A Bohr Phenomenon for Elliptic Equations

Lev Aizenberg
Department of Mathematics and Computer Science
Bar-Ilan University
52900 Ramat-Gan
Israel

Nikolai Tarkhanov
Institut für Mathematik
Universität Potsdam
Postfach 60 15 53
14415 Potsdam
Germany

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Abstract

In 1914 Bohr proved that there is an $r \in (0, 1)$ such that if a power series converges in the unit disk and its sum has modulus less than 1 then, for $|z| < r$, the sum of absolute values of its terms is again less than 1. Recently analogous results were obtained for functions of several variables. The aim of this paper is to comprehend the theorem of Bohr in the context of solutions to second order elliptic equations meeting the maximum principle.

Contents

1 Preliminaries **4**

2 Spaces with reproducing kernels **6**

3 Harmonic functions **9**

4 Separately harmonic functions **14**

5 Pluriharmonic functions **17**

6 Polyharmonic functions **18**

7 Solutions of elliptic equations **18**

References **23**

1 Preliminaries

It was in the spirit of function theory of the beginning of this century that Bohr [Boh14] published the following theorem.

Theorem 1.1 *There exists $r \in (0,1)$ with the property that if a power series $\sum_{\nu=0}^{\infty} c_\nu z^\nu$ converges in the unit disk and its sum has modulus less than 1, then $\sum_{\nu=0}^{\infty} |c_\nu z^\nu| < 1$ for all $|z| < r$.*

We don't know any motivation of Bohr's result but very classical subjects and coefficient estimates much more precise than the Cauchy inequalities. Perhaps, this called on such mathematicians as M. Riesz, I. Shur and F. Wiener who put Theorem 1.1 in final form by showing that one can take $r = 1/3$ and this constant cannot be improved.

We call the best constant r in Theorem 1.1 (i.e., 1/3) the *Bohr radius.* If regarded as a homothety coefficient, this concept extends easily to domains in $\mathbb{C}^n$. Let us review some of the recent generalisations of Bohr's theorem to functions of several complex variables.

Given a complete Reinhardt domain $\mathcal{D}$ in $\mathbb{C}^n$, we denote by $R(\mathcal{D})$ the largest non-negative number r with the property that if a power series $\sum_\alpha c_\alpha z^\alpha$ converges in $\mathcal{D}$ and its sum has modulus less than 1, then $\sum_\alpha |c_\alpha z^\alpha| < 1$ in the homothety $r\mathcal{D}$. Here, the sums are over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, $z = (z_1, \ldots, z_n)$ is the tuple of complex variables, and $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. In [BKh97] the following result is proved in case $\mathcal{D}$ is the unit polydisk

$$U^n = \{z \in \mathbb{C}^n : \ |z_j| < 1, \ j = 1, \ldots, n\}.$$

Theorem 1.2 *For $n > 1$, one has*

$$\frac{1}{3\sqrt{n}} < R(U^n) < \frac{2\sqrt{\log n}}{\sqrt{n}}.$$

We see from Theorem 1.2 that $R(U^n) \to 0$ when $n \to \infty$. If $\mathcal{D}$ is the hypercone $\mathcal{C} = \{z \in \mathbb{C}^n : \ |z_1| + \ldots + |z_n| < 1\}$, the situation is quite different, cf. [Aiz00].

Theorem 1.3 *The following estimates are true:*

$$\frac{1}{3\sqrt[3]{e}} < R(\mathcal{C}) \le \frac{1}{3}.$$

Moreover, if $z^0 \notin (1/3)\mathcal{C}$, then there exists a series $\sum_\alpha c_\alpha z^\alpha$ converging in $\mathcal{C}$ and such that $|\sum_\alpha c_\alpha z^\alpha| < 1$ is valid there, but $\sum_\alpha |c_\alpha z^\alpha| < 1$ fails at the point z^0.

For further estimates of $R(\mathcal{D})$ in domains $\{z \in \mathbb{C}^n : |z_1|^p + \ldots + |z_n|^p < 1\}$, $1 \leq p \leq \infty$, we refer the reader to [Boa98], for other generalisations of Bohr's theorem cf. [DR99].

We now discuss yet another natural multidimensional analogue of Bohr's theorem. Let $r(\mathcal{D})$ stand for the largest $r \geq 0$ such that if a power series $\sum_\alpha c_\alpha z^\alpha$ converges in $\mathcal{D}$ and its sum has modulus less than 1, then $\sum_\alpha \sup_{r\mathcal{D}} |c_\alpha z^\alpha| < 1$, $r\mathcal{D}$ being the homothety of $\mathcal{D}$. The following result is contained in [Aiz00].

Theorem 1.4 *For any bounded complete Reinhardt domain $\mathcal{D}$ in $\mathbb{C}^n$, the inequality holds*

$$1 - \sqrt[n]{\frac{2}{3}} < r(\mathcal{D}).$$

It is worth pointing out that this constant is near to the best one for the hypercone $\mathcal{C}$.

For holomorphic functions with positive real part, there is another result equivalent to Bohr's theorem, cf. [AAD98b].

Theorem 1.5 *If a function $f(z) = \sum_{\nu=0}^\infty c_\nu z^\nu$ has positive real part in the unit disk and $f(0) > 0$, then $\sum_{\nu=0}^\infty |c_\nu z^\nu| < 2f(0)$ for all $|z| < 1/3$ and the constant $1/3$ cannot be improved.*

The coincidence of the constants in Theorems 1.1 and 1.5 is not accidental, as is shown in [AAD98b]. These constants also coincide for holomorphic functions on complex manifolds.

In [AAD98a] the existence of a Bohr phenomenon is proved in $\mathrm{Hol}(M)$, the space of holomorphic functions on a complex manifold M.

Theorem 1.6 *If $(f_\nu)_{\nu=0,1,\ldots}$ is a basis in $\mathrm{Hol}(M)$ satisfying*

1) $f_0 \equiv 1$;
2) *all the functions f_ν, $\nu = 1, 2, \ldots$, vanish at a point $z^0 \in M$,*

then there exist a neighbourhood U of z^0 and a compact set $K \subset M$, such that, for all $f \in \mathrm{Hol}(M)$ with $f = \sum_{\nu=0}^\infty c_\nu f_\nu$,

$$\sum_{\nu=0}^{\infty} \sup_U |c_\nu f_\nu| \leq \sup_K |f|.$$

In other words, in this case we meet a Bohr phenomenon in M with Bohr radius $r(M)$, the more so with Bohr radius $R(M)$.

Though the proofs make essential use of the algebra structure of holomorphic functions, the underlying fact seems to be nothing but the maximum principle. More precisely, we need a substitute of Harnack's inequality, e.g. [Eva98, 6.4.3], which reads that, given a continuous function f in a domain $\mathcal{D} \subset \mathbb{R}^n$ with values in $\mathbb{R}^k$, such that $f(x^0) = 0$ for some $x^0 \in \mathcal{D}$, the supremum of $|f|$ over any set $\sigma \subset\subset \mathcal{D}$ is dominated by the supremum of f itself over $\mathcal{D}$. This amounts to saying that for every set $\sigma \subset\subset \mathcal{D}$ there is a constant $c > 0$ such that

$$\sup_{\sigma} |f - f(x^0)| \leq c \sup_{\mathcal{D}} \left(f - f(x^0) \right), \tag{1.1}$$

the inequality being uniform in f belonging to an appropriate function class and $x^0 \in \sigma$.

In this paper we show that, for bases in nuclear function spaces, the estimate (1.1) always implies a Bohr phenomenon.

The paper is organised as follows. In Section 2 we briefly discuss Bohr's phenomenon in Hilbert spaces with reproducing kernels, the functions obeying (1.1).

Many function classes of mathematical physics meet (1.1), among them are harmonic, separately harmonic and pluriharmonic functions. In Sections 3-5 we will be concerned with evaluating the Bohr radii for these classes of functions. The important point to note here is explicit formulas for the Bohr radii for harmonic and separately harmonic functions.

Polyharmonic functions fail to fulfil the maximum principle; in Section 6 it is shown that no Bohr phenomenon exists for positive polyharmonic functions.

Finally, in Section 7 we prove that, for second order elliptic equations, the estimate (1.1) reduces to classical Harnack's inequality, thus implying a Bohr phenomenon.

2 Spaces with reproducing kernels

It seems that an effective way of proving Bohr's phenomenon in concrete cases is trough establishing some coefficient estimates for the relevant classes of functions. In this section we show coefficient estimates for expansions in Hilbert spaces with reproducing kernels.

Let $\mathcal{D}$ be a relatively compact domain with C^∞ boundary in $\mathbb{R}^n$, and A a hypoelliptic operator in $\mathcal{D}$. This means that any distribution f in $\mathcal{D}$ satisfying $Af = 0$ is actually a C^∞ function.

We assume moreover that for each compact set $K \subset \mathcal{D}$ there is a constant $c > 0$ such that

$$\sup_{K} |f| \leq c \, \|f\|_{L^2(\partial\mathcal{D})} \tag{2.1}$$

for all $f \in C^\infty(\bar{\mathcal{D}})$ satisfying $Af = 0$ in $\mathcal{D}$.

Denote by H the Hilbert space obtained by completing the space of all $f \in C^\infty(\bar{\mathcal{D}})$ satisfying $Af = 0$ in $\mathcal{D}$, with respect to the norm $f \mapsto \|f\|_{L^2(\partial\mathcal{D})}$.

In this way we obtain what is known as a Hardy space of solutions to $Af = 0$ in $\mathcal{D}$.

From (2.1) we deduce that H is a Hilbert space with reproducing kernel. Namely, fix an orthonormal basis

$$(f_\alpha)_{\alpha \in Z_+^n}$$

in H, all the elements being smooth up to the boundary. Then, given any $x \in \mathcal{D}$, the series

$$K(x,y) = \sum_{\alpha \in \mathbb{Z}_+^n} f_\alpha(x) \otimes f_\alpha^*(y)$$

converges in the norm of $L^2(\partial\mathcal{D})$, where $f^*(y) = \overline{f(y)}$. Moreover, it converges uniformly in x on compact subsets of $\mathcal{D}$, so that $K(x,y)$ is actually a C^∞ function in the product $\mathcal{D} \times \mathcal{D}$. This function is called the *Szegö kernel* of $\mathcal{D}$ relative to A.

We now suppose that the basis $(f_\alpha)_{\alpha \in Z_+^n}$ has the following natural properties:

1) f_0 is a constant;
2) $|f_\alpha(x)| \le C\, a^{|\alpha|} |x - x^0|^{|\alpha|}$ for all $x \in \mathcal{D}$ and $\alpha \in \mathbb{Z}_+^n$, the constant C being independent of x and α.

Lemma 2.1 *If $f = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha f_\alpha$ is a solution to $Af = 0$ in $\mathcal{D}$ satisfying $f \le B$ on the boundary, then*

$$|c_\alpha| \le s(\partial\mathcal{D}) \sup_{\mathcal{D}} |f_\alpha| \left(B - f(x^0)\right) \tag{2.2}$$

for all $\alpha \ne 0$, where $s(\partial\mathcal{D})$ is the surface area of $\partial\mathcal{D}$.

Proof. Pick $\alpha \in \mathbb{Z}_+^n$ different from zero. Letting ds denote the area element of $\partial\mathcal{D}$, we have

$$c_\alpha = \int_{\partial\mathcal{D}} f_\alpha^* f \, ds.$$

Since $(f_\alpha)_{\alpha \in Z_+^n}$ is an orthogonal system, we get

$$\int_{\partial\mathcal{D}} f_\alpha^* \, ds = 0$$

whence

$$\begin{aligned} |c_\alpha| &= \left| \int_{\partial\mathcal{D}} f_\alpha^* (B - f) \, ds \right| \\ &\le \sup_{\partial\mathcal{D}} |f_\alpha| \int_{\partial\mathcal{D}} (B - f) \, ds \\ &= \sup_{\partial\mathcal{D}} |f_\alpha| \left(s(\partial\mathcal{D}) B - \frac{1}{f_0} c_0 \right), \end{aligned}$$

the second inequality being due to the fact that $B - f \geq 0$ on $\partial\mathcal{D}$. Taking into account that

$$\begin{aligned} \frac{1}{f_0} c_0 &= \frac{1}{f_0^2} f(x^0) \\ &= s(\partial\mathcal{D})\, f(x^0), \end{aligned}$$

we arrived at the desired inequality.

□

Coefficient estimates of the type (2.2) are of independent interest, especially for the expansion of solutions to homogeneous elliptic equations with constant coefficients, cf. [Tar97, 2.3].

Theorem 2.2 *There is an* $r > 0$ *with the property that if* $f = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha f_\alpha$ *fulfills* $|f| < 1$ *in* $\mathcal{D}$ *then* $\sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha f_\alpha| < 1$ *in the ball* $B(x^0, r)$.

Proof. Indeed, we can assume without loss of generality that $f(x^0) > 0$, for if not we replace f by $-f$. Then,

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha f_\alpha(x)| &= f(x^0) + \sum_{\alpha \neq 0} |c_\alpha|\, |f_\alpha(x)| \\ &\leq f(x^0) + s(\partial\mathcal{D}) \left(\sum_{\alpha \neq 0} |f_\alpha(x)| \sup_{\mathcal{D}} |f_\alpha| \right) \left(1 - f(x^0)\right), \end{aligned}$$

the last inequality being a consequence of Lemma 2.1. We now invoke Property 2) of the basis $(f_\alpha)_{\alpha \in Z_+^n}$ to obtain

$$\begin{aligned} \sum_{\alpha \neq 0} |f_\alpha(x)| \sup_{\mathcal{D}} |f_\alpha| &\leq C^2 \sum_{\alpha \neq 0} a^{2|\alpha|} \left(\operatorname{dist}(x^0, \partial\mathcal{D})\right)^{|\alpha|} |x - x^0|^{|\alpha|} \\ &= C^2 \left(\sum_{\alpha \in \mathbb{Z}_+^n} \left(a^2 \operatorname{dist}(x^0, \partial\mathcal{D})\, |x - x^0|\right)^{|\alpha|} - 1 \right) \\ &= C^2 \left(\left(\frac{1}{1 - a^2 \operatorname{dist}(x^0, \partial\mathcal{D})\, |x - x^0|} \right)^n - 1 \right). \end{aligned}$$

If $x \to x^0$ then the right-hand side tends monotonically to zero. Hence there is an $r > 0$ such that it is less than $1/s(\partial\mathcal{D})$ when $|x - x^0| < r$. In fact, we have

$$r = \frac{1}{a^2 \operatorname{dist}(x^0, \partial\mathcal{D})} \left(1 - \frac{1}{\sqrt[n]{1 + \frac{1}{C^2 s(\partial\mathcal{D})}}} \right).$$

Thus, for $x \in B(x^0, r)$, we get

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha f_\alpha(x)| &< f(x^0) + (1 - f(x^0)) \\ &= 1, \end{aligned}$$

as desired.

□

Loosely speaking, the theorem says that Bohr's phenomenon extends to bases in Hardy spaces of solutions to elliptic equations of order ≤ 2. The question arises on evaluating the Bohr radius for some concrete bases. We discuss this problem in the next sections.

3 Harmonic functions

To justify our formulation of Bohr's phenomenon for harmonic functions we need an analogue of Theorem 1.5. In the context of real-valued functions it is fairly simple.

Let Σ be a vector space of bounded real-valued functions in a domain $\mathcal{D} \subset \mathbb{R}^n$ satisfying some relations there, and let $1 \in \Sigma$. Suppose $(f_\nu)_{\nu=0,1,\ldots}$ is a basis in Σ, such that $f_0 \equiv 1$ and all the functions f_ν, $\nu = 1, 2, \ldots$, vanish at a point $x^0 \in \mathcal{D}$.

Lemma 3.1 *Given any neighbourhood $U \subset \mathcal{D}$ of x^0, the following assertions are equivalent:*

1) *If $f = \sum_{\nu=0}^\infty c_\nu f_\nu$ and $|f| < 1$ in $\mathcal{D}$, then $\sum_{\nu=0}^\infty |c_\nu f_\nu| < 1$ in U.*
2) *If $f = \sum_{\nu=0}^\infty c_\nu f_\nu$ and $f > 0$ in $\mathcal{D}$, then $\sum_{\nu=0}^\infty |c_\nu f_\nu| < 2f(x^0)$ in U.*

Proof. Suppose 1) holds, and $f = \sum_{\nu=0}^\infty c_\nu f_\nu$ is a positive bounded function in $\mathcal{D}$. Choose a constant $k > 0$ with the property that $0 < kf < 2$ in $\mathcal{D}$, and set $F = 1 - kf$. Then $F \in \Sigma$ satisfies $|F| < 1$ in $\mathcal{D}$. Since

$$F = (1 - kc_0) - \sum_{\nu=1}^\infty kc_\nu f_\nu,$$

it follows by 1) that

$$|1 - kc_0| + \sum_{\nu=1}^\infty |kc_\nu f_\nu| < 1$$

in U. Hence

$$\begin{aligned}\sum_{\nu=0}^{\infty}|c_\nu f_\nu| &= \frac{1}{k}\left((|1-kc_0|+\sum_{\nu=1}^{\infty}|kc_\nu f_\nu|)+(|kc_0|-|1-kc_0|)\right)\\ &< \frac{1}{k}(1+kc_0-|1-kc_0|)\\ &\leq 2c_0\end{aligned}$$

in U, as desired.

Conversely, let 2) be valid, and let $f=\sum_{\nu=0}^{\infty}c_\nu f_\nu$ satisfy $|f|<1$ in $\mathcal{D}$. We can certainly assume that $c_0 \geq 0$, for if not, we replace f by $-f$. Set $F=1-f$, then $F\in\Sigma$ fulfills $0<F<2$ in $\mathcal{D}$. As

$$F=(1-c_0)-\sum_{\nu=1}^{\infty}c_\nu f_\nu,$$

we deduce by 2) that

$$|1-c_0|+\sum_{\nu=1}^{\infty}|c_\nu f_\nu|<2(1-c_0)$$

in U. Hence it follows that

$$\begin{aligned}\sum_{\nu=0}^{\infty}|c_\nu f_\nu| &= (|1-c_0|+\sum_{\nu=1}^{\infty}|c_\nu f_\nu|)+(|c_0|-|1-c_0|)\\ &< 2(1-c_0)+(c_0-(1-c_0))\\ &= 1\end{aligned}$$

in U, which completes the proof.

□

Let $B=\{x\in\mathbb{R}^n:\ |x|<1\}$ be the unit ball in $\mathbb{R}^n$. On the sphere ∂B we have an orthonormal basis of spherical harmonics $(Y_{k,j}(\xi))$, where $|\xi|=1$, the degree of $Y_{k,j}$ is equal to k, and, given any $k\in\mathbb{Z}_+$, the index j varies from 1 to $\sigma(n,k)$ (cf. [SW71]). For example,

$$\sigma(n,k)=\binom{n+k-1}{k}-\binom{n+k-3}{k-2}$$

if $k\geq 2$.

Any spherical harmonics $Y_{k,j}(\xi)$ is the restriction to ∂B of a homogeneous harmonic polynomial $Y_{k,j}(x)$ in $\mathbb{R}^n$,

$$Y_{k,j}(x)=|x|^k\,Y_{k,j}\left(\frac{x}{|x|}\right).$$

These polynomials form a basis in $\mathrm{Har}(B)$, the space of harmonic functions in B. Each function $f \in \mathrm{Har}(B)$ expands as a series

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} c_{k,j}\, Y_{k,j}(x) \tag{3.1}$$

in B which converges uniformly on compact subsets of B.

By a *Bohr radius* for harmonic functions in B we will mean the largest number $r \geq 0$ with the property that if $f \in \mathrm{Har}(B)$ expands as (3.1) and $|f| < 1$ in B, then

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} |c_{k,j}\, Y_{k,j}(x)| < 1$$

in the ball $B_r = \{x \in \mathbb{R}^n : \ |x| < r\}$. We denote this radius by $R_{\mathrm{Har}}(B)$.

Theorem 3.2 *As defined above, $R_{\mathrm{Har}}(B)$ is equal to the root of the equation*

$$\frac{1+r}{(1-r)^{n-1}} = 2, \tag{3.2}$$

lying in the interval $(0,1)$.

Proof. According to Lemma 3.1, $R_{\mathrm{Har}}(B)$ is equal to the largest number $r \geq 0$ such that if f is a positive bounded harmonic function in B with expansion (3.1), then

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} |c_{k,j}\, Y_{k,j}(x)| < 2\, c_{0,1}\, Y_{0,1} \tag{3.3}$$

for all $x \in B_r$.

Note that

$$\int_{\partial B} ds = \omega_{n-1},$$

where ds is the area form of ∂B and ω_{n-1} the area of the unit sphere in $\mathbb{R}^n$. Hence

$$Y_{0,1}(\xi) = \frac{1}{\sqrt{\omega_{n-1}}},$$

and so (3.1) implies

$$f(0) = \frac{c_{0,1}}{\sqrt{\omega_{n-1}}}.$$

Further, we have

$$c_{k,j} = \lim_{r \to 1-0} \int_{\partial B_r} f(x)\, Y_{k,j}(x)\, ds_r$$

where ds_r is the area element of ∂B_r. Using the mean value theorem for harmonic functions we get

$$
\begin{aligned}
\sum_{j=1}^{\sigma(n,k)} |c_{k,j}|^2 &= \sum_{j=1}^{\sigma(n,k)} \lim_{r\to 1-0} \left(\int_{\partial B_r} f(x)\, Y_{k,j}(x)\, ds_r \right)^2 \\
&\leq \sum_{j=1}^{\sigma(n,k)} \lim_{r\to 1-0} \left(\int_{\partial B_r} f(x)\, ds_r \int_{\partial B_r} f(x)\, (Y_{k,j}(x))^2\, ds_r \right) \\
&= \lim_{r\to 1-0} \int_{\partial B_r} f(x)\, ds_r \ \lim_{r\to 1-0} \int_{\partial B_r} f(x) \sum_{j=1}^{\sigma(n,k)} |Y_{k,j}(x)|^2\, ds_r \\
&= \lim_{r\to 1-0} \int_{\partial B_r} f(x)\, ds_r \ \lim_{r\to 1-0} \int_{\partial B_r} f(x)\, r^k Z_k\, ds_r \\
&= Z_k \left(\lim_{r\to 1-0} \int_{\partial B_r} f(x)\, ds_r \right)^2 \\
&= Z_k\, c_{0,1}\, \omega_{n-1},
\end{aligned}
$$

Z_k being surface zonal harmonics, cf. [SW71].

For further estimates we need an expansion of the Poisson kernel $\wp(x,y)$ by the zonal harmonics

$$
Z_k(\xi,\eta) = \sum_{j=1}^{\sigma(n,k)} Y_{k,j}(\xi)\, Y_{k,j}(\eta)
$$

where $|\xi| = 1$ and $|\eta| = 1$. In this notation, we have $Z_k = Z_k(\xi,\xi)$ for all $k \in \mathbb{Z}_+$. Moreover,

$$
\begin{aligned}
\wp(r\xi,\eta) &= \frac{1}{\omega_{n-1}} \frac{1-r^2}{|r\xi-\eta|^n} \\
&= \sum_{k=0}^{\infty} r^k\, Z_k(\xi,\eta) \\
&= \sum_{k=0}^{\infty} r^k\, Z_k(\eta,\xi),
\end{aligned}
$$

cf. [SW71]. Thus,

$$
\sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} |c_{k,j}\, Y_{k,j}(r\xi)| = \frac{c_{0,1}}{\sqrt{\omega_{n-1}}} + \sum_{k=1}^{\infty} r^k \sum_{j=1}^{\sigma(n,k)} |c_{k,j}|\, |Y_{k,j}(\xi)|
$$

$$
\begin{aligned}
&\leq \frac{c_{0,1}}{\sqrt{\omega_{n-1}}} + \sum_{k=1}^{\infty} r^k \sqrt{\sum_{j=1}^{\sigma(n,k)} |c_{k,j}|^2} \sqrt{\sum_{j=1}^{\sigma(n,k)} |Y_{k,j}(\xi)|^2} \\
&\leq \frac{c_{0,1}}{\sqrt{\omega_{n-1}}} + c_{0,1}\sqrt{\omega_{n-1}} \sum_{k=1}^{\infty} r^k Z_k \\
&= \frac{c_{0,1}}{\sqrt{\omega_{n-1}}} + c_{0,1}\sqrt{\omega_{n-1}} \left(\frac{1}{\omega_{n-1}} \frac{1-r^2}{(1-r)^n} - \frac{1}{\omega_{n-1}} \right) \\
&= \frac{c_{0,1}}{\sqrt{\omega_{n-1}}} \frac{1+r}{(1-r)^{n-1}}.
\end{aligned}
$$

Hence it follows that $R_{\mathrm{Har}}(B)$ is not less than the root of the equation (3.2) lying in the interval $(0,1)$. It is a simple matter to check that (3.2) has a unique root in $(0,1)$.

Conversely, the Poisson kernel

$$
\wp(x, \eta^0) = \frac{1}{\omega_{n-1}} \frac{1-|x|^2}{|x-\eta^0|^n},
$$

where $\eta^0 = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$, gives us an extremal function. Indeed, it is harmonic and positive in B. After the homothety $x \mapsto \vartheta x$, $0 < \vartheta < 1$, the function $\wp(\vartheta x, \eta^0)$ is bounded in B, and $\wp(0, \eta^0) = 1/\omega_{n-1}$. Moreover,

$$
\begin{aligned}
\wp(\vartheta x, \eta^0) &= \sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} Y_{k,j}(\vartheta x)\, Y_{k,j}(\eta^0) \\
&= \sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} \vartheta^k Y_{k,j}(\eta^0)\, Y_{k,j}(x),
\end{aligned}
$$

i.e., in this case $c_{k,j} = \vartheta^k Y_{k,j}(\eta^0)$. Setting $x = r\eta^0$ yields

$$
\begin{aligned}
\sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(n,k)} |\vartheta^k Y_{k,j}(\eta^0)|\, |Y_{k,j}(r\eta^0)| &= \sum_{k=0}^{\infty} (\vartheta r)^k \sum_{j=1}^{\sigma(n,k)} \big(Y_{k,j}(\eta^0)\big)^2 \\
&= \frac{1}{\omega_{n-1}} \frac{1+\vartheta r}{(1-\vartheta r)^{n-1}}.
\end{aligned}
$$

Hence we conclude that if ϑr is the root of (3.2) in the interval $(0,1)$, then the inequality (3.3) fails. To finish the proof it remains to pass to the limit when $\vartheta \to 1$.

□

Note that if $n = 2$ then $R_{\mathrm{Har}}(B) = 1/3$, which agrees with the classical Bohr radius. Furthermore,

$$\begin{array}{rcll} R_{\mathrm{Har}}(B) & = & \frac{5-\sqrt{17}}{4} & \text{if} \quad n = 3; \\ R_{\mathrm{Har}}(B) & = & 1 - \sqrt[3]{\frac{1}{2} + \sqrt{\frac{55}{216}}} + \frac{1}{6\sqrt[3]{\frac{1}{2}+\sqrt{\frac{55}{216}}}} & \text{if} \quad n = 4. \end{array}$$

One can still write down $R_{\mathrm{Har}}(B)$ in radicals for $n = 5$, however, the formula is cumbersome. We just mention an asymptotic formula of $R_{\mathrm{Har}}(B)$ when $n \to \infty$, namely

$$R_{\mathrm{Har}}(B) = \frac{\log 2}{n} + O\left(\frac{1}{n^2}\right). \tag{3.4}$$

4 Separately harmonic functions

By a *separately harmonic function* in a domain $\mathcal{D} \subset \mathbb{C}^n$ is meant any function f harmonic in every variable $z_j = x_j + i x_{n+j}$, $j = 1, \ldots, n$, i.e., $\partial^2 f / \partial z_j \partial \bar{z}_j = 0$ in $\mathcal{D}$.

Let $\mathcal{D} = U^n$ be the unit polydisk. Each function f separately harmonic in U^n can be represented in any smaller polydisk rU^n, $0 < r < 1$, by the multiple Poisson integral where the integration is over the n-dimensional skeleton of rU^n, i.e., $s_r = \{z \in \mathbb{C}^n : \ |z_j| = r, \ j = 1, \ldots, n\}$. Expanding every Poisson kernel as a power series in z_j and $\bar{z}_j$, we obtain easily a basis in the space of functions separately harmonic in U^n, endowed with the topology of uniform convergence on compact subsets of U^n. This is $(1, w_I^\alpha)$ where I varies over all n-tuples consisting of ± 1 (there are 2^n such tuples), and α varies over all multi-indices in $\mathbb{Z}_+^n$. Write $I = (i_1, \ldots, i_n)$, then $w_I = (w_{I,1}, \ldots, w_{I,n})$ where $w_{I,j} = z_j$, if $i_j = +1$, and $w_{I,j} = \bar{z}_j$, if $i_j = -1$. Under this notation, we have $w_I^\alpha = w_{I,1}^{\alpha_1} \ldots w_{I,n}^{\alpha_n}$, and any function $f(z, \bar{z})$ separately harmonic in U^n expands as a series

$$f(z, \bar{z}) = c_0 + \sum_{\alpha \neq 0} \sum_I c_{\alpha,I} w_I^\alpha \tag{4.1}$$

where

$$c_{\alpha,I} = \lim_{r \to 1-0} \frac{1}{(2\pi)^n} \int_{s_r} f(z, \bar{z})\, \bar{w}_I^\alpha \, \frac{dz}{z}.$$

By a *Bohr radius* for functions separately harmonic in U^n we mean the largest number $r \geq 0$ with the property that if f expands as (4.1) and $|f| < 1$ in U^n, then

$$|c_0| + \sum_{\alpha \neq 0} \sum_I |c_{\alpha,I}\, w_I^\alpha| < 1 \tag{4.2}$$

in the polydisk rU^n. We denote this radius by $R_{\mathrm{SHar}}(U^n)$.

Theorem 4.1

$$R_{\text{SHar}}(U^n) = \frac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}} + 1}.$$

Proof. By Lemma 3.1, $R_{\text{SHar}}(U^n)$ is equal to the largest number $r \geq 0$ such that if f is a positive bounded separately harmonic function in U^n with expansion (4.1), then

$$c_0 + \sum_{\alpha \neq 0} \sum_I |c_{\alpha,I} w_I^\alpha| < 2\, c_0$$

for all $z \in rU^n$.

If $f(z, \bar{z}) > 0$ then we get, using the mean value theorem for the skeleton s_r,

$$\begin{aligned} |c_{\alpha,I}| &\leq \lim_{r \to 1-0} \frac{1}{(2\pi)^n} \int_{s_r} f(z, \bar{z})\, |\bar{w}_I^\alpha| \left| \frac{dz}{z} \right| \\ &= \lim_{r \to 1-0} r^{|\alpha|} \frac{1}{(2\pi)^n} \int_{s_r} f(z, \bar{z}) \left| \frac{dz}{z} \right| \\ &= f(0) \\ &= c_0 \end{aligned}$$

for all α and I. Furthermore, if $z \in rU^n$, then

$$\begin{aligned} c_0 + \sum_{\alpha \neq 0} \sum_I |c_{\alpha,I} w_I^\alpha| &< c_0 + c_0 \sum_{\alpha \neq 0} \sum_I r^{|\alpha|} \\ &= c_0 \left(1 + 2 \sum_{j=1}^{\infty} r^j \right)^n \\ &= c_0 \left(\frac{1+r}{1-r} \right)^n \end{aligned}$$

whence

$$R_{\text{SHar}}(U^n) \geq \frac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}} + 1}.$$

Conversely, the Poisson kernel in the complex variable z_j can be written in the form

$$\frac{1}{2\pi} \frac{1 - |z_j|^2}{|1 - z_j|^2},$$

the second point being $\zeta = 1$. Given any fixed $0 < \vartheta < 1$, we consider the function

$$f(z, \bar{z}) = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \frac{1 - |\vartheta z_j|^2}{|1 - \vartheta z_j|^2}$$

which is separately harmonic, positive and bounded in U^n. For this function, we have

$$c_{\alpha,I} = \frac{1}{(2\pi)^n}\,\vartheta^{|\alpha|}$$

for all α and I. If $z \in s_r$ then

$$\begin{aligned} c_0 + \sum_{\alpha\neq 0}\sum_{I} |c_{\alpha,I} w_I^\alpha| &= \frac{1}{(2\pi)^n}\left(1 + 2\,\frac{\vartheta r}{1-\vartheta r}\right)^n \\ &= \frac{1}{(2\pi)^n}\left(\frac{1+\vartheta r}{1-\vartheta r}\right)^n. \end{aligned}$$

Letting $\vartheta \to 1$, we deduce that

$$R_{\mathrm{SHar}}(U^n) \le \frac{2^{\frac{1}{n}}-1}{2^{\frac{1}{n}}+1},$$

which completes the proof.

□

Note that $R_{\mathrm{SHar}}(U) = 1/3$ which agrees with the harmonic Bohr radius of U.

It is easy to show an asymptotic formula for $R_{\mathrm{SHar}}(U^n)$ when $n \to \infty$, namely

$$R_{\mathrm{SHar}}(U^n) = \frac{\log 2}{2n} + O\left(\frac{1}{n^2}\right). \tag{4.3}$$

cf. (3.4). One must take into account that the real dimension of U^n is $2n$, hence the asymptotic formulas for the "harmonic" Bohr radius of the ball B in $\mathbb{R}^{2n}$ and the "separately harmonic" Bohr radius of the polydisk U^n coincide.

Let $\mathcal{D}$ be a complete Reinhardt domain in $\mathbb{C}^n$. We define the Bohr radius $R_{\mathrm{SHar}}(\mathcal{D})$ to be the largest number $r \ge 0$ such that if $f(z,\bar{z})$ expands as (4.1) and $|f(z,\bar{z})| < 1$ for all $z \in \mathcal{D}$, then (4.2) is fulfilled in the homothety $r\mathcal{D}$. Since $\mathcal{D}$ is the union of polydisks, we arrive at the following consequence of Theorem 4.1.

Corollary 4.2 *For any complete Reinhardt domain $\mathcal{D} \subset \mathbb{C}^n$, it follows that*

$$R_{\mathrm{SHar}}(\mathcal{D}) \ge \frac{2^{\frac{1}{n}}-1}{2^{\frac{1}{n}}+1}.$$

Theorem 3.2 and 4.1 gain in interest if we realise that the corresponding Bohr radii in these theorems are computed explicitly. Till now mere estimates of various Bohr radii were known in all cases, except for the classical Bohr theorem and the equivalent assertion for holomorphic functions with positive real part in the unit disk.

The proofs of these theorems give more, namely explicit inequalities for positive harmonic functions. Since $Z_k = \sigma(n,k)/\omega_{n-1}$, cf. [SW71], the proof of Theorem 3.2 contains an estimate of the root mean square of the coefficients in the expansion over the basis by homogeneous harmonic polynomials of the same degree,

$$\sqrt{\frac{\sum_{j=1}^{\sigma(n,k)} |c_{k,j}|^2}{\sigma(n,k)}} \le c_{0,1}$$

for all $k \in \mathbb{Z}_+$. On the other hand, the proof of Theorem 4.1 makes use of the inequality $|c_{\alpha,I}| \le c_0$, for each α and I. It is the inequalities that are the key tool of evaluating the Bohr radii for various function classes. They are analogues of the classical Carathéodory inequality [Car07], which states that if $f(z)$ expands as a series $\sum_{\nu=0}^{\infty} c_\nu z^\nu$ in the unit disk and $\Re f(z) > 0$, $\Im f(0) = 0$, then $|c_\nu| \le 2c_0$ for all ν.

5 Pluriharmonic functions

By *pluriharmonic functions* are meant real parts of holomorphic functions of several complex variables. The coincidence of $R_{\mathrm{Har}}(U)$ with the classical Bohr radius 1/3 is not accidental. Let $(f_\nu)_{\nu=0,1,\ldots}$ be a basis in the space of holomorphic functions in a domain $\mathcal{D} \subset \mathbb{C}^n$ with the natural topology of uniform convergence on compact subsets of $\mathcal{D}$. We assume that $f_0 \equiv 1$ and all the other functions f_ν vanish at a fixed point $z^0 \in \mathcal{D}$. Then $\left(1, f_\nu, \bar{f}_\nu\right)_{\nu=1,2,\ldots}$ form a basis in the space of pluriharmonic functions in $\mathcal{D}$. Any such function expands as a series

$$F(z) = C_0 + \sum_{\nu=1}^{\infty} \left(C_\nu f_\nu + \bar{C}_\nu \bar{f}_\nu\right) \tag{5.1}$$

which converges uniformly on compact sets in $\mathcal{D}$, the coefficients C_ν and $\bar{C}_\nu$ being conjugate to each other.

Theorem 5.1 *Given any neighbourhood $U \subset \mathcal{D}$ of z^0, the following assertions are equivalent:*

1) *If F is a positive pluriharmonic function in $\mathcal{D}$ with expansion* (5.1)*, then $C_0 + 2\sum_{\nu=1}^{\infty} |C_\nu f_\nu| < 2F(z^0)$ in U.*
2) *If $f = \sum_{\nu=0}^{\infty} c_\nu f_\nu$ is a holomorphic function with positive real part in $\mathcal{D}$ and $\Im f(z^0) = 0$, then $\sum_{\nu=0}^{\infty} |c_\nu f_\nu| < 2f(z^0)$ in U.*

Proof. The proof is straightforward. Each pluriharmonic function F has the form $(f + \bar{f})/2$, where f is holomorphic. Under the above assumption, we have

$$\begin{aligned} F(0) &= f(0), \\ C_\nu &= \frac{c_\nu}{2}, \end{aligned}$$

whence the desired conclusion follows.

□

Theorem 5.1 implies that all estimates of Bohr radii for holomorphic functions of [BKh97], [Aiz00], [Aiz99], [AAD98a], [AAD98b], [Boa98], [DR99] discussed in Section 1 remain valid for Bohr radii of pluriharmonic functions in the same domains.

6 Polyharmonic functions

Recall that by a *polyharmonic function* in a domain $\mathcal{D} \subset \mathbb{R}^n$ is meant any function f satisfying $\Delta^N f = 0$ in $\mathcal{D}$, where $N > 1$. For such functions, there is no Bohr phenomenon. Indeed, consider the function family $1 + c\,|x|^{2(N-1)}$ in the unit ball B, parametrised by a positive constant c. All the functions are polyharmonic and positive. Moreover, 1 and $|x|^{2(N-1)}$ enter into a simplest basis in the space of polyharmonic functions in B. Were the Bohr phenomenon available, one had $1+c\,|x|^{2(N-1)} < 2$ in a ball rB, for each $c > 0$. This amounts to saying that

$$|x| < \left(\frac{1}{c}\right)^{\frac{1}{2(N-1)}}$$

for all $|x| < r$, which contradicts to the fact that the right hand side is infinitesimal when $c \to \infty$.

7 Solutions of elliptic equations

The preceding section shows that the Bohr phenomenon does not extend to solutions of elliptic equations of order larger than 2. Moreover, it is not valid even for second order elliptic equations that do not meet the maximum principle. Indeed, $1 + c\,|z|^2$, $c > 0$, is a family of positive solutions to the elliptic equation $(\partial/\partial\bar{z})^2 f = 0$ in the unit disk $U \subset \mathbb{C}$ (the solutions are known as *bianalytic functions*). Since 1 and $|z|^2$ enter into a simplest basis in the space of bianalytic functions in U, we may repeat the same reasoning as that in Section 6.

Thus, we restrict ourselves to second order elliptic equation $Af = 0$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ whose solutions verify the maximum principle. As is known, e.g. [Eva98, 6.4], A should be of the form

$$A = -\sum_{i,j=1}^{n} a_{i,j}(x)\,\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\,\frac{\partial}{\partial x_i} + a(x),$$

the coefficients being C^∞ real-valued functions in $\mathcal{D}$ satisfying $a_{i,j} = a_{j,i}$, for each $i, j = 1, \ldots, n$, and $a \geq 0$ in $\mathcal{D}$. To ensure that $1 \in \mathrm{Sol}(\mathcal{D})$ we actually require $a \equiv 0$ in $\mathcal{D}$.

We continue to write $\mathrm{Sol}(\mathcal{D})$ for the space of all functions f satisfying $Af = 0$ in $\mathcal{D}$. By Weil's lemma, any "reasonable" solution of $Af = 0$ is actually a C^∞ function in $\mathcal{D}$. Moreover, any "reasonable" topology in the space $\mathrm{Sol}(\mathcal{D})$ coincides with that induced by the embedding into $C^\infty(\mathcal{D})$, cf. [Tar97, 3.1].

For a set $K \subset\subset \mathcal{D}$, we put

$$\|f\|_K = \sup_K |f(x)|$$

where $f \in \mathrm{Sol}(\mathcal{D})$. The system of seminorms $\|f\|_K$, $K \subset\subset \mathcal{D}$, defines the topology of uniform convergence on compact subsets of $\mathcal{D}$. When regarded with this topology, $\mathrm{Sol}(\mathcal{D})$ is a nuclear Fréchet space, e.g. [MV97].

A sequence $(f_\nu)_{\nu=0,1,\ldots}$ of functions in $\mathrm{Sol}(\mathcal{D})$ is said to be a *basis* in this space if for each solution $f \in \mathrm{Sol}(\mathcal{D})$ there is a unique sequence of numbers c_ν such that $f = \sum_{\nu=0}^{\infty} c_\nu f_\nu$, where the series converges uniformly on any compact subset of $\mathcal{D}$.

Lemma 7.1 *If* $(f_\nu)_{\nu=0,1,\ldots}$ *is a basis in the space* $\mathrm{Sol}(\mathcal{D})$ *then for each* $U \subset\subset \mathcal{D}$ *there exist* $K \subset\subset \mathcal{D}$ *and* $C > 0$ *such that, given any* $f = \sum_{\nu=0}^{\infty} c_\nu f_\nu$ *in* $\mathrm{Sol}(\mathcal{D})$, *we have*

$$\sum_{\nu=0}^{\infty} |c_\nu|\, \|f_\nu\|_U \leq C\ \|f\|_K.$$

Proof. This result is a straightforward consequence of the theorem on absoluteness of bases in nuclear spaces proved in [DM60], cf. also Theorem 28.12 in [MV97].

□

Certainly, the generic situation is when $U \subset\subset K$, which can be assumed without loss of generality.

Lemma 7.2 *For each point* $x^0 \in \mathcal{D}$ *and set* $\sigma \subset\subset \mathcal{D}$, *there is a constant* $c > 0$ *such that whenever* $f \in \mathrm{Sol}(\mathcal{D})$ *and* $f(x^0) = 0$ *we have*

$$\sup_\sigma |f| \leq c\ \sup_{\mathcal{D}} f.$$

Proof. We can assume, by enlarging σ if necessary, that σ is connected and contains x^0. Choose a domain $U \subset\subset \mathcal{D}$ such that $\overline{\sigma} \subset U$. Suppose $f \in \mathrm{Sol}(\mathcal{D})$ vanishes at x^0. Set

$$\begin{aligned} \inf_\sigma f &= -m_\sigma, \\ \sup_\sigma f &= M_\sigma \end{aligned}$$

and

$$\begin{aligned} \inf_U f &= -m_U, \\ \sup_U f &= M_U, \end{aligned}$$

so that all the m_σ, M_σ and m_U, M_U are non-negative. Then

$$\begin{array}{rcccll} -m_\sigma & \leq & f(x) & \leq & M_\sigma & \text{for all} \quad x \in \sigma, \\ -m_U & \leq & f(x) & \leq & M_U & \text{for all} \quad x \in U. \end{array}$$

Consider the function

$$u(x) = M_U - f(x)$$

in U. It is a simple matter to see that u is a non-negative solution to $Au = 0$ in U, and

$$\begin{aligned} \inf_\sigma u &= M_U - M_\sigma, \\ \sup_\sigma u &= M_U + m_\sigma. \end{aligned}$$

According to Harnack's inequality, e.g. [Eva98, 6.4.3], there exists a constant c depending only on σ and the coefficients of A, such that

$$M_U + m_\sigma \leq c\,(M_U - M_\sigma)\,.$$

Hence

$$\begin{aligned} \sup_\sigma |f| &\leq M_\sigma + m_\sigma \\ &\leq M_U + m_\sigma \\ &\leq c\,(M_U - M_\sigma) \\ &\leq c\ M_U \\ &\leq c\ \sup_{\mathcal{D}} f, \end{aligned}$$

as desired.

□

As we will shortly learn, Lemmas 7.1 and 7.2 already imply a Bohr phenomenon for solutions of $Af = 0$. While the first of the two extends easily to solutions of general hypoelliptic equations, the second one is a fine result on the geometry of graph hypersurfaces of solutions to second order elliptic equations. Indeed, an equivalent formulation of Lemma 7.2 is that for each set $\sigma \subset\subset \mathcal{D}$ there is a constant $c > 0$ depending only on σ, such that whenever $f \in \mathrm{Sol}(\mathcal{D})$ we have

$$f(x^0) - \inf_{\sigma} f \le c \left(\sup_{\mathcal{D}} f - f(x^0) \right)$$

for all $x^0 \in \sigma$. To see this it is sufficient to apply Lemma 7.2 to the functions $f(x) - f(x^0)$. Setting $t = 1/(1+c)$ yields

$$f(x^0) \le t \ \inf_{\sigma} f + (1-t) \ \sup_{\mathcal{D}} f \qquad (7.1)$$

for all $x^0 \in \sigma$, where $t \in (0,1)$ is independent of $f \in \mathrm{Sol}(\mathcal{D})$. The inequality (7.1) specifies the geometry of the hypersurface $y = f(x)$ in $\mathbb{R}^{n+1}$ through the rigid constant t.

Example 7.3 Take $\mathcal{D} = B(0,R)$, $R > 1$, and $\sigma = B(0,1)$, $x^0 = 0$. For the family of harmonic functions

$$f(x) = \frac{1}{2\pi} \log \left((x_1 - a)^2 + x_2^2 \right)$$

in $\mathcal{D}$, $a \ge R$, the inequality (7.1) reduces to

$$\frac{a}{a-1} \le \left(1 + \frac{R}{a} \right)^c,$$

for all $a \ge R$.

□

Having disposed of these preliminary steps, we can now extend Theorem 1.6 to solutions of $Af = 0$.

Theorem 7.4 *If* $(f_\nu)_{\nu=0,1,\ldots}$ *is a basis in* $\mathrm{Sol}(\mathcal{D})$ *satisfying*

1) $f_0 \equiv 1$;
2) *all the functions* f_ν, $\nu = 1,2,\ldots$, *vanish at a point* $x^0 \in \mathcal{D}$,

then there exist a neighbourhood U *of* x^0 *and a compact set* $K \subset \mathcal{D}$, *such that, for any* $f \in \mathrm{Sol}(\mathcal{D})$ *with* $f = \sum_{\nu=0}^{\infty} c_\nu f_\nu$,

$$\sum_{\nu=0}^{\infty} \sup_U |c_\nu f_\nu| \le \sup_K |f|.$$

Proof. According to the Grothendieck-Pietsch criterion for nuclearity, e.g. Theorem 28.15 of [MV97], for every set $U \subset\subset \mathcal{D}$ there is a set $\Sigma \subset\subset \mathcal{D}$ such that

$$\sum_{\nu=0}^{\infty} \frac{\|f_\nu\|_U}{\|f_\nu\|_\Sigma} < \infty. \tag{7.2}$$

Set $B_r = \{x \in \mathbb{R}^n : \ |x - x^0| < r\}$, for $r > 0$. Since $f_\nu(x^0) = 0$ for all $\nu \geq 1$, we have $\|f_\nu\|_{B_r} \searrow 0$ when $r \to 0$. Thus, from (7.2) it follows that

$$\sup_{\nu \geq 1} \frac{\|f_\nu\|_{B_r}}{\|f_\nu\|_\Sigma} \to 0 \tag{7.3}$$

as $r \to 0$, Σ corresponding to B_{r^0}.

On the other hand, Lemma 7.1 implies that for any $\Sigma \subset\subset \mathcal{D}$ there are $\sigma \subset\subset \mathcal{D}$ and $C > 0$ with the property that, given any $f = \sum_{\nu=0}^{\infty} c_\nu f_\nu$ in $\mathrm{Sol}(\mathcal{D})$, we have

$$\sum_{\nu=0}^{\infty} |c_\nu| \, \|f_\nu\|_\Sigma \leq C \, \|f\|_\sigma. \tag{7.4}$$

Fix $r^0 > 0$ such that $B_{r^0} \subset\subset \mathcal{D}$, choose $\Sigma \subset\subset \mathcal{D}$ such that (7.2) holds, and find $\sigma \subset\subset \mathcal{D}$ so that (7.4) holds. Finally, choose a compact set $K \subset \mathcal{D}$ whose interior contains $\overline{\sigma}$. By Lemma 7.2, there exists a constant $c > 0$ with the property that

$$\|f - f(x^0)\|_\sigma \leq c \, \sup_K \left(f - f(x^0)\right) \tag{7.5}$$

for all $f \in \mathrm{Sol}(\mathcal{D})$.

Now suppose $f \in \mathrm{Sol}(\mathcal{D})$. We may assume without loss of generality that $c_0 = f(x^0) \geq 0$, since otherwise we multiply f by -1. Then it follows from (7.4) and (7.5) that

$$\begin{aligned} \sum_{\nu=1}^{\infty} |c_\nu| \, \|f_\nu\|_\Sigma &\leq C \, \|f - f(x^0)\|_\sigma \\ &\leq C \, c \sup_K \left(f - f(x^0)\right). \end{aligned}$$

By (7.3) we can choose $r < r^0$ such that

$$\sup_{\nu \geq 1} \frac{\|f_\nu\|_{B_r}}{\|f_\nu\|_\Sigma} \leq \frac{1}{C \, c},$$

then

$$\sum_{\nu=0}^{\infty} \sup_{B_r} |c_\nu f_\nu| = f(x^0) + \sum_{\nu=1}^{\infty} |c_\nu| \, \|f_\nu\|_\Sigma \left(\frac{\|f_\nu\|_{B_r}}{\|f_\nu\|_\Sigma} \right)$$

$$
\begin{aligned}
&\leq \; f(x^0) + C\, c \sup_K \big(f - f(x^0)\big) \sup_{\nu \geq 1} \frac{\|f_\nu\|_{B_r}}{\|f_\nu\|_\Sigma} \\
&\leq \; \sup_K f \\
&\leq \; \sup_K |f|.
\end{aligned}
$$

Hence we obtain the statement with $U = B_r$.

□

In [AAD98a, Remark 3.4] it is shown that if the space $\mathrm{Sol}(\mathcal{D})$ has a basis then it has also a basis satisfying the hypotheses of Theorem 7.4. While the proof is given for holomorphic functions, the same arguments still work for the space $\mathrm{Sol}(\mathcal{D})$.

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