An Algebra of Boundary Value Problems Not Requiring Shapiro–Lopatinskij Conditions

B.–W. Schulze

Abstract

We construct an algebra of pseudo–differential boundary value problems that contains the classical Shapiro–Lopatinskij elliptic problems as well as all differential elliptic problems of Dirac type with APS boundary conditions, together with their parametrices. Global pseudo–differential projections on the boundary are used to define ellipticity and to show the Fredholm property in suitable scales of spaces.

Contents

Introduction

The ellipticity of a boundary value problem consists of both (i) the ellipticity of a given (say differential) operator A on a manifold X with boundary Y (here C^∞) and (ii) the ellipticity of boundary conditions. Classical elliptic boundary value problems are given in the form

$$Au = f \text{ on } X, \qquad Tu = g \text{ on } Y, \tag{0.0.1}$$

where $T = \{\mathrm{r}'\tilde{T}_1, \ldots, \mathrm{r}'\tilde{T}_N\}$ is a vector of trace operators, each component of which is the composition of a differential operator with r', the restriction to Y. When X is compact, the column matrix $\mathcal{A} = \binom{A}{T}$ represents a continuous operator $\mathcal{A} : H^s(X) \to H^{s-\mu}(X) \oplus \{\bigoplus_{j=1}^N H^{s-\mu_j-\frac{1}{2}}(Y)\}$ between appropiate Sobolev spaces; here $\mu = \operatorname{ord} A$, $\mu_j = \operatorname{ord} \tilde{T}_j$. The Shapiro–Lopatinskij condition for T with respect to

A is equivalent to the Fredholm property of $\mathcal{A}$ (for sufficiently large real s), and the index $\operatorname{ind}\mathcal{A}$ is independent of s. Elliptic boundary value problems of this class can be embedded as operators in an algebra $\mathcal{B}(X)$ of pseudo–differential boundary value problems with (pseudo–differential) trace and potential conditions; cf. Boutet de Monvel [4]. Each element $\mathcal{A} \in \mathcal{B}(X)$ then has a principal symbol $\sigma(\mathcal{A})$ consisting of two components $(\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$. The interior component $\sigma_\psi(\mathcal{A})(x,\xi)$ is the usual homogeneous principal symbol of the operator A on X, for $(x,\xi) \in T^*X \setminus 0$, and $\sigma_\partial(\mathcal{A})(y,\eta)$ is the (operator–valued, "twisted") homogeneous boundary symbol of $\mathcal{A}$, for $(y,\eta) \in T^*Y \setminus 0$. This latter component is a block–matrix family of operators; the upper left corner of this matrix is $\sigma_\partial(A)(y,\eta)$ plus a so–called "Green summand," and this sum acts in Sobolev spaces on $\mathbb{R}_+$, the inner normal to the boundary. The ellipticity of $\mathcal{A}$ in the algebra $\mathcal{B}(X)$ is precisely the bijectivity of both components of $\mathcal{A}$; that of $\sigma_\partial(\mathcal{A})$ is just the Shapiro–Lopatinskij condition, referred to here as the SL condition.

The class $\mathcal{B}(X)$ of (block–matrix) operators solves the problem of completing the set of differential boundary value problems into a pseudo–differential algebra that contains the parametrices of SL elliptic elements and in which an analogue of the Atiyah–Singer index theorem holds. Boutet de Monvel [4] proved the index theorem in his algebra and thereby extended a corresponding theorem of Atiyah and Bott [1] for elliptic differential boundary value problems.

It is well known that there exist elliptic differential operators A on X that are not upper left corners of certain SL elliptic elements $\mathcal{A} \in \mathcal{B}(X)$. In other words, for such A there do not exist SL elliptic boundary conditions. Dirac operators and many other geometric operators belong to this category.

Let A be an elliptic differential operator and consider the boundary symbol $\sigma_\partial(A)(y,\eta)$ for $(y,\eta) \in S^*Y$, where S^*Y is the cosphere bundle induced by T^*Y with the projection $\pi_1 : S^*Y \to Y$. The boundary symbol $\sigma_\partial(A)(y,\eta) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$ is a family of Fredholm operators for real $s-\operatorname{ord} A > -\frac{1}{2}$. (Kernels and cokernels are independent of s). This gives us an index element $\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in K(S^*Y)$ (with $K(\cdot)$ denoting the K–group on the space in the brackets). A necessary and sufficient condition for the existence of SL elliptic conditions to A is the relation $\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in \pi_1^* K(Y)$; for this, see Atiyah and Bott [1], Boutet de Monvel [4].

When this latter relation is not satisfied, the well–known machinery of the Atiyah–Patodi–Singer (APS) boundary conditions is often of great assistance. These are given in terms of certain global pseudo–differential projections on Y and have been elaborated for a wide class of elliptic differential operators. The index theory in this case connects the global data on the boundary with η–invariants of associated operators on Y; cf. [3] or [17], as well as the references cited there.

Related boundary value problems have also been treated in several papers with a geometric orientation. These are mainly devoted, however, to homogeneous problems $Au = f$, $Pu = 0$ for first order elliptic operators A that have the form $A = \partial_t + A'$ near the boundary Y, where A' is a self–adjoint operator on Y, t the normal variable, and P the spectral projection to the eigenspace belonging to the non–negative eigenvalues of A'. For this work cf. Gilkey and Smith [9], Booss–Bavnbek and Wojciechowski [3], or Brüning and Lesch [6] and the references there.

Non–homogeneous elliptic problems $Au = f$, $Bu = g$ for arbitrary elliptic differential operators A and suitable generalisations B of APS conditions have been studied in the author's joint papers with Nazaikinskij, Sternin and Shatalov [20], [21]. Index theory for this class of problems is formulated in Savin and Sternin [26], [27] in terms of a certain dimension function on subspaces of Sobolev spaces that are images of pseudo–differential projections on the boundary. This program is continued in joint work with Nazaikinskij, Savin and Sternin [24], [25], [19].

The main objective of the present paper is to complete the set of all elliptic differential boundary value problems of the above type to a new algebra of pseudo–differential boundary value problems that contains the parametrices of elliptic elements. (We call it an algebra, though compositions only belong to the structure when bundle and projection data of factors are compatible; in this sense we employ a similar terminology as it is customary in the standard pseudo–differential calculus of boundary value problems.) This will be a calculus of block matrix operators, where the upper left corners are the same as in $\mathcal{B}(X)$, namely (classical) pseudo–differential operators with the transmission property, plus Green operators. In particular, we shall see that each such operator that is elliptic with respect to σ_ψ admits elliptic conditions in the new algebra. We employ corresponding scales of spaces, in fact subspaces of the standard Sobolev spaces, for which ellipticity entails the Fredholm property, cf. [21], [20]. The subspaces are defined as the ranges of pseudo–differential projections on the boundary; in particular, Calderón–Seeley projections are of this type. Our new algebra $\mathcal{S}(X)$ contains a subalgebra $\mathcal{T}(Y)$ of (generalisations of) Toeplitz operators consisting of the lower right corners of the block matrices in $\mathcal{S}(X)$. Special operators of this kind are studied in Boutet de Monvel [5].

The present theory also includes an analogue of the classical reduction of elliptic boundary value problems in $\mathcal{S}(X)$ to the boundary, under which the resulting reduced operators are elliptic in $\mathcal{T}(Y)$. We also establish a formula that connects the index of elliptic elements in $\mathcal{S}(X)$ within our algebra with the index both of elliptic operators in $\mathcal{T}(Y)$ and of standard ones on $2X$, the double of X, as arise in the study of transmission problems.

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1 Operators on manifolds with boundary

1.1 The symbols of boundary value problems

Boundary value problems for pseudo–differential operators on a C^∞ manifold X with C^∞ boundary Y are generated by parametrices (or inverses) of elliptic differential boundary value problems. Let $\mathrm{Vect}(\cdot)$ be the set of all smooth complex vector bundles on a C^∞ manifold in the brackets. Consider an elliptic differential operator

$$A : C^\infty(X, E) \to C^\infty(X, F) \tag{1.1.1}$$

in spaces of C^∞ sections of $E, F \in \mathrm{Vect}(X)$.

A boundary value problem for A is traditionally regarded as an operator

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : C^\infty(X, E) \to \begin{matrix} C^\infty(X, F) \\ \oplus \\ C^\infty(Y, G) \end{matrix} \tag{1.1.2}$$

for a $G \in \mathrm{Vect}(Y)$, where $T : C^\infty(X, E) \to C^\infty(Y, G)$ is a trace operator that defines the boundary conditions in the problem $Au = f$, $Tu = g$.

In the simplest case T is a column matrix of operators $T_j = \mathrm{r}'\tilde{T}_j$, $j = 1, \ldots, N$, with the restriction $\mathrm{r}'u = u|_Y$ and differential operators $\tilde{T}_j : C^\infty(V, E) \to C^\infty(V, \tilde{G}_j)$

in a collar neighbourhood V of Y, with vector bundles $\tilde{G}_j$ on V and $G = \bigoplus_{j=1}^N G_j$, $G_j = \tilde{G}_j|_Y$.

The smooth complex vector bundles that arise are assumed to be equipped with Hermitian metrics. On X and Y we fix Riemannian metrics such that V corresponds to $[0,1) \times Y$ in the product metric. The canonical projections of the cotangent bundles minus zero sections are denoted by $\pi_X : T^*X \setminus 0 \to X$ and $\pi_Y : T^*Y \setminus 0 \to Y$, respectively.

Let $\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F$ be the homogeneous principal symbol of A of order $\mu = \operatorname{ord} A$, which is a map $\sigma_\psi(A)(x,\xi) : \mathbb{C}^k \to \mathbb{C}^l$ for each $(x,\xi) \in T^*X \setminus 0$, where k and l are the fibre dimensions of E and F, respectively. Locally near Y for $x = (t,y) \in \overline{\mathbb{R}}_+ \times \Omega$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, with the covariables $\xi = (\tau, \eta)$, we get an operator family, parametrised by $(y,\eta) \in T^*\Omega \setminus 0$,

$$\sigma_\partial(A)(y,\eta) = \sigma_\psi(A)(0,y,D_t,\eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^k \to \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^l \qquad (1.1.3)$$

for $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$, with the Schwartz space $\mathcal{S}(\mathbb{R})$. Globally on Y the system (1.1.3) represents a homomorphism

$$\sigma_\partial(A) : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \to \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F', \qquad (1.1.4)$$

$E' = E|_Y$, $F' = F|_Y$, called the homogeneous principal boundary symbol of A. Similarly the operators $\tilde{T}_j$ give rise to operator families

$$\sigma_\partial(\tilde{T}_j) : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \to \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes G_j,$$

$j = 1, \dots, N$. After composition with the operator r' of restriction to the boundary ($t = 0$) we obtain the boundary symbol of the trace operator T:

$$\sigma_\partial(T) = (\sigma_\partial T_j)_{j=1,\dots,N} : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \to \pi_Y^* G,$$

where $\sigma_\partial(T_j) = \mathrm{r}' \sigma_\partial(\tilde{T}_j)$.

We call

$$\sigma_\partial(\mathcal{A}) = \begin{pmatrix} \sigma_\partial(A) \\ \sigma_\partial(T) \end{pmatrix} : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \to \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F' \\ \oplus \\ G \end{pmatrix} \qquad (1.1.5)$$

the (homogeneous principal) boundary symbol and $\sigma_\psi(\mathcal{A}) = \sigma_\psi(A)$ the (homogeneous principal) interior symbol of $\mathcal{A}$.

Ellipticity of a boundary value problem $\mathcal{A}$ requires (by definition) the ellipticity of A in the usual sense, namely that $\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F$ is an isomorphism. The meaning of ellipticity (with respect to A) of boundary conditions is not a priori clear. In traditional boundary value problems (e.g., the Dirichlet or Neumann problem for the Laplace operator), ellipticity is guaranteed by the Shapiro–Lopatinskij condition, which is precisely the bijectivity of (1.1.5). However, it is well–known that there are elliptic differential operators A that do not possess SL elliptic boundary conditions, i.e., operators for which no corresponding T and G can be chosen to make (1.1.5) bijective. Examples include the Cauchy–Riemann operator in any smooth domain in the complex plane, Dirac operators on C^∞ manifolds with C^∞ boundaries (in even dimensions) and many other interesting geometric operators.

In fact, there is a topological obstruction whose presence or absence determines wether or not an elliptic operator A admits SL elliptic boundary conditions, cf. Seeley

[34], Atiyah and Bott [1], Boutet de Monvel [4]. Let us briefly recall this condition. Denote by S^*Y the unit sphere bundle of Y induced by T^*Y, with the projection $\pi_1 : S^*Y \to Y$. Assume for simplicity that Y is compact (otherwise we could consider any compact subsets K of Y). If A is elliptic, the restriction of (1.1.4) to S^*Y is a family of Fredholm operators parametrised by the compact space S^*Y. If we wish to work with Hilbert spaces, we can replace (1.1.4) by

$$\sigma_\partial(A) : \pi_1^* H^s(\mathbb{R}_+) \otimes E' \to \pi_1^* H^{s-\mu}(\mathbb{R}_+) \otimes F'$$

where the Sobolev spaces in question are defined by $H^s(\mathbb{R}_+) = H^s(\mathbb{R})|_{\mathbb{R}_+}$, for any sufficiently large s such that, say, $s - \mu > -\frac{1}{2}$. (The specific choice of s is not essential; kernels and cokernels are independent of s and coincide with those of the restrictions of the operators to the Schwartz spaces.) There is then an index element $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in K(S^*Y)$, and an elliptic operator A admits SL elliptic boundary conditions if and only if $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in \pi_1^* K(Y)$.

The operators of the form (1.1.2) belong to an algebra of block matrices

$$\begin{pmatrix} P+G & C \\ S & R \end{pmatrix} : \begin{array}{c} C^\infty(X,E) \\ \oplus \\ C^\infty(Y,J^-) \end{array} \to \begin{array}{c} C^\infty(X,F) \\ \oplus \\ C^\infty(Y,J^+) \end{array}. \tag{1.1.6}$$

Here P is a pseudo–differential operator on X with the transmission property with respect to Y, and G, S and C are Green, trace, and potential operators, respectively. The last component R is a pseudo–differential operator on Y, acting between sections of vector bundles $J^-, J^+ \in \mathrm{Vect}(Y)$. Such an algebra has been studied by Boutet de Monvel [4], cf. also the books of Rempel and Schulze [22], and Grubb [14]. For simplicity we consider the case of classical operators of integer order. By applying suitable elliptic pseudo–differential operators on Y we may (and will) assume that all orders are equal, say $\mu \in \mathbb{Z}$. By definition, Green and trace operators have a type $d \in \mathbb{N}$, cf. the notation below.

The space of all operators (1.1.6) of order μ and type d is denoted by $\mathcal{B}^{\mu,d}(X;E,F;J^-,J^+)$ and the space of upper left corners in the block matrices by $B^{\mu,d}(X;E,F)$. The latter space belongs to $L^\mu_{\mathrm{cl}}(\mathrm{int}\,X;E,F)$, the set of all classical pseudo–differential operators of order μ in $\mathrm{int}\,X$ (acting between sections in $E, F \in \mathrm{Vect}(X)$) with smooth (up to the boundary) local left symbols. More precisely, $B^{\mu,d}(X;E,F)$ is a subspace of all operators in $L^\mu_{\mathrm{cl}}(\mathrm{int}\,X;E,F)$ having the transmission property with respect to Y.

Each $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;E,F;J^-,J^+)$ induces continuous operators

$$\mathcal{A} : \begin{array}{c} H^s(X,E) \\ \oplus \\ H^s(Y,J^-) \end{array} \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y,J^+) \end{array} \tag{1.1.7}$$

for all $s \in \mathbb{R}$, $s - d > -\frac{1}{2}$; here $H^s(X,E), \ldots$, are the Sobolev spaces of distributional sections in the corresponding bundles. (Recall that X is assumed to be compact.) Notice that the orders of local "scalar" amplitude functions of the operators G, C, S are not μ but shifted by 1 or $\frac{1}{2}$, though the operator–valued symbols in the description below have precisely the order μ.

Let (1.1.2) be a boundary value problem for an elliptic differential operator A with SL elliptic boundary conditions, i.e., where (1.1.5) is an isomorphism. Composing $\mathcal{A}$ from the left by a reduction of orders we get an operator $\tilde{\mathcal{A}} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} A \\ T \end{pmatrix}$; here

R is a diagonal matrix of elliptic operators $R_j \in L^{\nu_j}_{\mathrm{cl}}(Y; G_j, G_j)$, $\nu_j = \mu - \operatorname{ord} \tilde{T}_j - \frac{1}{2}$ that induce isomorphisms $R_j : H^s(Y, G_j) \to H^{s-\nu_j}(Y, G_j)$ for all $s \in \mathbb{R}$. For simplicity we denote the operator $\tilde{\mathcal{A}}$ again by $\mathcal{A}$; this is then an elliptic element $\mathcal{A} \in \mathcal{B}^{\mu,0}(X; E, F; 0, G)$. (The argument 0 indicates the bundle of fibre dimension zero.) Boutet de Monvel [4] showed that $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,0}(X; E, F; G, 0)$, in the sense that the smoothing remainders $\mathcal{AP} - \mathcal{I}$, $\mathcal{PA} - \mathcal{I}$ are compact in Sobolev spaces. (Explicit definitions are given below.)

More generally, operators $\mathcal{A}$ in $\mathcal{B}^{\mu,d}(X; E, F; J^-, J^+)$ are characterised (modulo compact operators) by a pair of principal symbols $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$, namely interior and boundary symbols. Ellipticity of $\mathcal{A}$ means bijectivity of both the first component on $T^*X \setminus 0$ and of the second on $T^*Y \setminus 0$ (the SL condition). Each elliptic operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; E, F; J^-, J^+)$ has a parametrix $\mathcal{P} = \mathcal{B}^{-\mu,e}(X; F, E; J^+, J^-)$, for $e = (d - \mu)^+$, with $\varrho^+ = \max(\varrho, 0)$, $\varrho \in \mathbb{R}$, and

$$\sigma_\psi(\mathcal{P}) = \sigma_\psi(\mathcal{A})^{-1}, \qquad \sigma_\partial(\mathcal{P}) = \sigma_\partial(\mathcal{A})^{-1}.$$

The union of all $\mathcal{B}^{\mu,d}(X; E, F; J^-, J^+)$ over $E, F \in \operatorname{Vect}(X)$, $J^-, J^+ \in \operatorname{Vect}(Y)$ is denoted by $\mathcal{B}^{\mu,d}(X)$, $(\mu, d) \in \mathbb{Z} \times \mathbb{N}$. Algebraic operations are defined when the bundles fit together. Let

$$\operatorname{Ell} \mathcal{B}^{\mu,d}(X; E, F; J^-, J^+) \tag{1.1.8}$$

be the set of all $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; E, F; J^-, J^+)$ that are elliptic with respect to $\sigma(\cdot) = (\sigma_\psi(\cdot), \sigma_\partial(\cdot))$ and

$$\operatorname{Ell}_\psi B^{\mu,d}(X; E, F) \tag{1.1.9}$$

the set of all $A \in B^{\mu,d}(X; E, F)$ that are elliptic with respect to $\sigma_\psi(\cdot)$. Then, if we denote by $(\mathrm{SL}) \operatorname{Ell}_\psi B^{\mu,d}(X; E, F)$ the space of upper left corners of elements in (1.1.8), we have a proper inclusion

$$(\mathrm{SL}) \operatorname{Ell}_\psi B^{\mu,d}(X; E, F) \subset \operatorname{Ell}_\psi B^{\mu,d}(X; E, F).$$

As noted in the beginning there is also a proper inclusion

$$(\mathrm{SL}) \operatorname{Ell}_\psi \operatorname{Diff}^\mu(X; E, F) \subset \operatorname{Ell}_\psi \operatorname{Diff}^\mu(X; E, F),$$

for each order $\mu \in \mathbb{N}$. Here $\operatorname{Diff}^\mu(X; E, F)$ is the space of all differential operators of order μ on X with smooth coefficients up to $Y = \partial X$, acting in the corresponding spaces of sections, Ell_ψ denotes the elliptic elements, and $(\mathrm{SL}) \operatorname{Ell}_\psi$ those which admit Shapiro–Lopatinskij elliptic boundary conditions.

The examination of these boundary value problems leads to an important question: Does there exist an algebra $\mathcal{S}(X) = \bigcup_{\mu,d} \mathcal{S}^{\mu,d}(X)$ of block matrix operators that generalises $\mathcal{B}(X) = \bigcup_{\mu,d} \mathcal{B}^{\mu,d}(X)$ in such a way that each element $A \in \operatorname{Ell}_\psi \operatorname{Diff}^\mu(X; E, F)$ (or more generally each $A \in \operatorname{Ell}_\psi B^{\mu,d}(X; E, F)$) can be completed by "elliptic boundary conditions" to an operator $\mathcal{A} \in \mathcal{S}(X)$ that has a parametrix in the new algebra. The affirmative answer is given in Sections 1.4 and 2.1 below.

1.2 Pseudo–differential and Green operators

Our algebra $\mathcal{S}(X) = \bigcup_{\mu,d} \mathcal{S}^{\mu,d}(X)$ will be defined as a set of block matrix operators $\mathcal{A} = (A_{ij})_{i,j=1,2}$ with $\mathcal{B}(X) \subset \mathcal{S}(X)$ and

$$\mathrm{u.l.c.}\,\mathcal{B}(X) = \mathrm{u.l.c.}\,\mathcal{S}(X),$$

where u. l. c. indicates the corresponding spaces of upper left corners A_{11}. In contrast to (1.1.7) the operators $\mathcal{A} \in \mathcal{S}(X)$ will be continuous maps

$$\mathcal{A}: \begin{array}{c} H^s(X,E) \\ \oplus \\ P^s(Y,\boldsymbol{L}^+) \end{array} \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ P^{s-\mu}(Y,\boldsymbol{L}^-) \end{array}, \tag{1.2.1}$$

where

$$P^s(Y,\boldsymbol{L}^+) \subset H^s(Y,J^-), \qquad P^{s-\mu}(Y,\boldsymbol{L}^-) \subset H^{s-\mu}(Y,J^+)$$

are closed subspaces, for certain $J^-, J^+ \in \mathrm{Vect}(Y)$, related to prescribed sub–bundles $L^\pm$ of $\pi_Y^* J^\pm$ and pseudo–differential projections $P^\pm : H^s(Y,J^\pm) \to P^s(Y,\boldsymbol{L}^\pm)$. Precise definitions are given below. In order to develop the calculus, we prepare further material on the spaces $B^{\mu,d}(X;E,F)$, $E,F \in \mathrm{Vect}(X)$, that constitute u. l. c. $\mathcal{B}(X)$. First we have the space $B^{-\infty,0}(X;E,F)$ of smoothing operators of type 0; its elements $C : C^\infty(X,E) \to C^\infty(X,F)$ are nothing other than operators with C^∞ kernels up to the boundary. Moreover, the space $B^{-\infty,d}(X;E,F)$ of smoothing operators of type $d \in \mathbb{N}$ is defined to be the set of all $C = \sum_{j=0}^d C_j D^j$, where D^j are arbitrary elements in $\mathrm{Diff}^j(X;E,E)$ and $C_j \in B^{-\infty,0}(X;E,F)$. The space $B_G^{\nu,0}(X;E,F)$ of all Green operators of order ν and type zero is defined to be the set of all $G_0 + C$ for arbitrary $C \in B^{-\infty,0}(X;E,F)$ and smoothing operators G_0 on $\mathrm{int}\, X$, that are locally near Y in coordinates $(t,y) \in \overline{\mathbb{R}}_+ \times \Omega$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, of the form of pseudo–differential operators along Ω with operator–valued symbols $g(y,\eta)$, namely

$$C_0^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)) \ni u(y) \to \mathrm{Op}_y(g)u(y) = \iint e^{i(y-y')\eta} g(y,\eta) u(y')\, dy'\, đ\eta$$

for

$$g(y,\eta) \in S_{\mathrm{cl}}^\nu(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+) \otimes \mathbb{C}^k, \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^l),$$
$$g^*(y,\eta) \in S_{\mathrm{cl}}^\nu(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+) \otimes \mathbb{C}^l, \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^k).$$

Here k and l are the fibre dimensions of E and F, respectively. Details on pseudo–differential operators with operator–valued symbols in this context may be found in Schulze [30], Section 2.2.2, or [32], Section 1.3 (see also [32], Section 4.2.3). The present definition of the Green operators from [29], Theorem 3.1, is equivalent to the original one in [4]. Moreover, $B_G^{\mu,d}(X;E,F)$, the space of all Green operators of order μ and type d is defined to be the set of all

$$G = \sum_{j=0}^d G_{\mu-j} D^j + C \tag{1.2.2}$$

for arbitrary $G_{\mu-j} \in B_G^{\mu-j,0}(X;E,F)$, $D^j \in \mathrm{Diff}^j(X;E,E)$, $C \in B^{-\infty,d}(X;E,F)$. Finally, $B^{\mu,d}(X;E,F)$ is defined to be the set of all operators $P+G$ for arbitrary $G \in B_G^{\mu,d}(X;E,F)$ and $P \in L_{\mathrm{cl}}^\mu(\mathrm{int}\, X;E,F)$, where in local coordinates $x = (t,y) \in \overline{\mathbb{R}}_+ \times \Omega$ near Y the operator P has the form $\mathrm{r}^+ \mathrm{Op}_x(p) \mathrm{e}^+$ for a symbol $p(x,\xi) \in S_{\mathrm{cl}}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^n) \otimes \mathbb{C}^l \otimes \mathbb{C}^k$ with the transmission property at $t=0$. Here e^+ is the operator extending functions on $\mathbb{R}_+ \times \Omega$ by zero to $\mathbb{R} \times \Omega$ and r^+ is the restriction operator from $\mathbb{R} \times \Omega$ to $\mathbb{R}_+ \times \Omega$. Each $A \in B^{\mu,d}(X;E,F)$ has a homogeneous principal symbol $\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F$, given by $\sigma_\psi(A) = \sigma_\psi(P)$ for $A = P + G$. Further there is the homogeneous principal boundary symbol of A that is locally for $(y,\eta) \in T^*\Omega \setminus 0$ of the form

$$\sigma_\partial(A)(y,\eta) = \mathrm{r}^+\sigma_\psi(P)(0,y,D_t,\eta)\mathrm{e}^+ + \sigma_\partial(G)(y,\eta),$$

with $\sigma_\psi(P)(0,y,D_t,\eta)u(t) = \iint e^{i(t-t')\tau}\sigma_\psi(P)(0,y,\tau,\eta)u(t')\,dt' đ\tau$ and

$$\sigma_\partial(G)(y,\eta) = \sum_{j=0}^{d} \sigma_\partial(G_{\mu-j})(y,\eta)\sigma_\psi(D^j)(0,y,D_t,\eta), \tag{1.2.3}$$

with $\sigma_\partial(G_{\mu-j})(y,\eta)$ being the homogeneous principal part of the classical operator–valued symbol of $G_{\mu-j}$ in the representation (1.2.2) above.

(A decomposition like 1.2.3 is not unique, in contrast to the operator function $\sigma_\partial(G)(y,\eta)$ itself; throughout this paper multiplications of operator functions are interpreted as pointwise compositions, also indicated by $\circ$.) Globally we get a homomorphism

$$\sigma_\partial(A) : \pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \to \pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+) \otimes F'$$

or

$$\sigma_\partial(A) : \pi_Y^* H^s(\mathbb{R}_+) \otimes E' \to \pi_Y^* H^{s-\mu}(\mathbb{R}_+) \otimes F' \tag{1.2.4}$$

for (sufficiently large) $s \in \mathbb{R}$, $\pi_Y : T^*Y \setminus 0 \to Y$. (Clearly homomorphisms in this connection are understood in the standard fibre bundle sense, here with the corresponding infinite–dimensional fibres and invariance of local representations under transition maps that are inherited from the bundles E' and F'; for simplicity we always refer to an atlas near Y that keeps the normal variable t fixed.) We also consider the restriction

$$\sigma_\partial(A) : \pi_1^* H^s(\mathbb{R}_+) \otimes E' \to \pi_1^* H^{s-\mu}(\mathbb{R}_+) \otimes F' \tag{1.2.5}$$

to the cosphere bundle S^*Y with the projection $\pi_1 : S^*Y \to Y$. Each operator $A \in B^{\mu,d}(X;E,F)$ then induces continuous operators

$$A : H^s(X,E) \to H^{s-\mu}(X,F)$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$. We need the following result on reductions of orders:

Theorem 1.2.1 *Let E be a vector bundle on X. To every $\mu \in \mathbb{Z}$ there exists an element $R^\mu = R^\mu(E) \in B^{\mu,0}(X;E,E)$ which induces isomorphisms*

$$R^\mu : H^s(X,E) \to H^{s-\mu}(X,E) \tag{1.2.6}$$

for all $s \in \mathbb{R}$ where $(R^\mu)^{-1} \in B^{-\mu,0}(X;E,E)$.

The statement of Theorem 1.2.1 for all real s is first published in Grubb [12]. Reductions of orders for Sobolev spaces on a manifold with boundary exist in different versions, cf. Boutet de Monvel [4], Section 5, Rempel and Schulze [23], Section 3.3, Eskin [7], Lemma 4.6. The method of [4] (that works for $s > \mu^+ - \frac{1}{2}$, $\mu^+ = \max(\mu, 0)$) can be formulated in terms of a symbol with the transmission property $(\chi(\xi)\langle\eta\rangle - i\tau)^\mu$ with a suitable choice of a function χ. Grubb uses $(\chi(\xi)\langle\eta\rangle - i\tau)^\mu$ with a minus function χ in τ which implies the desired behaviour for all s (see also [7] for that point).

Let us now briefly recall the definition of the full algebra $\mathcal{B}(X)$. This is the union of spaces $\mathcal{B}^{\mu,d}(X;E,F;J^-,J^+)$ over $(\mu,d) \in \mathbb{Z} \times \mathbb{N}$ and $E,F \in \mathrm{Vect}(X)$, $J^-,J^+ \in \mathrm{Vect}(Y)$ (for more details, cf. [32]). Let us set

$$\boldsymbol{b} = (E,F;J^-,J^+).$$

The difference between $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ and $B^{\mu,d}(X;E,F)$, the space of upper left corners, lies only in a subclass $\mathcal{B}_G^{\mu,d}(X;\boldsymbol{b})$ of block matrix operators that is defined as follows. First the space $\mathcal{B}^{-\infty,0}(X;\boldsymbol{b})$ of smoothing operators of type 0 consists of operators $C^\infty(X,E)\oplus C^\infty(Y,J^-)\to C^\infty(X,F)\oplus C^\infty(Y,J^+)$ whose entries have C^∞ kernels (smooth up to the boundary in the arguments on X). Moreover, the space $\mathcal{B}^{-\infty,d}(X;\boldsymbol{b})$ of smoothing operators of type d is defined to be the set of all $\mathcal{C}=\mathcal{C}_0+\sum_{j=1}^d \mathcal{C}_j \begin{pmatrix} D^j & 0\\ 0 & 0\end{pmatrix}$ for arbitrary $D^j\in \mathrm{Diff}^j(X;E,E)$ and $\mathcal{C}_j\in\mathcal{B}^{-\infty,0}(X;E,F)$. Further $\mathcal{B}_G^{\nu,0}(X;\boldsymbol{b})$ is the set of all $\mathcal{G}_0+\mathcal{C}$ for $\mathcal{C}\in\mathcal{B}^{-\infty,0}(X;\boldsymbol{b})$ and operators $\mathcal{G}_0$ that are smoothing in $x\in\mathrm{int}\,X$, $y\in Y$ and have in local coordinates $(t,y)\in\overline{\mathbb{R}}_+\times\Omega$ near Y the form of pseudo–differential operators

$$\mathrm{Op}_y(g): C_0^\infty(\Omega,(\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^k)\oplus\mathbb{C}^{N_-})\to C^\infty(\Omega,(\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^l)\oplus\mathbb{C}^{N_+})$$

along Ω with operator–valued symbols. In this definition $N_\pm$ denotes the fibre dimensions of $J^\pm$, where

$$g(y,\eta)\in S_{\mathrm{cl}}^\nu(\Omega\times\mathbb{R}^{n-1};(L^2(\mathbb{R}_+)\otimes\mathbb{C}^k)\oplus\mathbb{C}^{N_-},(\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^l)\oplus\mathbb{C}^{N_+}),$$
$$g^*(y,\eta)\in S_{\mathrm{cl}}^\nu(\Omega\times\mathbb{R}^{n-1};(L^2(\mathbb{R}_+)\otimes\mathbb{C}^l)\oplus\mathbb{C}^{N_+},(\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^k)\oplus\mathbb{C}^{N_-}).$$

Moreover, for $\boldsymbol{b}=(E,F;J^-,J^+)$ the space $\mathcal{B}_G^{\mu,d}(X;\boldsymbol{b})$ is defined to consist of all operators

$$\mathcal{G}=\mathcal{G}_\mu+\sum_{j=1}^d \mathcal{G}_{\mu-j}\begin{pmatrix} D^j & 0\\ 0 & 0\end{pmatrix}+\mathcal{C} \tag{1.2.7}$$

for arbitrary $\mathcal{G}_{\mu-j}\in\mathcal{B}_G^{\mu-j,0}(X;\boldsymbol{b})$, $0\le j\le d$, $D^j\in\mathrm{Diff}^j(X;E,E)$, $\mathcal{C}\in\mathcal{B}^{-\infty,d}(X;\boldsymbol{b})$. Notice that $B_G^{\mu,d}(X;E,F)=\mathrm{u.l.c.}\,\mathcal{B}_G^{\mu,d}(X;\boldsymbol{b})$. Every $\mathcal{G}\in\mathcal{B}_G^{\mu,d}(X;\boldsymbol{b})$ has a homogeneous principal symbol $\sigma_\partial(G)$, locally defined by

$$\sigma_\partial(\mathcal{G})(y,\eta)=\sigma_\partial(\mathcal{G}_\mu)(y,\eta)+\sum_{j=1}^d \sigma_\partial(\mathcal{G}_{\mu-j})(y,\eta)\begin{pmatrix}\sigma_\psi(D^j)(0,y,D_t,\eta) & 0\\ 0 & 0\end{pmatrix}$$

with $\sigma_\partial(\mathcal{G}_{\mu-j})(y,\eta)$ being the homogeneous principal part of the classical operator–valued symbol of $\mathcal{G}_{\mu-j}$, $0\le j\le d$. Now $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ is the space of all operators of the form

$$\mathcal{A}=\begin{pmatrix} P & 0\\ 0 & 0\end{pmatrix}+G$$

for arbitrary $P\in\mathcal{B}^{\mu,d}(X;E,F)$ and $\mathcal{G}=(G_{ij})_{i,j=1,2}\in\mathcal{B}_G^{\mu,d}(X;\boldsymbol{b})$. We then set

$$\sigma_\psi(\mathcal{A})=\sigma_\psi(P),\qquad \sigma_\partial(\mathcal{A})=\begin{pmatrix}\sigma_\partial(P) & 0\\ 0 & 0\end{pmatrix}+\sigma_\partial(\mathcal{G})$$

and $\sigma(\mathcal{A})=(\sigma_\psi(\mathcal{A}),\sigma_\partial(\mathcal{A}))$. Globally, the boundary symbol $\sigma_\partial(\mathcal{A})$ of $\mathcal{A}$ is a homomorphism

$$\sigma_\partial(\mathcal{A}):\pi_Y^*\begin{pmatrix}\mathcal{S}(\overline{\mathbb{R}}_+)\otimes E'\\ \oplus\\ J^-\end{pmatrix}\to\pi_Y^*\begin{pmatrix}\mathcal{S}(\overline{\mathbb{R}}_+)\otimes F'\\ \oplus\\ J^+\end{pmatrix},$$

or

$$\sigma_\partial(\mathcal{A}):\pi_Y^*\begin{pmatrix}H^s(\mathbb{R}_+)\otimes E'\\ \oplus\\ J^-\end{pmatrix}\to\pi_Y^*\begin{pmatrix}H^{s-\mu}(\mathbb{R}_+)\otimes F'\\ \oplus\\ J^+\end{pmatrix}$$

for (sufficiently large) $s\in\mathbb{R}$. The following result is well–known, cf. [4], [22], [14]:

Theorem 1.2.2 *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; E_0, F; J_0, J^+)$ and $\mathcal{B} \in \mathcal{B}^{\nu,e}(X; E, E_0; J^-, J_0)$. Then for $h = \max(d+\nu, e)$ the product $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h}(X; E, F; J^-, J^+)$, where the symbol*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

is formed by componentwise multiplication.

1.3 Boundary symbols

We now develop the boundary symbol calculus for operators $A \in B^{\mu,d}(X; E, F)$ without referring to Shapiro–Lopatinskij conditions in the elliptic case. In other words we admit arbitrary operators $A \in B^{\mu,d}(X; E, F)$, elliptic with respect to σ_ψ, i.e., where $\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F$ is an isomorphism. In this context (1.2.5) is a family of Fredholm operators for each sufficiently large s. In fact, $p_\partial(y,\eta) := \mathrm{r}^+ \sigma_\psi^{-1}(A)(0, y, D_t, \eta)\mathrm{e}^+ : \pi_Y^* H^{s-\mu}(\mathbb{R}_+) \otimes F' \to \pi_Y^* H^s(\mathbb{R}_+) \otimes E'$ is a family of operators with

$$p_\partial(y,\eta) \circ \sigma_\partial(A)(y,\eta) = 1 + c(y,\eta)$$

for some family of Green operators, that is, $c(y,\eta) = \sigma_\partial(C)(y,\eta)$ for a Green operator C of order 0 and type $h = \max(-\mu, d)$. (Compare Theorem 1.2.2 above.) Since $c(y,\eta) : H^s(\mathbb{R}_+) \otimes E_y \to H^s(\mathbb{R}_+) \otimes E_y$ is compact for every $s > h - \frac{1}{2}$, the operator $\sigma_\partial(A)(y,\eta)$ is Fredholm for every $(y,\eta) \in T^*Y \setminus 0$. We then obtain an index element

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in K(S^*Y) \tag{1.3.1}$$

that is independent of the choice of s. Moreover, since $\sigma_\partial(G)(y,\eta) : H^s(\mathbb{R}_+) \otimes E_y \to H^{s-\mu}(\mathbb{R}_+) \otimes E_y$ is compact for every $G \in B_G^{\mu,d}(X; E, F)$ (for $s > d - \frac{1}{2}$), we have

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A+G) = \mathrm{ind}_{S^*Y}\, \sigma_\partial(A)$$

for (elliptic with respect to σ_ψ) $A \in B^{\mu,d}(X; E, F)$ and $G \in B_G^{\mu,d}(X; E, F)$.

In the following considerations we often employ the Schwartz space $\mathcal{S}(\overline{\mathbb{R}}_+)$ instead of Sobolev spaces on $\mathbb{R}_+$, though our operator families extend to appropriate Sobolev spaces of sufficiently large smoothness.

Lemma 1.3.1 *Let $A \in B^{\mu,d}(X; E, F)$ be elliptic with respect to σ_ψ. Then there exists an isomorphism*

$$\boldsymbol{a} = \begin{pmatrix} \sigma_\partial(A) & \lambda^- \\ \lambda^+ & \varrho \end{pmatrix} : \begin{pmatrix} \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \\ \oplus \\ L^- \end{pmatrix} \to \begin{pmatrix} \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F' \\ \oplus \\ L^+ \end{pmatrix} \tag{1.3.2}$$

for suitable $L^-, L^+ \in \mathrm{Vect}(T^*Y \setminus 0)$, where (1.3.2) is homogeneous in the sense

$$\boldsymbol{a}(y,\delta\eta) = \delta^\mu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \mathrm{id}_{L^+} \end{pmatrix} \boldsymbol{a}(y,\eta) \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \mathrm{id}_{L^-} \end{pmatrix}^{-1} \tag{1.3.3}$$

for $(\kappa_\delta u)(t) = \delta^{\frac{1}{2}} u(\delta t)$, $u \in \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E_y$, $\delta \in \mathbb{R}_+$, $(y,\eta) \in T^*Y \setminus 0$.

If we consider $\sigma_\partial(A)$ on S^*Y we can easily construct an isomorphism in block matrix form where

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) = [L^+|_{S^*Y}] - [L^-|_{S^*Y}]. \tag{1.3.4}$$

In the present context we wish the entries $\lambda^\pm$ in a similar form as in the standard boundary symbol calculus for Boutet de Monvel's algebra. Then, having $\lambda^\pm$ on S^*Y, we get corresponding operator families on $T^*Y \setminus 0$ by homogeneous extensions.

Using the fact that cokernels and kernels of (1.2.4) for elliptic A can be represented by vectors of elements in $\mathcal{S}(\overline{\mathbb{R}}_+)$, the operator functions λ^-, λ^+ can be chosen as sections in tensor products

$$(\pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+) \otimes F') \otimes (L^-)^* \qquad \text{and} \qquad L^+ \otimes (\pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+) \otimes E')^*,$$

respectively. In other words, λ^- acts on a vector in $L^-_{y,\eta}$ via the pairing with a corresponding element in $(L^-_{y,\eta})^*$ and thus maps to $(\mathcal{S}(\overline{\mathbb{R}}_+) \otimes F')_{y,\eta}$, while λ^+ acts via the pairing of functions by the scalar product in $L^2(\mathbb{R}_+)$, fibrewise for each $(y,\eta) \in T^*Y \setminus 0$ represented as

$$\left\{ \sum_{j=0}^{k} \int_0^\infty u_j(t,y,\eta)\lambda^+_{jm}(t,y,\eta)\, dt \right\}_{m=1,\dots,l^+}.$$

Here $(\lambda^+_{jm}(t,y,\eta))_{\substack{j=1,\dots,k\\ m=1,\dots,l^+}}$ is the value of the section λ^+ over y, η with respect to chosen trivialisations of L^+ and E', with l^+ and k being the fibre dimensions of L^+ and E', respectively. The argument from $(\pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+) \otimes E')_{y,\eta}$ is represented by the vector $(u_1(t,y,\eta), .., u_k(t,y,\eta))$.

In other words, the proof of Lemma 1.3.1 is analogous to the construction of elliptic boundary symbols using an elliptic operator $A \in B^{\mu,d}(X; E, F)$ in the context of [4] (concerning an explicit proof, cf., for instance, [32], Proposition 4.3.45). These considerations only employ (1.2.4) as a Fredholm homomorphism, together with the homogeneous extension to $T^*Y \setminus 0$, but does not use the condition $\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in \pi_1^* K(Y)$. Therefore, the bundles $L^\pm \in \operatorname{Vect}(T^*Y \setminus 0)$ that arise in the block matrix (1.3.2) are not necessarily lifted versions of the bundles $J^\pm$ on Y. Boundary symbols in the sense of [4] are characterised by $L^\pm = \pi_Y^* J^\pm$ for certain $J^\pm \in \operatorname{Vect}(Y)$. Clearly, in the general case, the bundles $L^\pm$ can be regarded as sub–bundles of liftings $M^\pm = \pi_Y^* J^\pm$ for certain $J^\pm \in \operatorname{Vect}(Y)$. To achieve this, it suffices to choose complementary bundles $(L^\pm)^\perp$ in trivial bundles $(T^*Y \setminus 0) \times \mathbb{C}^{m_\pm}$ for suitable $m_\pm$; then we may set $J^\pm = Y \times \mathbb{C}^{m_\pm}$.

Remark 1.3.2 *In the case* $\dim Y = 1$*, where the cosphere bundle* S^*Y *splits into two copies* Y_- *and* Y_+ *of* Y*, the bundles* $L^\pm \in \operatorname{Vect}(S^*Y)$ *(or similarly* $\in \operatorname{Vect}(T^*Y \setminus 0)$*) are admitted to be of different fibre dimensions over the corresponding plus or minus parts. We talk about subbundles of* $\pi_1^* J$ *(or* $\pi_Y^* J$*) for* $J \in \operatorname{Vect}(Y)$ *also in this situation. In particular, there is a natural decomposition* $\pi_Y^* J = L^- \oplus L^+$*, where* $L^\pm$ *are the restrictions of* $\pi_Y^* J$ *to the* $\pm$ *parts of* $T^*Y \setminus 0$*. Clearly, they are not pull–backs of bundles on* Y*.*

Let $J^\pm$ be arbitrary bundles on Y such that $L^\pm$ are sub–bundles of $M^\pm = \pi_Y^* J^\pm$. Choose projections $p^\pm : M^\pm \to L^\pm$ that are C^∞ in $(y,\eta) \in T^*Y \setminus 0$ and homogeneous of order zero in η, i.e., $p^\pm(y,\delta\eta) = p^\pm(y,\eta)$ for all $(y,\eta) \in T^*Y \setminus 0$ and all $\delta > 0$. It is well–known that $p^\pm(y,\eta)$ can be regarded as the homogeneous principal symbols of pseudo–differential operators

$$P^\pm \in L^0_{\mathrm{cl}}(Y; J^\pm, J^\pm).$$

The possible choices of such projections have been characterised in Gramsch [11] in a very general framework. Special choices of $P^{\pm}$ to $p^{\pm}$ can explicitly be written in terms of suitable integrals, using the holomorphic functional calculus for pseudo–differential operators, cf. Section 2.4 below. Examples are the Calderón–Seeley projections, cf. Seeley [34], Birman and Solomjak [2], or Wojciechowski [36].

Our class of general pseudo–differential boundary value problems will be denoted by

$$\mathcal{S}^{\mu,d}(X;\boldsymbol{v}), \qquad (\mu,d)\in\mathbb{Z}\times\mathbb{N}, \tag{1.3.5}$$

where the symbol $\boldsymbol{v}=(E,F;(J^-,L^-,P^-),(J^+,L^+,P^+))$ abbreviates the data entering into the above discussion. The components of $\boldsymbol{v}$ are as follows:

(i) $E,F\in\mathrm{Vect}(X)$,

(ii) sub–bundles $L^{\pm}\in\mathrm{Vect}(T^*Y\setminus 0)$ of lifted bundles $M^{\pm}=\pi_Y^*J^{\pm}$ for given $J^{\pm}\in\mathrm{Vect}(Y)$,

(iii) pseudo–differential projections $P^{\pm}\in L^0_{\mathrm{cl}}(Y;J^{\pm},J^{\pm})$ whose homogeneous principal symbols are projections $p^{\pm}:M^{\pm}\to L^{\pm}$.

To develop a calculus we introduce the boundary symbols of operators in (1.3.5). Denote by

$$r^{\pm}:L^{\pm}\to M^{\pm} \tag{1.3.6}$$

the embeddings as sub–bundles, homogeneous of order zero in the sense $r^{\pm}(y,\lambda\eta)=r^{\pm}(y,\eta)$ for all $(y,\eta)\in T^*Y\setminus 0$, $\lambda\in\mathbb{R}_+$. Then $p^{\pm}r^{\pm}=\mathrm{id}_{L^{\pm}}$ and $r^{\pm}p^{\pm}=p^{\pm}$.

Definition 1.3.3 *The space $\sigma_\partial\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ of boundary symbols is the set of all operator families*

$$\boldsymbol{a}=\begin{pmatrix} a & \lambda^- \\ \lambda^+ & \varrho \end{pmatrix} : \begin{pmatrix} \pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+)\otimes E' \\ \oplus \\ L^- \end{pmatrix} \to \begin{pmatrix} \pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+)\otimes F' \\ \oplus \\ L^+ \end{pmatrix} \tag{1.3.7}$$

that are C^∞ in $(y,\eta)\in T^*Y\setminus 0$ and homogeneous in η of order μ (in the sense of relation (1.3.7)). Here $a=\sigma_\partial(A)$ for arbitrary $A\in B^{\mu,d}(X;E,F)$ and

$$\lambda^+=p^+\sigma_\partial(T), \qquad \lambda^-=\sigma_\partial(K)r^-, \qquad \varrho=p^+\sigma_\partial(Q)r^- \tag{1.3.8}$$

for arbitrary elements

$$\begin{pmatrix} 0 & K \\ T & Q \end{pmatrix}\in\mathcal{B}^{\mu,d}(X;E,F;J^-,J^+).$$

Notice that the specific choices of the projections $P^{\pm}$ do not affect the operator families (1.3.7). Given an $\boldsymbol{a}\in\sigma_\partial\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ with upper left corner $a\in\sigma_\partial B^{\mu,d}(X;E,F)$ we set $\sigma_\psi(\boldsymbol{a})=\sigma_\psi(a)$.

Theorem 1.3.4 *$\boldsymbol{a}\in\sigma_\partial\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$, $\boldsymbol{b}\in\sigma_\partial\mathcal{S}^{\nu,e}(X;\boldsymbol{w})$ for $\mu,\nu\in\mathbb{Z}$, $d,e\in\mathbb{N}$ and*

$$\begin{aligned}\boldsymbol{v}&=(E_0,F;(J_0,L_0,P_0),(J^+,L^+,P^+)),\\ \boldsymbol{w}&=(E,E_0;(J^-,L^-,P^-),(J_0,L_0,P_0))\end{aligned}$$

implies $\boldsymbol{ab}\in\sigma_\partial\mathcal{S}^{\mu+\nu,h}(X;\boldsymbol{v}\circ\boldsymbol{w})$ for

$$h=\max(\nu+d,e), \qquad \boldsymbol{v}\circ\boldsymbol{w}=(E,F;(J^-,L^-,P^-),(J^+,L^+,P^+)),$$

and we have $\sigma_\psi(\boldsymbol{ab})=\sigma_\psi(\boldsymbol{a})\sigma_\psi(\boldsymbol{b})$.

Proof. By assumption there are elements

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} \in \mathcal{B}^{\mu,d}(X; E_0, F; J_0, J^+),$$

$$\mathcal{B} = \begin{pmatrix} B & H \\ S & R \end{pmatrix} \in \mathcal{B}^{\nu,e}(X; E, E_0; J^-, J_0)$$

such that for $\boldsymbol{a} = (a_{ij})_{i,j=1,2}$, $\boldsymbol{b} = (b_{ij})_{i,j=1,2}$ we have

$$\begin{aligned} a_{11} &= \sigma_\partial(A), & a_{12} &= \sigma_\partial(K)r_0, & a_{21} &= p^+\sigma_\partial(T), & a_{22} &= p^+\sigma_\partial(Q)r_0, \\ b_{11} &= \sigma_\partial(B), & b_{12} &= \sigma_\partial(H)r^-, & a_{21} &= p_0\sigma_\partial(S), & b_{22} &= p_0\sigma_\partial(R)r^-, \end{aligned}$$

with projections $p_0 : M_0 \to L_0$, $p^+ : M^+ \to L^+$, and embeddings $r_0 : L_0 \to M_0$, $r^- : L^- \to M^-$. Write $\boldsymbol{ab} = \boldsymbol{c} = (c_{ij})_{i,j=1,2}$.

We have $r_0 p_0 = p_0 : \pi_Y^* J_0 \to \pi_Y^* J_0$ which is the homogeneous principal symbol of order zero of a pseudo–differential operator $P_0 \in L^0_{\mathrm{cl}}(Y; J_0, J_0)$. From the rules for the composition of boundary symbols in standard pseudo–differential boundary value problems, i.e., of the class $\mathcal{B}(X)$, we get the following relations

$$\begin{aligned} c_{11} &= \sigma_\partial(A)\sigma_\partial(B) + \sigma_\partial(K)r_0p_0\sigma_\partial(S) \\ &= \sigma_\partial(AB + KP_0S), \\ c_{12} &= \sigma_\partial(A)\sigma_\partial(H)r^- + \sigma_\partial(K)r_0p_0\sigma_\partial(R)r^- \\ &= \sigma_\partial(AH + KP_0R)r^-, \\ c_{21} &= p^+\sigma_\partial(T)\sigma_\partial(B) + p^+\sigma_\partial(Q)r_0p_0\sigma_\partial(S) \\ &= p^+\sigma_\partial(TB + QP_0S), \\ c_{22} &= p^+\sigma_\partial(T)\sigma_\partial(H)r^- + p^+\sigma_\partial(Q)r_0p_0\sigma_\partial(R)r^- \\ &= p^+\sigma_\partial(TH + QP_0R)r^-, \end{aligned}$$

where

$$\begin{pmatrix} AB + KP_0S & AH + KP_0R \\ TB + QP_0S & TH + QP_0R \end{pmatrix} \in \mathcal{B}^{\mu+\nu,h}(X; E, F; J^-, J^+).$$

Thus $\boldsymbol{ab} \in \sigma_\partial \boldsymbol{\mathcal{S}}^{\mu+\nu,h}(X; \boldsymbol{v} \circ \boldsymbol{w})$. Note, in particular, that KP_0S is a Green operator in $B^{\mu+\nu,h}(X; E, F)$, i.e., $\sigma_\psi(KP_0S) = 0$. From the symbol rules in boundary value problems we get $\sigma_\psi(AB) = \sigma_\psi(A)\sigma_\psi(B)$. This completes the proof of the theorem. □

1.4 The algebra of boundary value problems

The operators $P^\pm$ induce continuous projections in $H^s(Y, J^\pm)$ for all $s \in \mathbb{R}$. Accordingly, we set

$$P^s(Y, \boldsymbol{L}^\pm) = \{g = P^\pm u : \ u \in H^s(Y, J^\pm)\} \qquad \text{for } \boldsymbol{L}^\pm = (J^\pm, L^\pm, P^\pm).$$

These are closed subspaces of $H^s(Y, J^\pm)$. Denote by

$$R^\pm : P^s(Y, \boldsymbol{L}^\pm) \to H^s(Y, J^\pm)$$

the canonical embeddings; the relevant degree of Sobolev smoothness $s \in \mathbb{R}$ will be clear in each concrete case, so we do not indicate it explicitly, neither for $P^\pm$ nor for $R^\pm$.

Given vector bundles $E, F \in \mathrm{Vect}(X)$, $J^-, J^+ \in \mathrm{Vect}(Y)$ and sub–bundles $L^\pm$ of $\pi_Y^* J^\pm$ we set

$$\boldsymbol{v} = (E, F; \boldsymbol{L}^-, \boldsymbol{L}^+) \qquad \text{for } \boldsymbol{L}^\pm = (J^\pm, L^\pm, P^\pm). \tag{1.4.1}$$

Here $P^\pm \in L^0_{\mathrm{cl}}(Y; J^\pm, J^\pm)$ are pseudo–differential projections with homogeneous principal symbols of order zero $p^\pm : \pi_Y^* J^\pm \to L^\pm$ which project to the sub–bundles.

Definition 1.4.1 *Define $\mathcal{S}^{\mu,d}(X; \boldsymbol{v})$ to be the space of all operators of the form*

$$\mathcal{A} = \begin{pmatrix} A & KR^- \\ P^+T & P^+QR^- \end{pmatrix} : \begin{matrix} H^s(X, E) \\ \oplus \\ P^s(Y, \boldsymbol{L}^-) \end{matrix} \to \begin{matrix} H^{s-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \boldsymbol{L}^+) \end{matrix}, \tag{1.4.2}$$

$(\mu, d) \in \mathbb{Z} \times \mathbb{N}$, $s \in \mathbb{R}$, $s > d - \frac{1}{2}$, where

$$\tilde{\mathcal{A}} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{b}) \tag{1.4.3}$$

for $\boldsymbol{b} = (E, F; J^-, J^+)$.

Let us write

$$\mathcal{A} = \mathcal{P}^+ \tilde{\mathcal{A}} \mathcal{R}^- \tag{1.4.4}$$

for $\mathcal{P}^+ = \left(\begin{smallmatrix} 1 & 0 \\ 0 & P^+ \end{smallmatrix}\right)$, $\mathcal{R}^- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R^- \end{smallmatrix}\right)$. Then we get a well–defined principal symbol

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$$

with the interior symbol $\sigma_\psi(\mathcal{A}) = \sigma_\psi(\tilde{\mathcal{A}}) = \sigma_\psi(A)$ and the boundary symbol

$$\sigma_\partial(\mathcal{A}) = \sigma_\partial(\mathcal{P}^+) \sigma_\partial(\tilde{\mathcal{A}}) \sigma_\partial(\mathcal{R}^-),$$

for $\sigma_\partial(\mathcal{P}^+) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^+ \end{smallmatrix}\right)$, $\sigma_\partial(\mathcal{R}^-) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & r^- \end{smallmatrix}\right)$, where $\sigma_\partial(\mathcal{A})$ is the boundary symbol of $\mathcal{A}$ in the sense of the class $\mathcal{B}^{\mu,d}(X; \boldsymbol{b})$.

Remark 1.4.2 *By definition there is an isomorphism*

$$\mathcal{S}^{\mu,d}(X; \boldsymbol{v}) \cong \{\tilde{\tilde{\mathcal{A}}} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{b}) : \ \tilde{\tilde{\mathcal{A}}} = \mathcal{P}^+ \tilde{\mathcal{A}} \mathcal{P}^- \text{ for some } \tilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{b})\}, \tag{1.4.5}$$

or, equivalently, $\mathcal{S}^{\mu,d}(X; \boldsymbol{v}) \cong \mathcal{B}^{\mu,d}(X; \boldsymbol{b})/\sim$, where $/\sim$ denotes the quotient map with respect to the equivalence relation $\tilde{\mathcal{A}}_1 \sim \tilde{\mathcal{A}}_2 \Leftrightarrow \mathcal{P}^+ \tilde{\mathcal{A}}_1 \mathcal{P}^- = \mathcal{P}^+ \tilde{\mathcal{A}}_2 \mathcal{P}^-$ in $\mathcal{B}^{\mu,d}(X; \boldsymbol{b})$. To study the properties of operators (1.4.2) *it suffices to represent $\mathcal{A}$ in the form*

$$\tilde{\tilde{\mathcal{A}}} := \mathcal{P}^+ \tilde{\mathcal{A}} \mathcal{P}^- : \begin{matrix} H^s(X, E) \\ \oplus \\ H^s(Y, J^-) \end{matrix} \to \begin{matrix} H^{s-\mu}(X, F) \\ \oplus \\ H^{s-\mu}(Y, J^+) \end{matrix} \tag{1.4.6}$$

where the specific choice of $\tilde{\mathcal{A}}$ is unessential.

Relation (1.4.5) is compatible with the behaviour of boundary symbols in the sense of $\mathcal{S}^{\mu,d}(X; \boldsymbol{v})$ and $\mathcal{B}^{\mu,d}(X; \boldsymbol{b})$, respectively. This allows us to treat the embedding operators R^- formally as operators with symbols structure. In the following considerations we prefer to employ embeddings, though everything can be translated to relations with projections.

Proposition 1.4.3 *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ and suppose $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2)$. Then $\mathcal{C} = \mathcal{A}_1 - \mathcal{A}_2$ is compact as an operator*

$$\mathcal{C} : \begin{array}{c} H^s(X,E) \\ \oplus \\ P^s(Y,\boldsymbol{L}^-) \end{array} \rightarrow \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ P^{s-\mu}(Y,\boldsymbol{L}^+) \end{array} \qquad (1.4.7)$$

for every $s > d - \frac{1}{2}$.

Proof. Let us represent $\mathcal{A}_i$ in the form $\tilde{\tilde{\mathcal{A}}}_i$, $i = 1, 2$, cf. (1.4.6). Then $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2)$ implies $\sigma(\tilde{\tilde{\mathcal{A}}}_1) = \sigma(\tilde{\tilde{\mathcal{A}}}_2)$ in the sense of corresponding pairs of principal symbols in $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$. From the corresponding known result for operators in this class we conclude that

$$\tilde{\tilde{\mathcal{A}}}_1 - \tilde{\tilde{\mathcal{A}}}_2 : \begin{array}{c} H^s(X,E) \\ \oplus \\ H^s(Y,J^-) \end{array} \rightarrow \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y,J^+) \end{array}$$

is a compact operator. It follows that the restriction to $H^s(X,E) \oplus P^s(Y,\boldsymbol{L}^-)$ is also compact. This restriction maps to $H^{s-\mu}(X,F) \oplus P^{s-\mu}(Y,\boldsymbol{L}^+)$. Thus $\mathcal{A}_1 - \mathcal{A}_2$ itself is compact in the sense of (1.4.7). □

In particular, the operators in $\mathcal{S}^{-\infty,d}(X;\boldsymbol{v}) = \bigcap_\mu \mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ are compact for all $s > d - \frac{1}{2}$.

Theorem 1.4.4 *Let $\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ and $\mathcal{B} \in \mathcal{S}^{\nu,e}(X;\boldsymbol{w})$. Suppose that $\mu, \nu \in \mathbb{Z}$, $d, e \in \mathbb{N}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ are as in Theorem 1.2.2. Then the product $\mathcal{AB} \in \mathcal{S}^{\mu+\nu,h}(X;\boldsymbol{v} \circ \boldsymbol{w})$, for $h = \max(\nu + d, e)$, and we have (with componentwise multiplication)*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B}).$$

This theorem is an immediate consequence of Theorem 1.2.2 above.

Consider an operator $\tilde{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ for $\boldsymbol{b} = (E, F; J^-, J^+)$ and fix the data $J^\pm$, $L^\pm$, $P^\pm$ in $\boldsymbol{v}$, cf. (1.4.1). Set

$$\boldsymbol{w} = (E, F; \boldsymbol{K}^-, \boldsymbol{K}^+) \qquad \text{for } \boldsymbol{K}^\pm = (J^\pm, K^\pm, Q^\pm),$$

where $K^+ = \ker p^\pm$ and $Q^\pm = 1 - P^\pm$. Then we can form the spaces

$$P^s(Y, \boldsymbol{K}^\pm) = \{f = Q^\pm u : \ u \in H^s(Y, J^\pm)\},$$

with the canonical embeddings $P^s(Y,\boldsymbol{K}^\pm) \rightarrow H^s(Y,J^\pm)$. By definition we then have

$$H^s(Y,J^\pm) = P^s(Y,\boldsymbol{L}^\pm) \oplus P^s(Y,\boldsymbol{K}^\pm).$$

Parallel to $\mathcal{A} = \mathcal{P}^+\tilde{\mathcal{A}}\mathcal{R}^-$ we have the operator $\mathcal{B} = \mathcal{Q}^+\tilde{\mathcal{A}}\mathcal{S}^-$ for $\mathcal{Q}^+ = \begin{pmatrix} 1 & 0 \\ 0 & Q^+ \end{pmatrix}$, $\mathcal{S}^- = \begin{pmatrix} 1 & 0 \\ 0 & S^- \end{pmatrix}$, and call $\mathcal{B}$ complementary to $\mathcal{A}$ (or $\mathcal{A}$ complementary to $\mathcal{B}$). Then $\mathcal{B} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{w})$.

To every $\mathcal{A}_i \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v}_i)$ $\boldsymbol{v}_i = (E_i, F_i; \boldsymbol{L}_i^-, \boldsymbol{L}_i^+)$ for arbitrary $\boldsymbol{L}_i^\pm = (J_i^\pm, L_i^\pm, P_i^\pm)$, $i = 1, 2$, we can define the direct sum

$$\mathcal{A}_1 \oplus \mathcal{A}_2 \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v}_1 \oplus \boldsymbol{v}_2)$$

in the sense of block matrices, where

$$\boldsymbol{v}_1 \oplus \boldsymbol{v}_2 = (E_1 \oplus E_2, F_1 \oplus F_2; \boldsymbol{L}_1^- \oplus \boldsymbol{L}_2^-, \boldsymbol{L}_1^+ \oplus \boldsymbol{L}_2^+)$$

for $\boldsymbol{L}_1^\pm \oplus \boldsymbol{L}_2^\pm = (J_1^\pm \oplus J_2^\pm, L_1^\pm \oplus L_2^\pm, P_1^+ \oplus P_2^\pm)$. In particular, if we take $\mathcal{A}_1 = \mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{v})$, $\mathcal{A}_2 = \mathcal{B} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{w})$ with the above $\boldsymbol{v}$, $\boldsymbol{w}$, where $\mathcal{B}$ is complementary to $\mathcal{A}$, then we get

$$\mathcal{A} \oplus \mathcal{B} \in \mathcal{B}^{\mu,d}(X, \boldsymbol{b}) \text{ for } \boldsymbol{b} = (E \oplus E, F \oplus F; J^-, J^+).$$

Let us now consider adjoints of operators $\mathcal{A} \in \mathcal{S}^{0,0}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; \boldsymbol{L}^-, \boldsymbol{L}^+)$, with respect to the fixed scalar products in the spaces

$$H^0(X, E) \oplus H^0(Y, J^-), \qquad H^0(X, F) \oplus H^0(Y, J^+)$$

and the induced ones in the subspaces defined by the projections $P^\pm$. (Recall that we fixed Riemannian metrics on X and Y and Hermitian metrics in the bundles.) The projections $P^\pm : H^0(Y, J^\pm) \to H^0(Y, J^\pm)$ give rise to adjoints $P^{\pm,*} \in L^0_{\text{cl}}(Y; J^\pm, J^\pm)$ that coincide with $P^\pm$ in the case of orthogonal projections. On account of the projection property, $P^{\pm,*}P^{\pm,*} = P^{\pm,*}$, the operators $P^{\pm,*}$ can be interpreted as embeddings $R^{\pm,*} : \operatorname{im} P^{\pm,*} \to H^0(Y, J^\pm)$, or, more generally, as $R^{\pm,*} : P^s(Y, \boldsymbol{L}^{\pm,*}) \to H^s(Y, J^\pm)$ for all $s \in \mathbb{R}$, where

$$\boldsymbol{L}^{\pm,*} = (J^\pm, \operatorname{im} p^{\pm,*}, P^{\pm,*}).$$

In the space $\sigma_\partial \mathcal{S}^{0,0}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; \boldsymbol{L}^-, \boldsymbol{L}^+)$ we can form the adjoint by the rule

$$\boldsymbol{a}^* = \begin{pmatrix} a^* & \lambda^{+,*} \\ \lambda^{-,*} & \varrho^* \end{pmatrix} : \begin{pmatrix} \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F' \\ \oplus \\ L^{+,(*)} \end{pmatrix} \to \begin{pmatrix} \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \\ \oplus \\ L^{-,(*)} \end{pmatrix},$$

where the upper left corner a^* has its usual meaning (as in the notation of Definition 1.3.3), while the other three components are given by

$$\lambda^{-,*} = p^{-,*}\sigma_\partial(K)^*, \qquad \lambda^{+,*} = \sigma_\partial(T)^* r^{+,*}, \qquad \varrho^* = p^{-,*}\sigma_\partial(Q)^* r^{+,*}.$$

Here $\sigma_\partial(K)^*$, $\sigma_\partial(T)^*$, $\sigma_\partial(Q)^*$ are taken in $\sigma_\partial \mathcal{B}^{0,0}(X; E, F; J^-, J^+)$ and $r^{+,*} : L^{+,(*)} \to J^+$ is the embedding of $L^{+,(*)} = \operatorname{im} p^{+,*}$.

Proposition 1.4.5 *Let $\mathcal{A} \in \mathcal{S}^{0,0}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; \boldsymbol{L}^-, \boldsymbol{L}^+)$. Then $\mathcal{A}^* \in \mathcal{S}^{0,0}(X; \boldsymbol{v}^*)$ for $\boldsymbol{v}^* = (F, E; \boldsymbol{L}^{+,*}, \boldsymbol{L}^{-,*})$, and*

$$\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*.$$

(The adjoints are understood to be taken in the corresponding symbol spaces).

The proof is an obvious consequence of the definition of the space $\mathcal{S}^{0,0}(X; \boldsymbol{v})$ and of the fact that adjoints of operators in $\mathcal{B}^{0,0}(X; E, F; J^-, J^+)$ belong to the space $\mathcal{B}^{0,0}(X; F, E; J^+, J^-)$.

Remark 1.4.6 *The operator space $\mathcal{S}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; \boldsymbol{L}^-, \boldsymbol{L}^+)$, induces a space of operators on the boundary, $\mathcal{T}^\mu(Y; \boldsymbol{l})$, $\boldsymbol{l} = (\boldsymbol{L}^-, \boldsymbol{L}^+)$ that appear as right lower corners*

$$P^+ Q R^- : P^s(Y, \boldsymbol{L}^-) \to P^{s-\mu}(Y, \boldsymbol{L}^+)$$

of (1.4.2), cf. Section 2.2 below. While our notation $\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ is derived from the role of Seeley projectors in special standard operators, that for $\mathcal{T}^{\mu}(Y;\boldsymbol{l})$ is motivated by Toeplitz operators. In fact, classical Toeplitz operators on (say) the unit circle $Y=S^1$ in $\mathbb{C}$ belong to $\mathcal{T}^0(S^1;\boldsymbol{l})$ for an evident choice of $\boldsymbol{l}$, namely, $\boldsymbol{l}=(\boldsymbol{L}^-,\boldsymbol{L}^+)$ for $\boldsymbol{L}^-=\boldsymbol{L}^+=(J,L^+,P^+)$ for $J=\pi_Y^*\mathbb{C}$ with the trivial line bundle $\mathbb{C}$ on $Y=S^1$, L^+ as in Remark 1.3.2 above, and P^+ the orthogonal projection of $L^2(S^1)$ to the subspace of all $u\in L^2(S^1)$ spanned by $\{z^k : k\in\mathbb{N}\}$ (which is the Hardy space and equals $P^0(S^1,\boldsymbol{L}^+)=P^0(S^1,\boldsymbol{L}^-)$ in our general notation). Moreover, the generalised Toeplitz operators in the sense of Boutet de Monvel [5] are of type $\mathcal{T}^{\mu}(Y;\boldsymbol{l})$ for appropriate Y and $\boldsymbol{l}$.

Intuitively, $\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ is a "Toeplitz–variant" of the algebra $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ of pseudo–differential boundary value problems.

2 Elliptic boundary value problems

2.1 Ellipticity

Definition 2.1.1 *An operator $\mathcal{A}\in\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ for $(\mu,d)\in\mathbb{Z}\times\mathbb{N}$, $\boldsymbol{v}=(E,F;\boldsymbol{L}^-,\boldsymbol{L}^+)$, $\boldsymbol{L}^{\pm}=(J^{\pm},L^{\pm},P^{\pm})$, is called elliptic if the mappings*

$$\sigma_{\psi}(\mathcal{A}):\pi_X^*E\to\pi_X^*F$$

and

$$\sigma_{\partial}(\mathcal{A}):\begin{pmatrix}\pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+)\otimes E'\\ \oplus\\ L^-\end{pmatrix}\to\begin{pmatrix}\pi_Y^*\mathcal{S}(\overline{\mathbb{R}}_+)\otimes F'\\ \oplus\\ L^+\end{pmatrix}$$

are both isomorphisms.

We say that an operator $\mathcal{B}\in\mathcal{S}^{-\mu,e}(X;\boldsymbol{v}^{-1})$ for some $e\in\mathbb{N}$ and $\boldsymbol{v}^{-1}=(F,E;\boldsymbol{L}^+,\boldsymbol{L}^-)$ is a parametrix of $\mathcal{A}$ if

$$\mathcal{B}\mathcal{A}-\mathcal{I}\in\mathcal{S}^{-\infty,d_l}(X;\boldsymbol{v}_l),\qquad \mathcal{A}\mathcal{B}-\mathcal{I}\in\mathcal{S}^{-\infty,d_r}(X;\boldsymbol{v}_r)\tag{2.1.1}$$

for certain $d_l,d_r\in\mathbb{N}$ and $\boldsymbol{v}_l=(E,E;\boldsymbol{L}^-,\boldsymbol{L}^-)$, $\boldsymbol{v}_r=(F,F;\boldsymbol{L}^+,\boldsymbol{L}^+)$.

Given $J\in\mathrm{Vect}(Y)$ and $L\in\mathrm{Vect}(T^*Y\setminus 0)$, where L is a subbundle of π_Y^*J and $p:\pi_Y^*J\to L$ a projection, homogeneous of order zero in the covariables $\eta\neq 0$, we say that an operator $P\in L^0_{\mathrm{cl}}(Y;J,J)$ is associated with L, if P is a projection and p its homogeneous principal symbol.

Theorem 2.1.2 *Let $\mathcal{A}\in\mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ be an elliptic operator where $\boldsymbol{v}=(E,F;\boldsymbol{L}^-,\boldsymbol{L}^+)$ and $\boldsymbol{L}^{\pm}=(J^{\pm},L^{\pm},P^{\pm})$. Then there exists an elliptic operator $\mathcal{B}\in\mathcal{S}^{\mu,d}(X;\boldsymbol{w})$ for $\boldsymbol{w}=(F,E;\boldsymbol{M}^-,\boldsymbol{M}^+)$, $\boldsymbol{M}^{\pm}=(\mathbb{C}^N,M^{\pm},Q^{\pm})$ with $L^-\oplus M^+\cong L^+\oplus M^-\cong\mathbb{C}^N$ for some $N\in\mathbb{N}$ such that $\mathcal{A}\oplus\mathcal{B}$ is elliptic in $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ for $\boldsymbol{b}=(E\oplus F,F\oplus E;\mathbb{C}^N,\mathbb{C}^N)$.*

Proof. Set $A=\mathrm{u.l.c.}\mathcal{A}$ which belongs to $B^{\mu,d}(X;E,F)$ and is elliptic with respect to σ_{ψ}. By construction the boundary symbol

$$\sigma_{\partial}(A):\pi_1^*H^s(\mathbb{R}_+)\otimes E'\to\pi_1^*H^s(\mathbb{R}_+)\otimes F'$$

represents a family of Fredholm operators on S^*Y; the specific choice of $s>d-\frac{1}{2}$ is unessential. Choose any $B\in B^{\mu,d}(X;F,E)$ that is elliptic with respect to σ_{ψ} such that

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial}(B)=-\,\mathrm{ind}_{S^*Y}\,\sigma_{\partial}(A)$$

holds. There are many ways to find such operators B. For the special case $\mu = d = 0$ we can form the adjoint $B = A^*$ that belongs to $\mathcal{B}^{0,0}(X; F, E)$ (adjoints always refer to corresponding L^2–scalar products). In fact, we then have $\sigma_\partial(A^*A) = \sigma_\partial(A)^*\sigma_\partial(A)$ which is self–adjoint in $L^2(\mathbb{R}_+) \otimes E'$ and have $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A^*A) = 0 = \mathrm{ind}\, \sigma_\partial(A)^* + \mathrm{ind}\, \sigma_\partial(A)$. For arbitrary μ and d we first observe that d can be ignored, since d is only involved in a Green summand that is compact on the level of boundary symbols and hence does not contribute to the index element on S^*Y. From Theorem 1.2.1 we have an order reducing element $R^\mu_E \in B^{\mu,0}(X; E, E)$ for every $E \in \mathrm{Vect}(X)$, i.e., isomorphisms $H^s(X, E) \to H^{s-\mu}(X, E)$ for all s (it is employed in this proof for $s > \mu^+ - \frac{1}{2}$). The operator R^μ_E is SL–elliptic without additional boundary conditions; in particular, we have $\mathrm{ind}_{S^*Y}\, \sigma_\partial(R^\mu_E) = 0$). Setting $A_0 = AR^{-\mu}_E$ for $\mu > 0$, $A_0 = R^{-\mu}_F A$ for $\mu < 0$ we get $A_0 \in B^{0,0}(X; E, F)$ where $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) = \mathrm{ind}_{S^*Y}\, \sigma_\partial(A_0)$. Using $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A_0^*) = -\mathrm{ind}_{S^*Y}\, \sigma_\partial(A_0)$ it suffices to set $B = R^\mu_E A_0^*$ because $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A_0^*) = \mathrm{ind}_{S^*Y}\, R^\mu_E + \mathrm{ind}\, \sigma_\partial(A_0^*) = \mathrm{ind}\, \sigma_\partial(A_0^*)$. By virtue of relation (1.3.1) we have $\mathrm{ind}_{S^*Y}\, \sigma_\partial(B) = [L^-|_{S^*Y}, L^+|_{S^*Y}] = [M^-|_{S^*Y}, M^+|_{S^*Y}]$, where $M^\pm$ denote complementing bundles of $L^\mp$ in a trivial bundle $\mathbb{C}^N$. From Lemma 1.3.1, applied to B and to the bundles $M^\pm \in \mathrm{Vect}(T^*Y \setminus 0)$, and from the constructions of the preceding section we get an elliptic operator $\mathcal{B} \in \mathcal{S}^{\mu,0}(X; \boldsymbol{w})$. Choosing $Q^\pm$ as the complementary projections to $P^\mp$ the operator $\mathcal{A} \oplus \mathcal{B}$ belongs to $\mathcal{B}^{\mu,d}(X; \boldsymbol{b})$ and is elliptic. □

Note that Grubb and Seeley [15] used a similar idea to embed an elliptic boundary value problem with projection into a standard one by means of the adjoint operator and the complementing projection.

Theorem 2.1.3 *Let $\mathcal{A} \in \mathcal{S}^{\mu,d}(X, \boldsymbol{v})$ be elliptic, $\boldsymbol{v} = (E, F; \boldsymbol{L}^-, \boldsymbol{L}^+)$. Then*

$$\mathcal{A} : \begin{array}{c} H^s(X, E) \\ \oplus \\ P^s(Y, \boldsymbol{L}^-) \end{array} \to \begin{array}{c} H^{s-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \boldsymbol{L}^+) \end{array} \tag{2.1.2}$$

is a Fredholm operator for each $s \in \mathbb{R}$, with $s > \max(\mu, d) - \frac{1}{2}$. Moreover, $\mathcal{A}$ has a parametrix $\mathcal{B} \in \mathcal{S}^{-\mu,e}(X; \boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} = (F, E; \boldsymbol{L}^+, \boldsymbol{L}^-)$ and $e = (d - \mu)^+$. More precisely, (2.1.1) holds for $d_l = \max(\mu, d)$ and $d_r = (d - \mu)^+$, and we have

$$\sigma(\mathcal{B}) = \sigma(\mathcal{A})^{-1} \tag{2.1.3}$$

(with componentwise inversion).

Proof. According to Theorems 2.1.2 there is an elliptic operator $\mathcal{A}^\perp \in \mathcal{S}^{\mu,d}(X; \boldsymbol{w})$ such that $\tilde{\mathcal{A}} := \mathcal{A} \oplus \mathcal{A}^\perp$ is elliptic in $\mathcal{B}^{\mu,d}(X; \boldsymbol{b})$. Applying a known result on elliptic operators in Boutet de Monvel's algebra we find a parametrix $\tilde{\mathcal{B}} \in \mathcal{B}^{-\mu,e}(X; \boldsymbol{b}^{-1})$ for $e = (d - \mu)^+$ where $\tilde{\mathcal{B}}\tilde{\mathcal{A}} - \mathcal{I} \in \mathcal{B}^{-\infty,d_l}(X; \boldsymbol{c})$, $\tilde{\mathcal{A}}\tilde{\mathcal{B}} - \mathcal{I} \in \mathcal{B}^{-\infty,d_r}(X; \boldsymbol{c})$ for $\boldsymbol{c} = (E \oplus F; \mathbb{C}^N, \mathbb{C}^N)$ and $d_l = \max(\mu, d)$, $d_r = (d - \mu)^+$. It is now sufficient to set $\mathcal{B} = \mathcal{P}^- \tilde{\mathcal{B}} \mathcal{R}^+$ with operators $\mathcal{P}^-$ and $\mathcal{R}^+$ of analogous meaning as in relation (1.4.4), here, associated with the data $\boldsymbol{L}^-$ and $\boldsymbol{L}^+$, respectively. □

Corollary 2.1.4 *Under the conditions of Theorem 2.1.3 we have elliptic regularity of solutions in the following sense. Suppose that $\mathcal{A}u = f \in H^{s-\mu}(X, F) \oplus P^{s-\mu}(Y, \boldsymbol{L}^+)$ and that $u \in H^{-\infty}(X, E) \oplus P^{-\infty}(Y, \boldsymbol{L}^-)$. Then $u \in H^s(X, E) \oplus P^s(Y, \boldsymbol{L}^-)$. This regularity holds for all real $s > d - \frac{1}{2}$ satisfying $s - \mu > (d - \mu)^+ - \frac{1}{2}$.*

In fact, since $\mathcal{A}$ has a parametrix $\mathcal{B} \in \mathcal{S}^{-\mu,e}(X;\boldsymbol{v}^{-1})$, for the composition we get $\mathcal{B}\mathcal{A}u = \mathcal{B}f \in H^s(X,E) \oplus P^s(Y,\boldsymbol{L}^+)$. As $\mathcal{B}\mathcal{A} = \mathcal{I} + \mathcal{C}$ for $\mathcal{C} \in \mathcal{S}^{-\infty,\max(\mu,d)}(X;\boldsymbol{v}_l)$, then $u = -\mathcal{C}u + \mathcal{B}f \in H^s(X,E) \oplus P^s(Y,\boldsymbol{L}^-)$.

Lemma 2.1.5 *Let $E \in \mathrm{Vect}(X)$, $\boldsymbol{L} = (J,L,P)$ for $L \in \mathrm{Vect}(T^*Y \setminus 0)$, and $J \in \mathrm{Vect}(Y)$, where L is a sub–bundle of $\pi_Y^* J$ and $P \in L^0_{\mathrm{cl}}(Y;J,J)$ is associated with L. Then for every finite–dimensional subspace $V \subset H^\infty(X,E) \oplus P^\infty(Y,\boldsymbol{L})$ the orthogonal projection $\tilde{\mathcal{P}}_V : H^0(X,E) \oplus H^0(Y,J) \to V$ (orthogonal with respect to the scalar product in $H^0(X,E) \oplus H^0(Y,J)$) induces an element $\mathcal{P}_V \in \mathcal{S}^{-\infty,0}(X;\boldsymbol{v})$ for $\boldsymbol{v} = (E,E;\boldsymbol{L},\boldsymbol{L})$, i.e.,*

$$\mathcal{P}_V = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P} \end{pmatrix} \tilde{\mathcal{P}}_V \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R} \end{pmatrix} \in \mathcal{S}^{-\infty,0}(X;\boldsymbol{v}) \tag{2.1.4}$$

for $\mathcal{P} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & P \end{smallmatrix}\right)$ and $\mathcal{R} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right)$ with the embedding $R : P^s(Y,\boldsymbol{L}) \to H^s(Y,J)$.

Proof. We can choose $\mathcal{P}_V$ as an element in $\mathcal{B}^{-\infty,0}(X;E,E;J,J)$ by setting

$$\mathcal{P}_V u = \sum_{j=1}^{N} (u, v_j) v_j$$

for $N = \dim V$, with an orthogonal base $v_1, \ldots, v_N$ in V. This gives us (2.1.4). □

If an operator $\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ in the notation of Theorem 2.1.3 is regarded as a map (2.1.2) for each given s, we also write $\ker_s \mathcal{A}$, $\mathrm{im}_s \mathcal{A}$, $\mathrm{coker}_s \mathcal{A}$ for the respective kernels, images, cokernels, etc.

Theorem 2.1.6 *Let $\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v})$ for $\boldsymbol{v} = (E,F;\boldsymbol{L}^-,\boldsymbol{L}^+)$, $\boldsymbol{L}^\pm = (J^\pm, L^\pm, P^\pm)$, be elliptic. Then (in the notation of Theorem 2.1.3) there exists a parametrix $\mathcal{B} \in \mathcal{S}^{-\mu,e}(X;\boldsymbol{v}^{-1})$ such that the remainders*

$$\mathcal{C}_l = \mathcal{I} - \mathcal{B}\mathcal{A} \qquad \textit{and} \qquad \mathcal{C}_r = \mathcal{I} - \mathcal{A}\mathcal{B} \tag{2.1.5}$$

are projections, where $\mathcal{C}_l$ projects to $\ker_s \mathcal{A}$ and $\mathcal{C}_r$ to a complement of $\mathrm{im}_s \mathcal{A}$, for each $s \in \mathbb{R}$ with $s > d - \frac{1}{2}$, $s - \mu > (d-\mu)^+ - \frac{1}{2}$. In particular, $\ker \mathcal{A}$, $\mathrm{coker}\, \mathcal{A}$ (and hence $\mathrm{ind}\, \mathcal{A}$) are independent of s.

Proof. First it is clear that $\ker_s \mathcal{A}$ is a finite–dimensional subspace of $H^\infty(X,E) \oplus P^\infty(Y,\boldsymbol{L}^-)$. In fact, since $\mathcal{A}$ has a parametrix $\mathcal{B}_0 \in \mathcal{S}^{-\mu,e}(X;\boldsymbol{v}^{-1})$, the relations $\mathcal{A}u = 0$, $u \in H^s(X,E) \oplus P^s(Y,\boldsymbol{L}^-)$ imply $\mathcal{B}_0\mathcal{A}u = (\mathcal{I} + \mathcal{C})u = 0$ for an $\mathcal{C} \in \mathcal{S}^{-\infty,\max(\mu,d)}(X;\boldsymbol{v}_l)$, and hence $u \in H^\infty(X,E) \oplus P^\infty(Y,\boldsymbol{L}^-)$. Let $V = \ker_0 \mathcal{A}$ and form $\mathcal{P}_V \in \mathcal{S}^{-\infty,0}(X;\boldsymbol{v}_l)$, according to Lemma 2.1.5. Then $\mathcal{B}_1 = (\mathcal{I} - \mathcal{P}_V)\mathcal{B}_0$ is also a parametrix of $\mathcal{A}$ in $\mathcal{S}^{-\mu,e}(X;\boldsymbol{v}^{-1})$ (because $\mathcal{B}_0$ is of type $e = (d-\mu)^+$ we have $e = \max(-\mu, e)$ for the type of $\mathcal{B}_1$).

The kernel of $\mathcal{B}_1$ is a finite–dimensional subspace W in $H^\infty(X,F) \oplus P^\infty(Y,\boldsymbol{L}^+)$. Let $\mathcal{P}_W$ be a projection in the sense of Lemma 2.1.5. Then $\mathcal{B}_2 = \mathcal{B}_1(\mathcal{I} - \mathcal{P}_W)$ maps a complement of W injectively to a complement of V. Now W can be decomposed into a direct sum $W = W_0 \oplus W_1$, where $W_1 = \mathrm{im}_0 \mathcal{A} \cap W$. We then find a subspace $U_1 \subset H^\infty(X,E) \oplus P^\infty(Y,\boldsymbol{L}^-)$ with $U_1 \cap \ker_0 \mathcal{A} = \{0\}$, such that $\mathcal{A}$ induces a bijection $U_1 \to W_1$. Let $D : W_1 \to U_1$ be its inverse, and form the operator $\mathcal{D} = D\mathcal{P}_{W_1}$. Then $\mathcal{B} = \mathcal{B}_2 + \mathcal{D}$ is another parametrix of $\mathcal{A}$ which has just the desired property. But this implies that $\mathcal{C}_l$ projects to $\ker_s \mathcal{A}$ and $\mathcal{C}_r$ to a complement of $\mathrm{im}_s \mathcal{A}$ for all admitted $s \in \mathbb{R}$. Thus kernel, cokernel and index of $\mathcal{A}$ are independent of s. □

2.2 Reduction to the boundary

The operator algebra $\mathcal{S}(X)$ of general boundary value problems contains the subalgebra of right lower corners. Ellipticity and index are also interesting in that subalgebra. Given the data $\boldsymbol{l} = (\boldsymbol{L}^-, \boldsymbol{L}^+)$ for $\boldsymbol{L}^\pm = (J^\pm, L^\pm, P^\pm)$ we denote by $\mathcal{T}^\mu(Y; \boldsymbol{l})$ the space of all operators of the form

$$A = P^+ \tilde{A} R^- : P^s(Y, \boldsymbol{L}^-) \to P^{s-\mu}(Y, \boldsymbol{L}^+)$$

with P^+, R^- from Definition 1.4.1 and $\tilde{A} \in L^\mu_{\mathrm{cl}}(Y; J^-, J^+)$. Symbols and composition results in this operator class are direct consequences of those in $\mathcal{S}(X)$. It is, of course, not essential here that μ is an integer. Ellipticity of an operator $A \in \mathcal{T}^\mu(Y; \boldsymbol{l})$ simply means that

$$p^+ \sigma_\psi(\tilde{A}) r^- : L_- \to L_+$$

is an isomorphism, cf. the notation of Definition 1.3.3. A corollary of Theorems 2.1.3 and 2.1.6 is that an elliptic $A \in \mathcal{T}^\mu(Y; \boldsymbol{l})$ induces Fredholm operators

$$A : P^s(Y, \boldsymbol{L}^-) \to P^{s-\mu}(Y, \boldsymbol{L}^+),$$

where $\ker A$, $\operatorname{coker} A$, $\operatorname{ind} A$ are independent of s, and that there is a parametrix $B \in \mathcal{T}^{-\mu}(Y; \boldsymbol{l}^{-1})$, $\boldsymbol{l}^{-1} = (\boldsymbol{L}^+, \boldsymbol{L}^-)$. Consider the case in which $J = J^- = J^+$ and $L = L^- = L^+$, but for which P^- and P^+ are arbitrary. Then $P^- - P^+$ is a compact operator, because P^- and P^+ have the same principal symbols. The relative index $\operatorname{ind}(P^-, P^+)$ is then defined as the index of the Fredholm operator $P^+ : \operatorname{im} P^- \to \operatorname{im} P^+$ (we realise the operators, for instance, in $H^0(Y, J) = L^2(Y, J)$), and $\operatorname{ind}(P^-, P^+) = -\operatorname{ind}(P^+, P^-)$. If $\tilde{A} \in \mathcal{T}^0(Y; J, J)$ is the identity modulo a compact operator, we have $A = P^+ R^- \in \mathcal{T}^0(Y; \boldsymbol{L}^-, \boldsymbol{L}^+)$ for $\boldsymbol{L}^\pm = (J, L, P^\pm)$ and $\operatorname{ind} A = \operatorname{ind}(P^-, P^+)$. Now let $D \in B^{\mu,d}(X; E, F)$ be an (σ_ψ–) elliptic operator for which we have two elliptic boundary value problems

$$\mathcal{A}_i = \begin{pmatrix} D \\ T_i \end{pmatrix} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{v}_i),$$

$\boldsymbol{v}_i = (E, F; \boldsymbol{0}, \boldsymbol{L}_i)$ for $\boldsymbol{L}_i = (J, L_i, P_i)$, $i = 1, 2$. The sub–bundles L_1, L_2 of $\pi_Y^* J$ are of course, isomorphic, but they may be different, including the projections P_1, P_2. According to Theorem 2.1.3 there are parametrices $\mathcal{B}_i \in \mathcal{S}^{-\mu,e}(X; \boldsymbol{v}_i^{-1})$, $i = 1, 2$. Set $\mathcal{B}_2 = (B_2, S_2)$ and consider the composition

$$\mathcal{A}_1 \mathcal{B}_2 = \begin{pmatrix} DB_2 & DS_2 \\ T_1 B_2 & T_1 S_2 \end{pmatrix} \tag{2.2.1}$$

Then we have $DB_2 = 1$ and $DS_2 = 0$ modulo compact operators, and $T_1 S_2 \in \mathcal{T}^0(Y; \boldsymbol{L}_2, \boldsymbol{L}_1)$ is elliptic.

Theorem 2.2.1 *We have*

$$\operatorname{ind} \mathcal{A}_1 - \operatorname{ind} \mathcal{A}_2 = \operatorname{ind}(T_1 S_2). \tag{2.2.2}$$

Proof. We have $\operatorname{ind} \mathcal{A}_2 = -\operatorname{ind} \mathcal{B}_2$, so the assertion is a direct consequence of (2.2.1). □

Remark 2.2.2 *Assume $L_1 = L_2$ and $T_i = P_i \tilde{T}$ for the same $\tilde{T}$, $i = 1, 2$. Then, as a consequence of the observations above, we have*

$$\operatorname{ind} \mathcal{A}_1 - \operatorname{ind} \mathcal{A}_2 = \operatorname{ind}(P_2, P_1).$$

Similarly to classical boundary value problems the operator T_1S_2 may be regarded as the reduction of the boundary conditions T_1 to the boundary, by means of a second elliptic boundary value problem $\mathcal{A}_2$ for the same elliptic operator D. In this sense (2.2.2) is an analogue of the Agranovich–Dynin formula. Also for elliptic operators $\mathcal{A}_i \in \mathcal{S}^{\mu,d}(X;\boldsymbol{v}_i)$, $i=1,2$, in general, with $\mathrm{u.l.c.}\mathcal{A}_1 = \mathrm{u.l.c.}\mathcal{A}_2$ there is a reduction to the boundary; the corresponding algebraic manipulations are similar to those in the book [22], pages 252–254 and left to the reader.

2.3 Transmission operators

Let $X = X_+$ and X_- be compact C^∞ manifolds with C^∞ boundaries, $\dim X_+ = \dim X_-$. Assume that M is a closed compact C^∞ manifold with $X_\pm \subset M$ such that

$$X_+ \cup X_- = M, \qquad X_+ \cap X_- = Y$$

where $Y = \partial X_\pm$. An example for such a situation is $M = 2X$, the double of X, where two copies of X are glued together along $Y = \partial X$. Given an elliptic operator $S \in L^\mu_{\mathrm{cl}}(M;V,W)$ for $V,W \in \mathrm{Vect}(M)$ we can ask for relations to elliptic boundary value problems for $S_\pm = S|_{X_\pm} \in L^\mu_{\mathrm{cl}}(\mathrm{int}\,X_\pm;V_\pm,W_\pm)$, $V_\pm = V|_{X_\pm}$, $W_\pm = W|_{X_\pm}$. This problem leads to a number of difficulties. The operators $S_\pm$ do not automatically have the transmission property with respect to Y. According to the orientation of this paper we assume that $S_\pm$ have the transmission property, though this may appear rather restrictive under the point of view of all possible operators $S \in L^\mu_{\mathrm{cl}}(M;V,W)$. The general case can be treated in the framework of the edge pseudo–differential machinery, especially concerning the analytic characterisation of transmission operators for arbitrary S, cf. [30], Section 2.1.10. Here, we are interested in expressions connecting the index of elliptic operators S with indices of boundary value problems for $S_\pm$ as they are always possible in the sense of Section 2.1, provided the transmission property is fulfilled. Note that we may always reach elliptic symbols with the transmission property, starting with arbitrary ones, by a stable homotopy through elliptic symbols.

Denote by $\mathrm{r}^\pm$ the operators of restriction from M to $\mathrm{int}\,X_\pm$ and by $\mathrm{e}^\pm$ the operators of extension by zero from $\mathrm{int}\,X_\pm$ to M. Given an $S \in L^\mu_{\mathrm{cl}}(M;V,W)$ with the transmission property with respect to Y,

$$S : H^s(M,V) \to H^{s-\mu}(M,W),$$

we can form the operators

$$\mathrm{r}^\pm S \mathrm{e}^\pm : H^s(X_\pm,V_\pm) \to H^{s-\mu}(X_\pm,W_\pm) \tag{2.3.1}$$

and

$$\mathrm{r}^\pm S \mathrm{e}^\mp : H^s(X_\mp,V_\mp) \to H^{s-\mu}(X_\pm,W_\pm), \tag{2.3.2}$$

for $s > -\frac{1}{2}$. The operators (2.3.1) belong to $B^{\mu,0}(X_\pm;V_\pm,W_\pm)$ while (2.3.2) have C^∞ kernels because of the pseudo–locality of S, though near Y they are singular. In the case $M = 2X$ we can define the reflection diffeomorphisms $\varepsilon : X_\pm \to X_\mp$ that map an $x \in X_\pm$ to the corresponding point on $X_\mp$ (recall that X_+ and X_- are copies of the same X). In this case we have

$$\mathrm{r}^+ S \mathrm{e}^- \varepsilon^*, \varepsilon^* \mathrm{r}^- S \mathrm{e}^+ \in B^{\mu,0}_G(X_+;V_+,W_+), \tag{2.3.3}$$

$$\mathrm{r}^- S \mathrm{e}^+ \varepsilon^*, \varepsilon^* \mathrm{r}^+ S \mathrm{e}^- \in B^{\mu,0}_G(X_-;V_-,W_-). \tag{2.3.4}$$

Relations of the latter kind that are valid in analogous for for arbitrary M are systematically employed in Myshkis [18] and proved in detail in Grubb [12]. Concerning more general results in this direction, cf. Eskin [7], Lemma 15.3, or Schulze [32], Remark 4.1.25.

For notational convenience we assume that $M = 2X$ in the sequel. The extension of the results to the general case is straightforward and left to the reader.

First let $\mu = 0$ and denote the H^0–spaces also by L^2. The ellipticity of $S \in L^0_{\mathrm{cl}}(M; V, W)$ is equivalent to the Fredholm property of the operator

$$S : L^2(M, V) \to L^2(M, W)$$

or of

$$\begin{pmatrix} \mathrm{r}^+ S\mathrm{e}^+ & \mathrm{r}^+ S\mathrm{e}^- \\ \mathrm{r}^- S\mathrm{e}^+ & \mathrm{r}^- S\mathrm{e}^- \end{pmatrix} : \begin{matrix} L^s(X_+, V_+) \\ \oplus \\ L^2(X_-, V_-) \end{matrix} \to \begin{matrix} L^s(X_+, W_+) \\ \oplus \\ L^2(X_-, W_-) \end{matrix}.$$

This is equivalent to the Fredholm property of

$$\mathbb{S} = \begin{pmatrix} \mathrm{r}^+ S\mathrm{e}^+ & \mathrm{r}^+ S\mathrm{e}^- \varepsilon^* \\ \varepsilon^* \mathrm{r}^- S\mathrm{e}^+ & \varepsilon^* \mathrm{r}^- S\mathrm{e}^- \varepsilon^* \end{pmatrix} : \begin{matrix} L^2(X, E) \\ \oplus \\ L^2(X, E) \end{matrix} \to \begin{matrix} L^2(X, F) \\ \oplus \\ L^2(X, F) \end{matrix}. \tag{2.3.5}$$

In the last mapping $X = X_+$, $E = V_+$, $F = W_+$. Note that we can write

$$\varepsilon^* \mathrm{r}^- S\mathrm{e}^- \varepsilon^* = \mathrm{r}^+ (\varepsilon^* S \varepsilon^*) \mathrm{e}^+ = \mathrm{r}^+ (\varepsilon_* S) \mathrm{e}^+,$$

where ε_* is the operator push–forward under the (involutive) diffeomorphism ε. Given a bundle H on a space we also write $2H = H \oplus H$. The Fredholm property of (2.3.5) means that $\mathbb{S} \in B^{0,0}(X; 2E, 2F)$ is elliptic; the Shapiro–Lopatinskij condition is automatically satisfied, without additional trace or potential entries with respect to Y. In other words

$$\sigma_\partial(\mathbb{S}) : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes 2E' \to \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes 2F'$$

is an isomorphism, $\pi_Y : T^*Y \setminus 0 \to Y$, or, equivalently,

$$\sigma_\partial(\mathbb{S}) : \pi_Y^* L^2(\mathbb{R}_+) \otimes 2E' \to \pi_Y^* L^2(\mathbb{R}_+) \otimes 2F' \tag{2.3.6}$$

is an isomorphism. Now

$$\sigma_\partial(\mathrm{r}^+ S\mathrm{e}^- \varepsilon^*), \sigma_\partial(\varepsilon^* \mathrm{r}^- S\mathrm{e}^+) : \pi_Y^* L^2(\mathbb{R}_+) \otimes E' \to \pi_Y^* L^2(\mathbb{R}_+) \otimes F'$$

are compact–operator–valued, since these entries are of Green type, cf. (2.3.3), (2.3.4). Thus, restricting (2.3.6) to S^*Y, we get

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(\mathbb{S}) = \mathrm{ind}_{S^*Y}\, \sigma_\partial(\mathrm{r}^+ S\mathrm{e}^+) + \mathrm{ind}_{S^*Y}\, \sigma_\partial(\mathrm{r}^+ (\varepsilon^* S) \mathrm{e}^+) = 0.$$

We can also consider the boundary symbol of $\mathrm{r}^- S\mathrm{e}^-$ with respect to X_- (if necessary, we denote by $\sigma_{\partial,(\pm)}$ the boundary symbols with respect to the $\pm$–sides). We have

$$\sigma_{\partial,(-)}(\mathrm{r}^- S\mathrm{e}^-) : \pi_Y^* L^2(\mathbb{R}_-) \otimes E' \to \pi_Y^* L^2(\mathbb{R}_-) \otimes F'.$$

It is clear that then

$$\mathrm{ind}_{S^*Y}\, \sigma_{\partial,(+)}(\mathrm{r}^+ (\varepsilon_* S) \mathrm{e}^+) = \mathrm{ind}_{S^*Y}\, \sigma_{\partial,(-)}(\mathrm{r}^- S\mathrm{e}^-).$$

Thus we proved the following result:

Proposition 2.3.1 *Let $S \in L^0_{\mathrm{cl}}(M;V,W)$ be an operator with the transmission property with respect to Y. Then*

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+S\mathrm{e}^+) = -\,\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-S\mathrm{e}^-).$$

Next consider an arbitrary operator $T \in L^\mu_{\mathrm{cl}}(M;V,W)$ with the transmission property with respect to Y. (Recall that this condition is symmetric with respect to the minus or the plus side of Y). Thus we get $T_+ = \mathrm{r}^+T\mathrm{e}^+ \in B^{\mu,0}(X_+;V_+,W_+)$ and $T_- = \mathrm{r}^-T\mathrm{e}^- \in B^{\mu,0}(X_-;V_-,W_-)$. Composing

$$T_+ : H^\mu(X_+,V_+) \to L^2(X_+,W_+)$$

from the right with an order reduction $R^{-\mu}(V_+) : L^2(X_+,V_+) \to H^\mu(X_+,V_+)$, cf. Theorem 1.2.1, we obtain $T_+R^{-\mu}(V_+) \in B^{0,0}(X_+;V_+,W_+)$ and

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(T_+R^{-\mu}(V_+)) = \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(T_+). \tag{2.3.7}$$

The operator $T_+R^{-\mu}(V_+)$ can also be interpreted as $\mathrm{r}^+S\mathrm{e}^+$ for an operator $S \in L^0_{\mathrm{cl}}(M;V,W)$ with the transmission property. For this S we then have

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-S\mathrm{e}^-) = \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(T_-) - \mu[\pi_1^*E']$$

where $E' = V|_Y$ and $\pi_1 : S^*Y \to Y$. Thus, using Proposition 2.3.1 and the fact that (2.3.7) equals $\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+S\mathrm{e}^+)$, we get the following theorem:

Theorem 2.3.2 *Let $T \in L^\mu_{\mathrm{cl}}(M;V,W)$ be an operator with the transmission property with respect to Y and which is elliptic with respect to σ_ψ. Set $T_\pm = \mathrm{r}^\pm T\mathrm{e}^\pm$. Then the boundary symbols $\sigma_{\partial,(\pm)}(T_\pm)$ of $T_\pm$, understood in the sense of the classes $B^{\mu,0}(X_\pm;V_\pm,W_\pm)$ satisfy the relation*

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(T_+) = -\,\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(T_-) + \mu[\pi_1^*E'].$$

Note that when T is an elliptic differential operator of order μ the assertion of Theorem 2.3.2 has a relation to well–known facts about Calderón–Seeley projectors associated with T. For an elliptic differential operator the boundary symbols

$$\sigma_{\partial,(+)}(T_+)(y,\eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E'_y \to \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F'_y,$$
$$\sigma_{\partial,(-)}(T_-)(y,\eta) : \mathcal{S}(\overline{\mathbb{R}}_-) \otimes E'_y \to \mathcal{S}(\overline{\mathbb{R}}_-) \otimes F'_y,$$

$E = V|_Y$, $F = W|_Y$, are surjective for all $(y,\eta) \in T^*Y \setminus 0$, and the kernels are isomorphic to sub–bundles $L^\pm$ of π_Y^*J for $J = E' \oplus \ldots \oplus E'$ (μ summands), i.e., $\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(\pm)}(T_\pm) = [L^\pm|_{S^*Y}]$.

Set $\boldsymbol{L}^\pm = (J, L^\pm, P^\pm)$, where the projections $P^\pm \in L^0_{\mathrm{cl}}(Y;J,J)$ are associated with $L^\pm$, and suppose that $P^+ + P^- = 1$. Then we have

$$H^s(Y,J) = P^s(Y,\boldsymbol{L}^+) \oplus P^s(Y,\boldsymbol{L}^-) \tag{2.3.8}$$

for each $s \in \mathbb{R}$. Let $\boldsymbol{v}_\pm = (V_\pm, W_\pm; \boldsymbol{0}, \boldsymbol{L}^\pm)$ for $\boldsymbol{0} = (0,0,0)$ and choose elliptic operators

$$\mathcal{T}_\pm = \begin{pmatrix} T_\pm \\ B_\pm \end{pmatrix} \in \mathcal{S}^{\mu,\mu}(X_\pm;\boldsymbol{v}_\pm);$$

recall the definition $B_\pm = P^\pm \tilde{B}_\pm$ for trace operators $\tilde{B}_\pm$ in the sense of $\mathcal{B}^{\mu,\mu}(X_\pm;V_\pm,W_\pm;0,J)$. (These may be, for instance, standard differential boundary operators composed with suitable reductions of orders on the boundary.)

In view of the ellipticity we have the following Fredholm operators

$$T : H^{\mu}(M,V) \to L^2(M,W) \tag{2.3.9}$$

and

$$\mathcal{T}_{\pm} : H^{\mu}(X_{\pm},V_{\pm}) \to \begin{array}{c} L^2(X_{\pm},W_{\pm}) \\ \oplus \\ P^0(Y,\boldsymbol{L}^{\pm}) \end{array}. \tag{2.3.10}$$

We want to derive a relation between their indices. To this end, consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{\mu}(M,V) & \xrightarrow{i} & H^{\mu}(X_+,V_+) \oplus H^{\mu}(X_-,V_-) & \xrightarrow{j} & L^2(Y,J) & \longrightarrow & 0 \\ & & \downarrow T & & \mathcal{T} \downarrow \uparrow \mathcal{L} & & \uparrow R & & \\ 0 & \longleftarrow & L^2(M,W) & \xleftarrow{b} & \begin{pmatrix} L^2(X_+,W_+) \\ \oplus \\ P^0(Y,\boldsymbol{L}^+) \end{pmatrix} \oplus \begin{pmatrix} L^2(X_-,W_-) \\ \oplus \\ P^0(Y,\boldsymbol{L}^-) \end{pmatrix} & \xleftarrow{a} & L^2(Y,J) & \longleftarrow & 0 \end{array} . \tag{2.3.11}$$

The maps i and j are defined as follows:

$$i(u) = u|_{X_+} \oplus u|_{X_-}, \qquad j(u_+ \oplus u_-) = \bigoplus_{k=0}^{\mu-1} \Delta_{E'}^{-\mu+k+\frac{1}{2}} (\gamma_+^k u_+ - \gamma_-^k u_-),$$

for $\gamma_{\pm}^k f = D_t^k f|_{Y_{\pm}}$, with D_t^k being the derivative in normal direction to Y and $|_{Y_{\pm}}$ the restriction to Y from the $\pm$ side. The symbol $\Delta_{E'}^{\nu} \in L_{\mathrm{cl}}^{\nu}(Y;E',E')$ denotes an order reduction that induces isomorphisms $H^s(Y,E') \to H^{s-\nu}(Y,E')$ for all $s \in \mathbb{R}$. The map a is the canonical embedding, where we use (2.3.8) for $s=0$; the map b is the canonical projection, where we use $L^2(M,W) = L^2(X_+,W_+) \oplus L^2(X_-,W_-)$, and

$$\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-, \qquad \mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-, \tag{2.3.12}$$

where $\mathcal{L}_{\pm} \in \mathcal{S}^{-\mu,0}(X_{\pm};\boldsymbol{v}_{\pm}^{-1})$ are parametrices of $\mathcal{T}_{\pm}$. Finally, we set

$$R = j \circ \mathcal{L} \circ a : L^2(Y,J) \to L^2(Y,J), \tag{2.3.13}$$

which is an elliptic pseudo–differential operator on Y and, as such, Fredholm.

The rows of the diagram (2.3.11) are then exact and we have $T = b \circ \mathcal{T} \circ i$. All in all, the assumptions of an abstract well–known lemma (see, e.g., Rempel and Schulze [22], Section 3.1.1.3) on indices of Fredholm operators are thus fulfilled so that the following conclusion is valid.

Theorem 2.3.3 *With the notation of* (2.3.12) *and* (2.3.13)*,*

$$\operatorname{ind} T = \operatorname{ind} \mathcal{T} + \operatorname{ind} R.$$

2.4 Pseudo–differential projections

This section has the character of an appendix. We give an explicit construction of pseudo–differential projections to corresponding principal symbols. The result is known, but we believe it may useful for the reader to see a brief proof.

Let M be a closed compact C^{∞} manifold with the space $L_{\mathrm{cl}}^{\mu}(M;E,F)$ of classical pseudo–differential operators of order μ, acting between distributional sections of vector bundles E and F on M. Recall that the homogeneous principal symbol of order μ of an operator $A \in L_{\mathrm{cl}}^{\mu}(M;E,F)$ is a bundle homomorphism $\sigma_{\psi}(A) : \pi^* E \to \pi^* F$ where $\pi : T^*M \setminus 0 \to 0$.

Theorem 2.4.1 *Let $p : \pi^* E \to \pi^* E$, $E \in \mathrm{Vect}(M)$, be a projection, i.e., $p^2 = p$, with $p(x, \lambda\xi) = p(x, \xi)$ for all $(x, \xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$. Then there exists an element $P \in L^0_{\mathrm{cl}}(M; E, E)$ with $P^2 = P$ and $\sigma_\psi(P) = p$.*

Moreover, if $p = p^2$ satisfies the condition $p = p^*$, there is a choice of $P = P^2 \in L^0_{\mathrm{cl}}(M; E, E)$ with $\sigma_\psi(P) = p$ and $P = P^*$.

The adjoint of p refers to a given Hermitian metric in E and the adjoint of P to a fixed scalar product in the space $L^2(M, E)$, with respect to a Riemannian metric on M and the Hermitian metric in E.

Let H be a (complex) Hilbert space, $\mathcal{L}(H)$ the space of linear continuous operators, $\mathcal{K}(H)$ the subspace of compact operators in H, $\mathcal{L}(H)/\mathcal{K}(H)$ the Calkin algebra, and $\pi : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{K}(H)$ the canonical map.

Lemma 2.4.2 *Let $p \in \mathcal{L}(H)/\mathcal{K}(H)$ be an element with $p^2 = p$ and choose any $Q \in \mathcal{L}(H)$ with $\pi Q = p$. Then the spectrum $\sigma_{\mathcal{L}(H)}(Q)$ of Q has the property that*

$$\sigma_{\mathcal{L}(H)}(Q) \cap (\mathbb{C} \setminus (\{0\} \cup \{1\}))$$

is discrete.

Proof. First observe that $p^2 = p$ implies $\sigma_{\mathcal{L}(H)/\mathcal{K}(H)}(p) \subseteq \{0\} \cup \{1\}$. In fact, for $\lambda \in \mathbb{C} \setminus (\{0\} \cup \{1\}) =: U$ there exists $(\lambda e - p)^{-1} = \frac{1}{\lambda - 1} p + \frac{1}{\lambda}(e - p)$, where $e \in \mathcal{L}(H)/\mathcal{K}(H)$ is the identity, $e = \pi I$ for the identity $I \in \mathcal{L}(H)$. Now $U \ni \lambda \to \lambda I - Q \in \mathcal{L}(H)$ is a holomorphic Fredholm family in U, and $\lambda I - Q$ is invertible in $\mathcal{L}(H)$ for $|\lambda| > \|Q\|_{\mathcal{L}(H)}$. A well–known invertibility result on holomorphic Fredholm families (a proof may be found in [28], Section 2.2.5) implies that $\lambda I - Q$ is invertible for all $\lambda \in U \setminus D$ for a certain discrete subset D (i.e., D is countable and $D \cap K$ finite for every compact subset $K \subset U$). □

Proof of Theorem 2.4.1. Lemma 2.4.2 implies that there exists a $0 < \delta < 1$ such that the circle $C_\delta := \{\lambda : \ |\lambda - 1| = \delta\}$ does not intersect $\sigma_{\mathcal{L}(H)}(Q)$. We set

$$P := \frac{1}{2\pi i} \int_{C_\delta} (\lambda I - Q)^{-1} \, d\lambda. \tag{2.4.1}$$

Then $P^2 = P$, and we have $P \in L^0_{\mathrm{cl}}(M; E, E)$ as a consequence of the holomorphic functional calculus for $L^0_{\mathrm{cl}}(M; E, E)$. Moreover, we have

$$\begin{aligned} \sigma_\psi(P) &= \frac{1}{2\pi i} \int_{C_\delta} (\lambda e - p)^{-1} \, d\lambda \\ &= \left\{ \frac{1}{2\pi i} \int_{C_\delta} \frac{1}{\lambda - 1} \, d\lambda \right\} p + \left\{ \frac{1}{2\pi i} \int_{C_\delta} \frac{1}{\lambda} \, d\lambda \right\} (e - p). \end{aligned}$$

The second summand on the right hand side vanishes, while the first one equals p by the Residue theorem.

To prove the second part of Theorem 2.4.1 we suppose $p = p^*$. Then, if $P_1 = P_1^2 \in L^0_{\mathrm{cl}}(M; E, E)$ is any choice with $\sigma_\psi(P_1) = p$, also $Q := P_1^* P_1 \in L^0_{\mathrm{cl}}(M; E, E)$ satisfies $\sigma_\psi(Q) = p^* p = p^2 = p$. For Q we have $Q = Q^* \geq 0$. Let η be the spectral measure of Q. Then the projection $P \in L^0_{\mathrm{cl}}(M; E, E)$ defined by formula (2.4.1) equals the spectral projection

$$\eta(B_\delta(1) \cap \sigma_{\mathcal{L}(L^2(M,E))}(Q)) \qquad \text{for } B_\delta = \{\lambda \in \mathbb{C} : \ |\lambda - 1| < \delta\}.$$

In particular, we have $P = P^* = P^2$, and $\sigma_\psi(P) = p$ as above. □

Remark 2.4.3 *The above construction of projections has a more general functional analytic background. If Ψ is a Fréchet operator algebra with a given ideal $\mathcal{I}$, there is a lifting of idempotent elements of $\Psi/\mathcal{I}$ to idempotent elements in Ψ, provided some natural assumptions on the operator algebra are satisfied, cf. Gramsch* [10]. *In particular, for $\Psi = L^0_{\text{cl}}(M; E, E)$ and $\mathcal{I} = L^{-1}_{\text{cl}}(M; E, E)$ the space $\Psi/\mathcal{I}$ is isomorphic to the space of homogeneous symbols of order zero. The general theory gives a characterisation of the space of all idempotent elements $P \in L^0_{\text{cl}}(M; E, E)$ that belong to the connected component of a given idempotent $P_1 \in L^0_{\text{cl}}(M; E, E)$ and have the same homogeneous principal symbol as P_1. The result says that all those P have the form GP_1G^{-1}, where G varies over the connected component of the identity in the group $\{I + K \in \Psi^{-1} : K \in L^{-1}_{\text{cl}}(M; E, E)\}$, where Ψ^{-1} is the group of invertible elements of $L^0_{\text{cl}}(M; E, E)$.*

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