\mathbb{R}^n)} \\ &\leq c\,\|S_s^{-1}(y,\eta)\mathrm{op}(a)(\lambda)S_s(y,\eta)u\|_{L_p(\mathbb{R}^n)} + \|\tilde{K}(\lambda)S_s(y,\eta)u\|_{L_p(\mathbb{R}^n)} \end{aligned} \tag{2}$$

with a constant $c \geq \|\varphi\tilde{B}(\lambda)\|_{\mathcal{L}(L_p(\mathbb{R}^n))} > 0$. For the second identity note that the support of $S_s(y,\eta)u$ shrinks to y as $s \to \infty$. Now let $0 \neq (\eta,\lambda)$ and $s_k \to \infty$ as in Lemma 1.4. Then

$$\|\tilde{K}(s_k\lambda)S_{s_k}(y,\eta)u\|_{L_p(\mathbb{R}^n)} \overset{k\to\infty}{\longrightarrow} 0,$$

since $\tilde{K}(s_k\lambda) \to 0$ for $\lambda \neq 0$ and $\tilde{K}(0)S_{s_k}(y,\eta)u \to 0$ by Lemma 1.3 and the compactness of $\tilde{K}(0)$. Replacing λ by $s_k\lambda$ and s by s_k in (2), and passing to the limit $s_k \to \infty$, Lemma 1.5 then implies $\|u\|_{L_p(\mathbb{R}^n)} \geq c|\beta(y)|\|u\|_{L_p(\mathbb{R}^n)}$, i.e.

$$\liminf_{|(\eta,\lambda)|\to\infty} |a(y,\eta,\lambda)| = |\beta(y)| \geq \frac{1}{c}.$$

This yields (1). ∎

1.7 Remark. *Theorem 1.6 extends to the case of parameter-dependent operators acting on sections in vector bundles over X. The proof is quite the same as before, by considering*

$$\liminf_{|v|=1,\,|(\eta,\lambda)|\to\infty} |a(y,\eta,\lambda)v|$$

for $v \in \mathbb{R}^L$, where L is the fiber dimension.

1.8 Corollary. *Suppose $A \in L^0_{(cl)}(X;\Lambda)$ is pointwise invertible in $\mathcal{L}(L_p(\mathbb{R}^n))$ and $A(\lambda)^{-1}$ is uniformly bounded in $\lambda \in \Lambda$. Then $A^{-1} \in L^0_{(cl)}(X;\Lambda)$.*

Proof: By Proposition 1.6 A is elliptic. Hence there is a parametrix $B \in L^0_{(cl)}(X;\Lambda)$ with $BA - 1 = R_L$ and $AB - 1 = R_R$ for $R_L, R_R \in L^{-\infty}(X;\Lambda) \cong \mathcal{S}(\Lambda, C^\infty(X \times X))$. Solving for A^{-1} yields

$$A^{-1} = B - R_LB + R_LA^{-1}R_R \;\in\; L^0_{(cl)}(X;\Lambda),$$

since $R_LA^{-1}R_R \in L^{-\infty}(X;\Lambda)$ in view of the assumptions on A. ∎

2 The cone algebra on L_p-Sobolev spaces

2.1 Pseudodifferential operators based on the Mellin transform

For real σ we set $\Gamma_\sigma = \{z \in \mathbb{C} \mid \operatorname{Re} z = \sigma\}$.

2.1 Definition. *For $\gamma \in \mathbb{R}$ let $\mathcal{T}_\gamma(\mathbb{R}_+ \times X)$ denote the Fréchet space of all $\varphi \in C^\infty(\mathbb{R}_+, C^\infty(X))$ with*

$$\sup\left\{ |||t^{1/2-\gamma}(t\partial_t)^k\varphi(t)|||\,\langle \log t\rangle^l \;\mid t \in \mathbb{R}_+ \right\} < \infty$$

for all $k, l \in \mathbb{N}_0$ and each continuous semi-norm $|||\cdot|||$ of $C^\infty(X)$.

Multiplication by t^ζ, $\zeta \in \mathbb{C}$, induces an isomorphism $\mathcal{T}_\gamma(\mathbb{R}_+ \times X) \to \mathcal{T}_{\gamma+\operatorname{Re}\zeta}(\mathbb{R}_+ \times X)$; this is also true for the map $S_\gamma : \mathcal{T}_\gamma(\mathbb{R}_+ \times X) \to \mathcal{S}(\mathbb{R}\times X) := \mathcal{S}(\mathbb{R}, C^\infty(X))$ defined by

$$S_\gamma\varphi(r) = e^{(\gamma-1/2)r}\varphi(e^{-r}). \tag{2.1}$$

We define the *(weighted) Mellin transform* of $\varphi \in \mathcal{T}_\gamma(\mathbb{R}_+ \times X)$ by

$$\mathcal{M}_\gamma \varphi(z) = \int_0^\infty t^z \varphi(t) \, \frac{dt}{t}, \qquad z \in \Gamma_{1/2-\gamma},$$

with convergence of the integral in $C^\infty(X)$. If we write $đ\tau = (2\pi)^{-1} d\tau$, then $\mathcal{M}_\gamma : \mathcal{T}_\gamma(\mathbb{R}_+ \times X) \to \mathcal{S}(\Gamma_{1/2-\gamma} \times X)$ is an isomorphism with inverse

$$\mathcal{M}_\gamma^{-1} \phi(t) = \int t^{-(1/2-\gamma+i\tau)} \phi(1/2 - \gamma + i\tau) \, đ\tau, \qquad t > 0.$$

2.2 Definition. *For $\mu \in \mathbb{R}$ let $ML^\mu(X, \mathbb{R}_+; \Gamma_\gamma)$ consist of all $h \in C^\infty(\mathbb{R}_+, L^\mu(X; \Gamma_\gamma))$ such that for each continuous semi-norm $|||\cdot|||$ of $L^\mu(X; \Gamma_\gamma)$ and each $k \in \mathbb{N}_0$*

$$\sup \left\{ |||(t\partial_t)^k h(t)||| \mid t \in \mathbb{R}_+ \right\} < \infty.$$

With each such symbol h we associate a Mellin operator $\mathrm{op}_M^\gamma(h) : \mathcal{T}_\gamma(\mathbb{R}_+ \times X) \to \mathcal{T}_\gamma(\mathbb{R}_+ \times X)$ by

$$[\mathrm{op}_M^\gamma(h)\varphi](t) = \int t^{-(1/2-\gamma+i\tau)} h(t, 1/2 - \gamma + i\tau)(\mathcal{M}_\gamma \varphi)(1/2 - \gamma + i\tau) \, đ\tau. \tag{2.2}$$

2.2 Meromorphic Mellin symbols

Let $\zeta \in \mathbb{C}$ and $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega \equiv 1$ near $t = 0$. Then the function

$$\varphi(t) = \varphi_{\zeta,k}(t) = \omega(t) \, t^{-\zeta} \, \log^k t, \qquad t > 0,$$

belongs to $\mathcal{T}_\gamma(\mathbb{R}_+)$ if and only if $\operatorname{Re} \zeta < 1/2 - \gamma$. In this case, $\mathcal{M}_\gamma \varphi \in \mathcal{S}(\Gamma_{1/2-\gamma})$ extends to a meromorphic function with exactly one pole in ζ of order $k+1$ that admits a decomposition

$$\psi_{\zeta,k}(z) := \mathcal{M}_\gamma \varphi(z) = \frac{(-1)^k k!}{(z - \zeta)^{k+1}} \; + \; g(z) \tag{2.3}$$

with an entire function g. Moreover, $\sigma \mapsto (\chi \mathcal{M}_\gamma \varphi)(\sigma + i\cdot) : \mathbb{R} \to \mathcal{S}(\mathbb{R})$ is continuous for any ζ-excision function χ.

2.3 Definition. *A set P is called a discrete asymptotic type for Mellin symbols if*

$$P = \{(p_j, n_j, N_j) \mid \operatorname{Re} p_j \to \pm\infty \text{ for } j \to \mp\infty, \, n_j \in \mathbb{N}_0, \, j \in \mathbb{Z}\},$$

with finite dimensional subspaces $N_j \subset L^{-\infty}(X)$ of finite rank operators. We also allow P to be a finite set. Let $\pi_{\mathbb{C}} P = \{p_j \mid j \in \mathbb{Z}\}$ and O the empty asymptotic type.

2.4 Definition. *With P as in Definition 2.3, the space $M_P^\mu(X)$ consists of all meromorphic functions $h : \mathbb{C} \setminus \pi_{\mathbb{C}} P \to L_{cl}^\mu(X)$ with poles in p_j at most of order $n_j + 1$ that satisfy:*

a) *The coefficients of the principal part of h at p_j are elements of N_j, i.e. for all $j \in \mathbb{Z}$ and $0 \le k \le n_j$*

$$r_{p_j,k}(h) := \frac{\partial^k}{\partial z^k} \left\{(z-p_j)^{n_j+1} h(z)\right\}\Big|_{z=p_j} \in N_j,$$

b) *if $\psi_{p_j,k}$ is as in (2.3) and*

$$h_N := h - \sum_{|\mathrm{Re}\, p_j| \le N} \sum_{k=0}^{n_j} r_{p_j,k}(h) \psi_{p_j,k},$$

then $\gamma \mapsto h_N(\gamma + i\cdot) : [-N,N] \to L^{\mu}_{cl}(X;\mathbb{R})$ is continuous for each $N \in \mathbb{N}$.

$M^{\mu}_{P}(X)$ is a Fréchet space if equipped with the projective limit topology under the maps

$$h \mapsto r_{p_j,k}(h) : M^{\mu}_P(X) \to N_j, \qquad h \mapsto h_N : M^{\mu}_P(X) \to C([-N,N], L^{\mu}_{cl}(X;\mathbb{R})).$$

2.3 The cone algebra

2.5 Definition. *By $C^{\infty}_{\gamma}(\mathbb{B})$ we denote the Fréchet space of all $\varphi \in C^{\infty}(\mathrm{int}\,\mathbb{B})$ such that $\omega\varphi \in \mathcal{T}_{\gamma-n/2}(\mathbb{R}_+ \times X)$ for a cut-off function $\omega \in C^{\infty}_0([0,1[)$.*

2.6 Definition. *i) Let $\gamma \in \mathbb{R}$ and $\Theta =]-k, 0]$ with $k \in \mathbb{N}$. A set Q is called an asymptotic type with respect to (γ, Θ) if*

$$Q = \left\{(q_j, l_j, L_j) \mid \tfrac{n+1}{2} - \gamma - k < \mathrm{Re}\, q_j < \tfrac{n+1}{2} - \gamma,\ l_j \in \mathbb{N}_0,\ j = 1, \ldots, N\right\}$$

for some $N \in \mathbb{N}_0$, and with finite dimensional subspaces $L_j \subset C^{\infty}(X)$. The projection of Q to the complex plane is denoted by $\pi_{\mathbb{C}}Q$, the set of all such asymptotic types by $As(\gamma,\Theta)$. We write O for the empty asymptotic type.

ii) With $Q \in As(\gamma, \Theta)$ as in i) and a fixed cut-off function $\omega \in C^{\infty}_0([0,1[)$ let

$$\mathcal{E}_Q(\mathbb{B}) = \left\{ t \mapsto \sum_{j=1}^{N} \sum_{l=0}^{l_j} f_{jl}\, \omega(t)\, t^{-q_j} \log^l t \mid f_{jl} \in L_j \right\}.$$

This is a finite dimensional subspace of $C^{\infty}_{\gamma}(\mathbb{B})$. Moreover, we set

$$C^{\infty}_{\gamma,Q}(\mathbb{B}) := \operatorname*{proj-lim}_{\varepsilon > 0} C^{\infty}_{\gamma+k-\varepsilon}(\mathbb{B}) \oplus \mathcal{E}_Q(\mathbb{B}),$$

which is a Fréchet subspace of $C^{\infty}_{\gamma}(\mathbb{B})$.

2.7 Definition. *For $\gamma, \mu \in \mathbb{R}$ the space $C_G(\mathbb{B}; (\gamma, \gamma-\mu, \Theta))$ of Green operators consists of all integral operators G that have a smooth kernel*

$$k_G \in (C^{\infty}_{\gamma-\mu,Q}(\mathbb{B}) \,\widehat{\otimes}_{\pi}\, C^{\infty}_{-\gamma}(\mathbb{B})) \cap (C^{\infty}_{\gamma-\mu}(\mathbb{B}) \,\widehat{\otimes}_{\pi}\, C^{\infty}_{-\gamma,Q'}(\mathbb{B}))$$

with asymptotic types $Q \in As(\gamma-\mu,\Theta)$ and $Q' \in As(-\gamma,\Theta)$ (depending on G). More precisely,

$$Gu(y) = \langle k_G(y,\cdot),\overline{u}\rangle_{L_2(B)} = \int_{\mathbb{B}} k_G(y,y')u(y')t(y')^n\, dy', \qquad u \in C_\gamma^\infty(\mathbb{B}).$$

It can be shown, cf. [33], that Green operators equivalently can be characterized by their mapping properties in corresponding Sobolev spaces (with asymptotics), as stated below in Theorem 2.14.

Definition 2.8 below, as well as the two subsequent theorems may be found in [7], Chapter 8.1.4.

2.8 Definition. *The cone algebra $C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ of order μ with respect to $(\gamma,\gamma-\mu,\Theta)$ is the space of all continuous operators $A : C_\gamma^\infty(\mathbb{B}) \to C_{\gamma-\mu}^\infty(\mathbb{B})$ of the form*

$$A = \omega_0\, A_M\, \omega_1 \;+\; (1-\omega_2)\, A_\psi\, (1-\omega_3) \;+\; M \;+\; G$$

with functions $\omega_j \in C_0^\infty([0,1[)$ that are identically 1 near $t=0$, and

i) a Mellin operator $A_M = t^{-\mu}\operatorname{op}_M^{\gamma-n/2}(h)$ with $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$,

ii) a pseudodifferential operator $A_\psi \in L_{cl}^\mu(2\mathbb{B})$,

iii) a so-called smoothing Mellin operator $M = \omega_0 \left\{ \sum_{l=0}^{k-1} t^{-\mu+l}\operatorname{op}_M^{\gamma_l-n/2}(h_l)\right\}\omega_1$ with $h_l \in M_{P_l}^{-\infty}(X)$, $\pi_{\mathbb{C}}P_l \cap \Gamma_{\frac{n+1}{2}-\gamma_l} = \emptyset$, and $\gamma - l \le \gamma_l \le \gamma$,

iv) a Green operator $G \in C_G(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$.

The conormal symbol of A is the element of $M_{P_0}^\mu(X)$ defined by

$$\sigma_M^\mu(A)(z) := h(0,z) + h_0(z).$$

2.9 Theorem. *The composition of cone operators yields mappings*

$$C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta)) \times C^{\mu'}(\mathbb{B};(\gamma+\mu',\gamma,\Theta)) \longrightarrow C^{\mu+\mu'}(\mathbb{B};(\gamma+\mu',\gamma-\mu,\Theta)).$$

The Green operators form an 'ideal' with respect to this composition. For the conormal symbol we obtain

$$\sigma_M^{\mu+\mu'}(AA')(z) = \sigma_M^\mu(A)(z+\mu')\sigma_M^{\mu'}(A')(z).$$

2.10 Theorem. *If $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$, its formal adjoint A^* with respect to the $L_2(B)$-scalar product is an element of $C^\mu(\mathbb{B};(-\gamma+\mu,-\gamma,\Theta))$. The conormal symbol of A^* is given by*

$$\sigma_M^\mu(A^*)(z) = \sigma_M^\mu(A)(n+1-\mu-\bar{z})^*,$$

where * on the right-hand side denotes the pointwise formal adjoint in $L_{cl}^\mu(X)$.

2.4 Extension to Sobolev spaces

Let $\{U_1, \ldots, U_N\}$ be an open covering of X with charts $\chi_j : U_j \to V_j \subset \mathbb{R}^n$, and $\phi_j, \psi_j \in C_0^\infty(U_j)$ functions with $\phi_j \psi_j = \phi_j$ and $\sum_{j=1}^N \phi_j \equiv 1$.

2.11 Definition. *Let $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ be the closure of $C_\gamma^\infty(\mathbb{B})$ with respect to the norm*

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} = \Big(\sum_{j=1}^N \|[S_{\gamma-n/2}(\phi_j \omega u)] \circ (1 \times \chi_j^{-1})\|^2_{H_p^s(\mathbb{R}^{1+n})} + \|(1-\omega)u\|^2_{H_p^s(2\mathbb{B})} \Big)^{1/2},$$

where $S_{\gamma-n/2}$ is as in (2.1). Note that different choices of cut-off functions $\omega \in C_0^\infty([0,1[)$ yield equivalent norms. For an asymptotic type $Q \in As(\gamma, \Theta)$,

$$\mathcal{H}_{p,Q}^{s,\gamma}(\mathbb{B}) := \operatorname*{proj\text{-}lim}_{\varepsilon>0} \mathcal{H}_p^{s,\gamma+k-\varepsilon}(\mathbb{B}) \oplus \mathcal{E}_Q(\mathbb{B})$$

is a Fréchet subspace of $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$.

2.12 Remark. *a) $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is a Banach space and, in case $p = 2$, a Hilbert space.*

b) If $m \in \mathbb{N}_0$ then $u \in \mathcal{H}_p^{m,\frac{n+1}{2}}(\mathbb{B})$ if and only if $(1-\omega)u \in H_p^m(2\mathbb{B})$ and $(t\partial_t)^k D_l(\omega u) \in L_p(\mathbb{R}_+ \times X; \frac{dt}{t}dx)$ for all $k + l \le m$ and all differential operators D_l on X of order at most l.

c) $L_p(B) = \mathcal{H}_p^{0,\gamma_p}(\mathbb{B})$ with $\gamma_p = (n+1)(\frac{1}{2} - \frac{1}{p})$.

d) The embedding $\mathcal{H}_p^{s,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{s',\gamma'}(\mathbb{B})$ is continuous if $s \ge s'$, $\gamma \ge \gamma'$, and compact if $s > s'$, $\gamma > \gamma'$.

e) The dual of $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ can be identified with $\mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{B})$ via the non-degenerate sesquilinear pairing

$$(u,v) \mapsto \langle u, v\rangle_{L_2(\mathbb{B})} : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \times \mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{B}) \to \mathbb{C}.$$

The following proposition was shown in [2], Remark 4.2, Proposition 6.8.

2.13 Proposition. *Each $A \in C^\mu(\mathbb{B}; (\gamma, \gamma-\mu, \Theta))$ induces continuous operators*

$$A : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$$

for all $s \in \mathbb{R}$. Moreover, to each asymptotic type $Q \in As(\gamma, \Theta)$ there exists a $Q' \in As(\gamma-\mu, \Theta)$ depending on A and Q such that

$$A : \mathcal{H}_{p,Q}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_{p,Q'}^{s-\mu,\gamma-\mu}(\mathbb{B}).$$

Sometimes it is important to know that Green operators can equivalently be characterized by their mapping properties in the Sobolev spaces rather than by the structure of their kernels. It is particularly useful that the analogue of the definition for the case $p = 2$, namely (2.4) below, yields the same class of operators for each choice of p.

2.14 Theorem. *An operator $G : C^\infty_\gamma(\mathbb{B}) \to C^\infty_{\gamma-\mu}(\mathbb{B})$ belongs to $C_G(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ if and only if G and its adjoint G^* extend for each $s \in \mathbb{R}$ to continuous operators*

$$G : \mathcal{H}^{s,\gamma}_p(\mathbb{B}) \to C^\infty_{\gamma-\mu,Q}(\mathbb{B}), \qquad G^* : \mathcal{H}^{s,\mu-\gamma}_{p'}(\mathbb{B}) \to C^\infty_{-\gamma,Q'}(\mathbb{B}) \tag{2.4}$$

for appropriate asymptotic types $Q \in As(\gamma-\mu,\Theta)$ and $Q' \in As(-\gamma,\Theta)$.

3 Ellipticity, Fredholm property, and spectral invariance

3.1 Definition. *A cone operator $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ is called elliptic, if*

i) the homogeneous principal symbol $\sigma^\mu_\psi(A) \in C^\infty(T^*(\operatorname{int}\mathbb{B}) \setminus 0)$ does not vanish,

ii) the conormal symbol $\sigma^\mu_M(A)(z) \in L^\mu_{cl}(X)$ is invertible for each $z \in \Gamma_{\frac{n+1}{2}-\gamma}$, and $\sigma^\mu_M(A)^{-1} \in L^{-\mu}_{cl}(X;\Gamma_{\frac{n+1}{2}-\gamma})$.

3.2 Theorem. *Let $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ be elliptic. Then there exists a parametrix $B \in C^{-\mu}(\mathbb{B};(\gamma-\mu,\gamma,\Theta))$ such that*

$$BA - 1 \in C_G(\mathbb{B};(\gamma,\gamma,\Theta)), \qquad AB - 1 \in C_G(\mathbb{B};(\gamma-\mu,\gamma-\mu,\Theta)).$$

Proof: The ellipticity condition in Definition 3.1 is (seemingly) stronger than the one employed by Schulze, cf. Section 3.3; it coincides with that given in [25], Chapter 3. The usual parametrix construction then yields the assertion. ∎

From this theorem and the established mapping properties of Green operators we immediately obtain the following results on the Fredholm property and the elliptic regularity in the L_p-Sobolev spaces.

3.3 Corollary. *An elliptic $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ induces for all $s \in \mathbb{R}$ Fredholm operators $A : \mathcal{H}^{s,\gamma}_p(\mathbb{B}) \to \mathcal{H}^{s-\mu,\gamma-\mu}_p(\mathbb{B})$.*

Proof: Green operators are compact in the corresponding Sobolev spaces by Remark 2.12.d) and Proposition 2.14. The result now immediately follows from Theorem 3.2. ∎

3.4 Corollary. *Let $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ be elliptic and $Au = f$ with $u \in \mathcal{H}^{-\infty,\gamma}_p(\mathbb{B})$.*

a) If $f \in \mathcal{H}^{s-\mu,\gamma-\mu}_p(\mathbb{B})$ then $u \in \mathcal{H}^{s,\gamma}_p(\mathbb{B})$.

b) If $f \in \mathcal{H}^{s-\mu,\gamma-\mu}_{p,Q'}(\mathbb{B})$ for some type Q', then $u \in \mathcal{H}^{s,\gamma}_{p,Q}(\mathbb{B})$ for a certain type Q depending on A and Q'.

Proof: Using a parametrix B, we get $u = Bf + Gu$ for some Green operator G. ∎

3.5 Corollary. *If $A \in C^\mu(\mathbb{B}; (\gamma, \gamma-\mu, \Theta))$ is elliptic and $A_p^s : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$ denotes the extension of A to the Sobolev spaces, then both* $\ker A_p^s \subset C^\infty_{\gamma,Q(A)}(\mathbb{B})$ *and* $\operatorname{ind} A_p^s$ *are independent of s and p.*

Proof: The statement concerning the kernel of A follows from the elliptic regularity. If A_p^{s*} the adjoint operator $\mathcal{H}_{p'}^{-s+\mu,-\gamma+\mu}(\mathbb{B}) \to \mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{B})$, then

$$\operatorname{ind} A_p^s = \dim\ker A_p^s - \dim\ker A_p^{s*}.$$

Furthermore, $A_p^{s*} = (A^*)_p^s$ for the formal adjoint A^* of A. Applying Theorem 2.10, we see that A^* is also elliptic, hence $\ker A_p^{s*}$ also does not depend on s and p. ∎

As we shall see in Section 3.4 below, both the Fredholm property and the index of A on $L_p(B)$ will depend on p.

3.1 Necessity of the ellipticity

3.6 Proposition. *Let $A \in C^\mu(\mathbb{B}; (\gamma, \gamma-\mu, \Theta))$, $\Theta =]-1, 0]$ be a Fredholm operator in $\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}), \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}))$ for some $s \in \mathbb{R}$. Then the homogeneous principal symbol of A does not vanish on $T^*(\operatorname{int}\mathbb{B}) \setminus 0$.*

Proof: Since A is a usual pseudodifferential operator in the interior of $\mathbb{B}$, and the cone Sobolev spaces are modelled over L_p away from the boundary of $\mathbb{B}$, the same argument as in the proof of Theorem 1.6 applies. ∎

In the following calculations we identify $L_p(\mathbb{R}_+ \times X; \frac{dt}{t}dx)$ and $L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$.

3.7 Definition. *For $\varepsilon > 0$, $\tau_0 \in \mathbb{R}$, and $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$ define*

$$(T_\varepsilon u)(t) = u(t/\varepsilon), \qquad (R_{\varepsilon,\tau_0} u)(t) = \varepsilon^{1/p} t^{-i\tau_0} u(t^\varepsilon).$$

Then T_ε, $R_{\varepsilon,\tau_0} : L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t}) \to L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$ are bijective isometries with inverses $T_\varepsilon^{-1} = T_{1/\varepsilon}$ and $R_{\varepsilon,\tau_0}^{-1} = R_{1/\varepsilon,\tau_0/\varepsilon}$.

3.8 Lemma. *If $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$, then $T_\varepsilon u \to 0$ weakly in $L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$ as $\varepsilon \to 0$.*

Proof: It suffices to show that $\langle T_\varepsilon u, v\rangle_{L_2(\mathbb{R}_+, L_2(X); \frac{dt}{t})} \xrightarrow{\varepsilon\to 0} 0$ for all $u \in C_0^\infty(\mathbb{R}_+, L_p(X))$ and $v \in C_0^\infty(\mathbb{R}_+, L_{p'}(X))$, since T_ε is uniformly bounded in $\varepsilon > 0$. But this is true, since if $\operatorname{supp} u \subset [a, b]$ then $\operatorname{supp} T_\varepsilon u \subset [\varepsilon a, \varepsilon b]$ and therefore $\langle T_\varepsilon u, v\rangle = 0$ for sufficiently small ε. ∎

3.9 Lemma. *Let $h \in ML^0(X, \mathbb{R}_+; \Gamma_0) \cap C(\overline{\mathbb{R}}_+, L^0(X; \Gamma_0))$ and $h_0(z) := h(0, z)$. Then*

$$T_\varepsilon^{-1} \operatorname{op}_M^{1/2}(h)\, T_\varepsilon u \longrightarrow \operatorname{op}_M^{1/2}(h_0) u$$

in $L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$ as ε tends to 0, for any $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$.

Proof: Since all T_ε are isometric, we can assume that $u \in C_0^\infty(\mathbb{R}_+, L_p(X))$. It is easy to verify that

$$T_\varepsilon^{-1} \operatorname{op}_M^{1/2}(h)\, T_\varepsilon = \operatorname{op}_M^{1/2}(h_\varepsilon) \quad \text{with} \quad h_\varepsilon(t,z) = h(\varepsilon t, z).$$

Noting that $\|t^{-i\tau} h_\varepsilon(t,i\tau)\mathcal{M}u(i\tau)\|_{L_p(X)} \leq C\,\|\mathcal{M}u(i\tau)\|_{L_p(X)} \in L_1(\mathbb{R}_\tau)$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}
[\operatorname{op}_M^{1/2}(h_\varepsilon)u](t) &= \int t^{-i\tau} h_\varepsilon(t,i\tau)\mathcal{M}u(i\tau)\, đ\tau \\
&\xrightarrow{\varepsilon\to 0} \int t^{-i\tau} h_0(i\tau)\mathcal{M}u(i\tau)\, đ\tau = [\operatorname{op}_M^{1/2}(h_0)u](t)
\end{aligned}$$

in $L_p(X)$ pointwise for each $t > 0$. Integration by parts yields

$$\langle \log t\rangle^2\, [\operatorname{op}_M^{1/2}(h_\varepsilon - h_0)u](t) = \sum_{k+l\leq 2} c_{kl} \int t^{-i\tau} \partial_\tau^l (h_\varepsilon(t,i\tau) - h_0(i\tau))\partial_\tau^k \mathcal{M}u(i\tau)\, d\tau$$

with universal constants c_{kl}. Therefore

$$\|[\operatorname{op}_M^{1/2}(h_\varepsilon - h_0)u](t)\|_{L_p(X)}^p \leq c\, \langle \log t\rangle^{-2p} \in L_1(\mathbb{R}_+; \tfrac{dt}{t}),$$

and a second application of the dominated convergence theorem gives the result. ∎

3.10 Lemma. *If $h \in L^0(X;\Gamma_0)$ and $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$ then*

$$R_{\varepsilon,\tau_0}^{-1} \operatorname{op}_M^{1/2}(h)\, R_{\varepsilon,\tau_0} u \longrightarrow h(i\tau_0)u$$

in $L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$ if ε tends to 0.

Proof: First observe that for $\varepsilon > 0$

$$R_{\varepsilon,\tau_0}^{-1} \operatorname{op}_M^{1/2}(h)\, R_{\varepsilon,\tau_0} = \operatorname{op}_M^{1/2}(h_\varepsilon) \quad \text{with} \quad h_\varepsilon(i\tau) = h(i\varepsilon\tau + i\tau_0).$$

Then proceed as in the proof of the previous Lemma 3.9. ∎

3.11 Lemma. *Let $h \in L^0(X;\Gamma_0)$ and suppose that for each $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$*

$$\|u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} \leq c\, \|\operatorname{op}_M^{1/2}(h)u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})}$$

for some constant c independent of u. Then for each $\varphi \in L_p(X)$ and all $\tau \in \mathbb{R}$ the estimate

$$\|\varphi\|_{L_p(X)} \leq c\, \|h(i\tau)\varphi\|_{L_p(X)}$$

is valid. In particular, $h(i\tau) \in \mathcal{L}(L_p(X))$ is injective and has closed range.

PROOF: Since R_{ε,τ_0} is an isometry, and due to Lemma 3.10

$$\begin{aligned}\|u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} &\le c\,\|R^{-1}_{\varepsilon,\tau_0}\operatorname{op}_M^{1/2}(h)\,R_{\varepsilon,\tau_0}u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})}\\ &\overset{\varepsilon\to 0}{\longrightarrow} c\,\|h(i\tau_0)u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})}\end{aligned}$$

for each $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$. Choosing $v \in C_0^\infty(\mathbb{R}_+)$ with $\|v\|_{L_p(\mathbb{R}_+;\frac{dt}{t})} = 1$ and inserting $u = v \otimes \varphi$ yields

$$\|\varphi\|_{L_p(X)} = \|u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} \le c\,\|v\otimes h(i\tau_0)\varphi\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} = c\,\|h(i\tau_0)\varphi\|_{L_p(X)}.$$

∎

3.12 Proposition. *Let $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$, $\Theta =]-1,0]$ be a Fredholm operator in $\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}),\mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}))$ for some $s\in\mathbb{R}$. Then the conormal symbol is invertible on $\Gamma_{\frac{n+1}{2}-\gamma}$ and the inverse belongs to $L_{cl}^{-\mu}(X;\Gamma_{\frac{n+1}{2}-\gamma})$.*

PROOF: *Step 1*: To show the pointwise invertibility, it suffices to show that, pointwise, the conormal symbol is injective and has closed range in $L_p(X)$. In fact, suppose we have verified this. Since with A also the adjoint $A^* \in C^\mu(\mathbb{B};(-\gamma+\mu,-\gamma,\Theta))$ is a Fredholm operator in $\mathcal{L}(\mathcal{H}_{p'}^{-s+\mu,-\gamma+\mu}(\mathbb{B}),\mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{B}))$, we then know that $\sigma_M^\mu(A^*)(z)$ is injective for each $z\in\Gamma_{\frac{n+1}{2}+\gamma-\mu}$, or equivalently by Theorem 2.10, $\sigma_M^\mu(A)(z)^*$ is injective for $z\in\Gamma_{\frac{n+1}{2}-\gamma}$. But if an operator is injective with closed range, and its adjoint is injective, then the operator is bijective.
Step 2: By the existence of order reducing cone operators (as they were constructed for instance in Theorem 2.4.49 of [30]), we can assume that $s=\mu=0$. Conjugation with an arbitrary smooth function on $\mathbb{B}$, which is positive and equals $t^{\frac{n+1}{2}-\gamma}$ near the boundary, allows us further to assume that $A\in C^0(\mathbb{B};(\frac{n+1}{2},\frac{n+1}{2},\Theta))$. Hence, let

$$A = \omega\operatorname{op}_M^{1/2}(h)\,\omega_0 + (1-\omega)\,P\,(1-\omega_1) + G$$

with $P\in L^0(2\mathbb{B})$, $G\in C_G(\mathbb{B};(\frac{n+1}{2},\frac{n+1}{2},\Theta))$, and $h(t,z)=a(t,z)+\tilde a(z)$ with $a\in C^\infty(\overline{\mathbb{R}}_+,M_O^0(X))$ and $\tilde a\in M_P^{-\infty}(X)$ for some asymptotic type P with $\pi_{\mathbb{C}}P\cap\Gamma_0=\emptyset$. In particular, $\sigma_M^0(A)(z)=h(0,z)=:h_0(z)$.
Step 2a: There exist operators $B,K\in\mathcal{L}(L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t}))$, K compact, and a cut-off function $\sigma\in C_0^\infty([0,1[)$ such that

$$(B\operatorname{op}_M^{1/2}(h)-1)\sigma = K \quad \text{on} \quad L_p(\mathbb{R}_+,L_p(X);\tfrac{dt}{t}). \tag{1}$$

In fact, there are by assumption $B_1,K_1\in\mathcal{L}(\mathcal{H}_p^{0,\frac{n+1}{2}}(\mathbb{B}))$, K_1 compact, such that $B_1A-1=K_1$. If we choose $\sigma,\sigma_1\in C^\infty([0,1[)$ with $\sigma\sigma_1=\sigma$ and $\sigma_1\omega_1=\sigma_1$ then

$$B_1\,\sigma_1\,A\,\sigma + B_1\,(1-\sigma_1)\,A\,\sigma - \sigma = K_1\,\sigma.$$

Now $(1-\sigma_1)\,A\,\sigma$ is a Green operator due to the disjoint supports of $(1-\sigma_1)$ and σ. Thus the second term on the left-hand side is compact and we obtain

$$\sigma_1\,B_1\,\sigma_1\,A\,\sigma - \sigma = \sigma_1\,K_2\,\sigma$$

with a compact K_2. Inserting $A\,\sigma = \omega\,\mathrm{op}_M^{1/2}(h)\,\sigma + G\,\sigma$ yields

$$\sigma_1\,B_1\,\sigma_1\,\mathrm{op}_M^{1/2}(h)\,\sigma - \sigma = \sigma_1\,(K_2 - B_1\,\sigma_1\,G)\,\sigma.$$

This together with Remark 2.12.b) implies (1).

Step 2b: Let $u \in C_0^\infty(\mathbb{R}_+, L_p(X))$. Since T_ε is an isometry and by (1),

$$\begin{aligned}\|u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} &= \|T_\varepsilon u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} = \|\sigma T_\varepsilon u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})}\\ &\leq \|B\|_{\mathcal{L}(L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t}))}\|T_{1/\varepsilon}\,\mathrm{op}_M^{1/2}(h)\,T_\varepsilon u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} + \\ &\quad + \|KT_\varepsilon u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})}.\end{aligned}$$

The second identity is true for sufficiently small ε, since the support of $T_\varepsilon u$ shrinks to zero with ε. If we pass to the limit $\varepsilon \to 0$ and use Lemmas 3.8, 3.9, we get the estimate

$$\|u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})} \leq c\,\|\mathrm{op}_M^{1/2}(h_0)u\|_{L_p(\mathbb{R}_+,L_p(X);\frac{dt}{t})}.$$

This estimate extends to $u \in L_p(\mathbb{R}_+, L_p(X); \frac{dt}{t})$. By Lemma 3.11, $h_0(z) \in \mathcal{L}(L_p(X))$ is injective and has closed range for each $z \in \Gamma_0$.

Step 3: By Steps 1, 2b, and Lemma 3.11, we now know that $h_0(i\tau)$ is invertible for each τ and $\|h_0(i\tau)^{-1}\|_{\mathcal{L}(L_p(X))}$ is uniformly bounded in τ. Hence $h_0^{-1} \in L^0(X;\Gamma_0)$ by Corollary 1.8. ■

3.13 Theorem. *A cone operator $A \in C^\mu_{cl}(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ is elliptic if and only if it is a Fredholm operator in $\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}), \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}))$ for some $s \in \mathbb{R}$.*

PROOF: Since $C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ is a subset of $C^\mu(\mathbb{B};(\gamma,\gamma-\mu,]-1,0]))$, the result follows from Propositions 3.6 and 3.12. ■

3.2 Spectral invariance

3.14 Theorem. *Let $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ be invertible as a bounded operator in $\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}), \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}))$ for some $s \in \mathbb{R}$. Then $A^{-1} \in C^{-\mu}(\mathbb{B};(\gamma-\mu,\gamma,\Theta))$.*

PROOF: Since A, in particular, is a Fredholm operator, Theorems 3.13 and 3.2 imply the existence of a parametrix $B \in C^{-\mu}(\mathbb{B};(\gamma-\mu,\gamma,\Theta))$ with

$$G_R := AB - 1 \;\in\; C_G(\mathbb{B};(\gamma-\mu,\gamma-\mu,\Theta)), \qquad G_L := BA - 1 \;\in\; C_G(\mathbb{B};(\gamma,\gamma,\Theta)).$$

Solving for A^{-1} yields

$$A^{-1} = B - BG_R + G_L A^{-1} G_R.$$

By the characterization of Green operators via mapping properties, cf. Theorem 2.14, the third term on the right-hand side belongs to $C_G(\mathbb{B};(\gamma-\mu,\gamma,\Theta))$, hence A^{-1} is an element of the cone algebra. ■

3.15 Corollary. *If $A \in C^\mu(\mathbb{B};(\gamma,\gamma-\mu,\Theta))$ is invertible as an operator $\mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$ for some $s \in \mathbb{R}$ and $1 < p < \infty$, then so it is for all $s \in \mathbb{R}$ and all $1 < p < \infty$.*

3.3 Other notions of ellipticity

Schulze uses a seemingly weaker definition of ellipticity, cf. for example [29], Definition 1.2.16. For $A \in C^\mu(\mathbb{B}, (\gamma, \gamma - \mu, \Theta))$ he asks that

1) $\sigma_\psi^\mu(A)$ is invertible on $T^*(\operatorname{int} \mathbb{B}) \setminus 0$ and, in coordinates $(t, x) \in \overline{\mathbb{R}}_+ \times \mathbb{R}^n$ near the boundary, that $t^\mu \sigma_\psi^\mu(A)(t, x, t^{-1}\tau, \xi)$ is invertible up to $t = 0$ for $(\tau, \xi) \neq 0$.

2) $\sigma_M^\mu(A)(z) : H_2^s(X) \to H_2^{s-\mu}(X)$ is invertible for all z with $\operatorname{Re} z = \frac{n+1}{2} - \gamma$ and any fixed $s \in \mathbb{R}$.

In view of Theorem 1.1, we may replace 2) by

2') $\sigma_M^\mu(A)(z) : H_p^s(X) \to H_p^{s-\mu}(X)$ is invertible for all z with $\operatorname{Re} z = \frac{n+1}{2} - \gamma$ and any choice of $s \in \mathbb{R}$ and $1 < p < \infty$.

We have the following result:

3.16 Proposition. *For $A \in C^\mu(\mathbb{B}, (\gamma, \gamma - \mu, \Theta))$ conditions 1) and 2) above are equivalent to conditions i) and ii) in Definition 3.1.*

Proof: We know from [29], (1.1.142), that, in the notation of Definition 2.8, for small $t > 0$

$$t^\mu \sigma_\psi^\mu(A)(t, x, t^{-1}\tau, \xi) = \sigma_\psi^\mu(h)(t, x, \beta + i\tau, \xi), \tag{1}$$

where the right-hand side denotes the parameter-dependent principal symbol of $h \in C^\infty(\overline{\mathbb{R}}_+, L_{cl}^\mu(X; \Gamma_\beta))$, which in fact does not depend on β. If A satisfies i) and ii) in 3.1, then 2) also holds. Moreover, as t tends to 0, the right-hand side of (1) approaches the parameter-dependent principal symbol of the conormal symbol of A, which itself is invertible by condition ii). Therefore $t^\mu \sigma_\psi^\mu(A)(t, x, t^{-1}\tau, \xi)$ stays invertible up to $t = 0$. If A satisfies 1) and 2), then i) also holds. Since the left-hand side of (1) is invertible up to $t = 0$, the conormal symbol $\sigma_M^\mu(A) \in L_{cl}^\mu(X; \Gamma_{\frac{n+1}{2}-\gamma})$ is parameter-dependent elliptic. To show ii), we may assume $\mu = 0$, which can be achieved by multiplication with a reduction of orders from $L_{cl}^{-\mu}(X; \Gamma_{\frac{n+1}{2}-\gamma})$. Due to ellipticity, there exists a parametrix $B \in L^0(X; \Gamma_{\frac{n+1}{2}-\gamma})$ such that $R = 1 - \sigma_M^0(A)B \in L^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma})$. In particular, $\|R(z)\|_{\mathcal{L}(L_2(X))} < 1$ for $|\operatorname{Im} z| \geq c$ sufficiently large. Therefore, $[\sigma_M^0(A)(z)]^{-1} = B(z) \sum_j R(z)^j$ is uniformly bounded in $\mathcal{L}(L_2(X))$ for $|\operatorname{Im} z| \geq c$. The continuity of inversion in $\mathcal{L}(L_2(X))$ in connection with 2) then implies the boundedness for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. Thus we can apply Corollary 1.8 to $\sigma_M^0(A)$. This yields ii). ∎

3.4 Fredholm property and invertibility in $L_p(B)$

We shall consider the Fredholm property respectively the index of a zero order cone operator A on different $L_p(B)$ spaces. Since

$$L_p(B) = \mathcal{H}_p^{0,\gamma_p}(\mathbb{B}), \qquad \gamma_p = (n+1)(\tfrac{1}{2} - \tfrac{1}{p}),$$

this amounts to a change of the weight γ_p. A priori A cannot be considered on different $L_p(B)$, since there are two obstructions: (i) the Green operators are only defined for a fixed choice of weight data, (ii) the conormal symbol may have a pole on the line $\Gamma_{\frac{n+1}{2}-\gamma_p} = \Gamma_{\frac{n+1}{p}}$. Concerning the Green operators we shall see below that we may change the weight; as long as we do not interfere with the poles in the asymptotic types, the Green operators stay defined and furnish the same map on, say, $C_0^\infty(\operatorname{int}\mathbb{B})$.

3.17 Lemma. *Let $G \in C_G(\mathbb{B};(\gamma,\gamma,\Theta))$, $\Theta =]-k,0]$, be a Green operator with kernel in $C^\infty_{\gamma,Q_0}(\mathbb{B}) \widehat{\otimes}_\pi C^\infty_{-\gamma,Q_1}(\mathbb{B})$ with $Q_j \in As((-1)^j\gamma,\Theta)$, cf. Definition 2.7. Then G extends to continuous operators $G \in \mathcal{L}(\mathcal{H}_p^{0,\varrho}(\mathbb{B}))$ for all $\gamma - \varepsilon_1 < \varrho < \gamma + \varepsilon_0$ and*

$$\varepsilon_j = \min\{k,\ \tfrac{n+1}{2} - (-1)^j\gamma - \operatorname{Re} q,\ \mid q \in \pi_{\mathbb{C}} Q_j\}.$$

Proof: The only point to note for this statement is that $C^\infty_{(-1)^j\gamma,Q_j}(\mathbb{B}) \hookrightarrow C^\infty_\varrho(\mathbb{B})$ for $\varrho > (-1)^j\gamma + \varepsilon_j$, and the duality of cone Sobolev spaces, cf. Remark 2.12.e). ∎

Let us now focus on operators without Green term. Let $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^0(X))$ be a holomorphic zero order Mellin symbol, $h_0 \in M_P^{-\infty}(X)$ for some asymptotic type P, and $A_\psi \in L^0_{cl}(\operatorname{int}\mathbb{B})$ a pseudodifferential operator. With these data we associate a family of cone operators $A_p : L_p(B) \to L_p(B)$ given by

$$A_p = \omega_0 \operatorname{op}_M^{\gamma_p - n/2}(h + h_0)\,\omega_1 + (1-\omega_2)\,A_\psi\,(1-\omega_3),$$

provided h_0 has no pole on the vertical line $\Gamma_{\frac{n+1}{p}}$. The residue theorem immediately shows:

3.18 Lemma. *A_{p_1} and A_{p_2} coincide on $C_0^\infty(\operatorname{int}\mathbb{B})$ if and only if h_0 has no singularity between the lines $\Gamma_{\frac{n+1}{p_1}}$ and $\Gamma_{\frac{n+1}{p_2}}$.*

Let us assume that A_p is elliptic for one p, in particular,

$$\sigma_M^0(A_p)(z) = h(0,z) + h_0(z), \qquad z \in \Gamma_{\frac{n+1}{2}-\gamma_p},$$

is invertible. As a matter of fact, cf. [30], Theorem 2.4.20, $h(0,z) + h_0(z)$ is then invertible on Γ_β for all $\beta \in \mathbb{R}\setminus D$, where D is a discrete subset of $\mathbb{R}$. More precisely,

$$\sigma_M^0(A_p)^{-1} = (h(0,\cdot) + h_0)^{-1} \in M_R^0(X)$$

for a certain asymptotic type R. Since $-\frac{n+1}{2} < \gamma_p < \frac{n+1}{2}$ this particularly implies:

3.19 Proposition. *If $A_p : L_p(B) \to L_p(B)$ is elliptic for some p, then it is elliptic for all but finitely many $1 < p < \infty$.*

Varying p, the index of A_p changes whenever p crosses a 'non Fredholm point' according to the following relative index formula:

3.20 Theorem. *If $A_{p_j} : L_{p_j}(B) \to L_{p_j}(B)$ are elliptic for $1 < p_1 < p_2 < \infty$, then*

$$\operatorname{ind} A_{p_1} - \operatorname{ind} A_{p_2} = \sum_{\frac{n+1}{p_2} < \operatorname{Re} z < \frac{n+1}{p_1}} M(\sigma_M^0(A), z).$$

Here, for an operator-valued function h that is holomorphic in a punctured neighborhood of z, $M(h,z)$ is the multiplicity of h at z in the sense of Gohberg, Sigal [13].

For certain classes of operators, an index formula thus can be deduced from the L_2-results by Brüning, Seeley [5], Fedosov, Schulze, Tarkhanov [9], Lesch [19], and Schulze, Shatalov, Sternin [32].
Let us point out that even if $\sigma_M^0(A)$ has no singularities between the lines $\Gamma_{\frac{n+1}{p_1}}$ and $\Gamma_{\frac{n+1}{p_2}}$, hence A_{p_1} and A_{p_2} coincide on $C_0^\infty(\operatorname{int}\mathbb{B})$, the indices $\operatorname{ind} A_{p_1}$ and $\operatorname{ind} A_{p_2}$ will in general be different due to the singularities of $\sigma_M^0(A)^{-1}$.
In fact, the above formula is immediate from Corollary 3.5 and the well-known formula for the change of the index under weight shift, cf. [21], Theorem 6.5, and [28], Section 2.2.3, Theorem 14. We shall illustrate these effects by treating an example. To make things easier, we consider the one-dimensional case ($n = 0$). Here we even have a simple index formula, cf. [8], [20].

3.21 Example. In the following, we view the unit circle $B = S^1$ as a manifold with one conical singularity in 1. We blow up, using the standard argument function, and obtain $\mathbb{B} \cong [0, 2\pi]$. The space $L_p(B)$ then corresponds to $L_p([0, 2\pi], dt)$. The function

$$h_0(z) = (\tfrac{1}{2} - z)^{-1} e^{(z-1)^2}$$

is meromorphic in $\mathbb{C}$ with a simple pole in $z = 1/2$. In fact, it is not difficult to check that $h_0 \in M_P^{-\infty}$, where P consists of the single element $(1/2, 0, \mathbb{C})$, and that $1 + h_0$ has no zero in the strip $\{0 \le \operatorname{Re} z \le 1\}$. Therefore,

$$(1 + h_0)^{-1} = 1 - h_0(1 + h_0)^{-1}$$

and $h_0(1 + h_0)^{-1} \in M_Q^{-\infty}$ is holomorphic in $\{0 \le \operatorname{Re} z \le 1\}$. We define on $[0, 2\pi]$ the operators

$$\begin{aligned} A_p &= 1 - \omega_0 \operatorname{op}_M^{\frac{1}{2}-\frac{1}{p}}(h_0(1+h_0)^{-1})\, \omega_1, \qquad 1 < p < \infty, \\ B_p &= 1 + \omega_0 \operatorname{op}_M^{\frac{1}{2}-\frac{1}{p}}(h_0)\, \omega_1, \qquad 1 < p < \infty,\ p \ne 2. \end{aligned}$$

Here, 1 is the identity map and $\omega_0, \omega_1 \in C_0^\infty([0, 2\pi[)$ are cut-off functions. In view of the holomorphy of $h_0(1+h_0)^{-1}$, the operators A_p coincide on $C_0(]0, 2\pi[)$ for $1 < p < \infty$. Moreover, A_p extends to a bounded operator on $L_p([0, 2\pi])$, and B_p has a continuous extension for $p \ne 2$. We have

$$\sigma_\psi^0(A_p) = \sigma_\psi^0(B_p) = 1, \qquad \sigma_M^0(A_p) = 1 - h_0(1+h_0)^{-1} = (1+h_0)^{-1} = \sigma_M^0(B_p)^{-1},$$

and therefore conclude that A_2 is not elliptic but A_p, B_p are elliptic, hence Fredholm, for all $p \neq 2$. Moreover, A_p is a parametrix to B_p since

$$A_pB_p = 1 + \omega_0 \operatorname{op}_M^{\frac{1}{2}-\frac{1}{p}}(h_0(1+h_0)^{-1})\,(1-\omega_0\omega_1)\operatorname{op}_M^{\frac{1}{2}-\frac{1}{p}}(h_0)\,\omega_1,$$

and the second term is known to be a Green operator, cf. [30], Lemma 2.3.73. In particular,

$$\operatorname{ind} A_p = -\operatorname{ind} B_p.$$

A result of Eskin, [8] Theorem 15.12, asserts that the index of an operator of the form $1+\omega_0 \operatorname{op}_M^0(f)\,\omega_1$, $f \in M_Q^{-\infty}$, acting on $L_2(\mathbb{R}_+)$ is given by the winding number around zero of $1+f$ along the line $\Gamma_{1/2}$, traversed in the upward direction. Since the index of $1+\omega_0 \operatorname{op}_M^{\gamma}(f)\,\omega_1$ on $t^\gamma L_2(\mathbb{R}_+)$ coincides with the index of $t^{-\gamma}(1+\omega_0 \operatorname{op}_M^{\gamma}(f)\,\omega_1)t^\gamma = 1+\omega_0 \operatorname{op}_M^0(f(\cdot-\gamma))\,\omega_1$ on $L_2(\mathbb{R}_+)$, it is given by the winding number of $1+f$ along $\Gamma_{1/2-\gamma}$. From this it is easily seen that the index of B_p on $L_p([0,2\pi])$ is given by the winding number of $(1+h_0)$ along $\Gamma_{1/p}$. It is straightforward to check that the winding number of $1+h_0$ along Γ_0 equals 0 while that along Γ_1 equals 1. By continuity, the winding number on Γ_β is 0 for $0 \le \beta < 1/2$ and -1 for $1/2 < \beta \le 1$. Summing up,

$$\operatorname{ind} A_p = \begin{cases} 1, & 1<p<2 \\ 0, & 2<p<\infty \end{cases}.$$

In particular, $\operatorname{ind} A_{p_1} - \operatorname{ind} A_{p_2} = 1$ for all $1 < p_1 < 2 < p_2 < \infty$, i.e. the index changes with varying p.

3.22 Theorem. *The set of all p such that the operator A_p is invertible, is open.*

Proof: Suppose A_p is invertible. According to Theorem 3.14 we can write

$$A_p^{-1} = \tilde\omega_0 \operatorname{op}_M^{\gamma_p}(g+g_0)\tilde\omega_1 + (1-\tilde\omega_2)B_\psi(1-\tilde\omega_3) + \tilde G$$

with $g \in C^\infty(\overline{\mathbb{R}}_+, M_O^0(X))$, $g_0 \in M_R^{-\infty}(X)$ for a suitable asymptotic type R, $B_\psi \in L_{cl}^0(\operatorname{int}\mathbb{B})$, and $\tilde G \in C_G(\mathbb{B};(\gamma,\gamma,\Theta))$, $\Theta =]-1,0]$, with asymptotic types Q, Q', in the sense of Proposition 2.14. Choose $0<\varepsilon<1$ such that the strip

$$\left\{z\in\mathbb{C} \mid \tfrac{n+1}{2} - \gamma_p - \varepsilon < \operatorname{Re} z < \tfrac{n+1}{2} - \gamma_p + \varepsilon\right\}$$

does not contain a singularity of g_0 or h_0, respectively, and such that

$$\operatorname{Re} q' < \tfrac{n+1}{2} + \gamma_p - \varepsilon \quad \forall\, q' \in \pi_{\mathbb{C}}Q', \qquad \operatorname{Re} q < \tfrac{n+1}{2} - \gamma_p - \varepsilon \quad \forall q \in \pi_{\mathbb{C}}Q.$$

Then $A_p = A_q$ on $C_0^\infty(\operatorname{int}\mathbb{B})$. Moreover, $A_p^{-1}|_{C_0^\infty(\operatorname{int}\mathbb{B})}$ extends to a bounded operator on $L_q(B) = \mathcal{H}_q^{0,\gamma_q}(\mathbb{B})$ which inverts A_q. ■

3.23 Remark. *We see from Theorem 3.20 that, in general, the spectrum of A_p will depend on p. If the Fredholm index jumps in a point p_0, then, for small ε, at most one of the operators $A_{p_0-\varepsilon}$ or $A_{p_0+\varepsilon}$ can be invertible.*

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