On the Homotopy Classification of Elliptic Operators on Manifolds with Singularities

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September 14, 1999

Abstract

We study the homotopy classification of elliptic operators on manifolds with singularities and establish necessary and sufficient conditions under which the classification splits into terms corresponding to the principal symbol and the conormal symbol.

Keywords: elliptic operators, homotopy classification, manifold with singularities, Atiyah–Singer theorem

1991 AMS classification: Primary 58G03, Secondary 58G10, 58G12, 58G25, 19K56

*Supported by Arbeitsgruppe "Differentialgleichungen und Komplexe Analysis" (Institut für Matematik, Universität Potsdam) and RFBR grants Nos. 97-01-00703 and 99-01-01100.

Contents

0 Introduction

The present paper deals with homotopy classification issues for elliptic operators on manifolds with conical singularities. Informative classifications and, accordingly, informative index formulas emerge for the case in which the principal symbol of the conormal symbol (e.g., see [1]) is subjected to constraints of a special form, for example, has certain symmetries.[1] Then the classification splits into summands corresponding to the principal symbol and the conormal symbol, and the index formula can essentially be obtained from the Atiyah–Singer theorem applied to the principal symbol (the contribution of the interior) and the relative index theorem (the contribution of the singular

[1] Unfortunately, the homotopy classification of *all* elliptic operators on manifolds with singularities is such that the corresponding most general index formula for elliptic operators D is necessary tautological:

$$\mathrm{ind}_a(D) = \mathrm{ind}_a(D).$$

points). In this paper we do not touch the index formulas themselves, and we refer the reader to, say, [2, 3] for that matter.

We establish *necessary and sufficient* classification splitting conditions in terms of the spectral flow of a periodic family of conormal symbols (for spectral boundary value problems, similar splitting conditions for the *index formula* were obtained in [4]) and describe the classification both for the case in which these conditions are satisfied and for the case in which they are violated. Next, we present simple *sufficient* symmetry type conditions under which the classification splits and give more specific classification formulas under these conditions.

1 Preliminaries

1.1 Elliptic operators on manifolds with singularities

All relevant definitions pertaining to manifolds with singularities and pseudodifferential operators on such manifolds can be found, e.g., in the monograph [1] (cf. also [5]). We consider a compact manifold M with conical singularities, and by Ω we denote the base of the corresponding cone. (If there are several conical points, we treat them as a single point with base Ω having several connected components. Furthermore, M will be identified with its stretched manifold, which is a C^∞ manifold with C^∞ boundary Ω.) Without loss of generality, we consider only zero-order pseudodifferential operators in weighted Sobolev spaces with the weight line $\operatorname{Im} p = 0$. These spaces will be denoted by $H^s(M)$ or $H^s(M, E)$; the latter form is used if we need to indicate the vector bundle explicitly. We consider only classical pseudodifferential operators, so that the principal symbol has an asymptotic expansion into a sum of functions homogeneous of orders $0, -1, -2, \ldots$ with respect to the cotangent variables.

The case of manifolds with conical singularities is different from the case of smooth compact manifolds without singularities in that the class of a given pseudodifferential operator D modulo compact operators (that is, an element of the Calkin algebra) is determined by a *pair* of symbols rather than one symbol, namely, by the *principal symbol* $\sigma(D)$, which is a function on the *compressed cotangent bundle* T^*M (see [6, 1]) outside the zero section, and the *conormal symbol* $\sigma_c(D)$. The conormal symbol is a one-parameter operator family in the spaces $H^s(\Omega)$; this family is defined and analytically depends on the parameter p in some neighborhood $|\operatorname{Im} p| < \varepsilon$ of the weight line. More precisely, it is a *pseudodifferential operator with parameter* p on Ω, where the parameter ranges in the above-indicated strip. Treated as a pseudodifferential operator with parameter, the conormal symbol, in turn, has a principal symbol, which will be denoted by $\sigma(\sigma_c(D))$ and called the *principal conormal symbol* and which is a homogeneous function of the variables (ξ, p) on $(T^*\Omega \times \mathbf{R})\setminus\{0\}$, where ξ is the variable in the fibers of $T^*\Omega$, $p \in \mathbf{R}$, and $\{0\}$ is the zero section $\{p = 0, \xi = 0\}$ of the vector bundle $T^*\Omega \times \mathbf{R}$ over Ω.

The principal and conormal symbols of an operator D satisfy the *consistency condition*

$$\sigma(\sigma_c(D)) = \sigma(D)|_{\partial T^*M} \qquad (1)$$

under the natural identification $T^*\Omega \times \mathbf{R} \simeq \partial(T^*M)$.

One of the main theorems of elliptic theory, namely, the finiteness theorem (e.g., see [1]) states that a pseudodifferential operator D on M is Fredholm in the Sobolev spaces $H^s(M)$ if and only if it is elliptic in the following sense:

1°. The principal symbol $\sigma(D)$ is invertible $T^*M\backslash\{0\}$.

2°. The conormal symbol $\sigma_c(D)$ is (boundedly) invertible in the Sobolev spaces $H^s(\Omega)$ for all $p \in \mathbf{R}$ (that is, for all p lying on the weight line).

Under the first condition, we say that the symbol $\sigma(D)$ is elliptic. Under the second condition, we say that $\sigma_c(D)$ is elliptic. It follows from condition (1) that if $\sigma(D)$ is elliptic, then $\sigma_c(D)$ is a family of operators on Ω elliptic with parameter p in the sense of Agranovich–Vishik [7] in a sufficiently narrow strip containing the weight line $\operatorname{Im} p = 0$. For brevity, in this case we say that D and $\sigma_c(D)$ are *formally elliptic*. If $\sigma_c(D)$ is formally elliptic, then the ellipticity condition 2° can *a priori* be violated at at most finitely many points (poles of the finitely meromorphic family $\sigma_c(D)^{-1}(p)$ on the weight line).

In what follows we need some simple properties of pseudodifferential operators on manifolds with singularities.

Proposition 1 *For any principal and conormal symbols satisfying the consistency condition* (1)*, there exists a pseudodifferential operator on M with these symbols. If the symbols continuously* (*resp., smoothly*) *depend on additional parameters, then the pseudodifferential operator can also be chosen to depend on these parameters continuously* (*smoothly*)*.*

The proof can be obtained by standard techniques related to partitions of unity.

Let a be a given elliptic symbol on $T^*M\backslash\{0\}$. A *trivial extension* of a is a symbol of the form $a \oplus \pi^*\gamma$, where $\gamma : E_1 \to E_2$ is a bundle isomorphism over M and $\pi : T^*M \to M$ is the natural projection.

Proposition 2 *Let a_t, $t \in [0,1]$, be a family of elliptic symbols on $T^*M\backslash\{0\}$ continuously depending on the parameter t. Then, possibly after a trivial extension and a substitution $t = f(\tau)$ in the homotopy parameter, this family can be lifted to a continuous family of elliptic pseudodifferential operators on M with principal symbols a_t.*

The proof will be given in § 5.

We shall also use further properties of elliptic pseudodifferential operators on M; they will be stated below, together with relevant references.

1.2 The spectral flow of a family of conormal symbols

The definitions and theorems given in this subsection pertain to the generalization, given in [3] (see also [8]), of the notion of spectral flow [9] to the case of arbitrary conormal symbols (as to polynomial symbols, e.g., see [4]).

Definition 1 Let $D(p)$ be a formally elliptic conormal symbol on a closed manifold Ω, and let p_0 be a singular point of $D(p)$, that is, a pole of the family $D^{-1}(p)$. The *multiplicity* of p_0 is the integer

$$m(p_0) \equiv m_D(p_0) = \operatorname{Trace} \operatorname*{Res}_{p=p_0} \left\{ D^{-1}(p) \frac{\partial D(p)}{\partial p} \right\}. \tag{2}$$

In what follows we consider only conormal symbols holomorphic in some strip $|\operatorname{Im} p| < \varepsilon$ around the weight line $\operatorname{Im} p = 0$.

Let $D_t(p)$, $t \in [0,1]$, be a continuous family of conormal symbols. Then for any sufficiently small $\delta > 0$ there exists a partition $0 = t_0 < t_1 < \ldots < t_N = 1$ of the interval $[0,1]$ and numbers $\gamma_i \in (-\varepsilon, \varepsilon)$, $i = 1, \ldots, N$, $\gamma_1 = \gamma_N = -\delta$, such that for $t \in [t_{i-1}, t_i]$ the operator $D_t(p)$ is invertible on the weight line $\operatorname{Im} p = \gamma_i$. For $j = 1, \ldots, N-1$, let p_{jk}, $k = 1, \ldots, N_j$, be the poles of the conormal symbol $D_{t_j}(p)$ in the strip between the weight lines $\operatorname{Im} p = \gamma_j$ and $\operatorname{Im} p = \gamma_{j+1}$. We set

$$g(\delta) = \sum_{j=1}^{N-1} \sum_{k=1}^{N_j} \pm m_{D_{t_j}}(p_{jk}), \tag{3}$$

where the sign "+" is taken for $\gamma_{j+1} < \gamma_j$ and the sign "−" otherwise.

Definition and Theorem 2 (a) *For sufficiently small $\delta > 0$, the number $g(\delta)$ is independent of δ, the partition $t_1, \ldots, t_{N-1}$, and the numbers $\gamma_2, \ldots, \gamma_{N-1}$. We set*

$$\operatorname{sf}\{D_t\} \stackrel{\text{def}}{=} g(\delta), \tag{4}$$

($\delta > 0$ is sufficiently small) and refer to this number as the spectral flow of the family D_t.

(b) *The spectral flow is a homotopy invariant of the family D_t with fixed endpoints and is also invariant under deformations of the family for which $D_0(p)$ and $D_1(p)$ always remain invertible on the weight line* $\operatorname{Im} p = 0$.

The main properties of spectral flow are expressed by the following three theorems.

Theorem 1 *Let $D_t(p)$, $t \in (-\infty, \infty)$, be a family of conormal symbols that smoothly depends on t and exponentially stabilizes as $t \to \pm\infty$ to some conormal symbols $D_{\pm\infty}(p)$ invertible on the weight line $\operatorname{Im} p = 0$. Then*

$$\operatorname{sf}\{D_t\} = -\operatorname{ind} D_t\left(-i\frac{\partial}{\partial t}\right), \tag{5}$$

where $D_t\left(-i\frac{\partial}{\partial t}\right)$ is treated as an elliptic operator in the Sobolev spaces $H^s(\Omega \times \mathbf{R})$ with weight exponent 0 on the infinite cylinder $\Omega \times \mathbf{R}_t$.

Theorem 2 *Let $D_t(p)$, $t \in S^t$, be a smooth periodic family of conormal symbols. Then*

$$\operatorname{sf}\{D_t\} = -\operatorname{ind} D_t\left(-i\frac{\partial}{\partial t}\right), \tag{6}$$

where the right-hand side is the index of the elliptic operator on $\Omega \times S^1$ generated by the family D_t.

The proofs of Theorems 1 and 2 can be found in [3].

Theorem 3 (a relative index formula) *Let M be a manifold with conical singularities, P_0 and P_1 elliptic operators in the Sobolev spaces $H^s(M)$, and P_t a homotopy between P_0 and P_1 in the class of formally elliptic operators, that is, operators with elliptic principal symbols. Next, let $D_t(p) = \sigma_c(P_t)$ be the conormal symbol of P_t. Then*

$$\operatorname{ind} P_0 - \operatorname{ind} P_1 = \operatorname{sf}\{D_t\}.$$

The proof will be given in § 5.

2 The classification of the set of all elliptic operators

2.1 The reduced classification

Prior to studying the general classification of elliptic operators on a manifold M with conical singularities, it is useful to find out how the reduced classification, that is, the classification of operators with *given principal symbol*, looks like. Thus, let an elliptic principal symbol a be given. We choose some elliptic operator A of order zero with principal symbol $\sigma(A) = a$.

Lemma 1 *Every elliptic operator D of order zero with principal symbol $\sigma(D) = a$ has the form*

$$D = NA + Q,$$

where Q is a compact operator and N is an elliptic operator with unit principal symbol.

The proof readily follows from the calculus of pseudodifferential operators on M (the principal and conormal symbols of a product of operators are equal to the products of the respective symbols of the factors); see [1].

Lemma 1 shows that the reduced homotopy classification of elliptic pseudodifferential operators on M is just the homotopy classification of operators with unit principal symbol. In turn, this classification is obviously equivalent to the classification of elliptic conormal symbols with unit principal symbol.[2] We identify the group of stable homotopy equivalence classes of elliptic operators on M with unit principal symbol and the group of stable homotopy equivalence classes of conormal symbols on Ω with unit principal symbol and denote either of these groups by $\Phi_0(\Omega)$.

Theorem 4 *The group $\Phi_0(\Omega)$ is isomorphic to $\mathbf{Z}$. Furthermore, one of the two possible isomorphisms*

$$\lambda : \Phi_0(\Omega) \to \mathbf{Z} \tag{7}$$

is given by the formula

$$\lambda([a(p)]) = \mathrm{sf}\,\{ta(p) + (1-t)E\}, \tag{8}$$

where $[a(p)] \in \Phi_0(\Omega)$ is the stable homotopy equivalence class of an elliptic conormal symbol $a(p)$ with principal symbol $\sigma(a(p)) = 1$ and E is the identity operator (the constant family).

Proof. Let us first prove that $\Phi_0 \simeq \mathbf{Z}$. Let $a(p)$ be an elliptic conormal symbol with unit principal symbol. Consider the family

$$tE + (1-t)a(p), \quad t \in [0,1].$$

Arguing as in the proof of Proposition 2 (see Subsection 5.1), we see that, possibly after a trivial extension of $a(p)$ and a change of the homotopy parameter, we can find a smooth family $b(p)$ of finite rank operators decaying at infinity and such that the operator

$$tE + (1-t)(a(p) + b(p))$$

[2]Of course, here we speak only of homotopies in which the principal symbol always remains constant.

is invertible for all $p \in \mathbf{R}$ and $t \in [0,1]$. Now the homotopy

$$a_t(p) = a(p)[tE + (1-t)(a(p)+b(p))]^{-1}$$

in the class of elliptic conormal symbols takes $a(p)$ to

$$a(p)(a(p)+b(p))^{-1} = 1 - b(p)(a(p)+b(p))^{-1} \equiv 1 + T(p),$$

where $T(p)$ is finite rank. Since $T(p)$ decays as $p \to \pm\infty$, we can view $1+T(p)$ as a continuous mapping from S^1 into the set of invertible operators in a finite-dimensional space, and now the desired assertion follows from the fact that

$$\pi_1(GL(n,\mathbf{C})) = \mathbf{Z}.$$

Next, the mapping (7) is well-defined by formula (8). Indeed, let

$$[b(p)] = [a(p)].$$

We can assume that there is a homotopy $a_t(p)$ such that

$$a_0(p) = a(p), \qquad a_1(p) = b(p).$$

Since $a_t(p)$ is elliptic for all t, it follows that $\operatorname{sf}\{a_t\} = 0$. Next, the family

$$tb(p) + (1-t)E$$

is homotopic to the concatenation of the families

$$a_t(p), \qquad ta(p) + (1-t)E,$$

and by Definition and Theorem 2, (b), we see that

$$\operatorname{sf}\{tb(p)+(1-t)E\} = \operatorname{sf}\{a_t\} + \operatorname{sf}\{ta(p)+(1-t)E\} = \operatorname{sf}\{ta(p)+(1-t)E\}.$$

Let us prove that the mapping (7) is an epimorphism. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of smooth functions in $L^2(\Omega)$, and let P_1 be the orthogonal projection on e_1. We set

$$a(p) = 1 + P_1\left(e^{i\varphi(p)} - 1\right),$$

where

$$\varphi(p) = \int_{-\infty}^{p} \psi(p)\, dp$$

and $\psi(p)$ is a function analytic in some strip containing the real axis, rapidly decaying as $\operatorname{Re} p \to \pm\infty$, and such that

$$\varphi(p) = \int_{-\infty}^{\infty} \psi(p)\, dp = 2\pi.$$

Since $e^{i\varphi(p)} - 1$ rapidly decays in the same strip as $\operatorname{Re} p \to \pm\infty$, it follows that $a(p)$ is an elliptic conormal symbol with $\sigma(a(p)) = 1$, and straightforward verification shows that

$$\operatorname{sf}\{ta(p) + (1-t)E\} = 1.$$

It remains to note that any epimorphism $\mathbf{Z} \to \mathbf{Z}$ is an isomorphism. The proof of the theorem is complete. □

2.2 The general classification

Now we can consider the usual stable homotopy classification of *all* elliptic pseudodifferential operators on M. Let $\operatorname{Ell}(M)$ be the set of stable homotopy equivalence classes of elliptic pseudodifferential operators on M. We equip $\operatorname{Ell}(M)$ with the structure of an abelian group induced by the direct sum of operators. Then the following theorems hold.

Theorem 5 *The mapping*

$$\chi : \operatorname{Ell}(M) \to K(T^*M) \oplus \mathbf{Z} \tag{9}$$

given by the formula

$$[D] \mapsto ([\sigma(D)], \operatorname{ind} D), \tag{10}$$

where $[\sigma(D)] \in K(T^*M)$ *is the element specified by the difference construction* [10] *and* $\operatorname{ind} D$ *is the analytical index of the operator* D*, is an isomorphism of groups.*

Here T^*M is the compressed cotangent bundle, and K is the K-functor with compact supports.

Theorem 6 *For a manifold* M *with conical singularities, there is no isomorphism*

$$\widetilde{\chi} : \operatorname{Ell}(M) \to G_1 \oplus G_2 \tag{11}$$

on the direct sum of any groups G_1 *and* G_2 *such that* $\widetilde{\chi}_1([D])$ *depends only on* $[\sigma(D)]$ *and* $\widetilde{\chi}_2([D])$ *depends only on* $\sigma_c(D)$*. Moreover, the stable homotopy equivalence class of an operator* D *is not determined by the pair* $[\sigma(D)]$*,* $\sigma_c(D)$*.*

We do not give a separate proof of Theorems 5 and 6, since these theorems are a special case of more general classification results proved below in § 3. The theorems themselves serve as the motivation of our subsequent considerations.

3 Splitting classifications

3.1 Homotopies and equivalence classes of elliptic pseudodifferential operators

Theorems 5 and 6 show that, generally speaking, there is no "good" homotopy classification of all elliptic operators on a manifold with singularities. Hence, instead of $\mathrm{Ell}(M)$, we consider subtler groups whose construction involves restrictions of a special form imposed on the operators and/or allowed homotopies. We deal only with restrictions specified in terms of the *principal conormal symbol*. By definition, an elliptic principal conormal symbol is a bundle isomorphism[3]

$$a: \pi^*E_1 \to \pi^*E_2,$$

where E_1 and E_2 are vector bundles over Ω and

$$\pi: S((T^*\Omega \times \mathbf{R})\backslash\{0\}) \to \Omega$$

is the natural projection (here by $S(F\backslash\{0\})$ we denote the sphere bundle associated with a vector bundle F).

Let Σ be some set of principal conormal symbols with the following properties:

(i) if $\alpha: E_1 \to E_2$ is a bundle isomorphism over Ω, then $\pi^*\alpha \in \Sigma$;

(ii) if $a \in \Sigma$, then $a^{-1} \in \Sigma$;

(iii) if $a, b \in \Sigma$, then $a \oplus b \in \Sigma$;

(iv) if $a, b \in \Sigma$ and the product ab is defined, then $ab \in \Sigma$.

Obviously, in the set of sets Σ satisfying conditions (i)–(iv) there is a maximal element $\overline{\Sigma}$ and a minimal element $\underline{\Sigma}$; the set $\overline{\Sigma}$ consists of all principal conormal symbols, and $\underline{\Sigma}$ consists of symbols of the form $\pi^*\alpha$, where α is a bundle isomorphism over Ω.

We choose some set Σ with properties (i)–(iv) and refer to elements of it as *admissible symbols*.

Next, let Γ be some set of continuous families $\{a\}_{t\in[0,1]}$ of principal conormal symbols with the following properties:

(v) if $\{a_t\} \in \Gamma$, then $a_t \in \Sigma$ for any $t \in [0,1]$;

(vi) if $a \in \Sigma$, then the family $a_t \equiv a$, $t \in [0,1]$, belongs to Γ;

[3] We do not use nonelliptic symbols, and for brevity we omit the adjective "elliptic" in what follows.

(vii) if $\{\alpha_t\}_{t\in[0,1]}$ is a continuous family of bundle isomorphisms over Ω, then $\{\pi^*\alpha_t\} \in \Gamma$;

(viii) if $\{a_t\}, \{b_t\} \in \Gamma$, then $\{a_t^{-1}\}$, $\{a_t \oplus b_t\} \in \Gamma$;

(ix) if $\{a_t\}, \{b_t\} \in \Gamma$ and the product $a_t b_t$ is defined, then $\{a_t b_t\} \in \Gamma$;

(x) if $\{a_t\}, \{b_t\} \in \Gamma$ and $a_1 = b_0$, then $\{(a \sqcup b)_t\}\Gamma$, where the *concatenation* $\{(a \sqcup b)_t\}$ of the families a_t and b_t is defined by the formula

$$(a \sqcup b)_t = \begin{cases} a_{2t}, & t \in [0, 1/2], \\ b_{2t-1}, & t \in [1/2, 1]; \end{cases}$$

(xi) if $\{a_t\} \in \Gamma$, $f : [0,1] \to [0,1]$ is a continuous mapping, and $b_t = a_{f(t)}$, then $\{b_t\} \in \Gamma$.

We choose some set Γ with properties (v)–(xi) and refer to elements of it as *admissible conormal homotopies*.

Remarks.

(a) Properties (i)–(xi) are natural if we intend to use Σ and Γ in the definition of stable homotopy equivalence classes.

(b) Properties (v)–(vi) are the compatibility conditions for Σ and Γ.

(c) For given Σ, a trivial example of Γ can be constructed as follows:

$$\Gamma = \{\{a_t\} \mid a_t \in \Sigma \ \ \forall t \in [0,1], \ a_t \text{ is a continuous family}\}.$$

The pair (Σ, Γ) thus obtained will be denoted merely by Σ.

(d) For a given Σ, we can construct a pair (Σ', Γ') as follows: $\Sigma' = \overline{\Sigma}$ consists of all conormal symbols, and

$$\Gamma' = \{\{a_t\} \mid a_t \text{ is a continuous family and } a_t a_0^{-1} \in \Sigma \quad \forall\, t \in [0,1]\}.$$

This pair (Σ', Γ') will be called the "dual" pair of Σ and will be denoted by Σ^*. One can readily verify that Σ^* satisfies conditions (i)–(xi). (We do not define the notion of a dual pair for an arbitrary pair (Σ, Γ).)

Let us now give a definition of the set of stable homotopy equivalence classes of elliptic pseudodifferential operators on M associated with a pair (Σ, Γ).

Definition 3 By $\mathrm{Ell}_{\Sigma,\Gamma}(M)$ we denote the quotient of the set of elliptic pseudodifferential operators D of order zero on M with principal conormal symbols $\sigma(D(p)) \in \Sigma$ modulo the following equivalence relation: $D_1 \approx D_2$ if there are isomorphisms β_1 and β_2 of vector bundles over M such that $\beta_1 \oplus D_1$ is homotopic to $\beta_2 \oplus D_2$ in the class of elliptic operators on M, and moreover, the principal conormal symbol of the homotopy belongs to Γ.

Proposition 3 (a) *Definition 3 is consistent. In other words,* $\approx$ *is indeed an equivalence relation.*

(b) *The direct sum of operators induces the structure of an abelian group on the set* $\mathrm{Ell}_{\Sigma,\Gamma}(M)$.

(c) *For* $(\Sigma,\Gamma)=\overline{\Sigma}$ *we have* $\mathrm{Ell}_{\Sigma,\Gamma}(M)=\mathrm{Ell}_{\overline{\Sigma}}(M)$.

(d) *The analytical index of an operator induces a well-defined homomorphism*

$$\mathrm{ind}:\mathrm{Ell}_{\Sigma,\Gamma}(M)\to\mathbf{Z}.$$

Proof. We only outline the proof, which closely follows that of the corresponding fact in ordinary elliptic theory [10].

(a) The symmetricity, reflexivity, and transitivity of $\approx$ follow from (xi), (vi), and (x), respectively, with regard to (vi), (viii), and (i).

(b) Obviously, the direct sum preserves the equivalence relation $\approx$. Next, for $a,b\in\Sigma$ the family

$$c_t=\begin{pmatrix}\cos\frac{\pi t}{2} & \sin\frac{\pi t}{2}\\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}\cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2}\\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{pmatrix}$$

satisfies the condition

$$c_0=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix},\quad c_1=\begin{pmatrix}b & 0\\ 0 & a\end{pmatrix}$$

and is, by virtue of (vi)–(ix), an admissible conormal homotopy, whence we can readily find that $A\oplus B\approx B\oplus A$ whenever A and B are admissible operators. Moreover, if A is a given admissible operator and B is an elliptic operator with principal symbol $\sigma(A)^{-1}$, then $AB=N$ is an operator with unit principal symbol. Let $\widetilde{N}$ be the class of N in $\Phi_0(\Omega)$, and let $N_1\in\widetilde{N}^{-1}$. Then $ABN_1\simeq 1$, so that $\mathrm{Ell}_{\Sigma,\Gamma}(M)$ is an abelian group.

Assertions (c) and (d) are obvious. The proof of the theorem is complete. □

Let us introduce yet another groups associated with the pair Σ,Γ. Namely, we denote the set of stable homotopy equivalence classes of admissible principal symbols by $K_{\Sigma,\Gamma}(T^*M)$, the set of stable homotopy equivalence classes of admissible principal conormal symbols by $K_{\Sigma,\Gamma}(\partial T^*M)$, and the set of stable homotopy equivalence classes of conormal symbols with admissible principal symbols by $\mathrm{Co}_{\Sigma,\Gamma}(\Omega)$ (in all three cases, only homotopies consistent with Γ are used in the definition of homotopy equivalence).

Theorem 7 (a) $K_{\Sigma,\Gamma}(T^*M)$, $K_{\Sigma,\Gamma}(\partial T^*M)$ *and* $\mathrm{Co}_{\Sigma,\Gamma}(\Omega)$ *are abelian groups (the addition in all cases is induced by the direct sum of representatives of the corresponding classes).*

(b) *The diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi_0(\Omega) & \xrightarrow{i} & \mathrm{Ell}_{\Sigma,\Gamma}(M) & \xrightarrow{\sigma} & K_{\Sigma,\Gamma}(T^*M) & \longrightarrow & 0 \\
 & & \| & & \Big\downarrow{\scriptstyle \sigma_c} & & \Big\downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & \Phi_0(\Omega) & \xrightarrow{i} & \mathrm{Co}_{\Sigma,\Gamma}(\Omega) & \xrightarrow{\sigma} & K_{\Sigma,\Gamma}(\partial T^*M) & \longrightarrow & 0,
\end{array}
\qquad (12)
$$

commutes. Here the mappings σ are induced by corresponding principal symbols, the mapping i in the upper row is induced by the mapping that takes each conormal symbol to an operator with unit principal symbol and with this conormal symbol, the mapping i in the lower row is induced by the embedding of conormal symbols with unit principal symbol in general conormal symbols, σ_c is induced by the passage to the conormal symbol, and j is induced by the restriction to the boundary of the compressed cotangent bundle.

(c) *The upper row in diagram* (12) *is exact, and the lower row is a complex and is exact in all terms except possibly for the first.*

Proof. All assertions of the theorem are a direct consequence of the corresponding definitions, except the exactness. To prove that the upper row is exact, we note that by Theorems 3 and 4, with regard for the obvious fact that the index of the identity operator is zero, one has

$$\operatorname{ind}(i(a)) = \lambda(a), \quad a \in \Phi_0(\Omega),$$

where λ is the isomorphism (7). Thus, $\lambda^{-1} \circ \operatorname{ind}$ is a left inverse of i, and the sequence is thereby exact in the first term. Exactness in the second term is obvious for both rows. Exactness in the third term follows from Proposition 2 (for the second row, more precisely, from the construction in the proof of that proposition).

The proof of the theorem is complete. □

Corollary 1 (the "universal" classification") *The mapping*

$$\widetilde{\chi} \equiv \widetilde{\chi}_{\Sigma,\Gamma} : \mathrm{Ell}_{\Sigma,\Gamma}(M) \to K_{\Sigma,\Gamma}(T^*M) \oplus \Phi_0(\Omega)$$

given by the formula

$$[D] \mapsto ([\sigma(D)], \lambda^{-1}(\operatorname{ind} D))$$

is an isomorphism of groups.

Indeed, we have seen from the proof that the mapping $\lambda^{-1} \circ \operatorname{ind}$ splits the upper row of (12). Note that we obtain Theorem 5 as a special case.

Remark 1 *The classification provided by Corollary 1, as well as the special case of it given by Theorem 5, has the following disadvantage: the second component of the mapping* $\widetilde{\chi}$ *depends both on the principal and the conormal symbol. In the following subsection we study conditions under which there exist splitting classifications, that is, classifications in which each of the components depends on only one component of the symbolic information.*

3.2 Necessary and sufficient conditions for the existence of a splitting classification

In this subsection we prove the following theorem.

Theorem 8 (a) *Let* $\Lambda \subseteq Q$ *be an arbitrary subring of the ring* Q *of rationals. The following condition is necessary for the existence of an isomorphism*

$$\chi = \chi_A \oplus \chi_B : \mathrm{Ell}_{\Sigma,\Gamma}(M) \otimes \Lambda \to A \oplus B$$

of Λ*-modules such that* χ_A *depends only on the principal symbol and* χ_B *depends only on the conormal symbol, that is,* $\chi_A = f \circ \sigma$ *and* $\chi_B = f \circ \sigma_c$ *for some homomorphisms*

$$f : K_{\Sigma,\Gamma}(T^*M) \otimes \Lambda \to A, \quad g : \mathrm{Co}_{\Sigma,\Gamma}(\Omega) \otimes \Lambda \to B.$$

> **Condition (SF)** *For any periodic admissible homotopy* $\{A_t\}$ *of conormal symbols (that is, a periodic family of conormal symbol such that* $\{\sigma(A_t)\} \in \Gamma$*), one has* $\mathrm{sf}\{A_t\} = 0$.

(b) *Suppose that condition*(SF) *is satisfied, and let* $\Lambda \subset Q$ *be a subring such that* $K_{\Sigma,\Gamma}(\partial T^*M) \otimes \Lambda$ *is a projective* Λ*-module. Then there exists a mapping*

$$g : \mathrm{Co}_{\Sigma,\Gamma}(\Omega) \otimes \Lambda \to \Phi_0(\Omega) \otimes \Lambda$$

such that

$$(\sigma, g \circ \sigma_c) : \mathrm{Ell}_{\Sigma,\Gamma}(M) \otimes \Lambda \to (K_{\Sigma,\Gamma}(M) \otimes \Lambda) \oplus (\Phi_0(\Omega) \otimes \Lambda)$$

is an isomorphism of Λ*-modules.*

Proof. Condition (SF) is equivalent to the special case of it in which A_0 is the identity operator. Indeed, without loss of generality we can assume that A_0 is invertible, and then

$$\mathrm{sf}\{A_t\} = \mathrm{sf}\{A_t A_0^{-1}\},$$

where the new family already satisfies the desired condition. Let us prove (a). Let condition (SF) be violated, and let $\{A_t\}$ be a periodic family of conormal symbols with $A_0 = A_1 = 1$ for which $\mathrm{sf}\{A_t\} \neq 0$. On the manifold M we consider two operators D_0 and D_1 such that $D_0 = 1$, the conormal symbol of D_1 is equal to unity, and the principal symbol $\sigma(D_1)$ is equal to 1 everywhere except for a collar neighborhood $U \simeq \partial T^*M \times [0,1)$ of the boundary ∂T^*M in TM, where it is equal to $\sigma(A_t)$, $t \in [0,1)$ (under the natural identification $\partial T^*M \simeq T^*\Omega \times \mathbf{R}$). By applying the relative index theorem 3, we see that

$$\operatorname{ind} D_1 = \operatorname{ind} D_0 + \mathrm{sf}\{A_t\} \neq \operatorname{ind} D_0,$$

and hence $[D_0] \neq [D_1]$ in $\mathrm{Ell}_{\Sigma,\Gamma}(M)$. On the other hand,

$$\begin{array}{rcl} [\sigma(D_0)] & = & [\sigma(D_1)] \quad \text{in} \quad K_{\Sigma,\Gamma}(M) \\ \sigma_c(D_0) & = & \sigma_c(D_1), \end{array}$$

whence we see that the isomorphism indicated in the theorem cannot exist, and moreover, the class of the operator D in $\mathrm{Ell}_{\Sigma,\Gamma}(M)$ is not uniquely determined by the classes of its principal and conormal symbols.

Now let us prove (b). Note that condition (SF) is equivalent to the exactness of the second row in (12). (Recall, that by Theorem 7, it suffices to verify the exactness in the first term.) Now if $K_{\Sigma,\Gamma}(\partial T^*M) \otimes \Lambda$ is a projective module, then, after tensorizing diagram (12) by Λ, we see that the second row splits. In other words, there is a mapping

$$g : \mathrm{Co}_{\Sigma,\Gamma}(\Omega) \otimes \Lambda \to \Phi_0(\Omega) \otimes \Lambda$$

that is a left inverse of i. But then the mapping $g \circ \sigma_c$ splits the upper row, which completes the proof of the theorem. □

Remark 2 In assertion (b) of the theorem, one can always take $\Lambda = Q$. On the other hand, we need not necessarily require that $K_{\Sigma,\Gamma}(\partial T^*M) \otimes \Lambda$ be projective. All we need is that the lower row in (12) splits after tensorizing by Λ.

4 Sufficient conditions for splitting

4.1 General symmetry conditions

It follows from Theorem 8 that to provide the existence of a splitting classification, we must ensure that the spectral flow is zero for any admissible periodic family of conormal symbols. We shall only deal with the situation in which Γ contains all homotopies with principal symbols in Σ and, according to the remark in § 3.1, omit the letter Γ. Note that the general conditions we write out in this section are transferred automatically to the dual pair Σ^*.

Thus, we shall describe sets Σ of a special form.

Let

$$\psi : T^*\Omega \times \mathbf{R} \to T^*\Omega \times \mathbf{R}$$

be an orientation-preserving automorphism of the vector bundle $T^*\Omega \times \mathbf{R} \to \Omega$ (the fiber of this bundle over a point $\omega \in \Omega$ is the vector space $T^*_\omega\Omega \times \mathbf{R}$). We specify $\Sigma = \Sigma_\psi$ as the set of symbols $a = a(\omega, \xi, p)$ elliptic with parameter $p \in \mathbf{R}$ on $(T^*\Omega \times \mathbf{R})\backslash\{0\}$ such that the symmetry condition

$$(\psi^* a)(\omega, \xi, p) = a(\omega, \xi, -p)$$

is satisfied (or, equivalently,

$$\beta^*\psi^* a = a, \tag{13}$$

where β is the mapping $p \mapsto -p$, which reverses the orientation).

Theorem 9 *Condition* (SF) *is satisfied for admissible symbols and homotopies specified by the set* Σ_ψ.

Proof. Let $\{A_t\}$ be a periodic family of conormal symbols with principal symbols $a_t = \sigma(A_t) \in \Sigma_\psi$. By Theorem 2,

$$sf\{A_t\} = -\mathrm{ind}\, A_t\left(-i\frac{\partial}{\partial t}\right) \equiv -\mathrm{ind}\, \hat{A},$$

where the right-hand side is the index of an elliptic operator on $\Omega \times S^1$. Let us express this index by the cohomological Atiyah–Singer formula:

$$\mathrm{ind}\, \hat{A} = \left\langle \mathrm{ch}[\sigma(\hat{A})]\ \pi^*\,\mathrm{Td}(T^*(\Omega \times S^1) \otimes \mathbf{C}),\ [T^*(\Omega \times S^1)]\right\rangle, \tag{14}$$

where

$$\pi: T^*(\Omega \times S^1) \longrightarrow \Omega \times S^1$$

is the natural projection. The cohomology classes occurring on the right-hand side in (14) are invariant with respect to the orientation-reversing mapping $\beta^*\psi^*$, whence it follows that $\mathrm{ind}\, \hat{A} = 0$ (cf. [4]). The proof of the theorem is complete. □

Remarks.

1) We can generalize the above construction to include automorphisms

$$\psi : T^*\Omega \times \mathbf{R} \to T^*\Omega \times \mathbf{R}$$

over a diffeomorphism

$$\alpha : \Omega \to \Omega$$

of the base Ω and to consider symmetry conditions of the form

$$\beta^* \psi^* a = \sigma_1 a \sigma_2,$$

where σ_1 and σ_2 are automorphisms of the bundles between which the conormal symbol a acts. (Confer [2], where index formulas were considered under a similar condition.)

2) Condition (13) includes the following special cases:

(i) symmetry under the reflection $p \mapsto -p$;

(ii) symmetry under the reflection $\xi \mapsto -\xi$ for odd-dimensional Ω;

(iii) symmetry under the reflection $(\xi, p) \mapsto (-\xi, -p)$ for even-dimensional Ω.

Thus, in all these cases there is an invariant homotopy classification.

4.2 Example

It is of interest to consider any cases in which one can explicitly describe the mapping

$$g : \mathrm{Co}_{\Sigma,\Gamma}(\Omega) \otimes \Lambda \to \Phi_0(\Omega) \otimes \Lambda \tag{15}$$

specifying the homotopy classification in assertion (b) of Theorem 8. The following proposition gives an appropriate example.

Proposition 4 *Suppose that ψ is homotopic to the identity, and moreover, the homotopy commutes with $\psi\beta$, that is, there is a continuous family of bundle automorphisms*

$$\psi_\tau : T^*\Omega \times \mathbf{R} \to T^*\Omega \times \mathbf{R},$$

such that $\psi_0 = id$, $\psi_1 = \psi$, and

$$\psi\beta\psi_\tau = \psi_\tau\psi\beta$$

for all $\tau \in [0, 1]$. Then the following assertions hold.

(a) *For any $a \in \Sigma$, the symbol $\psi_\tau^* a$ belongs to Σ for all $\tau \in [0, 1]$.*

(b) *Let $[A] \in \mathrm{Co}_\Sigma(\mu)$, and let $A_\tau(p)$ be a family of conormal symbols satisfying the following conditions:*

$$\begin{aligned} A_0 &= A(p), \quad A_1 = A(-p), \\ \sigma(A_\tau) &= \psi_\tau^* \sigma(A_0). \end{aligned}$$

The mapping (15) *given by the formula*

$$g([A]\otimes 1)=\frac{1}{2}\lambda^{-1}(\mathrm{sf}\,\{A_\tau\}) \tag{16}$$

splits the lower row in diagram (12).

Proof. Suppose that $[A(p)]\in\Phi_0(\Omega)$, that is, $\sigma(A)=1$. Then

$$\lambda([A(p)])=-\lambda([A(-p)]),$$

which readily follows from formula (8) and the definition of the spectral flow (under the substitution $p\to -p$, all residues change their signs). We take A_τ to be the concatenation of the families

$$ta(p)+(1-t)E, \qquad (1-t)a(-p)+tE;$$

then, with regard for the preceding, we have

$$\mathrm{sf}\,\{A_t\}=2\lambda([A]),$$

which completes the proof. □

Example 1 In particular, the assumptions of Proposition 4 are satisfied by the automorphism $\psi\equiv 1$. In this case, Σ is the set of symbols symmetric with respect to the reflection $p\mapsto -p$. Formula (16) in this case can be rewritten in the form

$$g([A(p)]\otimes 1)=\frac{1}{2}\mathrm{sf}\,\{tA(p)+(1-t)A(-p)\}$$

(see [3], where this expression was used in an index formula).

5 Proof of technical results

5.1 Proof of Proposition 2

We consider a continuous family D_t of pseudodifferential operators on M with principal symbols a_t. The corresponding conormal symbols, generally speaking, need not be invertible. However, they can be made invertible by an appropriate choice of weights on subintervals of the interval $[0,1]$ (see Fig. 1 and the definition of the spectral flow).

Joining the horizontal segments by vertical ones as shown in Fig. 1, we arrive at a new operator family, say $\overline{D}_\tau$, where $\tau\in[0,1]$ is the coordinate along the polygonal

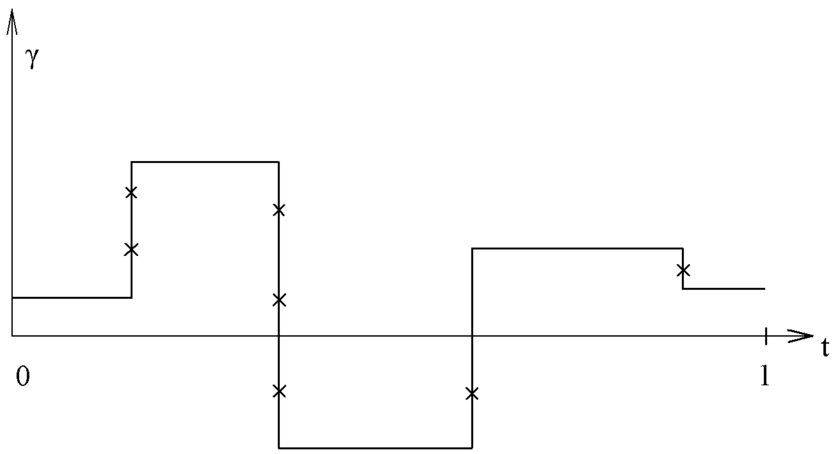

Figure 1: $\sigma_c(D_t)$ is invertible on the horizontal segments

line, with the following property: there are at most finitely many points (τ_j, p_j), $j = 1, \dots, N$, $\tau_j \in [0,1]$, $p_j \in \mathbf{R}$, where the family $\sigma_c(\widetilde{D}_\tau)(p)$ is not invertible. We wish to make $\sigma_c(\widetilde{D}_\tau)(p)$ invertible. This can be done as follows. Let

$$N_j = \mathrm{Ker}\sigma_c(\widetilde{D}_{\tau_j})(p_j), \quad n_j = \dim N_j, \quad n = \sum_{j=1}^{N} n_j.$$

Now let $\{e_k\}_{k=1}^{\infty}$ be a smooth basis in the space $L^2(\Omega)$, and let

$$\varphi_j : N_j \to L^2(\Omega)$$

be an isometric isomorphism onto the subspace generated by the vectors

$$\{e_k\}_{k=n_1+\dots+n_{j-1}+1}^{n_1+\dots+n_j}.$$

Furthermore, let E_1 and E_2 be the bundles in whose sections the conormal symbols act, and suppose that

$$\begin{array}{ll} P_j & \text{is the orthogonal projection on } \; N_j, \\ R & \text{is the linear span of the vectors } \; \{e_k\}_{k=1}^{n}. \end{array}$$

We define a mapping

$$Q_\tau(p) : L^2(\Omega, E) \to L^2(\Omega, E) \oplus R$$

by the formula

$$u \mapsto (\sigma_c(\widetilde{D}_\tau(p))u, \chi(p) \sum \varphi_j P_j u),$$

where $\chi(p)$ is a function that does not vanish for $p \in \mathbf{R}$, is analytic in a narrow strip near the real line, and is rapidly decaying as $\operatorname{Re} p \to \pm\infty$. Clearly, for all $p \in \mathbf{R}$ we have

$$\operatorname{Ker} Q_\tau(p) = \{0\} \quad \text{and} \quad \dim \operatorname{Coker} Q_\tau(p) = n.$$

Moreover, the cokernel $\operatorname{Coker} Q_\tau(p)$ can be identified with the orthogonal complement of $\operatorname{Im} Q_\tau(p)$, which is a subspace of dimension n in $C^\infty(\Omega, E) \oplus R$. Indeed,

$$(\operatorname{Im} Q_\tau(p))^\perp = \operatorname{Ker} Q_\tau(p)^*,$$

and any element

$$(v, w) \in (\operatorname{Im} Q_\tau(p))^\perp$$

satisfies the equation

$$\sigma_c(\widetilde{D}_\tau(p))^* v = -\bar{\chi}(p) \left(\sum \varphi_j P_j\right)^* w.$$

The right-hand side is C^∞, and so v is C^∞ as a solution of an elliptic equation with smooth right-hand side. For each $p \in \mathbf{R}$, let

$$Z(p) = (Z_1(p), Z_2(p)) : \ R \to C^\infty(\Omega, E) \oplus R$$

be an isomorphism onto this subspace. We can assume that $Z(p)$ rapidly tends to $(0, 1)$ as $p \to \pm\infty$, since for large $|p|$ the subspace $\operatorname{Im} Q_\tau(p)$ rapidly tends to $L^2(\Omega, E) \oplus 0$. We define a mapping

$$\widetilde{Q}_\tau(p) : \ L^2(\Omega, E) \oplus L^2(\Omega) \to L^2(\Omega, E) \oplus L^2(\Omega)$$

by setting

$$(u, v) \mapsto Q_\tau(p)u + Z(p)P_R v + (1 - P_R)v,$$

where P_R is the orthogonal projection on R in $L^2(\Omega)$. This mapping can be represented in the matrix form as

$$\begin{pmatrix} \sigma_c(\widetilde{D}_\tau(p)) & Z_1(p)P_R \\ \chi(p)\sum \phi_j P_j & 1 + (Z_2(p) - 1)P_R \end{pmatrix} = \begin{pmatrix} \sigma_c(\widetilde{D}_\tau(p)) & 0 \\ 0 & 1 \end{pmatrix} + \Phi,$$

where Φ is smoothing and rapidly decaying at infinity. By taking the convolution of $\widetilde{Q}_\tau(p)$ with the Fourier transform of a rapidly decaying function $\psi(\varepsilon t)$, $\psi(0) = 1$, we obtain a conormal symbol, which will be denoted by the same letter. For sufficiently small ε, this conormal symbol is invertible (elliptic) for all τ, and

$$\sigma(\widetilde{Q}_\tau(p)) = a_{f(\tau)} \oplus 1,$$

where $t = f(\tau)$ is the projection of the broken line in Fig. 1 on the t-axis.

This completes the proof of Proposition 2.

5.2 Proof of Theorem 3

Let $t_1, \ldots, t_{N-1}$ and $\gamma_1, \ldots, \gamma_N$ be the numbers used in the definition of the spectral flow of the family D_t. We can assume that $\gamma_1 = \gamma_N = 0$, since the families $D_0(p)$ and $D_1(p)$ are invertible on the weight line $\operatorname{Im} p = 0$. Let us consider continuous families of elliptic operators

$$Q_j(t), \quad t \in [t_{j-1}, t_j]$$

on M with principal symbols

$$\sigma(Q_j(t)) = \sigma(P_t)$$

and conormal symbols

$$\sigma_c(Q_j(t)) = D_t(p + i\gamma_j), \quad j = 1, \ldots N. \tag{17}$$

We can assume that

$$Q_1(0) = P_0 \quad \text{and} \quad Q_N(1) = P_1.$$

Then

$$\operatorname{ind} P_0 - \operatorname{ind} P_1 = \sum_{j=1}^{N-1} (\operatorname{ind} Q_j(t_j) - \operatorname{ind} Q_{j+1}(t_j)), \tag{18}$$

since the index of each of the $Q_j(t)$ is independent of $t \in [t_{j-1}, t_j]$. However, the operators $Q_{j+1}(t_j)$ and $Q_j(t_j)$ have the same principal symbols, whereas their conormal symbols differ only by the shift by $i(\gamma_{j+1} - \gamma_j)$ in the complex p-plane. By applying the standard relative index theorem for the change of the weight line (e.g., see [2]), we find that the corresponding term on the right-hand side in (18) is equal to $\pm \sum_{k=1}^{N_j} m_{D_{t_j}}(p_{jk})$ (in the notation adopted in (3)), whence the assertion of the theorem readily follows. □

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