The Green formula and layer potentials

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Abstract

The Green formula is proved for boundary value problems (BVPs), when "basic" operator is arbitrary partial differential operator with variable matrix coefficients and "boundary" operators are quasi–normal with vector–coefficients. If the system possesses the fundamental solution, representation formula for a solution is derived and boundedness properties of participating layer potentials from function spaces on the boundary (Besov, Zygmund spaces) into appropriate weighted function spaces on the inner and the outer domains are established. Some related problems are discussed in conclusion: traces of functions from weighted spaces, traces of potential–type functions, Plemelji formulae, Calderón projections, restricted smoothness of the underlying surface and coefficients. The results have essential applications in investigations of BVPs by the potential method, in apriori estimates and in asymptotics of solutions.

Contents

*Supported by grants: a) INTAS 96 – 0876; b)Academy of Sciences of Georgia 1.2.97

[1)]Keywords: Green formula, Layer potentials, Pseudodifferential equations, Calderón projections, Plemelji formulae, Apriory estimates

AMS classification: 47A68, 35J25, 35J55

Introduction

Let $\Omega^+ \subset \mathbb{R}^n$ be a domain with the smooth boundary $\partial\Omega^+ = \mathcal{S}$, $\Omega^- := \mathbb{R}^n \backslash \overline{\Omega^+}$ and $\vec{\nu}(t) = (\nu_1(t), \ldots, \nu_n(t))$, $t \in \mathcal{S}$ be the outer unit normal vector (see Fig.1).

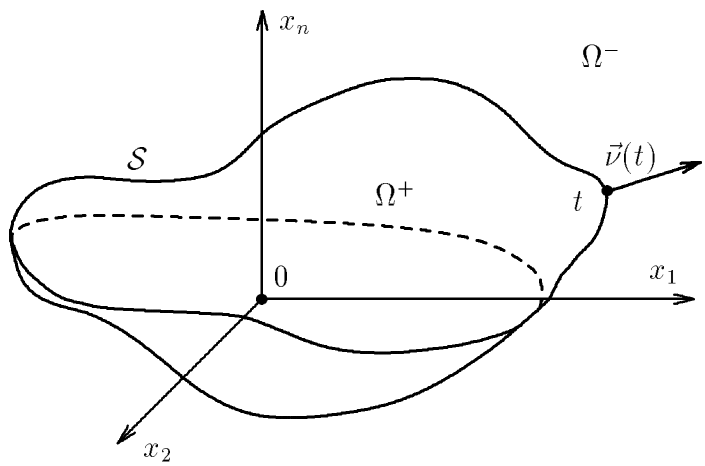

Fig. 1

Let $\gamma_{\mathcal{S}}^{\pm}$ denote the trace operators on the boundary:

$$\gamma_{\mathcal{S}}^{\pm} u(t) := \lim_{\substack{x\to t \\ x\in\Omega^{\pm},\, t\in\mathcal{S}}} u(x) .$$

We consider a boundary value problem

$$\begin{cases} \mathbf{A}(x, D_x)u(x) = f(x), & x \in \Omega^{\pm}, \\ \gamma_{\mathcal{S}}^{\pm}\mathbf{b}_j u(t) = g_j(t), & j = 0, \dots, \omega - 1, \ \ t \in \mathcal{S}\,, \quad \omega \leq m\,, \end{cases} \tag{0.1}$$

where $\mathbf{A}(x, D_x)$ is a partial differential operator with $N \times N$ matrix coefficients

$$\mathbf{A}(x, D_x) := \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha\,, \qquad a_\alpha \in C^\infty(\overline{\Omega}^{\pm}, \mathbb{C}^{N\times N}) \tag{0.2}$$

(we call it “basic”) and quasi–normal system of “boundary” operators

$$\mathbf{b}_j(t, D_t) = \sum_{|\alpha|\leq m_j} b_{j\alpha}(t)\partial_t^\alpha\,, \quad b_{j\alpha} \in C^\infty(\mathcal{S}, \mathbb{C}^N)\,,$$
$$m_j \leq m - 1\,, \quad j = 0, \dots, \omega - 1$$

with vector–row coefficients of length N. The Green formula

$$\int_{\Omega^+} ((\mathbf{A}u)^\top \overline{v} - u^\top \overline{\mathbf{A}^* v})dy = \pm \sum_{j=0}^{mN-1} \oint_{\mathcal{S}} \mathbf{b}_j u \overline{\mathbf{c}_j v} d_\tau \mathcal{S} \tag{0.3}$$

is proved (see Theorem 1.6), where $\mathbf{A}^*$ stands for the formally adjoint operator to (0.1), $\{\mathbf{b}_j\}_{j=0}^{mN-1}$ is a Dirichlet system, arbitrary extention of “boundary” operator system $\{\mathbf{b}_j\}_{j=0}^{\omega-1}$; another system $\{\mathbf{c}_j\}_{j=0}^{mN-1}$ of “boundary” differential operators is then defined uniquely and is a Dirichlet system if and only if the “basic” operator $\mathbf{A}(x, D_x)$ is normal. If the “basic” operator is normal, it is possible to prescribe parts of both systems $\{\mathbf{b}_j\}_{j=0}^{kN-1}$ and $\{\mathbf{c}_j\}_{j=kN}^{mN-1}$ if they are Dirichlet systems and find another parts (which are unique) such that the Green formula (0.3) holds.

For a formally self–adjoint operators of even order $m = 2\ell$ is proved a simplified Green formula (see Theorem (1.7)).

The Green formula (0.3) is proved in [Ta1, Ta2] for a “rectangular” system of “basic” operator with $\ell \times k$ matrix coefficients when the principal symbol is injective (see [LM1, Ch.2, Theorem 6.1] for scalar $N = 1$ elliptic operators and [Ro1, RS2] for elliptic Douglis–Nirenberg systems; see also the survey [Ag1, § 4]). All the mentioned investigations in [LM1, Ro1, RS2, Ta1, Ta2] are based on local diffeomorphisms which replaces the domain $\Omega^{\pm}$ by the half–space $\mathbb{R}^{+}$. The present approach is direct and applies partial integration formulae (1.22)–(1.25), which follow from the Gauss formula on divergence and the Stokes formula on differential forms. Other important ingredients are the special Green formula with the normal derivatives $\mathbf{B}_j = \partial^j_{\vec{\nu}}$ as "boundary" operators (see Theorem 1.10; similar formulae see in [CP1, CW1, Di1, Se1]) and Lemma 4.7, which is a matrix analogue of [LM1, Ch. 2, Lemma 2.1] (see also [RS2, (11)]).

Moreover, the approach is constructive and allows us to write the "boundary" differential operators $\{\mathbf{c}_j(x, D_x)\}_{j=0}^{mN-1}$ in explicit form (see Theorem 1.11) provided the "boundary" operators $\{\mathbf{b}_j(x, D_x)\}_{j=0}^{mN-1}$ are fixed. The algorithm is pure algebraic and invokes only coefficients of the differential operators $\mathbf{A}(x, D_x)$ and $\mathbf{B}_j(x, D_x)$.

Let us note that only for symbols of operators $\mathbf{c}_j(x, D_x)$, $j = 0, \ldots, mN-1$, there existed explicit formulae (see [Ta1, § 8.33]).

Let us assume $\mathbf{A}(x, D_x)$ has the double–sided inverse on the entire space $\mathbb{R}^n$

$$\mathbf{A}(x, D_x)\mathbf{F}_{\mathbf{A}}(x, D_x) = \mathbf{I}, \quad \mathbf{F}_{\mathbf{A}}(x, D_x)\mathbf{A}(x, D_x) = I,$$

i.e. the operator has the fundamental solution; then $\mathbf{A}(x, D_x)$ is elliptic and (for $n > 2$) has even order $m = \mathbf{ord}\,\mathbf{A} = 2\ell$. We "insert" the distributional Schwartz kernel $v_{\varepsilon,x}(y) = \chi_\varepsilon(x-y)\mathcal{K}_{\mathbf{A}}(x, y)$ of the fundamental solution $\mathbf{F}_{\mathbf{A}}(x, D_x)$ with "cutted–off" singularities on the diagonal set $x = y$ into the Green formula (0.3). Sending $\varepsilon \to 0$ we get a representation of the solution $u(x)$ to the elliptic equation $\mathbf{A}(x, D_x)u(x) = f(x)$ in the domain $\Omega^{\pm}$

$$\chi_{\Omega^{\pm}}(x)u(x) = \mathbf{N}_{\Omega^{\pm}} f(x) \pm \sum_{j=0}^{2\ell-1} \mathbf{V}_j \gamma^{\pm}_{\mathcal{S}} \mathbf{B}_j u(x), \tag{0.4}$$

where $\chi_{\Omega^{\pm}}$ stands for the characteristic function of $\Omega^{\pm} \subset \mathbb{R}^n$. The operators

$$\begin{aligned} \mathbf{N}_{\Omega^{\pm}}\varphi(x) &:= \int\limits_{\Omega^{\pm}} \left[\overline{\mathcal{K}_{\mathbf{A}^*}(y, x)}\right]^{\top} \varphi(y)dy = \int\limits_{\Omega^{\pm}} \mathcal{K}_{\mathbf{A}}(x, y)\varphi(y)dy, \\ \mathbf{V}_j\psi(x) &:= \oint\limits_{\mathcal{S}} \left[\overline{\mathbf{C}_j(\tau, D_\tau)}\mathcal{K}^{\top}_{\mathbf{A}}(x, \tau)\right]^{\top} \varphi(\tau)\, d_\tau\mathcal{S}, \quad j = 0, \ldots, 2\ell \end{aligned} \tag{0.5}$$

are the volume (Newton) and the layer potentials, respectively (see (3.3)–(3.7)).

The layer potentials $\mathbf{V}_0, \ldots, \mathbf{V}_{2\ell-1}$ extend functions defined on the boundary into the domain and their continuity properties have essential applications in many investigations. Namely, in the potential method (see [CW1, DNS1, Gu1, KGBB1, Lo1, MMT1, Se1] etc.), in a priori estimates of solutions to BVPs (see [CW1, DNS1, DN1, Gr3, LM1] etc. and Corollary 3.4), in full asymptotic expansion of solutions to crack–type and mixed BVPs for elliptic partial differential equations (see [CD2]).

As a particular case of Theorem 3.2 we can formulate the following (see §1 for the definition of the BESSEL potential $\mathbb{H}^s_{p,loc}(\overline{\Omega^\pm})$, BESOV $\mathbb{B}^s_{p,p}(\mathcal{S})$ and other spaces).

Theorem 0.1 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $\mu_j = \mathbf{ord}\,\mathbf{C}_j < \mathbf{ord}\,\mathbf{A} = 2\ell$. The layer potentials*

$$\mathbf{V}_j \;:\; \mathbb{B}^s_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-j+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{0.6}$$

are continuous for $j = 0, \ldots, 2\ell - 1$.

Theorem 0.1 is proved with the help of Lemma 4.8, which has independent interest. It allows representation of layer potentials in (0.5) in the form of volume potentials, i.e. pseudodifferential operators (PsDOs). Here is a slightly particular case of this lemma.

Lemma 0.2 *Let $s > 0$, $s \notin \mathbb{N}$, $k = 0, 1, \ldots$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $\mathbf{A}(x, D_x)$ in (0.2) be normal $\mathbf{det}\,\mathcal{A}(\mathrm{t}, \xi) \neq 0$ for all $t \in \mathcal{S}$, $|\xi| = 1$, $\mathbf{ord}\,\mathbf{A} = \mathrm{m}$.*

For a DIRICHLET system $\{\mathbf{B_j}\}_{j=0}^{m-1}$ of "boundary" differential operators of the order $m-1$ with C^∞–smooth $N \times N$ matrix coefficients there exists a continuous linear operator

$$\mathcal{P} : \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-1-j}_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+m-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{0.7}$$

such that

$$\gamma^+_{\mathcal{S}} \mathbf{B}_j \mathcal{P}\Phi = \varphi_j, \quad \mathbf{A}\mathcal{P}\Phi \in \widetilde{\mathbb{H}}^{s-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{0.8}$$

for $j = 0, \ldots, m-1$ and arbitrary $\Phi = (\varphi_0, \ldots, \varphi_{m-1}) \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-1-j}_{p,p}(\mathcal{S})$.

A similar assertion is proved in [LM1, Ch.2, Theorem 6.1] for the scalar case (see also [LM1, Ch.2, Lemmata 2.1 and 2.2] and [Hr2, Theorem 1.2.6]. The proof exposed below is carried out for the matrix–operators, is more transparent and the spaces are more general (we consider weighted spaces $\mathbb{H}^{s,m}_{p,loc}(\overline{\Omega^\pm})$ as well).

Theorem 0.1 can be derived from the results on PsDOs with the transmission property (see [Bo1, Gr1, Gr2, Jo2, RS1] and the survey [BS1, Theorems 2.17, 2.21]). The approach suggested here is different, works for weighted spaces and seems to be simpler. It has consequences which are perhaps difficult to obtain within the approach suggested earlier (see e.g. §§ 6.3–6.5 below).

In § 1.1 we discuss the Green formula (0.3) and related topics. Namely we recall definitions of normal operators, Dirichlet systems of operators and formal adjoint BVPs (see [LM1]); we define systems of quasi–normal operators. BVPs with quasi–normal "boundary" operators cover mixed–type problems of elasticity, diffraction of electromagnetic waves and many other problems of the mathematical physics. The principal Theorems 1.6 and 1.7 on the Green formula are formulated. The proofs are deferred to §§ 5.1, 5.2. In § 1.2 we prepare tools for the investigation: the Günter and the Stokes tangent derivatives, partial integration formulae on the domain and on the surface, based on the Gauss formula on divergence and the Stokes formula on differential forms (see Lemma 1.8). The special Green formula is proved for arbitrary "basic" operator when "boundary" operators are given by the normal derivatives $\mathbf{B}_j = \partial^j_{\overline{\nu}}$. In § 1.3 we write the explicit formulae for the "boundary" operators $\{\mathbf{c}_j(x, D_x)\}_{j=0}^{mN-1}$ in the Green formula (0.3) when the extended Dirichlet system $\{\mathbf{b}_j(x, D_x)\}_{j=0}^{mN-1}$ is fixed (see Theorem 1.11).

In § 2 we expose definitions of weighted Bessel potential $\mathbb{H}^{r,k}_{p,loc}(\overline{\Omega^\pm})$, Besov $\mathbb{B}^{r,k}_{p,q,loc}(\overline{\Omega^\pm})$ and Zygmund $\mathbb{Z}^{r,k}(\overline{\Omega^\pm})$ spaces.

In § 3, based on the Green formula, representation of solutions to elliptic differential equation is derived, provided the "basic" operator has a fundamental solution. The result on continuity of layer potentials, participating in the representation of solutions, and of more general potential–type operators is formulated (cf. Theorem 0.1). Namely, continuity of layer potentials is proved from the Besov spaces on the boundary $\mathbb{B}^s_{p,p}(\mathcal{S})$, $\mathbb{B}^s_{p,q}(\mathcal{S})$ (including the Zygmund spaces $\mathbb{Z}^s(\mathcal{S}) = \mathbb{B}^s_{\infty,\infty}(\mathcal{S})$) into the appropriate weighted Bessel potential $\mathbb{H}^{r,k}_{p,loc}(\overline{\Omega^\pm})$ and Besov $\mathbb{B}^{r,k}_{p,q,loc}(\overline{\Omega^\pm})$ spaces in the outer Ω^- and inner Ω^+ domains (see Theorem 3.2). In conclusion of the section a priori estimates for solutions to BVP (0.1) is written when the "basic" operator is hypoelliptic (see Corollary 3.4 and Remark 3.5).

In § 4 a central auxiliary result–Lemma 4.8 is proved. This result plays a crucial role in the proof of Theorem 3.2 in § 5.3.

In § 5 we expose the proofs of Theorems 1.6, 1.7 and 3.2.

In § 6.1 is proved that generalized layer potentials, representing integral operators with supersingular kernels on the boundary surface, have correctly defined traces on the boundary of the domain, interpreted as classical PsDOs. The assertion is obviously false if a simple layer potential is missing and we deal with a pure differential operator of non–tangential type; moreover, one can not apply such operators to functions defined only on the surface.

In § 6.2 we extend the trace theorem (see also Theorem 4.6) and the basic Lemma 4.8 to functions in weighted spaces.

In § 6.3 we prove the theorem on the CALDERÓN projections, related to the GREEN formula (0.3) and the corresponding layer potentials (0.5). Namely, the operators $\mathbf{P}^{\pm}_{\mathbf{A,j}} := \pm\gamma^{\pm}_{\mathcal{S}}\mathbf{B}_j\mathbf{V}_j$ for $j = 0, \ldots, 2\ell - 1$ are proved to be projections $(\mathbf{P}^{\pm}_{\mathbf{A,j}})^2 = \mathbf{P}^{\pm}_{\mathbf{A,j}}$, $\mathbf{P}^{-}_{\mathbf{A,j}} + \mathbf{P}^{+}_{\mathbf{A,j}} = I$ in the spaces $\mathbb{H}^s_p(\mathcal{S})$ and $\mathbb{B}^s_{p,q}(\mathcal{S})$.

In § 6.4 we get the PLEMELJI formulae (the jump relations) for the layer potentials.

In § 6.5 we substantially weaken smoothness restrictions on the boundary $\mathcal{S} = \partial\Omega^{\pm}$ of the domain and on coefficients of differential operators. Such results gain special interest recently due to the progress in the theory of BVPs for differential equations in a domains with a LIPSCHITZ boundary. Such investigations are based on results for layer potentials on LIPSCHITZ surfaces (see [Ke1,Ke2,MMP1,MMT1,MT1] and the literature cited therein).

Most of the above–mentioned results on the GREEN formula, layer potentials, the PLEMELJI formulae under minimal restrictions on the boundary manifold and coefficients are known for the second order equations (see [MMT1, MT1] for recent results). Less is done for higher order equations (see [CP1, CW1, Di1, Gr1, LM1, Ro1, Se1]). CALDERÓN projections were investigated in [Se1] (see also [CP1, CW1, Gr1, Di1]).

Acknowledgement: The author thanks E.Shargorodsky who suggested the first version of Theorem 2.2 and Lemma 4.8, as well as the results of § 6.4. He also has made many valuable remarks while reading the manuscript.

1 The Green formula and boundary value problems

1.1. The GREEN formula for quasi–normal BVP. Let Ω^+, $\partial\Omega = \mathcal{S}$ and $\vec{\nu}(t)$ br the same as in Introduction [1)] and consider a partial

[1)]Optimal smoothness constraints on $\partial\Omega = \mathcal{S}$ will be discussed later on in § 6.5.

differential operator with $N \times N$ matrix coefficients

$$\mathbf{A}(x, D_x) := \sum_{|\alpha| \le m} a_\alpha(x) \partial_x^\alpha, \quad a_\alpha \in C^\infty(\overline{\Omega}^\pm, \mathbb{C}^{N \times N}). \tag{1.1}$$

The operator

$$\mathbf{A}^*(x, D_x) = \sum_{|\alpha| \le m} (-1)^\alpha \partial_x^\alpha \overline{[a_\alpha(x)]}^\top I\,, \tag{1.2}$$

where $\mathcal{B}^\top$ denotes the transposed matrix to $\mathcal{B}$, is the formal adjoint to $\mathbf{A}(x, D_x)$ with respect to the sesquilinear form

$$(u, v) := \int_{\Omega^\pm} [u(y)]^\top \overline{v(y)} dy.$$

Definition 1.1 (see [LM1, Ch.2, § 1.4]). *The operator* $\mathbf{A}(x, D_x)$ *in* (1.1) *is called* **normal** *on* $\mathcal{S}$ *if*

$$\inf |\,\mathbf{det}\, \mathcal{A}_0(t, \vec{\nu}(t))| \neq 0, \quad t \in \mathcal{S}\,, \quad |\xi| = 1\,, \tag{1.3}$$

where $\mathcal{A}_0(x, \xi)$ *denotes the* **homogeneous principal** *symbol of* $\mathbf{A}$

$$\mathcal{A}_0(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x)(-i\xi)^\alpha, \quad x \in \overline{\Omega^\pm}\,, \quad \xi \in \mathbb{R}^n. \tag{1.4}$$

Condition (1.3) means that the surface $\mathcal{S}$ is not characteristic for the operator $\mathbf{A}(x, D_x)$.

Normal operators contain, as a subclass, elliptic operators on the surface

$$\inf |\,\mathbf{det}\, \mathcal{A}_0(\mathrm{t}, \xi)| \neq 0 \quad \text{for all} \quad \mathrm{t} \in \mathcal{S}, \quad \xi \in \mathrm{S}^{\mathrm{n}-1}, \tag{1.5}$$

where $S^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ is the unit sphere in $\mathbb{R}^n$; these two definitions coincide for operators with constant coefficients since the unit normal vector $\vec{\nu}(t)$ runs the entire unit sphere if t ranges through the closed smooth surface $\mathcal{S}$. In fact, the surface $\mathcal{S} = \partial\Omega^+$ is the boundary of the domain Ω^+ and thus any connected part of this boundary can be continuously deformed to the unit sphere. If we suppose that the unit normal, while ranging through the surface $\mathcal{S}$, leaves some (obviously open) domain on the unit sphere free, we run into the contradiction.

Let us consider a boundary value problem (BVP in short) with mixed conditions

$$\begin{cases} \mathbf{A}(x, D_x) u(x) = f(x), & x \in \Omega^\pm, \\ \gamma_{\mathcal{S}}^\pm \mathbf{b}_j u(t) = g_j(t), & j = 0, \ldots, \omega - 1, \;\; t \in \mathcal{S}\,, \quad \omega \le m\,, \end{cases} \tag{1.6}$$

where $\mathbf{A}(x, D_x)$ is a "basic" operator, defined in (1.1) and

$$\mathbf{b}_j(t, D_t) = \sum_{|\alpha| \le m_j} b_{j\alpha}(t) \partial_t^\alpha, \qquad b_{j\alpha} \in C^\infty(\mathcal{S}, \mathbb{C}^N) \tag{1.7}$$

are "boundary" differential operators with vector–row coefficients of length N and $\mathbf{ord}\, \mathbf{b}_\mathrm{j} = \mathrm{m_j} \le \mathrm{m} - 1$.

Together with (1.6) we will consider the boundary value problem with the formal adjoint "basic" operator

$$\begin{cases} \mathbf{A}^*(x, D_x) v(x) = d(x), & x \in \Omega^\pm, \\ \gamma_\mathcal{S}^\pm \mathbf{c}_{mN-j-1} v(t) = h_j(t), \quad j = 0, \dots, \omega^* - 1, & t \in \mathcal{S} \end{cases} \tag{1.8}$$

(see (1.2)); here $\omega^* \le mN$, $\mathbf{ord}\, \mathbf{c}_\mathrm{j} = \mu_\mathrm{j} \le \mathrm{m} - 1$. and $\mathbf{c}_j(t, D_t)$ are some "boundary" differential operators

$$\mathbf{c}_j(t, D_t) = \sum_{|\alpha| \le \mu_j} c_{j,\alpha}(t) \partial_t^\alpha, \qquad c_{j,\alpha} \in C^\infty(\mathcal{S}, \mathbb{C}^N) \tag{1.9}$$

with vector–row coefficients of length N.

A particular case of BVP (1.6) is the following BVP

$$\begin{cases} \mathbf{A}(x, D_x) u(x) = f(x), & x \in \Omega^\pm, \\ \gamma_\mathcal{S}^\pm \mathbf{B}_j u(t) = G_j(t), \qquad j = 0, \dots, \ell - 1, & t \in \mathcal{S}, \end{cases} \tag{1.10}$$

where

$$\mathbf{B}_j(t, D_t) = \sum_{|\alpha| \le m_j} b_{j\alpha}(t) \partial_t^\alpha, \qquad b_{j\alpha} \in C^\infty(\mathcal{S}, \mathbb{C}^{N \times N})$$

are "boundary" operators with $N \times N$ matrix coefficients and $\mathbf{ord}\, \mathbf{B}_j = m_j \le m - 1$. The formal adjoint BVP to (1.10) acquires the form

$$\begin{cases} \mathbf{A}^*(x, D_x) v(x) = d(x), & x \in \Omega^\pm, \\ \gamma_\mathcal{S}^\pm \mathbf{C}_{m-j-1} v(t) = H_j(t), \quad j = 0, \dots, \ell^* - 1, & t \in \mathcal{S} \end{cases} \tag{1.11}$$

(see (1.2)), where $\ell^* \le m$ and $\mathbf{C}_j(t, D_t)$ are some "boundary" differential operators

$$\mathbf{C}_j(t, D_t) = \sum_{|\alpha| \le \mu_j} c_{j,\alpha}(t) \partial_t^\alpha, \qquad c_{j,\alpha} \in C^\infty(\mathcal{S}, \mathbb{C}^{N \times N})$$

with $\mathbf{ord}\, \mathbf{C}_\mathrm{j} = \mu_\mathrm{j} \le \mathrm{m} - 1$.

BVPs (1.10) encounter e.g. in elasticity, when the displacement ($I^{\pm}$ BVP) or the stress ($II^{\pm}$ BVP) fields are prescribed. BVPs (1.6) cover as well mixed problems of elasticity when the normal component of the displacement and both tangent components of the stress fields ($III^{\pm}$ BVP) or the normal component of the stress and both tangent components of the displacement fields ($IV^{\pm}$ BVP) are prescribed (see [KGBB1, §§ 1.8–1.10]).

Definition 1.2 *A system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{k-1}$ of differential operators with matrix $N\times N$ coefficients is called a* DIRICHLET *system of order k if all participating operators are normal on $\mathcal{S}$ (see Definition* 1.1*) and, after renumbering, $\mathbf{ord}\,\mathbf{B}_{\mathrm{j}}=\mathrm{j}$, $j=0,1,\dots,k-1$.*

A system of differential operators $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1}$ with row–vector coefficients of length N is quoted to as a DIRICHLET *system of the order k if*

$$\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1}=\mathcal{H}_0\,\{\mathbf{B}_j(t,D_t)\}_{j=0}^{k-1}$$

where $\{\mathbf{b}_j(t,D_t)\}_{j=\omega}^{kN-1}$ is a DIRICHLET *system and $\mathcal{H}_0$ is a constant $kN\times kN$ matrix, interchanging rows.*

Definition 1.3 *A system $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$ is quoted to as a* **quasi–normal** *if:*

i. the principal homogeneous symbols $\mathbf{b}_{j,0}(t,\vec{\nu}(t))$, $j=0,\dots,\omega-1$ evaluated at the normal vectors $\xi=\vec{\nu}(t)$ are linearly independent vector–rows for all $t\in\mathcal{S}$ on the boundary;

ii. amount of operators with equal order among $\mathbf{b}_0(t,D_t),\dots,\mathbf{b}_{\omega-1}(t,D_t)$ does not exceeds N.

Lemma 1.4 *For arbitrary quasi–normal system of operators $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$, $\mathbf{ord}\,\mathbf{b}_{\mathrm{j}}\le\mathrm{m}-1$, there exists a non–unique extension up to a* DIRICHLET *system*

$$\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}=\mathcal{H}_0\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$$

of the order m with some constant $mN\times mN$ matrix $\mathcal{H}_0$.

Proof. Le us select among "boundary" row–operators $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$ those with equal orders m_j and add to the selected rows new rows of differential operators of the same order in such a way that the obtained $N\times N$ matrix–operator $\mathbf{B}_j(t,D_t)$ would have linearly independent rows in the principal homogeneous symbol $\mathcal{B}_{j,0}(t,\vec{\nu}(t))$, i.e. would be normal. Next we extend the system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{\ell}$ up to a DIRICHLET system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}$ of the order m by adding normal operators with missing orders (say, $\partial_{\vec{\nu}(t)}^{m_k}$, $k=\ell+1,\dots,m-1$).

As the last step we rearrange rows in extended system $\{\mathbf{b}_j(t, D_t)\}_{j=0}^{mN-1}$ with the help of some matrix $\mathcal{H}_0$ which has entries 0 and 1 to get a DIRICHLET system $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{m-1}$. ■

Definition 1.5 (1.8) *is called* **formally adjoint** *to BVP* (1.6) *if there exist two systems of "boundary" differential operators*

$$\mathbf{b}_j(t, D_t) = \sum_{|\alpha|\le m_j} b_{j\alpha}(t)\partial_t^\alpha, \qquad \mathbf{c}_k(t, D_t) = \sum_{|\alpha|\le \mu_k} c_{k\alpha}(t)\partial_t^\alpha\,,$$

$$b_{j\alpha}, c_{k\alpha} \in C^\infty(\mathcal{S}, \mathbb{C}^N), \quad j, k = 0, \dots, mN-1\,,$$

which are extensions of systems $\{\mathbf{b}_j(t, D_t)\}_{j=0}^{\omega-1}$ *and* $\{\mathbf{c}_j(t, D_t)\}_{j=0}^{\omega^*-1}$, *such that the* GREEN *formula*

$$\int_{\Omega^\pm} ((\mathbf{A}u)^\top \overline{v} - u^\top \overline{\mathbf{A}^* v})dy = \pm \sum_{j=0}^{mN-1} \oint_{\mathcal{S}} \mathbf{b}_j u \overline{\mathbf{c}_j v} d_\tau \mathcal{S} \tag{1.12}$$

holds[2)] *with* $u, v \in C^\infty(\overline{\Omega^\pm}, \mathbb{C}^N)$.

For BVP (1.10) and its formal adjoint (1.11) the GREEN formula (1.12) acquires the form

$$\int_{\Omega^\pm} ((\mathbf{A}u)^\top \overline{v} - u^\top \overline{\mathbf{A}^* v})dy = \pm \sum_{j=0}^{m-1} \oint_{\mathcal{S}} (\mathbf{B}_j u)^\top \overline{\mathbf{C}_j v} d_\tau \mathcal{S}\,, \tag{1.13}$$

where "boundary" differential operators $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{m-1}$ and $\{\mathbf{C}_j(t, D_t)\}_{j=0}^{m-1}$ have $N \times N$ matrix coefficients. If (1.8) is formally adjoint to BVP (1.8), then [3)]

$$m_j + \mu_j = m - 1, \qquad j = 0, \dots . \omega - 1. \tag{1.14}$$

Since the DIRICHLET systems participating in the GREEN formulae (1.13) and (1.12) coincide up to rearrangement of rows (cf. (5.1)), we will mostly adress formula (1.13).

Theorem 1.6 *If either* $\{\mathbf{b}_j(t, D_t)\}_{j=0}^{mN-1}$ *or* $\{\mathbf{c}_j(t, D_t)\}_{j=0}^{mN-1}$ *is a fixed* DIRICHLET *system of "boundary" operators, the* GREEN *formula* (1.12) *holds,*

[2)]The integral $\oint_{\mathcal{S}}$ is used to underline that integration is performed over the closed surface $\mathcal{S}$.

[3)](1.14) follows e.g. from the formulae (1.38) for "boundary" operators $\{\mathbf{C}_j(t, D_t)\}_{j=0}^{m-1}$.

another system (respectively,V*'*$\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ *or* $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$*) is defined in a unique way and* BVP (1.8) *is formally adjoint to* (1.6).

The system $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ *(the system* $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$*) would be a* DIRICHLET *system if and only if the "basic" operator* $\mathbf{A}(x,D_x)$ *is normal.*

If the "basic" operator $\mathbf{A}(x,D_x)$ *is normal,* $\omega = kN$, $\omega^* = (m-k)N$, *the systems* $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1}$, $\{\mathbf{c}_{mN-j-1}(t,D_t)\}_{j=0}^{(m-k)N-1}$ *are fixed and one of them is quasi–normal, the* GREEN *formula* (1.12) *holds if and only if both are* DIRICHLET *systems* $\mathbf{ord}\,\mathbf{b}_\mathrm{j} = \mathbf{ord}\,\mathbf{c}_{\mathrm{mN-j-1}} = \mathrm{j}$ *(of order* k *and* $m-k$*, respectively). Then the extended systems* $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$ *and* $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ *in* (1.12) *are* DIRICHLET *systems (of order* m*) and are unique.*

Proof is deferred to § 5.1. The first part of the Theorem for scalar $N=1$ elliptic operators has been proved earlier (see [LM1, Ch. 2, Theorem 2.1]) and for elliptic DOUGLIS–NIRENBERG systems–in [Ro1, RS1]. Most general case, on our knowledge, is considered in [Ta1, Ta2], where the "basic" and "boundary" operators have "rectangular" $k \times \ell$ matrix coefficients and an injective principal symbol of the "basic" operator.

It is well–known that if $\mathbf{A}(x,D_x)$ is scalar ($N=1$), elliptic and has real valued matrix–coefficients (or complex valued coefficients and $n>2$) than it is proper elliptic and has even order $\mathbf{ord}\,\mathbf{A}(\mathrm{x},\mathrm{D_x}) = \mathrm{m} = 2\ell$ (see [LM1, Ch.2, §§1.1]). Although for non–scalar case $N=2,3,\dots$ matters are different (see §6.6), many elliptic systems in applications (e.g. in elasticity, thermoelasticity, hydrodynamics) have even order. Let us consider some simplification of the GREEN formula for such systems, especially when the system is formally self–adjoint.

Assume the operator in (1.1) has even order $m=2\ell$. It can be represented in the form

$$A(x,D_x) = \sum_{|\alpha|,|\beta|\le\ell} (-1)^{|\alpha|}\partial_x^\alpha a_{\alpha,\beta}(x)\partial_x^\beta, \quad a_{\alpha,\beta} \in \mathbf{C}^\infty(\overline{\Omega^\pm}, \mathbb{C}^{N\times N}) \tag{1.15}$$

(representation is not unique) and with it one associates the following sesquilinear form

$$\mathcal{A}(u,v) := \int_{\Omega^\pm} \sum_{|\alpha|,|\beta|\le\ell} \left[a_{\alpha,\beta}(y)\partial_y^\beta u(y)\right]^\top \overline{\partial_y^\alpha v(y)}\,dy, \tag{1.16}$$

$$u,v \in C_0^\infty(\overline{\Omega^\pm}, \mathbb{C}^N)\,.$$

Theorem 1.7 *For arbitrary "basic" differential operator* (1.15) *of the even order* 2ℓ *and arbitrary* DIRICHLET *system* $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{\ell-1}$ *of the order* ℓ *of "boundary" differential operators with matrix* $N\times N$ *coefficients there exists a* *system* $\{\mathbf{C}_j(t, D_t)\}_{j=0}^{\ell-1}$ *of "boundary" operators with* $\mathbf{ord}\,\mathbf{B}_{\rm j} + \mathbf{ord}\,\mathbf{C}_{\rm j} = 2\ell - 1$ *such that*

$$\mathcal{A}(u,v) = \int\limits_{\Omega^\pm} (\mathbf{A}u)^\top \overline{v} dy \pm \sum_{j=0}^{\ell-1} \int\limits_{\mathcal{S}} (\mathbf{C}_j u)^\top \overline{\mathbf{B}_j v} d_\tau \mathcal{S}\,, \tag{1.17}$$
$$u, v \in C_0^\infty(\overline{\Omega^\pm}, \mathbb{C}^N)\,.$$

$\{\mathbf{C}_j(t, D_t)\}_{j=0}^{\ell-1}$ *would be a* Dirichlet *system if and only if the "basic" operator* $\mathbf{A}(x, D_x)$ *is normal.*

If $\mathbf{A}$ *is formally self–adjoint* $\mathbf{A} = \mathbf{A}^*$ *we get the following simplified* Green *formula*

$$\int\limits_{\Omega^\pm} \left[(\mathbf{A}u)^\top \overline{v} - u^\top \overline{\mathbf{A}v}\right] dy = \pm \sum_{j=0}^{\ell-1} \int\limits_{\mathcal{S}} \left[(\mathbf{C}_j u)^\top \overline{\mathbf{B}_j v} - (\mathbf{B}_j u)^\top \overline{\mathbf{C}_j v}\right] d_\tau \mathcal{S}. \tag{1.18}$$

Proof is deferred to § 5.2. For scalar $N = 1$ elliptic operators a slightly different proof see in [LM1, Ch. 2, § 2.4].

1.2. Partial integration and the special Green formula. Let us consider "extended" normal derivatives

$$\partial_{\vec{\nu}(x)} := \sum_{k=1}^{n} \nu_k(x)\partial_k, \quad x \in \mathbb{R}^n, \quad j = 0, 1, \dots\,, \tag{1.19}$$

where $\vec{\nu}(x) = (\nu_1(x), \dots, \nu_n(x))$, $x \in \mathbb{R}^n$ is some C^∞–smooth vector field which coincides with the unit normal vector field on $\mathcal{S}$ and stabilizes to the identity in the vicinity of infinity: $\vec{\nu}(x) = 1$ for sufficiently large $|x| > R$.

We will apply the Günter $\mathcal{D}_j$ and the Stokes $\mathcal{M}_{j,k}$ derivatives, which are defined as follows[4)]:

$$\mathcal{D}_x := (\mathcal{D}_1, \dots, \mathcal{D}_n)\,, \quad \mathcal{D}_j := \partial_j - \nu_j(x)\partial_{\vec{\nu}(x)} = \vec{d}_j \cdot \nabla\,, \quad \nabla := (\partial_1, \dots, \partial_n)\,,$$
$$\mathcal{M}_x := [\mathcal{M}_{j,k}]_{n\times n}\,, \quad \mathcal{M}_{j,k} := \nu_j(x)\partial_k - \nu_k(x)\partial_j = \vec{m}_{j,k} \cdot \nabla\,. \tag{1.20}$$

These derivatives are tangent to $\mathcal{S}$, i.e. the directing vectors $\vec{d}_j(t)$ and $\vec{m}_{j,k}(t)$ are orthogonal to $\vec{\nu}(t)$:

$$\vec{\nu}(t) \cdot \vec{d}_j(t) \equiv \vec{\nu}(t) \cdot \vec{m}_{j,k}(t) \equiv 0, \quad t \in \mathcal{S}\,.$$

[4)]The tangent derivatives $\mathcal{D}_j$ were introduced in [Gu1, §1.3]) while $\mathcal{M}_{j,k}$ for $n = 3$ in [KGBB1, Ch. V] (for $n > 3$ see [BD1]). The derivatives $\mathcal{M}_{j,k}$ are natural entries of the Stokes formula (1.27).

Therefore the derivative $\mathcal{D}_j$ can be applied to a function $\varphi(t)$ defined only on the boundary $\mathcal{S}$

$$\mathcal{D}_j\varphi(t) := \lim_{\lambda\to 0}\frac{\varphi(t+\lambda\vec{d}_j(t))}{\lambda} = \lim_{\lambda\to 0}\frac{\varphi(t+\lambda\vec{d}_{j,0}(t))}{\lambda}, \quad \varphi\in C^1(\mathcal{S})$$

where $\lambda\vec{d}_{j,0}(t)$ is the projection of the tangent vector $\lambda\vec{d}_j(t)$ onto the surface $\mathcal{S}$ (the projection is correctly defined for small $|\lambda|<\varepsilon$). Similarly can be interpreted $\mathcal{M}_{j,k}\varphi(t)$.

Only $n-1$ out of n derivatives $\mathcal{D}_1,\ldots,\mathcal{D}_n$ and out of n^2 derivatives $\mathcal{M}_{1,1},\ldots,\mathcal{M}_{n,n}$ are linearly independent and the following relations are valid:

$$\begin{gathered}\mathcal{D}_j := -\sum_{k=1}^n \nu_k\mathcal{M}_{j,k}\,, \quad \sum_{k=1}^n \nu_k\mathcal{D}_k = 0\,, \\ \mathcal{M}_{j,k} = \nu_j\mathcal{D}_k - \nu_k\mathcal{D}_j\,, \quad \mathcal{M}_{j,j}=0\,, \quad \mathcal{M}_{j,k} = -\mathcal{M}_{k,j}\,.\end{gathered} \tag{1.21}$$

Lemma 1.8 *For the partial derivatives ∂_k ($k=1,\ldots,n$), arbitrary "tangent" differential operator*

$$\mathbf{G} = \sum_{|\alpha|\le k} g_\alpha(x)\mathcal{D}_x^\alpha = \sum_{|\beta|\le k} g_\beta(x)\mathcal{M}_x^\beta, \qquad x\in\Omega^\pm$$

(see (1.28)–(1.21)*) and the normal derivative $\partial_{\vec{\nu}(t)}$ (see* (1.19)*) there hold the formulae*

$$\int_{\Omega^\pm} [\partial_k u(y)]^\top \overline{v(y)}dy = \pm\oint_{\mathcal{S}} \nu_k(\tau)\,[u(\tau)]^\top\,\overline{v(\tau)}d_\tau\mathcal{S} - \int_{\Omega^\pm}[u(y)]^\top\overline{\partial_k v(y)}dy\,, \tag{1.22}$$

$$\int_{\Omega^\pm} [\mathbf{G}u(y)]^\top\,\overline{v(y)}dy = \int_{\Omega^\pm}[u(y)]^\top\,\overline{\mathbf{G}^* v(y)}dy\,, \tag{1.23}$$

$$\int_{\Omega^\pm} \left[\partial_{\vec{\nu}(y)}u(y)\right]^\top\,\overline{v(y)}dy = \pm\oint_{\mathcal{S}}[u(\tau)]^\top\,\overline{v(\tau)}d_\tau\mathcal{S} - \int_{\Omega^\pm}[u(y)]^\top\,\overline{\partial^*_{\vec{\nu}(y)}v(y)}dy\,, \tag{1.24}$$

$$\oint_{\mathcal{S}} [\mathbf{G}u(\tau)]^\top\,\overline{v(\tau)}\,d_\tau\mathcal{S} = \oint_{\mathcal{S}} u^\top(\tau)\overline{\mathbf{G}^*_{\mathcal{S}}v(\tau)}\,d_\tau\mathcal{S}\,, \tag{1.25}$$

where[5)]

$$\mathbf{G}^* = \sum_{|\alpha|\le k}[\mathcal{D}_x^*]^\alpha\left[\overline{g_\alpha(x)}\right]^\top = \sum_{|\beta|\le k}[\mathcal{M}_x^*]^\beta\left[\overline{g_\beta(x)}\right]^\top\,,$$

[5)] It is worth to underline that the formally adjoint operators $\mathcal{D}_j^*$, $\mathcal{M}_{j,k}^*$ on the domains $\Omega^\pm$ (see (1.23)) and the "surface" adjoints $(\mathcal{D}_j)^*_{\mathcal{S}}$, $(\mathcal{M}_{j,k})^*_{\mathcal{S}}$ (see (1.26)) are essentially different, although the difference has lower order $(\mathcal{D}_j)^*_{\mathcal{S}}u = \mathcal{D}^*_j u + h_j u$, $\quad(\mathcal{M}_{j,k})^*_{\mathcal{S}}u = \mathcal{M}^*_{j,k}u + f_{j,k}u$, where h_j and $f_{j,k}$ are functions.

$$\mathbf{G}^*_{\mathcal{S}} = \sum_{|\alpha|\leq k} [(\mathcal{D}_x)^*_{\mathcal{S}}]^\alpha \left[\overline{g_\alpha(x)}\right]^\top = \sum_{|\beta|\leq k} [(\mathcal{M}_x)^*_{\mathcal{S}}]^\beta \left[\overline{g_\beta(x)}\right]^\top ,$$

$$\mathcal{D}^*_j u(x) = -\partial_j u(x) - \partial^*_{\vec{\nu}(x)}\nu_j u(x) , \quad \mathcal{M}^*_{j,k} u(x) = -\partial_k \nu_j u(x) + \partial_j \nu_k u(x) ,$$

$$(\mathcal{D}_j)^*_{\mathcal{S}} u(x) = -\sum_{k=1}^n \nu_k \partial_j \nu_k u(x) + \nu_j \partial^*_{\vec{\nu}} u(x) , \tag{1.26}$$

$$(\mathcal{M}_{j,k})^*_{\mathcal{S}} u(x) = -\mathcal{M}_{j,k} u(x) = \mathcal{M}_{k,j} u(x) .$$

Proof. Formula (1.22) is a direct consequence of the Gauss formula on divergence

$$\int\limits_{\Omega^\pm} \partial_k u(y) dy = \pm \oint\limits_{\mathcal{S}} \nu_k(\tau) u(\tau) d_\tau \mathcal{S} , \qquad k = 1, 2, \ldots, n$$

(see [Di1], [Si1, 4.13(4)]). In fact,

$$\int\limits_{\Omega^\pm} [\partial_k u(y)]^\top \overline{v(y)} dy = \int\limits_{\Omega^\pm} \partial_k [u^\top(y) \overline{v(y)}] dy - \int\limits_{\Omega^\pm} [u(y)]^\top \overline{\partial_k v(y)} dy$$
$$= \pm \oint\limits_{\mathcal{S}} \nu_k(\tau) \, [u(\tau)]^\top \, \overline{v(\tau)} d_\tau \mathcal{S} - \int\limits_{\Omega^\pm} [u(y)]^\top \overline{\partial_k v(y)} dy .$$

To prove (1.24) we apply (1.22) and proceed as follows:

$$\int\limits_{\Omega^\pm} \left[\partial_{\vec{\nu}(y)} u(y)\right]^\top \overline{v(y)} dy = \sum_{k=1}^n \int\limits_{\Omega^\pm} \left[\nu_k(y) \partial_k u(y)\right]^\top \overline{v(y)} dy$$
$$= \sum_{k=1}^n \int\limits_{\Omega^\pm} [\partial_k u(y)]^\top \overline{\nu_k(y) v(y)} dy$$
$$= \pm \sum_{k=1}^n \oint\limits_{\mathcal{S}} [u(\tau)]^\top \overline{\nu_k^2(\tau) v(\tau)} d_\tau \mathcal{S} - \int\limits_{\Omega^\pm} [u(y)]^\top \overline{\partial^*_{\vec{\nu}(y)} v(y)} dy$$
$$= \pm \oint\limits_{\mathcal{S}} [u(\tau)]^\top \overline{v(\tau)} d_\tau \mathcal{S} - \int\limits_{\Omega^\pm} [u(y)]^\top \overline{\partial^*_{\vec{\nu}(y)} v(y)} dy .$$

It suffices to prove (1.23) only for the generators $\mathcal{D}_j$. For this purpose we apply (1.22), (1.24) and continue as follows:

$$\int\limits_{\Omega^\pm} [\mathcal{D}_j u(y)]^\top \overline{v(y)} dy = \int\limits_{\Omega^\pm} [\partial_j u(y)]^\top \overline{v(y)} dy - \int\limits_{\Omega^\pm} \left[\partial_{\vec{\nu}(y)} u(y)\right]^\top \overline{\nu_j(y) v(y)} dy$$

$$= \pm \int_{\mathcal{S}} \nu_j(\tau) u^\top(\tau) \overline{v(\tau)} d_\tau \mathcal{S} - \int_{\Omega^\pm} [u(y)]^\top \overline{\partial_j v(y)} dy \mp \int_{\mathcal{S}} \nu_j(\tau) u^\top(\tau) \overline{v(\tau)} d_\tau \mathcal{S}$$

$$+ \int_{\Omega^\pm} [u(y)]^\top \overline{\partial_{\vec{\nu}}^* \nu_j(y) v(y)} dy = \int_{\Omega^\pm} [u(y)]^\top \overline{\mathcal{D}_j^* v(y)} dy \,.$$

To prove (1.25) we follow [BD1] and note it is sufficient to prove the formula for the generators $\mathcal{D}_j$ and $\mathcal{M}_{j,k}$. For this let us recall the STOKES formula

$$\oint_{\mathcal{S}} (\mathcal{M}_{j,k} u)(\tau) d_\tau \mathcal{S} = \oint_{\mathcal{S}} [\nu_j(\tau)(\partial_k u)(\tau) - \nu_k(\tau)(\partial_j u)(\tau)] d_\tau \mathcal{S} = 0 \,, \qquad (1.27)$$

$$j, k = 1, \ldots, n \,.$$

This formula is well–known for $n = 2, 3$ (see e.g. [Di1, Si1]). In general, for $n = 2, 3, \ldots$, (1.27) follows from another STOKES formula on external differential forms

$$\oint_{\mathcal{S}} d\omega = 0 \,, \qquad \mathbf{ord}\, \omega = \mathbf{dim}\, \mathcal{S} - 1$$

(see [Sc1, (VI.7;3], [Ca1, Ch. III, § 4.10]). In fact, It is easy to ascertain that

$$\nu_j d\mathcal{S} = (-1)^{j-1} \bigwedge_{m \neq j} dx_m$$

(see [Sc1, (VI.6;48)] for a detailed proof). With this formula at hand the integrand in (1.27) can be represented as a total differential

$$\mathcal{M}_{j,k} u d\mathcal{S} = (-1)^{j-1} (\partial_k u) \bigwedge_{m \neq j} dx_m - (-1)^{k-1} (\partial_j u) \bigwedge_{m \neq k} dx_m = d \left((-1)^{j+k} u \bigwedge_{m \neq j,k} dx_m \right)$$

for $j > k$ and we get (1.27). Since $\mathcal{M}_{k,j} = -\mathcal{M}_{j,k}$, $\mathcal{M}_{k,k} = 0$ (see (1.21)) (1.27) is proved for all $j, k = 1, \ldots n$.

From (1.27) we derive the following rule of partial integration for the generator $\mathcal{M}_{j,k}$

$$\oint_{\mathcal{S}} [\mathcal{M}_{j,k} u(\tau)]^\top \, \overline{v(\tau)} d_\tau \mathcal{S} = \oint_{\mathcal{S}} \mathcal{M}_{j,k} \left[u^\top(\tau) \overline{v(\tau)} \right] d_\tau \mathcal{S} - \oint_{\mathcal{S}} u^\top(\tau) \overline{\mathcal{M}_{j,k} v(\tau)} d_\tau \mathcal{S}$$

$$= \oint_{\mathcal{S}} u^\top(\tau) \overline{(\mathcal{M}_{j,k})_{\mathcal{S}}^* v(\tau)} \, d_\tau \mathcal{S} \,, \qquad j, k = 0, \ldots, n$$

where $(\mathcal{M}_{jk})^*_{\mathcal{S}} = -\mathcal{M}_{jk} = \mathcal{M}_{kj}$ and (1.25), (1.26) are proved for the generators $\mathcal{M}_{jk}$. Invoking relations (1.21) we find

$$(\mathcal{D}_j)^*_{\mathcal{S}} = -\sum_{k=1}^{n} (\mathcal{M}_{j,k})^*_{\mathcal{S}} \nu_k = -\sum_{k=1}^{n} \nu_k \partial_j \nu_k + \nu_j \partial^*_{\vec{\nu}} ,$$

which yields (1.25) for another generator $\mathcal{D}_j$. ■

Example 1.9 *Let*

$$\mathbf{A}(x, D_x) := \sum_{j,k=0}^{n} a_{j,k}(x) \partial_j \partial_k, \quad a_{j,k} \in C^\infty(\overline{\Omega^\pm}, \mathbb{C}^{N \times N})$$

be arbitrary second order operator with variable coefficients and consider the Dirichlet *problem ($Au = f$ in $\Omega^\pm$ and $\gamma^\pm_{\mathcal{S}} u = g$ on $\mathcal{S}$) or the* Neumann *problem ($Au = f$ in $\Omega^\pm$ and $\sum_{j,k=0}^{n} a_{j,k} \nu_j \gamma^\pm_{\mathcal{S}} \partial_k u = g$ on $\mathcal{S}$). Applying the partial integration* (1.22)*, we get the* Green *formula* (1.13) *with*

$$\mathbf{B}_0(x, D_x) = I , \quad \mathbf{B}_1(x, D_x) u(x) = \sum_{j,k=0}^{n} a_{j,k}(x) \nu_j(x) \partial_k u(x) ,$$

$$\mathbf{C}_0(x, D_x) = I , \quad \mathbf{C}_1(x, D_x) u(x) = -\sum_{j,k=0}^{n} \nu_k(x) \partial_j a^*_{j,k}(x) u(x) ,$$

for the Dirichlet *problem and with*

$$\mathbf{B}_0(x, D_x) u(x) = \sum_{j,k=0}^{n} a_{j,k}(x) \nu_j(x) \partial_k u(x) , \quad \mathbf{B}_1(x, D_x) = I ,$$

$$\mathbf{C}_0(x, D_x) u(x) = -\sum_{j,k=0}^{n} \nu_k(x) \partial_j a^*_{j,k}(x) u(x) , \quad \mathbf{C}_1(x, D_x) = I ,$$

for the Neumann *problem.*

Thus, the partial integration (see (1.22)) can be used to get the special Green formula for arbitrary "basic" operator (not necessarily elliptic; cf. the foregoing Example 1.9). But it is not for sure that thus we get normal "boundary" operators even if the "basic" operator is elliptic. On the other hand normality of one of two systems of "boundary" operators is necessary to replace them by arbitrary system of "boundary" operators of our choice (see

§ 5.1). For this purpose we derive the special GREEN formula in Theorem 1.10.

The operator $\mathbf{A}(x, D_x)$ in (1.1) can be written in the form

$$\mathbf{A}(x, D_x) = \mathcal{A}_0(x, \vec{\nu}(x))\partial^m_{\vec{\nu}(x)} + \sum_{j=0}^{m-1} \mathbf{A}_{m-j}(x, \mathcal{D}_x)\partial^j_{\vec{\nu}(x)}$$

$$= \mathcal{A}_0(x, \vec{\nu}(x))\partial^m_{\vec{\nu}(x)} + \sum_{j=0}^{m-1} \widetilde{\mathbf{A}}_{m-j}(x, \mathcal{M}_x)\partial^j_{\vec{\nu}(x)}\,, \tag{1.28}$$

$$\mathbf{A}_k(x, \mathcal{D}_x) = \sum_{|\alpha|\leq k} a^0_{k,\alpha}(x)\mathcal{D}^\alpha_x = \widetilde{\mathbf{A}}_k(x, \mathcal{D}_x) = \sum_{|\beta|\leq k} \widetilde{a}^0_{k,\beta}(x)\mathcal{M}^\beta_x\,,$$

$$\mathcal{D}^\alpha_x := \mathcal{D}^{\alpha_1}_1 \cdots \mathcal{D}^{\alpha_n}_n\,, \quad \mathcal{M}^\beta_x := \mathcal{M}^{\beta_{1,1}}_{1,1} \cdots \mathcal{M}^{\beta_{n,n}}_{n,n}\,,$$

$$\alpha \in \mathbb{N}^n_0\,, \quad \beta \in \mathbb{N}^{n\times n}_0\,, \quad x \in \Omega^\pm\,, \quad k = 1, 2, \ldots, m\,,$$

where $\mathcal{A}_0(x, \xi)$ is the homogeneous principal symbol (see (1.4)) and the derivatives $\partial_{\vec{\nu}(x)}$, $\mathcal{D}_j$, $\mathcal{M}_{j,k}$ are defined in (1.19)–(1.20).

Theorem 1.10 *Let* $\mathbf{A}(x, D_x)$ *be defined in* (1.1) *and*

$$\mathbf{B}_k(t, D_t) := \partial^k_{\vec{\nu}(t)}\,, \quad \mathbf{C}_k(t, D_t) := \sum_{j=k+1}^{m} (\partial^*_{\vec{\nu}(t)})^{j-k-1}\mathbf{A}^*_{m-j}(t, \mathcal{D}_t) \tag{1.29}$$

$$= \sum_{j=0}^{m-k-1} (\partial^*_{\vec{\nu}(t)})^j\mathbf{A}^*_{m-j-k-1}(t, \mathcal{D}_t)\,, \quad \partial^*_{\vec{\nu}(t)}u(t) := -\sum_{k=1}^{n} \partial_{t_k}\nu_k(t)u(t)\,.$$

Then the GREEN *formula* (1.13) *is valid.*

Proof. Applying (1.23) and (1.24) we find the following:

$$\int_{\Omega^\pm} (\mathbf{A}u)^\top\overline{v}dy = \pm\sum_{k=0}^{m-1}\sum_{j=k+1}^{m}\oint_{\mathcal{S}} \left[\gamma^\pm_{\mathcal{S}}\partial^k_{\vec{\nu}}u\right]^\top \overline{\gamma^\pm_{\mathcal{S}}(\partial^*_{\vec{\nu}})^{j-k-1}\mathbf{A}^*_{m-j}v}d_\tau\mathcal{S} + \int_{\Omega^\pm} u^\top\overline{\mathbf{A}^*v}dy$$

$$= \pm\sum_{k=0}^{m-1}\oint_{\mathcal{S}}[\gamma^\pm_{\mathcal{S}}\partial^k_{\vec{\nu}}u]^\top\overline{\gamma^\pm_{\mathcal{S}}\mathbf{C}_kv}d_\tau\mathcal{S} + \int_{\Omega^\pm} u^\top\overline{\mathbf{A}^*v}dy\,.$$

The GREEN formula (1.13) for the BVPs (1.10), (1.11) with operators (1.29) is proved. ∎

BVP (1.10) with the normal "boundary" operators $\mathbf{B}_k = \partial^k_{\vec{\nu}}$, $k = 0, \ldots, m-1$, is called the DIRICHLET problem.

The GREEN formulae (1.13) with operators (1.29) can be found in [Se1, (5.3)], [Tv1, Ch.III, (5.41)], [CW1, (1.5)], [CP1, Di1]. This special formula is a crucial component of the proof of Theorem 1.6.

1.3. About "boundary" operators in the Green formula. Next we will discuss the problem of finding "boundary" differential operators $\{\mathbf{C}_j(t, D_t)\}_{j=0}^{m-1}$ in the GREEN formula (1.13) in explicit form, provided the DIRICHLET system $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{m-1}$ is fixed.

Similar formulae holds, obviously, for the "boundary" differential operators $\{\mathbf{c}_j(t, D_t)\}_{j=0}^{mN-1}$ in the GREEN formula (1.12).

Since $\{\mathbf{B}_j(x, D_x)\}_{j=0}^{m-1}$ is a DIRICHLET system, to simplify the representations formulae hereafter we will suppose (cf. (1.14))

$$\mathbf{ord}\,\mathbf{B}_\mathrm{j} = \mathrm{j}\,, \quad \mathbf{ord}\,\mathbf{C}_\mathrm{j} = \mathrm{m} - 1 - \mathrm{j}\,, \quad \mathrm{j} = 0, \dots, \mathrm{m} - 1\,. \tag{1.30}$$

Let us introduce, for convenience, the following vector–operators of length m:

$$\begin{aligned} \vec{\mathbf{D}}^{(m)}(x, D_x) &:= \left\{\partial_{\vec{\nu}(x)}^{m-1}, \dots, \partial_{\vec{\nu}(x)}, I, \right\}^\top, \\ \vec{\mathbf{B}}^{(m)}(x, D_x) &:= \{\mathbf{B}_0(x, D_x), \dots, \mathbf{B}_{m-1}(x, D_x)\}^\top, \\ \vec{\mathbf{C}}^{(m)}(x, D_x) &:= \{\mathbf{C}_0(x, D_x), \dots, \mathbf{C}_{m-1}(x, D_x)\}^\top; \end{aligned} \tag{1.31}$$

applied to a vector–function they produce longer vector–function, e.g.

$$\vec{\mathbf{B}}^{(m)}(x, D_x)u := \{\mathbf{B}_j(x, D_x)u\}_{j=0}^{m-1}\,.$$

Then the GREEN formula (1.13) acquires the form

$$\int_{\Omega^\pm} ((\mathbf{A}u)^\top \cdot \overline{v} - u^\top \cdot \overline{\mathbf{A}^* v})dy = \pm \oint_{\mathcal{S}} (\vec{\mathbf{B}}^{(m)}u)^\top \cdot \overline{\vec{\mathbf{C}}^{(m)}v} d_\tau\mathcal{S}\,, \tag{1.32}$$

while the representation (1.28)–the form

$$\mathbf{A}(x, D_x) = \left[\vec{\mathbf{A}}^{(m+1)}(x, \mathcal{D}_x)\right]^\top \cdot \vec{\mathbf{D}}^{(m+1)}(x, D_x)\,, \tag{1.33}$$

$$\vec{\mathbf{A}}^{(m+1)}(x, D_x) := \{\mathcal{A}_0(x, \vec{\nu}(x)), \mathbf{A}_1(x, \mathcal{D}_x), \dots, \mathbf{A}_m(x, \mathcal{D}_x), \}^\top\,,$$

where "$\cdot$" designates the formal scalar product of vectors. For the DIRICHLET system $\vec{\mathbf{B}}^{(m)}(t, D_t)$ we introduce the $m \times m$ lower–triangular matrix-operator

$$\mathbf{b}^{(m\times m)}(x, \mathcal{D}_x) =$$

$$
= \begin{bmatrix} \mathcal{B}_{0,0}(x,\vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{B}_{1,0}(x,\mathcal{D}_x) & \mathcal{B}_{1,0}(x,\vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots, \\ \mathbf{B}_{m-1,0}(x,\mathcal{D}_x) & \mathbf{B}_{m-1,1}(x,\mathcal{D}_x) & \cdots & \mathcal{B}_{m-1,0}(x,\vec{\nu}(x)) \end{bmatrix} \tag{1.34}
$$

with the entries $\mathbf{B}_{j,k}(x, \mathcal{D}_x)$ representing "tangent" differential operators of the order $j - k$ with matrix coefficients from the representations

$$
\mathbf{B}_j(x, D_x) = \mathcal{B}_{j,0}(x, \vec{\nu}(x))\partial^j_{\vec{\nu}(x)} + \sum_{k=0}^{j-1} \mathbf{B}_{j,k}(x, \mathcal{D}_x)\partial^k_{\vec{\nu}(x)} \tag{1.35}
$$

and $\mathcal{B}_{j,0}(x, \xi)$ standing for the principal homogeneous symbol of $\mathbf{B}_j(x, D_x)$ ($j = 0, \dots m - 1$; cf. (1.28)).

Invertible block matrix–operators of type (1.34) will be referred to as admissible (cf. [Ag1, § 4]).

Since the entries of the principal diagonal in (1.34) are non–degenerate in the vicinity of $\mathcal{S}$

$$
\mathbf{det}\, \mathcal{B}_{j,0}(x, \vec{\nu}(x)) \neq 0\,, \qquad j = 0, \dots, m-1
$$

(we remind that the operators $\mathbf{B}_j(t, D_t)$ are normal), $\mathbf{b}^{(m\times m)}(x, \mathcal{D}_x)$ is admissible on $\mathcal{S}$:

$$
\left[\mathbf{b}^{(m\times m)}(x, \mathcal{D}_x)\right]^{-1} =
$$

$$
= \begin{bmatrix} \mathcal{B}^{-1}_{0,0}(x,\vec{\nu}(x)) & 0 & \cdots & 0 \\ \widetilde{\mathbf{B}}_{1,0}(x,\mathcal{D}_x) & \mathcal{B}^{-1}_{1,0}(x,\vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots, \\ \widetilde{\mathbf{B}}_{m-1,0}(x,\mathcal{D}_x) & \widetilde{\mathbf{B}}_{m-1,1}(x,\mathcal{D}_x) & \cdots & \mathcal{B}^{-1}_{m-1,0}(x,\vec{\nu}(x)) \end{bmatrix}, \tag{1.36}
$$

$$
\widetilde{\mathbf{B}}_{j,k} := -\mathcal{B}^{-1}_{k,0}(x, \vec{\nu}(x))\mathbf{B}_{j,k}(x, \mathcal{D}_x)\mathcal{B}^{-1}_{j,0}(x, \vec{\nu}(x))\,.
$$

The set of admissible matrix–operators is an algebra: finite sums, products and even the inverse (if it exists) of admissible matrix–operators are admissible again.

The representations (1.35), in above introduced notations, can be written in the form

$$
\vec{\mathbf{B}}^{(m)}(x, D_x) = \mathbf{b}^{(m\times m)}(x, \mathcal{D}_x)\vec{\mathbf{D}}^{(m)}(x, D_x)\,. \tag{1.37}
$$

Theorem 1.11 *Let the* Dirichlet *system* $\vec{\mathbf{B}}^{(m)}(x, D_x)$ *be fixed and convention* (1.30) *hold. Then the system* $\vec{\mathbf{C}}^{(m)}(x, D_x)$ *in the* Green *formula* (1.13) *(see* (1.31)*) is found as follows*

$$\vec{\mathbf{C}}^{(m)}(x,D_x)=\left[\left(\mathbf{b}^{(m\times m)}\right)^*_{\mathcal{S}}(x,\mathcal{D}_x)\right]^{-1}\left[\left(\vec{\mathbf{D}}^{(m)}\right)^*(x,D_x)\right]^{\top}$$
$$\times\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathcal{D}_x)\mathbb{S}_m\,, \tag{1.38}$$

where $\left(\mathbf{b}^{(m\times m)}\right)^*_{\mathcal{S}}(x,\mathcal{D}_x)$ *denotes the "surface" adjoint to* $\mathbf{b}^{(m\times m)}(x,\mathcal{D}_x)$ *(see* (1.25), (1.26)*), while* $\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathcal{D}_x)$ *is the formally adjoint (see* (1.23), (1.26)*) to the following lower–triangular matrix–operator*

$$\mathbf{A}^{(m\times m)}(x,\mathcal{D}_x)=\begin{bmatrix}\mathcal{A}_0(x,\vec{\nu}(x)) & 0 & \cdots & 0\\ \mathbf{A}_1(x,\mathcal{D}_x) & \mathcal{A}_0(x,\vec{\nu}(x)) & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{A}_{m-2}(x,\mathcal{D}_x) & \mathbf{A}_{m-3}(x,\mathcal{D}_x) & \cdots & 0\\ \mathbf{A}_{m-1}(x,\mathcal{D}_x) & \mathbf{A}_{m-2}(x,\mathcal{D}_x) & \cdots & \mathcal{A}_0(x,\vec{\nu}(x))\end{bmatrix} \tag{1.39}$$

(cf. [Se1, (7a)], [Gr1]) compiled of "tangent" differential operators of the representation (1.28) *(see also* (1.33)*), where* $\mathbb{S}_m$ *is the skew–identity matrix of order m:*

$$\mathbb{S}_m=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 1 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 1 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\end{bmatrix}. \tag{1.40}$$

Proof is a byproduct of the proof of Theorem 1.6 (see § 5.1). ■

Remark 1.12 *Using representations* (1.28) *for the "basic" operator and* (1.35) *for a "boundary" operator* $\mathbf{B}_j(x,D_x)$ *with* $\mathbf{ord}\,\mathbf{B}_j > m-1$ *boundary values* $\gamma_0^{\pm}\mathbf{B}_j(t,D_t)u(t)$ *of a solution to "basic" equation* $\mathbf{A}(x,D_x)u=f$ *in* (1.10) *can be found if the boundary values of normal derivatives* $\left\{\gamma_0^{\pm}\partial_{\vec{\nu}(t)}u(t)\right\}_{j=0}^{m-1}$ *are known (or, due to Lemma* 4.7*, if the datae* $\left\{\gamma_0^{\pm}\mathbf{C}_j(x,D_x)u(t)\right\}_{j=0}^{m-1}$ *are known for some* DIRICHLET *system* $\{\mathbf{C}_j(x,D_x)\}_{j=0}^{m-1}$*). Details can be found in [Hr2, § 20.1]. Therefore orders of "boundary" operators* $\mathbf{B}_j(x,D_x)$ *in* (1.10) *are restricted* $\mathbf{ord}\,\mathbf{B}_j \le m-1$.

2 Spaces

We recall definitions and some properties of function spaces from [CD1, Tr1, Tr2], needed for further exposition.

$\mathbb{S}(\mathbb{R}^n)$ denotes the SCHWARTZ space of all rapidly decaying functions and $\mathbb{S}'(\mathbb{R}^n)$ – the dual space of tempered distributions. Since the FOURIER transform and its inverse, defined by

$$\begin{aligned} \mathcal{F}\varphi(\xi) = & \int\limits_{\mathbb{R}^n} e^{i\xi x}\varphi(x)dx \qquad \text{and} \\ \mathcal{F}^{-1}\psi(x) = & \ (2\pi)^{-n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}\psi(\xi)d\xi, \quad x,\xi\in\mathbb{R}^n \end{aligned} \tag{2.1}$$

are continuous in both spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$, the **convolution operator**

$$\mathbf{a}(D)\varphi = W_a^0\varphi := \mathcal{F}^{-1}a\mathcal{F}\varphi \quad \text{with} \quad a\in\mathbb{S}'(\mathbb{R}^n), \quad \varphi\in\mathbb{S}(\mathbb{R}^n) \tag{2.2}$$

is a continuous transformation from $\mathbb{S}(\mathbb{R}^n)$ into $\mathbb{S}'(\mathbb{R}^n)$ (see [Du1, DS1]).

The BESSEL potential space $\mathbb{H}^s_p(\mathbb{R}^n)$ is defined as a subset of $\mathbb{S}'(\mathbb{R}^n)$ and is endowed with the following norm (see [Tr1, Tr2]):

$$||u|\mathbb{H}^s_p(\mathbb{R}^n)|| := ||\langle D\rangle^s u|L_p(\mathbb{R}^n)||, \text{ where } \langle\xi\rangle^s := (1+|\xi|^2)^{\frac{s}{2}}\,. \tag{2.3}$$

For the definition of the BESOV space $\mathbb{B}^s_{p,q}(\mathbb{R}^n)$ ($1\le p\le\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$) see [Tr1]: the space $\mathbb{B}^s_{p,p}(\mathbb{R}^n)$ ($1<p<\infty$, $s>0$) coincides with the trace space $\gamma^+_{\mathbb{R}^n}\mathbb{H}_p^{s+\frac{1}{p}}(\mathbb{R}^{n+1}_+)$ ($\mathbb{R}^{n+1}_+ := \mathbb{R}^n\otimes\mathbb{R}^+$) and is known also as the SOBOLEV–SLOBODEČKII space $W^s_p(\mathbb{R}^n)$.

The space $\mathbb{B}^s_{\infty,\infty}(\mathbb{R}^n)$ for $s>0$ coincides with the well known ZYGMUND space $\mathbb{Z}^s(\mathbb{R}^n)$, while for $s\in\mathbb{R}^+\setminus\mathbb{N}$ both $\mathbb{B}^s_{\infty,\infty}(\mathbb{R}^n)$ and $\mathbb{Z}^s(\mathbb{R}^n)$ coincide with the HÖLDER space $C^s(\mathbb{R}^n)$.

The space $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+)$ is defined as the subspace of $\mathbb{H}^s_p(\mathbb{R}^n)$ of those functions $\varphi\in\mathbb{H}^s_p(\mathbb{R}^n)$, which are supported in the half space $\mathbf{supp}\,\varphi\subset\overline{\mathbb{R}^n_+}$ wheareas $\mathbb{H}^s_p(\mathbb{R}^n_+)$ denotes the quotient space $\mathbb{H}^s_p(\mathbb{R}^n_+) = \mathbb{H}^s_p(\mathbb{R}^n)\big/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_-)$, $\mathbb{R}^n_- := \mathbb{R}^n\setminus\overline{\mathbb{R}^n_+}$ and can be identified with the space of distributions φ on $\mathbb{R}^n_+$ which admit extensions $\ell\varphi\in\mathbb{H}^s_p(\mathbb{R}^n)$. Therefore $r_+\mathbb{H}^s_p(\mathbb{R}^n) = \mathbb{H}^s_p(\mathbb{R}^n_+)$, where $r_+ = r_{\mathbb{R}^n_+}$ denotes the restriction to the half–space $\mathbb{R}^n_+$ from $\mathbb{R}^n$.

The spaces $\widetilde{\mathbb{B}}^s_{p,q}(\mathbb{R}^n_+)$ and $\mathbb{B}^s_{p,q}(\mathbb{R}^n_+)$ are defined similarly [Tr1, Tr2].

Next we define BESSEL potential spaces with weight, (see [CD1, §1.3], [Es1, §§23 and 26]).

Let $s\in\mathbb{R}$, $m\in\mathbb{N}_0$ and $1<p<\infty$; by $\mathbb{H}^{s,m}_p(\mathbb{R}^n_+)$ we denote the space of functions (of distributions for $s<0$) endowed with the norm

$$\|u|\mathbb{H}^{s,m}_p(\mathbb{R}^n_+)\| := \sum_{k=0}^m \|x^k_n u|\mathbb{H}^{s+k}_p(\mathbb{R}^n_+)\|. \tag{2.4}$$

Obviously, $\mathbb{H}_p^{s,0}(\mathbb{R}^n) = \mathbb{H}_p^s(\mathbb{R}^n)$. The space $\mathbb{B}_{p,q}^{s,m}(\mathbb{R}_+^n)$ is defined in a similar way:

$$||u|\mathbb{B}_{p,q}^{s,m}(\mathbb{R}_+^n)|| := \sum_{k=0}^{m} \|x_n^k u|\mathbb{B}_{p,q}^{s+k}(\mathbb{R}_+^n)||$$

Let

$$\mathbb{H}_p^{s,\infty}(\mathbb{R}_+^n) := \bigcap_{m\in\mathbb{N}_0} \mathbb{H}_p^{s,m}(\mathbb{R}_+^n), \quad \mathbb{B}_{p,q}^{s,\infty}(\mathbb{R}_+^n) := \bigcap_{m\in\mathbb{N}_0} \mathbb{B}_{p,q}^{s,m}(\mathbb{R}_+^n) \tag{2.5}$$

with an appropriate topology which makes them into Frechet spaces.

Let $\mathcal{M}$ be a compact, C^∞–smooth n–dimensional manifold with a smooth boundary $\Gamma := \partial\mathcal{M} \neq \emptyset$. The spaces $\mathbb{H}_p^s(\mathcal{M})$, $\widetilde{\mathbb{H}}_p^s(\mathcal{M})$, $\mathbb{B}_{p,q}^s(\mathcal{M})$, $\widetilde{\mathbb{B}}_{p,q}^s(\mathcal{M})$, $\mathbb{H}_p^{s,m}(\mathcal{M})$, $\widetilde{\mathbb{H}}_p^{s,m}(\mathcal{M})$, $\mathbb{B}_{p,q}^{s,m}(\mathcal{M})$ and $\widetilde{\mathbb{B}}_{p,q}^{s,m}(\mathcal{M})$ can be defined by a partition of the unity $\{\psi_j\}_{j=1}^{\ell}$ subordinated to some covering $\{Y_j\}_{j=1}^{\ell}$ of $\mathcal{M}$ and local coordinate diffeomorphisms

$$æ_j : X_j \to Y_j, \qquad X_j \subset \mathbb{R}_+^n.$$

In particular, for a compact domain $\Omega^+ \subset \mathbb{R}^n$ and non–compact $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}^+$ the spaces $\mathbb{H}_p^{s,m}(\Omega^\pm)$, $\widetilde{\mathbb{H}}_p^{s,m}(\Omega^\pm)$, $\mathbb{H}_{p,com}^{s,m}(\overline{\Omega^\pm})$, $\widetilde{\mathbb{B}}_{p,q,loc}^{s,m}(\overline{\Omega^\pm})$ etc. are defined as described above. For compact Ω^+ the subscripts *com* and *loc* can be omitted.

From the embedding theorems of Sobolev we get that

$$\begin{aligned}
&\mathbb{H}_{p,loc}^{s,\infty}(\overline{\Omega^\pm}), \mathbb{B}_{p,q,loc}^{s,\infty}(\overline{\Omega^\pm}) \subset C^\infty(\Omega^\pm) \ \ (\text{but} \not\subset C^\infty(\overline{\Omega^\pm}),\\
&\varphi(x) = o(1) \quad \text{as} \quad x \in \Omega^-, \quad |x| \to \infty
\end{aligned} \tag{2.6}$$

whatever the parameters $s \in \mathbb{R}$ and $1 < p < \infty$ are.

Let $\mathbb{L}(\mathbb{X}_1, \mathbb{X}_2)$ denote the space of all linear bounded operators operating between the Banach spaces $\mathbf{A} : \mathbb{X}_1 \to \mathbb{X}_2$.

Next two theorems summarize some results on interpolation (see [BL1, Tr2]), which will be used later on in the paper.

Theorem 2.1 *Let* $\mathbf{Int}[\mathbb{X}_1, \mathbb{X}_2]$ *denote one of the interpolation methods either the real* $[\mathbb{X}_1, \mathbb{X}_2]_{\vartheta,q}$ *or the complex* $(\mathbb{X}_1, \mathbb{X}_2)_\vartheta$ *(see [BL1, Tr2]). Then*

$$\mathbb{X}' = [\mathbb{X}_1', \mathbb{X}_2'], \qquad \mathbb{X}'' = [\mathbb{X}_1'', \mathbb{X}_2'']$$

imply

$$\mathbb{L}(\mathbb{X}', \mathbb{X}'') \subset \mathbb{L}(\mathbb{X}_1', \mathbb{X}_1'') \cap \mathbb{L}(\mathbb{X}_2', \mathbb{X}_2''). \tag{2.7}$$

Theorem 2.2 *(see [BL1, §§ 6.2,6.4]). Let*

$$
\begin{aligned}
s = \vartheta s_1 + (1-\vartheta)s_0\,, \quad s, s_0, s_1 \in \mathbb{R}\,, \quad 0 < \vartheta < 1\,, \\
\frac{1}{p} = \frac{\vartheta}{p_1} + \frac{1-\vartheta}{p_0}\,, \quad 1 \le p, p_0, p_1 \le \infty\,, \\
\frac{1}{q} = \frac{\vartheta}{q_1} + \frac{1-\vartheta}{q_0}\,, \quad 1 \le q, q_0, q_1 \le \infty
\end{aligned}
\tag{2.8}
$$

and $\dfrac{\vartheta}{r} := 0$ *if* $r = \infty$. *Then*

$$
\left(\mathbb{H}^{s_0}_{p_0}(\mathbb{M}), \mathbb{H}^{s_1}_{p_1}(\mathbb{M})\right)_\vartheta = \mathbb{H}^s_p(\mathbb{M})\,, \tag{2.9}
$$

$$
\left[\mathbb{H}^{s_0}_{p_0}(\mathbb{M}), \mathbb{H}^{s_1}_{p_1}(\mathbb{M})\right]_{\vartheta,q} = \mathbb{B}^s_{p,q}(\mathbb{M})\,, \tag{2.10}
$$

$$
\left(\mathbb{B}^{s_0}_{p_0,q_0}(\mathbb{M}), \mathbb{B}^{s_1}_{p_1,q_1}(\mathbb{M})\right)_\vartheta = \mathbb{B}^s_{p,q}(\mathbb{M})\,, \tag{2.11}
$$

where $\mathbb{M} = \Omega^\pm \subset \mathbb{R}^n$ *or* $\mathbb{M} = \mathcal{M}$ *is a smooth manifold.*

The same interpolation results (2.9)–(2.11) *hold for the spaces* $\widetilde{\mathbb{H}}^s_p(\mathbb{M})$ *and* $\widetilde{\mathbb{B}}^s_{p,q}(\mathbb{M})$ *if* $\mathbb{M}$ *has the boundary* $\partial\mathbb{M} \neq \emptyset$.

Slight modification of the proof allows to prove the foregoing theorem for weighted spaces $\mathbb{H}^{s,k}_p(\mathbb{M})$, $\widetilde{\mathbb{H}}^{s,k}_p(\mathbb{M})$, $\mathbb{B}^{s,k}_{p,q}(\mathbb{M})$ and $\widetilde{\mathbb{B}}^{s,k}_{p,q}(\mathbb{M})$.

Let us agree the following: if we use the notation $\mathbb{X}^{s,m}_p(\mathbb{M})$ ($\widetilde{\mathbb{X}}^{s,m}_p(\mathbb{M})$) the following spaces will be meant:

$$
\text{either} \quad \mathbb{H}^{s,m}_p(\mathbb{M}) \quad \text{or} \quad \mathbb{B}^{s,m}_{p,q}(\mathbb{M}) \tag{2.12}
$$

$$
(\text{either} \quad \widetilde{\mathbb{H}}^{s,m}_p(\mathbb{M}) \quad \text{or} \quad \widetilde{\mathbb{B}}^{s,m}_{p,q}(\mathbb{M}))
$$

with arbitrary $1 \le q \le \infty$.

3 Representation of solutions and layer potentials

Throughout the present section we will assume that the differential operator $\mathbf{A}(x, D_x)$ in (1.1) is invertible on $\mathbb{R}^n$ or, in other words, has a fundamental solution (see [Hr2, § 4.4]), which is understood either as the inverse

$$
\begin{aligned}
&\mathbf{F_A} = \mathbf{A}^{-1}(x, D_x) : C^\infty_{\text{com}}(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)\,, \\
&\mathbf{A}(x, D_x)\mathbf{F_A}\varphi = \mathbf{F_A}\mathbf{A}(x, D_x)\varphi = \varphi\,, \qquad \varphi \in C^\infty_0(\Omega^\pm)\,,
\end{aligned}
$$

or as the distribution SCHWARTZ kernel $\mathcal{K}_{\mathbf{A}}(x,y) : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ of the operator $\mathbf{F}_{\mathbf{A}}$ (see [Hr2, Theorem 5.2.1]):

$$\mathbf{A}(x, D_x)\mathcal{K}_{\mathbf{A}}(x,y) = \delta(x-y) \tag{3.1}$$

with the DIRAC function $\delta(x)$.

We will suppose that $\mathbf{A}(x, D_x)$ is elliptic and the order $\mathbf{ord}\,\mathrm{A} = \mathrm{m} = 2\ell$ is even (see § 6.6). Then the inverse $\mathbf{F}_{\mathbf{A}} = \mathbf{F}_{\mathbf{A}}(x, D_x)$ is a pseudodifferential operator[6)] with the symbol from the HÖRMANDER class $\mathbb{S}^{-m}(\Omega^\pm, \mathbb{R}^n)$ (see e.g. [EgS1, Hr2, Sb1, Tv1]). This yields the inclusion $\mathbf{sing\,supp}\,\mathcal{K}_{\mathbf{A}} = \Delta_{\mathbb{R}^n}$ or, in other notation, $\mathcal{K}_{\mathbf{A}} \in C^\infty((\mathbb{R}^n \otimes \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n})$.

Moreover, if $\mathbf{A}(x, D_x)$ is hypoelliptic (see § 4.1) a fundamental solution $\mathbf{F}_{\mathbf{A}}(x, D_x)$ is PsDO again and solution $\mathbf{sing\,supp}\,\mathcal{K}_{\mathbf{A}} = \Delta_{\mathbb{R}^n}$ [7)] (see [Hr1, § 4.1], [Tv1, Ch.1, Theorem 2.2]).

Since $\mathbf{A}(x, D_x)$ has a fundamental solution $\mathbf{F}_{\mathbf{A}}$, the adjoint operator $\mathbf{A}^*(x, D_x)$ in (1.2) has it as well and

$$\mathbf{F}_{\mathbf{A}^*} = \mathbf{F}^*_{\mathbf{A}}\,, \qquad \mathcal{K}_{\mathbf{A}^*}(x,y) = [\overline{\mathcal{K}_{\mathbf{A}}(y,x)}]^\top\,, \tag{3.2}$$

where $\mathcal{K}_{\mathbf{A}^*}(x,y)$ is the SCHWARTZ kernel of the fundamental solution $\mathbf{F}_{\mathbf{A}^*}$ of the adjoint operator.

As a first application of the GREEN formula (1.13) we can get the representation of a solution of BVP (1.10). For this purpose let us consider $v_{\varepsilon,x}(y) = \chi_\varepsilon(x-y)\mathcal{K}_{\mathbf{A}^*}(y,x)$, where $\mathcal{K}_{\mathbf{A}^*}(x,y)$ is the kernel of the fundamental solution $\mathbf{F}_{\mathbf{A}^*}$ (see (3.2)) and $\chi_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\chi_\varepsilon(x) = 1$, $\chi_\varepsilon(x) = 0$ for $|x| > \varepsilon$ and $|x| < \varepsilon/2$, respectively. Inserting $v_{\varepsilon,x}(y)$ into the GREEN formula (1.13) and sending $\varepsilon \to 0$ we find the following

$$\chi_{\Omega^\pm}(x)u(x) = \mathbf{N}_{\Omega^\pm}f(x) \pm \sum_{j=0}^{2\ell-1} \mathbf{V}_j\gamma_{\mathcal{S}}^\pm\mathbf{B}_j u(x)\,, \quad x \in \Omega^\pm\,, \tag{3.3}$$

$$\mathbf{N}_{\Omega^\pm}\varphi(x) := \int\limits_{\Omega^\pm} \left[\overline{\mathcal{K}_{\mathbf{A}^*}(y,x)}\right]^\top \varphi(y)dy = \int\limits_{\Omega^\pm} \mathcal{K}_{\mathbf{A}}(x,y)\varphi(y)dy\,, \tag{3.4}$$

where $\chi_{\Omega^\pm}$ is the characteristic function of $\Omega^\pm \subset \mathbb{R}^n$ and

$$\mathbf{V}_j\psi(x) := \oint\limits_{\mathcal{S}} \left[\overline{\mathbf{C}_j(\tau, D_\tau)\mathcal{K}_{\mathbf{A}^*}(\tau,x)}\right]^\top \varphi(\tau)d_\tau\mathcal{S}$$

[6)]See § 4.1 for some elementary information about PsDOs.

[7)]Almost all results of the present Section and further hold valid for hypoelliptic operators, but operators have not even order $m = 2\ell$. Operators with odd order $m = 2\ell + 1$ exist among properly elliptic (terminology from [Ag1, LM1, Ro1]) systems as well (see § 6.6).

$$= \oint\limits_{\mathcal{S}} \left[\overline{\mathbf{C}_j(\tau, D_\tau)}\mathcal{K}^\top_{\mathbf{A}}(x,\tau)\right]^\top \varphi(\tau) d_\tau \mathcal{S}$$

$$= \sum_{|\alpha|\le\mu_j} \oint\limits_{\mathcal{S}} \partial^\alpha_\tau \mathcal{K}_{\mathbf{A}}(x,\tau)\overline{c^\top_{j\alpha}(\tau)}\varphi(\tau) d_\tau\mathcal{S}\,, \quad j = 0,\cdots, 2\ell-1 \tag{3.5}$$

(cf. (1.9), (1.11)) are the layer potentials.

Integrals in (3.3)–(3.5) and similar ones later ((3.3) etc.) are understood as the functionals $\mathcal{K}_{\mathbf{A}}(x,\cdot)$, $(\partial^\alpha_t \mathcal{K}^{(\beta)}_{\mathbf{A}}(x,\cdot)$ etc.) with the parameter $x \in \mathbb{R}^n$ applied to the test function $\varphi(\tau)$ (to $\overline{c^\top_\alpha(\tau)}\varphi(\tau)$).

Summing up (3.3) for the domains $\Omega^\pm$ we get

$$u(x) = \mathbf{F}_{\mathbf{A}} f(x) + \sum_{j=0}^{2\ell-1} \mathbf{V}_j \mathbf{B}_j u(x)\,, \tag{3.6}$$

$$[v](t) := \gamma^+_{\mathcal{S}} v(t) - \gamma^-_{\mathcal{S}} v(t)\,, \quad t \in \mathcal{S}\,, \quad x \in \mathbb{R}^n \setminus \mathcal{S} = \Omega^+ \cup \Omega^-\,,$$

where $f = \mathbf{A}u\Big|_{\Omega^+\cap\Omega^-} = \mathbf{A}u\Big|_{\mathbb{R}^n\setminus\mathcal{S}}$ and

$$\mathbf{F}_{\mathbf{A}}\varphi(x) := \mathbf{N}_{\Omega^-} v(x) + \mathbf{N}_{\Omega^+} v(x) = \int\limits_{\mathbb{R}^n} \mathcal{K}_{\mathbf{A}}(x,y)\varphi(y)dy \tag{3.7}$$

is the fundamental solution of $\mathbf{A}(x, D_x)$.

Pseudodifferential operators $\mathbf{a}(x, D)$ and $\mathbf{b}(x, D)$ are called local equivalent at $x_0 \in \mathbb{R}^n$ (recorded as $\mathbf{a}(x, D) \overset{x_0}{\sim} \mathbf{b}(x, D)$) if

$$\inf_\chi \|\chi\,[\mathbf{a}(\cdot, D) - \mathbf{b}(\cdot, D)]\,|\mathbb{H}^s_p(\mathbb{R}^n)\| = 0\,, \tag{3.8}$$

where the infimum is taken over all smooth functions $\chi \in C^\infty_0(\mathbb{R}^n)$ equal identity $\chi(x) \equiv 1$ in a neighbourhood of x_0 (see [Du1] for elementary properties of local equivalence).

Lemma 3.1 *Assume* $\mathbf{A}(x, D_x)$ *is defined in* (1.1).

If $\mathbf{A}(x, D_x)$ *has constant matrix-coefficients* $a_\alpha =$ **const** *, the fundamental solution* $\mathbf{F}_{\mathbf{A}} = \mathbf{F}_{\mathbf{A}}(D)$ *exists provided* [8)] $\mathbf{det}\,\mathbf{A}(\mathrm{D}) \neq 0$. *In addition, if* $\mathbf{det}\,\mathcal{A}(\xi) \neq 0$, $\xi \neq 0$, *where*

$$\mathcal{A}(\xi) := \sum_{|\alpha|\le m} a_\alpha(-i\xi)^\alpha, \quad \xi \in \mathbb{R}^n$$

8) Fundamental solutions exist also for operators with analytic coefficients $a_\alpha(x)$ (see [Jo1].

is the symbol of $\mathbf{A}(D)$, *the fundamental solution* $\mathbf{F_A} = \mathbf{F_A}(D)$ *represents a convolution*

$$\mathbf{F_A}(D) = \mathcal{F}^{-1}_{\xi\to x}\left[\mathcal{A}^{-1}(\xi)\right] . \tag{3.9}$$

The SCHWARTZ *kernel of* $\mathbf{F_A}(D)$ *depends on the difference of arguments* $\mathcal{K}_\mathbf{A}(x,y) = \mathcal{K}_\mathbf{A}(x-y)$.

In general case of non–constant coefficients if $\mathbf{A}(x,D_x)$ *has a fundamental solution* $\mathbf{F_A} : C_0^\infty(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$) *for arbitrary* $x_0 \in \mathbb{R}^n$ *there holds the following local equivalence* (*cf.* (3.8))

$$\mathbf{F_A} \overset{x_0}{\sim} \mathbf{F}_{\mathbf{A}_0}(x_0, D_x)\,, \tag{3.10}$$

where the convolution operator $\mathbf{F}_{\mathbf{A}_0}(x_0,D)$ *is the fundamental solution of the principal part* $\mathbf{A}_0(x_0,D)$ (*see* (1.4)) *with coefficients frozen at* x_0 (*cf.* (3.9)).

Proof. All claims, except (3.10), can be found in [Hr1, §§ 3,4], [Hr2, § 11].

The local equivalence (3.10) follows from the obvious equivalence $\mathbf{A}(x,D_x) \overset{x_0}{\sim} \mathbf{A}(x_0,D_x)$ (see [Du1]) and from the elementary property: if operators are local equivalent and invertible, the inverses are local equivalent as well. ■

If $\mathbf{A}(x,D_x)$ is hypoelliptic and has the fundamental solution, we can only indicate the symbol of the fundamental solution, which is the symbol of a parametrix (see § 4.1). In particular, the principal symbol of the fundamental solution coincides with the inverse $\mathcal{A}_0^{-1}(x,\xi)$ of the principal symbol of $\mathbf{A}(x,D_x)$.

If $\mathbf{A}(x,D_x)$ has constant matrix-coefficients and is not elliptic, the condition **sing supp** $\mathcal{K}_\mathrm{A} = \Delta_{\mathbb{R}^n}$ might be violated (see [Hr2, § 10.2]) which means that the fundamental solution $\mathbf{F_A}$ can not be a pseudodifferential operator.

Let $\beta \in \mathbb{N}_0^n$ and consider the following generalized layer potentials

$$\begin{aligned}
\mathbf{V}^{(\beta)}_\mathbf{G}\varphi(x) &:= \oint_\mathcal{S} \left[\overline{\mathbf{G}(\tau,D_\tau)}\mathcal{K}^{(\beta)}_{\mathbf{A}^*}(\tau,x)\right]^\top \varphi(\tau)d_\tau\mathcal{S} \\
&= \sum_{|\alpha|\le\mu}\oint_\mathcal{S} \partial^\alpha_\tau \mathcal{K}^{(\beta)}_\mathbf{A}(x,\tau)\overline{c^\top_\alpha(\tau)}\varphi(\tau)d_\tau\mathcal{S}\,,
\end{aligned} \tag{3.11}$$

$$\mathbf{G}(t,D_t) = \sum_{|\alpha|\le\mu} c_\alpha(t)\partial^\alpha_t\,, c_\alpha \in C^\infty(\mathcal{S})\,,$$

$$\mathcal{K}^{(\beta)}_\mathbf{A}(x,y) := (x-y)^\beta\mathcal{K}_\mathbf{A}(x,y).$$

If $\mathbf{G}_0(t,D_t) = I$ and $\beta = 0$

$$\mathbf{V}\psi(x) := \mathbf{V}_0\psi(x) = \oint_\mathcal{S} \mathcal{K}_\mathbf{A}(x,\tau)\psi(\tau)d_\tau\mathcal{S} \tag{3.12}$$

is the single layer potential.

Theorem 3.2 *Let $\beta \in \mathbb{N}_0^n$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $m = 2l$, $\mathbf{A}(x, D_x)$ be elliptic with a fundamental solution $\mathcal{K}_A(x, y)$.*

The generalized layer potentials

$$\mathbf{V}_{\mathbf{G}}^{(\beta)} \ : \ \mathbb{B}^s_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-1-\mu+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^{\pm}}), \tag{3.13}$$

$$: \ \mathbb{B}^s_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+2\ell-1-\mu+|\beta|+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^{\pm}}) \tag{3.14}$$

with $\mu = \mathbf{ord}\,\mathbf{G} < 2\ell$, are continuous for all $k = 0, 1, \ldots, \infty$.

The result holds for $s > 0$ and for $1 \leq p, q \leq \infty$. In particular, it holds for the Zygmund *spaces (the case $p = q = \infty$):*

$$\mathbf{V}_{\mathbf{G}}^{(\beta)} : \mathbb{Z}^s(\mathcal{S}) \longrightarrow \mathbb{Z}^{s+2\ell-1-\mu+|\beta|,k}(\overline{\Omega^{\pm}})\,. \tag{3.15}$$

The proof is deferred to § 5.3.

Remark 3.3 *If the operator* (1.1) *has constant matrix coefficients $a_\alpha(x) =$* **const** *, the restriction $\mu =$ $\mathbf{ord}\,\mathbf{G} < 2\ell$ in Theorem* 3.2 *turns out superfluous.*

In fact, the potential operators $\mathbf{V}^{(\beta)}_{\partial^k_{y_j}} = \partial^k_{x_j}\mathbf{V}^{(\beta)}$ are well defined even for $k \geq 2\ell + |\beta|$. Moreover, a potential–type operator $\mathbf{G}(x, D_x)\mathbf{V}^{(\beta)}$ (see (3.11) *for $\mathbf{G}(x, D_x)$) is well defined for arbitrary $\mu =$ $\mathbf{ord}\,\mathbf{G} \in \mathbb{N}$ and restricted to the surface $\gamma^{\pm}_{\mathcal{S}}\mathbf{G}(x, D_x)\mathbf{V}^{(\beta)}$ can be interpreted as a pseudodifferential operator of the order $-2\ell + 1 + \mu - |\beta|$ on $\mathcal{S}$, although has a hypersingular kernel when $-2\ell + 1 + \mu - |\beta| > 0$ (see § 6.1).*

Corollary 3.4 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $m = 2\ell$, $k = 0, 1, \ldots, \infty$, $\mathbf{A}(x, D_x)$ be elliptic with a fundamental solution.*

Then any solution $u(x)$ of the system

$$\mathbf{A}(x, D_x)u = f\,, \quad f \in \mathbb{H}^{s-2\ell,k}_p(\overline{\Omega^{\pm}})\,, \tag{3.16}$$

$f \in \mathbb{B}^{s-2\ell,k}_{p,q}(\overline{\Omega^{\pm}})$ (or $f \in \mathbb{Z}^{s-2\ell,k}(\overline{\Omega^{\pm}})$ with $s - 2\ell > 0$) satisfies a priori estimates

$$\|u|\mathbb{H}^{s,k}_{p,loc}(\overline{\Omega^{\pm}})\| \leq M\left[\|f|\mathbb{H}^{s-2\ell,k}_p(\overline{\Omega^{\pm}})\| + \sum_{j=0}^{2\ell-1}\|\gamma^{\pm}_{\mathcal{S}}\partial^j_{\vec{\nu}}u|\mathbb{B}^{s-\frac{1}{p}-j,k}_{p,p,loc}(\mathcal{S})\|\right],$$

$$\|u|\mathbb{B}^{s,k}_{p,q,loc}(\overline{\Omega^{\pm}})\| \leq M\left[\|f|\mathbb{B}^{s-2\ell,k}_{p,q,loc}(\overline{\Omega^{\pm}}\| + \sum_{j=0}^{2\ell-1}\|\gamma^{\pm}_{\mathcal{S}}\partial^j_{\vec{\nu}}u|\mathbb{B}^{s-\frac{1}{p}-j,k}_{p,q}(\mathcal{S})\|\right] \tag{3.17}$$

$$\left(\|u|\mathbb{Z}^{s,k}(\overline{\Omega^{\pm}})\| \leq M\left[\|f|\mathbb{Z}^{s-2\ell,k}(\overline{\Omega^{\pm}})\| + \sum_{j=0}^{2\ell-1}\|\gamma^{\pm}_{\mathcal{S}}\partial^j_{\vec{\nu}}u|\mathbb{Z}^{s-j,k}(\mathcal{S})\|\right]\right).$$

Proof follows from Theorem 3.2 and the representation formula (3.3). ■

Remark 3.5 *If the operator* $\mathbf{A}(x, D_x)$ *is hypoelliptic and has not a fundamental solution, parametrix* $\mathbf{R_A}(x, D_x)$ *can be applied (see § 4.1 below). Namely, inserting the truncated* SCHWARTZ *kernel of the parametrix into the* GREEN *formulae similarly to* (3.3) *we get the following representation for the solution of BVP* (1.14)*:*

$$\chi_{\Omega^\pm}(x)u(x) = \mathbf{N}_{\Omega^\pm}f(x) \pm \sum_{j=0}^{2\ell-1} \mathbf{V}_j\gamma_{\mathcal{S}}^\pm \mathbf{B}_j u(x) + \mathbf{T}u(x)\,, \quad x \in \Omega^\pm\,, \tag{3.18}$$

where the operator $\mathbf{T}$ *has order* $-\infty$*. From Theorem* 3.2 *and the representation formula* (3.18) *we get the following a priori estimate*

$$\|u|\mathbb{H}^{s,k}_{p,loc}(\overline{\Omega^\pm})\| \leq M\Bigg[\|f|\mathbb{H}^{s-2\ell,k}_p(\overline{\Omega^\pm})\| + \sum_{j=0}^{2\ell-1}\|\gamma_{\mathcal{S}}^\pm\partial^j_{\vec{\nu}}u|\mathbb{B}^{s-\frac{1}{p}-j,k}_{p,p,loc}(\mathcal{S})\| + \|u|\mathbb{H}^{s-m,k}_{p,loc}(\overline{\Omega^\pm})\|\Bigg] \tag{3.19}$$

for arbitrary $m = 1, 2, \ldots$ *(cf.* (3.17)*), which holds for the space* $\mathbb{B}^{s-2\ell,k}_{p,q}(\overline{\Omega^\pm})$ *and for* $\mathbb{Z}^{s-2\ell,k}(\overline{\Omega^\pm})$ *(with* $s - 2\ell > 0$*) as well.*

Remark 3.6 *A priory estimates proved, e.g. in [LM1, Ch.2,§ 4], are different. In contrast to* (3.19) *they contain twice less traces* $\gamma_{\mathcal{S}}^\pm\partial^{m_j}_{\vec{\nu}}u$*,* $j = 0, \ldots, m-1$ *in the right–hand side. They are applied in [LM1, Ch.2,§ 5] to establish the* FREDHOLM *property of BVP* (1.14) *provided the* SHAPIRO–LOPATINSKII *conditions hold. For this purpose we will apply the potential method (see a forthcoming paper).*

4 Auxiliary propositions

4.1. On pseudodifferential operators. If the convolution operator in (2.2) admits the continuous extension

$$W^0_a : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n),$$

we write $a \in M_p(\mathbb{R}^n)$ and call $a(\xi)$ a (FOURIER) L_p–multiplier. Let

$$M^{(\nu)}_p(\mathbb{R}^n) = \{\langle\xi\rangle^\nu a(\xi) : a \in M_p(\mathbb{R}^n)\}\,, \quad \nu \in \mathbb{R}\,,$$

where $\langle\xi\rangle$ is defined in (2.3). It is easy to observe, that the operator

$$W_a^0 : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-\nu}(\mathbb{R}^n)$$

is continuous if and only if $a \in M_p^{(\nu)}(\mathbb{R}^n)$ (cf. e.g. [DS1, CD1]).

If $\partial_{\xi_n}^k a \in M_p^{(\nu-k)}(\mathbb{R}^n)$ for all $k = 0, \dots, m$, then W_a^0 is continuous between weighted spaces

$$W_a^0 : \mathbb{H}_p^{s,m}(\mathbb{R}^n) \to \mathbb{H}_p^{s-\nu,m}(\mathbb{R}^n)$$

(see [CD1, Theorem 1.6]).

As an example we consider the Bessel potential operators

$$\begin{aligned} W_{\langle\xi\rangle^r} = \langle D\rangle^r \quad &: \quad \mathbb{X}_p^{s,m}(\mathbb{R}^n) \longrightarrow \mathbb{X}_p^{s-r,m}(\mathbb{R}^n)\,, \\ W_{(\xi_n - i\langle\xi'\rangle)^r} = r_+(D_n - i\langle D'\rangle)^r \ell \quad &: \quad \mathbb{X}_p^{s,m}(\mathbb{R}_+^n) \longrightarrow \mathbb{X}_p^{s-r,m}(\mathbb{R}_+^n)\,, \qquad (4.1) \\ W_{(\xi_n + i\langle\xi'\rangle)^r} = (D_n + i\langle D'\rangle)^r \quad &: \quad \widetilde{\mathbb{X}}_p^{s,m}(\mathbb{R}_+^n) \longrightarrow \widetilde{\mathbb{X}}_p^{s-r,m}(\mathbb{R}_+^n)\,, r \in \mathbb{R} \end{aligned}$$

(cf. (2.12)), where r_+ is the restriction operator (from $\mathbb{R}^n$ to $\mathbb{R}_+^n$), while ℓ is arbitrary extension of a function $\varphi \in \mathbb{X}_p^{s,m}(\mathbb{R}_+^n)$ to $\ell\varphi \in \mathbb{X}_p^{s,m}(\mathbb{R})$ (a right inverse to r_+) ; although extensions can be chosen differently, applying the restriction the final result is independent of a choice. In fact, $r_+(D_n - i\langle D'\rangle)^r r_-\varphi = 0$ due to the Paley–Wiener theorem on the Fourier transforms of functions supported on half spaces.

Operators in (4.1) are isomorphisms for arbitrary $r \in \mathbb{R}$ and the inverse isomorphisms are $\langle D\rangle^{-r}$ and $(D_n \pm i\langle D'\rangle)^{-r}$ (see e.g. [CD1, §1.3]).

The next theorem is a slight modification of the Mikhlin–Hörmander–Lizorkin multiplier theorem. The proof can be found in [Hr2, Theorem 7.9.5] and [Sr1].

Theorem 4.1 *If the inequality*

$$|\xi^\beta \partial^\beta a(\xi)| \le M\langle\xi\rangle^\nu, \quad \xi \in \mathbb{R}^n, \quad |\beta| \le \left[\frac{n}{2}\right] + 1, \quad \beta \le 1\,,$$

holds for some $M > 0$, then $a \in \bigcap_{1<p<\infty} M_p^{(\nu)}(\mathbb{R}^n)$. ■

Let $a \in M_p^{(\nu)}(\mathbb{R}^n)$. Then the operator

$$W_a := r_+\mathbf{a}(D) : \widetilde{\mathbb{X}}_p^s(\mathbb{R}_+^n) \to \mathbb{X}_p^{s-\nu}(\mathbb{R}_+^n)$$

is continuous.

If a symbol $a(x, \xi)$ depends on the variable x, $a \in C(\mathbb{R}^n, \mathbb{S}'(\mathbb{R}^n))$, the corresponding convolution operator (see (2.2))

$$\mathbf{a}(x, D)\varphi(x) = W^0_{a(x,\cdot)}\varphi(x) := \left(\mathcal{F}^{-1}_{\xi\to x} a(x, \xi)\mathcal{F}_{y\to\xi}\varphi(y)\right)(x), \qquad \varphi \in \mathbb{S}(\mathbb{R}^n) \tag{4.2}$$

is called **pseudodifferential**. Here $C(\Omega, \mathcal{B})$ denotes the set of all continuous functions $a : \Omega \to \mathcal{B}$. Let $M_p^{(s,s-\nu)}(\mathbb{R}^n, \mathbb{R}^n)$ denote the class of symbols $a(x, \xi)$ for which the operator in (4.2) extends to a continuous mapping

$$\mathbf{a}(x, D) : \mathbb{H}^s_p(\mathbb{R}^n) \to \mathbb{H}^{s-\nu}_p(\mathbb{R}^n)$$

and $M_p^{(\nu)}(\mathbb{R}^n, \mathbb{R}^n) := \bigcup\limits_{s\in\mathbb{R}} M_p^{(s,s-\nu)}(\mathbb{R}^n, \mathbb{R}^n)$.

Theorem 4.2 *Let $\mathbb{N}_0 := \{0, 1, \ldots\}$. If the estimates*

$$\int\limits_{\mathbb{R}^n} |\xi^\beta \partial^\alpha_x \partial^\beta_\xi a(x, \xi)| dx \leq M_\alpha \langle\xi\rangle^\nu, \quad \xi \in \mathbb{R}^n \tag{4.3}$$

hold for some $M_\alpha > 0$ and all $\alpha, \beta \in \mathbb{N}^n_0$, $|\beta| \leq \left[\frac{n}{2}\right] + 1$, $\beta \leq 1$, then $a \in \bigcap\limits_{1<p<\infty} M_p^{(\nu)}(\mathbb{R}^n, \mathbb{R}^n)$.

Moreover, if (4.3) holds for all $\beta_n = 0, 1, \ldots$ and $|\beta'| \leq \left[\frac{n}{2}\right] + 1$, the PsDO

$$\mathbf{a}(x, D) : \mathbb{H}^{s,m}_p(\mathbb{R}^n) \to \mathbb{H}^{s-\nu,m}_p(\mathbb{R}^n)$$

extends continuously for arbitrary $m \in \mathbb{N}_0$.

Proof. The first part is proved in [Sh2, Theorems 4.1 and 5.1], while the second part–in [CD1, Theorems 1.6]. ∎

If the estimates

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha,\beta,K} \langle\xi\rangle^{\nu-|\beta|}, \quad \nu \in \mathbb{R}, \quad x \in K, \quad \xi \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{N}^n_0$$

hold for all compact $K \subset \overline{\Omega^\pm}$, we write $a \in \mathbb{S}^\nu(\Omega^\pm, \mathbb{R}^n)$ and call $\mathbb{S}^\nu(\Omega^\pm, \mathbb{R}^n)$ the Hörmander class. The operator

$$r_{\Omega^\pm}\mathbf{a}(x, D_x) : \widetilde{\mathbb{X}}^{s,m}_{p,com}(\overline{\Omega^\pm}) \longrightarrow \mathbb{X}^{s-\nu,m}_{p,loc}(\overline{\Omega^\pm}), \quad s \in \mathbb{R}, \ 1 < p < \infty, \tag{4.4}$$

(see (2.12)), where r_Ω is the restriction to $\Omega \subset \mathbb{R}^n$ and $a \in \mathbb{S}^\nu(\Omega^\pm, \mathbb{R}^n)$, is continuous.

The matrix–symbol $\mathcal{A}(x,\xi)$ (and the corresponding operator $\mathbf{A}(x,D_x)$) is called hypoelliptic $\mathcal{A}\in \mathbb{HS}^{\nu,\nu_0}(\Omega^\pm,\mathbb{R}^n)=\mathbb{HS}^{\nu,\nu_0}_{1,0}(\Omega^\pm,\mathbb{R}^n)$ if the following holds:

$$\begin{aligned} &a)\ \ C_{1,K}\,|\xi|^{\nu_0}\le|\sigma(x,\xi)|\le C_{2,K}\,|\xi|^{\nu}\ , \quad x\in K\,,\\ &b)\ \ \left|\left[\partial_\xi^\alpha\partial_x^\beta\sigma(x,\xi)\right]\sigma^{-1}(x,\xi)\right|\le C_{\alpha,\beta,K}|\xi|^{-|\alpha|}\,,\ \ \xi\in\mathbb{R}^n\end{aligned}$$

for all multiindices $\alpha,\beta\in\mathbb{N}_0^n$ and all compact $K\subset\overline{\Omega^\pm}$ (see [Hr1, § 4.1], [Sb1, § 5]). If hypoelliptic, $\mathbf{A}(x,D_x)$ has the parametrix

$$\mathbf{R}_\mathbf{A}(x,D_x)\mathbf{A}(x,D_x)=I-\mathbf{T}_1(x,D_x)\,,\quad \mathbf{A}(x,D_x)\mathbf{R}_\mathbf{A}(x,D_x)=I-\mathbf{T}_2(x,D_x)\,,$$

where the PsDOs $\mathbf{T}_1(x,D_x)$ and $\mathbf{T}_2(x,D_x)$ have order $-\infty$, i.e. are continuous from $\mathbb{X}^s_{p,com}(\overline{\Omega^\pm})$ into $C^\infty(\Omega^\pm)$.

In [Hr2, § 7], [Sb1, § 5] symbol of the parametrices are written explicitely, expecially for classical PsDOs (see [Sb1, § 5.5]). We remind only that the principal homogeneous symbol of the parametrix coincides with the inverse to the principal homogeneous symbol of the operator $(\mathcal{R}_\mathbf{A})_{pr}(x,\xi)=\mathcal{A}^{-1}_{pr}(x,\xi)=\mathcal{A}^{-1}_0(x,\xi)$.

Corollary 4.3 *Let $\mathbf{A}(x,D_x)$ be hypoelliptic $\mathcal{A}\in\mathbb{HS}^{\nu,\nu_0}(\mathbb{R}^n,\mathbb{R}^n)$ with a fundamental solution.*

The generalized fundamental solution

$$\mathbf{F}^{(\beta)}_\mathbf{A}u(x):=\int\limits_{\mathbb{R}^n}\mathcal{K}^{(\beta)}_\mathbf{A}(x,y)u(y)dy$$

(*cf.* (3.7)) *is continuous*

$$\mathbf{F}^{(\beta)}_\mathbf{A}:\mathbb{X}^s_{p,com}(\mathbb{R}^n)\longrightarrow\mathbb{X}^{s+2\ell+|\beta|}_{p,loc}(\mathbb{R}^n)\tag{4.5}$$

(*see* (2.12)) *provided*

$$\beta\in\mathbb{N}_0^n,\quad \mu,s\in\mathbb{R},\quad m\in\mathbb{N}_0,\quad 1<p<\infty.$$

Proof. The symbol of PsDO $\mathbf{F}^{(\beta)}_\mathbf{A}(D)$ reads as

$$\mathcal{F}^{(\beta)}_\mathbf{A}(x,\xi)=(-i\partial_\xi)^\beta\mathcal{R}_\mathbf{A}(x,\xi)$$

where $\mathcal{R}_\mathbf{A}(x,\xi)$ is the symbol of a parametrix $\mathbf{R}_\mathbf{A}(x,D_x)$ of the hypoelliptic operator $\mathbf{A}(x,D_x)$ and $\mathcal{R}_\mathbf{A}\in\mathbb{S}^{-\nu}(\mathbb{R}^n,\mathbb{R}^n)$ (see [Sb1, § 5.5]). Therefore $\mathcal{F}^{(\beta)}_\mathbf{A}\in\mathbb{S}^{-\nu-|\beta|}(\mathbb{R}^n,\mathbb{R}^n)$ and continuity (4.6) follows from Theorem 4.2. ∎

Remark 4.4 *The generalized volume potentials*

$$\mathbf{N}^{(\beta)}_{\Omega^\pm} u(x) := \int\limits_{\Omega^\pm} \mathcal{K}^{(\beta)}_{\mathbf{A}}(x,y)u(y)dy \tag{4.6}$$

(cf. (3.6)*) are continuous, as usual PsDOs of the order* $-2\ell - |\beta|$*, between spaces*

$$\mathbf{N}^{(\beta)}_{\Omega^\pm} : \widetilde{\mathbb{X}}^{s,m}_{p,com}(\overline{\Omega^\pm}) \longrightarrow \mathbb{X}^{s+2\ell+|\beta|,m}_{p,loc}(\overline{\Omega^\pm}).$$

Since the symbol of these operators are rational functions, they possess the transmission proiperty and are continuous between spaces

$$\mathbf{N}^{(\beta)}_{\Omega^\pm} : \mathbb{X}^{s,m}_{p,com}(\overline{\Omega^\pm}) \longrightarrow \mathbb{X}^{s+2\ell+|\beta|,m}_{p,loc}(\overline{\Omega^\pm})$$

(see [BS1, Bo1, GH1, RS1] for details).

Lemma 4.5 *Let* $\mathcal{S} = \partial\Omega^+$ *be* C^∞*–smooth and*

$$a(x,\xi) = a_\nu(x,\xi) + a_{\nu-1}(x,\xi) + \cdots + a_{\nu-k}(x,\xi) + \cdots,$$

$$a_{\nu-k}(x,\lambda\xi) = \lambda^{\nu-k} a_{\nu-k}(x,\xi) \quad x \in \Omega^\pm, \quad \xi \in \mathbb{R}^n, \quad \lambda > 0$$

be a classical $N \times N$ *matrix–symbol* $a \in \mathcal{S}^\nu(\Omega^\pm, \mathbb{R}^n)$ *with* $\nu \leq -1$*. Let* $\mathcal{K}_{\mathbf{a}}(x,y)$ *be the* Scwartz *kernel of the corresponding PsDO* $\mathbf{a}(x,D)$ *and*

$$\mathbf{V}_{\mathbf{a}}\varphi(x) := \oint\limits_{\mathcal{S}} \mathcal{K}_{\mathbf{a}}(x,\tau)\varphi(\tau)d_\tau\mathcal{S}, \quad x \in \Omega^\pm \tag{4.7}$$

be the corresponding potential–type operator, i.e. restriction of the domain of definition of PsDO $\mathbf{a}(x,D)$ *to the boundary* $\mathcal{S} = \partial\Omega^\pm$*.*

If $\nu < -1$ *the traces*

$$\gamma^\pm_{\mathcal{S}} \mathbf{V}_{\mathbf{a}}\varphi(t) = \mathbf{a}_{\mathcal{S}}(t,D)\varphi(t) = \int\limits_{\mathcal{S}} \mathcal{K}_{\mathbf{a}}(t,\tau)\varphi(\tau)d_\tau\mathcal{S}, \quad t \in \mathcal{S}$$

from the domains Ω^+ *and* Ω^- *coincide with the direct value of the potential–type operator* (4.7) *(i.e. with the full restriction of PsDO* $\mathbf{a}(x,D)$ *to* $\mathcal{S}$*) and represent a pseudodifferential operator*

$$\mathbf{a}_{\mathcal{S}}(t,D) : \widetilde{\mathbb{H}}^{s,m}_p(\mathcal{S}) \to \mathbb{H}^{s-\nu-1,m}_p(\mathcal{S}). \tag{4.8}$$

with the full classical symbol

$$a_{\mathcal{S}}(t,\xi') = \sum_{k=0}^{\infty} a_{\mathcal{S},\nu+1-k}(t,\xi'), \quad a_{\mathcal{S},\nu+1-k} \in \mathbb{S}^{\nu+1-k}(\mathcal{S}, \mathbb{R}^n)$$

and the principal symbol

$$a_{\mathcal{S},pr}(\varkappa_j(x),\xi) := a_{\mathcal{S},\nu+1}(\varkappa_j(x),\xi') \\ = \frac{\mathcal{G}_{\varkappa_j}(x)}{2\pi\,\mathbf{det}\,\mathcal{J}_{\varkappa_j}(0,x)}\int\limits_{-\infty}^{\infty} a_\nu\left(\varkappa_j(x), \mathcal{J}_{\varkappa_j}^{-1}(0,x)^\top(\xi',\lambda)\right)d\lambda\,, \quad x\in U_j\,.$$

Here $\mathcal{J}_{\varkappa_j}(t)$ *denotes the* JACOBIAN *and*

$$\mathcal{G}_{\varkappa_j} := (\,\mathbf{det}\,\|(\partial_k\varkappa_j,\partial_l\varkappa_j)\|_{(n-1)\times(n-1)})^{\frac{1}{2}} \ \textit{with}\ \partial_k\varkappa_j := (\partial_k\varkappa_{j1},\dots,\partial_k\varkappa_{jn})^\top$$

denotes the square root of the GRAM *determinant of the local (coordinate) diffeomorphisms* $\varkappa_j : U_j \to V_j$, $j=1,2,\dots,N$ *of* $U_j\subset\mathbb{R}^{n-1}$ *to* $V_j\subset\mathcal{S}$.

If $\nu=-1$ *the direct value of the potential–type operator* (4.7)

$$\mathbf{a}_{\mathcal{S}}(t,D_t)\varphi(t) := \oint\limits_{\mathcal{S}} \mathcal{K}_{\mathbf{a}}(t,\tau)\varphi(\tau)d_\tau\mathcal{S}\,, \quad t\in\mathcal{S}$$

on $\mathcal{S}$ *is a* CALDERÓN–ZYGMUND *singular integral operator (i.e. is a PsDO of the order 0); the integral is understood in the* CAUCHY *principal value sence (cf.* (6.30) *below). The traces* $\gamma_{\mathcal{S}}^{\pm}\mathbf{V}_{\mathbf{a}}$ *and the direct value* $\mathbf{a}_{\mathcal{S}}(t,D_t)$ *are related as follows*

$$\gamma_{\mathcal{S}}^{\pm}\mathbf{V}_{\mathbf{a}}\varphi(t) := \pm\frac{1}{2}ia_{\rm pr}(t,\vec{\nu}(t))\varphi(t) + \mathbf{a}_{\mathcal{S}}(t,D_t)\varphi(t)\,, \quad t\in\mathcal{S}\,, \tag{4.9}$$

where $\vec{\nu}(t)$ *is the outer unit normal vector at* $t\in\mathcal{S}$ *and* $a_{\rm pr}(t,\xi)$, $\xi\in\mathbb{R}^n$, *denotes the homogeneous principal symbol of* $\mathbf{a}(t,D)$.

Proof, including a detailed description of the lower order terms of asymptotic expansion of the symbol of PsDO on the manifold $\mathcal{S}$ can be found in [CD1, § 1.4, Example 2] with two differences. First the proof in [CD1] is carried out for pure convolution operators with symbols $a(\xi)$ but can be extended to the case of PsDOs with classical symbols $a(x,\xi)$ with minor modifications. Second for the coefficient in (4.9) there was quoted different formula from [Es1, (3.26)].

Different proof of (4.9), including the formula for coefficient, can be found in [MT1, Appendix C]. ∎

4.2. On traces of functions. Let us recall the following theorem on traces, which will be generalized later in Theorem 6.4 for weighted spaces.

Theorem 4.6 *The trace operator*

$$\mathcal{R}_k u := \{\gamma_{\mathcal{S}}^{\pm} u, \gamma_{\mathcal{S}}^{1} u, \ldots, \gamma_{\mathcal{S}}^{k} u\}, \quad \gamma_{\mathcal{S}}^{j} u := \gamma_{\mathcal{S}}^{\pm} \partial_{\vec{\nu}}^{j} u, \quad u \in C_0^{\infty}(\overline{\Omega^{\pm}}) \tag{4.10}$$

is a retraction

$$\mathcal{R}_k \ : \ \mathbb{H}_{p,loc}^{s}(\overline{\Omega^{\pm}}) \longrightarrow \bigotimes_{j=0}^{k} \mathbb{B}_{p,p}^{s-\frac{1}{p}-j}(\mathcal{S}), \quad 1 < p < \infty,$$

$$: \ \mathbb{B}_{p,q,loc}^{s}(\overline{\Omega^{\pm}}) \longrightarrow \bigotimes_{j=0}^{k} \mathbb{B}_{p,q}^{s-\frac{1}{p}-j}(\mathcal{S}), \quad 1 \le p, q \le \infty,$$

provided $m \in \mathbb{N}_0$, $k < s - 1/p$, *i.e. is continuous and has a continuous inverse from the right (a coretraction)*

$$\begin{aligned} \mathcal{R}_k^{-1} \ &: \ \bigotimes_{j=0}^{k} \mathbb{B}_{p,p}^{s-\frac{1}{p}-j}(\mathcal{S}) \longrightarrow \mathbb{H}_{p,loc}^{s}(\overline{\Omega^{\pm}}), \\ &: \ \bigotimes_{j=0}^{k} \mathbb{B}_{p,q}^{s-\frac{1}{p}-j}(\mathcal{S}) \longrightarrow \mathbb{B}_{p,q,loc}^{s}(\overline{\Omega^{\pm}}). \end{aligned} \tag{4.11}$$

Proof see in [Tr1, § 2.7.2] ■

The next lemma generalizes [LM1, Ch2, Lemma 2.1], proved there only for the scalar case (see also [RS2, (11)].

Lemma 4.7 *Let*

$$\begin{aligned} \vec{\mathbf{Q}}^{(m)}(x, D_x) &:= \{\mathbf{Q}_0(x, D_x), \ldots, \mathbf{Q}_{m-1}(x, D_x)\}^{\top}, \\ \vec{\mathbf{G}}^{(m)}(x, D_x) &:= \{\mathbf{G}_0(x, D_x), \ldots, \mathbf{G}_{m-1}(x, D_x)\}^{\top} \end{aligned} \tag{4.12}$$

be two Dirichlet *systems on* $\mathcal{S}$. *Then*

$$\vec{\mathbf{Q}}^{(m)}(x, D_x) = \mathbf{Q}_{\mathbf{G}}^{(m\times m)}(x, \mathcal{D}_x)\vec{\mathbf{G}}^{(m)}(x, D_x), \tag{4.13}$$

where $\mathbf{Q}_{\mathbf{G}}^{(m\times m)}(x, \mathcal{D}_x)$ *is the admissible matrix and thus invertible (see* (4.15), (4.16), (4.18) *below).*

Proof. Next representaions are similar to (1.28), (1.33):

$$\mathbf{Q}_j(x, D_x) = \vec{\mathbf{Q}}_j^{(j+1)}(x, \mathcal{D}_x) \cdot \vec{\mathbf{D}}^{(j+1)}(x, D_x),$$

$$\mathbf{G}_j(x, D_x) = \vec{\mathbf{G}}_j^{(j+1)}(x, \mathcal{D}_x) \cdot \vec{\mathbf{D}}^{(j+1)}(x, D_x), \quad j = 0, \ldots, m-1, \tag{4.14}$$

where

$$\vec{\mathbf{Q}}_j^{(j+1)}(x,\mathcal{D}_x) := \{\mathbf{Q}_{j,j}(x,\mathcal{D}_x),\dots,\mathbf{Q}_{j,1}(x,\mathcal{D}_x),\mathcal{Q}_j(x,\vec{\nu}(x))\}^\top ,$$

$$\vec{\mathbf{G}}_j^{(j+1)}(x,\mathcal{D}_x) := \{\mathbf{G}_{j,j}(x,\mathcal{D}_x),\dots,\mathbf{G}_{j,1}(x,\mathcal{D}_x),\mathcal{G}_j(x,\vec{\nu}(x))\}^\top .$$

Then the lower–triangular matrix–operators

$$\mathbf{q}^{(m\times m)}(x,\mathcal{D}_x) = \begin{bmatrix} \mathcal{Q}_{0,0}(x,\vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{Q}_{1,0}(x,\mathcal{D}_x) & \mathcal{Q}_{1,0}(x,\vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_{m-1,0}(x,\mathcal{D}_x) & \mathbf{Q}_{m-1,1}(x,\mathcal{D}_x) & \cdots & \mathcal{Q}_{m-1,0}(x.\vec{\nu}(x)) \end{bmatrix},$$

$$\mathbf{det}\, \mathcal{Q}_{\mathrm{j}}(\mathrm{x}) \neq 0\,, \quad \mathrm{t} \in \mathcal{S}\,, \tag{4.15}$$

$$\mathbf{g}^{(m\times m)}(x,\mathcal{D}_x) = \begin{bmatrix} \mathcal{G}_{0,0}(x,\vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{G}_{1,0}(x,\mathcal{D}_x) & \mathcal{G}_{1,0}(x,\vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{G}_{m-1,0}(x,\mathcal{D}_x) & \mathbf{G}_{m-1,1}(x,\mathcal{D}_x) & \cdots & \mathcal{G}_{m-1,0}(x.\vec{\nu}(x)) \end{bmatrix},$$

$$\mathbf{det}\, \mathcal{G}_{\mathrm{j}}(\mathrm{x}) \neq 0\,, \quad \mathrm{t} \in \mathcal{S}\,, \quad \mathrm{j} = 0,\dots,\mathrm{m}-1 \tag{4.16}$$

are admissible (see (1.34), (1.36)) and

$$\vec{\mathbf{Q}}^{(m)}(x,D_x) = \mathbf{q}^{(m\times m)}(x,\mathcal{D}_x)\vec{\mathbf{D}}^{(m)}(x,D_x)\,,$$

$$\vec{\mathbf{G}}^{(m)}(x,D_x) = \mathbf{g}^{(m\times m)}(x,\mathcal{D}_x)\vec{\mathbf{D}}^{(m)}(x,D_x)\,. \tag{4.17}$$

From (4.15)–(4.17) we get (4.13) with the following admissible matrix–operator

$$\mathbf{Q}_{\mathbf{G}}^{(m\times m)}(x,\mathcal{D}_x) := \mathbf{q}^{(m\times m)}(x,\mathcal{D}_x)\left[\mathbf{g}^{(m\times m)}(x,\mathcal{D}_x)\right]^{-1} \tag{4.18}$$

(cf. (1.35), (1.37)). ∎

Lemma 4.8 *Let $s>0$, $s\notin\mathbb{N}$, $1<p<\infty$, $1\leq p,q\leq\infty$ and $\mathbf{A}(x,D_x)$ in (1.1) be a normal (not necessarily elliptic) operator; let further $\{\mathbf{B}_j(x,D_x)\}_{j=0}^{m-1}$ be a* DIRICHLET *system of the order $m-1$.*

There exists a continuous linear operator

$$\mathcal{P}: \bigotimes_{j=0}^{m-1} \mathbb{B}_{p,p}^{s+m-1-j}(\mathcal{S}) \longrightarrow \mathbb{H}_{p,loc}^{s+m-1+\frac{1}{p}}(\overline{\Omega^\pm}) \tag{4.19}$$

$$\left(\mathcal{P}: \bigotimes_{j=0}^{m-1} \mathbb{B}_{p,q}^{s+m-1-j}(\mathcal{S}) \longrightarrow \mathbb{B}_{p,q,loc}^{s+m-1+\frac{1}{p}}(\overline{\Omega^\pm})\right)$$

such that

$$\gamma^{\pm}_{\mathcal{S}}\mathbf{B}_j\mathcal{P}\Phi = \varphi_j, \quad j = 0, 1, \dots, m-1, \tag{4.20}$$

$$\mathbf{A}\mathcal{P}\Phi \in \widetilde{\mathbb{H}}^{s-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^{\pm}}) \qquad (\mathbf{A}\mathcal{P}\Phi \in \widetilde{\mathbb{B}}^{s-1+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^{\pm}})) \tag{4.21}$$

for arbitrary

$$\Phi = (\varphi_0, \dots, \varphi_{m-1}) \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-1-j}_{p,p}(\mathcal{S}) \quad \left(\Phi \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-1-j}_{p,q}(\mathcal{S})\right).$$

Proof. Let us recall the following property of the space $\widetilde{\mathbb{X}}^s_p(\Omega^{\pm})$:

$$\widetilde{\mathbb{X}}^{\mu}_p(\Omega^{\pm}) = \{u \in \mathbb{X}^{\mu}_p(\Omega^{\pm}) : \mathcal{R}_\ell u = 0\} \tag{4.22}$$

(cf. (2.12)) which holds under the following constraints $\frac{1}{p} + \ell < \mu < \frac{1}{p} + \ell + 1$ (see [Tr1] and [Sh1, Lemma 1.15]).

On behalf of (4.22) condition (4.21) can be reformulated as follows

$$\begin{gathered} \mathcal{R}_k A\mathcal{P}\Phi = \{\gamma^{\pm}_{\mathcal{S}} A\mathcal{P}\Phi, \dots, \gamma^{k}_{\mathcal{S}} A\mathcal{P}\Phi\} = 0, \\ 0 < s - k - \frac{1}{p} < 1, \quad k \in \mathbb{N}_0 \end{gathered} \tag{4.23}$$

(cf. (4.10)). For $0 < s \leq \frac{1}{p}$ condition (4.23) fall away. The operators

$$\mathbf{B}_{m+j}(x, D_x) := \partial^j_{\vec{\nu}(x)}\mathbf{A}(x, D_x), \quad \mathbf{ord}\,\mathbf{B}_{\mathrm{m+j}} = \mathrm{m} + \mathrm{j}, \quad \mathrm{j} = 0, \dots, \mathrm{k}$$

are normal

$$\begin{gathered} \mathcal{B}_{m+j,0}(t, \vec{\nu}(t)) = \left(-i\sum_{s=1}^{n} \nu^2_s(t)\right)^j \mathcal{A}_0(t, \vec{\nu}(t)) = (-i)^j \mathcal{A}_0(t, \vec{\nu}(t)), \\ \mathbf{det}\,\mathcal{B}_{\mathrm{m+j,0}}(\mathrm{t}, \vec{\nu}(\mathrm{t})) \neq 0, \quad \mathrm{t} \in \mathcal{S}, \quad \mathrm{j} = 0, \dots, \mathrm{k} \end{gathered}$$

and combined with the above Dirichlet system $\vec{\mathbf{B}}^{(m)}(x, D_x)$ extends to a new Dirichlet system $\vec{\mathbf{B}}^{(m+k+1)}(x, D_x)$. Then

$$\vec{\mathbf{B}}^{(m+k+1)}(x, D_x) = \mathbf{b}^{((m+k+1)\times(m+k+1))}(x, \mathcal{D}_x)\vec{\mathbf{D}}^{(m+k+1)}(x, D_x), \tag{4.24}$$

and $\mathbf{b}^{((m+k+1)\times(m+k+1))}(x, \mathcal{D}_x)$ is admissible (see (1.34), (1.36)). On defining

$$\Phi_0 := (\varphi_0, \dots, \varphi_{m-1}, \underbrace{0, \dots, 0}_{(k+1)-times}) \in \bigotimes_{j=0}^{m+k} \mathbb{B}^{s+m-1-j}_{p,q}(\mathcal{S}), \tag{4.25}$$

we can match conditions (4.20) and (4.23) (which replaces (4.21)) and reformulate the problem as follows: let us look for a continuous linear operator

$$\mathcal{P}_0 : \bigotimes_{j=0}^{m+k} \mathbb{B}^{s+m-1-j}_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+m+k+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{4.26}$$

$$\left(\mathcal{P}_0 : \bigotimes_{j=0}^{m+k} \mathbb{B}^{s+m-1-j}_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+m+k+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm})\right)$$

such that

$$\gamma^\pm_{\mathcal{S}}\vec{\mathbf{B}}^{(m+k+1)}\mathcal{P}_0\Phi_0 = \mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathcal{D}_x)\gamma^\pm_{\mathcal{S}}\vec{\mathbf{D}}^{(m+k+1)}\mathcal{P}_0\Phi_0$$
$$= \mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathcal{D}_x)\mathcal{R}_{m+k+1}\mathcal{P}\Phi_0 = \Phi_0\,. \tag{4.27}$$

Here we applied that $\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathcal{D}_x)$ is a "tangent" differential operator and $\mathcal{R}_{m+k+1} = \gamma^\pm_{\mathcal{S}}\vec{\mathbf{D}}^{(m+k+1)}$ (cf. (4.10)). Thus,

$$\mathcal{R}_{m+k+1}\mathcal{P}_0\Phi_0 = \left[\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathcal{D}_x)\right]^{-1}\Phi_0$$

and it remains to apply a coretraction (4.3): the function

$$\mathcal{P}_0\Phi_0 = \mathcal{R}^{-1}_{m+k+1}\left[\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathcal{D}_x)\right]^{-1}\Phi_0 \in \mathbb{H}^{s+m+k+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{4.28}$$

(in $\mathbb{B}^{s+m+k+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm})$) solves equation (4.27). ■

Let us consider the following surface δ-function

$$(g\otimes\delta_{\mathcal{S}}, v)_{\mathcal{S}} := \int_{\mathcal{S}} g(\tau)\gamma^\pm_{\mathcal{S}}v(\tau)d_\tau\mathcal{S}, \tag{4.29}$$
$$g\in C^\infty_0(\mathcal{S}), \quad v\in C^\infty_0(\mathbb{R}^n).$$

Obviously, $\mathbf{supp}\,(\mathrm{g}\otimes\delta_{\mathcal{S}}) = \mathbf{supp}\,\mathrm{g}\subset\mathcal{S}$.

Definition (4.29) can be extended to non C^∞-smooth functions. Namely there holds the following.

Lemma 4.9 *Let* $1<p<\infty$ $(1\le q\le\infty)$, $s<0$, $\varphi\in\mathbb{B}^s_{p,p}(\mathcal{S})$ $(\varphi\in\mathbb{B}^s_{p,q}(\mathcal{S}))$. *Then*

$$\varphi\otimes\delta_{\mathcal{S}}\in\widetilde{\mathbb{H}}^{s-\frac{1}{p'},\infty}_{p,com}(\overline{\Omega^\pm}), \quad \left(\varphi\otimes\delta_{\mathcal{S}}\in\widetilde{\mathbb{B}}^{s-\frac{1}{p'},\infty}_{p,q,com}(\overline{\Omega^\pm})\right),$$

where $p' = p/(p-1)$.

Proof. The inclusion $\varphi \otimes \delta_{\mathcal{S}} \in \widetilde{\mathbb{H}}^{s-\frac{1}{p'}}_{p,com}(\overline{\Omega^\pm})$ $(\varphi \otimes \delta_{\mathcal{S}} \in \widetilde{\mathbb{B}}^{s-\frac{1}{p'}}_{p,q,com}(\overline{\Omega^\pm}))$ is based on the duality and proved in [Es1] (L_2–case) and in [DW1, Gr3, Sh1] (L_p–case). For the weighted spaces the result follows because

$$\partial^k_\nu \rho^m \cdot (\varphi \otimes \delta_{\mathcal{S}}) = 0 \quad \text{provided} \quad k < m\,,$$

$$\partial^m_\nu \rho^m \cdot (\varphi \otimes \delta_{\mathcal{S}}) = m!\, \varphi \otimes \delta_{\mathcal{S}} \quad \text{by definition}\,, \tag{4.30}$$

where $\rho = \rho(x) := \operatorname{dist}(x, \mathcal{S}), \quad x \in \Omega^\pm$.

To justify the second definition in (4.30) let $v \in C^\infty(\mathbb{R})$ and define

$$\langle \partial^m \delta, v\rangle := \lim_{t\to 0} \frac{\Delta^m_0 v(t)}{t^m}, \quad \Delta^k_0 = \Delta^{k-1}_0 \Delta_0\,, \quad \Delta_0 v(t) := v(t) - v(0).$$

Then, obviously,

$$\langle \partial^m \delta, t^m v\rangle = m! v(0) = m! \langle \delta, v\rangle\,. \qquad \blacksquare$$

As a direct application of definition (4.29) we can write the generalized layer potential (3.11) as a volume potential

$$\mathbf{V}^{(\beta)}_{\mathbf{G}} \varphi(x) = \int\limits_{\mathbb{R}^n} \left[\overline{\mathbf{G}(y, D_y)} \mathcal{K}^{(\beta)}_{\mathbf{A}^*}(y, x)\right]^\top (\varphi \otimes \delta_{\mathcal{S}})(y) dy$$

$$=: \mathbf{F}^{(\beta)}_{\mathbf{A},\mathbf{G}} (\varphi \otimes \delta_{\mathcal{S}})(x), \qquad x \in \Omega^\pm. \tag{4.31}$$

Representation (4.31) has only one shortcoming: $\varphi \otimes \delta_{\mathcal{S}} \notin \mathbb{X}^s_{p,loc}(\overline{\Omega^\pm})$ for $s > -\dfrac{1}{p'}$ even for $\varphi \in C^\infty(\mathcal{S})$ (i.e. Lemma 4.9 is precise). In fact, locally $\mathcal{S}$ can be interpreted as $\mathbb{R}^{n-1}$ and $\Omega^\pm$–as $\mathbb{R}^n_+$. Then $1 \otimes \delta_{\mathbb{R}^{n-1}} = \delta(x_n) \notin \mathbb{X}^s_{p,loc}(\overline{\mathbb{R}^n_+})$ if $s > -1/p'$ (see [Es1] for $p = 2$ and [Tr1, Tr2] for $1 < p < \infty$).

5 Proofs

5.1. Proof of Theorem 1.6. It suffices to prove the theorem for particular case–for BVP (1.10) and the corresponding Green formula (1.13). First we extend the system $\{\mathbf{b}_j(t, D_t)\}^{\omega-1}_{j=0}$ of "boundary" differential operators up to a Dirichlet system

$$\{\mathbf{B}_j(t, D_t)\}^{m-1}_{j=0} = \mathcal{H}_0 \{\mathbf{b}_j(t, D_t)\}^{mN-1}_{j=0}$$

of the order m (see Lemma 1.4). If the Green formula (1.13) is proved we get

$$\int_{\Omega^\pm}((\mathbf{A}u)^\top\overline{v}-u^\top\overline{\mathbf{A}^*v})dy=\pm\sum_{j=0}^{m-1}\oint_{\mathcal{S}}(\mathbf{B}_ju)^\top\overline{\mathbf{C}_jv}d_\tau\mathcal{S}$$
$$=\pm\sum_{j=0}^{mN-1}\oint_{\mathcal{S}}(\mathbf{b}_ju)^\top\overline{\mathbf{c}_jv}d_\tau\mathcal{S}\,, \tag{5.1}$$

where $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}=\mathcal{H}_0^\top\{\mathbf{C}_j(t,D_t)\}_{j=0}^{m-1}$ is a decomposition in rows.

Thus, we can concentrate on BVP (1.10) and the corresponding Green formula (1.13). Moreover, we suppose that the choice is made and $\vec{\mathbf{B}}^{(m)}(x,D_x)$ (see (1.31)) is the fixed Dirichlet system of the order $m-1$. Without restricting generality we can suppose $\mathbf{ord}\,\mathbf{B}_{\rm j}={\rm j}$, $j=0,\dots,m-1$; otherwise we have just to renumber these operators.

In Theorem 1.10 we have already proved the Green formula

$$\int_{\Omega^\pm}\left[(\mathbf{A}u)^\top\overline{v}-u^\top\overline{\mathbf{A}^*v}\right]dy=\pm\oint_{\mathcal{S}}(\vec{\mathbf{D}}^{(m)}u)^\top\cdot\overline{\vec{\mathbf{G}}^{(m)}v}\;d_\tau\mathcal{S} \tag{5.2}$$

(see (1.32)) with the special operators $\vec{\mathbf{D}}^{(m)}(x,D_x)$ defined in (1.31) and

$$\vec{\mathbf{G}}^{(m)}(x,D_x):=\{\mathbf{G}_0(x,D_x),\dots,\mathbf{G}_{m-1}(x,D_x)\}^\top$$
$$=\left[\left(\vec{\mathbf{D}}^{(m)}\right)^*(x,D_x)\right]^\top\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathcal{D}_x)\mathbb{S}_m\,, \tag{5.3}$$

(see (1.29)) with formally self–adjoint skew identity matrix $\mathbb{S}_m^*=\mathbb{S}_m$ (see (1.40)) and the formally adjoint matrix–operator $\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathcal{D}_x)$ to (1.39). $\vec{\mathbf{B}}^{(m)}=\left\{\partial_{\vec\nu}^j\right\}_{j=0}^{m-1}$ is a Dirichlet systems.

Due to Lemma 4.7

$$\vec{\mathbf{D}}^{(m)}(t,D_t)=\left[\left(\mathbf{b}^{(m\times m)}\right)(t,\mathcal{D}_t)\right]^{-1}\vec{\mathbf{B}}^{(m)}(t,D_t)\,,\quad t\in\mathcal{S} \tag{5.4}$$

(see (1.36)). Inserting (5.4) into (5.2), taking into account (5.3) and applying the partial integration formula (1.25) we get

$$\int_{\Omega^\pm}\left[(\mathbf{A}u)^\top\overline{v}-u^\top\overline{\mathbf{A}^*v}\right]dy=\pm\oint_{\mathcal{S}}(\vec{\mathbf{D}}^{(m)}u)^\top\cdot\overline{\vec{\mathbf{G}}^{(m)}v}\;d_\tau\mathcal{S}$$

$$=\pm\oint_{\mathcal{S}}\left[\left(\mathbf{b}^{(m\times m)}\right)\right]^{-1}\vec{\mathbf{B}}^{(m)}u)^\top\cdot\overline{\left[\left(\vec{\mathbf{D}}^{(m)}\right)^*\right]^\top\left(\mathbf{A}^{(m\times m)}\right)^*\mathbb{S}_mv}\;d_\tau\mathcal{S}$$

$$= \pm \oint_{\mathcal{S}} (\vec{\mathbf{B}}^{(m)} u)^\top \cdot \overline{\vec{\mathbf{C}}^{(m)} v} \, d_\tau \mathcal{S} \,, \tag{5.5}$$

where $\vec{\mathbf{C}}^{(m)}(x, D_x)$ are defined by (1.38) and are unique. Due to this formula the operators $\mathbf{C}_k(t, D_t)$, $k = 0, 1, \ldots, m-1$ are normal iff the matrix $\overline{\mathcal{A}_0^\top(t, \vec{\nu}(t))}$ on the main diagonal of the block–matrix $\left(\mathbf{A}^{(m\times m)}(x, \mathcal{D}_x)\right)^*$ is invertible for all $t \in \mathcal{S}$, i.e. iff the "basic" operator $\mathbf{A}(x, D_x)$ is normal (see Definition 1.1).

If the Dirichlet system $\{\mathbf{C}_j(t, D_t)\}_{j=0}^{m-1}$ is fixed (instead of $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{m-1}$), the proof proceeds similarly with a single difference–instead of $\mathbf{A}(x, D_x)$ the proof starts with the formally adjoint operator $\mathbf{A}^*(x, D_x)$.

Now let us suppose the "basic" operator is normal and the systems $\{\mathbf{b}_j(t, D_t)\}_{j=0}^{k-1}$ and $\{\mathbf{C}_{m-j-1}(t, D_t)\}_{j=0}^{m-k-1}$ are fixed.

If one of them is a Dirichlet system (of order k or $m-k$, respectively,), we extend it up to a Dirichlet system $\{\mathbf{B}_{j,0}(t, D_t)\}_{j=0}^{m-1}$ (or $\{\mathbf{C}_{j,0}(t, D_t)\}_{j=0}^{m-1}$) of order m and write the Green formula (1.13) (see (5.5)). Next we replace the system $\{\mathbf{C}_{m-j-1,0}(t, D_t)\}_{j=0}^{m-k-1}$, $\mathbf{ord}\ \mathbf{C}_{\mathrm{m-j-1,0}} = \mathrm{j}$ (or $\{\mathbf{B}_{j,0}(t, D_t)\}_{j=0}^{k-1}$, $\mathbf{ord}\ \mathbf{B}_{\mathrm{j,0}} = \mathrm{j}$; see (1.14), (1.30)) by the fixed system $\{\mathbf{C}_{m-j-1}(t, D_t)\}_{j=0}^{m-k-1}$ (by $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{k-1}$) with the help of a matrix $\left[c^{((m-k)\times((m-k))}(t, \mathcal{D}_t)\right]^\top$ transposed to an admissible[9] (an admissible matrix $b^{(k\times k)}(t, \mathcal{D}_t)$, respectivelyl; see Lemma 4.7) and leave another part of the system unchanged. Then connection between entire systems has the form

$$\vec{\mathbf{C}}_0^{(m)}(t, D_t) = c^{(m\times m)}(t, \mathcal{D}_t)\vec{\mathbf{C}}^{(m)}(t, D_t)$$

$$(\vec{\mathbf{B}}_0^{(m)}(t, D_t) = b^{(m\times m)}(t, \mathcal{D}_t)\vec{\mathbf{B}}^{(m)}(t, D_t)) \,,$$

where the participating block–matrices are defined as follows

$$c^{(m\times m)}(t, \mathcal{D}_t) = \begin{bmatrix} I_k & 0 \\ 0 & \left[c^{(m-k)\times(m-k))}(t, \mathcal{D}_t)\right]^\top \end{bmatrix}$$

$$\left(b^{(m\times m)}(t, \mathcal{D}_t) = \begin{bmatrix} b^{(k\times k)}(t, \mathcal{D}_t) & 0 \\ 0 & I_{m-k} \end{bmatrix}\right) \,,$$

where I_ℓ denotes the identity matrix of order ell.

[9]It is easy to ascertain that connection between Dirichlet systems with diminishing orders is established by a transposed (and therefore upper triangular) admissible matrix; see Lemma 4.7.

Inserting the obtained representations into the GREEN formula we find

$$\int\limits_{\Omega^\pm} \left[(\mathbf{A}u)^\top \overline{v} - u^\top \overline{\mathbf{A}^* v}\right] dy = \pm \oint\limits_{\mathcal{S}} (\vec{\mathbf{B}}_0^{(m)} u)^\top \cdot \overline{\vec{\mathbf{C}}_0^{(m)} v}\, d_\tau \mathcal{S}$$

$$= \pm \oint\limits_{\mathcal{S}} (\vec{\mathbf{B}}_0^{(m)} u)^\top \cdot \overline{c^{(m\times m)} \vec{\mathbf{C}}^{(m)}}\, d_\tau \mathcal{S} = \pm \oint\limits_{\mathcal{S}} \left(\vec{\mathbf{B}}^{(m)} u\right)^\top \cdot \overline{\vec{\mathbf{C}}^{(m)}}\, d_\tau \mathcal{S}\,.$$

Due to the structure of the connection matrix $c^{(m\times m)}(t, \mathcal{D}_t)$ the first part of the transformed system $\vec{\mathbf{B}}^{(m)} := \left[c^{(m\times m)}\right]^\top \vec{\mathbf{B}}_0^{(m)}$ resists the transformation and coincides with the one $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{k-1}$ fixed at the begining.

Similarly, if the system $\{\mathbf{B}_{j,0}(t, D_t)\}_{j=0}^{k-1}$ is changed, the second part of the transformed system $\vec{\mathbf{C}}^{(m)} := \left[b^{(m\times m)}\right]^\top \vec{\mathbf{C}}_0^{(m)}$ in the GREEN formula resists the transformation and coincides with the second part $\{\mathbf{C}_{m-j-1}(t, D_t)\}_{j=m-k}^{m}$ of the system fixed at the begining.

As for uniqueness of the full systems $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{m-1}$ and $\{\mathbf{C}_j(t, D_t)\}_{j=0}^{m-1}$, although we choose arbitrary extention of one system at the begining, another system is defined uniquely (see the first part of the Theorem) and the operators chosen arbitrarily are replaced by new ones, which are unique; we might have doubts only about second half of the system which are extensions of a fixed DIRICHLET system up to a DIRICHLET system of order m. But uniqueness of this part becomes evident if we reverse the choice of system subject to extension.

Assume $\{\mathbf{b}_j(t, D_t)\}_{j=0}^{kN-1}$ (or $\{\mathbf{c}_{mN-j}(t, D_t)\}_{j=0}^{(m-k)N-1}$) is not a DIRICHLET system. Then (see Definition 1.2):

i. if linear independence of rows is missing the GREEN formula (1.14) can not be valid because by the first part of the Theorem both systems of "boundary" operators must be DIRICHLET systems;

ii. if one or several orders are missing, then the structure of the connection matrix $(c^{(m\times m)}(t, \mathcal{D}_t)$ (of $b^{(m\times m)}(t, \mathcal{D}_t)$, respectively) does not allows to maintain fixed parts of "boundary" systems in the GREEN formula. ■

5.2. Proof of Theorem 1.7. Let us apply representation (1.28);

$$\partial_x^\beta = \mathbf{b}_{\beta,0}(x)\partial_{\vec{\nu}(x)}^{|\beta|} + \sum_{j=1}^{|\beta|} \mathbf{b}_{\beta,|\beta|-j}(x, \mathcal{D}_x)\partial_{\vec{\nu}(x)}^{j}\,, \tag{5.6}$$

$$\mathbf{b}_{\beta,0}(x) = \vec{\nu}^\beta(x) := \nu_1^{\beta_1}(x)\ldots\nu_n^{\beta_n}(x)\,.$$

Inserting (5.6) into (1.16) and applying (1.23), (1.24) we proceed as follows:

$$
\begin{aligned}
\mathcal{A}(u,v) &:= \int\limits_{\Omega^\pm} \sum_{|\alpha|,|\beta|\le\ell} \left[a_{\alpha,\beta}(x)\partial_y^\alpha u(y)\right]^\top \overline{\sum_{j=0}^{|\beta|} \mathbf{b}_{\beta,|\beta|-j}(y,\mathcal{D}_y)\partial^j_{\vec{\nu}(y)} v(y)dy} \\
&= \sum_{|\alpha|,|\beta|\le\ell} \sum_{j=0}^{|\beta|} \int\limits_{\Omega^\pm} \left[\mathbf{b}^*_{\beta,|\beta|-j}(y,\mathcal{D}_y) a_{\alpha,\beta}(x)\partial_y^\alpha u(y)\right]^\top \overline{\partial^j_{\vec{\nu}(y)} v(y)} dy \\
&= \sum_{j=0}^{\ell} \sum_{k=0}^{j-1} \Bigg[\pm \int\limits_{\mathcal{S}} (-1)^k \left[\mathbf{C}_{1,j}(\tau, D_\tau)u(\tau)\right]^\top \overline{\partial^{j-k-1}_{\vec{\nu}(\tau)} v(\tau)} d_\tau\mathcal{S} \\
&\quad - \int\limits_{\Omega^\pm} \left[(-1)^{j-1}\partial^j_{\vec{\nu}(y)} \mathbf{C}^*_{1,j}(y, D_y)u(y)\right]^\top \overline{v(y)}dy \Bigg] \qquad (5.7)\\
&= \sum_{j=0}^{\ell-1} \pm \int\limits_{\mathcal{S}} \left[\mathbf{C}_{2,j}(\tau, D_\tau)u(\tau)\right]^\top \overline{\partial^j_{\vec{\nu}(\tau)} v(\tau)} d_\tau\mathcal{S} - \int\limits_{\Omega^\pm} \left[\mathbf{A}(y, D_y)u(y)\right]^\top \overline{v(y)}dy\,.
\end{aligned}
$$

Thus, we get the GREEN formula (1.17) with special "boundary" operators $\mathbf{C}_j = \mathbf{C}_{2,j}$ and $\mathbf{B}_j = \partial^j_{\vec{\nu}(y)}$ $(j = 0, \dots, \ell-1)$. Now we can apply (5.4) with $m = \ell$ and replace $\{\partial^j_{\vec{\nu}(y)}\}_{j=0}^{\ell-1}$ in (5.7) by another DIRICHLET system $\{\mathbf{B}_j\}_{j=0}^{\ell-1}$ (see (5.5)), which gives us the claimed formula (1.17).

For the system $\{\mathbf{C}_j\}_{j=0}^{\ell-1}$ the formula, similar to (1.38), can be derived. Based on this formulae, similarly to the foregoing case (see § 5.1) can be proved that $\{\mathbf{C}_j\}_{j=0}^{\ell-1}$ is a DIRICHLET system if and only if $\mathbf{A}(x, D_x)$ is normal.

If $\mathbf{A}$ is formally self–adjoint $\mathbf{A} = \mathbf{A}^*$ then $\mathcal{A}(u,v) = \overline{\mathcal{A}(v,u)}$ and from (1.17) written for pairs u,v and v,u we get the simplified GREEN formula (1.18). ∎

5.3. Proof of Theorem 3.2. Due to Theorem 1.6 we can suppose that the GREEN formula (1.13) holds and let $\{\mathbf{C}_j(x, D_x)\}_{j=0}^{2\ell-1}$ be the DIRICHLET system, participating in the formula (1.13). Without restricting generality $\mathbf{ord}\,\mathbf{C}_j = 2\ell - \mathbf{ord}\,\mathbf{B}_j - 1 = 2\ell - j - 1$ (see (1.14)).

Due to Lemma 4.7

$$
\mathbf{G}(x, D_x) = \sum_{j=0}^{\mu} \mathbf{G}_{\mu-j}(x, \mathcal{D}_x)\mathbf{C}_{2\ell-j-1}(x, D_x)\,, \qquad (5.8)
$$

$$
\mathbf{G}_k(x, \mathcal{D}_x) = \sum_{|\alpha|\le k} c^0_{k\alpha}(x)\mathcal{D}_x^\alpha, \quad x \in \Omega^\pm, \quad k = 1, 2, \dots, \mu\,.
$$

Then (see Lemma 1.8)

$$\mathbf{V}^{(\beta)}_{\mathbf{G}}\varphi(x) = \sum_{j=0}^{\mu} \oint_{\mathcal{S}} \left[\overline{\mathbf{G}_{\mu-j}(\tau,\mathcal{D}_\tau)\gamma^{\pm}_{\mathcal{S}}\mathbf{C}_{2\ell-j-1}(\tau,D_\tau)\mathcal{K}^{(\beta)}_{\mathbf{A}^*}(x,\tau)}\right]^{\top} \varphi(\tau)d_\tau\mathcal{S}$$

$$= \sum_{j=0}^{\mu} \oint_{\mathcal{S}} \left[\overline{\gamma^{\pm}_{\mathcal{S}}\mathbf{C}_{2\ell-j-1}(\tau,D_\tau)\mathcal{K}^{(\beta)}_{\mathbf{A}^*}(x,\tau)}\right]^{\top} (\mathbf{G}_{\mu-j})^*_{\mathcal{S}}(\tau,\mathcal{D}_\tau)\varphi(\tau)d_\tau\mathcal{S}$$

and it suffices to prove the theorem for the generalized layer potentials

$$\mathbf{V}^{(\beta)}_{j}\varphi(x) := \oint_{\mathcal{S}} \left[\overline{\gamma^{\pm}_{\mathcal{S}}\mathbf{C}_{j}(\tau,D_\tau)\mathcal{K}^{(\beta)}_{\mathbf{A}^*}(x,\tau)}\right]^{\top} \varphi(\tau)d_\tau\mathcal{S} = \mathbf{F}^{(\beta)}_{\mathbf{A},\mathbf{j}}(\varphi\otimes\delta_{\mathcal{S}})(x)\,,$$

$$\mathbf{F}^{(\beta)}_{\mathbf{A},\mathbf{j}}\psi(x) := \int_{\Omega^{\pm}} \left[\overline{\gamma^{\pm}_{\mathcal{S}}\mathbf{C}_{j}(\tau,D_\tau)\mathcal{K}^{(\beta)}_{\mathbf{A}^*}(x,\tau)}\right]^{\top} \psi(y)dy\,, \qquad x\in\Omega^{\pm} \tag{5.9}$$

(see (3.5), (4.31)). Let us consider the symbol of the PsDO $\mathbf{F}^{(\beta)}_{\mathbf{A},\mathbf{j}}$:

$$\mathcal{F}^{(\beta)}_{\mathbf{A},\mathbf{j}}(x,\xi) = (-i\partial_\xi)^{\beta}\mathcal{F}_{\mathbf{A}}(x,\xi)\left[\overline{\mathcal{C}_j(x,\xi)}\right]^{\top}, \quad \mathcal{F}^{(\beta)}_{\mathbf{A},\mathbf{j}} \in \mathbb{S}^{-2\ell+j-|\beta|}(\mathbb{R}^n,\mathbb{R}^n)$$

(see Corollary 4.3), where $\mathcal{F}_{\mathbf{A}}(x,\xi)$ is the symbol of the fundamental solution of the operator $\mathbf{A}(x,D_x)$.

If $\varphi\in\mathbb{B}^s_{p,p}(\mathcal{S})(\varphi\in\mathbb{B}^s_{p,q}(\mathcal{S}))$ and $s<0$, then

$$\varphi\otimes\delta_{\mathcal{S}} \in \widetilde{\mathbb{H}}^{s-\frac{1}{p'},\infty}_{p,com}(\overline{\Omega^{\pm}}) \quad \left(\varphi\otimes\delta_{\mathcal{S}} \in \widetilde{\mathbb{B}}^{s-\frac{1}{p'},\infty}_{p,q,com}(\overline{\Omega^{\pm}})\right),$$

where $p'=p/(p-1)$. From (5.9) and (4.4) we derive the continuity results (3.13), (3.14).

Next we take $s>0$, $s\notin\mathbb{N}$. Defining the operator

$$\mathcal{P}_j\varphi := \mathcal{P}\Psi_j, \quad \Psi_j := (0,\ldots,0,\varphi,0,\ldots,0)\,,$$

where φ stands at j-th place and $\mathcal{P}$ is from Lemma 4.8 (see (4.12), (4.14)) we get

$$\begin{aligned} \mathcal{P}_j \;:\; & \mathbb{B}^{s+2\ell-m_j-1}_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^{\pm}}), \\ :\; & \mathbb{B}^{s+2\ell-m_j-1}_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+2\ell-1+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^{\pm}}), \end{aligned} \tag{5.10}$$

$$\gamma^{\pm}_{\mathcal{S}}\mathbf{B}_k\mathcal{P}_j = 0, \quad k\neq j\,, \qquad \gamma^{\pm}_{\mathcal{S}}\mathbf{B}_j\mathcal{P}_j = I\,,$$

$$
\begin{aligned}
\mathbf{A}\mathcal{P}_j \quad &: \quad \mathbb{B}^{s+2\ell-m_j-1}_{p,p}(\mathcal{S}) \longrightarrow \widetilde{\mathbb{H}}^{s-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}), \\
&: \quad \mathbb{B}^{s+2\ell-m_j-1}_{p,q}(\mathcal{S}) \longrightarrow \widetilde{\mathbb{B}}^{s-1+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm}).
\end{aligned} \tag{5.11}
$$

Let us consider $v^{(\beta)}_{\varepsilon,x}(y) := \chi_\varepsilon(x-y)\mathcal{K}^{(\beta)}_{\mathbf{A}^*}(y,x)$, where $\mathcal{K}_{\mathbf{A}^*}(x,y)$ is the kernel of the fundamental solution $\mathbf{F}_{\mathbf{A}^*}(x,D_x)$ and $\chi_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\chi_\varepsilon(x)=1$, $\chi_\varepsilon(x)=0$ for $|x|>\varepsilon$ and $|x|<\varepsilon/2$, respectively. Inserting

$$
v(y) = v^{(\beta)}_{\varepsilon,x}(y)\,, \quad u = \mathcal{P}_j\varphi\,, \quad \varphi \in \mathbb{B}^{s+\mu_j}_{p,p}(\mathcal{S}) \quad \big(\varphi \in \mathbb{B}^{s+\mu_j}_{p,q}(\mathcal{S})\big)
$$

into the Green formula (1.13) and sending $\varepsilon \to 0$, similarly to (3.3) we find the following

$$
\begin{aligned}
\pm\mathbf{V}^{(\beta)}_j\varphi(x) &= \chi^{(\beta)}_{\Omega^\pm}(x)\mathcal{P}_j\varphi(x) - \mathbf{N}^{(\beta)}_{\Omega^\pm}\mathbf{A}\mathcal{P}_j\varphi(x) \\
&+ \sum_{\substack{\alpha+\gamma\le 2\ell \\ 0\ne\alpha\le\beta}} \int_{\Omega^\pm} c^1_{\alpha\beta\gamma}(y)(x-y)^{\beta-\alpha}(\partial^\gamma_y\mathcal{K}_{\mathbf{A}})(x,y)c^2_{\alpha\beta\gamma}(y)\mathcal{P}_j\varphi(y)dy
\end{aligned} \tag{5.12}
$$

(see (4.6) for $\mathbf{N}^{(\beta)}_{\Omega^\pm}$), where $c^1_{\alpha\beta\gamma}$, $c^2_{\alpha\beta\gamma} \in C^\infty(\mathbb{R}^n)$, and $\chi^{(\beta)}_\pm(x) = 0$ for $\beta \ne 0$, $\chi^{(0)}_\pm(x) = \chi_\pm(x)$, $x \in \Omega^\pm$ $(j = 0,\dots,2\ell-1)$.

Applying Remark 4.4 and Lemma 4.8 from (5.12) we derive the following continuity

$$
\begin{aligned}
\mathbf{V}^{(\beta)}_j \quad &: \quad \mathbb{B}^{s+2\ell-m_j-1}_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm})\,, \\
&: \quad \mathbb{B}^{s+2\ell-m_j-1}_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm})\,.
\end{aligned} \tag{5.13}
$$

Since $m_j + \mu_j = 2\ell - 1$ (see (1.14)) (5.13) implies the continuity

$$
\begin{aligned}
\mathbf{V}^{(\beta)}_j \quad &: \quad \mathbb{B}^{s}_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm})\,, \\
&: \quad \mathbb{B}^{s}_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm}) \quad j = 0,\dots,2\ell-1\,,
\end{aligned} \tag{5.14}
$$

provided

$$
s > 2\ell - \mu^0 - 1, \quad s \ne 2\ell - m_j + k, \quad k \in \mathbb{N}, \qquad \mu^0 := \min\{\mu_0,\dots,\mu_{2\ell-1}\}. \tag{5.15}
$$

(5.14), in its turn, implies the continuity

$$\begin{aligned} \mathbf{V}_j^{(\beta)} &: \mathbb{B}^s_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm}), \\ &: \mathbb{B}^s_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm}), \end{aligned} \tag{5.16}$$

because $\beta_n = 0, 1, \cdots$ is arbitrary integer in (5.14) and $\rho^k \mathbf{V}_j^{(\beta)} = \mathbf{V}_j^{(\beta', \beta_n + k)}$ in local coordinates, in which $\rho^k(x) := [\,\mathrm{dist}\,(x, \mathcal{S})]^k = x_n^k$.

Continuity (3.13) and (3.14) for the cases $s < 0$ and (5.15) is proved. The missing cases are filled in with the interpolation (2.9)–(2.11). ■

6 Consequences and related results

6.1. Traces of generalized potentials on the boundary.

Let $\mathbf{A}(x, D_x)$ in (1.1) be an elliptic differential operator with even order $m = 2\ell$ and with a fundamental solution $\mathbf{F_A} = \mathbf{F_A}(x, D)$. $\mathcal{K}_\mathbf{A}(x, y)$ will denote the corresponding SCHWARTZ kernel.

Let us consider a **Potential–type** operator

$$\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x, D_x) := \mathbf{B}(x, D_x)\mathbf{V}^{(\beta)}\mathbf{C}(t, \mathcal{D}_t), \quad x \in \Omega^\pm, \quad t \in \mathcal{S} = \partial\Omega^+ \tag{6.1}$$

where $\mathbf{V}^{(\beta)}$, $\beta \in \mathbb{N}^n_0$ is a generalized single layer potential

$$\mathbf{V}^{(\beta)}\psi(x) := \oint_{\mathcal{S}} \mathcal{K}^{(\beta)}_\mathbf{A}(x, \tau)\varphi(\tau) d_\tau\mathcal{S} \tag{6.2}$$

(cf. (3.12), (3.11)) and

$$\begin{aligned} \mathbf{B}(x, D_x) &= \sum_{|\alpha|\le m} b_\alpha(x)\partial^\alpha_x, \quad b_\alpha \in C^\infty(\Omega^\pm, \mathbb{C}^{N\times N}), \quad x \in \Omega^\pm, \\ \mathbf{C}(t, \mathcal{D}_t) &= \sum_{|\alpha|\le \mu} c_\alpha(t)\mathcal{D}^\alpha_t, \quad c_\alpha \in C^\infty(\mathcal{S}, \mathbb{C}^{N\times N}), \quad t \in \mathcal{S} \end{aligned} \tag{6.3}$$

are some differential operators of orders $m, \mu = 0, 1, \cdots$. $\mathbf{C}(t, \mathcal{D}_t)$ is a tangent differential operator and can be restricted to the boundary $\mathcal{S}$ (see (1.20)–(1.21)).

Theorem 6.1 *Let* $\beta \in \mathbb{N}^n_0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $m, \mu \in \mathbb{N}_0$.

The potential–type operators

$$\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x) \;:\; \mathbb{B}^s_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+2\ell-1-m-\mu+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm}), \tag{6.4}$$

$$\;:\; \mathbb{B}^s_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+2\ell-1-m-\mu+|\beta|+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm}) \tag{6.5}$$

are continuous for all $k=0,1,\dots,\infty$*.*

The tracess $\gamma^\pm_{\mathcal{S}}\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x)$ *exist and represent classical pseudodifferential operators with symbols*

$$\mathcal{V}^{(\beta)}_{\mathcal{B},\mathcal{C}}(t,\xi) \simeq \sum_{k=0}^{N} \mathcal{V}^{(\beta)}_{\mathcal{B},\mathcal{C},k}(t,\xi) + \widetilde{\mathcal{V}}^{(\beta)}_{\mathcal{B},\mathcal{C}.N+1}(t,\xi)\,, \quad t\in\mathcal{S},\;\; \xi\in\mathbb{R}^n\,, \tag{6.6}$$

$$\widetilde{\mathcal{V}}_{\mathcal{B},\mathcal{C},N+1} \in \mathbb{S}^{-2\ell+1+m+\mu-|\beta|-N-1}(\mathcal{S})\,,$$

where $N\in\mathbb{N}_0$ *is arbitrary and* $\mathcal{V}^{(\beta)}_{\mathcal{B},\mathcal{C},k}(t,\xi)$ *are homogeneous of order* $-2\ell+1+m+\mu-|\beta|-k$ $(k=0,1,\cdots,N)$*.*

The result holds for $s>0$ *and* $1\le p,q\le\infty$*. In particular, it holds for the* Zygmund *spaces (the case* $p=q=\infty$*):*

$$\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x) : \mathbb{Z}^s(\mathcal{S}) \longrightarrow \mathbb{Z}^{s+2\ell-1-m-\mu+|\beta|,k}(\Omega^\pm)\,. \tag{6.7}$$

Proof. Continuity in (6.4), (6.5) and (6.7) follow from Theorem 3.2 and we shall concentrate on the traces $\gamma^\pm_{\mathcal{S}}\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x)$.

Without restricting generality we can suppose $\mathbf{C}(x,\mathcal{D}_x)=I$ because composition of classical PsDOs is classical.

Representing $\mathbf{B}(x,D_x)$ similarly to (1.28)

$$\mathbf{B}(x,D_x) = \sum_{k=0}^{m} \mathbf{B}^{(2\ell-k)}(x,\mathcal{D}_x)\partial^k_{\vec{\nu}(x)}\,,$$

where $\mathbf{B}^{(k)}(x,\mathcal{D}_x)$ is a tangent differential operator of the order k, we find

$$\mathbf{V}^{(\beta)}_{\mathbf{B}}(x,D_x) = \sum_{k=0}^{m} \mathbf{B}^{(2\ell-k)}(x,\mathcal{D}_x)\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)\,,$$

$$\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x) := \mathbf{V}^{(\beta)}_{\partial^k_{\vec{\nu}(x)}}(x,D_x) := \partial^k_{\vec{\nu}(x)}\mathbf{V}^{(\beta)}(x,D_x)\,. \tag{6.8}$$

If $k=0,1,\cdots,2\ell-1$ the generalized potentials $\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)$ are PsDOs due to Lemma 4.5: the restriction $\gamma^\pm_{\mathcal{S}}\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)$ is a well defined classical PsDO on $\mathcal{S}$.

Let us consider the representation

$$\mathbf{A}(x, D_x) = \mathcal{A}_0(x, \vec{\nu}(x))\partial^{2\ell}_{\vec{\nu}(x)} + \sum_{k=0}^{2\ell-1} \mathbf{A}_{2\ell-k}(x, \mathcal{D}_x)\partial^k_{\vec{\nu}(x)} \tag{6.9}$$

(cf. (1.28)), where $\mathcal{A}_0(x, \xi)$ is the principal symbol of $\mathbf{A}(x, D_x)$ (cf. (1.13)) and

$$\mathbf{A}_j(t, \mathcal{D}_t) = \sum_{|\alpha|\leq j} a_{j,\alpha}(t)\mathcal{D}^\alpha_t\,, \quad t \in \mathcal{S} \quad j = 0, 1, \dots, 2\ell - 1$$

are tangent differential operators. Since $\mathcal{K}_{\mathbf{A}}(x, y)$ is the kernel of the fundamental solution, we get

$$\begin{aligned}\mathbf{A}(x, D_x))\mathcal{K}^{(\beta)}_{\mathbf{A}}(x, y) &= \mathbf{A}(x, D_x)(x - y)^\beta\mathcal{K}_{\mathbf{A}}(x, y)\\ &= (x - y)^\beta\mathbf{A}(x, D_x)\mathcal{K}_{\mathbf{A}}(x, y) + \mathbf{E}(x, D_x)\mathcal{K}_{\mathbf{A}}(x, y)\\ &= (x - y)^\beta\delta(x - y) + \mathbf{E}(x, D_x)\mathcal{K}_{\mathbf{A}}(x, y)\\ &= \delta_{|\beta|,0}\delta(x - y) + \mathbf{E}(x, D_x)\mathcal{K}_{\mathbf{A}}(x, y)\end{aligned} \tag{6.10}$$

(cf. (3.1) and (6.9)), where

$$\mathbf{E}(x, D_x) = \begin{cases} 0\,, & \text{if} \quad \beta = 0\,,\\ \displaystyle\sum_{k=0}^{2\ell-1} \mathbf{E}_{2\ell-k-1}(x, \mathcal{D}_x)\partial^k_{\vec{\nu}(x)}\,, & \text{if} \quad \beta \neq 0\end{cases}$$

and $\mathbf{ord}\,\mathbf{E}_{\mathrm{j}} = \mathrm{j}$. On the other hand, by invoking (6.9), we find

$$\begin{aligned}\mathbf{A}(x, D_x)\mathcal{K}^{(\beta)}_{\mathbf{A}}(x, y) &= \mathcal{A}_0(x, \vec{\nu}(x))\partial^{2\ell}_{\vec{\nu}(x)}\mathcal{K}^{(\beta)}_{\mathbf{A}}(x, y)\\ &\quad + \sum_{k=0}^{2\ell-1}\sum_{\gamma\leq\beta} \mathbf{G}_{2\ell-k,\gamma}(x, \mathcal{D}_x)\partial^k_{\vec{\nu}(x)}\mathcal{K}^{(\gamma)}_{\mathbf{A}}(x, y)\\ &= \delta_{|\beta|,0}\,\delta(x - y) + \mathbf{E}(x, D_x)\mathcal{K}_{\mathbf{A}}(x, y)\,.\end{aligned} \tag{6.11}$$

Now we recall that $\mathbf{A}(x, D_x)$ is elliptic, which implies $\mathbf{det}\,\mathcal{A}_0(x, \vec{\nu}(x)) \neq 0$ in the neighbourhood of the boundary $\mathcal{S}$ (see (1.5)). This ensures solvability of equation (6.11) and we find:

$$\begin{aligned}\partial^{2\ell}_{\vec{\nu}(x)}\mathcal{K}^{(\beta)}_{\mathbf{A}}(x, y) &= \delta_{|\beta|,0}\,\delta(x - y)\,[\mathcal{A}_0(x, \vec{\nu}(x))]^{-1}\\ &\quad + \sum_{k=0}^{2\ell-1}\sum_{\gamma\leq\beta} \mathbf{H}^{(2\ell)}_{2\ell-k,\gamma}(x, \mathcal{D}_x)\partial^k_{\vec{\nu}(x)}\mathcal{K}^{(\gamma)}_{\mathbf{A}}(x, y)\,.\end{aligned} \tag{6.12}$$

Applying the mathematical induction and invoking (6.12) we obtain the representation

$$\partial^m_{\overline{\nu}(x)}\mathcal{K}^{(\beta)}_{\mathbf{A}}(x,y)=\sum_{k=0}^{2\ell-1}\sum_{\gamma\leq\beta}\mathbf{H}^{(m)}_{m-k,\gamma}(x,\mathcal{D}_x)\partial^k_{\overline{\nu}(x)}\mathcal{K}^{(\gamma)}_{\mathbf{A}}(x,y) \tag{6.13}$$

for arbitrary $m=2\ell,2\ell+1,\cdots$.

Representation (6.13), inserted into (6.8), shows that all generalized potentials $\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)$ possess traces on $\mathcal{S}$ which are classical PsDOs (see [CD1, § 1, Example 2]). ■

Remark 6.2 *The representation* (6.12) *for* $\beta=0$ *is well known in the literature and was exploited e.g. in [KGBB1, § 6.7] and in [Na1].*

Remark 6.3 *In the definition of potential–type operators* $\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x)$ *in* (6.1) *the operator* $\mathbf{C}(t,\mathcal{D}_t)$ *can be replaced by arbitrary classical pseudodifferential operator on the boundary* $\mathcal{S}$.

6.2. The trace theorem for weighted spaces.

Next Theorem generalizes Theorem 4.6.

Theorem 6.4 *The trace operator*

$$\mathcal{R}_k u:=\{\gamma^{\pm}_{\mathcal{S}}u,\gamma^1_{\mathcal{S}}u,\ldots,\gamma^k_{\mathcal{S}}u\},\quad \gamma^j_{\mathcal{S}}u:=\gamma^{\pm}_{\mathcal{S}}\partial^j_{\nu}u,\ \ u\in C^\infty_0(\overline{\Omega^\pm})$$

is a retraction

$$\begin{aligned}\mathcal{R}_k\ \ &:\ \ \mathbb{H}^{s,m}_{p,loc}(\overline{\Omega^\pm})\longrightarrow \bigotimes_{j=0}^{k}\ \mathbb{B}^{s-\frac{1}{p}-j}_{p,p}(\mathcal{S}),\\ &:\ \ \mathbb{B}^{s,m}_{p,q,loc}(\overline{\Omega^\pm})\longrightarrow \bigotimes_{j=0}^{k}\ \mathbb{B}^{s-\frac{1}{p}-j}_{p,q}(\mathcal{S}),\end{aligned} \tag{6.14}$$

provided $1\leq p,q\leq\infty$, $m\in\mathbb{N}_0$, $k<s-\dfrac{1}{p}$ *and has a coretraction.*

We will expose two different proofs of this assertion.

Proof 1. If $m=1,2,\ldots$ continuity of the trace operator (6.14) follows directly from Theorem 4.6 since $\mathbb{H}^{s,m}_{p,loc}(\overline{\Omega^\pm})$ and $\mathbb{B}^{s,m}_{p,q,loc}(\overline{\Omega^\pm})$ are subspaces of $\mathbb{H}^{s}_{p,loc}(\overline{\Omega^\pm})$ and of $\mathbb{B}^{s}_{p,q,loc}(\overline{\Omega^\pm})$, respectively.

To construct a continuous coretraction $\mathcal{R}_k^{-1}$ we use the representation formulae (3.6), setting there $\mathbf{A}u(x)=0$:

$$\mathcal{R}_{2\ell}^{-1}(\gamma_{\mathcal{S}}^{\pm}u)(x) := u(x) = \sum_{j=0}^{\ell-1}\left\{[\mathbf{V}_{\ell+j}\mathbf{B}_j]\,u(x) - [\mathbf{V}_j\mathbf{B}_{\ell+j}]\,u(x)\right\} \tag{6.15}$$

for $x\in\Omega^+\cup\Omega^-$. Now the continuity of $\mathcal{R}_{2\ell}^{-1}$ follows from Theorem 3.2.

Proof 2. Let us dwell on the case of the half–spaces $\Omega^{\pm}=\mathbb{R}^n_{\pm}$ and $k=0$, because the cases $k\neq 0$ and of arbitrary domains $\Omega^{\pm}$ are treated as in [Tr1, Theorem 2.7.2, Steps 6–7] and [Tr1, Theorem 3.3.3].

Let us recall an alternative definition of (equivalent) norms in the spaces $\mathbb{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbb{H}^s_p(\mathbb{R}^n)=\mathbb{F}^s_{p,2}(\mathbb{R}^n)$:

$$\|\varphi|\mathbb{B}^s_{p,q}(\mathbb{R}^n)\| = \left\|\left\{2^{sj}\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\right\}_{j=0}^{\infty}\Big|\ell_q\left(L_p(\mathbb{R}^n)\right)\right\|,$$

$$\|\varphi|\mathbb{F}^s_{p,q}(\mathbb{R}^n)\| = \left\|\left\{2^{sj}\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\right\}_{j=0}^{\infty}\Big|L_p(\mathbb{R}^n,\ell_q)\right\| \tag{6.16}$$

(see [Tr1, §§ 2.3.1,2.5.6], where

$$\chi_j\in C_0^{\infty}(\mathbb{R}^n)\,,\quad \mathbf{supp}\,\chi_0\subset\{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}\;:\;|\mathrm{x}|\leq 2\}\;,$$

$$\mathbf{supp}\,\chi_j\subset\left\{x\in\mathbb{R}^n\;:\;2^{j-1}<|x|<2^{j+1}\right\}\;,\quad \sum_{j=0}^{\infty}\chi_j(x)\equiv 1\,.$$

In [Tr1, § 2.3.1, Step 5] the coretraction $\mathcal{R}_0^{-1}$ is defined as follows

$$\mathcal{R}_0^{-1}\varphi(x',x_n) = \sum_{j=0}^{\infty}2^{-j}\mathcal{F}^{-1}_{\lambda_n\to x_n}\psi_j(\lambda_n)\mathcal{F}^{-1}_{\lambda'\to x'}\chi_j(\lambda')\mathcal{F}_{y'\to\lambda'}[\varphi(y')]\,, \tag{6.17}$$

where

$$\psi_j(\lambda_n)=\psi(2^{-j}\lambda_n)\,,\quad j\in\mathbb{N},\quad \psi_0,\psi\in C_0^{\infty}(\mathbb{R})\,,$$

$$\mathbf{supp}\;\psi_0\in(0,1)\,,\quad \mathrm{supp}\,\psi\in(1,2)\,,\quad \mathcal{F}^{-1}\psi_0(0)=\mathcal{F}^{-1}\psi(0)=1\,.$$

Then $\mathcal{F}^{-1}\psi_j(0)=2^j$ which yields $(\mathcal{R}_0^{-1}\varphi)(x',0)=\psi(x',0)$. We proceed as in [Tr1, § 2.7.2–(30)]:

$$\|x_n^m\mathcal{R}_0^{-1}\varphi|\mathbb{B}^{s+m+\frac{1}{p}}_{p,q}(\mathbb{R}^n)\| \leq C_1\Big\|\Big\{2^{(s+m+\frac{1}{p})j}\mathcal{F}^{-1}_{\lambda_n\to x_n}\left[(-i\partial_{\lambda_n})^m\psi_j(\lambda_n)\right]$$
$$\times\mathcal{F}^{-1}_{\lambda'\to x'}\chi_j(x)\mathcal{F}_{y'\to\lambda'}[\varphi(y')]\Big\}_{j=0}^{\infty}\Big|\ell_q\left(L_p(\mathbb{R}^n)\right)\Big\|$$

$$= C_1 \Big\| \Big\{ 2^{(s+\frac{1}{p})j} \mathcal{F}^{-1}_{\lambda_n \to x_n} \psi_j^{(m)}(\lambda_n) \mathcal{F}^{-1}_{\lambda' \to x'} \chi_j(x) \mathcal{F}_{y' \to \lambda'}[\varphi(y')] \Big\}_{j=0}^{\infty} \ell_q\left(L_p(\mathbb{R}^n)\right) \Big\|$$

$$\leq C_2 \| \left\{ \mathcal{F}^{-1} \chi_j \mathcal{F} \varphi \right\}_{j=0}^{\infty} \Big| \ell_q\left(L_p(\mathbb{R}^n)\right) \| = \left\| \varphi | \mathbb{B}^s_{p,q}(\mathbb{R}^n) \right\|$$

where $\psi^{(m)}(t) := \partial_t^m \psi(t)$. Similarly we find

$$\| x_n^m \mathcal{R}_0^{-1} \varphi | \mathbb{H}_p^{s+m+\frac{1}{p}}(\mathbb{R}^n) \| \leq C_3 \| \left\{ 2^{sj} \mathcal{F}^{-1} \chi_j \mathcal{F} \varphi \right\}_{j=0}^{\infty} | L_p(\mathbb{R}^n, \ell_2) \|$$

$$\leq C_3 \| \varphi | \mathbb{H}_p^s(\mathbb{R}^n) \| . \qquad \blacksquare$$

Corollary 6.5 *Let $s > 0$, $s \notin \mathbb{N}$, $1 < p < \infty$ ($1 \leq p,q \leq \infty$) and $\mathbf{A}(x, D_x)$ in* (1.1) *be a normal (not recessarily elliptic) operator; let further $\{\mathbf{B}_j(x, D_x)\}_{j=0}^{m-1}$ be a* DIRICHLET *system of the order m (see Definition* 1.2*).*

For arbitrary $k \in \mathbb{N}_0$ there exists a continuous linear operator

$$\mathcal{P}^{(k)} : \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-j-1}_{p,p}(\mathcal{S}) \longrightarrow \mathbb{H}^{s+m-1+\frac{1}{p},k}_{p,loc}(\overline{\Omega^{\pm}})$$

$$\left(\mathcal{P}^{(k)} : \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-j-1}_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s+m-1+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^{\pm}}) \right)$$

such that

$$\gamma_{\mathcal{S}}^{\pm} \mathbf{B}_j \mathcal{P}^{(k)} \Phi = \varphi_j, \quad j = 0, 1, \ldots, m-1,$$

$$\mathbf{A}\mathcal{P}^{(k)}\Phi \in \widetilde{\mathbb{H}}^{s-1+\frac{1}{p},k}_{p,loc}(\overline{\Omega^{\pm}}) \qquad (\mathbf{A}\mathcal{P}^{(k)}\Phi \in \widetilde{\mathbb{B}}^{s-1+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^{\pm}}))$$

for arbitrary

$$\Phi = (\varphi_0, \ldots, \varphi_{m-1}) \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-j-1}_{p,p}(\mathcal{S}) \quad \left(\Phi \in \mathbb{B}^{s+m-j-1}_{p,q}(\mathcal{S}) \right).$$

Proof: applies Theorem 6.4 and proceeds as in Lemma 4.8. $\blacksquare$

6.3. The Calderón projections. Throughout this subsection it is assumed that conditions of Theorem 1.6 hold and the GREEN formula (1.13) is valid. Let

$$\begin{aligned} \mathbb{H}_p^{s,\pm}(\mathbf{A}, \mathcal{S}) &:= \left\{ \gamma_{\mathcal{S}}^{\pm} \mathbf{B}_j \varphi \ : \ \varphi \in \mathbb{H}_p^{s+j+\frac{1}{p}}(\Omega^{\pm}), \ \ \mathbf{A}(x, D_x)\varphi = 0 \right\}, \\ \mathbb{B}_{p,q}^{s,\pm}(\mathbf{A}, \mathcal{S}) &:= \left\{ \gamma_{\mathcal{S}}^{\pm} \mathbf{B}_j \varphi \ : \ \varphi \in \mathbb{B}_{p,q}^{s+j+\frac{1}{p}}(\Omega^{\pm}), \ \ \mathbf{A}(x, D_x)\varphi = 0 \right\} \end{aligned} \qquad (6.18)$$

for $j=0,\dots,2\ell-1\,,\quad s\in\mathbb{R}\,,\ 1<p<\infty\,,\ 1\le q\le\infty$, where $\gamma^{\pm}_{\mathcal{S}}u$ denote the traces (see Introduction).

Theorem 6.6 *There hold the following decompositions*

$$\begin{aligned}\mathbb{H}^s_p(\mathcal{S}) &= \mathbb{H}^{s,-}_p(\mathbf{A},\mathcal{S})\oplus\mathbb{H}^{s,+}_p(\mathbf{A},\mathcal{S})\,,\\ \mathbb{B}^s_{p,q}(\mathcal{S}) &= \mathbb{B}^{s,-}_{p,q}(\mathbf{A},\mathcal{S})\oplus\mathbb{B}^{s,+}_{p,q}(\mathbf{A},\mathcal{S})\,,\end{aligned}\tag{6.19}$$

$$\mathbb{H}^{s,-}_p(\mathbf{A},\mathcal{S})\cap\mathbb{H}^{s,+}_p(\mathbf{A},\mathcal{S})=\emptyset\,,\quad \mathbb{B}^{s,-}_{p,q}(\mathbf{A},\mathcal{S})\cap\mathbb{B}^{s,+}_{p,q}(\mathbf{A},\mathcal{S})=\emptyset\tag{6.20}$$

and the corresponding CALDERÓN *projections*

$$\begin{aligned}\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}\ &:\ \mathbb{H}^s_p(\mathcal{S})\longrightarrow\mathbb{H}^{s,\pm}_p(\mathbf{A},\mathcal{S})\,,\\ &:\ \mathbb{B}^s_{p,q}(\mathcal{S})\longrightarrow\mathbb{B}^{s,\pm}_{p,q}(\mathbf{A},\mathcal{S})\end{aligned}\tag{6.21}$$

are defined as follows

$$\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}=\pm\gamma^{\pm}_{\mathcal{S}}\mathbf{B}_j\mathbf{V}_j\quad\text{for}\quad j=0,\dots,2\ell-1\,.\tag{6.22}$$

Proof (see [Se1, Lemmata 5 and 6] for a simpler case). We will prove (6.19)–(6.20) for the BESOV space. For the BESSEL potential spaces we have to prove only the continuity property (6.21) while others (including (6.22)) follow due to the embedding $\mathbb{B}^s_{p,q}(\mathcal{S})\subset\mathbb{H}^s_r(\mathcal{S})$ for $1<r<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$.

First we note that $\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}$ are PsDOs of the order 0 (see Lemma 4.5). Continuity (6.21) follows from usual boundedness of PsDOs (see e.g. Theorem 4.2) if we would have the inclusions

$$\begin{aligned}\mathbf{Im}\,\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}&\subset\mathbb{H}^{\mathrm{s},\pm}_{\mathrm{p}}(\mathbf{A},\mathcal{S})\subset\mathbb{H}^{\mathrm{s}}_{\mathrm{p}}(\mathcal{S})\,,\\ \mathbf{Im}\,\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}&\subset\mathbb{B}^{\mathrm{s},\pm}_{\mathrm{p,q}}(\mathbf{A},\mathcal{S})\subset\mathbb{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathcal{S})\,,\end{aligned}\tag{6.23}$$

where $\mathbf{Im}\,\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}$ denotes the image in appropriate spaces. Inclusions (6.23) follows because $\mathbf{A}\mathbf{V}_j\varphi(x)=0$ for $x\in\Omega^-\cup\Omega^+$ and $j=0,\dots,2\ell-1$.

Inserting $u=\mathcal{P}_j\varphi$, $f=\mathbf{A}u=\mathbf{A}\mathcal{P}_j\varphi$ (cf. (5.10), (5.11)) into (3.3) we get

$$\chi_{\Omega^+}\mathcal{P}_j\varphi(x)=\mathbf{N}_{\Omega^+}\mathbf{A}\mathcal{P}_j\varphi(x)+\sum_{k=0}^{2\ell-1}\mathbf{V}_k\mathbf{B}_k\mathcal{P}_j\varphi(x)\tag{6.24}$$

$$=\mathbf{N}_{\Omega^+}\mathbf{A}\mathcal{P}_j\varphi(x)+\mathbf{V}_j\varphi(x)\,,\quad j=0,\dots,2\ell-1\,,\quad x\in\Omega^-\cup\Omega^+\,.$$

Since the first summand in (6.24) and its derivatives are continuous across the surface $\mathcal{S}$

$$(\gamma_{\mathcal{S}}^{-}\partial_x^\alpha \mathbf{N}_{\Omega^+}\mathbf{A}\mathcal{P}_j\varphi)(t) = (\gamma_{\mathcal{S}}^{+}\partial_x^\alpha \mathbf{N}_{\Omega^+}\mathbf{A}\mathcal{P}_j\varphi)(t) \quad t\in\mathcal{S}\,, \quad \alpha\in\mathbb{N}_0^n\,,$$

by invoking (5.10), (5.11) we get

$$(\gamma_{\mathcal{S}}^{+}\mathbf{B}_k\mathbf{V}_j\varphi)(t) - (\gamma_{\mathcal{S}}^{-}\mathbf{B}_k\mathbf{V}_j\varphi)(t) = \mathbf{B}_k\mathcal{P}_j\varphi(t) = \delta_{kj}\varphi(t)\,, \tag{6.25}$$

where $j,k = 0,\cdots,2\ell-1$. The obtained formulae (6.25) yields

$$\mathbf{P}^{-}_{\mathbf{A},\mathbf{j}}\varphi + \mathbf{P}^{+}_{\mathbf{A},\mathbf{j}}\varphi = \gamma_{\mathcal{S}}^{+}\mathbf{B}_j\mathbf{V}_j\varphi - \gamma_{\mathcal{S}}^{-}\mathbf{B}_j\mathbf{V}_j\varphi = \varphi\,, \quad \varphi\in\mathbb{B}^s_{p,q}(\mathcal{S})\,. \tag{6.26}$$

By virtue of (6.21) this proves (6.19).

To prove (6.20) (for the Besov spaces) let us apply formula (6.24), written for the homogeneous equation $f = \mathbf{A}u = \mathbf{A}\mathcal{P}_j\varphi = 0$ and a similar one for the outer domain Ω^-:

$$\chi_{\Omega^\pm}\mathcal{P}_j\varphi(x) = \pm\mathbf{V}_j\varphi(x)\,, \quad j = 0,\ldots,2\ell-1\,, \quad x\in\Omega^-\cup\Omega^+\,.$$

Taking the sum, applying the operator $\mathbf{B}_j$ and invoking (5.10), (5.11) we find the representation of a function $\varphi\in\mathbb{B}^{s,-}_{p,q}(\mathbf{A},\mathcal{S})\cap\mathbb{B}^{s,+}_{p,q}(\mathbf{A},\mathcal{S})$:

$$\varphi(x) = \mathbf{B}_j\mathbf{V}_j[\varphi](x)\,, \quad j = 0,\ldots,2\ell-1\,, \quad x\in\Omega^-\cup\Omega^+\,. \tag{6.27}$$

where $[\varphi](t) := \gamma_{\mathcal{S}}^{+}\varphi(t) - \gamma_{\mathcal{S}}^{-}\varphi(t)$. Thus $[\varphi](t) = 0$ on $\mathcal{S}$ implies $\varphi(x) = 0$ for all $x\in\mathbb{R}^n$.

From (6.20), (6.23) and (6.26) there follows that $\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}$ are projections:

$$\left(\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}\right)^2 = \mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}\left(\mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}} + \mathbf{P}^{\mp}_{\mathbf{A},\mathbf{j}}\right) = \mathbf{P}^{\pm}_{\mathbf{A},\mathbf{j}}\,. \qquad \blacksquare$$

Example 6.7 . *If in Example* 1.9 *we take the* Laplacian $\mathbf{A}(x,D_x)u(x) = \mathbf{\Delta}u(x) = 0$ *in the plane domains* $\Omega^\pm\subset\mathbb{R}^2$ *(see* (1.1)*), the spaces* $\mathbb{H}^s_p(\mathcal{S})$ *and* $\mathbb{B}^s_{p,q}(\mathcal{S})$ *are decomposed into the spaces of harmonic functions in* Ω^+ *and in* Ω^-.

6.4. The Plemelji formulae for layer potentials. Let

$$\mathbf{V}_{j,k}(t,D_t)\varphi(t) := \oint_{\mathcal{S}}\mathbf{B}_j(t,D_t)\left[\overline{\mathbf{C}_k(\tau,D_\tau)}\mathcal{K}^\top_{\mathbf{A}}(t,\tau)\right]^\top\varphi(\tau)d_\tau\mathcal{S} \tag{6.28}$$

for $j = 0, \dots, 2\ell - 1$ denote the direct value of the potential–type operator $\mathbf{B}_j \mathbf{V}_k$ on the surface $t \in \mathcal{S}$ (see (3.5)). According to Theorems 3.2 and 6.1 $\mathbf{V}_{j,k}$ is a pseudodifferential operator and

$$\begin{aligned} \mathbf{V}_{j,k} \quad &: \quad \mathbb{H}^s_p(\mathcal{S}) \longrightarrow \mathbb{H}^{s-m_j-\mu_k+2\ell-1}_p(\mathcal{S}), \\ &: \quad \mathbb{B}^s_{p,q}(\mathcal{S}) \longrightarrow \mathbb{B}^{s-m_j-\mu_k+2\ell-1}_{p,q}(\mathcal{S}) \end{aligned} \tag{6.29}$$

is continuous provided $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ ($1 \le p \le \infty$, $1 \le q \le \infty$, $s > 0$).

We have already explained in § 6.1 what is meant under this operator when the order is positive $\mathbf{ord}\,\mathbf{V}_{\rm j,k} = \rm m_j + \mu_k - 2\ell + 1 > 0$. Since $\mathbf{ord}\,\mathbf{V}_{\rm j,j} = 0$ (see (1.14)) $\mathbf{V}_{j,j}$ becomes an usual Calderón–Zygmund singular integral operator and the integral in (6.28) is understood in the Cauchy principal value sence:

$$\begin{aligned} &\mathbf{V}_{j,j}(t, D_t)\varphi(t) := \\ &\quad = \lim_{\varepsilon \to 0} \oint\limits_{\mathcal{S} \setminus \mathcal{S}(t,\varepsilon)} \mathbf{B}_j(t, D_t) \left[\overline{\mathbf{C}_j(\tau, D_\tau)} \mathcal{K}^\top_{\mathbf{A}}(t, \tau) \right]^\top \varphi(\tau) d_\tau \mathcal{S}\,, \end{aligned} \tag{6.30}$$

where $\mathcal{S}(t, \varepsilon) := S^{n-1}(t, \varepsilon) \cap \mathcal{S}$ is the portion of the surface $\mathcal{S}$ inside the sphere $S^{n-1}(t, \varepsilon)$ with radius ε centered at $t \in \mathcal{S}$. Then $\mathbf{V}_{j,j}$ is continuous in the spaces $\mathbb{H}^s_p(\mathcal{S})$ and $\mathbb{B}^s_{p,q}(\mathcal{S})$ (see (6.29)).

Theorem 6.8 *Let BVP* (1.11) *be formally adjoint to* (1.10) *and the* Green *formula* (1.13) *hold.*

For the traces $\gamma^\pm_{\mathcal{S}} \mathbf{B}_j \mathbf{V}_k$ *there hold the following* Plemelji *formulae*

$$(\gamma^-_{\mathcal{S}} \mathbf{B}_j(x, D_x) \mathbf{V}_k \varphi)(t) = (\gamma^+_{\mathcal{S}} \mathbf{B}_j(x, D_x) \mathbf{V}_k \varphi)(t) \quad \text{for} \quad k \ne j\,, \tag{6.31}$$

$$(\gamma^\pm_{\mathcal{S}} \mathbf{B}_j(x, D_x) \mathbf{V}_j \varphi)(t) = \pm \frac{1}{2} \varphi(t) + \mathbf{V}_{j,j}(t, D_t)\varphi(t)\,, \quad t \in \mathcal{S}\,, \tag{6.32}$$

$$k, j = 0, \dots 2\ell - 1\,, \quad \varphi \in \mathbb{H}^s_p(\mathcal{S}) \quad .$$

Proof. (6.31) follows from (6.25).

Let $t \in \mathcal{S}$ be the projection of $x \in \Omega^\pm$, i.e. $x \in \mp \vec{\nu}(t)$ (we remind that the normal $\vec{\nu}(t)$ is outer, directed into Ω^-).

The potential–type operator

$$\begin{aligned} &\mathbf{V}_{j,j}\varphi(x) := \oint\limits_{\mathcal{S}} \mathcal{K}_{j,\mathbf{A}}(x, x - \tau) \varphi(\tau) d_\tau \mathcal{S}\,, \\ &\mathcal{K}_{j,\mathbf{A}}(x, x - y) := \mathbf{B}_j(x, D_x) \left[\overline{\mathbf{C}_j(\tau, D_\tau)} \mathcal{K}^\top_{\mathbf{A}}(x, y) \right]^\top\,, \quad x, y \in \Omega^\pm \end{aligned} \tag{6.33}$$

has a CALDERÓN–ZYGMUND kernel (is of order 0 if restricted to $\mathcal{S}$):

$$\mathcal{K}_{j,\mathbf{A}} \in C^\infty(\mathbb{R}^n \otimes \mathbb{R}^n \setminus \Delta_{\mathbb{R}^n}), \tag{6.34}$$

$$|\mathcal{K}_{j,\mathbf{A}}(x, x-y)| \leq M_0 |x-y|^{1-n}, \qquad x, y \in \mathbb{R}^n, \quad x \neq y. \tag{6.35}$$

Then the truncated potential–type operator

$$\mathbf{V}^0_{j,j,\varepsilon}\varphi(x) := \oint\limits_{\mathcal{S}\setminus\mathcal{S}(t,\varepsilon)} \mathcal{K}_{j,\mathbf{A}}(x, x-\tau)\varphi(\tau)d_\tau\mathcal{S}, \quad \varepsilon > 0 \tag{6.36}$$

(see (6.30)) has C^∞-smooth kernel (see (6.34)) and

$$\lim_{\varepsilon\to 0}\left(\gamma^-_{\mathcal{S}}\mathbf{V}^0_{j,j,\varepsilon}\varphi\right)(t) = \lim_{\varepsilon\to 0}(\gamma^+_{\mathcal{S}}\mathbf{V}^0_{j,j,\varepsilon}\varphi)(t). \tag{6.37}$$

Due to the definition (6.30) and to the continuity property (6.37),

$$\left(\gamma^\pm_{\mathcal{S}}\mathbf{B}_j(x, D_x)\mathbf{V}_j\varphi\right)(t) = \left(\mathbf{V}_{j,j}(t, D_t)\varphi\right)(t) + \lim_{\varepsilon\to 0}\left(\gamma^\pm_{\mathcal{S}}\mathbf{V}_{j,j,\varepsilon}\varphi\right)(t), \tag{6.38}$$

$$\mathbf{V}_{j,j,\varepsilon}\varphi(x) = \oint\limits_{\mathcal{S}(t,\varepsilon)} \mathcal{K}_{j,\mathbf{A}}(x, x-\tau)\varphi(\tau)d_\tau\mathcal{S}, \quad x \in \Omega^\pm, \quad \varphi \in C^\infty(\mathcal{S}).$$

Since $\varepsilon > 0$ is sufficiently small there exists a diffeomorphism

$$\begin{gathered}\varkappa : \mathcal{S}_0(t,\varepsilon) \longrightarrow \mathcal{S}(t,\varepsilon), \quad \varkappa(x') = (x', g(x')) \in \mathcal{S}(t,\varepsilon) \subset \mathcal{S}, \\ x' = (x_1, \dots, x_{n-1}) \in \mathcal{S}_0(t,\varepsilon) \subset \mathbb{R}^{n-1}_t, \\ g(t) = t \in \mathcal{S}, \quad (\partial_k g)(t) = 0, \quad k = 1, \dots, n-1\end{gathered} \tag{6.39}$$

and $\mathcal{S}_0(t,\varepsilon)$ is the projection of the piece $\mathcal{S}(t,\varepsilon)$ onto the tangent plane $\mathbb{R}^{n-1}_t$ to $\mathcal{S}$ at $t \in \mathcal{S}$. By changing the variable $\tau = \varkappa(y')$, $y' \in \mathcal{S}_0(t,\varepsilon)$ in the integral (6.38) we find the following

$$\mathbf{V}_{j,j,\varepsilon}\varphi(x) := \oint\limits_{\mathbb{R}^{n-1}} \mathcal{K}_{j,\mathbf{A}}(x, x-\varkappa(y'))\mathcal{G}_\varkappa(y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy',$$

$$|x-t| < 2\varepsilon, \quad x \notin \mathcal{S}_0(t,\varepsilon),$$

where χ_ε is the the characteristic function of the piece $\mathcal{S}_0(t,\varepsilon) \subset \mathbb{R}^{n-1}$ and

$$\mathcal{G}_\varkappa(y') := \sqrt{|\,\mathbf{grad}\ \mathrm{g}(\mathrm{y}')|^2 + 1} = 1 + O(|y'-t|) \tag{6.40}$$

is the GRAM determinant (see [Sc1, §IV.10.38], [Si1, §3.6]).

Next we note that

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \oint\limits_{\mathbb{R}^{n-1}} \mathcal{K}_{j,\mathbf{A}}(x, x-y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy' + o(1) \tag{6.41}$$

as $\varepsilon \to 0$ uniformly for $x \in \mathbb{R}^n$ in the vicinity of $\mathcal{S}_0(t,\varepsilon)$.

In fact, the remainder kernel

$$\mathcal{K}^0_{j,\mathbf{A}}(x, y') := \mathcal{K}_{j,\mathbf{A}}(x, x-\varkappa(y'))\mathcal{G}_\varkappa(y') - \mathcal{K}_{j,\mathbf{A}}(x, x-y')$$

is weak singular

$$\left|\mathcal{K}^0_{j,\mathbf{A}}(x, y')\right| \le M_1|x-y|^{2-n}, \qquad x, y \in \mathbb{R}^n, \quad x \ne y\,. \tag{6.42}$$

(cf. (6.34); see (6.37) and [CD1, § 1.4]) and it is almost obvious that

$$\lim_{\varepsilon\to 0} \gamma^{\pm}_{\mathcal{S}} \oint\limits_{\mathcal{S}_0(t,\varepsilon)} \mathcal{K}^0_{j,\mathbf{A}}(x, x-y')\mathcal{G}_\varkappa(y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy' = 0$$

for arbitrary $\varphi \in C^\infty(\mathcal{S})$. By the same rasons

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \varphi(t) \oint\limits_{\mathcal{S}_0(t,\varepsilon)} \mathcal{K}_{j,\mathbf{A}}(x, x-y')dy' + o(1) \quad \text{as} \quad \varepsilon \to 0\,, \tag{6.43}$$

because $|\varphi(\varkappa(y')) - \varphi(t)| \le M_2|y'-t|$.

If in the definition of the kernel $\mathcal{K}_{j,\mathbf{A}}(x, x-y')$ in (6.33) the differential operators $\mathbf{B}_j(x, D_x)$, $\mathbf{C}_j(x, D_x)$ and $\mathbf{A}(x, D_x)$ are replaced by their principal parts $\mathbf{B}_{j,0}(t, D_x)$, $\mathbf{C}_{j,0}(t, D_x)$ and $\mathbf{A_0}(t, D_x)$, respectively, the remainder kernel is weak singular and admits an estimate similar to (6.42). Therefore, as in (6.43),

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \varphi(t) \oint\limits_{\mathcal{S}_0(t,\varepsilon)} \mathcal{K}_{j,0,\mathbf{A}}(x, x-y')dy' + o(1) \quad \text{as} \quad \varepsilon \to 0\,, \tag{6.44}$$

where the kernel is homogeneous of order $1-n$:

$$\mathcal{K}_{j,0,\mathbf{A}}(x, \lambda z) = \lambda^{1-n}\mathcal{K}_{j,0,\mathbf{A}}(x, z)\,, \quad x, z \in \mathbb{R}^n, \quad z \ne 0\,. \tag{6.45}$$

We can simplify the integral (6.44) further:

1. First we replace the domain of integration $\mathcal{S}_0(t,\varepsilon)$ by the ball $|y'-t| \le \varepsilon$, $y' \in \mathbb{R}^{n-1}$; these domains have difference of the order ε and the difference is estimated as $o(1)$.

2. Next it is possible, freezing coefficients at $t_0 \in \mathcal{S}$ as $\varepsilon \to 0$, to consider a pure convolution kernel $\mathcal{K}_{j,0,\mathbf{A}}(t_0, x-y')$ which is translation invariant; the remainder has weak singularity and contributes $o(1)$ in (6.44).

3. Due to the described simplifications the domain of integration $|y'-t| \leq \varepsilon$ can be translated (shifted) to the origin and stretched up to the unit ball $|y'| \leq 1$; the integral is invariant with respect to translation and dilation (stretching).

Finally, taking the traces, we get the following

$$\left(\gamma_{\mathcal{S}}^{\pm}\mathbf{V}_{j,\varepsilon}\varphi\right)(t) := \pm c_0\, \varphi(t) + o(1) \quad \text{as} \quad \varepsilon \to 0\,. \tag{6.46}$$

where the signs "$\pm$" are due to traces, which approach different faces of the surface; the integral

$$c_0 := \oint\limits_{|y'|\leq 1} \mathcal{K}_{j,0,\mathbf{A}}(t_0, y')dy'$$

is independent of $\varepsilon > 0$ and $t_0 \in \mathcal{S}$. Invoking (6.26) we find $c_0 = \dfrac{1}{2}$. Now (6.38) and (6.46) yield (6.32). ∎

Remark 6.9 *Applyed to the operator* $\mathbf{B}_j(x, D_x)\mathbf{V}_j$ (4.9) *gives*

$$(\gamma_{\mathcal{S}}^{\pm}\mathbf{B}_j(x, D_x)\mathbf{V}_j\varphi)(t) = \pm\frac{c_0(t)}{2}\varphi(t) + \mathbf{V}_{j,j}(t, D_t)\varphi(t) \quad t \in \mathcal{S}\,, \tag{6.47}$$

where $c_0(t) = i\mathcal{B}_j(t, \vec{\nu}(t))\mathcal{N}_{j,j}(t, \vec{\nu}(t))$ *and* $\mathcal{N}_{j,j}(t, \vec{\nu}(t))$ *is the symbol of the pseudodifferential operator on* $\mathbb{R}^n$

$$\mathbf{N}_{j,j}(x, D_x)\varphi(t) := \int\limits_{\mathbb{R}^n} \mathbf{B}_j(x, D_x)\left[\overline{\mathbf{C}_j(y, D_y)}\mathcal{K}_{\mathbf{A}}^{\top}(x, y)\right]^{\top}\varphi(y)dy\,, \tag{6.48}$$

associated with the potential operator $\mathbf{V}_{j,j}$ *in* (6.30). *From* (6.26) *we find* $c_0(t) \equiv 1$.

It is possible to find the symbol $\mathcal{B}_j(t, \vec{\nu}(t))\mathcal{N}_{j,j}(t, \vec{\nu}(t))$ *directly by invoking* (1.38).

6.5. On smoothness of solutions and coefficients. It is possible to relax substantially smoothness requirements on coefficients and on the boundary imposed in § 2. We need only to ensure invariant definition of the relevant spaces $\mathbb{H}_p^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathcal{S})$, $\mathbb{B}_{p,q}^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathcal{S})$ etc. and continuity of operator

(1.1) and of its formal adjoint (1.2) in appropriate spaces. For more refined results for the second order equations on domains with LIPSCHITZ boundary we quote [MMT1, MT1] and the literature cited therein.

Let the boundary $\partial\Omega = \mathcal{S}$ be C^ω-smooth.

If integers $\omega, \ell_0, \dots, \ell_m$ and coefficients $a_\alpha(x)$ of the operator $\mathbf{A}(x, D_x)$ in (1.1) satisfy the following conditions

$$\omega > \left|\vartheta + \frac{m}{2} - \frac{1}{p}\right| > 0, \quad a_\alpha \in C^{\ell_{|\alpha|}}(\mathbb{R}^n, \mathbb{C}^{N\times N}), \tag{6.49}$$

$$\ell_k \begin{cases} > |\vartheta + \frac{m}{2} - k| & \text{for} \quad \vartheta - \frac{m}{2} \geq 0\,, \\ = 0 & \text{for} \quad \vartheta + \frac{m}{2} - k \geq 0\,, \;\; \vartheta - \frac{m}{2} \leq 0\,, \\ > k - \vartheta - \frac{m}{2} & \text{for} \quad \vartheta + \frac{m}{2} - k < 0 \end{cases} \tag{6.50}$$

for all $k = 0, 1, \dots, m$, then the spaces $\mathbb{H}_p^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathcal{S})$, $\mathbb{B}_{p,q}^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathcal{S})$ are well–defined, the traces $\mathbb{B}_{p,q}^{\vartheta+\frac{m}{2}-\frac{1}{p}}(\mathcal{S}) = \gamma_{\mathcal{S}}^\pm \mathbb{B}_{p,q,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm})$ exist and the operators

$$\begin{aligned} \mathbf{A}(x, D_x) \;\; &: \;\; \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm}) \longrightarrow \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})\,, \\ &: \;\; \mathbb{B}_{p,q,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm}) \longrightarrow \mathbb{B}_{p,q,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm}) \end{aligned} \tag{6.51}$$

are continuous.

In fact, let $\vartheta - \dfrac{m}{2} \geq 0$. Since $\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm})$ we get $a_\alpha\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$ for $a_\alpha \in C^{\vartheta+\frac{m}{2}-|\alpha|}(\mathbb{R}^n, C^{N\times N})$. $|\alpha| \leq m$ (we remind that a multiplication oprator aI is continuous in $\mathbb{H}_p^\nu(\mathcal{L})$, $\mathbb{B}_{p,q}^\nu(\mathcal{L})$ provided $a \in C^\mu(\mathcal{L})$ and $\mu > \nu$; see [Tr1, Corollary 2.8.2]).

Now let $\vartheta - \dfrac{m}{2} < 0$. If $\vartheta + \dfrac{m}{2} - |\alpha| \geq 0$ we have $\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset L_{p,loc}(\Omega^\pm)$ and $a_\alpha\partial^\alpha\varphi \in L_{p,loc}(\Omega^\pm) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$ for $a_\alpha \in C(\mathbb{R}^n, C^{N\times N})$. If $\vartheta + \dfrac{m}{2} - |\alpha| < 0$, then $a_\alpha\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$ for $a_\alpha \in C^{|\alpha|-\vartheta-\frac{m}{2}}(\mathbb{R}^n, C^{N\times N})$, $|\alpha| \leq m$. This yields the boundedness result (6.51).

Condition (6.50) can be slightly improved, provided the condition $\omega > \left|\vartheta + \dfrac{m}{2} - \dfrac{1}{p}\right| > 0$ holds: if $s + \dfrac{m}{2} + \dfrac{1}{p} \geq 0$ and $s - \dfrac{m}{2} < 0$ we can take

$$c_\alpha \in \mathbb{H}_{p,loc}^\mu(\overline{\Omega^\pm}) \quad \text{for} \;\; \vartheta + \frac{m}{2} - |\alpha| > \frac{n}{2}\,, \;\; \vartheta - \frac{m}{2} \leq 0\,, \tag{6.52}$$

$$\mu := \max\left\{-\vartheta - \frac{m}{2} + |\alpha| + \frac{n}{p}, \vartheta - \frac{m}{2}\right\}.$$

In fact, under conditions (6.52) and $\varphi \in \mathbb{H}^{\vartheta+\frac{m}{2}}_{p,loc}(\overline{\Omega^\pm})$ we get $\partial^\alpha\varphi \in \mathbb{H}^{\vartheta+\frac{m}{2}-|\alpha|}_{p,loc}(\overline{\Omega^\pm}) \subset C^{\vartheta+\frac{m}{2}-|\alpha|-\frac{n}{p}+\varepsilon}(\overline{\Omega^\pm})$ for a small $\varepsilon > 0$. Therefore $a_\alpha\partial^\alpha\varphi \in \mathbb{H}^{\mu}_{p,loc}(\overline{\Omega^\pm}) \subset \mathbb{H}^{\vartheta-\frac{m}{2}}_{p,loc}(\overline{\Omega^\pm})$.

Under conditions (6.49) and (6.50) Lemma 4.8 with $s = \vartheta - \dfrac{m}{2} - \dfrac{1}{p} > 0$ (which implies $\vartheta > \dfrac{m}{2} + \dfrac{1}{p}$ and $\omega > m$) remain valid.

Theorem 3.2 can also be extended, based on Lemma 4.8 with relaxed smoothness constraints. These results we leave for forthcoming publications.

6.6. Concluding Remarks. As we have already mentioned if $\mathbf{A}(x, D_x)$ in (1.1) is scalar ($N = 1$), elliptic and has real valued matrix–coefficients (or complex valued coefficients and $n > 2$) than it is proper elliptic and has even order $\mathbf{ord}\,\mathbf{A}(\mathrm{x}, \mathrm{D_x}) = \mathrm{m} = 2\ell$ (see [LM1, Ch.2, §§1.1]).

For non–scalar case $N = 2, 3, \dots$ matters are different. The operator

$$\mathbf{A}(D_x) = \begin{pmatrix} i\partial_3 & -i\partial_1 - \partial_2 \\ i\partial_1 - \partial_2 & i\partial_3 \end{pmatrix} \tag{6.53}$$

is elliptic

$$\mathcal{A}(\xi) = \begin{pmatrix} \xi_3 & -\xi_1 + i\xi_2 \\ \xi_1 + i\xi_2 & \xi_3 \end{pmatrix}, \qquad \mathbf{det}\ \mathcal{A}(\xi) = |\xi|^2 \neq 0 \quad \text{for} \ \ \xi \neq 0$$

and has order 1.

Let us consider BVP (1.10) with elliptic "basic" operator $\mathbf{A}(x, D_x)$, $\mathbf{ord}\,\mathbf{A} = m$, with quasi–normal "boundary" operators $\mathbf{b}_0(x, D_x), \dots, \mathbf{b}_{\omega-1}(x, D_x)$ and the following constraints:

$$u \in \mathbb{H}^s_p(\Omega^\pm), \quad f \in \mathbb{H}^{s-m}_p(\Omega^\pm), \quad s \in \mathbb{R}\,,\ 1 < p, \infty\,,\ s - \frac{1}{p} > m - 1\,. \tag{6.54}$$

The Fredholm properties and solvability of BVP (1.10) is completely dependent on the factorization of the "lifted" principal homogeneous symbol

$$\mathcal{A}_{(m)}(t, \xi', \lambda) := (\lambda - i|\xi'|)^{-m}\mathcal{A}_0(t, \xi' + \lambda\vec{\nu}(t)), \tag{6.55}$$

$$t \in \mathcal{S}\,, \ \ \xi' \in \mathcal{T}(t, \mathcal{S})\,, \ \ \lambda \in \mathbb{R}\,,$$

where $\mathcal{T}(t, \mathcal{S}) := \{\xi' \in \mathbb{R}^n \ : \ \xi' \cdot \vec{\nu}(t) = 0\}$ is the tangent space to $\mathcal{S}$ at $t \in \mathcal{S}$ and $\mathcal{A}_0(x, \xi)$ is the principal homogeneous symbol of $\mathbf{A}(x, D_x)$ (see (1.4)).

The symbol $\mathcal{A}_{(m)}(t,\xi',\lambda)$ admits the following factorization

$$\mathcal{A}_{(m)}(t,\xi',\lambda) = \mathcal{A}_{-}(t,\xi',\lambda)\left(\frac{\lambda + i|\xi'|}{\lambda - i|\xi'|}\right)^{\frac{m}{2}}\mathcal{A}_{+}(t,\xi',\lambda), \tag{6.56}$$

$$t\in\mathcal{S}, \quad \xi'\in\mathcal{T}(t,\mathcal{S}), \quad \lambda\in\mathbb{R},$$

where $\mathcal{A}^{\pm}_{-}(t,\xi',\lambda)$ and $\mathcal{A}^{\pm}_{+}(t,\xi',\lambda)$ are rational, uniformly bounded (with derivatives) and have analytic continuation in the lower $\mathbf{Im}\,\lambda < 0$ and the upper $\mathbf{Im}\,\lambda > 0$ complex half–planes, respectively (see [Du1, Es1, Lo2, Sh1] and most recent [CD1, § 1.7]). The factors $\mathcal{A}^{\pm 1}_{\pm}$ in (6.56) does not influence the Fredholm and solvability properties of equation and we are left with the middle factor. This leads locally to the problem of invertibility of PsDO (of convolution) with the symbol $\left(\frac{\lambda + i|\xi'|}{\lambda - i|\xi'|}\right)^{\frac{m}{2}}$ in the space $\mathbb{H}^s_p(\mathbb{R}^n_+)$ (details see e.g. in [CD1, § 1.7], [Es1, Sh1]). If $m = 2$ this PsDO has kernel which is eliminated by the Schapiro–Lopatinskii condition; this condition in the scalar case $N = 1$ can be written as follows

$$\mathbf{det}\,[\mathrm{b_j}(\mathrm{t},\xi',\lambda^+_{\mathrm{k}})]_{\frac{m}{2}\times\frac{m}{2}} \neq 0, \quad \mathrm{t}\in\mathcal{S}, \quad \xi'\in\mathcal{T}(\mathrm{t},\mathcal{S}), \tag{6.57}$$

where $\lambda^+_0,\dots,\lambda^+_{\frac{m}{2}-1}$ are all roots of the Polynomial equation $\mathcal{A}_0(t,\xi',\lambda) = 0$, $\mathbf{Im}\,\lambda > 0$ (see e.g. [LM1, Ro1] and [EgS1, Ch.2, § 2]). As we see the amount of boundary conditions in BVP (1.10) in the scalar case coincides with $\frac{m}{2}$ and is independent of the space where BVP is considered.

For the matrix case conditions are formulated in terms of unique solvability of the initial boundary value problem for ordinary differential equations (see [Ag1, Es1, Hr2, Ro1]). If "basic" operator in BVP (1.10) has even order (see (6.53), there arise problem: the values of parameters

$$s - \frac{1}{p} = \text{integer} + \frac{1}{2} \tag{6.58}$$

are critical and BVP (1.10) under constraints (6.54) is not Fredholm ($\mathbf{A}(x,D_x)$ has non–closed range; see [CD1, § 1.5]). If (6.58) is not the case, the amount of boundary conditions ω in (1.10) also depends on the space parameters $s - \frac{1}{p}$. The details will be discussed in further publications.

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