Boundary Value Problems in Domains with Corners

V. Rabinovich
B.-W. Schulze
N. Tarkhanov

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Abstract

We describe Fredholm boundary value problems for differential equations in domains with intersecting cuspidal edges on the boundary.

Contents

Introduction

The main aim of this paper is the Fredholm theory of elliptic boundary value problems in domains with intersecting cuspidal edges on the boundary.

We consider a domain $\mathcal{D}$ in $\mathbb{R}^{n+1}$ whose boundary $\partial\mathcal{D}$ is a C^∞ submanifold of $\mathbb{R}^{n+1}$ outside of a set of "singular points" $S \subset \partial\mathcal{D}$. We shall sometimes write $\operatorname{sing} \partial\mathcal{D}$ for S. We assume that S is the union of a finite number of one-dimensional cuspidal edges E_ν, $\nu = 1, \ldots, N$. The edges E_ν themselves can have a finite number of cuspidal points at which they meet each other. Thus, any point of the intersection of edges is a cuspidal (perhaps, conical) point of the boundary, the cross-section (link) of $\mathcal{D}$ near the point being a compact manifold B of dimension n with a finite set of cuspidal points on the boundary ∂B.

Boundary value problems in domains with isolated singular points on the boundary were studied by many authors. The classical work here is that by Kondrat'ev [Kon67]. For a recent account of the theory and the complete bibliography we refer the reader to [MKR97] and [RST97].

The paper of Feigin [Fei72] was the starting point of our investigation on boundary value problems in domains with smooth cuspidal edges on the boundary, cf. [RST98]. The geometry gives rise to a natural class of typical operators. Under an ellipticity condition, the Fredholm property of these operators can be described in "naive" weighted Sobolev spaces.

Yet another approach to analysis on manifolds with cuspidal edges is developed in [ST99]. It goes back as far as [Sch91], and completes the results of [RST98] in the case of operators with rather singular "coefficients."

Domains with edges on the boundary intersecting at non-zero angles belong to the next step in the hierarchy of manifolds with singularities that consists of manifolds with corners. These arise in various ways. Constructions leading to manifolds with corners include the desingularisation of singular varieties (blow-up) and the compactification of non-compact spaces. Stability under products is of primary importance, enough alone to justify the detailed study of manifolds with corners, because of the Schwartz kernel theorem.

In the literature there are several approaches to the analysis on manifolds with corners. We mention three of them.

The first approach is worked out by Maz'ya and Plamenevskii [MP77]. It usually applies to boundary value problems for differential equations though generalisations to differential analysis on compact closed manifolds with corners are straightforward.

In a number of papers of Plamenevskii and Senichkin, cf. for instance [PS95], the spectra of C^*-algebras of pseudodifferential operators on manifolds with intersecting edges on the boundary ("polyhedrons") are studied. Thus, the authors restrict themselves to operators with homogeneous symbols of order 0 acting in L^2-spaces.

The second approach is due to Melrose [Mel87, Mel96]. It deals with so-called b-pseudodifferential operators on manifolds with corners. While originating from geometry this theory does not apply, however, to many interesting elliptic operators, e.g., the Laplace operator in the corner $(\mathbb{R}_+)^n$, $n \geq 3$.

From the point of view of analysis, the most informative and richest in content calculus of pseudodifferential operators on manifolds with corners is due to Schulze [Sch92]. This approach relies on two basic theories, namely, 'conifications' and 'edgifications' of an operator algebra.

The singularities are generated successively by 'conifications' and 'edgifications' of given geometric objects, starting with $\mathbb{R}_+$ and a C^∞ compact closed manifold X. The conification of X is then $X^\wedge = \mathbb{R}_+ \times X$ which is thought of as a cone with base X, the vertex being deleted. The edgification is defined by $\mathbb{R}^q \times X^\wedge$ which is the local model of a wedge.

Now we can pass to further conification $\mathbb{R}_+ \times \mathbb{R}^q \times X^\wedge$ which is the local model of a corner, and so on. This gives rise to evident global definitions of "manifolds" with conical points, edges, corners, etc.

The program is, parallel to the geometric picture, to realise function spaces and operator algebras with symbolic structures for studying the solvability for natural classes of differential operators on the underlying spaces with singularities. Compared with the situation on smooth manifolds, the calculi on spaces with singularities require new concepts and a refined analysis.

The theory of [Sch92] includes a concept of ellipticity for corner operators in terms of leading symbols, which is equivalent to the Fredholm property of those operators.

It is an important feature in the analysis on manifolds with singularities that for a calculus to be useful, it should be 'iterative'. This means that given a manifold M, possibly with singularities, one can pass step by step to the treatment of higher singularities by conification and edgification. As a necessary intermediate tool one requires parameter-dependent versions of calculi for lower order singularities. The results of the present paper are based on those of [RST97, RST98] and develop them.

It is organised as follows. In Chapter 1 we study boundary value problems depending on a parameter $\lambda + i\delta$, $\delta \in \mathbb{R}$ being fixed, in a domain $\mathcal{D} \subset \mathbb{R}^{n+1}$ with a finite number of cuspidal points on the boundary. As function spaces we take special weighted Sobolev spaces with norms depending on the parameter.

We show conditions for the problem to be invertible for large values of the parameter. Moreover, we prove that the norm of the inverse operator acting in the above spaces with parameter is estimated by a constant independent of $\lambda + i\delta$. This is a crucial fact because the problem under consideration degenerates at the singular points of the boundary. Thus, we deal with a non-standard theory of boundary value problems depending on a parameter, cf. Agranovich and Vishik [AV64].

In Chapter 2 we treat boundary value problems in cylindrical domains whose cross-sections are domains with a finite set of cuspidal points on the boundary. We prove that a problem is Fredholm if it is uniformly elliptic outside of the set of singularities, the edge symbols of [RST98] are invertible over smooth parts of the edges, and an additional condition at either of the two points at infinity is fulfilled. This latter just amounts to the invertibility of a family of boundary value problems in the cross-section, parametrised by $\lambda + i\delta$.

Chapter 3 is devoted to boundary value problems in domains $\mathcal{D}$ with cusps on the boundary, the cross-sections (links) having cuspidal points, too. Thus, $\partial\mathcal{D}$ contains a finite number of cuspidal edges of dimension 1 intersecting at zero angles. A blow-up process near any point v of the intersection of edges reduces the problem to that in a cylinder whose cross-section is the link of $\mathcal{D}$ close to v.

Chapter 1

Boundary Value Problems with Parameter

1.1 Cuspidal singular points

We say that (r, ω) is a polar system of coordinates with centre at the origin in $\mathbb{R}^{n+1}$ if each point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ can be written in the form

$$x = r\, S(\omega),$$

for $(r, \omega) \in \mathbb{R}_+ \times \Omega$, where Ω is a domain in $\mathbb{R}^n$ star-shaped with respect to the origin, and S a diffeomorphism of Ω to the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

By a canonical *oscillating cusp* we mean a domain in $\mathbb{R}^{n+1}$ given in the form

$$C_{x^0} = \{x^0 + r\, S(\phi(r) f(r) \theta) : \ r \in \mathbb{R}_+,\, \theta \in B\}$$

where B is a subdomain of Ω, and $f(r)$, $\phi(r)$ are C^∞ functions on $\mathbb{R}_+$ specifying the *degeneracy* and the *oscillation* of the cusp at x^0, respectively.

To exclude the case where C_{x^0} possesses a tangent plane at x^0, we require $f(r)$ to be bounded near $r = 0$. More precisely, we assume that

1) $f(r) < 0$ for all $r \in \mathbb{R}_+$;
2) $|r^j f^{(j)}(r)| \le c_j$ near $r = 0$, for every $j \in \mathbb{Z}_+$.

From 1) and 2) it follows that the singularity of $1/rf(r)$ at $r = 0$ is not integrable. Moreover, we may modify f away from any neighbourhood of $r = 0$. Thus, setting

$$\mathfrak{d}(r) = \int_{r^0}^{r} \frac{d\vartheta}{\vartheta f(\vartheta)}$$

for $r > 0$, we get a diffeomorphism of $\mathbb{R}_+$ onto the entire real axis $\mathbb{R}$, such that $\mathfrak{d}'(r) = 1/rf(r)$.

The following example including both power-like and exponential cusps is typical for our theory.

Example 1.1.1 Let

$$f(r) = \begin{cases} -r^p \exp(-1/r^q), & r \in (0,1]; \\ -1/r, & r \in [2,+\infty), \end{cases}$$

where $p, q \in \mathbb{R}$. Suppose $q \geq 0$. If $q = 0$, we moreover require $p \geq 0$. When appropriately extended to the interval $(1,2)$, this function bears Properties 1) and 2).

□

On the other hand, $\phi \in C^\infty_{\text{loc}}(\mathbb{R}_+)$ is required to meet the following conditions:

a) $\inf_{r\in\mathbb{R}_+} \phi(r) > 0$;
b) $|D^\beta_{\mathfrak{d}} \phi(r)| \leq c_\beta$, for every $\beta \in \mathbb{Z}_+$; and
c) $\lim\limits_{r\to 0} D_{\mathfrak{d}}\phi(r) = 0$

where

$$D_{\mathfrak{d}} = \frac{1}{\mathfrak{d}'(r)} \frac{1}{i} \frac{\partial}{\partial r}. \tag{1.1.1}$$

As but one example of $\phi(r)$ satisfying a)–c) we show

$$\phi(r) = 1 - \frac{1}{2} \sin(\mathfrak{d}(r))^\epsilon \omega(r),$$

where $\epsilon \in [0,1)$ and ω is a cut-off function on $\mathbb{R}_+$ vanishing near $r^0 = \mathfrak{d}^{-1}(0)$, cf. [RST98].

Pick a cut-off function $\chi(t)$ at $t = +\infty$, i.e., any C^∞ function on $\mathbb{R}$, such that $\chi(t) \equiv 0$ for $t < a$ and $\chi(t) \equiv 1$ for $t > b$, where $0 < a < b < \infty$. For $\varepsilon > 0$, set

$$\mathfrak{d}^*\chi_\varepsilon(r) = \chi(\varepsilon\, \mathfrak{d}(r)),$$

thus obtaining a cut-off function at the point $r = 0$ on the semi-axis. It is easy to check that

$$\begin{aligned} D^j_{\mathfrak{d}}\, \mathfrak{d}^*\chi_\varepsilon &= \varepsilon^j\, \mathfrak{d}^* \left(D^j \chi\right)_\varepsilon \\ &= 0\left(\varepsilon^j\right) \end{aligned}$$

for all $j \in \mathbb{Z}_+$, the constant depending on χ and j but not on ε.

In the sequel $\mathcal{D}$ stands for a bounded domain in $\mathbb{R}^{n+1}$ with a finite number of singular points on the boundary, $\operatorname{sing}\partial\mathcal{D} = \{x^1, \ldots, x^N\}$. We assume that $\overline{\mathcal{D}} \setminus \operatorname{sing}\partial\mathcal{D}$ is a C^∞ manifold with boundary. Moreover, $\mathcal{D}$ has a cuspidal singularity at each point x^ν, as defined above. Thus, $\mathcal{D}$ is a canonical oscillating cusp near x^ν, and we write $f_\nu(r)$ and $\phi_\nu(r)$ for the specifying functions of this cusp. Set

$$\mathfrak{d}_\nu(r) = \int_{r^0}^r \frac{d\vartheta}{\vartheta f_\nu(\vartheta)},$$

for $r > 0$.

1.2 Admissible differential operators

Let we be given a differential operator A in $\mathcal{D}$ of the form

$$A = \sum_{|\gamma| \leq m} a_{\gamma}(x) D^{\gamma}$$

the coefficients a_{γ} being C^{∞} functions near $\overline{\mathcal{D}} \setminus \operatorname{sing} \partial\mathcal{D}$. We also put some restrictions on the behaviour of the coefficients near every point $x^{\nu} \in \operatorname{sing} \partial\mathcal{D}$, namely

$$\begin{array}{rcl} |D^G a_{\gamma}(x)| & \leq & c_G(a_{\gamma}) \left(-\mathfrak{d}'_{\nu}(|x - x^{\nu}|)\right)^{|G|}, \\ \lim\limits_{x \to x^{\nu}} \nabla a_{\gamma}(x) \,/\, \mathfrak{d}'_{\nu}(|x - x^{\nu}|) & = & 0 \end{array} \qquad (1.2.1)$$

for all multi-indices $G \in \mathbb{Z}_+^{n+1}$, the constants $c_G(a_{\gamma})$ being independent of x near x^{ν} in $\mathcal{D}$.

As is shown in [RST98], the estimates (1.2.1) just amount to saying that the coefficients $a_{\gamma}(x)$ are slowly varying as $x \to x^{\nu}$, i.e.,

$$\lim_{x \to x^{\nu}} \left(-\mathfrak{d}'_{\nu}(|x - x^{\nu}|)\right)^{-|G|} D^G a_{\gamma}(x) = 0$$

for all $G \in \mathbb{Z}_+^{n+1} \setminus \{0\}$.

Introducing polar coordinates in $\mathcal{D} \cap B(x^{\nu}, \varepsilon)$ by

$$\begin{array}{rcl} x & = & \pi_{\nu}(r, \theta) \\ & = & x^{\nu} + r\, S(\phi_{\nu}(r) f_{\nu}(r) \theta), \end{array}$$

with $r \in (0, \varepsilon)$ and $\theta \in B_{\nu}$, we get for the pull-back $\pi_{\nu}^* A$ of the operator A an equality

$$\pi_{\nu}^* A = \left(\mathfrak{d}'_{\nu}(r)\right)^m \sum_{j + |\alpha| \leq m} a_{j,\alpha}(r, \theta)\, D_{\mathfrak{d}_{\nu}}^j D_{\theta}^{\alpha} + \left(\mathfrak{d}'_{\nu}(r)\right)^m R_{\nu}$$

where $a_{j,\alpha}$ have explicit expressions through the coefficients of A, cf. (3.2.10) in [RST98]. From (1.2.1) it follows that

$$\begin{array}{rcl} |D_{\mathfrak{d}_{\nu}}^J D_{\theta}^A a_{j,\alpha}(r, \theta)| & \leq & c_{J,A}(a_{j,\alpha}), \\ \lim\limits_{r \to 0} D_{\mathfrak{d}_{\nu}} D_{\theta}^A a_{j,\alpha}(r, \theta) & = & 0 \end{array} \qquad (1.2.2)$$

for all $J \in \mathbb{Z}_+$ and $A \in \mathbb{Z}_+^n$, with $c_{J,A}(a_{j,\alpha})$ a constant independent of $r \in (0, \varepsilon)$ and $\theta \in B_{\nu}$. On the other hand, the coefficients of R_{ν} meet stronger conditions than (1.2.2), namely, they are infinitesimal along with all derivatives as $r \to \infty$. Therefore, the operator R_{ν} has a small norm near $r = 0$ in relevant function spaces.

Now, by an *admissible* differential operator in $\mathcal{D}$ with parameter $\lambda + i\delta$ we mean

$$A(\lambda + i\delta) = \sum_{k+|\gamma|\le m} a_{k,\gamma}(x)\,(\lambda + i\delta)^k D_x^\gamma$$

where $a_{k,\gamma}(x)$ satisfy the conditions in the domain $\mathcal{D}$ discussed above.

1.3 Function spaces

Let C_{x^0} be a canonical oscillating cusp. Given any $s \in \mathbb{Z}_+$ and $\mu \in \mathbb{R}$, we define the space $H^{s,\mu}(C_{x^0})$ [1] to be the completion of C^∞ functions in a neighbourhood of the closure of C_{x^0} vanishing near x^0, with respect to the norm

$$\|u\|_{H^{s,\mu}(C_{x^0})} = \left(\int_{\mathbb{R}_+} (\mathfrak{d}'(r))^{2\mu} \sum_{j=0}^{s} \|D_{\mathfrak{d}}^j\,\pi^* u\|^2_{H^{s-j}(B)}\, dm(r) \right)^{\frac{1}{2}} \tag{1.3.1}$$

where π^* means the pull-back under the coordinates $x = x^0 + r\,S(\phi(r)f(r)\theta)$, $H^{s-j}(B)$ are the usual Sobolev spaces on the link, and the measure dm is induced by the Lebesgue measure dt on the real axis under $t = \mathfrak{d}(r)$, i.e., $dm(r) = |\mathfrak{d}'(r)|dr$.

For each singular point $x^\nu \in \operatorname{sing}\partial\mathcal{D}$, we fix a ball $B(x^\nu, \varepsilon)$ such that $\mathcal{D} \cap B(x^\nu, \varepsilon)$ admits polar coordinates with centre x^ν and $B(x^1, \varepsilon), \ldots, B(x^N, \varepsilon)$ are pairwise non-overlapping. Moreover, we choose a cut-off function with a support in $B(x^\nu, \varepsilon)$ which is of the form $\chi_{\nu,\varepsilon}(r) = \mathfrak{d}_\nu^* \chi_\varepsilon(r)$. To this end, it suffices to take

$$a = \max_{\nu=1,\ldots,N} \varepsilon\,\mathfrak{d}_\nu(\varepsilon),$$

for we can certainly assume, by decreasing ε if necessary, that all the $\mathfrak{d}_\nu(\varepsilon)$ are positive. Set

$$\chi_{0,\varepsilon} = 1 - \sum_{\nu=1}^{N} \chi_{\nu,\varepsilon}$$

regarded as a function over the whole domain $\mathcal{D}$.

Let $s \in \mathbb{Z}_+$ and $\mu = (\mu_1, \ldots, \mu_N)$, with $\mu_\nu \in \mathbb{R}$. We denote by $H^{s,\mu}(\mathcal{D})$ the completion of C^∞ functions in a neighbourhood of $\overline{\mathcal{D}}$ vanishing near every point x^ν, with respect to the norm

$$\|u\|_{H^{s,\mu}(\mathcal{D})} = \|\chi_{0,\varepsilon} u\|_{H^s(\mathcal{D})} + \sum_{\nu=1}^{N} \|\chi_{\nu,\varepsilon} u\|_{H^{s,\mu_\nu}(C_{x^\nu})}.$$

By interpolation and duality, this definition actually extends to arbitrary real values s.

[1] In the notation of [RST97], this is $H^{s,0,\mu}(C_{x^0})$.

Finally, we write $H^{s-\frac{1}{2},\mu}(\partial\mathcal{D})$ for the space of traces on the smooth part of $\partial\mathcal{D}$ of all elements in $H^{s,\mu}(\mathcal{D})$, $s > 1/2$.

1.4 Function spaces depending on a parameter

Suppose $s \in \mathbb{R}_+$. For any value of the parameter $\lambda + i\delta$ along the weight line $\Gamma_\delta = \{\zeta \in \mathbb{C} : \Im\zeta = \delta\}$, we introduce the space $H^{s,\mu}_{\lambda+i\delta}(C_{x^0})$ as the completion of C^∞ functions near the closure of C_{x^0} vanishing close to x^0, with respect to the norm

$$\|u\|_{H^{s,\mu}_{\lambda+i\delta}(C_{x^0})} = \left(\int_{\mathbb{R}_+} (\mathfrak{d}'(r))^{2\mu} \sum_{k+j\le s} \|(\frac{1}{\mathfrak{d}'(r)}(\lambda+i\delta))^k D^j_{\mathfrak{d}}\, \pi^* u\|^2_{H^{s-(k+j)}(B)} dm(r) \right)^{\frac{1}{2}}. \tag{1.4.1}$$

When localised away from a neighbourhood of the vertex x^0, the spaces $H^{s,\mu}_{\lambda+i\delta}(C_{x^0})$ coincide with the usual Sobolev spaces with parameter $\lambda + i\delta$, cf. Agranovich and Vishik [AV64].

Having disposed of this preliminary step, we can now define, just as above, the space $H^{s,\mu}_{\lambda+i\delta}(\mathcal{D})$ with the norm

$$\|u\|_{H^{s,\mu}_{\lambda+i\delta}(\mathcal{D})} = \|\chi_{0,\varepsilon} u\|_{H^{s}_{\lambda+i\delta}(\mathcal{D})} + \sum_{\nu=1}^{N} \|\chi_{\nu,\varepsilon} u\|_{H^{s,\mu_\nu}_{\lambda+i\delta}(C_{x^\nu})},$$

$H^{s}_{\lambda+i\delta}(\mathcal{D})$ being the usual Sobolev space with parameter $\lambda + i\delta$. By interpolation and duality, we extend this scale of spaces to all real s.

1.5 Local invertibility of boundary value problems with parameter

In the domain $\mathcal{D}$, we consider a boundary value problem depending on the parameter $\lambda + i\delta$, namely

$$\begin{cases} A(\lambda+i\delta)\, u = f & \text{in} \quad \mathcal{D}, \\ B_i(\lambda+i\delta)\, u = u_i & \text{on} \quad \partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}, \end{cases} \tag{1.5.1}$$

where A is an admissible differential operator in $\mathcal{D}$ and (B_i) a system of admissible differential operators in a neighbourhood of $\partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}$. We denote m the order of A and m_i the order of B_i.

We assign to (1.5.1) an operator

$$\mathcal{A}(\lambda+i\delta) = \begin{pmatrix} A(\lambda+i\delta) \\ \oplus\, r_{\partial\mathcal{D}} B_i(\lambda+i\delta) \end{pmatrix}$$

acting as

$$\mathcal{A}(\lambda+i\delta)\colon\ H^{s,\mu}_{\lambda+i\delta}(\mathcal{D}) \to \begin{array}{c} H^{s-m,\mu-m}_{\lambda+i\delta}(\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i}_{\lambda+i\delta}(\partial\mathcal{D}) \end{array}, \qquad (1.5.2)$$

where $r_{\partial\mathcal{D}}$ means the restriction to the smooth part of the boundary of $\mathcal{D}$ and s is any integer with $s > \max m_i$.

Under the change of variables $x = \pi_\nu(r,\theta)$ close to any singular point $x^\nu \in \partial\mathcal{D}$, the operators $A(\lambda+i\delta)$ and $(B_i(\lambda+i\delta))$ transform into operators

$$\begin{aligned} \pi_\nu^\sharp A(\lambda+i\delta) &= (\mathfrak{d}'_\nu(r))^m \sum_{k+j+|\alpha|\le m} a_{k,j,\alpha}(r,\theta)\,(\lambda+i\delta)^k D^j_{\mathfrak{d}_\nu} D^\alpha_\theta \\ \pi_\nu^\sharp B_i(\lambda+i\delta) &= (\mathfrak{d}'_\nu(r))^{m_i} \sum_{k+j+|\alpha|\le m_i} b_{i,k,j,\alpha}(r,\theta)\,(\lambda+i\delta)^k D^j_{\mathfrak{d}_\nu} D^\alpha_\theta \end{aligned}$$

over the semicylinder $\mathbb{R}_+ \times B_\nu$, up to “small” remainders which do not affect the local invertibility. The coefficients $a_{k,j,\alpha}$ and $b_{i,k,j,\alpha}$ are required to satisfy (1.2.2) uniformly in $\theta \in B_\nu$.

Put

$$\sigma_{\mathfrak{d}_\nu}(\mathcal{A}(\lambda+i\delta))(x^\nu;r,\varrho) = \begin{pmatrix} \displaystyle\sum_{k+j+|\alpha|\le m} a_{k,j,\alpha}(r,\theta)\,(\lambda+i\delta)^k \varrho^j D^\alpha_\theta \\ \oplus \displaystyle\sum_{k+j+|\alpha|\le m_i} r_{\partial B_\nu}\, b_{i,k,j,\alpha}(r,\theta)\,(\lambda+i\delta)^k \varrho^j D^\alpha_\theta \end{pmatrix},$$

for $r > 0$ small enough and $\varrho \in \mathbb{R}$. In this way we obtain what is usually referred to as the *conormal symbol* of $\mathcal{A}(\lambda+i\delta)$ at the point x^ν. It is a family of boundary value problems on the domain B_ν parametrised by (in general, complex) ϱ,

$$\sigma_{\mathfrak{d}_\nu}(\mathcal{A}(\lambda+i\delta))(x^\nu;r,\varrho):\ H^s(B_\nu) \to \begin{array}{c} H^{s-m}(B_\nu) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2}}(\partial B_\nu) \end{array}. \qquad (1.5.3)$$

We say that $\mathcal{A}(\lambda+i\delta)$ is a uniformly elliptic boundary value problem with parameter in $\overline{\mathcal{D}} \setminus \operatorname{sing}\partial\mathcal{D}$ if: 1) $A(\lambda+i\delta)$ is a uniformly elliptic differential operator in $\overline{\mathcal{D}} \setminus \operatorname{sing}\partial\mathcal{D}$; and 2) the Lopatinskii condition with parameter, cf. [AV64], is fulfilled uniformly at the points of $\partial\mathcal{D} \setminus \operatorname{sing}\partial\mathcal{D}$. The main result of this chapter is the following theorem.

Theorem 1.5.1 *Suppose that:*

1) $\mathcal{A}(\lambda+i\delta)$ *is a uniformly elliptic boundary value problem with parameter in* $\overline{\mathcal{D}} \setminus \operatorname{sing}\partial\mathcal{D}$*;*
2) *at each point* $x^\nu \in \operatorname{sing}\partial\mathcal{D}$*, the conormal symbol* (1.5.3) *is invertible for all* $r \in (0,\varepsilon)$*,* ε *being small enough, and* $(\lambda,\varrho) \in \mathbb{R}^2$*, and*

$$\sup_{(0,\varepsilon)\times\mathbb{R}^2} \|\sigma_{\mathfrak{d}_\nu}(\mathcal{A}(\lambda+i\delta))^{-1}(x^\nu;r,\varrho)\| < \infty.$$

Then, there is an $R > 0$ *such that the operator* (1.5.2) *is invertible if* $|\lambda| > R$*, and*

$$\sup_{|\lambda|>R} \|\mathcal{A}^{-1}(\lambda + i\delta)\| < \infty.$$

Proof. Let $\chi_{\nu,\varepsilon}$ be a cut-off function at x^ν, as constructed in Section 1.3, and $\chi_{0,\varepsilon} = 1 - \sum \chi_{\nu,\varepsilon}$. From 1) and the theory of boundary value problems with parameter, cf. [AV64], it follows that, for any $\varepsilon > 0$, there exist $R(\varepsilon) > 0$ and operators

$$\mathcal{B}^{(L)}_{0,\varepsilon}(\lambda + i\delta),$$
$$\mathcal{B}^{(R)}_{0,\varepsilon}(\lambda + i\delta)$$

with the property that

$$\begin{array}{rcl} \mathcal{B}^{(L)}_{0,\varepsilon}(\lambda + i\delta)\mathcal{A}(\lambda + i\delta)\chi_{0,\varepsilon'} & = & \chi_{0,\varepsilon'}, \\ \chi_{0,\varepsilon'}\mathcal{A}(\lambda + i\delta)\mathcal{B}^{(R)}_{0,\varepsilon}(\lambda + i\delta) & = & \chi_{0,\varepsilon'} \end{array} \tag{1.5.4}$$

provided $|\lambda| > R(\varepsilon)$, where $\varepsilon' = \varepsilon'(\varepsilon)$ will be determined later. Note that both $\mathcal{B}^{(L)}_{0,\varepsilon}(\lambda + i\delta)$ and $\mathcal{B}^{(R)}_{0,\varepsilon}(\lambda + i\delta)$ act as

$$\begin{array}{c} H^{s-m}_{\lambda+i\delta,\mathrm{comp}}(\overline{\mathcal{D}} \setminus \operatorname{sing} \partial\mathcal{D}) \\ \oplus \\ \oplus\, H^{s-m_i-\frac{1}{2}}_{\lambda+i\delta,\mathrm{comp}}(\partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}) \end{array} \to H^{s}_{\lambda+i\delta,\mathrm{loc}}(\overline{\mathcal{D}} \setminus \operatorname{sing} \partial\mathcal{D}).$$

On the other hand, Condition 2), if combined with Corollary 4.2.2 of [RST98], implies that, for every point $x^\nu \in \operatorname{sing} \partial\mathcal{D}$, there are $\varepsilon > 0$ and operators

$$\mathcal{B}^{(L)}_{\nu,\varepsilon}(\lambda + i\delta),$$
$$\mathcal{B}^{(R)}_{\nu,\varepsilon}(\lambda + i\delta)$$

such that

$$\begin{array}{rcl} \mathcal{B}^{(L)}_{\nu,\varepsilon}(\lambda + i\delta)\mathcal{A}(\lambda + i\delta)\chi_{\nu,\varepsilon} & = & \chi_{\nu,\varepsilon}, \\ \chi_{\nu,\varepsilon}\mathcal{A}(\lambda + i\delta)\mathcal{B}^{(R)}_{\nu,\varepsilon}(\lambda + i\delta) & = & \chi_{\nu,\varepsilon}. \end{array} \tag{1.5.5}$$

We can assume, by decreasing ε if necessary, that $\mathcal{B}^{(L)}_{\nu,\varepsilon}(\lambda + i\delta)$ and $\mathcal{B}^{(R)}_{\nu,\varepsilon}(\lambda + i\delta)$ act as

$$\begin{array}{c} H^{s-m,\mu_\nu}_{\lambda+i\delta,\mathrm{comp}}(\overline{\mathcal{D}} \cap B(x^\nu,\varepsilon)) \\ \oplus \\ \oplus\, H^{s-m_i-\frac{1}{2},\mu_\nu}_{\lambda+i\delta,\mathrm{comp}}(\partial\mathcal{D} \cap B(x^\nu,\varepsilon)) \end{array} \to H^{s,\mu_\nu}_{\lambda+i\delta,\mathrm{loc}}(\overline{\mathcal{D}} \cap B(x^\nu,\varepsilon)).$$

For every $\nu = 1, \ldots, N$, pick a cut-off function $\tilde{\chi}_{\nu,\varepsilon}$ at x^ν, as above, such that

$$\tilde{\chi}_{\nu,\varepsilon}\chi_{\nu,\varepsilon} = \tilde{\chi}_{\nu,\varepsilon},$$

i.e., $\chi_{\nu,\varepsilon} \equiv 1$ on the support of $\tilde{\chi}_{\nu,\varepsilon}$. Moreover, we can assume, by choosing $\varepsilon' > 0$ small enough, that $\tilde{\chi}_{0,\varepsilon} = 1 - \sum \tilde{\chi}_{\nu,\varepsilon}$ meets the condition

$$\tilde{\chi}_{0,\varepsilon}\chi_{0,\varepsilon'} = \tilde{\chi}_{0,\varepsilon}.$$

Set

$$\mathcal{B}_{\varepsilon}^{(L)}(\lambda + i\delta) = \mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)\tilde{\chi}_{0,\varepsilon} + \sum_{\nu=1}^{N} \mathcal{B}_{\nu,\varepsilon}^{(L)}(\lambda + i\delta)\tilde{\chi}_{\nu,\varepsilon},$$

for $\varepsilon > 0$. We have

$$\begin{aligned}
&\mathcal{B}_{\varepsilon}^{(L)}(\lambda + i\delta)\mathcal{A}(\lambda + i\delta) \\
&= \mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)\tilde{\chi}_{0,\varepsilon}\mathcal{A}(\lambda + i\delta) + \sum_{\nu=1}^{N} \mathcal{B}_{\nu,\varepsilon}^{(L)}(\lambda + i\delta)\tilde{\chi}_{\nu,\varepsilon}\mathcal{A}(\lambda + i\delta) \\
&= \mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)\tilde{\chi}_{0,\varepsilon}\mathcal{A}(\lambda + i\delta)\chi_{0,\varepsilon'} + \sum_{\nu=1}^{N} \mathcal{B}_{\nu,\varepsilon}^{(L)}(\lambda + i\delta)\tilde{\chi}_{\nu,\varepsilon}\mathcal{A}(\lambda + i\delta)\chi_{\nu,\varepsilon} \\
&= \mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)\mathcal{A}(\lambda + i\delta)\tilde{\chi}_{0,\varepsilon} + \sum_{\nu=1}^{N} \mathcal{B}_{\nu,\varepsilon}^{(L)}(\lambda + i\delta)\mathcal{A}(\lambda + i\delta)\tilde{\chi}_{\nu,\varepsilon} + \mathcal{R}_{\varepsilon}^{(L)}(\lambda + i\delta),
\end{aligned}$$

the remainder $\mathcal{R}_{\varepsilon}^{(L)}(\lambda + i\delta)$ being

$$\mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)[\tilde{\chi}_{0,\varepsilon}, \mathcal{A}(\lambda + i\delta)]\chi_{0,\varepsilon'} + \sum_{\nu=1}^{N} \mathcal{B}_{\nu,\varepsilon}^{(L)}(\lambda + i\delta)[\tilde{\chi}_{\nu,\varepsilon}, \mathcal{A}(\lambda + i\delta)]\chi_{\nu,\varepsilon}.$$

It follows that

$$\mathcal{B}_{\varepsilon}^{(L)}(\lambda + i\delta)\mathcal{A}(\lambda + i\delta) = 1 + \mathcal{R}_{\varepsilon}^{(L)}(\lambda + i\delta), \tag{1.5.6}$$

Our next objective is to estimate the norm of the operator $\mathcal{R}_{\varepsilon}^{(L)}(\lambda + i\delta)$ in $\mathcal{L}(H_{\lambda+i\delta}^{s,\mu}(\mathcal{D}))$. To this end, we first invoke the standard theory of [AV64] showing that

$$\lim_{\lambda\to\infty} \|\varphi\mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)[\tilde{\chi}_{0,\varepsilon}, \mathcal{A}(\lambda + i\delta)]\chi_{0,\varepsilon'}\|_{\mathcal{L}(H_{\lambda+i\delta}^{s}(\mathcal{D}))} = 0, \tag{1.5.7}$$

for any fixed $\varepsilon > 0$ and φ, a C^∞ function near $\overline{\mathcal{D}}$ vanishing close to any singular point x^ν. Indeed,

$$\|[\tilde{\chi}_{0,\varepsilon}, \mathcal{A}(\lambda + i\delta)]\chi_{0,\varepsilon'}\| = 0\left(\frac{1}{|\lambda|}\right),$$

the norm being in $\mathcal{L}(H_{\lambda+i\delta}^{s}(\mathcal{D}), H_{\lambda+i\delta}^{s-m}(\mathcal{D}) \oplus (\oplus H_{\lambda+i\delta}^{s-m_i-1/2}(\partial\mathcal{D})))$, while the operator $\varphi\mathcal{B}_{0,\varepsilon}^{(L)}(\lambda + i\delta)$ is bounded in the spaces with parameter, uniformly in

$\lambda + i\delta$. On the other hand, a straightforward verification shows, for every $\nu = 1, \ldots, N$, that

$$\lim_{\varepsilon \to 0} \|\varphi_\nu \mathcal{B}^{(L)}_{\nu,\varepsilon}(\lambda + i\delta)[\tilde{\chi}_{\nu,\varepsilon}, \mathcal{A}(\lambda + i\delta)]\chi_{\nu,\varepsilon}\|_{\mathcal{L}(H^{s,\mu_\nu}_{\lambda+i\delta}(C_{x^\nu}))} = 0 \qquad (1.5.8)$$

uniformly in $\lambda \in \mathbb{R}$, $\varphi_\nu(r)$ being an arbitrary cut-off function at x^ν with a support in $B(x^\nu, \varepsilon)$. Indeed, the norm of the operator $[\tilde{\chi}_{\nu,\varepsilon}, \mathcal{A}(\lambda + i\delta)]\chi_{\nu,\varepsilon}$ acting as $H^{s,\mu_\nu}_{\lambda+i\delta}(C_{x^\nu}) \to H^{s-m,\mu_\nu}_{\lambda+i\delta}(C_{x^\nu}) \oplus (\oplus H^{s-m_i-1/2,\mu_\nu}_{\lambda+i\delta}(\partial C_{x^\nu}))$ is infinitesimal as $\varepsilon \to 0$, uniformly in $\lambda \in \mathbb{R}$, while the operator $\varphi_\nu \mathcal{B}^{(L)}_{\nu,\varepsilon}(\lambda + i\delta)$ is bounded in the opposite direction, uniformly in ε and $\lambda + i\delta$.

The estimates (1.5.7) and (1.5.8) imply that there exist $\varepsilon > 0$ and $R > 0$ such that

$$\|\mathcal{R}^{(L)}_\varepsilon(\lambda + i\delta)\|_{\mathcal{L}(H^{s,\mu}_{\lambda+i\delta}(\mathcal{D}))} \le \frac{1}{2}$$

if $|\lambda| > R$. Hence it follows that the operator $\mathcal{A}(\lambda + i\delta)$ is invertible from the left provided $|\lambda| > R$.

In the same way we prove that $\mathcal{A}(\lambda + i\delta)$ is invertible from the right for $|\lambda|$ large enough. This completes the proof.

□

Chapter 2

Boundary Value Problems in Cylinders with Wedges

2.1 Differential operators with slowly varying coefficients

Suppose $\mathcal{D}$ is a bounded domain in $\mathbb{R}^{n+1}$ with a finite number of cuspidal points $x^1, \ldots, x^N$ on the boundary. We write (t, x) for the coordinates in the infinite cylinder $\mathbb{R} \times \mathcal{D}$ with cross-section $\mathcal{D}$.

For $\nu = 1, \ldots, N$, denote by $E_\nu = \mathbb{R} \times \{x^\nu\}$ the cuspidal edge on the boundary of $\mathbb{R} \times \mathcal{D}$ induced by the singular point x^ν of $\partial\mathcal{D}$. In a cylindrical neighbourhood of E_ν the cylinder $\mathbb{R} \times \mathcal{D}$ coincides with the canonical oscillating wedge $W_\nu = \mathbb{R} \times C_{x^\nu}$, cf. [RST98, 3.1].

An arbitrary differential operator A of order m in $\mathbb{R} \times \mathcal{D}$ can be written in the form

$$A = \sum_{k+|\gamma| \le m} a_{k,\gamma}(t,x) D_t^k D_x^\gamma,$$

the coefficients $a_{k,\gamma}$ being C^∞ functions in the cylinder. Moreover, we put the following restrictions on the coefficients:

1) $a_{k,\gamma}$ are actually C^∞ functions up to the boundary of $\mathbb{R} \times \mathcal{D}$ away from the edges (E_ν);
2) $|D_t^K D_x^G a_{k,\gamma}(t,x)| \le c_{K,G}(a_{k,\gamma}) \left(-\mathfrak{d}'_\nu(|x - x^\nu|)\right)^{|G|}$ close to every edge E_ν, i.e., for all $(t,x) \in \mathbb{R} \times (\mathcal{D} \cap B(x^\nu, \varepsilon))$;
3) $\lim\limits_{x \to x^\nu} \nabla_x a_{k,\gamma}(t,x) \,/\, \mathfrak{d}'_\nu(|x - x^\nu|) = 0$, the limit being achieved uniformly in $t \in \mathbb{R}$;
4) when $t \to \pm\infty$, every $a_{k,\gamma}(t,x)$ has a limit $a_{k,\gamma}(\pm\infty, x)$ satisfying 2) and 3).

Note that Condition 4) is imposed for simplicity. In fact, we only need to require the coefficients $a_{k,\gamma}(t,x)$ to be *slowly varying*, as $t \to +\infty$.

We say that the operator A is *admissible* if all the conditions 1)-4) are fulfilled.

2.2 Function spaces in an infinite cylinder

The cylinder $\mathbb{R} \times \mathcal{D}$ can be thought of as a manifold with two singular points at $t = \pm\infty$. When treating boundary value problems on $\mathbb{R} \times \mathcal{D}$ we will restrict our attention to a neighbourhood of $t = +\infty$. Therefore, the weighted Sobolev spaces on $\mathbb{R} \times \mathcal{D}$ to be introduced are intended for the analysis near $t = +\infty$. For the global analysis, we should glue them together with analogous spaces at $t = -\infty$ in a familiar way.

Given any $s \in \mathbb{Z}_+$ and $\mu, \delta \in \mathbb{R}$, we denote by $H^{s,\mu,\delta}(\mathbb{R} \times C_{x^0})$ the completion of C^∞ functions in a neighbourhood of the closure of $\mathbb{R} \times C_{x^0}$ vanishing for large $|t|$ and close to $\mathbb{R} \times \{x^0\}$, with respect to the norm

$$\|u\|_{H^{s,\mu,\delta}(\mathbb{R}\times C_{x^0})} = \left(\iint_{\mathbb{R}\times\mathbb{R}_+} e^{2\delta t} (\mathfrak{d}')^{2\mu} \sum_{k+j\le s} \|(\frac{1}{\mathfrak{d}'(r)} D_t)^k D_{\mathfrak{d}}^j \pi^* u\|^2_{H^{s-(k+j)}(B)} |\mathfrak{d}'| dt\, dm \right)^{\frac{1}{2}}$$

(cf. (3.3.3) in [RST98]). As usual, we make use of duality and interpolation to extend this scale to all real s.

On the whole cylinder $\mathbb{R} \times \mathcal{D}$, the spaces $H^{s,\mu,\delta}(\mathbb{R} \times \mathcal{D})$ are defined with the help of a suitable partition of unity, namely

$$1_{\mathbb{R}} \otimes \chi_{0,\varepsilon} + \sum_{\nu=1}^{N} 1_{\mathbb{R}} \otimes \chi_{\nu,\varepsilon} = 1$$

where $1_{\mathbb{R}}$ stands for the function on $\mathbb{R}$ identically equal to 1. Thus, the norm in $H^{s,\mu,\delta}(\mathbb{R} \times \mathcal{D})$ is

$$\|u\|_{H^{s,\mu,\delta}(\mathbb{R}\times\mathcal{D})} = \|(1_{\mathbb{R}} \otimes \chi_{0,\varepsilon})u\|_{e^{-\delta t} H^s(\mathbb{R}\times\mathcal{D})} + \sum_{\nu=1}^{N} \|(1_{\mathbb{R}} \otimes \chi_{\nu,\varepsilon})u\|_{H^{s,\mu_\nu,\delta}(W_\nu)}$$

where $\mu = (\mu_1, \ldots, \mu_N)$.

Proposition 2.2.1 *Any admissible differential operator A of order m on $\mathbb{R} \times \mathcal{D}$ extends to a bounded mapping $H^{s,\mu,\delta}(\mathbb{R} \times \mathcal{D}) \to H^{s-m,\mu-m,\delta}(\mathbb{R} \times \mathcal{D})$.*

Proof. This follows from [RST98, 3.3].

□

2.3 Boundary value problems

Consider a boundary value problem in the cylinder $\mathbb{R} \times \mathcal{D}$,

$$\begin{cases} Au = f & \text{in} \quad \mathbb{R} \times \mathcal{D}, \\ B_i u = u_i & \text{on} \quad \mathbb{R} \times (\partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}), \end{cases} \tag{2.3.1}$$

where

$$\begin{aligned} A &= \sum_{k+|\gamma|\le m} a_{k,\gamma}(t,x) D_t^k D_x^\gamma, \\ B_i &= \sum_{k+|\gamma|\le m_i} b_{i,k,\gamma}(t,x) D_t^k D_x^\gamma \end{aligned}$$

are admissible differential operators in $\mathbb{R} \times \mathcal{D}$ and near $\mathbb{R} \times (\partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D})$, respectively.

As usual, we assign to (2.3.1) an operator

$$\mathcal{A} = \begin{pmatrix} A \\ \oplus\, r_{\mathbb{R}\times\partial\mathcal{D}} B_i \end{pmatrix}$$

acting as

$$\mathcal{A}:\ H^{s,\mu,\delta}(\mathbb{R}\times\mathcal{D}) \to \begin{array}{c} H^{s-m,\mu-m,\delta}(\mathbb{R}\times\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i,\delta}(\mathbb{R}\times\partial\mathcal{D}) \end{array}, \tag{2.3.2}$$

where by $r_{\mathbb{R}\times\partial\mathcal{D}}$ is meant the restriction to the smooth part of the boundary of $\mathbb{R} \times \mathcal{D}$, and $s > \max m_i$.

Our concern will be the local invertibility of the operator $\mathcal{A}$ at the point at infinity $t = +\infty$. To this end, we fix a C^∞ function χ on $\mathbb{R}$ satisfying $\chi(t) = 0$, for $t \le 1$, and $\chi(t) = 1$, for $t \ge 2$. Set $\chi_R(t) = \chi(t/R)$, for $R > 0$. We may regard $\chi_R(t)$ as a function on $\mathbb{R} \times \mathcal{D}$ by tensoring it with $1_\mathcal{D}$, the function of $x \in \mathcal{D}$ identically equal to 1.

Let us now introduce an operator $\sigma(\mathcal{A})(+\infty;\zeta)$ depending on a parameter $\zeta = \lambda + i\delta$ varying over Γ_δ,

$$\sigma(\mathcal{A})(+\infty;\lambda+i\delta) = \begin{pmatrix} \displaystyle\sum_{k+|\gamma|\le m} a_{k,\gamma}(+\infty,x)\,(\lambda+i\delta)^k D_x^\gamma \\ \oplus \displaystyle\sum_{k+|\gamma|\le m_i} r_{\partial\mathcal{D}}\, b_{i,k,\gamma}(+\infty,x)\,(\lambda+i\delta)^k D_x^\gamma \end{pmatrix}.$$

By the above, $\sigma(\mathcal{A})$ is said to be the *conormal symbol* of $\mathcal{A}$ at the singular point $t = +\infty$. It is a family of boundary value problems on the domain $\mathcal{D}$ parametrised by ζ,

$$\sigma(\mathcal{A})(+\infty;\lambda+i\delta):\ H^{s,\mu}(\mathcal{D}) \to \begin{array}{c} H^{s-m,\mu-m}(\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i}(\partial\mathcal{D}) \end{array}. \tag{2.3.3}$$

Theorem 2.3.1 *Assume that:*

1) $\sigma(\mathcal{A})(+\infty;\lambda+i\delta)$ *satisfies the conditions of Theorem 1.5.1; and*
2) (2.3.3) *is invertible for all* $\lambda\in\mathbb{R}$.

Then, $\mathcal{A}$ *is locally invertible at* $t=+\infty$, *i.e., there exist an* $R>0$ *and operators* $\mathcal{B}^{(L)}$, $\mathcal{B}^{(R)}$ *such that*

$$\begin{array}{rcl}\mathcal{B}^{(L)}\mathcal{A}\,(\chi_R\otimes 1_{\mathcal{D}}) &=& \chi_R\otimes 1_{\mathcal{D}},\\ (\chi_R\otimes 1_{\mathcal{D}})\,\mathcal{A}\mathcal{B}^{(R)} &=& \chi_R\otimes 1_{\mathcal{D}}.\end{array} \qquad (2.3.4)$$

Proof. Denote by $\mathcal{A}(+\infty,x,D_t,D_x)$ the operator of the boundary value problem in $\mathbb{R}\times\mathcal{D}$ obtained from $\mathcal{A}=\mathcal{A}(t,x,D_t,D_x)$ by freezing the coefficients at $t=+\infty$. Thus, it is defined by the operators

$$\begin{array}{rcl}A(+\infty,x,D_t,D_x) &=& \sum\limits_{k+|\gamma|\le m} a_{k,\gamma}(+\infty,x)D_t^kD_x^\gamma,\\ B_i(+\infty,x,D_t,D_x) &=& \sum\limits_{k+|\gamma|\le m_i} b_{i,k,\gamma}(+\infty,x)D_t^kD_x^\gamma\end{array}$$

in $\mathbb{R}\times\mathcal{D}$.

From Condition 4) of Section 2.1 it follows that

$$\begin{array}{rcl}\lim\limits_{R\to\infty}\|\left(\mathcal{A}(t,x,D_t,D_x)-\mathcal{A}(+\infty,x,D_t,D_x)\right)\chi_R\otimes 1_{\mathcal{D}}\| &=& 0,\\ \lim\limits_{R\to\infty}\|\chi_R\otimes 1_{\mathcal{D}}\left(\mathcal{A}(t,x,D_t,D_x)-\mathcal{A}(+\infty,x,D_t,D_x)\right)\| &=& 0\end{array}$$

where $\|\cdot\|$ stands for the norm of operators acting as is shown in (2.3.2). Hence we deduce easily that the local invertibility at $t=+\infty$ of the operator $\mathcal{A}(t,x,D_t,D_x)$ is equivalent to that of $\mathcal{A}(+\infty,x,D_t,D_x)$. However, the operator $\mathcal{A}(+\infty,x,D_t,D_x)$ is translation invariant with respect to the shifts along the cylinder axis $t\in\mathbb{R}$. Therefore, the boundary value problem $\mathcal{A}(t,x,D_t,D_x)$ is locally invertible at the point $t=+\infty$ if and only if the operator $\mathcal{A}(+\infty,x,D_t,D_x)$ is invertible.

Applying the Fourier transform in $t\in\mathbb{R}$ we reduce $\mathcal{A}(+\infty,x,D_t,D_x)$ to the conormal symbol of $\mathcal{A}$ at $t=+\infty$ acting in spaces with parameter,

$$\sigma(\mathcal{A})(+\infty;\lambda+i\delta):\ H^{s,\mu}_{\lambda+i\delta}(\mathcal{D})\to\begin{array}{c}H^{s-m,\mu-m}_{\lambda+i\delta}(\mathcal{D})\\ \oplus\\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i}_{\lambda+i\delta}(\partial\mathcal{D})\end{array},$$

cf. (2.3.3).

Note that, given any fixed $\lambda+i\delta$, the norms in $H^{s,\mu}_{\lambda+i\delta}(\mathcal{D})$ and $H^{s,\mu}(\mathcal{D})$ are equivalent. Indeed, away from a neighbourhood of singular points this fact is well known, cf. [AV64]. On the other hand, close to any singular point $x^\nu\in\partial\mathcal{D}$ the norms of $H^{s,\mu}_{\lambda+i\delta}(\mathcal{D})$ and $H^{s,\mu}(\mathcal{D})$ differ by an infinitesimal term provided that the support of the cut-off function $\chi_{\nu,\varepsilon}(x)$ shrinks to x^ν, cf. (1.3.1) and (1.4.1). Hence, Condition 2) of the theorem implies the existence of the inverse operator

$$\sigma(\mathcal{A})^{-1}(+\infty;\lambda+i\delta): \begin{array}{c} H^{s-m,\mu-m}_{\lambda+i\delta}(\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i}_{\lambda+i\delta}(\partial\mathcal{D}) \end{array} \to H^{s,\mu}_{\lambda+i\delta}(\mathcal{D}),$$

for each $\lambda \in \mathbb{R}$.

The norm of this operator is estimated uniformly in λ on intervals of finite length in $\mathbb{R}$. Furthermore, Condition 1) yields, by Theorem 1.5.1, the existence of $R > 0$ such that

$$\|\sigma(\mathcal{A})^{-1}(+\infty;\lambda+i\delta)\| \le C$$

for all $\lambda \in \mathbb{R}$ with $|\lambda| > R$, the constant C being independent of λ.

Thus, using the Fourier transform proves the existence of a bounded inverse operator for $\mathcal{A}(+\infty, x, D_t, D_x)$, as desired.

□

Recall, cf. Theorem 3.10.1 in [RST97], that in order that (2.3.3) be Fredholm, for any $\lambda \in \mathbb{R}$, it is necessary and sufficient that, for each $\nu = 1, \ldots, N$, the symbol

$$\sigma_{\mathfrak{d}_\nu}(\sigma(\mathcal{A})(+\infty;\lambda+i\delta))(x^\nu;r,\varrho):\ H^s(B_\nu) \to \begin{array}{c} H^{s-m}(B_\nu) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2}}(\partial B_\nu) \end{array}$$

be invertible for all $(r,\varrho) \in (0,\varepsilon) \times \mathbb{R}$, and the inverse is bounded uniformly in r close to $r = 0$.

2.4 Local principle

Before discussing the Fredholm property of elliptic boundary value problems in domains with corners, we explain the abstract framework of our approach.

In the 1960's Simonenko [Sim65a, Sim65b] proposed the so-called *local principle* for investigating whether an operator of local type is Fredholm. This is in a certain sense analogous to the method of freezing coefficients, well known to specialists in partial differential equations. With the help of the local principle he gave a different interpretation of the theory of multidimensional singular integral equations on smooth compact closed manifolds and treated the case of compact manifolds with a smooth boundary.

In the latter situation the local principle reduces the study of the Fredholm property for a multidimensional singular integral equation to the investigation of invertibility of two types of operators. The first type of operators arising is identical to the operators that appear in the case of compact manifolds without boundary, while the second type is an one-dimensional singular integral operator with a parameter, which can be handled by using the machinery of the Riemann boundary value problem.

This method turned out to be fruitful in the theory of pseudodifferential operators on compact manifolds with a smooth boundary, which was developed thoroughly enough by Vishik and Eskin [VE67], where the concept of a factorisation of an elliptic symbol was introduced.

In the 1970's the local principle received an algebraic interpretation in the book of Gokhberg and Krupnik [GK79], and the construction and study of algebras of multidimensional pseudodifferential operators and Wiener-Hopf operators became more preferable to mathematicians from this time.

A third problem arises in the situation when the manifold is piecewise smooth and has, for example, singular points and edges of different dimensions. The Fredholm property for a multidimensional pseudodifferential operator on a manifold with singularities is equivalent to the invertibility of operators constructed from the singularities according to the local principle.

We recall some concepts associated with the local principle in a form we shall need below.

Let V be a C^∞ "manifold with singularities," e.g., a Thom-Mather stratified variety, cf. [GM88]. The points at infinity of V, if there are any, are also thought of as being singular.

The concept of a C^∞ manifold with singularities starts by describing a model object, M, which bears the information on the singularities involved along with C^∞ structures. As usual, M is embedded to a Euclidean space $\mathbb{R}^N$, and by a C^∞ structure on M is meant that induced from the surrounding space. Thus, we have a collection of C^∞ functions on M, among them the so-called cut-off functions which are identically equal to 1 near a given point or set.

By definition, each point $p \in V$ has a neighbourhood O_p homeomorphic to an open set Ω_p on a model object M_p. Then, we endow V with a C^∞ structure with singularities close to p if we fix an equivalence class of homeomorphisms (h_p) with the property that $g_p \circ h_p^{-1}$ is a C^∞ mapping of Ω_p.

We restrict ourselves to those V which are paracompact and countable at infinity. Moreover, in case V is not compact, we require a finite covering $(O_\nu)_{\nu\in\mathcal{N}}$ of V by "coordinate neighbourhoods." This should correspond to a finite number of different singularities under study.

The model objects M are given in a way which allows one to blow up every M to a direct product $U \times [0,1) \times B$, where U is an open subset of $\mathbb{R}^q$ and B a C^∞ compact manifold with singularities. We call $U \times [0,1) \times B$ the *stretched* model object. Gluing together V close to any singularity with the corresponding stretched model object yields a C^∞ manifold with corners on the boundary, $\mathcal{V}$, that is called the *stretched manifold* associated to V. The analysis on V takes really place in local coordinates of $\mathcal{V}$.

We may also move $\rho = 0$ to the point at infinity thus arriving at a manifold with cylindrical ends.

Typical differential operators on V close to a singularity modelled by M are of the form

$$A = (\mathfrak{D}'(\rho))^m \sum_{|\beta|+k\le m} a_{\beta,k}(y,\rho) \left(\frac{1}{\mathfrak{D}'(\rho)} D_y\right)^\beta \left(\frac{1}{\mathfrak{D}'(\rho)} D_\rho\right)^k \tag{2.4.1}$$

where $a_{\beta,k} \in C^\infty_{\mathrm{loc}}(U \times (0,1), \mathrm{Diff}^{m-(|\beta|+k)}(B))$, and the function $t = \mathfrak{D}(\rho)$ with non-vanishing derivative on $(0,1)$ comes from geometry of the singularity. A natural domain of typical operators is the scale of weighted Sobolev spaces with norms

$$\|u\|_{H^{s,(\gamma,\mu),(\delta,\nu)}(\mathbb{R}^q\times\mathbb{R}_+\times B)}$$
$$= \left(\iint_{\mathbb{R}^q\times\mathbb{R}_+} e^{2\delta\mathfrak{D}}|\mathfrak{D}'|^{2\nu} \sum_{|\beta|+k\le s} \|(\frac{1}{\mathfrak{D}'}D_y)^\beta(\frac{1}{\mathfrak{D}'}D_\rho)^k \pi^* u\|^2_{H^{s-(|\beta|+k),\gamma,\mu}(B)} |\mathfrak{D}'|^{q+1} dy d\rho\right)^{\frac{1}{2}}.$$

If $\mathfrak{D}'(\rho)$ is smooth up to $\rho = 0$ and $\mathfrak{D}'(0) \neq 0$, then we obtain the usual Sobolev spaces and elliptic problems with data at $\rho = 0$ unless the link B itself has singularities. In contrast to this situation, the cuspidal geometry is specified by the observation that the function $t = \mathfrak{D}(\rho)$ tends to $+\infty$ sufficiently fast, when $\rho \to 0$.

Typical differential operators give rise to a local pseudodifferential algebra on V. It is formed by operators of the form $A = \mathrm{op}(a)$, with specifically degenerated symbols $a = a(y,\eta)$ on T^*U taking their values in a lower order algebra $\Psi^m(\mathbb{R}_+\times B; w)$ in the fibres of V over an edge $E \subset V$, U being identified with a coordinate patch along E. The local algebra $\Psi^m(U \times \mathbb{R}_+ \times B; w)$ is invariant under natural diffeomorphisms of V near E which preserve the singular structure of V. Moreover, it is a module over C^∞ functions of compact support.

The structure of the leading symbols of operators (2.4.1) does depend on the nature of the coefficients $a_{\beta,k}(y,\rho)$ and the function $t = \mathfrak{D}(\rho)$. Let us assume that $a_{\beta,k}(y,\rho)$ behave well enough when $\rho \to 0$, in particular, they are continuous up to $\rho = 0$, cf. [RST98]. Over any interior point (y,ρ,θ), the operator A bears *principal interior symbol*

$${}^b\sigma^m(A)(y,\rho,\theta;\tilde\eta,\tilde\tau,\xi) = \sum_{|\beta|+k\le m} {}^b\sigma^{m-(|\beta|+k)}\left(a_{\beta,k}(y,\rho)\right)(\theta,\xi)\ \tilde\eta^\beta\tilde\tau^k,$$

here in the *compressed* form defined actually up to $\rho = 0$. To read off the symbol of A along the edge E, we use the Fourier transform in $y \in U$ and freeze the coefficients $a_{\beta,k}(y,\rho)$ at $\rho=0$. This results in

$$\sigma_{\mathrm{edge}}(A)(y,\eta)=(\mathfrak{D}'(\rho))^m\sum_{|\beta|+k\le m}a_{\beta,k}(y,0)\left(\frac{1}{\mathfrak{D}'(\rho)}\eta\right)^\beta\left(\frac{1}{\mathfrak{D}'(\rho)}D_\rho\right)^k$$

acting as $H^{s,(\gamma,\mu),(\delta,\nu)}(\mathbb{R}_+\times B)\to H^{s-m,(\gamma,\mu),(\delta,\nu-m)}(\mathbb{R}_+\times B)$. Were $t=\mathfrak{D}(\rho)$ "sufficiently good" up to $\rho=0$, we would freeze the factor $\mathfrak{D}'(\rho)$ at $\rho=0$, too, thus obtaining an ordinary differential operator with constant coefficients on $\mathbb{R}_+$, cf. *Lopatinskii condition.* In the general case, we are concerned about the powers of $\mathfrak{D}'(\rho)$. However, for cuspidal singularities we substitute a new symbol for $\sigma_{\mathrm{edge}}(A)$, now acting in Sobolev spaces on the link B. This is the family

$$\sum_{|\beta|+k\le m}a_{\beta,k}(y,0)\,\tilde{\eta}^\beta\tilde{\tau}^k$$

parametrised by $\tilde{\eta}\in\mathbb{R}^q$ and $\tilde{\tau}\in\Gamma_{\gamma_\iota}$. Finally, if $q=0$, then the point $\rho=0$ corresponds to a corner with link B. Such a singular point contributes to the collection of leading symbols by the so-called *conormal symbol*

$$\sigma_{\mathfrak{D}}(A)(0,\zeta)=\sum_{k=0}^m a_k(0)\,\zeta^k$$

acting as $H^{s,\gamma,\mu}(B)\to H^{s-m,\gamma,\mu-m}(B)$. It is parametrised by $\zeta\in\Gamma_\delta$. Let us emphasise that all the leading symbols are compatible in a natural way and behave properly under composition of operators and passing to formal adjoints.

Summarising, we conclude that each point $p\in V$ contributes to the tuple of leading symbols of A, written simply $\sigma(A)(p,q)$. In general, this is an operator family parametrised by q. By the above, $\sigma(A)(p,q)$ is obtained from A by passing to Fourier images and freezing all the coefficients that vary sufficiently slowly in the scale of suitable weighted spaces. If $\sigma(A)(p,q)$ is a family of Fredholm mappings, which is often the case for operators A elliptic in the interior, then a familiar trick is to pose additional conditions along the edges to guarantee the invertibility of $\sigma(A)$.

For a point $p\in V$, if the symbol $\sigma(A)(p,q)$ is invertible in spaces with parameter q, uniformly in q, then the operator $A=\mathrm{op}(\sigma(A))+R$ is locally invertible near $p\in V$, too, because the local norm of $R=\mathrm{op}(a-\sigma(A))$ at p is small. On the other hand, the uniform boundedness of $\sigma(A)^{-1}(p,q)$ in q is a consequence of the mere invertibility of $\sigma(A)(p,q)$, for any fixed q, and the uniform invertibility of the symbol, for $|q|>R$, this latter being a consequence of the ellipticity with parameter q.

We now define a pseudodifferential algebra on the whole manifold V, denoted $\Psi^m(V;w)$, by gluing together the local algebras. To this end, we fix a covering O_p, $p \in V$, of V by coordinate neighbourhoods with local algebras $\Psi^m(O_p; w_p)$. We may actually choose a finite subcovering $O_\iota = O_{p^\iota}$, for $\iota = 1, \dots, J$. Pick a partition of unity (φ_ι) on V, subordinate to this covering, and a system $\psi_\iota \in C^\infty_{\mathrm{comp}}(O_\iota)$ covering (φ_ι) in the sense that $\varphi_\iota \psi_\iota = \varphi_\iota$, for every $\iota = 1, \dots, J$. We introduce $\Psi^m(V; w)$ to consist of all operators of the form

$$A = \sum_\iota \varphi_\iota A_\iota \psi_\iota$$

where $A_\iota \in \Psi^m(O_\iota; w_\iota)$. They act in weighted Sobolev spaces obtained by gluing together local spaces,

$$H^{s,\gamma,\mu}(V) = \sum_\iota \varphi_\iota \, H^{s,\gamma_\iota,\mu_\iota}_{\mathrm{loc}}(O_\iota).$$

Thus, both γ and μ are tuples of real numbers in general, the sum being understood in the sense of non-direct sums of Fréchet spaces.

Recall that an operator $A \in \mathcal{L}(H^{s,\gamma,\mu}(V), H^{s-m,\gamma,\mu-m}(V))$ is said to be of *local type* if, for any two C^∞ functions φ, ψ on V with disjoint supports, the operator $\varphi A \psi$ is compact. Obviously, the operators $A \in \Psi^m(V; w)$ are of local type.

The local principle of [Sim65a, Sim65b] establishes the equivalence of the Fredholm property for an operator $A \in \mathcal{L}(L^2(V))$ of local type and the Fredholm property of all its local representatives A_p, p varying over V. It is clear that there can be as many local representatives as desired, but they are either all Fredholm or all not Fredholm. In the case when these local representatives are homogeneous operators, they are Fredholm if and only if they are invertible.

Our approach is a further development of the local principle. It is based on the following two theorems which hold under much more general assumptions than our setting here. However, we shall not attempt any proof and bring these results merely to highlight our method.

Theorem 2.4.1 *Let $A \in \Psi^m(V; w)$, and $p \in V$. If the symbol $\sigma(A)(p, q)$ is invertible for all q, then A is locally invertible at the point p.*

Proof. Indeed, the local invertibility means the existence of operators $B^{(L)}$ and $B^{(R)}$ such that

$$\begin{aligned} B^{(L)} A \chi &= \chi, \\ \chi A B^{(R)} &= \chi, \end{aligned}$$

for some cut-off function χ at p. To prove this, we choose a family of cut-off functions χ_ε, $\varepsilon > 0$, whose support shrinks to p and whose derivatives behave "well" when $\varepsilon \to 0$. Write B for the operator associated with the inverse symbol $\sigma(A)^{-1}(p,q)$. Then

$$\begin{aligned} BA\,\chi_\varepsilon &= \chi_\varepsilon - (1-BA)\,\chi_\varepsilon \\ &= (1-(1-BA)\chi_{\varepsilon'})\,\chi_\varepsilon, \end{aligned}$$

for all $\varepsilon > 0$, with $\varepsilon' > \varepsilon$ satisfying $\chi_{\varepsilon'}\chi_\varepsilon = \chi_\varepsilon$. If $\varepsilon \to 0$, then the norm of $(1-BA)\chi_{\varepsilon'}$ tends to zero. Hence it follows that the operator $1-(1-BA)\chi_{\varepsilon'}$ is invertible if ε is small enough. Setting $B^{(L)} = (1-(1-BA)\chi_{\varepsilon'})^{-1}B$ yields the desired local left inverse of A. In the same way we prove the existence of a right inverse.

□

It is to be expected that the inverse theorem is also valid, i.e., the local invertibility of an operator A at p just amounts to the invertibility of the symbol of A at p, for all v. Thus, we call an operator A *elliptic* if $\sigma(A)(p,q)$ is invertible at each point $p \in V$, for all v. Our second result reads as follows.

Theorem 2.4.2 *Every elliptic operator A on V induces a Fredholm mapping $H^{s,\gamma,\mu}(V) \to H^{s-m,\gamma,\mu-m}(V)$, for each $s \in \mathbb{R}$.*

Proof. Since the set of singularities $S \subset V$ has zero measure, the local invertibility of A at each point of S actually implies the local invertibility of A in a neighbourhood of S of small measure. This follows by the same method as in the proof of Theorem 2.4.1. Thus, there exist $\varepsilon^0 > 0$ and operators $B^{(L)}_{\varepsilon^0}$ and $B^{(R)}_{\varepsilon^0}$ such that

$$\begin{aligned} B^{(L)}_{\varepsilon^0} A\,\chi_\varepsilon &= \chi_\varepsilon, \\ \chi_\varepsilon\, AB^{(R)}_{\varepsilon^0} &= \chi_\varepsilon \end{aligned}$$

for all $\varepsilon \in (0,\varepsilon^0]$, where χ_ε is a cut-off function in a collar neighbourhood of S.

We now proceed by pasting together this local inverse with a parametrix of A on the smooth part of V. Namely, the usual ellipticity of A away from the set S on V implies the existence of classical pseudodifferential operators $Q^{(L)}_\varepsilon$ and $Q^{(R)}_\varepsilon$ such that

$$\begin{aligned} Q^{(L)}_\varepsilon A\,(1-\chi_\varepsilon) &= (1-\chi_\varepsilon) + K'_\varepsilon, \\ (1-\chi_\varepsilon)\,AQ^{(R)}_\varepsilon &= (1-\chi_\varepsilon) + K''_\varepsilon, \end{aligned}$$

both K'_ε and K''_ε being compact operators. Set

$$R^{(L)}_\varepsilon = B^{(L)}_{\varepsilon^0}\chi_\varepsilon + Q^{(L)}_\varepsilon(1-\chi_\varepsilon),$$

then

$$\begin{aligned} R^{(L)}_\varepsilon A &= B^{(L)}_{\varepsilon^0} A\,\chi_\varepsilon + Q^{(L)}_\varepsilon A\,(1-\chi_\varepsilon) + B^{(L)}_{\varepsilon^0}\,[\chi_\varepsilon, A] + Q^{(L)}_\varepsilon\,[1-\chi_\varepsilon, A] \\ &= 1 + K_\varepsilon + S_\varepsilon \end{aligned}$$

where

$$\begin{aligned} K_\varepsilon &= K'_\varepsilon + Q^{(L)}_\varepsilon [1-\chi_\varepsilon, A], \\ S_\varepsilon &= B^{(L)}_{\varepsilon^0} [\chi_\varepsilon, A]. \end{aligned}$$

It is clear that K_ε is a compact operator, for each $\varepsilon \in (0, \varepsilon^0]$. Furthermore, $1 + S_\varepsilon$ is invertible, for $\varepsilon > 0$ small enough, because

$$\lim_{\varepsilon \to 0} \| [\chi_\varepsilon, A] \|_{\mathcal{L}(H^{s,\gamma,\mu}(V), H^{s-m,\gamma,\mu-m}(V))} = 0.$$

Hence it follows that $(1 + S_\varepsilon)^{-1} R^{(L)}_\varepsilon$ is a left regulariser of A if $\varepsilon > 0$ is sufficiently small.

The same reasoning applies to prove the existence of a right regulariser, which completes the proof.

□

2.5 Fredholm property

We now turn to the boundary value problem (2.3.1) in the infinite cylinder $\mathbb{R} \times \mathcal{D}$.

Following [RST98, (4.2.3)], we introduce a so-called "compressed" symbol of $\mathcal{A}$ along an edge $E_\nu = \mathbb{R} \times \{x^\nu\}$ by

$$\sigma_{E_\nu}(\mathcal{A})(t, r; \lambda, \varrho) = \begin{pmatrix} \sum\limits_{k+j+|\alpha| \le m} a_{k,j,\alpha}(t, r, \theta)\, \lambda^k \varrho^j D^\alpha_\theta \\ \oplus \sum\limits_{|\beta|+j \le m_i} r_{\partial B_\nu} b_{i,k,j,\alpha}(t, r, \theta)\, \lambda^k \varrho^j D^\alpha_\theta \end{pmatrix},$$

for $(t, r; \lambda, \varrho) \in T^*(\mathbb{R} \times (0, \varepsilon))$. It follows that

$$\sigma_{E_\nu}(\mathcal{A})(t, r; \lambda, \varrho) : \ H^s(B_\nu) \to \begin{array}{c} H^{s-m}(B_\nu) \\ \oplus \\ \oplus H^{s-m_i-1/2}(\partial B_\nu) \end{array} \qquad (2.5.1)$$

is a C^∞ function on $T^*(\mathbb{R} \times (0, \varepsilon))$ taking its values in the space of boundary value problems on the link B_ν, cf. (1.5.3).

Definition 2.5.1 *We say that $\mathcal{A}$ meets the ellipticity condition at the point (t, x^ν) along the edge E_ν if the mapping* (2.5.1) *is invertible for all $r > 0$ small enough, and all $(\lambda, \varrho) \in \mathbb{R}^2$, and the inverse is bounded uniformly in r and (λ, ϱ).*

In more detail, the latter condition of Definition 2.5.1 just amounts to saying that

$$\lim_{\varepsilon \to 0} \sup_{\substack{r \in (0,\varepsilon) \\ (\lambda,\varrho) \in \mathbb{R}^2}} \| \sigma_{E_\nu}(\mathcal{A})^{-1}(t, r; \lambda, \varrho) \| < \infty. \qquad (2.5.2)$$

Theorem 2.5.2 *Let the following conditions hold:*

1) *the boundary value problem $\mathcal{A}$ is uniformly elliptic in $(\mathbb{R} \times \overline{\mathcal{D}}) \setminus \cup E_\nu$;*
2) *$\mathcal{A}$ meets the ellipticity condition along every edge E_ν, $\nu = 1, \ldots, N$;*
3) *the conormal symbol of $\mathcal{A}$ at the corners $t = \pm\infty$ is invertible on some weight lines $\Gamma_{\delta\pm}$, cf.* (2.3.3).

Then, $\mathcal{A}$ is a Fredholm operator in the corresponding weighted Sobolev spaces on $\mathbb{R} \times \mathcal{D}$.

Proof. Indeed, Condition 1) provides the local invertibility of the operator $\mathcal{A}$ at each point $(t, x) \in (\mathbb{R} \times \overline{\mathcal{D}}) \setminus \cup E_\nu$. Condition 2) implies, by Corollary 4.2.2 of [RST98], the local invertibility of $\mathcal{A}$ at each singular point of the edges E_ν, $\nu = 1, \ldots, N$. Finally, Condition 3) yields, by Theorem 2.3.1, the local invertibility of $\mathcal{A}$ at the points at infinity $t = \pm\infty$. It remains to glue together the local inverses, thus obtaining a global parametrix of $\mathcal{A}$.

□

Chapter 3

Boundary Value Problems in Domains with Intersecting Edges

3.1 Geometry near a singular point

Consider a point $x^0 \in \mathbb{R}^{n+1}$ belonging to the intersection of one-dimensional cuspidal edges. Thus, x^0 is a cuspidal point on the boundary of some polyhedral domain $\mathcal{D}$ in $\mathbb{R}^{n+1}$, whose link at x^0 is a domain $B \subset \mathbb{R}^n$ with a finite number of cuspidal points.

We will describe the geometry of $\mathcal{D}$ in a polar system of coordinates with centre x^0 in much the same way as in Section 1.1. More precisely, we say that

$$C_{x^0} = \{x^0 + \rho\, S(\Phi(\rho)F(\rho)\theta) : \ \rho \in \mathbb{R}_+,\ \theta \in B\}$$

is a canonical *non-smooth cusp* if both $F(\rho)$ and $\Phi(\rho)$ satisfy the conditions of Section 1.1 and $B \subset\subset \Omega$ bears a finite number of cuspidal points on the boundary, $\operatorname{sing} \partial B = \{\theta^1, \ldots, \theta^N\}$.

By the above, each point θ^ν, $\nu = 1, \ldots, N$, gives rise to an one-dimensional edge E_ν through x^0 and θ^ν on the boundary of $\mathcal{D}$. We assume that the singularities of B meet the assumptions of Section 1.1, hence the theory of [RST98] is applicable locally near smooth points of the edges $\cup E_\nu$, i.e., away from the corners.

Fix any $\rho^0 \in (0, 1]$. Set

$$\mathfrak{D}(\rho) = \int_{\rho^0}^{\rho} \frac{d\vartheta}{\vartheta F(\vartheta)}$$

for $\rho > 0$. Modifying F away from a neighbourhood of $\rho = 0$, if necessary, we may actually assume that $t = \mathfrak{D}(\rho)$ is a diffeomorphism of $\mathbb{R}_+$ onto the entire real axis $\mathbb{R}$, satisfying

$$\begin{array}{rcl} \lim\limits_{\rho\to 0} \mathfrak{D}(\rho) & = & +\infty, \\ \lim\limits_{\rho\to \infty} \mathfrak{D}(\rho) & = & -\infty. \end{array}$$

Changing the variables by $t = \mathfrak{D}(\rho)$ transforms C_{x^0} to the infinite cylinder $\mathbb{R} \times B$ over B, the corner x^0 passing to the point at infinity $t = +\infty$.

3.2 Transformation of differential operators

Let

$$A = \sum_{|\gamma|\le m} a_\gamma(x)\, D_x^\gamma$$

be a differential operator in a canonical *non-smooth cusp* C_{x^0}. In the polar coordinates $\pi(\rho,\theta) = x^0 + \rho\, S(\Phi(\rho)F(\rho)\theta)$ it takes the form

$$\pi^\sharp A = (\mathfrak{D}'(\rho))^m \sum_{k+|\alpha|\le m} a_{k,\alpha}(\rho,\theta) D_{\mathfrak{D}}^k D_\theta^\alpha,$$

cf. (2.4.1). Substituting $t = \mathfrak{D}(\rho)$ in turn yields a differential operator on the cylinder $\mathbb{R} \times B$,

$$\left(\pi \circ \mathfrak{D}^{-1}\right)^\sharp A = \left(\mathfrak{D}'(\mathfrak{D}^{-1}(t))\right)^m \sum_{k+|\alpha|\le m} a_{k,\alpha}(\mathfrak{D}^{-1}(t),\theta) D_t^k D_\theta^\alpha.$$

We say that A is *admissible* if

$$\sum_{k+|\alpha|\le m} a_{k,\alpha}(\mathfrak{D}^{-1}(t),\theta) D_t^k D_\theta^\alpha$$

is an admissible differential operator on the cylinder $\mathbb{R} \times B$ in the sense of Section 2.1.

On the other hand, the factor $(\mathfrak{D}'(\mathfrak{D}^{-1}(t)))^m$ will be handled through suitable weighted Sobolev spaces.

3.3 Function spaces in a canonical corner

Recall that the scale $H^{s,\mu,\delta}(\mathbb{R} \times B)$ of weighted Sobolev spaces over the infinite cylinder is introduced in Section 2.2. Here, $\mu = (\mu_1,\dots,\mu_N)$ is a multi-index of weights corresponding to singular points $\theta^\nu \in \partial B$, respectively, while $\delta \in \mathbb{R}$ corresponds to x^0.

Now, the space $H^{s,\mu,(\delta,\nu)}(C_{x^0})$ is defined to consists of all distributions u on C_{x^0}, such that

$$\left(\mathfrak{D}'(\mathfrak{D}^{-1}(t))\right)^\nu \left(\pi \circ \mathfrak{D}^{-1}\right)^* u \in H^{s,\mu,\delta}(\mathbb{R} \times B),$$

where $\nu \in \mathbb{R}$.

For $s > 1/2$, we denote $H^{s-\frac{1}{2},\mu,(\delta,\nu)}(\partial C_{x^0})$ the space of traces of all functions $u \in H^{s,\mu,(\delta,\nu)}(C_{x^0})$ on the smooth part of ∂C_{x^0}, endowed with the quotient topology.

3.4 Local invertibility of boundary value problems

Consider a boundary value problem in a canonical non-smooth cusp C_{x^0},

$$\begin{cases} Au = f & \text{in} \quad C_{x^0}, \\ B_i u = u_i & \text{on} \quad \partial C_{x^0} \setminus \cup E_\nu, \end{cases} \tag{3.4.1}$$

where A and (B_i) are admissible differential operators in C_{x^0} and near ∂C_{x^0}, respectively.

Let us assign to (3.4.1) an operator

$$\mathcal{A} = \begin{pmatrix} A \\ \oplus\, r_{\partial C_{x^0}} B_i \end{pmatrix}$$

acting as

$$\mathcal{A}: \ H^{s,\mu,(\delta,\nu)}(C_{x^0}) \to \begin{array}{c} H^{s-m,\mu-m,(\delta,\nu-m)}(C_{x^0}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i,(\delta,\nu-m_i)}(\partial C_{x^0}) \end{array}, \tag{3.4.2}$$

where $s > \max m_i$.

As is explained in Section 2.4, the local invertibility of the operator (3.4.2) at x^0 is controlled by its conormal symbol,

$$\sigma_{\mathfrak{D}}(\mathcal{A})(x^0;\lambda + i\delta) = \begin{pmatrix} \sum\limits_{k+|\alpha|\le m} a_{k,\alpha}(0,\theta)\,(\lambda+i\delta)^k D_\theta^\alpha \\ \oplus \sum\limits_{k+|\alpha|\le m_i} r_{\partial B}\, b_{i,k,\alpha}(0,\theta)\,(\lambda+i\delta)^k D_\theta^\alpha \end{pmatrix}.$$

This latter is a boundary value problem with parameter $\lambda + i\delta \in \Gamma_\delta$ on the link of C_{x^0} at x^0, namely

$$\sigma_{\mathfrak{D}}(\mathcal{A})(x^0;\lambda+i\delta): \ H^{s,\mu}(B) \to \begin{array}{c} H^{s-m,\mu-m}(B) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i}(\partial B) \end{array}. \tag{3.4.3}$$

Theorem 3.4.1 *If*

1) $\sigma(\mathcal{A})(x^0;\lambda+i\delta)$ *fulfills the conditions of Theorem 1.5.1, and*
2) (3.4.3) *is invertible for all* $\lambda \in \mathbb{R}$,

then $\mathcal{A}$ *is locally invertible at the corner* x^0.

Proof. This is a straightforward consequence of Theorem 2.3.1.

□

3.5 Fredholm property

In this last section we briefly sketch the Fredholm property of boundary value problems in a domain $\mathcal{D} \subset\subset \mathbb{R}^{n+1}$ with a finite number of cuspidal edges on the boundary, intersecting at possibly zero angles. We continue to write E_ν, $\nu = 1, \ldots, N$, for the edges, and we assume that they meet each other at a finite number of points, called corners.

In the domain $\mathcal{D}$, we consider a boundary value problem with data on the smooth part of $\partial\mathcal{D}$, namely

$$\begin{cases} Au = f & \text{in} \quad \mathcal{D}, \\ B_i u = u_i & \text{on} \quad \partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}, \end{cases} \tag{3.5.1}$$

A being an admissible differential operator in $\mathcal{D}$ and (B_i) a system of admissible differential operators near $\partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}$. Let m stand for the order of A and m_i for that of B_i.

We now apply the standard reasoning, cf. Section 2.4, to introduce weighted Sobolev spaces $H^{s,\mu,(\delta,\nu)}(\mathcal{D})$ in $\mathcal{D}$ as well as the corresponding spaces on the boundary $\partial\mathcal{D}$.

We specify (3.5.1) as an operator

$$\mathcal{A} = \begin{pmatrix} A \\ \oplus\, r_{\partial\mathcal{D}} B_i \end{pmatrix}$$

acting as

$$\mathcal{A}: \ H^{s,\mu,(\delta,\nu)}(\mathcal{D}) \to \begin{array}{c} H^{s-m,\mu-m,(\delta,\nu-m)}(\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\mu-m_i,(\delta,\nu-m_i)}(\partial\mathcal{D}) \end{array}, \tag{3.5.2}$$

where s is any integer with $s > \max m_i$.

Theorem 3.5.1 *Let the following conditions be fulfilled:*

1) *the boundary value problem $\mathcal{A}$ is uniformly elliptic in $\overline{\mathcal{D}} \setminus \cup E_\nu$;*
2) *$\mathcal{A}$ meets the ellipticity condition along every edge E_ν, $\nu = 1, \ldots, N$;*
3) *the conormal symbol of $\mathcal{A}$ at each corner is invertible, as is required in Theorem 3.4.1.*

Then, (3.5.2) *is a Fredholm operator.*

Proof. This follows from Corollary 4.2.2 of [RST98] and Theorem 3.4.1 by the same method as in the proof of Theorem 2.4.2.

□

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(V. Rabinovich) Departamento de Telecomunicaciones, ESIME del I.P.N., Unidad Zacatenco, Av. I.P.N., s/n Ed. 1, 07738 México, D.F., México

E-mail address: rabinov@maya.esimez.ipn.mx

(B.-W. Schulze) Universität Potsdam, Institut für Mathematik, Postfach 60 15 53, 14415 Potsdam, Germany

E-mail address: schulze@math.uni-potsdam.de

(N. Tarkhanov) Institute of Physics, Russian Academy of Sciences, Akademgorodok, 660036 Krasnoyarsk, Russia

E-mail address: tarkhan@math.uni-potsdam.de