

Noncommutative Residues, Dixmier's Trace, and Heat Trace Expansions on Manifolds with Boundary

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ABSTRACT. For manifolds with boundary, we define an extension of Wodzicki's noncommutative residue to boundary value problems in Boutet de Monvel's calculus. We show that this residue can be recovered with the help of heat kernel expansions and explore its relation to Dixmier's trace.

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Introduction

In 1984, M. Wodzicki discovered a trace on the algebra $\Psi_{cl}(\Omega)$ of all classical pseudodifferential operators on a closed compact manifold Ω , [36]. He called it the *noncommutative residue*; meanwhile the notion *Wodzicki residue* is also widely used. This trace vanishes if the order of the operator is less than $-\dim(\Omega)$. For one thing this shows that the noncommutative residue is not an extension of the usual operator trace; it also implies that it is zero on the ideal $\Psi^{-\infty}(\Omega)$ of all regularizing operators and therefore yields a trace on $\Psi_{cl}(\Omega)/\Psi^{-\infty}(\Omega)$. In fact it turns out to be the unique trace on this algebra up to multiples.

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Wodzicki's residue has found a wide range of applications both in mathematics and mathematical physics. It plays a prominent role, for example, in Connes' noncommutative geometry. This is mainly due to the fact that it coincides with Dixmier's trace on pseudodifferential operators of order $-\dim(\Omega)$ as observed by Connes [4]. Certain abstract expressions of noncommutative geometry thus can be computed explicitly for the classical differential geometric situation. Moreover, while Dixmier's trace in general depends on the choice of an averaging procedure, this is not the case for these operators.

The noncommutative residue is closely related to zeta functions of operators and generalized heat trace asymptotics. In fact this is how Wodzicki originally defined it: Given a pseudodifferential operator P , one may choose an invertible pseudodifferential operator A of order larger than that of P and consider, for small u , the zeta function $\zeta_{A+uP}(s)$. According to classical results of Seeley, this is a holomorphic function for large $\operatorname{Re} s$; it extends meromorphically to the whole complex plane. The formula $\operatorname{res} P = \operatorname{ord} A \frac{d}{du} (\operatorname{Res}_{s=1} \zeta_{A+uP}(s))|_{u=0}$ then defines Wodzicki's residue (note that the right hand side is independent of the choice of the auxiliary operator A).

Standard formulas link the zeta function of an operator A and the so-called 'heat trace' trace e^{-tA} . The analysis of both objects is therefore similar, and one obtains corresponding results for the noncommutative residue in terms of coefficients of heat trace expansions. For practical purposes, however, an expression in terms of the symbol of P , also derived by Wodzicki, is often more convenient, cf. Theorem 1.5.

The present survey focuses on manifolds with boundary. Wodzicki noticed right away that there is no trace on the algebra of classical pseudodifferential symbols whenever the underlying manifold is noncompact or has a boundary. One might argue, however, that this algebra is not the natural object to consider, since the standard symbol composition does not 'feel' the presence of the boundary.

The natural analogue of the algebra of pseudodifferential operators on a closed manifold is rather Boutet de Monvel's algebra of boundary value problems. And indeed, it turned out that on this algebra one can find a trace which is an extension of Wodzicki's in the sense that both coincide when the boundary is empty, cf. Fedosov, Gölse, Leichtnam, Schrohe [8]. It is therefore also called a noncommutative residue. It vanishes on operators of sufficiently low order, thus carries over to the quotient modulo regularizing elements. Very much like in the classical case it turns out to be the unique trace; for technical reasons, continuity is required here.

It is a natural question whether this trace is related to Dixmier's. In fact, one can show that operators in Boutet de Monvel's calculus of order $-\dim \Omega$ also belong to Dixmier's ideal. It is a little surprising that both traces do not coincide. It is possible, however, to prove a formula that computes Dixmier's trace in terms of the symbols of the operators; the ingredients in this formula are the same as in the case of the noncommutative residue. In particular, the expression is local, and Dixmier's trace is independent of the averaging procedure, cf. Nest and Schrohe [27].

It has been an open problem to link the noncommutative residue for manifolds with boundary to heat trace expansions. Of course, asymptotics for trace e^{-tB} , where B is an elliptic boundary value problem in Boutet de Monvel's calculus, had been established for example by Grubb in her 1986 book [9]. Due to technical difficulties, however, no complete expansion could be obtained. The remainder

term was $O(t^{1-\varepsilon})$ for some $\varepsilon > 0$, and the noncommutative residue therefore could not be detected.

In connection with their work on Atiyah-Patodi-Singer boundary problems, Grubb and Seeley [12] overcame similar difficulties by introducing a parameter-dependent pseudodifferential calculus based on so-called weakly parametric symbols. It recently became clear that this technique could also be applied here. In joint work with Grubb, we fix a class of ‘nice’ auxiliary second order boundary value problems, say A , and derive an asymptotic expansion for trace Pe^{-tA} for every operator P in Boutet de Monvel’s calculus. Moreover, we show how the coefficient of the logarithmic term in this expansion relates to the noncommutative residue.

The paper is organized as follows. For the benefit of the reader who is not familiar with the subject we start with a survey of the situation in the boundaryless case. There follows a short introduction to Boutet de Monvel’s calculus and the noncommutative residue on manifolds with boundary. Next we analyze the relation to Dixmier’s trace and give the trace formula. The subsequent section explains how the weakly parametric calculus can be used to obtain heat trace asymptotics and the link between the noncommutative residue and the logarithmic term. The final part gives references to various applications and related work.

1. The Classical Results for Closed Manifolds

This section contains a short introduction to the theorems of Wodzicki and Connes for the boundaryless case. In addition, the calculus of Grubb and Seeley [12] and its relation to Wodzicki’s residue are sketched. Let us first fix the notation.

DEFINITION 1.1. Let \mathcal{A} be an algebra over \mathbb{C} . A linear map $\tau : \mathcal{A} \rightarrow \mathbb{C}$ is called a *trace*, if it vanishes on commutators, i.e., if

$$\tau[P, Q] = \tau(PQ - QP) = 0 \text{ for all } P, Q \in \mathcal{A}.$$

Clearly, if τ is a trace, then $\lambda\tau$ is a trace for each λ in \mathbb{C} ; moreover, the zero map is always a trace. When we speak of a unique trace, we shall mean that it is non-zero and the only one up to multiples.

EXAMPLE 1.2. On $M_r(\mathbb{C})$, the algebra of $r \times r$ matrices over \mathbb{C} , there is a unique trace, namely the standard one, $\text{Tr} : A \mapsto \sum_j A_{jj}$. Indeed, let E_{jk} denote the matrix having a single 1 at position j, k (and zeros else). Then the statement is immediate from the observation that $[E_{jk}, E_{kk}] = E_{jk}$ for $j \neq k$ and $[E_{jk}, E_{kj}] = E_{jj} - E_{kk}$.

NOTATION 1.3. Ω is an n -dimensional compact manifold, E a vector bundle, and $\Psi_{cl}(\Omega)$ the algebra of all classical pseudodifferential operators on Ω , acting on sections of E . By $\Psi^{-\infty}(\Omega)$ we denote the ideal of regularizing elements and by \mathcal{A} the quotient $\Psi_{cl}(\Omega)/\Psi^{-\infty}(\Omega)$.

Let $A \in \Psi_{cl}(\Omega)$ be of order m . Over each coordinate neighborhood U its symbol a has an asymptotic expansion $a \sim \sum_{j=0}^{\infty} a_{m-j}$ into terms $a_{m-j} = a_{m-j}(x, \xi) \in S^{m-j}(\mathbb{R}^n \times \mathbb{R}^n)$ that are homogeneous in ξ of degree $m-j$ for $|\xi| \geq 1$. Changing these terms smoothly for small ξ results in regularizing terms. The equivalence class of A in \mathcal{A} therefore can be identified with a formal sum of homogeneous functions $\sum_{j=0}^{\infty} a_{m-j}(x, \xi)$, where now the $a_{m-j} \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ are homogeneous in ξ of degree $m-j$ taking values in square matrices. This is what we shall do in the following.

1.4 THE FORM σ . On \mathbb{R}^n , $n \geq 2$, define the $(n-1)$ -form

$$\sigma(\xi) = \sum_{j=1}^n (-1)^{j+1} \xi_j d\xi_1 \wedge \dots \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \dots \wedge d\xi_n.$$

Introducing polar coordinates, one can check that the restriction of σ to the unit sphere S^{n-1} yields the surface measure.

We can now introduce Wodzicki's noncommutative residue:

THEOREM 1.5. *Let $n \geq 2$, $A \in \Psi_{cl}(\Omega)$ as in 1.3, $x \in \Omega$. Denote by tr_E the trace on $\text{Hom}(E)$ and set*

$$\text{res}_x A = \left(\int_{S^{n-1}} \text{tr}_E a_{-n}(x, \xi) \sigma(\xi) \right) dx_1 \wedge \dots \wedge dx_n.$$

This defines a density on Ω . Moreover,

$$(1.1) \quad \text{res } A = \int_{\Omega} \text{res}_x A$$

has the following properties:

- (a) *It only depends on the equivalence class of A in \mathcal{A} .*
- (b) *It is a trace: $\text{res}[A, B] = 0$ for all $A, B \in \mathcal{A}$.*
- (c) *If Ω is connected, then any other trace on \mathcal{A} is a multiple of res .*

For $n = 1$, the sphere S consists of the two points -1 and 1 . One lets $\text{res } A = \int_{\Omega} (a_{-1}(x, -1) + a_1(x, 1)) dx$. Again this is the unique trace on \mathcal{A} up to multiples.

The local density $a_{-n}(x, \xi) \sigma(\xi) \wedge dx_1 \wedge \dots \wedge dx_n$ can be patched to a global density Ω_A with $\text{res } A = \int_{S^* \Omega} \Omega_A$: Denoting by ω the canonical symplectic form on $T^* \Omega$ and by ρ the radial vector field one has

$$a_{-n} \sigma \wedge dx_1 \wedge \dots \wedge dx_n = (-1)^{n(n-1)/2} \frac{1}{n!} (a \rho \rfloor \omega^n)_0,$$

where $(\dots)_0$ is the homogeneous component of degree 0 in an asymptotic expansion of $a \rho \rfloor \omega^n$ into homogeneous forms (\rfloor stands for the contraction of forms with vector fields).

A simple proof of Theorem 1.5 was given in [8]. Note that no continuity assumption is necessary for the proof of uniqueness.

REMARK 1.6. In fact, the proof of Theorem 1.5 shows even more: We may consider the algebra of all classical symbols with support in an open set of Euclidean space with the Leibniz product. Then any trace on this algebra is a multiple of Wodzicki's.

EXAMPLE 1.7. Let $A = (I - \Delta)^{-n/2}$. Then $a_{-n}(x, \xi) = |\xi|^{-n}$; hence

$$\text{res } A = \int_{\Omega} \int_{S^{n-1}} 1 \sigma(\xi) dx = \text{vol } S^{n-1} \cdot \text{vol } \Omega.$$

So the volume of Ω can be found as a noncommutative residue.

Note that the noncommutative residue vanishes on differential operators. Also, if the order of A is $< -n$, then $\text{res } A = 0$; so res is *not* an extension of the usual operator trace. In fact, Connes showed that Wodzicki's residue coincides with Dixmier's trace on pseudodifferential operators of order $-n$ and therefore *vanishes* on trace class operators. We shall see more, below.

Before, however, let us explore the relation to zeta functions and heat kernel expansions.

1.8 COMPLEX POWERS. Assume in addition that A is invertible of order $m > 0$. Then a is elliptic, but we impose a slightly stronger condition: There exists a ray $R_\theta = \{z \in \mathbb{C} : z = re^{i\theta}, r \geq 0\}$ in \mathbb{C} with no eigenvalue of the principal symbol $a_m(x, \xi)$ on R_θ for $\xi \neq 0$. The spectrum of A is discrete. Shifting θ slightly, R_θ will not intersect it. Seeley [33] showed that:

- (a) The norm of $(A - \lambda)^{-1}$ is $O(\lambda^{-1})$ on R_θ , and there exists a family of complex powers $\{A^s : s \in \mathbb{C}\}$, defined by

$$\begin{aligned} A^s &= \frac{i}{2\pi} \int_{\mathcal{C}} \lambda^s (A - \lambda)^{-1} d\lambda, \quad \operatorname{Re} s < 0; \\ A^{s+k} &= A^s A^k, \quad \operatorname{Re} s < 0, k \in \mathbb{N}. \end{aligned}$$

Here \mathcal{C} is the path in \mathbb{C} going from infinity along R_θ to a small circle around 0, clockwise about the circle, and back along R_θ .

- (b) A^s is a pseudodifferential operator of order $m \operatorname{Re} s$; $s \mapsto A^s$ is analytic.
(c) For $\operatorname{Re} s < -n/m$, A^s is an integral operator with a continuous integral kernel $k_s(x, y)$. For each $x \in \Omega$, $s \mapsto k_s(x, x)$ extends to a meromorphic map with at most simple poles in $s_j = \frac{j-n}{m}$, $j = 0, 1, \dots$. There is no pole in $s = 0$; the residue in s_j is given by an explicit formula. If A is a *differential* operator, then the residues at the positive integers vanish.

Since A^s is trace class for $\operatorname{Re} s < -n/m$ we may define the zeta function

$$\zeta_A(s) = \operatorname{trace} A^{-s}, \quad \operatorname{Re} s > n/m.$$

This is a holomorphic function. It coincides with $\int_{\Omega} k_{-s}(x, x) dx$ hence has a meromorphic extension to \mathbb{C} with at most simple poles in the points s_j .

Wodzicki realized that Seeley's explicit formulas yield

$$(1.2) \quad \operatorname{Res}_{s=-1} \zeta_A = (2\pi)^{-n} \operatorname{res} A / \operatorname{ord} A,$$

where $\operatorname{ord} A$ is the order of A , and, more generally,

$$(1.3) \quad \operatorname{Res}_{s=s_j} \zeta_A = (2\pi)^{-n} \operatorname{res} A^{-s_j} / \operatorname{ord} A.$$

He used this relation to define the noncommutative residue via zeta functions: Let P be an arbitrary pseudodifferential operator. Choose A as above with $\operatorname{ord} A > \operatorname{ord} P$. Then also $A + uP$, $u \in \mathbb{R}$, will meet the above requirements for small $|u|$, and (1.2) suggests to let

$$(1.4) \quad \operatorname{res} P = \frac{d}{du} \operatorname{res} (A + uP) \Big|_{u=0} = (2\pi)^n \operatorname{ord} A \frac{d}{du} \operatorname{Res}_{s=-1} \zeta_{A+uP}(s) \Big|_{u=0}.$$

Moreover, Wodzicki deduced that the latter implies

$$(1.5) \quad \operatorname{res} P = (2\pi)^n \operatorname{ord} A \operatorname{Res}_{s=0} \operatorname{trace} (PA^{-s}).$$

1.9 'HEAT' TRACE EXPANSIONS. In addition to the assumptions in 1.8 we demand that the eigenvalues of the principal symbol matrix a_m lie in a subsector of the right half-plane. Then one can define

$$e^{-tA} = \frac{i}{2\pi} \int_{\mathcal{C}'} e^{-t\lambda} (A - \lambda)^{-1} d\lambda,$$

where \mathcal{C}' is a suitable contour around the spectrum. The operator e^{-tA} solves the equation $\partial_t + A = 0$, which is a generalization of the classical heat equation $\partial_t - \Delta = 0$, where Δ is the Laplace-Beltrami operator. It is easy to see that e^{-tA} is trace class. The identity

$$A^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-tA} dt$$

shows that $\Gamma(s)\zeta_A(s) = \int_0^\infty t^{s-1} \text{trace}(e^{-tA}) dt$ is the Mellin transform of $\text{trace} e^{-tA}$.

The Mellin transform of a function which is $\sim t^{-s_j} \ln^k t$ near $t = 0$ has a pole in s_j of order $k + 1$ and vice versa. Seeley's analysis of the zeta function therefore implies that

$$(1.6) \quad \text{trace} e^{-tA} \sim \sum_{j=0}^{\infty} \alpha_j(A) t^{\frac{j-n}{m}} + \sum_{k=1}^{\infty} (\beta_k(A) \ln t + \beta'_k(A)) t^k.$$

There is no term $t^0 \ln t$, since ζ_A is regular in 0 while the Gamma function has a simple pole. For the same reason there are no terms $t^k \ln t$ if A is a *differential* operator.

So we get $\text{res} A = (2\pi)^n \text{ord} A \cdot \beta_1(A)$. Moreover, we can define the noncommutative residue for a general pseudodifferential operator by choosing an operator A with the above properties and $\text{ord} A > \text{ord} P$, then letting

$$(1.7) \quad \text{res} P = -(2\pi)^n \text{ord} A \frac{d}{du} \beta_1(A + uP)|_{u=0}.$$

Alternatively, one can establish an expansion for $\text{trace}(Pe^{-tA})$ of the form

$$\sum_{j=0}^{\infty} \tilde{\alpha}_j(A) t^{\frac{j-n-\text{ord} P}{m}} + \sum_{k=0}^{\infty} (\tilde{\beta}_k(A) \ln t + \tilde{\beta}'_k(A)) t^k$$

and define

$$(1.8) \quad \text{res} P = -(2\pi)^n \text{ord} A \tilde{\beta}_0.$$

Relations (1.5) and (1.8) can be deduced rather easily using a symbolic calculus introduced by Grubb and Seeley [12] which relies on pseudodifferential symbols that depend on a complex parameter in a special way:

DEFINITION 1.10. Let Γ be a sector in $\mathbb{C} \setminus \{0\}$. $S^{m,0}(\mathbb{R}^n \times \mathbb{R}^n; \Gamma)$ is the space of all functions $p = p(x, \xi, \mu) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$ that are holomorphic with respect to μ whenever $\mu \in \text{int} \Gamma$ and $|\xi, \mu| > \epsilon$ for some $\epsilon > 0$, and which satisfy, for all $j \in \mathbb{N}_0$,

$$\partial_z^j p(\cdot, \cdot, 1/z) \in S^{m+j}(\mathbb{R}^n \times \mathbb{R}^n), \quad 1/z \in \Gamma,$$

with uniform estimates for $|z| \leq 1$ and $1/z$ varying over a closed subcone of $\text{int} \Gamma$. Closure refers to the topology of $\mathbb{C} \setminus \{0\}$. Finally, $S^{m,l}(\mathbb{R}^n \times \mathbb{R}^n; \Gamma) = \mu^l S^{m,0}(\mathbb{R}^n \times \mathbb{R}^n; \Gamma)$.

A symbol p in $S^{m,l}$ is said to be weakly polyhomogeneous provided that there exists a sequence of symbols $p_j \in S^{m_j-l,l}$, $m_j \searrow -\infty$, $j = 1, 2, \dots$, with p_j homogeneous in (ξ, μ) for $|\xi| \geq 1$ of degree m_j such that $p \sim \sum p_j$. In particular we then have $p \in S^{m_1-l,l}$. Write $p \in S_{\text{wphg}}^{m,l}$.

Grubb and Seeley call a symbol p strongly polyhomogeneous of degree m with respect to $(\xi, \mu) \in \mathbb{R}^n \times (\Gamma \cup \{0\})$, when $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (\Gamma \cup \{0\}))$ and there

is a sequence of functions $p_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (\Gamma \cup \{0\}))$ that, for $|\xi, \mu| \geq 1$, are homogeneous in (ξ, μ) of degree $m - j$, with p and the p_j holomorphic in $\mu \in \text{int } \Gamma$ and

$$D_\xi^\alpha D_x^\beta D_\mu^k (p - \sum_{j=0}^J p_j) = O(\langle \xi, \mu \rangle^{m-J-|\alpha|-k}),$$

uniformly for μ in closed subsectors of $\Gamma \cup \{0\}$; as usual, $\langle y \rangle = (1 + |y|^2)^{1/2}$ for $y \in \mathbb{R}^k$. They write $p \in S_{\text{sphg}}^m$.

If p is strongly polyhomogeneous of degree $m \leq 0$, then $p \in S^{m,0} \cap S^{0,m}$ by [12, Theorem 1.16].

An important result is the following theorem [12, Theorem 2.1]:

THEOREM 1.11. *Let $p \sim \sum_{j=0}^\infty p_j$ in $S_{\text{wphg}}^{\infty,d}$, where p_j is homogeneous of degree $m_j, m_j \downarrow -\infty$, and has μ -exponent d . Assume further that p and the p_j with $m_j - d \geq -n$ are in $S^{m',d'}$ for some $m' < n'$, some $d' \in \mathbb{R}$. Then opp has a kernel $K_p(x, y, \mu)$ with an expansion along the diagonal*

$$K_p(x, x, \mu) \sim \sum_{j=0}^\infty c_j(x) \mu^{m_j+n} + \sum_{k=0}^\infty [c'_k(x) \ln \mu + c''_k(x)] \mu^{d-k}$$

for $|\mu| \rightarrow \infty$, uniformly for μ in closed subsectors of Γ .

The coefficients $c_j(x)$ and $c'_{d-m_j-n}(x)$ are determined by $p_j(x, \xi, \mu)$ for $|\xi| \geq 1$ (are “local”), while the $c''_k(x)$ are not in general determined by the homogeneous parts of the symbol (are “global”).

In addition, one easily deduces from the proof of this theorem that the coefficient of $\mu^{d-k} \ln \mu$ is given by

$$(1.9) \quad (2\pi)^{-n} \int_{|\xi|=1} \frac{1}{k!} \partial_z^k (z^d p_j(x, \xi, \frac{1}{z}))|_{z=0} \sigma(\xi),$$

where j is such that $k = d - m_j - n$.

Only one p_j contributes to this coefficient. For $d = 0$, the integrand is the coefficient of μ^{-k} in a Taylor expansion of $p_j(x, \xi, \mu)$ into powers of μ^{-1} ; it is homogeneous of degree $-n$ in ξ .

1.12 APPLICATION TO WODZICKI'S RESIDUE. The above classes are particularly suited for the analysis of $P(A-\lambda)^{-1}$, where P is an arbitrary pseudodifferential operator and A is as above, its order being a positive integer. The reason is the following: Letting $\mu^m = \lambda$, the symbol of $(A-\lambda)^{-1} = (A-\mu^m)^{-1}$ is strongly (hence weakly) polyhomogeneous. The composition with a μ -independent symbol stays weakly polyhomogeneous of the same order. Applying Theorem 1.11 one obtains an expansion

$$(1.10) \quad \text{trace } (P(A-\lambda)^{-k}) \sim \sum_{j=0}^\infty c_j \lambda^{\frac{n+\text{ord } P-j}{m}-k} + \sum_{l=0}^\infty (c'_l \ln \lambda + c''_l) \lambda^{-l-k},$$

[12, Theorem 2.7]. Here, k is chosen larger than $(n + \text{ord } P)/m$ so that the operator under consideration is indeed trace class. In fact, the operator $P(A-\lambda)^k$ is an integral operator with a continuous kernel $K = K(x, y, \lambda)$, and the corresponding expansion holds for the values $K(x, x, \lambda)$ along the diagonal.

We are interested in the coefficient of $\lambda^{-k} \ln \lambda = m\mu^{-mk} \ln \mu$ in (1.10). We just saw that we obtain the coefficient of $\mu^{-mk} \ln \mu$ by integrating the Taylor coefficient of μ^{-mk} in the expansion of the symbol of $P(A - \mu^m)^{-k}$, more precisely the component of homogeneity $-n$. This is the highest power of μ that can possibly occur. The asymptotic expansion formulae show that it only is involved in the term

$$[p(x, \xi)(a(x, \xi) - \mu^m)^{-k}]_{-n}.$$

Writing

$$(a - \mu^m)^{-k} = (-1)^k \mu^{-mk} (1 - a/\mu^m)^{-k} = (-1)^k \mu^{-mk} (1 + \dots (\text{powers of } a/\mu^m))$$

we see that the result is independent of a and that the coefficient of $\lambda^{-k} \ln \lambda$ is

$$(1.11) \quad (2\pi)^{-n} \frac{(-1)^k}{\text{ord}(A)} \int_{\Omega} \int_{|\xi|=1} p_{-n}(x, \xi) \sigma(\xi) dx.$$

One next notes the identities

$$(1.12) \quad \begin{aligned} A^{-s} &= \frac{ik!}{2\pi(s-1) \dots (s-k)} \int_{\mathcal{C}} \lambda^{k-s} (A - \lambda)^{-k-1} d\lambda \\ &= \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} e^{-tA} dt \end{aligned}$$

$$(1.13) \quad e^{-tA} = \frac{ik!}{2\pi} (-t)^{-k} \int_{\mathcal{C}'} e^{-t\lambda} (A - \lambda)^{-k-1} d\lambda.$$

They are obtained from the usual Dunford integrals via integration by parts and the relation

$$\partial_{\lambda}^k (A - \lambda)^{-1} = k! (A - \lambda)^{-k-1}.$$

They imply that

$$(1.14) \quad \text{trace}(Pe^{-tA}) \sim \sum_{j=0}^{\infty} \tilde{c}_j t^{\frac{j-n-\text{ord } P}{m}} + \sum_{l=0}^{\infty} (\tilde{c}'_l \ln t + \tilde{c}''_l) t^l,$$

$$(1.15) \quad \Gamma(s) \text{trace}(PA^{-s}) \sim \sum_{j=0}^{\infty} \frac{\tilde{c}_j}{s + \frac{j-n-\text{ord } P}{m}} + \sum_{l=0}^{\infty} \left(\frac{-\tilde{c}'_l}{(s+l)^2} + \frac{\tilde{c}''_l}{s+l} \right).$$

It is no longer necessary that the order of A be large. In (1.15), the tilde indicates that the left hand side is meromorphic with poles as indicated by the right hand side. The coefficients \tilde{c}_j , \tilde{c}'_l , and \tilde{c}''_l are multiples of the corresponding c_j , c'_l , and c''_l , the factors are universal constants independent of A and P . In particular, we see from (1.11), (1.12), and (1.15) that the coefficient \tilde{c}'_0 is given by

$$(1.16) \quad \tilde{c}'_0 = -\frac{(2\pi)^{-n}}{\text{ord}(A)} \int_{\Omega} \int_S p_{-n}(x, \xi) \sigma(\xi) dx = -\frac{(2\pi)^{-n}}{\text{ord}(A)} \text{res}(P).$$

DIXMIER'S TRACE

Let us now turn to the analysis of Dixmier's trace. By $\mathcal{K}(H)$ denote the ideal of compact operators on the Hilbert space H . In view of Example 1.2 it had been an open question, whether every completely additive trace is proportional to the standard trace on the subset of $\mathcal{K}(H)$ where it is finite. Dixmier [6] showed that the answer is 'no' by explicitly constructing counter-examples. We give a short review, following Connes [5] in presentation and terminology.

1.13 THE SPACES $\mathcal{L}^{(1,\infty)}(H)$ AND $\mathcal{L}_0^{(1,\infty)}(H)$. Let H be an infinite-dimensional Hilbert space, $T \in \mathcal{K}(H)$, and $|T| = (T^*T)^{1/2}$. Let $\mu_0(T) \geq \mu_1(T) \geq \dots$ be the sequence of the singular values of T , i.e. eigenvalues of $|T|$, repeated according to their multiplicity.

We define $\sigma_N(T) = \sum_{j=0}^N \mu_j(T)$ and let

$$\mathcal{L}^{(1,\infty)}(H) = \{T \in \mathcal{K}(H) : \sigma_N(T) = O(\ln N)\},$$

endowed with the norm

$$\|T\|_{1,\infty} = \sup_{N \geq 2} \frac{\sigma_N(T)}{\ln N}.$$

This clearly is a two-sided ideal in $\mathcal{L}(H)$.

1.14 CESÀRO MEAN. We define the Cesàro mean Mf for $f \in L^\infty(1,\infty)$ by

$$(Mf)(t) = \frac{1}{\ln t} \int_1^t f(s) \frac{ds}{s}.$$

The function Mf is continuous and bounded; $M : L^\infty(1,\infty) \rightarrow C_b(1,\infty)$ is continuous. Moreover, $M1 = 1$ and, for $\lambda > 0$, $M(f(\lambda \cdot)) - Mf \in C_{b(0)}(1,\infty)$. The subscript (0) indicates that the function vanishes at infinity.

1.15 THE ‘LIMIT’ \lim_ω AND DIXMIER’S TRACE. We embed $\mathcal{L}^\infty(H)$ into $L^\infty(1,\infty)$ by associating to the sequence $\{a_j\}$ the function $f_{\{a_j\}}$ which has the value a_j on the interval $[j, j+1[$, $j = 1, 2, \dots$. Next we choose a linear form ω on $C_b(1,\infty)$ with (i) $\omega \geq 0$, (ii) $\omega(1) = 1$, and (iii) $\omega(f) = 0$ for $f \in C_{b(0)}(1,\infty)$. Then we define $\lim_\omega \{a_j\} = \omega(Mf_{\{a_j\}})$ with the help of Cesàro’s mean.

Note that \lim_ω coincides with the usual limit on convergent sequences by (ii) and (iii).

For a positive operator $T \in \mathcal{L}^{(1,\infty)}(H)$ we now let

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\ln N} \sum_{n=0}^N \mu_n(T).$$

As Proposition 1.16(a), below, states, Tr_ω is additive on positive T ’s. We can therefore extend it uniquely to a linear map on $\mathcal{L}^{(1,\infty)}(H)$, also denoted Tr_ω .

PROPOSITION 1.16. Let $T, T_1, T_2 \in \mathcal{L}^{(1,\infty)}(H), S \in \mathcal{L}(H)$.

- (a) $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$ for positive T_1, T_2 .
- (b) $\mathrm{Tr}_\omega(T) \geq 0$ if $T \geq 0$.
- (c) If S is invertible, then $\mathrm{Tr}_\omega(STS^{-1}) = \mathrm{Tr}_\omega(T)$. In particular, Tr_ω is independent of the inner product in H .
- (d) $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$.
- (e) Tr_ω vanishes on trace class operators.

EXAMPLE 1.17. Consider the operator $(1 - \Delta)^{-n/2} : L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n)$, where Δ is the Laplacian. The eigenvalues of Δ are known to be the lengths $|k|^2$ as k varies over \mathbb{Z}^n , so the eigenvalues of $(1 - \Delta)^{-n/2}$ are $(1 - |k|^2)^{-n/2}$.

Let us show that $(1 - \Delta)^{-n/2} \in \mathcal{L}^{(1,\infty)}$ and $\mathrm{Tr}_\omega(1 - \Delta)^{-n/2} = \Omega_n/n$, independent of ω with $\Omega_n = \mathrm{vol}S^{n-1}$: We let N_R denote the number of lattice points in

B_R , the ball of radius R . Clearly, $N_R \sim \text{vol } B_R$, hence $\ln N_R \sim n \ln R$. Moreover,

$$\begin{aligned} \sum_{|k| \leq R} (1 + |k|)^{n/2} &\sim \Omega_n \int_0^R (1 + r^2)^{-n/2} r^{n-1} dr \\ &\sim \Omega_n \int_1^R r^{-1} dr = \Omega_n \ln R. \end{aligned}$$

We conclude that

$$(\ln N_R)^{-1} \sum_{|k| \leq R} (1 + |k|)^{n/2} \sim \frac{\Omega_n \ln R}{n \ln R} = \frac{\Omega_n}{n}.$$

Recall that for $(1 - \Delta)^{-n/2} : L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n)$ we had computed in 1.16 that

$$\text{res } (1 - \Delta)^{-n/2} = \text{vol } S^{n-1} \text{vol } \mathbb{T}^n = \Omega_n (2\pi)^n.$$

In one special case we therefore have checked the following result:

THEOREM 1.18. (CONNES 1988) *Let $P \in \Psi_{cl}(\Omega)$ be a pseudodifferential operator of order $-n$. Then*

- (a) $P \in \mathcal{L}^{(1, \infty)}(L^2(\Omega, E))$.
- (b) $\text{res } P = (2\pi)^n \text{Tr}_\omega P$, and the right hand side is independent of ω .

Proofs can be found in Connes' original paper [4] as well as in [34] and [31].

2. Boutet de Monvel's Calculus

In the following, X is an n -dimensional manifold with boundary ∂X embedded in Ω . We assume that $\dim X > 1$.

- DEFINITION 2.1.**
- (a) $\mathcal{S}(\mathbb{R}_\pm^n)$ denotes the rapidly decreasing functions on \mathbb{R}_\pm^n , i.e., $\mathcal{S}(\mathbb{R}^n)|_{\mathbb{R}_\pm^n}$.
 - (b) e^\pm is the operator of extension (by zero) of functions on \mathbb{R}_\pm^n to functions on \mathbb{R}^n , while r^\pm is the restriction operator from \mathbb{R}^n to \mathbb{R}_\pm^n . We also write e^+ for extension by zero from X to Ω and r^+ for restriction from Ω to X .
 - (c) $H^s(X)$ is the Sobolev space of order s on X , i.e. the restrictions of distributions in $H^s(\Omega)$ to X , while $H^s(\partial X)$ is the Sobolev space at the boundary.
 - (d) $H^+ = \{(e^+ u)^\wedge : u \in \mathcal{S}(\mathbb{R}_+^n)\}$; $H_0^- = \{(e^- u)^\wedge : u \in \mathcal{S}(\mathbb{R}_-^n)\}$. The hat denotes the Fourier transform:

$$\hat{f}(\tau) = (2\pi)^{-1/2} \int e^{-it\tau} f(t) dt.$$

H' is the space of all polynomials. We let $H = H^+ \oplus H_0^- \oplus H'$. The functions in $H^+ \oplus H_0^-$ satisfy $h(z) = O(\langle z \rangle^{-1})$ as $z \rightarrow \infty$. We let $H_d^- = \{h \in H_0^- \oplus H' : h(z) = O(\langle z \rangle^{d-1})\}$.

- (e) We define the operator $\Pi' : H \rightarrow \mathbb{C}$ as follows: Given a function $h = h_+ \oplus h_- \oplus p$ in H with $h_+ = (e^+ u)^\wedge$ we let $\Pi' h = (2\pi)^{-1/2} u(0)$. It is easy to see that, for $h \in H \cap L^1$,

$$(2.1) \quad \Pi' h = \frac{1}{2\pi} \int h(\tau) d\tau.$$

2.2 BOUTET DE MONVEL'S ALGEBRA. For detailed introductions see Boutet de Monvel [2], Rempel-Schulze [29], or Grubb [9]. A short version can be found in Schrohe and Schulze [32].

We consider matrices of operators acting on sections of vector bundles E_1, E_2 over X and F_1, F_2 over ∂X . An operator of order $m \in \mathbb{Z}$ and type d is a matrix

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{ccc} C^\infty(X, E_1) & & C^\infty(X, E_2) \\ \oplus & \rightarrow & \oplus \\ C^\infty(\partial X, F_1) & & C^\infty(\partial X, F_2) \end{array}.$$

$\mathcal{B}^{m,d}(X)$ denotes the collection of all operators of order m and type d , and \mathcal{B} is the union over all m and d .

Convention: In order to avoid superfluous notation, we shall omit ‘‘classical’’. All operators, however, are assumed to be classical.

Let me now give a short account of the various entries.

- P is a classical pseudodifferential operator of order m on Ω . The subscript ‘‘+’’ indicates that we consider $P_+ = r^+ P e^+$. The operator P is supposed to have the transmission property; this means that, for all j, k, α , the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies

$$\partial_{x_n}^k \partial_{\xi'}^\alpha p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|} \partial_{x_n}^k \partial_{\xi'}^\alpha p_j(x', 0, 0, -1).$$

- G is a singular Green operator (s.G.o.) of order m and type d . Recall that a singular Green symbol of order m and type d is a function

$$g = g(x', \xi', \xi_n, \eta_n) \text{ on } \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$$

such that

$$(2.2) \quad g(x', \xi', \langle \xi' \rangle \xi_n, \langle \xi' \rangle \eta_n) \in S^{m-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes}_\pi H^+ \hat{\otimes}_\pi H_d^-.$$

Being classical, it has an asymptotic expansion

$$g \sim \sum_{k=1}^{\infty} g_{m-k},$$

where g_j is homogeneous of degree j in (ξ', ξ_n, η_n) for $|\xi'| \geq 1$.

Such a singular Green symbol introduces an operator $\text{op}_G g$ on $\mathcal{S}(\mathbb{R}_+^n)$ by

$$(2.3) \quad \begin{aligned} & (\text{op}_G g)u(x', x_n) \\ &= (2\pi)^{-\frac{n-1}{2}} \int e^{ix\xi} \Pi'_{\eta_n}(g(x', \xi', \xi_n, \eta_n)(e^+ u)^\wedge(\xi', \eta_n)) d\xi. \end{aligned}$$

The subscript η_n only indicates that Π' acts with respect to this variable. This reduces to $(2\pi)^{-\frac{n+1}{2}} \iint e^{ix\xi} g(x', \xi', \xi_n, \eta_n)(e^+ u)^\wedge(\xi', \eta_n) d\eta_n d\xi$ if $d = 0$. A s.G.o. of order m and type d is an operator on $C^\infty(X)$ which is locally given by such an $\text{op}_G g$.

- T is a trace operator of order m and type d . Locally, it is given by a type d trace symbol, i.e., a smooth function $t = t(x', \xi', \xi_n)$ such that $t(x', \xi', \langle \xi' \rangle \xi_n) \in S^{m-1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes}_\pi H_d^-$. The symbol t induces a trace operator $\text{op}_T t : \mathcal{S}(\mathbb{R}_+^n) \rightarrow \mathcal{S}(\mathbb{R}^{n-1})$ by

$$(\text{op}_T t)u(x) = (2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} \Pi'_{\xi_n}(t(x', \xi', \xi_n)(e^+ u)^\wedge(\xi', \xi_n)) d\xi'.$$

The classicality of t here refers to the fact that t has an expansion

$$t \sim \sum_{k=0}^{\infty} t_{m-k-\frac{1}{2}}$$

where t_j is homogenous of degree j in (ξ', ξ_n) for $|\xi'| \geq 1$.

- K is a potential operator (or Poisson operator) of order m . It is given locally in the form $\text{op}_K k$, where $k = k(x', \xi', \xi_n)$ is a potential symbol of order m , i.e. satisfies $k(x', \xi', (\xi') \xi_n) \in S^{m-1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes}_{\pi} H^+$, and where $\text{op}_K k : \mathcal{S}(\mathbb{R}^{n-1}) \rightarrow \mathcal{S}(\mathbb{R}_+^n)$ is given by

$$(\text{op}_K k)u(x', x_n) = (2\pi)^{-\frac{n-1}{2}} \iint e^{ix\xi} k(x', \xi', \xi_n) \hat{u}(\xi') d\xi_n d\xi'.$$

Similarly as above we have an asymptotic expansion $k \sim \sum_{j=0}^{\infty} k_{m-j-1/2}$ with k_j homogeneous of degree j in (ξ', ξ_n) for $|\xi'| \geq 1$.

- Finally, S is a pseudodifferential operator of order m along the boundary.

Regularizing Elements. The intersection over the space of s.G.o. of all orders m is called the space of regularizing s.G.o.s (the type may be arbitrary). The regularizing s.G.o.s of type zero are integral operators with a smooth integral kernel. Similarly we obtain the notions of regularizing trace and potential operators. The space of matrices A as above with all entries regularizing is denoted by $\mathcal{B}^{-\infty}$.

2.3 BASIC PROPERTIES. We shall rely on a few facts.

- The sum in the upper left corner is direct up to regularizing pseudodifferential operators. In other words, if $P_+ = G$ for a pseudodifferential operator P and a singular Green operator G , then P is regularizing. Indeed this is a consequence of the following:
- Given a function ω which is equal to one near the boundary and a s.G.o. G of order m and type d , the composition $(1 - \omega)G$ is a regularizing s.G.o. of type d , while $G(1 - \omega)$ is regularizing of type zero.
- $\mathcal{B}^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map

$$(2.4) \quad \mathcal{B}^{m,d} \times \mathcal{B}^{m',d'} \rightarrow \mathcal{B}^{m+m', \max\{m'+d, d'\}}$$

Here we assume that the bundles the operators act on are such that the composition makes sense. In order to illustrate this somewhat more write

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \in \mathcal{B}^{m,d}, \quad A' = \begin{pmatrix} P'_+ + G' & K' \\ T' & S' \end{pmatrix} \in \mathcal{B}^{m',d'}$$

The composition AA' is obtained by multiplication of the matrices. The resulting compositions of the entries then are of the expected kind; the order of all entries is $m + m'$. TK' , for example, is a pseudodifferential operator on the boundary; KT' , P_+G' , and GG' are s.G.o.s of type d' , while GP'_+ is a s.G.o. of type $m' + d$. For the composition of the pseudodifferential parts we have

$$P_+P'_+ = (PP')_+ + L(P, P').$$

Here PP' is the usual composition of pseudodifferential operators, while $L(P, P')$, the so-called *leftover term*, is a s.G.o. of type $m' + d$.

- (d) It follows that the space $\mathcal{B}_0^{m,d}$ of all operator matrices, where the pseudo-differential part P_+ is zero, forms a two-sided ideal in the sense that (2.4) restricts to mappings

$$\begin{aligned} \mathcal{B}_0^{m,d} \times \mathcal{B}^{m',d'} &\rightarrow \mathcal{B}_0^{m+m', \max\{m'+d, d'\}} \quad \text{and} \\ \mathcal{B}^{m,d} \times \mathcal{B}_0^{m',d'} &\rightarrow \mathcal{B}_0^{m+m', \max\{m'+d, d'\}}. \end{aligned}$$

- (e) The linear span of all compositions KT' where K is a potential operator of order m and T' is a trace operator of order m' and type d' is dense in the space of all s.G.o.s of order $m+m'$ and type d' . The compositions TK' similarly span the space of pseudodifferential symbols of order $m+m'$.
- (f) By (2.3), a singular Green symbol g of type zero defines, for fixed x', ξ' , an operator $g(x', \xi', D_n) : \mathcal{S}(\mathbb{R}_+) \rightarrow \mathcal{S}(\mathbb{R}_+)$ by

$$g(x', \xi', D_n)u(x_n) = \frac{1}{2\pi} \int e^{ix_n \xi_n} g(x', \xi', \xi_n, \eta_n) \hat{u}(\eta_n) d\eta_n.$$

It extends to a trace class operator on $L^2(\mathbb{R}_+)$; the trace is given by

$$\begin{aligned} &\text{trace } g(x', \xi', D_n) \\ &= (2\pi)^{-1/2} \int g(x', \xi', \xi_n, \xi_n) d\xi_n = (2\pi)^{1/2} \Pi'_{\xi_n} g(x', \xi', \xi_n, \xi_n), \end{aligned}$$

where the subscript ξ_n indicates the variable Π' is acting on. For convenience we slightly change the norming factor and define the functional

$$(2.5) \quad \text{tr } g(x', \xi') = \Pi'_{\xi_n} g(x', \xi', \xi_n, \xi_n).$$

In this way it even extends to the case where g is of type d .

Note: If $g(x', \xi', \xi_n, \eta_n)$ is homogeneous of degree j in (ξ', ξ_n, η_n) for $|\xi'| \geq 1$, then $\text{tr } g(x', \xi')$ is homogenous in ξ' of degree $j+1$.

- (g) An operator $A \in \mathcal{B}^{m,d}$ extends to a bounded operator

$$A : \begin{array}{ccc} H^s(X, E_1) & & H^{s-m}(X, E_2) \\ \oplus & \rightarrow & \oplus \\ H^s(\partial X, F_1) & & H^{s-m}(\partial X, F_2) \end{array}$$

for each $s > d - 1/2$.

3. The Noncommutative Residue for Manifolds with Boundary

We use the notation of the previous section. In view of the fact that we are dealing with endomorphisms, we assume that $E_1 = E_2 = E$ and $F_1 = F_2 = F$.

It was already mentioned in the introduction that it is not sufficient to consider merely the algebra of usual classical pseudodifferential symbols (with the Leibniz product as composition), if one intends to find a trace functional for boundary value problems. Wodzicki made the following observation:

LEMMA 3.1. *There is no non-zero trace on $\Psi_{cl}(\Omega)/\Psi^{-\infty}(\Omega)$ whenever Ω is noncompact or has a boundary.*

Proof. We know from Remark 1.6 that the only candidate for a trace is a multiple of the noncommutative residue. For noncompact manifolds, however, formula (1.1) does not make sense.

Next suppose that Ω has a boundary and there exists a trace. Consider the symbols with support in $U \times \mathbb{R}^n$, where U is a neighborhood of the boundary of

the form $U = U_0 \times [0, 1)$ with U_0 open in \mathbb{R}^{n-1} . Every trace necessarily vanishes on symbols $p = p(x, \xi)$ of the form $p = \partial_{x_k} q$ for another symbol q , since $\partial_{x_k} q$ is the symbol of the commutator $[\partial_{x_k}, \text{op } q]$. On the other hand, each symbol with support in $U \times \mathbb{R}^n$ has a primitive with respect to x_n . Thus it is a derivative and hence has trace zero. The trace therefore vanishes. \triangleleft

For more details see [8, Theorem 3.2].

Note, in particular, that the expression

$$\tau(P) = \int_X \int_S \text{tr}_{EP-n}(x, \xi) \sigma(\xi) dx$$

does *not* define a trace if we admit pseudodifferential operators of all orders (it does define a trace on the algebra of operators of order $-n$).

On a manifold with boundary, however, it is much more natural to work with Boutet de Monvel's algebra. There, one obtains the following theorem [7], [8].

THEOREM 3.2. *Let Ω be connected of dimension ≥ 2 , let*

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \in \mathcal{B},$$

and denote by p, g , and s the local symbols of P, G , and s respectively. Define

$$(3.1) \quad \begin{aligned} \text{res } A &= \int_X \int_S \text{tr}_{EP-n}(x, \xi) \sigma(\xi) dx \\ &+ 2\pi \int_{\partial X} \int_{S'} \{ \text{tr}_E(\text{tr } g_{-n})(x', \xi') + \text{tr}_F s_{1-n}(x', \xi') \} \sigma'(\xi') dx'. \end{aligned}$$

Then

- (a) *res is a trace: $\text{res}[A, B] = 0$ for all $A, B \in \mathcal{B}$.*
- (b) *It is the unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.*

Here, σ' is the analog of the form σ in the $(n-1)$ -dimensional case (for $n=2$, we replace the integral $\int_S f(x', \xi') \sigma'(\xi')$ of a function f over S^{n-1} by $f(x', 1) + f(x', -1)$). The subscripts $-n$ and $1-n$ indicate that we are considering the components of the corresponding degree of homogeneity. Note that tr is the trace on singular Green symbols introduced in (2.5) and that $\text{tr } g_{-n}$ is homogeneous of degree $1-n$ in ξ' .

This trace 'res' reduces to Wodzicki's noncommutative residue if $\partial X = \emptyset$ and was therefore called the noncommutative residue for manifolds with boundary in [7], [8].¹

A proof can be found in [8]. Let me give a sketch of at least part of the argument: Using 2.3(a), the algebra $\mathcal{B}/\mathcal{B}^{-\infty}$ can be decomposed

$$\mathcal{B}/\mathcal{B}^{-\infty} = \Psi_{\text{tr}}(X)/\Psi^{-\infty}(X) \oplus \mathcal{B}_0/\mathcal{B}^{-\infty}$$

into the algebra $\Psi_{\text{tr}}(X)/\Psi^{-\infty}(X)$ of all classical pseudodifferential symbols with the transmission property, and the quotient of \mathcal{B}_0 – i.e. the matrices in Boutet de Monvel's calculus that have zero pseudodifferential part – modulo the regularizing elements.

¹The sign between the two terms in (3.1) should indeed be '+', not '-'.

On symbols supported in the interior, any trace will be a multiple of the non-commutative residue. This suggests that the trace is of the above form on the pseudodifferential part.

Next, on the algebra $\mathcal{B}_0/\mathcal{B}^{-\infty}$, one may apply the following lemma:

LEMMA 3.3. *Let $\tau : \mathcal{B}_0/\mathcal{B}^{-\infty} \rightarrow \mathbb{C}$ be linear. Then τ is a trace on $\mathcal{B}_0/\mathcal{B}^{-\infty}$ if and only if there exist a trace τ_1 on the algebra of singular Green symbols and a trace τ_2 on $\Psi(\partial M)/\Psi^{-\infty}(\partial M)$ such that for all trace symbols t and all potential symbols k*

$$(3.2) \quad \tau_1(k \circ t) = \tau_2(t \circ k)$$

and

$$(3.3) \quad \tau \left\{ \begin{pmatrix} g & k \\ t & s \end{pmatrix} \right\} = \tau_1(g) + \tau_2(s).$$

Proof. Given τ , define $\tau_1(g) = \tau \left\{ \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $\tau_2(s) = \tau \left\{ \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \right\}$. The identities

$$(3.4) \quad 0 = \tau \left\{ \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \right] \right\} = \tau \left\{ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \right\};$$

$$(3.5) \quad 0 = \tau \left\{ \left[\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\} = \tau \left\{ \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \right\};$$

$$(3.6) \quad 0 = \tau \left\{ \left[\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \right] \right\} = \tau \left\{ \begin{pmatrix} k \circ t & 0 \\ 0 & -t \circ k \end{pmatrix} \right\}$$

imply the assertion. Conversely, given traces τ_1 and τ_2 satisfying the compatibility condition, it is easily checked that the above τ defines a trace. \square

COROLLARY 3.4. We know that the zero map is a trace and that the span of the $k \circ t$ is dense in the space of all singular Green symbols while the span of the $t \circ k$ are the pseudodifferential symbols. Hence every continuous trace τ_1 on the singular Green symbols induces a trace on $\Psi_{cl}(\partial X)/\Psi^{-\infty}(\partial X)$. The latter must be a multiple of Wodzicki's residues on the boundary, $\text{res}_{\partial X}$.

So we do not have much choice: Suppose we have a trace τ and that $\tau_2 = c \text{res}_{\partial X}$. According to the above formulae, the composition $k \circ t$ has the singular Green symbol $g(\xi_n, \eta_n) = k(\xi_n)t(\eta_n)$. We conclude that

$$\tau_1(g) = \tau_1(k \circ t) = c \text{res}_{\partial X}(t \circ k) = c \text{res}_{\partial X} \Pi'(kt) = c \text{res}_{\partial X} \text{tr} g(x', \xi')$$

with the trace tr introduced in (2.5). For simplicity we have ignored the (x', ξ') variables in the composition – they only cause notational complications.

On the subalgebra $\mathcal{B}_0/\mathcal{B}^{-\infty}$ we therefore have precisely one continuous trace up to multiples, namely

$$\text{res} \begin{pmatrix} G & K \\ T & S \end{pmatrix} = \int_{\partial X} \int_{S'} \{ \text{tr}_E(\text{tr} g_{-n})(x', \xi') + \text{tr}_{FS_{1-n}}(x', \xi') \} \sigma'(\xi') dx'.$$

These two simple considerations show that any candidate for a trace must be of the form in Theorem 3.2. Yet it is far from obvious that (3.1) indeed defines a trace; in fact, this requires rather delicate computations, cf. [8].

4. Dixmier's Trace for Boundary Problems

We shall now analyze the connection between the noncommutative residue and Dixmier's trace. The exposition follows the original paper, Nest and Schrohe [27]. We first show that there is indeed a subclass of Boutet de Monvel's algebra on which Dixmier's trace makes sense.

Like in the previous section, the operators are supposed to be endomorphisms, hence $E_1 = E_2 = E$ and $F_1 = F_2 = F$. We shall work on the Hilbert space $H = L^2(X, E) \oplus L^2(\partial X, F)$. Operators of type $d > 0$ can not be defined on this space in general, we shall therefore work with type zero operators.

It is well-known that a bounded operator $L^2(\partial X) \rightarrow H^{n-1}(\partial X)$ defines an element of $\mathcal{L}^{(1,\infty)}(L^2(\partial X))$. This operator is even trace class on $L^2(\partial X)$, if its range is contained in $H^n(\partial X)$. The following proposition is not very difficult and shows the analog for the case with boundary.

PROPOSITION 4.1. *A bounded operator on $L^2(X)$ with range in $H^n(X)$ is an element of $\mathcal{L}^{(1,\infty)}(L^2(X))$; if its range even is contained in $H^{n+1}(X)$ then it is trace class.*

DEFINITION 4.2. *We shall denote by \mathcal{D}^m the space of all operator matrices*

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix}$$

in Boutet de Monvel's calculus, where P is of order m , G is of order m and type zero, K is of order m , T is of order $m+1$ and type zero, and S is of order $m+1$.

The following lemma is now obvious from 2.3(g):

COROLLARY 4.3. *An operator $A \in \mathcal{D}^{-n}$ defines a bounded map*

$$A : H \rightarrow H^{-n}(X) \oplus H^{1-n}(\partial X).$$

Hence $\mathcal{D}^{-n} \hookrightarrow \mathcal{L}^{(1,\infty)}(H)$. Similarly $\mathcal{D}^{-n-1} \hookrightarrow \mathcal{L}^1(H)$. In particular, Dixmier's trace applies to elements in \mathcal{D}^{-n} ; it vanishes on \mathcal{D}^{-n-1} .

We will show the following theorem:

THEOREM 4.4. *For an operator*

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \in \mathcal{D}^{-n}$$

acting on $H = L^2(X, E) \oplus L^2(\partial X, F)$, $n = \dim X > 2$, we have

$$\begin{aligned} \mathrm{Tr}_\omega(A) &= \frac{1}{(2\pi)^{n-1}} \int_X \int_S \mathrm{tr}_E P_{-n}(x, \xi) \sigma(\xi) dx \\ &\quad + \frac{1}{(2\pi)^{n-1}(n-1)} \int_{\partial X} \int_{S'} \mathrm{tr}_F s_{-n+1}(x', \xi') \sigma'(\xi') dx'. \end{aligned}$$

Here, S is the sphere $\{|\xi| = 1\}$ in $T^*\Omega$ over a point $x \in X$, σ is the form introduced in 1.4, while S' is the corresponding sphere $\{|\xi'| = 1\}$ in $T^*\partial X$ over a point x' in ∂X and σ' is the corresponding $(n-2)$ -form.

For $\dim X = 2$ the second summand becomes

$$\frac{1}{2\pi} \int_{\partial X} \mathrm{tr}_F (s_{-1}(x', 1) + s_{-1}(x', -1)) dx'.$$

We start with the following observation.

LEMMA 4.5. *Let $L \in \mathcal{K}(L^2(X, E))$. Then, apart from zero, the spectrum of L in $\mathcal{L}(L^2(X, E))$ including multiplicities coincides with the spectrum of $\begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{L}(H)$ and the spectrum of $1_X L 1_X$ in $\mathcal{L}(L^2(\Omega, E))$.*

Here 1_X denotes the characteristic function of X in Ω , and we identify L^2 -functions on X with their extensions by zero.

We obtain a first result:

LEMMA 4.6. *Let $\begin{pmatrix} G & K \\ T & S \end{pmatrix} \in \mathcal{D}^{-n}$ have zero pseudodifferential part. Then*

$$\mathrm{Tr}_\omega \begin{pmatrix} G & K \\ T & S \end{pmatrix} = \frac{\mathrm{res}_{\partial X} S}{(2\pi)^{n-1}(n-1)}$$

with Wodzicki's residue on ∂X . There is no contribution from G, T , or K .

Proof. Tr_ω is a trace on $\mathcal{L}^{(1, \infty)}(H)$, therefore $\mathrm{Tr}_\omega([B, C]) = 0$ whenever $B \in \mathcal{D}^{-n}(X)$ and $C \in \mathcal{L}(H)$, cf. Proposition 1.16 (d). In particular, we easily deduce from commutator identities (3.4) and (3.5) (with $\tau = \mathrm{Tr}_\omega$) that Dixmier's trace vanishes on the off-diagonal entries. We know from (an analog of) Lemma 4.5 and Theorem 1.18 that

$$(4.1) \quad \mathrm{Tr}_\omega \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} = \mathrm{Tr}_\omega(S) = \frac{\mathrm{res}_{\partial X} S}{(2\pi)^{n-1}(n-1)}.$$

Next let K' be a potential operator of order zero. Then $K' : L^2(\partial X, F) \rightarrow L^2(X, E)$ and $K' : H^n(\partial X, F) \rightarrow H^n(X, E)$ are bounded by 2.3(g). Pick also a trace operator T' of order $-n$ and type zero. The operator $T' : L^2(X, E) \rightarrow H^n(\partial X, F)$ is bounded, and $T'K'$ a trace class operator on $L^2(\partial X, F)$, for $\dim \partial X = n - 1$. Moreover, $B = \begin{pmatrix} 0 & K' \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(H)$, and $C = \begin{pmatrix} 0 & 0 \\ T' & 0 \end{pmatrix} \in \mathcal{D}^{-n}$. We conclude from (4.1) that

$$0 = \mathrm{Tr}_\omega([B, C]) = \mathrm{Tr}_\omega \begin{pmatrix} K'T' & 0 \\ 0 & -T'K' \end{pmatrix} = \mathrm{Tr}_\omega \begin{pmatrix} K'T' & 0 \\ 0 & 0 \end{pmatrix}.$$

The span of the operators $K'T'$ of this form is dense in the space of singular Green operators of order $-n$ and type zero. As a consequence, Dixmier's trace vanishes on the upper left corner, since it is continuous on $\mathcal{L}^{1, \infty}(H)$, and the topology on the singular Green operators is stronger than the operator topology. By linearity the proof is complete. \triangleleft

The task of determining Dixmier's trace for the matrix A in Definition 4.2 reduces to the case where G, K, T , and S are zero. By Lemma 4.5 this amounts to finding $\mathrm{Tr}_\omega(P_+)$ in $\mathcal{L}(L^2(X, E))$ or $\mathrm{Tr}_\omega(1_X P 1_X)$ in $\mathcal{L}(L^2(\Omega, E))$. Since Tr_ω is a trace, the latter equals $\mathrm{Tr}_\omega(1_X P)$.

PROPOSITION 4.7. *For a pseudodifferential operator P of order $-n$, acting on $L^2(\Omega, E)$,*

$$(4.2) \quad \mathrm{Tr}_\omega(1_X P) = \frac{1}{(2\pi)^{nn}} \int_X \int_S \mathrm{tr}_{EP-n}(x, \xi) \sigma(\xi) dx.$$

Here, P need not have the transmission property.

Proof. Let us first assume that P is positive. Choose sequences of smooth functions $\{\varphi_k\}, \{\psi_k\}$ with $0 \leq \varphi_k \leq 1_X \leq \psi_k \leq 1$ and $\varphi_k, \psi_k \rightarrow 1_X$ pointwise.

Since φ_k, ψ_k are smooth, $\varphi_k P$ and $\psi_k P$ are the pseudodifferential operators with symbols $\varphi_k p$ and $\psi_k p$, respectively. For them, Dixmier's trace is given by Theorem 1.18. In view of the positivity of Tr_ω ,

$$\text{Tr}_\omega(1_X P - \varphi_k P) = \text{Tr}_\omega((1_X - \varphi_k)P) = \text{Tr}_\omega(\sqrt{1_X - \varphi_k} P \sqrt{1_X - \varphi_k}) \geq 0.$$

Arguing similarly for ψ_k , we conclude that

$$\begin{aligned} \frac{1}{(2\pi)^n n} \int_\Omega \int_S \text{tr}_E \varphi_k(x) p_{-n}(x, \xi) \sigma(\xi) dx &= \text{Tr}_\omega(\varphi_k P) \leq \text{Tr}_\omega(1_X P) \leq \text{Tr}_\omega(\psi_k P) \\ &= \frac{1}{(2\pi)^n n} \int_\Omega \int_S \text{tr}_E \psi_k(x) p_{-n}(x, \xi) \sigma(\xi) dx. \end{aligned}$$

Lebesgue's theorem on dominated convergence then gives (4.2).

Next assume that P is selfadjoint. Let Δ be the Laplace-Beltrami operator associated with an arbitrary Riemannian metric. Define $T = (I - \Delta)^{-n/4}$. This is a positive selfadjoint pseudodifferential operator of order $-n/2$. It is invertible, and T^{-1} is of order $n/2$. The operator $T^{-1} P T^{-1}$ is of order zero and therefore bounded; it also is selfadjoint. For large $t \in \mathbb{R}$, we conclude that $T^{-1} P T^{-1} + tI$ is positive. Hence also $P + tT^2 = T(T^{-1} P T^{-1} + tI)T$ is positive. We can therefore write P as the difference of the positive operators $P + tT^2$ and tT^2 . In view of the additivity of the integral we obtain the assertion.

Finally, in the general case we write $P = \frac{1}{2}(P + P^*) - \frac{i}{2}(iP - iP^*)$. Since $P \pm P^*$ is pseudodifferential of order $-n$, additivity shows the assertion. \triangleleft

The proof of Theorem 4.4 is now complete.

In the more traditional literature one adopts slightly different way of looking at boundary value problems. We shall now sketch how the results apply to this situation.

Let $P : C^\infty(X, E) \rightarrow C^\infty(X, E)$ be a differential operator of order $m > 0$ and $T : C^\infty(X, E) \rightarrow C^\infty(Y, F)$ a trace operator. One usually assumes that

$$F = F_1 \oplus \dots \oplus F_m,$$

where the dimension of each F_j might be zero (in a standard application, m would be even and half of the F_j would be zero). Correspondingly one writes $T = (T_1, \dots, T_m)$, asking that T_k be a differential boundary operator of order $m_k < m$ involving at most $k - 1$ normal derivatives. In local coordinates near the boundary, T_k can be written in the form

$$T_k = \sum_{j < k, j + |\alpha| \leq m_k} a_{j\alpha}^{[k]}(x') D_{x'}^\alpha \gamma_j.$$

Here, $\gamma_j = \gamma_0 \partial_{x_n}^j$ is the operator of evaluation of the j -fold normal derivative at the boundary. Of course, T_k can be neglected when the dimension of F_k is zero.

In order to treat the homogeneous problem

$$(4.3) \quad Pu = f \quad \text{on } X; \quad Tu = 0 \quad \text{on } \partial X$$

one studies the "realization" P_T which is defined as the unbounded operator P_T on $L^2(X, E)$ with domain

$$(4.4) \quad \mathcal{D}(P_T) = \{u \in H^m(X, E) : Tu = 0\}$$

and acting like P rather than the operator

$$(4.5) \quad \begin{pmatrix} P \\ T \end{pmatrix} : H^m(X, E) \rightarrow \begin{matrix} L^2(X, E) \\ \oplus \\ \bigoplus_{k=1}^m H^{m-m_k-1/2}(\partial X, F_k) \end{matrix}.$$

In both cases one is interested in the question whether the corresponding operator has the Fredholm property. The link between the two approaches is given by the following proposition.

PROPOSITION 4.8. *Let H , H_1 and H_2 be Hilbert spaces over \mathbb{C} , and let $P : H \rightarrow H_1$ as well as $T : H \rightarrow H_2$ be linear operators. Then the following are equivalent:*

- (a) $\begin{pmatrix} P \\ T \end{pmatrix} : H \rightarrow \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix}$ is a Fredholm operator.
- (b) $P_T = P|_{\ker T} : \ker T \rightarrow H_1$ is a Fredholm operator, and $\text{im} T$ is finite codimensional in H_2 .

In standard cases one will often have T surjective so that the second condition in (ii) is easily fulfilled.

4.9 ELLIPTICITY. The operator in 4.5 is a Fredholm operator if and only if the following two conditions are fulfilled:

- (i) For all $(x, \xi) \in S^*\Omega|_X$ the principal symbol p_m of P is invertible as an endomorphism of E , and
- (ii) for all $(x', \xi') \in S^*\partial X$ the principal boundary symbol

$$\begin{pmatrix} p_m(x', 0, \xi', D_n) \\ t_{m-1/2}(x', \xi', D_n) \end{pmatrix} : \mathcal{S}(\mathbb{R}_+, E) \rightarrow \begin{matrix} \mathcal{S}(\mathbb{R}_+, E) \\ \oplus \\ F_{(x', \xi')} \end{matrix}$$

is invertible.

Here $t_{m-1/2}$ is the operator-valued principal symbol of the trace operator T . Locally near the boundary, this is the vector with entries $\sum_{j < k, j+|\alpha|=m_k} a_{j\alpha}^{[k]}(x') \xi'^\alpha \gamma_j$.

The orders of the entries in this boundary value problem are not as required in Definition 2.2. This, however, need not worry us: We may replace each T_k by $\tilde{T}_k = \Lambda^{m-m_k-1/2} T_k$, where $\Lambda = (1 - \Delta_{\partial X})^{1/2}$ is order-reducing along ∂X . In view of the fact that the powers of Λ are invertible, this affects neither (4.4) nor the ellipticity of $\begin{pmatrix} P \\ T \end{pmatrix}$, but it allows us to use the space $L^2(\partial X, F)$ instead of $\bigoplus_{k=1}^m H^{m-m_k-1/2}(\partial X, F_k)$ on the right hand side of (4.5).

The ellipticity implies that there is a parametrix to $A = \begin{pmatrix} P \\ T \end{pmatrix}$ in Boutet de Monvel's calculus. It is of the form $B = (Q_+ + G K)$; the pseudodifferential part Q is a parametrix to P , while G is a singular Green operator of order $-m$ and type zero and K is a potential operator of order $-m$. Being a parametrix here means that $AB - I = S_1$ and $BA - I = S_2$ are regularizing operators; their types are 0 and m , respectively. As a consequence the operator S_1 is an integral operator with a smooth kernel section.

Multiplication by $\text{diag}(1, \Lambda_1, \dots, \Lambda_m)$ with $\Lambda_k = \Lambda^{m-m_k-1/2}$, furnishes a parametrix for $\begin{pmatrix} P \\ T \end{pmatrix}$; note that this only affects the entry K of B .

The Theorem, below, follows from a construction by Grubb and Geymonat, see e.g. [9], Section 1.4.

THEOREM 4.10. Let $(\frac{P}{T})$ be the above elliptic boundary value problem and $B = (Q_+ + G - K)$ as above. Then there is a regularizing singular Green operator G_0 of type zero, i.e. an integral operator with smooth kernel on $X \times X$, such that $R = Q_+ + G + G_0$ has the following properties:

- (a) R maps $L^2(X, E)$ to $\mathcal{D}(P_T)$ and
- (b) $RP_T - I$ and $P_T R - I$ are finite rank operators whose range consists of smooth functions.

Any other parametrix for P_T differs from R by a regularizing s.G.o..

COROLLARY 4.11. Suppose that $m = n$ and $(\frac{P}{T})$ is elliptic. Then Theorem 4.4 shows that Dixmier's trace for an arbitrary parametrix R to the operator P_T is given by

$$(4.6) \quad \mathrm{Tr}_\omega R = \frac{1}{(2\pi)^{n_n}} \int_X \int_S \mathrm{tr}_{EP_n}(x, \xi)^{-1} \sigma(\xi) dx.$$

In particular, it is the same for all parametrices and independent of the boundary condition T . It coincides with the noncommutative residue for R .

For certain classes of parameter-elliptic boundary value problems one can deduce this result from eigenvalue asymptotics for the operator P_T . Results of this type were first established by H. Weyl [35] for the Dirichlet problem in the plane and subsequently by numerous authors for many operators. Theorem 4.4 on the other hand covers more general situations, in particular, non-elliptic operators. For example, we get the result of 4.11 for all elliptic boundary value problems $(\frac{P+G}{T})$ of order n in Boutet de Monvel's calculus (i.e., without requiring parameter-ellipticity and with additional singular Green terms as well as more general trace operators); formula (4.6) continues to hold.

5. Heat Trace Asymptotics

It is an obvious question whether there is an analog of the expansion (1.14) and relation (1.16). In fact one might be tempted to introduce the noncommutative residue for manifolds with boundary in this way. There is, however, the following obstacle:

5.1 EARLIER RESULTS. Let $B = (\tilde{P}_+ + \tilde{G})_{\tilde{T}}$ be the realization of an elliptic boundary value problem $(\frac{\tilde{P}_+ + \tilde{G}}{\tilde{T}})$ in Boutet de Monvel's calculus. Here, \tilde{P} and \tilde{T} are as in the previous section; \tilde{G} is a s.G.o.. Under suitable assumptions on \tilde{P} , \tilde{G} and \tilde{T} , Grubb showed in her book [9] that e^{-tB} is a trace class operator on $L^2(X, E)$ and studied the behavior of trace $\exp(-tB)$ as $t \rightarrow 0^+$. She obtained an expansion similar to (1.6). For technical reasons, however, this expansion was finite and the remainder term was $O(t^{1-\varepsilon})$, with $\varepsilon > 0$ determined by the so-called regularity number of the operator.

Given an operator $P_+ + G$ in Boutet de Monvel's calculus (for simplicity let $F = 0$) one would have liked to choose an elliptic boundary value problem $(\frac{\tilde{P}_+ + \tilde{G}}{\tilde{T}})$ of higher order as above and to define $\mathrm{res}(P_+ + G)$ by the derivative $d/du|_{u=0}$ of the coefficient of $t \ln t$ in the expansion of trace $\exp(-t(\tilde{P}_+ + \tilde{G} + u(P_+ + G)))_{\tilde{T}}$ for small $|u|$, just as in (1.7). Due to the uncertainty $O(t^{1-\varepsilon})$, however, this does not make sense.

The difficulties one encounters when considering trace $(P_+ + G) \exp(-t(\tilde{P}_+ + \tilde{G})_{\tilde{T}})$ are the same. Hence (1.8) can not be employed, either. It even does not help to restrict \tilde{P} , \tilde{G} , and \tilde{T} to suitable classes of operators.

Progress was possible as a result of the work of Grubb and Seeley. Already in [12] and [11] they applied the weakly parametric calculus in order to study trace expansions for operators arising from Atiyah-Patodi-Singer boundary value problems.

In [10] we now employ this calculus in order to obtain the complete expansion for

$$\text{trace}(P_+ + G) \exp(-t(\tilde{P}_+ + \tilde{G})_{\tilde{T}}),$$

for a certain class of auxiliary operators. This yields the link between the noncommutative residue and heat trace expansions.

5.2 OUTLINE. We proceed in the following way. We pick an elliptic second order pseudodifferential operator P_1 on Ω whose principal symbol $p_2 = p_2(x, \xi)$ has its eigenvalues in a sector in the right half-plane. We let $P_{1,Dir}$ be the Dirichlet realization of P_1 , i.e. the unbounded operator on $L^2(X)$ with domain $\mathcal{D}(P_{1,Dir}) = \{u \in H^2(X) : \gamma_0 u = 0 \text{ on } \partial X\}$.

Moreover, we choose an elliptic second order pseudodifferential operator S_1 on ∂X , asking again that the eigenvalues of the principal symbol lie in a sector of the right half-plane.

Just to give an example, we might fix arbitrary Riemannian metrics g on X and g' on ∂X and let $P_1 = \Delta_g \text{Id}_E$, $S_1 = \Delta_{g'} \text{Id}_F$ be the associated Laplace-Beltrami operators.

We then let

$$A = \begin{pmatrix} P_{1,Dir} & 0 \\ 0 & S_1 \end{pmatrix}.$$

From the analysis in [9, Section 4.2] one immediately obtains the following theorem.

THEOREM 5.3. *The spectrum of A lies in an obtuse key hole region*

$$W = \{\lambda \in \mathbb{C} : |\lambda| \leq r \text{ or } |\arg \lambda| \leq \pi/2 - \varepsilon\},$$

where $r > 0$ and $0 < \varepsilon < \pi/2$ are fixed. The operator A generates a strongly continuous semigroup e^{-tA} on $L^2(X, E) \oplus L^2(\partial X, F)$. For each $t > 0$, the operator e^{-tA} is given by the Dunford integral

$$e^{tA} = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (A - \lambda)^{-1} d\lambda,$$

where \mathcal{C} is the counter-clockwise oriented boundary of W . It maps $L^2(X, E) \oplus L^2(\partial X, F)$ to $C^\infty(X, E) \oplus C^\infty(\partial X, F)$.

For every operator

$$\mathcal{P} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix}$$

in Boutet de Monvel's calculus, the composition $\mathcal{P} e^{-tA}$ therefore is a trace class operator for $t > 0$.

Using identity (1.13) we obtain the following:

COROLLARY 5.4. Let \mathcal{P} be an operator of order ν and type d . Assume w.l.o.g. that $d \leq \max\{\nu, 0\}$. Fix $k > \frac{n+\nu}{2}$. Then

$$\begin{aligned}
\text{trace } \mathcal{P}e^{-tA} &= \frac{ik!}{2\pi} (-t)^k \int_{\mathcal{C}} e^{t\lambda} \text{trace} \{ \mathcal{P}(A - \lambda)^{-k} \} d\lambda \\
(5.1) \qquad &= \frac{ik!}{2\pi} (-t)^k \int_{\mathcal{C}} e^{t\lambda} \text{trace} \{ (P_+ + G)(P_{1,Dir} - \lambda)^{-k} \} d\lambda \\
&\quad + \frac{ik!}{2\pi} (-t)^k \int_{\mathcal{C}} e^{t\lambda} \text{trace} \{ S(S_1 - \lambda)^{-k} \} d\lambda
\end{aligned}$$

We know from Section 1 that there is an expansion for $\text{trace } S(S_1 - \lambda)^{-k}$ as $|\lambda| \rightarrow \infty$ which in turn leads to an expansion of the second integral

$$\text{trace } Se^{-tS_1} \sim \sum_{j=0}^{\infty} \tilde{e}_j t^{\frac{j-n-\nu}{2}} + \sum_{l=0}^{\infty} (\tilde{e}'_l \ln t + \tilde{e}''_l) t^l.$$

The coefficient \tilde{e}'_0 of $\ln t$ is given by

$$\tilde{e}'_0 = \frac{-1}{2(2\pi)^{n-1}} \text{res}_{\partial X} S.$$

It therefore remains to deal with the first integral. To this end we let $\lambda = \mu^2$ outside W . We recall from 4.10 that the inverse of $P_{1,Dir} - \mu^2$ is of the form

$$(P_1 - \mu^2)_+^{-1} + G_\mu$$

where $\{G_\mu : \mu^2 \in \mathbb{C} \setminus W\}$ is a family of s.G.o.s (it can be described rather precisely, but we shall presently not go into the details). We immediately deduce that

$$(P_{1,Dir} - \mu^2)^{-k} = [(P_1 - \mu^2)^{-k}]_+ + G_\mu^{(k)}$$

with the inverse $(P_1 - \mu^2)^{-k}$ on $L^2(\Omega, E)$ and

$$G_\mu^{(k)} = [(P_1 - \mu^2)^{-k}]_+ - [(P_1 - \mu^2)_+^{-1}]^k + R_\mu;$$

where R_μ is a polynomial in the non-commuting operators $(P_1 - \mu^2)_+$ and G_μ .

As a result, we write

$$\begin{aligned}
(P_+ + G)(P_{1,Dir} - \mu^2)^{-k} &= [P(P_1 - \mu^2)^{-k}]_+ - L(P, (P_1 - \mu^2)^{-k}) + P_+ G_\mu^{(k)} \\
&\quad + G[(P_1 - \mu^2)^{-k}]_+ + GG_\mu^{(k)}.
\end{aligned}$$

It remains to consider the traces of these operators. The first term is easily analyzed as a consequence of the fact that P_1 is an operator over the full manifold, satisfying the assumptions in Section 1. Recalling that relation (1.14) holds for the kernel, pointwise along the diagonal, we obtain the following assertion:

LEMMA 5.5. *There is an expansion of the form (1.10) for $\text{trace } P(P_1 - \lambda)^{-k}$, hence an expansion of $\text{trace } Pe^{-tP_1}$ of the form (1.14). The coefficient \tilde{c}'_0 of $\ln t$ is given by (recall that $\text{ord } P_1 = 2$)*

$$\tilde{c}'_0 = \frac{-1}{2(2\pi)^n} \int_X \int_S \text{tr}_{\mathbb{E}} p_{-n}(x, \xi) \sigma(\xi) dx.$$

The analysis of the remaining four terms is much less trivial. It relies on a careful study of the symbols and their compositions. The general strategy is to

reduce the analysis to the boundary and to show that all the arising pseudodifferential symbols belong to the weakly parametric calculus of Grubb and Seeley, the expansion formulas are a consequence of Theorem 1.11. We eventually show the following, provided the type is zero:

PROPOSITION 5.6. *The traces of the operators $L(P, (P_1 - \mu^2)^{-k})$, $P_+ G_\mu^{(k)}$, and $G G_\mu^{(k)}$ have an expansion*

$$\sum_{j=0}^{\infty} a_j \mu^{n-2k+\nu-1-j} + \sum_{l=0}^{\infty} (a'_l \ln \mu + a''_l) \mu^{-1-2k-l}$$

as $|\mu| \rightarrow \infty$. In particular, the coefficient of $\mu^{-2k} \ln \mu$ is zero. Performing the integration in (5.1) therefore produces no contribution to the coefficient of $\ln t$.

PROPOSITION 5.7. *The trace of $G[(P_1 - \mu^2)^{-k}]_+$ has an expansion*

$$\sum_{j=0}^{\infty} b_j \mu^{n-2k+\nu-1-j} + \sum_{l=0}^{\infty} (b'_l \ln \mu + b''_l) \mu^{-2k-l}.$$

The coefficient b'_0 of $\mu^{-2k} \ln \mu$ is given by

$$\frac{(-1)^k}{(2\pi)^{n-1}} \int_{\partial X} \int_{S'} \operatorname{tr}_E(\operatorname{tr} g_{-n})(x', \xi') \sigma'(\xi') dx'.$$

This is precisely what we expect; note that the missing factor $2 = \operatorname{ord} P_1$ is due to the fact that we consider μ instead of λ and that $\ln \lambda = 2 \ln \mu$.

In conclusion we obtain the following result:

THEOREM 5.8. *For an operator \mathcal{P} of order ν and type zero in Boutet de Monvel's calculus and for $k > (n + \nu)/2$, we have expansions*

$$\begin{aligned} \operatorname{trace} \mathcal{P}(A - \lambda)^{-k} &\sim \sum_{j=0}^{\infty} c_j \lambda^{\frac{n+\nu-j}{2}-k} + \sum_{l=0}^{\infty} (c'_l \ln \lambda + c''_l) \lambda^{-l/2-k}, \\ \operatorname{trace} \mathcal{P} e^{-tA} &\sim \sum_{j=0}^{\infty} \tilde{c}_j t^{\frac{j-n-\nu}{2}} + \sum_{l=0}^{\infty} (\tilde{c}'_l \ln t + \tilde{c}''_l) t^{l/2}, \\ \Gamma(s) \operatorname{trace} (\mathcal{P} A^{-s}) &\sim \sum_{j=0}^{\infty} \frac{\tilde{c}_j}{s + \frac{j-n-\nu}{m}} + \sum_{l=0}^{\infty} \left(\frac{-\tilde{c}'_l}{(s+l/2)^2} + \frac{\tilde{c}''_l}{s+l/2} \right), \end{aligned}$$

for the latter assuming A invertible. The coefficient \tilde{c}'_0 of $\ln t$ satisfies the relation

$$\begin{aligned} -2(2\pi)^n \tilde{c}'_0 &= \int_X \int_S \operatorname{tr}_E p_{-n}(x, \xi) \sigma(\xi) dx \\ &+ 2\pi \int_{\partial X} \int_{S'} \{ \operatorname{tr}_E(\operatorname{tr} g_{-n})(x', \xi') + \operatorname{tr}_{FS_{1-n}}(x', \xi') \} \sigma'(\xi') dx' = \operatorname{res} \mathcal{P}. \end{aligned}$$

As pointed out before, only the first of these expansions has to be established, the other two follow from (1.12) and (1.13).

6. Remarks and References to Further Work

In the one-dimensional case, Wodzicki's residue had been known before; it had been used by Manin [24] and Adler [1] in their work on algebraic aspects of the Korteweg-de Vries equation.

In 1987 Wodzicki gave a more detailed account of the noncommutative residue and several related topics, among them 'higher' residues cf. [37]. A very good survey was compiled by Kassel in 1989 for the Séminaire Bourbaki, [17].

Guillemin discovered the noncommutative residue independently in his so-called 'soft' proof of Weyl's formula on the asymptotic distribution of eigenvalues of operators [13]: For a self-adjoint operator P , given as $\text{op}^w p$ for a classical Weyl symbol p , the counting function $N_P(\lambda)$ of all eigenvalues of P which are less than λ , satisfies $N_P(\lambda) \sim \gamma \text{vol} \{p \leq \lambda\}$ as $\lambda \rightarrow \infty$. Here, γ is a universal constant and vol the symplectic volume.

In [4], Connes proved Theorem 1.18 and used the coincidence of the noncommutative residue and Dixmier's trace in roughly the following way: For an algebra \mathcal{A} and a p -summable Fredholm module (\mathcal{H}, F) over \mathcal{A} , he introduces a 'curvature' θ and an abstract Yang-Mills action $I(\theta)$, using Dixmier's trace. In the case of a 4-dimensional smooth compact Riemannian Spin^c manifold he shows that the classical Yang-Mills action $YM(A)$ for a connection A can be recovered by $YM(A) = 16\pi^2 \inf I(\theta)$ with the infimum taken over a suitable class of connections related to A .

In conformal field theory, Wodzicki's residue has been employed to construct central extensions of the Lie algebra of (pseudo-)differential symbols, cf. Khesin and Kravchenko [21], Radul [28]. It also has been used to derive an action for gravity in the framework of noncommutative geometry, Kalau and Walze [16], Kastler [18].

There are extensions in various directions. Guillemin [15], [14] defined traces on algebras of Fourier integral operators. Lesch [22] extended the residue to symbols including logarithmic terms in their expansion. Melrose [25] as well as Lesch and Pflaum [23] considered certain classes of parameter-dependent operators. Traces on operator algebras on manifolds with conical singularities have been studied in the paper [30] by the author. While new traces arise, there also is the noncommutative residue defined on an ideal. Melrose and Nistor [26] proved an index theorem for an algebra of cusp pseudodifferential operators on manifolds with boundary. They computed the Hochschild cohomology groups and expressed the index in terms of various trace functionals on ideals, among them Wodzicki's. Their work is partly based on the computation of the homology of the algebra of pseudodifferential operators by Brylinsky and Getzler [3], where Wodzicki's residue as well as the higher analogs naturally arise. Kontsevich and Vishik [19], cf. also [20], finally introduced a trace functional TR on classical pseudodifferential operators on noninteger order. For a holomorphic family of classical pseudodifferential operators $\{A(z) : z \in \mathbb{C}\}$ with $\text{ord } A(z) = z$ they showed that $\text{TR } A(z)$ is a meromorphic function on \mathbb{C} with at most simple poles in the integers and $\text{Res}_{z=m} \text{TR } A(z) = -\text{res } A(z)$.

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