Surgery and the Relative Index in Elliptic Theory

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Abstract

We prove a general theorem on the local property of the relative index for a wide class of Fredholm operators, including relative index theorems for elliptic operators due to Gromov–Lawson, Anghel, Teleman, Booß-Bavnbek–Wojciechowski, et al. as special cases. In conjunction with additional conditions (like symmetry conditions) this theorem permits one to compute the analytical index of a given operator. In particular, we obtain new index formulas for elliptic pseudodifferential operators and quantized canonical transformations on manifolds with conical singularities as well as for elliptic boundary value problems with a symmetry condition for the conormal symbol.

Keywords: elliptic operators, index theory, surgery, relative index, manifold with singularities, boundary value problems

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Contents

Introduction

In several cases, when trying to construct index formulas for elliptic operators on manifolds with boundary, manifolds with singularities, or noncompact manifolds with a special structure at infinity (manifolds with cylindrical ends), one faces the problem of separating the contributions to the index from the "interior" part of the manifold and, respectively, from the boundary, singular points, or a neighborhood of infinity. The possibility of such a separation is suggested by the well-known property that the index is "local" (e.g., see [1] and the papers cited therein). As to the separation itself, it was apparently carried out for the first time (for Dirac operators on noncompact Riemannian manifolds) by Gromov and Lawson [2], who obtained the corresponding relative index theorem.

Let us recall their result. Let X_0 and X_1 be complete even-dimensional Riemannian manifolds, and let D_0 and D_1 be generalized Dirac operators on X_0 and X_1, respectively, acting on sections of vector bundles S_1 and S_2. We say that D_0 and D_1 *coincide at*

infinity if there exist compact subsets $K_0 \subset X_0$ and $K_1 \subset X_1$, an isometry

$$F : (X_0 \backslash K_0) \overset{\approx}{\to} (X_1 \backslash K_1),$$

and an isometry

$$\widetilde{F} \colon S_0|_{X_0 \backslash K_0} \to S_1|_{X_1 \backslash K_1}$$

of vector bundles such that

$$D_1 = \widetilde{F} \circ D_0 \circ \widetilde{F}^{-1} \quad \text{on} \quad X_1 \backslash K_1.$$

To simplify the notation, we identify $X_0 \backslash K_0$ with $X_1 \backslash K_1$ and write

$$D_0 = D_1 \quad \text{on} \quad \Omega = X_0 \backslash K_0 \cong X_1 \backslash K_1.$$

In this situation we can define the *topological relative index* $\mathrm{ind}_t(D_1^+, D_0^+)$ of the operators

$$D_1^+ : \Gamma(S_1^+) \to \Gamma(S_1^-) \quad \text{and} \quad D_0^+ : \Gamma(S_0^+) \to \Gamma(S_0^-)$$

as follows. If X_0 and X_1 are compact, then we simply set

$$\mathrm{ind}_t(D_1^+, D_0^+) = \mathrm{index}(D_1^+) - \mathrm{index}(D_0^+).$$

(The expression on the right-hand side is simply the difference of the usual analytical=topological indices of the operators D_1^+ and D_0^+, expressed, say by the Atiyah–Singer formula). If X_0 (and hence X_1) is noncompact, then we use the following procedure. We cut the manifolds X_0 and X_1 along some compact hypersurface $H \subset \Omega$ and compactify them by attaching some compact manifold with boundary H. The operators D_0^+ and D_1^+ can be extended to elliptic operators $\widetilde{D}_0^+$ and $\widetilde{D}_1^+$ on the compact manifolds thus obtained. Now we set

$$\mathrm{ind}_t(D_1^+, D_0^+) = \mathrm{index}(\widetilde{D}_1^+) - \mathrm{index}(\widetilde{D}_0^+). \tag{1}$$

Using the formula

$$\mathrm{index}(A) = \mathrm{Trace}(1 - RA) - \mathrm{Trace}(1 - AR)$$

for the index of an elliptic operator A (here R is a parametrix of A), the localization of kernels of pseudodifferential operators in a neighborhood of the diagonal, and a partition of unity, Gromov and Lawson proved that the right-hand side of (1) is independent of the arbitrariness in the above construction.

Next, let the operators D_0^+ and D_1^+ be *positive at infinity* (the precise definition is given in [2]; roughly speaking, this condition means that the free terms in the operators

3

$(D_0^+)^* D_0$ and $(D_1^+)^* D_1$ expressed via covariant derivatives are positive). Then the operators D_0^+ and D_1^+ are Fredholm, and one can define the *analytical relative index*

$$\mathrm{ind}_a(D_1^+, D_0^+) = \mathrm{index}_a(D_1^+) - \mathrm{index}_a(D_0^+). \tag{2}$$

The Gromov–Lawson relative index theorem states that *the topological and analytical relative indices coincide*:

$$\mathrm{ind}_a(D_1^+, D_0^+) = \mathrm{ind}_t(D_1^+, D_0^+). \tag{3}$$

In [2] one can also find a more general theorem pertaining to the case in which the operators D_0 and D_1 coincide only on some of the "ends" of X_0 and X_1 at infinity. In this case, one again has a formula like (3), where the right-hand side is no longer the "topological relative index," but it is rather the analytical index of some elliptic Fredholm operator on a (generally speaking, noncompact) manifold obtained from X_0 and X_1 by cutting away the "common" ends along some hypersurface followed by gluing along that hypersurface. The proof uses the same technique.

We can conclude (as is easily seen from the second theorem) that the *topological index* actually has nothing to do with the Gromov–Lawson relative index theorem: this theorem states the equality of the *analytical* relative indices for two pairs of operators obtained from each other by simultaneous surgery on a part of the manifold where they coincide; the topological index occurs in the answer only if the newly obtained operators fall within the scope of the Atiyah–Singer theorem. (On the other hand, naturally, the *applications* of theorems of that type are just related to transforming the original operators to new operators such that the Atiyah–Singer theorem or any other theorem expressing the index in topological terms can be used.)

Later, the results obtained by Gromov and Lawson were generalized in various directions. Anghel [3] generalized them to arbitrary self-adjoint elliptic operators of the first order on a complete Riemannian manifold. "Local" relative index theorems (in terms of the kernels of fundamental solutions of the corresponding heat equations) were obtained by Donnelly [4] for the signature operator, Bunke [5] for arbitrary Dirac operators, and Lesch [6] for a class of symmetric elliptic operators of the first order in the equivariant situation and in the presence of conical singularities in the interior part of the manifold. One should also mention the results of Booß-Bavnbek–Wojciechowski [7] and Teleman [8], which essentially pertain to the case of Dirac operators.

It is not a mere occasion that the above-mentioned theorems are stated in the form of relative index theorems. An attempt to restate them, say, in the form

the index of an operator = the contribution of the interior part of the manifold

+ the contribution of the neighborhood of infinity $\qquad$ (4)

would result in an index formula that generally contains homotopy noninvariant terms. Indeed, as is shown in [9] for the example of spectral boundary value problems, there

exists a *topological obstruction* to such representations. However, it turns out that under additional assumptions like some symmetry conditions the decomposition of the index into homotopy invariant terms of such kind is possible, which results in simple and elegant index formulas for the corresponding classes of problems. For the first time, such results were obtained for manifolds with conical singularities in [10], and then the paper [11] were published, which dealt with index formula for quantized canonical transformations.

In the present paper we introduce a general functional-analytic model in which the locality principle for the index, or, which is the same, the relative index theorem, holds. All the above-mentioned results are covered by this model. By way of example, we obtain an index theorem on manifolds with singularities, including the results of [10], [11] (as well as, e.g., [12]) as special cases. We also state an index theorem for elliptic boundary value problems with a symmetry condition for the main operator.

The paper is organized as follows. In § 1 we introduce some definitions and then state and prove the main result, namely, the abstract relative index theorem. This theorem is applied in § 2 to prove the index increment formula for a special kind of surgery on manifolds and bundles where the elliptic operator in question acts. Examples of applications are considered in §3.

Preliminary publications. The results of the paper were reported at the international conference "Operator Algebras and Asymptotics on Manifolds with Singularities," Warsaw, [13]. A brief exposition of these results will appear in [14].

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1 Bottleneck spaces and an abstract relative index theorem

In the theory of ψDO and FIO, (micro)localization is usually performed with the help of a partition of unity on the configuration (or the phase) space. Bottleneck spaces are a convenient abstract model in which one can use partitions of unity and specific realizations of which, in particular, cover all cases mentioned in the introduction.

1.1 Bottleneck spaces. Examples

Definition 1 A *bottleneck space* is a Hilbert space H equipped with the structure of a $C^\infty([0,1])$-module.[1]

($C^\infty([0,1])$ is a topological algebra with unit, which is the function identically equal to 1, and we assume that the action of $C^\infty([0,1])$ on H is continuous and the unit is represented by the identity operator.)

Let $h \in H$ be an arbitrary element. We define the *support* of h as the closed subset

$$\operatorname{supp} h = \bigcap_{\substack{\varphi \in C^\infty([0,1]) \\ \varphi h = 0}} \varphi^{-1}(0) \tag{5}$$

of $[0,1]$. Next, for an arbitrary closed subset $K \subset [0,1]$ we set

$$H_K = \{ h \in H \mid \operatorname{supp} h \subset K \}. \tag{6}$$

Lemma 1 *The following assertions hold*:

1. $\operatorname{supp}(h_1 + h_2) \subset \operatorname{supp} h_1 \cup \operatorname{supp} h_2$;

2. H_K *is a closed linear subspace in* H ;

3. *if* $K_1, K_2 \in [0,1]$ *are disjoint closed subsets, then one has the direct (not necessarily orthogonal) sum decomposition*

$$H_{K_1 \cup K_2} = H_{K_1} \oplus H_{K_2}. \tag{7}$$

Proof. First, let us show that if $\varphi = 0$ in the ε-neighborhood $U_\varepsilon(\operatorname{supp} h)$ of the set $\operatorname{supp} h$, then $\varphi h = 0$. Indeed, consider the closed set $Z = [0,1] \setminus U_\varepsilon(\operatorname{supp} h)$. For every $\tau \in Z$ there exists a function $\varphi_\tau \in C^\infty([0,1])$ such that $\varphi_\tau(\tau) = 1$ and $\varphi_\tau h = 0$. We set

$$U_\tau = \left\{ t \in [0,1] \ \middle| \ \varphi_\tau(t) > \frac{1}{2} \right\}.$$

The sets U_τ form an open cover of the compact subset Z, and hence we can choose a finite subcover $\{U_{\tau_j}\}|_{j=1,N}$. The smooth function $f_0 = \sum_j |\varphi_{\tau_j}|^2$ does not vanish on the closed set Z. Hence there exists a function $f \in C^\infty([0,1])$ such that $f = f_0^{-1}$ on Z (it

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[1]One can replace the interval $[0,1]$ by an arbitrary compact manifold (possibly, with boundary); many of the subsequent constructions can be extended to this more general case.

can be constructed with the help of a partition of unity subordinate to the open cover of $[0, 1]$ by the set $U_\varepsilon(\operatorname{supp} h)$ and a sufficiently small neighborhood of Z). Let

$$\psi = \sum_j \left|\varphi_{\tau_j}\right|^2 f.$$

Then

$$\psi h = f \sum_j \overline{\varphi}_{\tau_j} \varphi_{\tau_j} h = 0.$$

Moreover, $\psi = 1$ on Z, so that $\psi\varphi = \varphi$, and hence $\varphi h = \varphi\psi h = 0$, as desired.

Now let us prove a). Let $t \notin \operatorname{supp} h_1 \cup \operatorname{supp} h_2$. Then there exists a function $\varphi \in C^\infty([0, 1])$ such that $\varphi(t) = 1$ and $\varphi h_1 = \varphi h_2 = 0$. We have $\varphi(h_1 + h_2) = \varphi h_1 + \varphi h_2 = 0$, whence $t \notin \operatorname{supp}(h_1 + h_2)$. Next, let us prove b). The fact that H_K is a linear manifold is already obvious by virtue of a). Let $h_n \in H_K$ and $h_n \to h$. Let us show that $h \in H_K$. Suppose the opposite: $t \notin K$. Then by a) there exists a function $\varphi \in C^\infty([0, 1])$ such that $\varphi(t) = 1$ and $\varphi g = 0$ for all $g \in H_K$. We have $\varphi h = \lim \varphi h_n = 0$, so that $t \notin \operatorname{supp} h$. Thus H_K is closed, and we have proved b). To prove c), let us construct functions $e_i \in C^\infty([0, 1])$, $i = 1, 2$, such that $e_i = \delta_{ij}$ in a neighborhood of the set K_j, $i, j = 1, 2$. For an arbitrary $h \in H_K$, we have $h = h_1 + h_2$, $h_i = e_i h \in H_{K_i}$, $i = 1, 2$, so that $H_K = H_{K_1} + H_{K_2}$. Next, if $h \in H_{K_1} \cap H_{K_2}$, then $h = e_1 h = e_1 e_2 h = 0$, so that the expansion into a sum is unique, and by Banach's closed graph theorem we obtain the topological isomorphism $H_K \simeq H_{K_1} \oplus H_{K_2}$. $\qquad\square$

Let us give some examples of bottleneck spaces.

1. Let M be a C^∞ manifold, and let $G \subset M$ be a compact subset. There exists a real-valued function $\chi \in C^\infty(M)$ such that $0 \le \chi \le 1$, $\chi|_G \equiv 1$, and $\chi \equiv 0$ outside some neighborhood of G with compact closure. Let $H = H^s(M, d\mu)$ be the Sobolev space of functions on M with respect to some smooth measure on M. Then H can be equipped with the structure of a bottleneck space by setting

$$[\varphi h](x) = \varphi(\chi(x))h(x), \qquad x \in M, \tag{8}$$

for any $h \in H$ and $\varphi \in C^\infty([0, 1])$. The set of $x \in M$ for which $0 < \chi(x) < 1$ will be referred to as the *bottleneck*.

2. Let M be a manifold (possibly, with singularities), and let $G \subset M$ be a two-sided compact submanifold of codimension 1 lying in the smooth part of M. We assume that the cut along G divides M into two disjoint parts (each of which may consist of several connected components). Let $U \simeq G \times [0, 1]$ be a closed collar neighborhood of G (the embedding $G \subset U$ is given by the formula $g \mapsto (g, \frac{1}{2})$). Let $\chi: [0, 1] \to [0, 1]$ be a smooth monotone increasing function such that $\chi(0) = 0$, $\chi(1) = 1$, $\chi'(t) > 0$ for $t \in (0, 1)$, and $\chi^{(k)}(0) = \chi^{(k)}(1) = 0$ for all $k = 1, 2, \ldots$. Next, let $H = H^s(M, \rho)$ be

the weighted Sobolev space on M with weight ρ (the specific choice of the weight near the singular points is irrelevant to us). For any function $\varphi \in C^\infty([0,1])$, we define a smooth function $\widetilde{\varphi}$ on M by setting

$$\widetilde{\varphi}\,(x) = \begin{cases} \varphi(\chi(t)) & \text{if } x = (g,t) \in U, \\ 0 & \text{if } x \text{ lies in the part of } M \text{ adjacent to } t = 0, \\ 1 & \text{if } x \text{ lies in the part of } M \text{ adjacent to } t = 1. \end{cases} \qquad (9)$$

Then the formula $[\varphi h](x) = \widetilde{\varphi}\,(x)h(x)$ defines the structure of a bottleneck space on H.[2]

3. Let M be a compact smooth manifold with boundary ∂M. We define H as the direct sum of Sobolev spaces on M and ∂M, that is,

$$H = H^s(M) \oplus H^l(\partial M).$$

The structure of a bottleneck space on H can be defined as follows. Let $U \equiv \partial M \times [0,1]$ be a collar neighborhood of the boundary ∂M. (The boundary corresponds to the value $t = 0$.) We define the action of $C^\infty([0,1])$ on H by the formula

$$\varphi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \widetilde{\varphi}\,u \\ \varphi(0)v \end{pmatrix}, \qquad (10)$$

where $\widetilde{\varphi}$ is the same function as in the preceding example.

In all these examples, function spaces can be replaced by spaces of sections of vector bundles.

Let H_1 and H_2 be bottleneck spaces, and let $D \colon H_1 \to H_2$ be a bounded linear operator. Let also δ be a given (sufficiently small) number.

We say that the operator D *extends supports by at most δ* (or is *δ-narrow*) if supp Dh is contained in the δ-neighborhood of supp h for any $h \in H_1$.

1.2   Properly Fredholm operators

Definition 2 An operator

$$D : H_1 \to H_2$$

_____

[2]Note that $\widetilde{\varphi}\,(x)$ is constant outside a compact set containing no singular points, so that the operator of multiplication by $\widetilde{\varphi}\,(x)$ is bounded in H regardless of the choice of the weight in the definition of Sobolev spaces. In more specific examples, one can drop the assumption that G is compact and even admit that G has singular points; the only requirement is that the operator of multiplication by $\widetilde{\varphi}\,(x)$ be bounded.

in bottleneck spaces is said to be *properly Fredholm* if it is Fredholm and if for any[3] $\delta > 0$ the operator D differs by a trace class operator from some operator $\widetilde{D}$ that extends supports by at most δ and that has a regularizer

$$\widetilde{R} \colon H_2 \to H_1$$

such that the operators

$$1 - \widetilde{R}\widetilde{D} \quad and \quad 1 - \widetilde{D}\widetilde{R}$$

belong to the trace class, and moreover, $\widetilde{R}$ also extends supports by at most δ.

Examples of properly Fredholm operators in the bottleneck spaces considered in Examples 1–3 include elliptic differential operators, elliptic pseudodifferential operators, and operators corresponding to elliptic boundary value problems or Sobolev problems [15, 16].

We intend to study how the index of a properly Fredholm operator in bottleneck spaces varies as the spaces and the operator itself vary. Let us introduce the corresponding notions.

Let H_1 and H_2 be bottleneck spaces. Let also $F \subset [0,1]$ be a given subset. We say that H_1 *coincides with* H_2 *on* F if for any closed subset $K \subset F$ an isometric isomorphism $\gamma_K \colon H_{1K} \to H_{2K}$ of Hilbert spaces is given, an moreover, the diagram

$$\begin{array}{ccc} H_{1K_1} & \stackrel{\gamma_{K_1}}{\longrightarrow} & H_{2K_1} \\ \downarrow & & \downarrow \\ H_{1K_2} & \stackrel{\gamma_{K_2}}{\longrightarrow} & H_{2K_2}, \end{array} \tag{11}$$

where the vertical arrows are the natural embeddings, commutes for any closed subsets $K_1 \subset K_2 \subset F$. We identify the isomorphic spaces H_{1K} and H_{2K} and do not write out the isomorphism explicitly each time.

Let $H_1, \ldots H_N$ be a family of bottleneck spaces. We say that they coincide on a subset $F \subset [0,1]$ if they coincide in the sense indicated above and if, moreover, the isometric isomorphisms specifying the coincidence are natural in the sense that the composition of the morphisms $H_{iK} \to H_{jK}$ and $H_{jK} \to H_{lK}$ is equal to the morphism $H_{iK} \to H_{lK}$ for any $i, j, l \in \{1, \ldots, n\}$ and any closed subset $K \subset F$.

If bottleneck spaces H_1 and H_2 coincide on F, then we also say that they are *modifications* of each other on $[0,1] \setminus F$.

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[3]In some applications, this property is only satisfied for some sufficiently small $\delta > 0$ rather than for *all* $\delta > 0$. The generalization of the subsequent theorems to this case is obvious, and we can always indicate exactly how small δ must be; however, in general theorems we use this definition just to avoid awkward statements.

Suppose that $F \subset [0,1]$, H_1 coincides on F with H_2, and G_1 coincides on F with G_2. Next, let

$$P_1 \colon H_1 \to G_1, \qquad P_2 \colon H_2 \to G_2$$

be δ-narrow operators, and let $K \subset [0,1]$ be a closed subset such that $U_\delta(K) \subset F$. Then for every $h \in H_{1K} \equiv H_{2K}$ we have

$$\operatorname{supp} P_1 h = K_1, \qquad \operatorname{supp} P_2 h = K_2$$

for some $K_1, K_2 \subset F$ (in general, depending on h). We set $\widetilde{K} = K_1 \cup K_2$. The elements $P_1 h$ and $P_2 h$ lie in the same (by virtue of the corresponding isometric isomorphism) space $G_{1\widetilde{K}} \equiv G_{2\widetilde{K}}$, and the following notion is well defined: we say that P_1 *coincides with P_2 on K* (briefly, $P_1 = P_2$ on K) if $P_1 h = P_2 h$ for all $h \in H_{1K}$. In this case we also say that P_1 and P_2 are modifications of each other on $[0,1] \setminus K$.

1.3 The relative index formula

Now we can state the main assertion of this subsection.

Suppose that there are two quadruples H, H_0, H_1, H_{01} and G, G_0, G_1, G_{01} of bottleneck spaces satisfying the following conditions.

1. The spaces H, H_0, H_1, H_{01} coincide on the open interval $(0,1)$.

2. The space H_0 is a modification of H at the point 0.

3. The space H_1 is a modification of H at the point 1.

4. The space H_0 is a modification of H_{01} at the point 1.

5. The space H_1 is a modification of H_{01} at the point 0.

6. The conditions similar to a)–e) are satisfied for the quadruple G, G_0, G_1, G_{01}.

Schematically, these conditions can be represented by the diagram

$$
\begin{array}{ccc}
H & \overset{0}{\longleftrightarrow} & H_0 \\
\updownarrow 1 & & \updownarrow 1 \\
H_1 & \overset{0}{\longleftrightarrow} & H_{01}
\end{array} \ ,
\qquad
\begin{array}{ccc}
G & \overset{0}{\longleftrightarrow} & G_0 \\
\updownarrow 1 & & \updownarrow 1 \\
G_1 & \overset{0}{\longleftrightarrow} & G_{01}
\end{array} \ ,
$$

where the vertical and horizontal arrows stand for modifications and the numbers 0 and 1 next to the arrows indicate the points at which the modifications are carried out.

Next, suppose that four operators

$$A \colon H \to G, \qquad A_0 \colon H_0 \to G_0, \qquad A_1 \colon H_1 \to G_1, \qquad A_{01} \colon H_{01} \to G_{01}$$

are given such that the following conditions are satisfied.

1. The operators A, A_0, A_1, A_{01} are properly Fredholm and coincide on the interval $[\delta, 1 - \delta]$ for some sufficiently small $\delta > 0$.

2. The operator A_0 is a modification of A in a neighborhood of 0.

3. The operator A_1 is a modification of A in a neighborhood of 1.

4. The operator A_0 is a modification of A_{01} in a neighborhood of 1.

5. The operator A_1 is a modification of A_{01} in a neighborhood of 0.

Schematically, these conditions can also be represented by the diagram

$$
\begin{array}{ccc}
A & \xleftrightarrow{\;0\;} & A_0 \\
\updownarrow 1 & & \updownarrow 1 \\
A_1 & \xleftrightarrow{\;0\;} & A_{01}
\end{array}
,
$$

where the vertical and horizontal arrows stand for modifications, and the numbers 0 and 1 next to the arrows indicate the points in neighborhoods of which the modifications occur.

Theorem 1 (the relative index formula) *Under the above conditions, the following relative index formula holds*:

$$
\operatorname{ind} A_0 - \operatorname{ind} A = \operatorname{ind} A_{01} - \operatorname{ind} A_1. \tag{12}
$$

This theorem essentially states that for a properly Fredholm operator in bottleneck spaces, *the index increment due to a modification of the operator on one side of the bottleneck is independent of the structure of the operator on the other side of the bottleneck.* The importance of this assertion is already seen from the above examples of bottleneck spaces and operators in such spaces.

Proof. By assumption, all operators A, A_0, A_1, A_{01} are properly Fredholm. By subtracting appropriate trace class operators, we can ensure, without violating any of the previous conditions, that these operators themselves are δ-narrow and have δ-narrow regularizers modulo smoothing operators. Without loss of generality, we can assume that all the regularizers coincide on the interval $[3\delta, 1 - 3\delta]$. Indeed, let $\widetilde{A}$ be any of the operators A_0, A_1, A_{01}, and let R and $\widetilde{R}$ be given regularizers of A and $\widetilde{A}$, respectively. Thus, R and $\widetilde{R}$ are δ-narrow operators, and

$$
AR = 1 + Q, \qquad RA = 1 + Q', \quad \widetilde{A}\widetilde{R} = 1 + \widetilde{Q}, \qquad \widetilde{R}\widetilde{A} = 1 + \widetilde{Q}',
$$

where $Q, Q', \widetilde{Q}$, and $\widetilde{Q}'$ are trace class. Let e_1, e_2 be a smooth partition of unity on the interval $[0, 1]$ such that

$$\operatorname{supp} e_1 = [0, 3\delta] \cup [1 - 3\delta, 1], \qquad \operatorname{supp} e_2 = [2\delta, 1 - 2\delta].$$

We construct a new regularizer $\widetilde{R}_{\text{new}}$ of $\widetilde{A}$ by setting

$$\widetilde{R}_{\text{new}} = \widetilde{R} \, e_1 + R e_2. \tag{13}$$

The operator (13) is well-defined: since $\operatorname{supp} e_2 = [2\delta, 1 - 2\delta]$ and the operator R is δ-narrow, we have $e_2 h \in \widetilde{G}_{[2\delta, 1-2\delta]} \equiv G_{[2\delta, 1-2\delta]}$ for any $h \in \widetilde{G}$, so that $e_2 h$ can be viewed as an element of G; next, $\operatorname{supp} R e_2 h \subset [\delta, 1 - \delta]$, and hence $R e_2 h$ can be interpreted as a well-defined element of H. (We omit similar arguments in the subsequent considerations.) Obviously, $\widetilde{R}_{\text{new}}$ is a δ-narrow operator, and $\widetilde{R}_{\text{new}} = R$ on $[3\delta, 1 - 3\delta]$. Let us show that $\widetilde{R}_{\text{new}}$ is a regularizer of $\widetilde{A}$ modulo trace class operators. We have

$$\widetilde{A} \widetilde{R}_{\text{new}} = \widetilde{A} \widetilde{R} \, e_1 + \widetilde{A} \, R e_2 = \widetilde{A} \widetilde{R} \, e_1 + A R e_2,$$

since $A = \widetilde{A}$ on $[\delta, 1 - \delta]$. By continuing the computation, we obtain

$$\widetilde{A} \widetilde{R}_{\text{new}} = (1 + \widetilde{Q}) e_1 + (1 + Q) e_2 = 1 + \{\widetilde{Q} \, e_1 + Q e_2\},$$

where the braced operator is trace class. Thus, $\widetilde{R}_{\text{new}}$ is a right regularizer of $\widetilde{A}$. Since $\widetilde{A}$ has a two-sided regularizer $\widetilde{R}$, we can show in the standard way that $\widetilde{R}_{\text{new}}$ is also a left regularizer:

$$
\begin{aligned}
\widetilde{R}_{\text{new}} \widetilde{A} &= (\widetilde{R} \widetilde{A} - \widetilde{Q}') \widetilde{R}_{\text{new}} \widetilde{A} = \widetilde{R} \widetilde{A} \widetilde{R}_{\text{new}} \widetilde{A} - \widetilde{Q}' \widetilde{R}_{\text{new}} \widetilde{A} = \\
&= \widetilde{R} (1 + \widetilde{Q} \, e_1 + Q e_2) \widetilde{A} - \widetilde{Q}' \widetilde{R}_{\text{new}} \widetilde{A} \\
&= 1 + \{\widetilde{Q}' (1 - \widetilde{R}_{\text{new}} \widetilde{A}) + \widetilde{R} (\widetilde{Q} \, e_1 + Q e_2) A\},
\end{aligned}
$$

where the operator in braces is again trace class.

Thus we have shown that the regularizers R, R_0, R_1, R_{01} of all four operators A, A_0, A_1, A_{01} can be assumed to coincide on $[3\delta, 1 - 3\delta]$. We fix the regularizers R and R_{01} and do not change them any more. But we modify the regularizers R_0 and R_1 in a way such that the following conditions be satisfied:

1. The operator R_0 is obtained from R by a modification near 0 and from R_{01} by a modification near 1.

2. The operator R_1 is obtained from R by a modification near 1 and from R_{01} by a modification near 0.

In other words, for the regularizers we have the diagram of modifications

$$
\begin{array}{ccc}
R & \xleftrightarrow{\ 0\ } & R_0 \\
\updownarrow{\scriptstyle 1} & & \updownarrow{\scriptstyle 1} \\
R_1 & \xleftrightarrow{\ 0\ } & R_{01}
\end{array} \quad ,
$$

completely similar to the corresponding diagram for the operators themselves.

Let us show that this construction is possible for R_1 (for R_0, the argument is similar). Let f_1, f_2 be a smooth partition of unity on the interval $[0,1]$ such that $\operatorname{supp} f_1 = [0, 4\delta]$ and $\operatorname{supp} f_2 = [3\delta, 1]$. We set $\widetilde{R}_1 = Rf_1 + R_1 f_2$. Just as above, we can show that $\widetilde{R}_1$ is a regularizer for A_1. Furthermore, it follows from the properties of the operators R and R_1 that $\widetilde{R}$ is a δ-narrow operator, and moreover,

$$
\widetilde{R}_1 = R_1 \quad \text{on } [4\delta, 1], \qquad \widetilde{R}_1 = R \quad \text{on } [0, 1 - 3\delta]. \tag{14}
$$

By modifying $\widetilde{R}$ similarly near 1, we obtain a new δ-narrow regularizer $R_{1\text{new}}$ with the following properties:

$$
R_{1\text{new}} = \widetilde{R}_1 \quad \text{on } [0, 1 - 4\delta], \qquad R_{1\text{new}} = R_{01} \quad \text{on } [3\delta, 1]. \tag{15}
$$

By combining (14) with (15), we conclude that

$$
R_{1\text{new}} = R \quad \text{on } [0, 1 - 4\delta], \qquad R_{1\text{new}} = R_{01} \quad \text{on } [3\delta, 1],
$$

as desired.

Let us now proceed to the proof of the relative index formula (12). It is well known that the index of an arbitrary Fredholm operator A can be expressed by the formula

$$
\operatorname{ind} A = \operatorname{Trace} Q' - \operatorname{Trace} Q, \qquad \text{where } Q = 1 - AR, \quad Q' = 1 - RA,
$$

and R is an arbitrary regularizer of A modulo trace class operators. Let us apply this formula to the four operators in question; we shall use regularizers with the above properties. Then

$$
\begin{aligned}
\operatorname{ind} A &= \operatorname{Trace} Q' - \operatorname{Trace} Q, & \operatorname{ind} A_0 &= \operatorname{Trace} Q_0' - \operatorname{Trace} Q_0, \\
\operatorname{ind} A_1 &= \operatorname{Trace} Q_1' - \operatorname{Trace} Q_1, & \operatorname{ind} A_{01} &= \operatorname{Trace} Q_{01}' - \operatorname{Trace} Q_{01},
\end{aligned}
$$

and, by construction, we have the modification diagrams

$$
\begin{array}{ccc}
Q & \xleftrightarrow{\ 0\ } & Q_0 \\
\updownarrow{\scriptstyle 1} & & \updownarrow{\scriptstyle 1} \\
Q_1 & \xleftrightarrow{\ 0\ } & Q_{01}
\end{array}
\qquad\qquad
\begin{array}{ccc}
Q' & \xleftrightarrow{\ 0\ } & Q_0' \\
\updownarrow{\scriptstyle 1} & & \updownarrow{\scriptstyle 1} \\
Q_1' & \xleftrightarrow{\ 0\ } & Q_{01}'
\end{array}
$$

(the width of the neighborhoods where the modifications occur does not exceed 5δ).

We have

$$\begin{aligned}
\operatorname{ind} A_0 - \operatorname{ind} A &= \operatorname{Trace} Q - \operatorname{Trace} Q_0 + \dots, \\
\operatorname{ind} A_{01} - \operatorname{ind} A_1 &= \operatorname{Trace} Q_1 - \operatorname{Trace} Q_{01} + \dots,
\end{aligned}$$

where the ellipses stand for the corresponding primed terms. Let us prove that

$$\operatorname{Trace} Q - \operatorname{Trace} Q_0 = \operatorname{Trace} Q_1 - \operatorname{Trace} Q_{01}$$

(for the primed terms, the proof is just the same). Let φ_1, φ_2 be a smooth partition of unity on $[0, 1]$ such that

$$\operatorname{supp} \varphi_1 = \left[0, \frac{1}{2} + \delta\right], \qquad \operatorname{supp} \varphi_2 = \left[\frac{1}{2} - \delta, 1\right].$$

We have

$$\begin{aligned}
\operatorname{Trace} Q - \operatorname{Trace} Q_0 &= \operatorname{Trace} Q(\varphi_1 + \varphi_2) - \operatorname{Trace} Q_0(\varphi_1 + \varphi_2) = \\
&= \operatorname{Trace} Q\varphi_1 - \operatorname{Trace} Q_0\varphi_1,
\end{aligned}$$

since $Q\varphi_2$ coincides with $Q_0\varphi_2$ by our assumptions.[4] By continuing this chain of equalities, for similar reasons we obtain

$$\begin{aligned}
\operatorname{Trace} Q\varphi_1 - \operatorname{Trace} Q_0\varphi_1 &= \operatorname{Trace} Q_1\varphi_1 - \operatorname{Trace} Q_{01}\varphi_1 = \\
&= \operatorname{Trace} Q_1\varphi_1 - \operatorname{Trace} Q_{01}\varphi_1 + \\
&\quad + \operatorname{Trace} Q_1\varphi_2 - \operatorname{Trace} Q_{01}\varphi_2 = \\
&= \operatorname{Trace} Q_1 - \operatorname{Trace} Q_{01},
\end{aligned}$$

as desired. The proof of the theorem is complete. $\square$

A formula of the form (12) for the special case of signature operators on Lipschitz manifolds was proved by N. Teleman ([8], § 12, the excision theorem). The proof given there uses the special structure of signature operators in an essential way.

2 The relative index for a special surgery on the manifold

In the present section we study how the index of an operator that is "locally" an elliptic ψDO on a manifold is changed if one performs surgery on the manifold where the

---

[4]Recall that δ was assumed to be small; now we see that it suffices to take $\delta < \frac{1}{12}$.

operator is defined and some associated surgery on the bundles. The index increment formula can naturally be treated as a *relative index formula*. The informal motivation is as follows: the symbol before and after the surgery is essentially the same; we deal with different realizations of the same symbol, which depend on the way in which the manifold was glued from pieces.[5]

2.1 Surgery on manifolds and bundles

Let M be an orientable manifold, and let $S \subset M$ be an embedded smooth compact two-sided submanifold of codimension 1 contained in the smooth "interior" part of M. Next, let U be a collar neighborhood of S contained in the smooth part of M. We choose and fix some trivialization $U = (-1, 1) \times S$ of this neighborhood and use the coordinates (t, s), $t \in (0, 1)$, $s \in S$, there. Let $g : S \to S$ be a given diffeomorphism. We perform the following operation: we cut M along S and glue together again, identifying each point $(-0, s)$ on the left coast of the cut with the corresponding point $(+0, g(s))$ on the right coast. The resultant smooth manifold (the smooth structure is well defined, since we have chosen and fixed the trivialization) will be denoted by M_g and called the *surgery of M via g*.

Let E be a vector bundle over M. Suppose that we are given an isomorphism of vector bundles

$$\mu : E|_S \to g^* \left(E|_S \right).$$

Then over M_g there is a naturally defined vector bundle $E_{g,\mu}$ (by attaching along S with the help of μ), which will be called the *surgery of E via the pair (g, μ)*.

Now let E, F be two vector bundles over M, and let

$$a : \pi^* E \to \pi^* F,$$

where $\pi : T_0^* M \to T_0^* M$ is the natural projection, be an elliptic symbol of some order m defined at least in $\pi^{-1}(U)$. By choosing the representation

$$E|_U = \widetilde{\pi}^{-1} \left(E|_S \right), \qquad F|_U = \widetilde{\pi}^{-1} \left(F|_S \right)$$

of the bundles E, F over $U = [0, 1] \times S$, where $\widetilde{\pi} : [0, 1] \times S \to S$ is the natural projection, and by passing to a homotopic symbol if necessary, we can assume that a is independent of the coordinate t in a sufficiently small neighborhood of S (that is, $a \equiv a_0$ in that neighborhood). Consider the mapping (denoted by the same letter)

$$a_0 \stackrel{\text{def}}{=} a|_{\pi^{-1}S} : \pi^* E|_{\pi^{-1}S} \longrightarrow \pi^* F|_{\pi^{-1}S}. \tag{16}$$

---

[5]We consider only the geometrically simplest surgery (see below).

With regard to the trivialization chosen, this mapping can be rewritten in the form

$$a_0(p, s, \xi) \colon E_s \longrightarrow F_s, \qquad p^2 + |\xi|^2 \neq 0, \quad s \in S,$$

where p is the dual variable of t and ξ is a point in the fiber of T^*S over s.

2.2   The relative index theorem

Suppose that a surgery g of M and associated surgeries μ_E and μ_F of the bundles E and F are given. If the diagram

$$
\begin{array}{ccc}
E_s & \xrightarrow{\quad a_0(p, s, \xi) \quad} & F_s \\
\downarrow \mu_E(s) & & \downarrow \mu_F(s) \\
E_{g(s)} & \xrightarrow{\quad a_0(p, g(s), {}^t g_s(s)^{-1}\xi) \quad} & F_{g(s)}
\end{array}
\qquad (17)
$$

where ${}^t g_s(s)$ is the transposed Jacobi matrix of the mapping g at the point s, commutes, then the surgery takes the original symbol a to a new smooth symbol $\widetilde{a}$ on the cotangent bundle $T_0^* M_g$. (The smoothness of the newly obtained symbol is guaranteed by the independence of a on the coordinate t in a neighborhood of S.) We intend to find out how the surgery affects the index of an operator that locally coincides with a pseudodifferential operator with principal symbol a. Since actually this index increment (the relative index) depends only on the surgery on S, we use the results of the preceding section to pass to the corresponding "local" model.

Thus, let A be a properly Fredholm operator in bottleneck spaces coiciding in a neighborhood of zero with an elliptic pseudodifferential operator with principal symbol a. Next, let $\widetilde{A}$ be a properly Fredholm operator obtained from A by a modification in a neighborhood of zero and such that in this neighborhood $\widetilde{A}$ coincides with an elliptic operator with principal symbol $\widetilde{a}$. The problem is to find the relative index $\operatorname{ind} \widetilde{A} - \operatorname{ind} A$.

It follows from Theorem 1 that the relative index is independent of the structure of the operators in question outside a small neighborhood of S. Hence we can use the simplest model for the computations. Namely, consider the manifold $M = S \times S^1$ and the elliptic pseudodifferential operator

$$A_0 : H^s(M, E) \to H^{s-m}(M, F)$$

(the bundles E and F are lifted to M with the help of the natural projection $M = S \times S^1 \to S$) with principal symbol a_0 independent of $\varphi \in S^1$. Next, let M_g be the surgery of M with the help of g, let E_{g,μ_E} and E_{g,μ_F} be the associated surgeries of the bundles E and F, and let

$$\hat{A}_0 \colon H^s(M_g, E_{g,\mu_E}) \to H^{s-m}(M_g, E_{g,\mu_F})$$

be the new elliptic pseudodifferential operator with principal symbol $\widetilde{a}_0$ coinciding with A_0 outside a neighborhood of the set S, where the surgery is done.

The operators A_0 and $\widetilde{A}_0$ are elliptic operators on compact manifolds, and their index can be calculated by the Atiyah–Singer theorem. The index of A_0 is zero, since its symbol is independent of $\varphi \in S^1$. Hence in this model only one term in the expression for the relative index is nontrivial:

$$\operatorname{ind} \widetilde{A} - \operatorname{ind} A = \operatorname{ind} \widetilde{A}_0 \ .$$

Let us state the assertion that we have just proved in the from of a theorem.

Theorem 2 *Let S be a smooth compact manifold without boundary, and let H and G be bottleneck spaces coinciding in a neighborhood of zero with the spaces $H^s(C, E)$ and $H^{s-m}(C, F)$, respectively, where E and F are vector bundles over the manifold $C = (-1, 1) \times S$. Next, let*

$$A \colon H \longrightarrow G$$

be a properly Fredholm operator coinciding in a neighborhood of zero with an elliptic pseudodifferential operator with principal symbol a of order m on $T_0^* C$. Let b be the restriction (16) of the symbol a to $\pi^{-1} S$. Suppose that we are given some surgery of C via a diffeomorphism $g \colon S \to S$ and associated surgeries μ_E and μ_F of the bundles E and F, and moreover, the diagram (17) commutes. Let $\widetilde{A}$ be the corresponding surgery of the operator A. Then the relative index of the surgery is given by the formula

$$\operatorname{ind} \widetilde{A} - \operatorname{ind} A = \operatorname{ind} \widetilde{A}_0,$$

where $\widetilde{A}_0$ is the operator with symbol $\widetilde{a}_0$ on $T_0^*(S \times S^1)_g$ obtained by surgery from the homomorphism a_0 lifted to $T_0^*(S \times S^1)$ with the help of the natural projection along S^1.

2.3 Some remarks

Remark 1 Needless to say, the (analytical) index of the operator $\widetilde{A}_0$, which is an elliptic pseudodifferential operator on the compact closed manifold $(S \times S^1)_g$, can be represented as the topological index of its symbol via the Atiyah–Singer theorem.

Remark 2 The assertion of Theorem 2 for the case in which A is a Dirac type operator on a compact closed manifold M divided into two parts M_+ and M_- by a smooth hypersurface S was proved by Booß-Bavnbek and Wojciechowski ([7, Chap. 25, pp. 276–281]). The assumption that A is a Dirac type operator is not essential to their proof.

However, it *is* essential that A is an elliptic operator on the *manifold M*, since their proof is based on the "local index formula" in this situation:

$$\operatorname{ind}(A) = \int\limits_M \alpha(x),$$

where $\alpha(x)$ is a differential form that depends only on the coefficients of A (and there derivatives) at x (see Gilkey [1]). In our theorem, it is only assumed that A is a pseudodifferential operator "locally" (in a neighborhood of S), while globally it need not be an operator on any smooth manifold. All we need to "localize" the index is proper Fredholm property; we do not have to pre-suppose the existence of any "local" expression of the index via the principal symbol (which is in the general situation defined only as an element of the Calkin algebra).

Remark 3 We can also readily prove the more general surgery theorem in which the operator A is locally (say, in a neighborhood of $t = 0$) a pseudodifferential operator on a manifold M in the vicinity of some domain $G \subset M$ with smooth boundary $S = \partial G$, and the surgery cuts away the domain G along S and replaces it by another domain G_1 with the same boundary $S = \partial G_1$. For the case in which A is a Dirac operator or an arbitrary formally self-adjoint elliptic differential operator of the first order, such theorems were proved, respectively, by Gromov and Lawson [2] and Anghel [3] (see also Donnelly [4], Bunke [5], and Lesch [6]).

3 The spectral flow for conormal families

Here we generalize the notion of the spectral flow [17] to arbitrary conormal families. (For the case of polynomial familes, e.g., see [9].)

3.1 The multiplicity of a singular point of an operator family

Let $H_{1,2}$ be Hilbert spaces, and let

$$D(p) \colon H_1 \to H_2$$

be a family of bounded linear operators with parameter p ranging in an open subset $\mathcal{U}$ of the complex plane. To shorten lengthy statements, we introduce the following definition.

Definition 3 The family $D(p)$ is said to be *strongly finitely meromorphic* if it is finitely meromorphic and finite-meromorphically invertible (that is, $D(p)$ is a meromorphic function of $p \in \mathcal{U}$, and the principal part of the Laurent series at each of the poles is

a finite rank operator; the same is assumed to be true of $D^{-1}(p)$). The *singular points* of a strongly finitely meromorphic family $D(p)$ are the poles of $D(p)$ and the poles of $D^{-1}(p)$.

Now we can give the definition of multiplicity.

Definition 4 The *multiplicity* of a singular point p_0 of a strongly finitely meromorphic family $D(p)$ is the number

$$m_D(p_0) = \operatorname{Trace} \operatorname*{Res}_{p=p_0} \left\{ D^{-1}(p) \frac{\partial D(p)}{\partial p} \right\}. \tag{18}$$

For convenience, we extend the definition of multiplicity to an arbitrary point p_0 by setting $m_D(p_0) = 0$ if p_0 is a pole of neither D nor D^{-1}.

If the family D in question is clear from the context, then instead of $m_D(p_0)$ we write simply $m(p_0)$.

Theorem 3 *The following assertions hold.*

1. *The multiplicity of a singular point of a strongly finitely meromorphic family is always an integer.*

2. *The multiplicities of singular points are homotopy invariant in the following sense. Let $\mathcal{B} \subset\subset \mathcal{U}$ be a domain with smooth boundary $\partial\mathcal{B}$. If D_t, $t \in [0,1]$ is a continuous homotopy of strongly finitely meromorphic families such that $\partial\mathcal{B}$ is free of singular points of D_t for every t, then the sum of multiplicities of the singular points of D_t lying in $\mathcal{B}$ is independent of t.*

3. *The multiplicity possesses the "logarithmic property"*

$$m_{D_1 D_2}(p) = m_{D_1}(p) + m_{D_2}(p). \tag{19}$$

   In particular, the multiplicity of a singular point does not change if we multiply the family by an operator function that is holomorphic and holomorphically invertible at that point.

4. *Along with (18), one has the formulas*

$$
\begin{aligned}
m_D(p_0) &= \operatorname{Trace} \operatorname*{Res}_{p=p_0} \left\{ \frac{\partial D(p)}{\partial p} D^{-1}(p) \right\} \tag{20}\\
&= -\operatorname{Trace} \operatorname*{Res}_{p=p_0} \left\{ \frac{\partial D^{-1}(p)}{\partial p} D(p) \right\} = -\operatorname{Trace} \operatorname*{Res}_{p=p_0} \left\{ D(p) \frac{\partial D^{-1}(p)}{\partial p} \right\}.
\end{aligned}
$$

19

Proof. Proof. Assertion 2) follows from 1) and the Cauchy integral formula. To prove 1), 3), and 4), note that the following assertion holds.

Lemma 2 *Let $\mathcal{A}$ be the algebra of meromorphic operator Laurent series at the point $p = 0$ with finite-dimensional principal parts. Let us define a functional tr on $\mathcal{A}$ by the formula*

$$\mathrm{tr}\, a = \mathrm{Trace}\, \underset{p=0}{\mathrm{Res}}\, a, \qquad a \in \mathcal{A}.$$

Then

$$\mathrm{tr}(ab) = \mathrm{tr}(ba)$$

for every $a, b \in A$.

The validity of the lemma can be verified by a simple calculation. Now the first equality in (20) is obvious, and the following two are obtained by differentiating the identity $DD^{-1} = D^{-1}D = 1$. Let us verify 3). Let $D = D_1 D_2$. Then

$$D^{-1}\frac{\partial D}{\partial p} = D_2^{-1} D_1^{-1}\left(\frac{\partial D_1}{\partial p} D_2 + D_1 \frac{\partial D_2}{\partial p}\right) = D_2^{-1}\left(D_1^{-1}\frac{\partial D_1}{\partial p}\right)D_2 + D_2^{-1}\frac{\partial D_2}{\partial p},$$

and we obtain the desired assertion by applying Lemma 2.

It remains to prove assertion 1). We carry out the proof by two different methods. The first method reveals the relationship between $m_D(p_0)$ and the (obviously integer) traces of finite-dimensional spectral projections. (For the case in which $D(p)$ is a polynomial operator pencil, our construction is essentially standard; it is however interesting that everything can be done in the nonpolynomial case as well.) However, this method fails if the point p_0 is a pole of $D_0(p)$ and $D_0^{-1}(p)$ simultaneously. The second method can handle this case equally well; it expresses $m_D(p_0)$ via the variation of the argument of some finite-dimensional determinant.

We start from the first method. Without loss of generality, we can assume that $H_1 = H_2 = H$ and the singular point is $p_0 = 0$. Suppose that this is a pole of D^{-1} (for the case in which p_0 is a pole of D, the proof is similar in view of assertion 4)).

Suppose first that the family $D(p)$ has the special form

$$D(p) = p - A. \tag{21}$$

Then

$$\underset{p=0}{\mathrm{Res}}\left\{D^{-1}(p)\frac{\partial D(p)}{\partial p}\right\} = \frac{1}{2\pi i}\oint_{|p|=\varepsilon}\frac{dp}{p - A}$$

is a projection on the finite-dimensional root space of A corresponding to the eigenvalue 0, and $m(0)$ is the trace of this projection and hence is equal to the dimension of this root space. Next, let $D(p)$ be a polynomial family of the form

$$D(p) = D_0 + pD_1 + \ldots + p^{k-1}D_{k-1} + p^k \tag{22}$$

20

(the leading coefficient is the identity operator). This case can be reduced to case (21) by the following standard trick: we consider the operator family

$$\tilde{D}(p) = \begin{pmatrix} p & -1 & 0 & \dots & 0 \\ 0 & p & -1 & \dots & 0 \\ \dots & \dots & \ddots & \ddots & \dots \\ 0 & 0 & 0 & \dots & -1 \\ D_0 & D_1 & D_2 & \dots & D_{k-1}+p \end{pmatrix} \tag{23}$$

in the space $H \oplus H \oplus \dots \oplus H$ (k summands). This family already has the form (21), and routine (although lengthy) computations show that

$$\mathrm{Trace}\,\mathop{\mathrm{Res}}_{p=0}\left\{ D^{-1}\frac{\partial D}{\partial p} \right\} = \mathrm{Trace}\,\mathop{\mathrm{Res}}_{p=0}\left\{ \tilde{D}^{-1}\frac{\partial \tilde{D}}{\partial p} \right\},$$

which proves that the multiplicity is an integer. Finally, let us get rid of the assumption that $D(p)$ is a polynomial. Since $D^{-1}(p)$ has a finite-order pole at $p = 0$, we can write

$$D^{-1}(p) = p^{-N}B(p)$$

where $B(p)$ is holomorphic in a neighborhood of zero. Let us expand $D(p)$ in the Taylor series with remainder:

$$D(p) = \sum_{j=0}^{N} p^j D_j + p^{N+1}R_{N+1}(p) \equiv C(p) + p^{N+1}R^{N+1}(p)$$

(here $C(p)$ is a polynomial of degree N). Then

$$C(p) = \left[D(p) - p^{N+1}R_{N+1}(p) \right] D^{-1}(p)D(p) = \left[1 - pR_{N+1}(p)B(p) \right] D(p)$$

where the operator function $[1 - pR_{N+1}(p)B(p)]$ is holomorphic and holomorphically invertible in a neighborhood of zero. By 3), $m_D(0) = m_C(0)$. The same reasoning shows that $m_C(0) = m_F(0)$, where $F(p) = C(p) + p^{N+1}$. We have thus arrived to the already known case (22) of a monic polynomial.

Let us now carry out the proof by the second method. Consider the Laurent expansions

$$D(p) = \frac{D_{-N}}{p^N} + \dots + \frac{D_{-1}}{p} + D_0 + D_1 p + \dots,$$

$$D^{-1}(p) = \frac{C_{-M}}{p^M} + \dots + \frac{C_{-1}}{p} + C_0 + C_1 p + \dots$$

21

(one of the numbers M and N may be zero). Consider the decompositions

$$H = G_0 \oplus G_1 = H_0 \oplus H_1,$$

where the finite-dimensional spaces G_0 and H_0 are the linear spans of the ranges of the principal parts of the Laurent series for $D(p)$ and $D^{-1}(p)$, respectively. In these decompositions, the families $D(p)$ and $D^{-1}(p)$ are given by the block matrices

$$D(p) = \begin{pmatrix} D_{00}(p) & D_{01}(p) \\ D_{10}(p) & D_{11}(p) \end{pmatrix}, \quad D^{-1}(p) = \begin{pmatrix} C_{00}(p) & C_{01}(p) \\ C_{10}(p) & C_{11}(p) \end{pmatrix},$$

where only the entries in the first row (finite rank operators) may be nonholomorphic at zero. Multiplying these two matrices in the two different orders, in the right lower corner we obtain the identities

$$D_{11}(p)C_{11}(p) = 1 - D_{10}(p)C_{01}(p) \equiv 1 + Q(p),$$

$$C_{11}(p)D_{11}(p) = 1 - C_{10}(p)D_{01}(p) \equiv 1 + \widetilde{Q}(p). \tag{24}$$

Obviously, $Q(p)$ and $\widetilde{Q}(p)$ are of finite rank. Moreover, they are holomorphic at zero, since so are the left-hand sides in (24). Thus, $D_{11}(p)$ is holomorphic at zero and Fredholm for small $|p|$. Consider the new decompositions

$$H = \widetilde{H}_0 \oplus \widetilde{H}_1 = \widetilde{G}_0 \oplus \widetilde{G}_1,$$

where

$$H_1 = \operatorname{Ker} D_{11} \oplus \widetilde{H}_1, \qquad \widetilde{H}_0 = H_0 \oplus \operatorname{Ker} D_{11};$$

$$G_1 = \operatorname{Coker} D_{11} \oplus \widetilde{G}_1, \quad \widetilde{G}_0 = G_0 \oplus \operatorname{Coker} D_{11}.$$

In these decompositions, the family $D(p)$ is given by the block matrix

$$D(p) = \begin{pmatrix} \widetilde{D}_{00}(p) & \widetilde{D}_{01}(p) \\ \widetilde{D}_{10}(p) & \widetilde{D}_{11}(p) \end{pmatrix},$$

where the entries in the lower row are holomorphic at zero, the entries in the upper row can have poles there, and the family $\widetilde{D}_{11}(p)$ is invertible in a neighborhood of zero.

Consider the LU-decomposition

$$D(p) = \begin{pmatrix} \widetilde{D}_{00}(p) - \widetilde{D}_{01}(p)\,\widetilde{D}_{11}^{-1}(p)\,\widetilde{D}_{10}(p) & \widetilde{D}_{01}(p) \\ 0 & \widetilde{D}_{11}(p) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \widetilde{D}_{11}^{-1}(p)\,\widetilde{D}_{10}(p) & 1 \end{pmatrix}.$$

Note that the second factor is holomorphic and has a holomorphic inverse. Simple calculations with the use of the logarithmic property yield

$$m_D(0) = m_B(0),$$

22

where
$$B(p) = \tilde{D}_{00}(p) - \tilde{D}_{01}(p)\,\tilde{D}_{11}(p)\,\tilde{D}_{10}(p)$$
is a family of *finite-dimensional* operators. Then we have
$$
\begin{aligned}
m_B(0) & = \ \mathrm{Trace}\,\mathrm{Res}_{p=0}\,B^{-1}(p)\frac{\partial B(p)}{\partial p} = \mathrm{Res}_{p=0}\,\mathrm{Trace}\,B^{-1}(p)\frac{\partial B(p)}{\partial p} \\
& = \ \frac{1}{2\pi i}\int\limits_{|p|=\varepsilon}\mathrm{Trace}\left(B^{-1}(p)\frac{\partial B(p)}{\partial p}\right)\,dp.
\end{aligned}
$$

Lemma 3 *The following formula is valid*:
$$\mathrm{Trace}\ \mathrm{Res}\ B^{-1}(p)\frac{\partial B(p)}{\partial p} = \frac{1}{2\pi}\mathop{\mathrm{Var}}_{|p|=\varepsilon}\ \arg \det B(p).$$

To prove the formula, we use noncommutative analysis [18]. For every p on the circle $|p| = \varepsilon$, on the Riemann surface of the function $f(\lambda) = \ln\lambda$ we choose a domain containing the spectrum of $B(p)$ in a way such that this domain varies continuously as $\arg p$ ranges over $[0, 2\pi]$. Then the function $f(B(p))$ defined by the Cauchy integral over the boundary of this domain depends on p smoothly on the circle everywhere except for the point $p = 1$, where it can undergo a jump. We have

$$
\begin{aligned}
\frac{\partial}{\partial p}\mathrm{Trace}\ln B(p) & = \ \mathrm{Trace}\frac{\partial}{\partial p}\ln(B(p)) \\
& = \ \mathrm{Trace}\frac{\partial B(p)}{\partial p}\frac{\delta\ln}{\delta\lambda}\left(\overset{1}{B}(p), \overset{3}{B}(p)\right) \\
& = \ \{\text{by the cyclic invariance of the trace}\} \\
& = \ \mathrm{Trace}\ \frac{\partial\overset{2}{B}(p)}{\partial p}\overset{3}{B}(p)^{-1} = \mathrm{Trace}\left(B^{-1}(p)\frac{\partial B}{\partial p}\right).
\end{aligned}
$$

Since $\mathrm{Trace}\ln B(p) = \ln\det B(p)$, we eventually obtain

$$\mathrm{Trace}\ \mathrm{Res}_{p=0}\ B^{-1}(p)\frac{\partial B(p)}{\partial p} = \frac{1}{2\pi}\mathop{\mathrm{Var}}_{\substack{|p|>\varepsilon \\ \arg p\in[0,2\pi]}}\arg\ \det B(p), \tag{25}$$

whence it readily follows that the multiplicity is an integer.

Formula (25) has a strongly finitely meromorphic interpretation if $D(p)$ itself is holomorphic at zero. Then

$$B(p)\ :\ \mathrm{Ker}\,D(0) \to \mathrm{Coker}\,D(0)$$

is none other than the noncommutative determinant associated with $D(p)$. $\qquad\square$

3.2 The definition of the spectral flow of a family of conormal symbols

The facts given in this item are closely related to results obtained in [19].

Let Ω be a smooth compact manifold without boundary, and let

$$D(p)\colon H^s(\Omega, E) \to H^{s-m}(\Omega, F)$$

be a family of mth-order pseudodifferential operators on Ω holomorphically[6] (for simplicity) depending on the parameter $p \in \mathbf{C}$ and elliptic in the sense of Agranovich–Vishik in some double sector of nonzero angle containing the real axis (the weight of the parameter in the ellipticity condition is equal to m). Then, obviously, this family is strongly finitely meromorphic and the multiplicities of its singular points (the poles of the inverse family) are well-defined integers. In what follows such families will be referred to as *conormal symbols* on Ω. For simplicity, everywhere except for examples we assume that $m = 0$. This restriction is unessential.

Let $D_t = D_t(p)$, $t \in [0,1]$, be a continuous family of conormal symbols on Ω. We intend to define the notion of the *spectral flow* sf $D_t \equiv \text{sf}_{t=0,1} D_t$ of the family D_t. The intuitive definition is as follows: sf D_t is the net number of singular points (with regard to multiplicity) of the family D_t that crossed the real axis upwards (more precisely, that passed from the open lower half-plane to the closed upper half-plane) as t varies from 0 to 1. This is a natural generalization of the notion of the spectral flow of a family A_t of normally elliptic operators; the latter is a special case of the former for $D_t = p - A_t$. However, the problem is to give the precise definition. In our case, the notion of a spectral section (for $D_t \neq p - A_t$) is not known, and we use a construction different from that in [9] to define the spectral flow.

Suppose first (to simplify the formulas) that for $t = 0$ and $t = 1$ the conormal symbol $D_t(p)$ has no singular points on the real axis. (In what follows this restriction will be removed). We divide the interval $[0,1]$ into subintervals by some points

$$0 = t_0 < t_1 < \ldots < t_N = 1$$

and choose real numbers γ_i, $i = 1, \ldots, N$ so that the operator $D_t(p)$ be invertible on the "weight line" $\text{Im}\, p = \gamma_i$ for all $t \in [t_{i-1}, t_i]$. Moreover, we must have $\gamma_1 = \gamma_N = 0$. The existence of this partition and the numbers γ_i follows from the ellipticity with parameter in a sector of nonzero angle and from the continuous dependence of the conormal symbol on t.

Definition 5 The *spectral flow* of the family $\{D_t\}_{t \in [0,1]}$ of conormal symbols is the number

$$\underset{t=0,1}{\text{sf}}\, D_t = \sum_{i=1}^{N-1} g_i, \tag{26}$$

---

[6]Needless to say, all we need is strongly finite meromorphy in some strip containing the real axis.

where

$$g_i \;=\; \begin{cases} -\displaystyle\sum_{\operatorname{Im} p_j \in (\gamma_i,\gamma_{i+1})} m_{D_{t_i}}(p_j), & \text{if } \gamma_i < \gamma_{i+1}; \\[2mm] \displaystyle\sum_{\operatorname{Im} p_j \in (\gamma_{i+1},\gamma_i)} m_{D_{t_i}}(p_j), & \text{if } \gamma_i \geq \gamma_{i+1}. \end{cases}$$

Here the p_j are the singular points of the conormal symbol D_{t_i} in that strip. (Note that there are finitely many such points, so that the sum is well defined.)

Instead of $\operatorname*{sf}_{t=0,1} D_t$ we usually write simply sf D_t.

Theorem 4 *The following assertions hold.*

1. *Definition 5 is independent of the choice of the partition of the interval $[0,1]$ and the numbers γ_j.*

2. *sf D_t depends only on the homotopy class of the path D_t (with fixed beginning and end). Moreover, sf D_t remains constant also for deformations (homotopies) of the path D_t such that the beginning and the end are not fixed, but the conormal symbols D_0 and D_1 have no singular points on the real axis for any value of the homotopy parameter.*

Proof. Let us prove 1). Suppose that we have two partitions of the interval $[0,1]$ and two sets of numbers γ_j. By passing to refinements, we can assume that the partitions coincide. Then the passage from one expression for sf D_t to the other is given by a series of elementary operations, each of which is the replacement of the number γ_j on some interval $[t_{j-1},t_j]$ by the new number $\widetilde{\gamma}_j$, where the conormal symbol D_t is invertible on the lines $\operatorname{Im} p = \gamma_j$ and $\operatorname{Im} p = \widetilde{\gamma}_j$ for all $t \in [t_{j-1},t_j]$. Let us show that this operation does not change the value of the expression for sf D_t. Indeed, let, for example, $\gamma_j < \widetilde{\gamma}_j$. Then the expression for sf D_t acquires the increment

$$\Delta(t_j) - \Delta(t_{j-1}), \qquad \text{where } \Delta(t) = \sum_{\operatorname{Im} p_k \in (\gamma_j,\widetilde{\gamma}_j)} m_{D_t}(p_k),$$

and the p_k are the singular points of the conormal symbol D_t (which, of course, depend on t). The ellipticity condition with parameter implies that there exists an $R > 0$ such that for $|\operatorname{Re} p| > R$ and $\operatorname{Im} p \in [\gamma_j,\widetilde{\gamma}_j]$ the operator $D_t(p)$ is invertible for all $t \in [t_{j-1},t_j]$. Then, by the cauchy theorem, we can write

$$\Delta(t) = \oint_\Gamma D_t^{-1}(p)\frac{\partial D_t(p)}{\partial p}\,dp, \qquad t \in [t_{j-1},t_j],$$

25

where Γ is the closed contour formed by the four segments

$$\{\operatorname{Re} p = R,\ \operatorname{Im} p \in [\gamma_j, \tilde{\gamma}_j]\},\ \{\operatorname{Re} p = -R,\ \operatorname{Im} p \in [\gamma_j, \tilde{\gamma}_j]\},$$
$$\{\operatorname{Im} p = \gamma_j,\ \operatorname{Re} p \in [-R, R]\}\ \text{and}\ \{\operatorname{Im} p = \tilde{\gamma}_j,\ \operatorname{Re} p \in [-R, R]\}.$$

This expression is continuous in t and integer-valued, so that $\Delta(t_j) = \Delta(t_{j-1})$, as desired.

The validity of assertion 2) follows in a standard way from assertion 2) of Theorem 3.
$\square$

Now we can generalize Definition 5 to the case in which $D_0(p)$ and $D_1(p)$ may have singular points on the real axis. The condition of ellipticity with parameter implies that there is an $\varepsilon > 0$ such that the open strip $-\varepsilon < \operatorname{Im} p < 0$ contains no singular points of the conormal symbols $D_0(p)$ and $D_1(p)$. Consider the family of conormal symbols

$$D_{t,\tau}(p) = D_t(p - i\tau)$$

with additional real parameter $\tau \in (0, \varepsilon)$. Then $D_{0,\tau}$ and $D_{1,\tau}$ have no singular points on the real axis, and by Theorem 4, 2), the spectral flow sf $D_{t,\tau}$ is independent of $\tau \in (0, \varepsilon)$. By definition, we set

$$\text{sf } D_t = \text{sf } D_{t,\varepsilon/2}.$$

3.3 Properties of the spectral flow

Let us establish some properties of the spectral flow for families of conormal symbols.

Theorem 5 *Let*
$$D_t(p)\colon H^s(\Omega, E) \to H^s(\Omega, F), \qquad t \in [0, 1]$$

be a family of conormal symbols on Ω such that D_0 and D_1 have no singular points on the real axis. Let $\chi\colon (-\infty, \infty) \to [0, 1]$, $\chi(-\infty) = 0$, $\chi(\infty) = 1$, be a smooth function whose derivatives rapidly decay at infinity. We set

$$D(\tau, p) = D_{\chi(\tau)}(p)$$

and consider the operator

$$\hat{D} \overset{\text{def}}{=} D\left(\overset{2}{\tau}, -i\,\frac{\overset{1}{\partial}}{\partial\tau}\right) : H^s(C, E) \longrightarrow H^s(C, F)$$

26

in the Sobolev spaces[7] on the infinite cylinder $C = \Omega \times (-\infty, \infty)$. Then

$$\text{sf } D_t = -\text{ind } \hat{D} .$$

Proof. Let us try to deform the family D_t into a constant (independent of t) family $\bar{D}_t \equiv D_0$ by setting

$$\tilde{D}_{t,\lambda} = D_{t\lambda}$$

(here $\lambda \in [0, 1]$ is the homotopy parameter). For $\lambda = 0$ the index and the spectral flow are both zero, and for $\lambda = 1$ we obtain the index and the spectral flow of the desired operators. Next, this homotopy changes neither the index of the operator on the cylinder nor the spectral flow as long as the conormal symbol $D_\lambda(p)$ is invertible on the weight line $\{\text{Im } p = 0\}$. To bypass the points where the singularities of the conormal symbol sit on the real axis, we change the weight line, essentially following our definition of the spectral flow. Since in this case both the spectral flow (which follows from our definition) and the index of the operator on the infinite cylinder change by the sum of multiplicities of singular points of the conormal family $D_\lambda(p)$ in the corresponding strip, but with opposite signs. This completes the proof. $\square$

In the following theorem we calculate the spectral flow of a *periodic* family of conormal symbols.

Theorem 6 *Let D_t, $t \in [0, 2\pi]$, be a periodic family of conormal symbols on a manifold Ω, that is, $D_0 = D_{2\pi}$. Then*

$$\underset{t=0,2\pi}{\text{sf}} D_t = -\text{ind } \hat{D},$$

where

$$\hat{D} = D_{\frac{2}{t}} \left(\frac{\overset{1}{\partial}}{\partial t} \right)$$

is the operator on the product $\Omega \times S^1$ naturally obtained from the periodic (in t) operator-valued symbol given by the family D_t.

Proof. Since the index of a periodic family is homotopy invariant, we can assume without loss of generality that D_0 has no singular points on the real axis (otherwise, we can apply a small deformation of the form $D_t(p) \to D_t(p + i\varepsilon)$). Let

$$\chi \colon (-\infty, \infty) \longrightarrow [0, 2\pi]$$

---

[7] We denote the lifts of the bundles E and F to C by the same letters. The weight line occurring in the definition of weighted Sobolev spaces is $\{\text{Im } p = 0\}$ in our case, and so we omit the weights in the notation for spaces.

be a smooth mapping such that $\chi(\tau) = 0$ on $(-\infty, -1]$, $\chi(\tau) = 2\pi$ on $[1, \infty)$, and $\chi(\tau)$ is monotone increasing on $(-1, 1)$. By the preceding theorem, the operator $\hat{B}$ on the infinite cylinder $C = \Omega \times (-\infty, \infty)$ with operator-valued symbol $B(\tau, p) = D_{\chi(\tau)}(p)$ has the index

$$\text{ind } \hat{B} = - \underset{t=0,2\pi}{\text{sf}} \; D_t.$$

Now let us use Theorem 2. Let $S \subset C$ be the submanifold

$$S = [\Omega \times \{-\pi\}] \cup [\Omega \times \{\pi\}].$$

We cut C along Ω, glue the resulting one-way infinite cylinders into a new infinite cylinder C_1, and glue together the ends of the finite cylinder $\Omega \times [-\pi, \pi]$ to obtain the product $\Omega \times S^1$. The bundles are glued with the help of the identity homomorphism (note that our bundles are independent of t, that is, constant along the cylinder). The operator thus obtained on the "torus" is homotopic to $\hat{D}$, and the coefficients of the operator on C_1 are independent of τ, so that its index is zero. By Theorem 2, the index of the original operator on C is equal to the sum of indices of the newly obtained operators on C_1 and $\Omega \times S^1$, which completes the proof. $\qquad\square$

A similar theorem holds for an operator family with conjugate beginning and end.

4   Some applications

Here the results of the preceding sections are applied to index calculations for operators on manifolds with conical singularities and for boundary value problems with the main operator satisfying the symmetry condition.

4.1   The index theorem for elliptic pseudodifferential operators on manifolds with conical singularities

Let us now apply the above results to prove an index theorem for elliptic pseudodifferential operators on manifolds with conical singularities. This theorem includes the previously known ones as special cases.

Let M be a compact manifold without boundary with conical singularities (without loss of generality, we assume that there is only one conical point α; the base Ω of the corresponding cone is not assumed to be connected), and let

$$\hat{D} \colon H^{s,0}(M, E) \longrightarrow H^{s-m,0}(M, F)$$

be an elliptic pseudodifferential operator on M with conormal symbol $D_0(p)$.

Next, let $g \colon \Omega \to \Omega$ be a diffeomorphism, and let

$$\mu_E : E|_\Omega \longrightarrow g^*(E_\Omega), \qquad \mu_F : F|_\Omega \longrightarrow g^*(F_\Omega)$$

be bundle homomorphisms. Suppose that the following condition is satisfied.

Condition A. *The family $D_0(p)$ is homotopic to the family $\mu_F^{-1} g^* D_0(-p)(g^*)^{-1} \mu_E$ in the class of conormal families.*[8]

We denote the homotopy by D_{0t},

$$D_{00}(p) = D_0(p), \qquad D_{01}(p) = \mu_F^{-1} g^* D_0(-p)(g^*)^{-1} \mu_E.$$

Theorem 7 *Under the above conditions, the following index formula is valid:*

$$\operatorname{ind} \hat{D} = \frac{1}{2} \left\{ \operatorname{ind} \mathcal{D} + \operatorname{sf} D_{0t} \right\},$$

where $\mathcal{D}$ is an operator on the closed manifold (the "double") $2M$ obtained by gluing two copies of the operator $\hat{D}$ to two ends of the cylinder $\Omega \times [0,1]$, on which the principal symbol of the operator $\mathcal{D}$ is given by the principal symbol of the homotopy D_{0t}.

For the case in which the homotopy has the form $D_{0\tau}(p) = B(p + i\tau)$, $\tau \in [\gamma_1, \gamma_2]$, and the mapping g is the identity, we obtain the previously known result from [10]. Indeed, the index theorem of [10] uses the symmetry condition (in our notation)

$$D_0(p + p_0) = \mu_F^{-1} D_0(-p + p_0)\mu_E$$

or, after the substitution $p + p_0 \to p$,

$$D_0(p) = \mu_F^{-1} D_0(2p_0 - p)\mu_E.$$

Thus the homotopy in Condition A has the form

$$D_{0t}(p) = D_0(p - 2tp_0).$$

Next, obviously,

$$sf D_{0t}(p) = sf D_0(p - 2ti\operatorname{Im} p_0),$$

since no singular points arrive at the real axis under real shifts. Computing the spectral flow on the right-hand side of the last identity directly by Definition 5, we obtain

$$sf D_{0t}(p) = - \sum_{\operatorname{Im} p_j \in (0, -2\gamma_0)} m_j,$$

---

[8]The case of symmetry with respect to a point $p_0 \neq 0$ can be considered in a completely similar way.

where $\gamma_0 = \operatorname{Im} p_0$. This formula coincides (for $\gamma = 0$) with the formula in [10]. The case $\gamma \neq 0$ is reduced to this by multiplying the operator $\widetilde{D}$ on the right and on the left by $r^{\pm\gamma}$. If g is an involution, then, in just the same way as was explained above, we obtain the corresponding result of [12]. Using the results of [20], one can readily transfer the assertion of Theorem 7 to the case of general spectral boundary value problems with the symmetry condition (the theorem on the existence of such formulas was proved in [9]).

The assertion of the theorem remains valid if $\hat{D}$ is a quantized canonical transformation (see [21]) on M (the definition of quantized canonical transformation on manifolds with conical singularities can be found, say, in [11]) whose conormal symbol satisfies Condition A. Thus we obtain a generalization of the index theorem for quantized canonical transformations from [11]) to the case in which the symmetry condition involves a nontrivial automorphism g of the base of the cone. The only important difference from the case of pseudodifferential operators is as follows. Quantized canonical transformations *are not properly Fredholm,* but they can be deformed in the class of Fredholm operators to operators satisfying the condition indicated in Definition 2 for an arbitrarily small *predefined $\delta > 0$.* As was indicated in the footnote to this definition, all proofs go in this case without any modifications provided δ is sufficiently small. The theorem for quantized canonical transformations coincides with Theorem 7 with the words "pseudodifferential operator" replaced by the words "quantized canonical transformation."

Proof of Theorem 7. The proof of Theorem 7 is as follows. We consider two configurations (see Fig. 1). The original configuration consists of two copies of M with the operator $\hat{D}$ and the infinite cylinder $(-\infty, \infty) \times \Omega$ on which the operator is specified with the help of the operator-valued symbol $D_{0\chi(\tau)}$, where $\chi: (-\infty, \infty) \to [0, 1]$ increases from 0 to 1 and is constant outside a compact set. The final configuration consists of the manifold $\widetilde{M}$ equipped with the operator $\mathcal{D}$ and two infinite cylinders with operators with constant coefficients (along the cylinder) corresponding to the endpoints of the homotopy. The final configuration is obtained from the original one by surgery of the form described in § 2 (cutting along subsets diffeomorphic to Ω followed by gluing in other combination with the use of g), and the passage from the original operators to the final ones can be interpreted as a modification of operators in bottleneck spaces. The relative index theorem for operators in bottleneck spaces permits one to replace noncompact manifolds by compact manifolds (more precisely, M is replaced by the product $\Omega \times S^1$) and show that the index in the initial and the final configuration is the same, whence the assertion of the theorem follows. $\square$
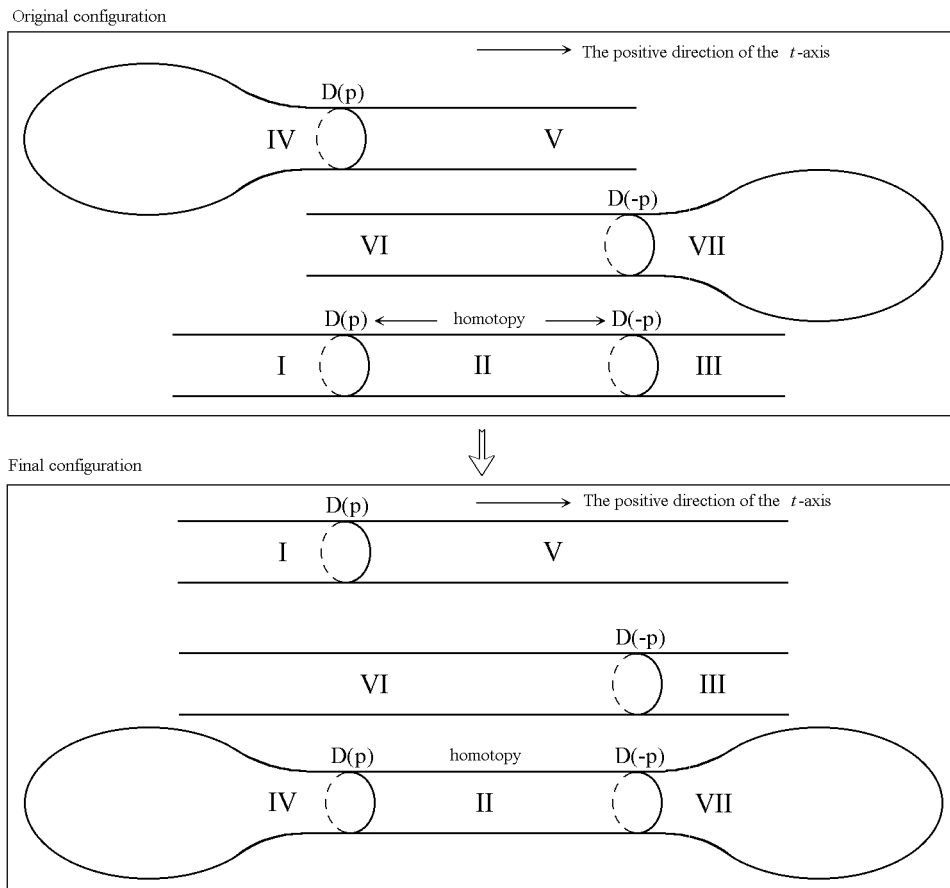
Figure 1: Surgery used in the proof of Theorem 7

4.2 Boundary value problems

Let M be a smooth manifold with boundary $X = \partial M$. On M we consider the classical boundary value problem

$$\begin{cases} \hat{D} u = f, \\ \hat{B} j_X^{m-1} u = g, \end{cases} \tag{27}$$

satisfying the Lopatinskii condition (e.g., see [22]). Here $\hat{D}$ is an mth-order elliptic differential operator on X, $\hat{B}$ is a pseudodifferential operator of zero order (for simplicity) on X, and j_X^{m-1} is the operator of taking the $(m-1)$st jet on X. The problem defines a continuous operator in the spaces

$$(\hat{D}, \hat{B} j_X^{m-1}) : H^s(M, E) \to H^{s-m}(M, E) \oplus \left(\bigoplus_{k=0}^{m-1} H^{s-1/2-k}(X, F_k) \right),$$

$s > m$, where E and F are bundles over M and the F_k, $k = 0, \ldots, m-1$, are bundles over X.

In a collar neighborhood U of the boundary ∂M, we consider some fixed trivialization

$$U \simeq X \times [0, 1), \qquad X \times \{0\} = \partial M.$$

In this neighborhood, $\hat{D}$ is represented in the form

$$\hat{D} = \sum_{k=0}^{m} \hat{D}_k(t) \left(-i \frac{\partial}{\partial t} \right)^k,$$

where the $\hat{D}_k(t)$ are differential operators on X, and the conormal symbol

$$\hat{D}_0(p) = \sum_{k=0}^{m} D_k(0) p^k$$

(corresponding to the given trivialization) is well defined. Without loss of generality, we can assume that the $\hat{D}_k(t)$ (and hence $\hat{D}(p)$) are independent of t for $t \in [0, 1/2]$.

Suppose that the conormal symbol satisfies the symmetry condition

$$\hat{D}(p) = \hat{D}(-p)$$

(we can also readily consider the more general case in which the symmetry includes a diffeomorphism $g : X \to X$ and associated bundle isomorphisms

$$\mu_E : E \to g^* E \qquad \mu_F : F \to g^* F;$$

32

Original configuration

M

a collar neighborhood of the boundary

X   X

M

the cut
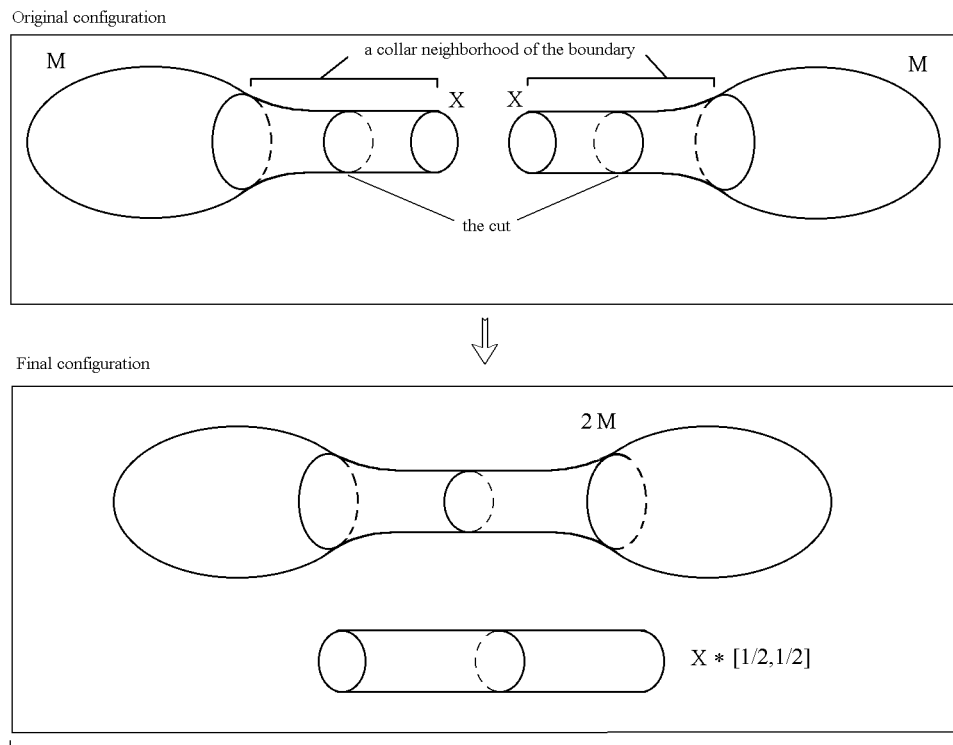
Final configuration

2 M

X * [1/2,1/2]

Figure 2: Surgery for the boundary value problem in Theorem 8

see the preceding example).

In this situation we can use the above-proved general surgery principle to obtain for the boundary value problem (27) an index formula simpler than the general Atiyah–Bott formula [23]. Indeed, consider the surgery shown in Fig. 2. This surgery takes two copies of the operator $\{\hat{D}, \hat{B} j_X^{n-1}\}$ to some new operator $\overset{\sim}{\hat{D}}$ on the closed manifold (double) $2M$ and the operator $\hat{D}_0 = \hat{D}_0 (-i\partial/\partial t)$ with coefficients independent of t on the cylinder $X \times [-1/2, 1/2]$ and with the boundary conditions

$$\hat{B} j_{\{-1/2\}}^{m-1} u = g_1,$$

$$\hat{B} j_{\{+1/2\}}^{m-1} u = g_2$$

on the bases of the cylinder.

According to [20], the index of the problem on the cylinder is equal to the index of the spectral boundary value problem for $\hat{D}_0$ plus the index of the operator

$$
\begin{pmatrix} \hat{B} \\ \hat{B}{}^0 \end{pmatrix} : \begin{array}{c} \hat{L}_+ \\ \oplus \\ \hat{L}_- \end{array} \to \begin{array}{c} \overset{m-1}{\underset{k=0}{\bigoplus}} H^{s-1/2-k}(X, F_k) \\ \oplus \\ \overset{m-1}{\underset{k=0}{\bigoplus}} H^{s-1/2-k}(X, F_k), \end{array}
$$

where the $\hat{L}_\pm \subset \overset{m-1}{\underset{k=0}{\bigoplus}} H^{s-1/2-k}(X, E|_X)$ are the Calderón subspaces corresponding to the two bases of the cylinder and $\hat{B}{}^0$ is obtained from $\hat{B}$ by changing the sign of odd components. Without loss of generality (for example, by adding a constant to the operator $\hat{D}$, which does not affect the index), we can assume that $\hat{D}_0 (p)$ is invertible for real p. Then the index of the spectral boundary value problem for $\hat{D}_0$ is zero, and

$$\hat{L}_+ \oplus \hat{L}_- = \overset{m-1}{\underset{k=0}{\bigoplus}} H^{s-1/2-k}(X, E|_X)$$

by the symmetry condition. Eventually, we obtain the following theorem.

Theorem 8

$$\mathrm{index}\,(\hat{D}, \hat{B} \circ j_X^{m-1})$$

$$= \frac{1}{2}\left\{ \mathrm{index}\,(\overset{\sim}{\hat{D}}) + \mathrm{index}\,\left(\hat{B} \oplus \hat{B}{}^0 : \overset{m-1}{\underset{k=0}{\bigoplus}} H^{s-\frac{1}{2}-k}(X, E|_X) \to 2 \overset{m-1}{\underset{k=0}{\bigoplus}} H^{s-\frac{1}{2}-k}(X, F_k) \right) \right\}.$$

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