Edge-degenerate Boundary Value Problems on Cones

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Abstract: We consider edge-degenerate families of pseudodifferential boundary value problems on a semi-infinite cylinder and study the behavior of their push-forwards as the cylinder is blown up to a cone near infinity. We show that the transformed symbols belong to a particularly convenient symbol class.

This result has applications in the Fredholm theory of boundary value problems on manifolds with edges.

Introduction

Pseudodifferential boundary value problems on manifolds with edges form an algebra in which parametrices to elliptic elements can be constructed by inverting the components of an associated symbol hierarchy. Operators of this kind arise not only in concrete edge situtations, but also in mixed problems, crack theory, and in the solvability theory of boundary value problems in corner domains or in configurations with higher order singularities. Pseudodifferential calculi for edge operators in the boundaryless case have been developed by Mazzeo [4] and Schulze [12, 3].

A crucial step is the inversion of the operator-valued edge symbol. Essentially, this a family of pseudodifferential boundary value problems on an infinite cone with boundary, parametrized by the edge covariable which degenerates in a typical way.

Requiring its invertibility is an analog of the Lopatinskij-Shapiro condition for boundary value problems; the inner normal is here replaced by the cone. Similarly as in the theory of boundary value problems, invertibility can only be achieved by imposing additional trace and potential conditions. They have to be included in the full calculus, similarly as this is done in Boutet de Monvel's concept.

In order to come close to an inverse for the edge-degenerate family, one has to perform a careful analysis both near the tip of the cone and on the infinite part. The analysis near the tip relies on the Mellin calculus for manifolds with conical singularities; it is a central topic in the paper [10] by the authors.

Here, however, we shall deal with the problems arising from the non-compactness of the cone at infinity. We show that the special structure of these symbols allows us to treat them within the framework of the SG-calculus of boundary value problems introduced by the first author; cf. the papers by Shubin [13] and Parenti [5] for earlier use of this symbol class in the boundaryless case.

The main result of this note is Theorem 3.10: Under a blow-up of the cylinder, the pseudodifferential boundary value problems stemming from edge-degenerate families transform into SG-boundary value problems. The blow-up preserves the grading of the algebra so that invertibility of the principal usual symbol will allow an SG-parametrix construction on the infinite part of the cone.

Together with the analysis near the tip this will provide a Fredholm inverse to the edge-degenerate family arising from an elliptic edge symbol and thus an essential step in the inversion process for elliptic edge symbols. This in turn is crucial for the parametrix construction in the edge boundary calculus, cf. [11].

1 Operator-valued Symbols. Sobolev Spaces

Throughout this article let X be an n-dimensional C^∞ manifold with boundary $Y = \partial X$, embedded in an n-dimensional manifold Ω without boundary. $E_1, E_2, \ldots$ are vector bundles over Ω and $F_1, F_2, \ldots$ vector bundles over Y.

On Ω we fix a Riemannian metric; moreover we endow the vector bundles with Hermitian structures so that we can speak of L^2-sections.

By ∂_r we denote an operator coinciding with the normal derivative in a neighborhood of the boundary and vanishing outside a slightly larger neighborhood.

We also fix a function $[\cdot] : \mathbb{R}^n \to \mathbb{R}_+$ such that $[x] > 0$ for all x and $[x] = |x|$ for $|x| \geq 1$. In connection with pseudodifferential operators, one often uses $\langle x \rangle = (1 + |x|^2)^{1/2}$. Clearly, there are constants c_1, c_2 such that $c_1\langle x\rangle \leq [x] \leq c_2\langle x\rangle$, so that both are comparable. This allows us to obtain the usual estimates like Peetre's inequality or the fact that $[t\xi] \leq ct[\xi]$ for $t \geq 1$ and suitable $c > 0$.

$H^s(\mathbb{R}^n)$ is the usual Sobolev space on $\mathbb{R}^n$, while $H^s(\mathbb{R}^n_+) = \{u|_{\mathbb{R}^n_+} : u \in H^s(\mathbb{R}^n)\}$ and $H^s_0(\overline{\mathbb{R}}^n_+)$ is the set of all $u \in H^s(\mathbb{R}^n)$ whose support is contained in $\overline{\mathbb{R}}^n_+$.

For $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$, we let $H^\sigma(\mathbb{R}^n_+) = \{[x]^{-\sigma_2}u : u \in H^{\sigma_1}(\mathbb{R}^n_+)\}$, and $H^\sigma_0(\overline{\mathbb{R}}^n_+) = \{[x]^{-\sigma_2}u : u \in H^{\sigma_1}_0(\overline{\mathbb{R}}^n_+)\}$; here x is the variable in $\overline{\mathbb{R}}^n_+$.

Finally, $\mathcal{S}(\mathbb{R}^n_+) = \{u|_{\mathbb{R}^n_+} : u \in \mathcal{S}(\mathbb{R}^n)\}$. We have $\mathcal{S}(\mathbb{R}^n_+) = \text{proj} - \lim_{\sigma\in\mathbb{R}^2} H^\sigma(\mathbb{R}^n_+)$ and $\mathcal{S}'(\mathbb{R}^n_+) = \text{ind} - \lim_{\sigma\in\mathbb{R}^2} H^{-\sigma}_0(\overline{\mathbb{R}}^n_+)$.

Operator-valued SG-symbols

1.1 Group actions. A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbb{R}_+\}$ of isomorphisms in $\mathcal{L}(E)$ such that $\kappa_\lambda\kappa_\mu = \kappa_{\lambda\mu}$ and the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous for all $e \in E$.

For all the above Sobolev spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$, we will use the group action

$$(\kappa_\lambda f)(x) = \lambda^{n/2} f(\lambda x). \tag{1.1}$$

This action extends to distributions by $\kappa_\lambda u(\varphi) = u(\kappa_{\lambda^{-1}}\varphi)$.

On $E = \mathbb{C}^l$, $l \in \mathbb{N}$, we use the trivial group action $\kappa_\lambda = \text{id}$. Sums of spaces of this kind will be endowed with the sum of the group actions.

1.2 Operator-valued SG-symbols. Let E, F be Banach spaces with strongly continuous group actions $\kappa, \tilde{\kappa}$, let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k, \mathcal{L}(E, F))$, and $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$. We shall write $a \in SG^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^k; E, F)$ provided that, for all multi-indices α, β, γ, there is a constant $C = C(\alpha, \beta, \gamma)$ with

$$||\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta D_{\tilde{y}}^\gamma a(y, \tilde{y}, \eta) \kappa_{[\eta]}||_{\mathcal{L}(E,F)} \le C[\eta]^{\mu_1 - |\alpha|}[y]^{\mu_2 - |\beta|}[\tilde{y}]^{\mu_3 - |\gamma|}. \tag{1.2}$$

We shall simply write SG^μ, if the arguments are obvious from the context. The choice of the best constants C provides a Fréchet topology for SG^μ. In case a is independent of $\tilde{y}$ we shall write $a \in SG^\mu(\mathbb{R}^n, \mathbb{R}^k; E, F)$ with $\mu \in \mathbb{R}^2$.

Extension to projective and inductive limits: Let $\tilde{E}, \tilde{F}$ be Banach spaces with group actions. If $F_1 \hookleftarrow F_2 \hookleftarrow \dots$ and $E_1 \hookrightarrow E_2 \hookrightarrow \dots$ are sequences of Banach spaces with the same group action, and $F = \text{proj} - \lim F_k$, $E = \text{ind} - \lim E_k$, then let $SG^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^k; \tilde{E}, F) = \text{proj} - \lim_k SG^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^k; \tilde{E}, F_k)$ and define $SG^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^k; E, \tilde{F})$ as well as $SG^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^k; E, F)$ similarly as projective limits. We shall use this concept particularly with $E = \mathcal{S}'(\mathbb{R}_+)$ and $F = \mathcal{S}(\mathbb{R}_+)$.

Pseudodifferential operators: For $a \in SG^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^l; E, F)$, the parameter-dependent pseudodifferential operator $\text{op}\, a$ with parameter space $\mathbb{R}^l$ is the operator family $\{\text{op}\, a(\lambda) : \lambda \in \mathbb{R}^l\}$ defined by

$$(\text{op}\, a(\lambda) f)(y) = \int e^{i(y - \tilde{y})\eta} a(y, \tilde{y}, \eta, \lambda) f(\tilde{y}) d\tilde{y} đ\eta, \quad f \in \mathcal{S}(\mathbb{R}^n, E), y \in \mathbb{R}^n. \tag{1.3}$$

This reduces to $(\text{op}\, a(\lambda) f)(y) = \int e^{iy\eta} a(y, \eta, \lambda) \hat{f}(\eta) đ\eta$ for symbols that are independent of $\tilde{y}$. Here, $\hat{f}(\eta) = \mathcal{F}_{y \to \eta} f(\eta) = \int e^{-iy\eta} f(y) dy$ is the vector-valued Fourier transform of f, and $đ\eta = (2\pi)^{-q} d\eta$.

One obtains a full symbolic calculus for operators with SG-symbols. In particular: (i) There is asymptotic summation; (ii) an operator with 'double' symbol in $SG^{(\mu_1, \mu_2, \mu_3)}$ has a 'left' symbol in $SG^{(\mu_1, \mu_2 + \mu_3)}$ and a 'right' symbol in $SG^{(\mu_1, 0, \mu_2 + \mu_3)}$, (iii) the composition of two SG-pseudodifferential operators is an SG-operator; its order is the sum of the orders.

1.3 General operator-valued symbols. We use the above notation. For $\mu \in \mathbb{R}$, we shall say that $a \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^k; E, F)$ if

$$||\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta D_{\tilde{y}}^\gamma a(y, \tilde{y}, \eta) \kappa_{[\eta]}||_{\mathcal{L}(E,F)} \le C[\eta]^{\mu - |\alpha|}.$$

We then have the above properties replacing $SG^\mu, \mu \in \mathbb{R}^3$ or $\mu \in \mathbb{R}^2$, by $S^\mu, \mu \in \mathbb{R}$.

1.4 Definition. Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space. The exterior direct sum $E \oplus F$ is Fréchet and has the closed subspace $\mathcal{N} = \{(a, -a) : a \in E \cap F\}$. The non-direct sum of E and F then is the Fréchet space $E + F := E \oplus F / \mathcal{N}$.

2 A Boutet de Monvel Type Calculus with Weighted Symbols

We review the concept of Boutet de Monvel's calculus with weighted symbols introduced in [6]. More details can be found in [7, 8]. We start with a review of the relevant spaces and terminology.

2.1 Definition. (a) Given a function f on $\mathbb{R}^n_+$ we denote by e^+f the function on $\mathbb{R}^n$ which is equal to f on $\mathbb{R}^n_+$ and zero otherwise. In the same way we let e^+g be the extension by zero of a function g on X to a function on Ω. By r^+f we denote the restriction of a function f on $\mathbb{R}^n$ to $\mathbb{R}^n_+$. Similarly we denote the restriction of a function g on Ω to X by r^+g.

(b) Let $H^+ = \{(e^+f)\hat{}\,: f \in \mathcal{S}(\mathbb{R}_+)\}, H_0^- = \{(e^-f)\hat{}\,: f \in \mathcal{S}(\mathbb{R}_-)\}$; the hat $\hat{}$ indicates the Fourier transform on $\mathbb{R}$. H' denotes the space of all polynomials. Then define

$$H = H^+ \oplus H_0^- \oplus H'.$$

(c) A symbol $p \in SG^\mu(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^l)$ has the SG-transmission property at $x_n = \tilde{x}_n = 0$ if, for all $k, k' \in \mathbb{N}$,

$$D^k_{x_n} D^{k'}_{\tilde{x}_n} p(x', 0, \tilde{x}', 0, \xi', [\xi', \lambda]\xi_n, \lambda) \in SG^\mu(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{\tilde{x}'}, \mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}^l_\lambda)\hat{\otimes}_\pi H_{\xi_n}.$$

Write $p \in SG^\mu_{tr}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^l)$. The subscripts indicate the variables with respect to which the corresponding properties hold.

2.2 Weighted parameter-dependent operators and symbols. A parameter-dependent SG-operator of order $\mu \in \mathbb{R}^2$ and type $d \in \mathbb{N}$ in Boutet de Monvel's calculus on $\overline{\mathbb{R}}^n_+$ is a family of operators

$$A(\lambda): \begin{array}{c}\mathcal{S}(\mathbb{R}^n_+)^{n_1}\\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1})^{m_1}\end{array} \to \begin{array}{c}\mathcal{S}(\mathbb{R}^n_+)^{n_2}\\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1})^{m_2}\end{array}, \quad \lambda \in \mathbb{R}^l \tag{2.4}$$

of the form $A(\lambda) = \begin{pmatrix} P_+(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sum_{j=0}^d \operatorname{op} g_j(\lambda)\partial_r^j & \operatorname{op} k(\lambda) \\ \sum_{j=0}^d \operatorname{op} t_j(\lambda)\partial_r^j & \operatorname{op} s(\lambda) \end{pmatrix}$, where ∂_r is the normal derivative and

(i) $P(\cdot) = \operatorname{op} p(\cdot)$ with $p \in SG^\mu_{tr}(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^l)$, $P_+ = r^+Pe^+$.

(ii) The symbols g_j, t_j, k, and s belong to the following spaces:

$$\begin{array}{ll} g_j & \in SG^{\mu-(j,0)}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \times \mathbb{R}^l; \mathcal{S}'(\mathbb{R}_+)^{n_1}, \mathcal{S}(\mathbb{R}_+)^{n_2}), \\ t_j & \in SG^{\mu-(j,0)}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \times \mathbb{R}^l; \mathcal{S}'(\mathbb{R}_+)^{n_1}, \mathbb{C}^{m_2}), \\ k & \in SG^\mu(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \times \mathbb{R}^l; \mathbb{C}^{m_1}, \mathcal{S}(\mathbb{R}_+)^{n_2}), \text{ and} \\ s & \in SG^\mu(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \times \mathbb{R}^l; \mathbb{C}^{m_1}, \mathbb{C}^{m_2}). \end{array}$$

We shall write $A \in \mathcal{B}^{\mu,d}_{SG}(\overline{\mathbb{R}}^n_+; \mathbb{R}^l)$; moreover, we write $A \in \mathcal{B}^{-\infty,d}_{SG}(\overline{\mathbb{R}}^n_+; \mathbb{R}^l)$ for a regularizing parameter-dependent operator of type d, i.e. for an operator in the intersection $\bigcap_{\mu \in \mathbb{R}^2} \mathcal{B}^{\mu,d}_{SG}(\overline{\mathbb{R}}^n_+; \mathbb{R}^l)$.

The decomposition $P_+ + G$ is not unique. The topology on $\mathcal{B}^{\mu,d}_{SG}(\overline{\mathbb{R}}^n_+; \mathbb{R}^l)$ and $\mathcal{B}^{-\infty,d}_{SG}(\overline{\mathbb{R}}^n_+; \mathbb{R}^l)$ is that of a non-direct sum of Fréchet spaces.

2.3 Definition. (a) (Erkip & Schrohe [3]) Let Ω be an n-dimensional manifold without boundary. Call Ω *SG-compatible* if conditions (SG1) – (SG3) hold.

(SG1) Ω has a finite covering by coordinate charts: $\Omega = \cup_{j=1}^{J} \Omega_j$.
(SG2) This cover has a good shrinking.
(SG3) All the changes of coordinates χ satisfy $\partial^\alpha \chi(x) = \mathrm{O}([x]^{1-\alpha})$.

The existence of a good shrinking in (SG2) means that Ω may also be written as the union of sets $\Omega_j' \subseteq \Omega_j$, such that there is an $\epsilon > 0$ with $B(x, \epsilon[x]) \subseteq \kappa_j(\Omega_j)$ for every $x \in \kappa_j(\Omega_j')$.

Clearly, $\mathbb{R}^n$ is SG-compatible with its standard coordinates and so is every compact manifold. A more elaborate example will be given in 2.5, below.

Let X be an n-dimensional submanifold of Ω with boundary $\partial X = Y$, where Y is an $(n-1)$-dimensional submanifold without boundary. Assume additionally that

(SG4) The functions $\kappa_j : \Omega_j \to \mathbb{R}^n$ map $X \cap \Omega_j$ to $\overline{\mathbb{R}}^n_+$, $Y \cap \Omega_j$ to $\partial \mathbb{R}^n_+$, and $\Omega_j \cap (\Omega \backslash X)$ to $\mathbb{R}^n_-$.
(SG5) There is a Riemannian metric g on Ω whose tensor in local coordinates, g_{ij}, satisfies the estimates $\partial^\alpha g_{ij}(x) = \mathrm{O}([x]^{-\alpha})$, $(g_{ij})(x)^{-1} = \mathrm{O}(1)$.

We then call the quadruple (Ω, X, Y, g) an *SG-manifold with boundary* or simply *SG-compatible*. A simple example is given by ($\mathbb{R}^n, \overline{\mathbb{R}}^n_+, \mathbb{R}^{n-1}$, Euclidean metric).

It is easy to see that pull-backs of the Euclidean metric on $\kappa_j(\Omega_j)$ can be patched together using a partition of unity to yield a metric with property (SG5).

(b) $\mathcal{S}(\Omega)$ and $\mathcal{S}(X)$ denote the spaces of all smooth functions on Ω and X, respectively, that satisfy the estimates for rapidly decreasing functions in all local coordinates. All notions are justified by (SG3). Given a vector bundle E over Ω with all transition functions satisfying *SG*-estimates of order zero (*SG*-vector bundles), we define $\mathcal{S}(\Omega, E)$, $\mathcal{S}(X, E)$, $H^s(\Omega, E)$, etc. in the obvious way.

2.4 Boutet de Monvel's algebra on an SG-manifold. Let $\Omega, \{\Omega_j\}, X, Y$ be as above, let E_1, E_2 be *SG*-vector bundles over Ω, and let F_1, F_2 be *SG*-vector bundles over Y. We shall say that a family $A = \{A(\lambda) : \lambda \in \mathbb{R}^l\}$

$$A(\lambda) : \begin{array}{c} \mathcal{S}(X, E_1) \\ \oplus \\ \mathcal{S}(Y, F_1) \end{array} \to \begin{array}{c} \mathcal{S}(X, E_2) \\ \oplus \\ \mathcal{S}(Y, F_2) \end{array}$$

is an element of $\mathcal{B}^{\mu,d}_{SG}(X; \mathbb{R}^l), \mu \in \mathbb{R}^2, d \in \mathbb{N}_0$, provided that

(i) For every choice of functions φ, ψ supported in the same coordinate neighborhood and satisfying zero order *SG* estimates, the push-forward of $\Phi A \Psi$ is an element of $\mathcal{B}^{\mu,d}_{SG}(\overline{\mathbb{R}}^n_+; \mathbb{R}^l)$. Here Φ and Ψ denote the operators of multiplication by diag $\{\varphi, \varphi|_Y\}$ and diag $\{\psi, \psi|_Y\}$, respectively.
(ii) If φ, ψ are as before, but the coordinate chart does not intersect the boundary, then all entries in the matrix $(M_\varphi A(\lambda) M_\psi)_*$ – except for the pseudodifferential part – are parameter-dependent regularizing.
(iii) Given two functions φ, ψ which satisfy the estimates for an SG^0-function in all local coordinates and additionally have disjoint support, the operator $\Phi A \Psi$, defined as above, is a parameter-dependent regularizing operator.

Recall that a regularizing operator of type 0 is an integral operator with a kernel section in $\big(\mathcal{S}(X,E_2)\oplus\mathcal{S}(Y,F_2)\big)\hat{\otimes}_\pi\big(\mathcal{S}(X,E_1)\oplus\mathcal{S}(Y,F_1)\big)$. A regularizing operator of type d is a sum $R=\sum_{j=0}^d R_j\begin{bmatrix}\partial_r^j & 0\\ 0 & I\end{bmatrix}$ with all R_j regularizing of type zero. The parameter-dependent regularizing operators of type d, denoted by $\mathcal{B}^{-\infty,d}(X;\mathbb{R}^l)$, are Schwartz functions on $\mathbb{R}^l$ with values in the regularizing elements of type d.

2.5 Example. Let $\Omega,\{\Omega_j\},X,Y$ be as before; assume additionally that Ω is compact, hence so are X and Y. By $\kappa_j:\Omega_j\to U_j\subseteq\mathbb{R}^n$ denote the coordinate maps. We then define the manifold $\Omega^{\asymp}$ by introducing on $\Omega\times\mathbb{R}$ coordinate maps

$$\chi_j:\Omega_j\times\mathbb{R}\to\mathbb{R}^{n+1},\quad \chi_j(x,t)=([t]\kappa_j(x),t).$$

Topologically, $\Omega^{\asymp}\cong\Omega\times\mathbb{R}$; intuitively, these coordinates make $\Omega^{\asymp}$ look like an outgoing cone with two ends.

It is easily checked that $\Omega^{\asymp}$ is an SG-compatible manifold. Restricting the coordinate maps to X we obtain $X^{\asymp}$. This is an SG-manifold with boundary $Y^{\asymp}$ when endowed with the metric induced from Euclidean space via the χ_j.

2.6 The standard version of Boutet de Monvel's calculus. We obtain the usual version of the calculus, if we ask for uniform boundedness of all x-derivatives instead of estimating by $[x]^{\mu_2-|\beta|}$. More precisely, we shall write $A\in\mathcal{B}^{\mu,d}(\overline{\mathbb{R}}^n_+;\mathbb{R}^l)$ if A is of the form (2.2) with

$$\begin{array}{lcll}
p & \in & S^\mu_{tr}(\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^l)\ ((n_1\times n_2)\text{-matrix-valued}); & \\
g_j & \in & S^\mu(\mathbb{R}^{n-1},\mathbb{R}^{n-1}\times\mathbb{R}^l;\mathcal{S}'(\mathbb{R}_+)^{n_1},\mathcal{S}(\mathbb{R}_+)^{n_2})\ , & j=0,\dots,d\\
t_j & \in & S^\mu(\mathbb{R}^{n-1},\mathbb{R}^{n-1}\times\mathbb{R}^l;\mathcal{S}'(\mathbb{R}_+)^{n_1},\mathbb{C}^{m_2})\ , & j=0,\dots,d\\
k & \in & S^\mu(\mathbb{R}^{n-1},\mathbb{R}^{n-1}\times\mathbb{R}^l;\mathbb{C}^{m_1},\mathcal{S}(\mathbb{R}_+)^{n_2})\ ; & \\
s & \in & S^\mu(\mathbb{R}^{n-1},\mathbb{R}^{n-1}\times\mathbb{R}^l;\mathbb{C}^{m_1},\mathbb{C}^{m_2})\ . &
\end{array}$$

We then can define operators on open manifolds; we ask that, in local coordinates, all operators be of the above form.

3 Behavior under Blow-up

3.1 Relation to edge-degenerate boundary value problems. In the Fredholm theory of pseudodifferential boundary value problems on manifolds with edges, we have an analog of the classical Lopatinskij-Shapiro condition, namely the invertibility of the principal edge symbol. Apart from the trace and potential conditions (which will be taken care of later) this is a family of boundary value problems in Boutet de Monvel's algebra on an infinite cone, the so-called model cone. It is parametrized by the edge-variable (which is omitted here, since it plays a minor role) and the edge-covariable, η.

In what follows, t is the axial variable of this cone, τ the corresponding covariable; x is the variable along the base X of the cone, ξ its covariable. Geometrically, the use of the variables (t,x) gives the picture of an infinite cylinder. The fact that we are dealing with a cone is reflected in the degeneracy of the axial covariable: τ only appears as $t\tau$. Similarly, the edge covariable only comes up in the degenerate form $t\eta$. As in the case of classical boundary value problems, the t-variable is 'frozen' at the edge, i.e. $t = 0$, and the edge-covariable $\eta \neq 0$ is fixed. The boundary value problem therefore depends on t only implicitly, via $t\tau$ and $t\eta$.

In order to establish the invertibility of the full edge symbol (including trace and potential conditions) one needs the Fredholm property of this edge-degenerate family of boundary value problems for each $\eta \neq 0$ on the Sobolev spaces over the infinite cone. Establishing it splits into two tasks: The analysis near the tip, which will be performed in [10], and the analysis on the cone near infinity. There the cone over X coincides with the manifold $X^{\asymp}$, which is a particularly simple SG-manifold.

What we shall establish in this section is the following: The blow-up which makes the cylinder $X \times \mathbb{R}$ the SG 'double cone' $X^{\asymp}$ induces a push-forward on the level of operators which transforms these edge-degenerate families of boundary value problems into SG-operators, cf. Theorem 3.14. The mapping properties on the cone Sobolev spaces are therefore immediate. Moreover, the natural grading of the algebras is preserved: The push-forward of lower order terms is of lower order in the SG sense. This will play an important role in the construction of a Fredholm inverse.

We start by computing the behavior of the symbol under the push-forward; the main technical result is Theorem 3.8, which allows us to deal with all components in Boutet de Monvel's calculus at the same time.

3.2 Definition and Lemma. We shall use the diffeomorphism

$$\chi : \mathbb{R}^n \times \overline{\mathbb{R}}_+ \to \mathbb{R}^n \times \overline{\mathbb{R}}_+, \qquad \chi(x,t) = ([t]x, t).$$

Its inverse is given by $\chi^{-1}(y,t) = (y/[t], t)$. We have

$$D\chi^{-1}(y,t) = \begin{pmatrix} [t]^{-1}I & -y\partial_t[t]/[t]^2 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

We define the function M on $\overline{\mathbb{R}}^{n+1}_+ \times \overline{\mathbb{R}}^{n+1}_+$ by

$$M(y,t,\tilde{y},\tilde{t}) \;=\; \int_0^1 D\chi^{-1}(y+\sigma(\tilde{y}-y), t+\sigma(\tilde{t}-t))\, d\sigma; \tag{3.6}$$

M is an invertible $(n+1)\times(n+1)$-matrix. For $t, \tilde{t} \geq 1$,

$$M(y,t,\tilde{y},\tilde{t}) = \begin{pmatrix} \frac{\ln t - \ln \tilde{t}}{t-\tilde{t}} I & -\int_0^1 \frac{y+\sigma(\tilde{y}-y)}{(t+\sigma(\tilde{t}-t))^2}\, d\sigma \\ 0 & 1 \end{pmatrix}$$

by (3.5). Its determinant only depends on t and $\tilde{t}$. Using the abbreviation

$$T(t,\tilde{t}) = \frac{t-\tilde{t}}{\ln t - \ln \tilde{t}}$$

we see that, for t, $\tilde{t} \geq 1$, the inverse of the adjoint is

$$M^{-\mathrm{T}} = \begin{pmatrix} TI & 0 \\ T\int_0^1 \frac{y+\sigma(\tilde{y}-y)}{(t+\sigma(\tilde{t}-t))^2}\, d\sigma & 1 \end{pmatrix}, \tag{3.7}$$

and $\det M^{-\mathrm{T}}(t,\tilde{t}) = T(t,\tilde{t})^n$.

3.3 Lemma. *Let $\tilde p \in C_b^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, S^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{n+1+q}; E, F))$. Fix $\eta \neq 0$, and define $p(t, x, \tilde t, \tilde x, \tau, \xi, \eta) = \tilde p(t, \tilde t; x, \tilde x, \xi, t\tau, t\eta)$.*

The symbol of the push-forward $\chi_*(\operatorname{op} p)$ is given by

$$q(y, t, \tilde y, \tilde t, \xi, \tau) = \tilde p\left(t, \tilde t; y/[t], \tilde y/[\tilde t], M^{-\mathrm{T}}\begin{pmatrix}\xi\\ t\tau\end{pmatrix}, t\eta\right) T(t, \tilde t)^n [\tilde t]^{-n}.$$

For $t, \tilde t \geq 1$ this expression simplifies to

$$\tilde p\left(t, \tilde t; y/t, \tilde y/\tilde t, T\xi, c\xi + t\tau, t\eta\right) T(t, \tilde t)^n \tilde t^{-n}.$$

Here $c = c(y, t, \tilde y, \tilde t) = T(t, \tilde t)\int_0^1 \frac{y+\sigma(\tilde y - y)}{(t+\sigma(\tilde t - t))^2}d\sigma$, and T is as above.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^{n+1})$ and $(y, t) \in \mathbb{R}^{n+1}$. Then

$$\begin{aligned}(\chi_* \operatorname{op} p) f(y, t) &= \operatorname{op} p(\chi^* f)(\chi^{-1}(y, t)) \qquad (3.8)\\ &= \iiiint e^{i(\frac{y}{[t]} - \frac{\tilde y}{[\tilde t]})\xi + i(t-\tilde t)\tau}\tilde p\left(t, \tilde t; \frac{y}{[t]}, \frac{\tilde y}{[\tilde t]}, \xi, t\tau, t\eta\right) f(\tilde y, \tilde t)[\tilde t]^{-n}\, d\tilde y\, d\tilde t\, đ\xi\, đ\tau.\end{aligned}$$

Here we made the substitution $\tilde x = \tilde y/[\tilde t]$. Now we note that

$$\begin{aligned}(y/[t] - \tilde y/[\tilde t])\,\xi + (t - \tilde t)\tau &= \left(\chi^{-1}(y, t) - \chi^{-1}(\tilde y, \tilde t)\right)\begin{pmatrix}\xi\\ \tau\end{pmatrix}\\ &= ((y, t) - (\tilde y, \tilde t)) M^{\mathrm{T}}(y, t, \tilde y, \tilde t)\begin{pmatrix}\xi\\ \tau\end{pmatrix}.\end{aligned}$$

The integral (3.8) therefore equals

$$\iiiint e^{i((y,t)-(\tilde y, \tilde t))\binom{\xi}{\tau}}\tilde p\left(t, \tilde t; \frac{y}{[t]}, \frac{\tilde y}{[\tilde t]}, M^{-\mathrm{T}}\begin{pmatrix}\xi\\ t\tau\end{pmatrix}, t\eta\right) f(\tilde y, \tilde t)\frac{\det M^{-\mathrm{T}}}{[\tilde t]^n}\, d\tilde y\, d\tilde t\, đ\xi\, đ\tau.$$

Now the first assertion follows from (3.7). The second is immediate. ◁

3.4 Proposition. *Let $\varphi, \tilde\varphi \in C_0^\infty(\mathbb{R}^n)$ with disjoint support and $\omega, \tilde\omega \in C_0^\infty(\mathbb{R}_+)$ with $\omega(t) = \tilde\omega(t) = 1$ for $t \leq 1$. Moreover let $\tilde p \in C_b^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, S^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^{1+q}; E, F))$. Fix $0 \neq \eta \in \mathbb{R}^q$, and define*

$$p(t, x, \tilde t, \tilde x, \tau, \xi) = \tilde p(t, \tilde t; x, \tilde x, \xi, t\tau, t\eta).$$

Then the push-forward

$$\chi_*(\varphi(x)(1 - \omega(t))(\operatorname{op}_{x,t} p)(1 - \tilde\omega(t))\tilde\varphi(x)) \qquad (3.9)$$

is an integral operator with a rapidly decreasing kernel taking values in $\mathcal{L}(E, F)$.

Proof. We abbreviate $F(x, \tilde x, t, \tilde t) = \varphi(x)\tilde\varphi(\tilde x)(1-\omega(t))(1-\tilde\omega(\tilde t))$. For $u \in \mathcal{S}(\mathbb{R}^{n+1})$, $K \in \mathbb{N}_0$ with $\mu - K < -n - 1$ we have

$$\begin{aligned}&(\varphi(1-\omega)[\operatorname{op} p](1-\tilde\omega)\tilde\varphi)u(x, t) =\\ &= \iiiint e^{i(x-\tilde x)\xi + i(t-\tilde t)\tau} F(x, \tilde x, t, \tilde t)\tilde p(t, \tilde t; x, \tilde x, \xi, t\tau, t\eta) u(\tilde x, \tilde t)\, d\tilde x\, d\tilde t\, đ\xi\, đ\tau\\ &= \iiiint e^{i(x-\tilde x)\xi + i(t-\tilde t)\tau}\frac{F(x, \tilde x, t, \tilde t)}{|x - \tilde x|^{2K}}\Delta_\xi^K \tilde p(t, \tilde t; x, \tilde x, \xi, t\tau, t\eta) u(\tilde x, \tilde t) d\tilde x d\tilde t đ\xi đ\tau.\end{aligned}$$

The integral exists and we may rewrite $\varphi(1-\omega)(\operatorname{op} p)(1-\tilde{\omega})\tilde{\varphi}$ as the integral operator with the kernel $k = k(x,t,\tilde{x},\tilde{t})$ given by

$$t^{-1}\iint e^{i(x-\tilde{x})\xi + i\frac{(t-\tilde{t})}{t}\tau} F(x,\tilde{x},t,\tilde{t})|x-\tilde{x}|^{-2K}(\Delta_\xi^K \tilde{p})(t,\tilde{t};x,\tilde{x},\xi,\tau,t\eta)\,d\xi\,d\tau.$$

Using integration by parts we conclude that

$$\begin{aligned} t\left[\frac{t-\tilde{t}}{t}\right]^{2N}|k(x,t,\tilde{x},\tilde{t})| &\leq c_1\iint [\xi,\tau,t\eta]^{\mu-2K-2N}\,d\xi\,d\tau\; F(x,\tilde{x},t,\tilde{t})|x-\tilde{x}|^{-2K} \\ &\leq c_2\iint [\xi,\tau]^{\mu-K}[t\eta]^{-K-2N}d\xi d\tau\, F(x,\tilde{x},t,\tilde{t})|x-\tilde{x}|^{-2K} \\ &\leq c_3\,[t\eta]^{-K-2N}\,F(x,\tilde{x},t,\tilde{t})|x-\tilde{x}|^{-2K}. \end{aligned}$$

For $t\geq 1$ we may estimate $[\frac{t-\tilde{t}}{t}]^{-2N}$ by $t^{2N}[t-\tilde{t}]^{-2N}$. Moreover, for $\eta\neq 0$ we have $[t\eta]^{-K-2N} = O([t]^{-K-2N})$.

Applying Peetre's inequality and using that $\varphi,\tilde{\varphi}$ have compact disjoint support we deduce that, for arbitrary L, we have

$$|k(x,t,\tilde{x},\tilde{t})| \leq C[x]^{-L}[\tilde{x}]^{-L}[t]^{-L}[\tilde{t}]^{-L}.$$

Differentiation under the integral sign yields the same estimate for the derivatives of k. The push-forward of k under χ is the integral kernel $l(y,t,\tilde{y},\tilde{t}) = [t]^{-n}k(y/[t],t,\tilde{y}/[\tilde{t}],\tilde{t})$. It is obviously also rapidly decreasing with respect to all variables. $\triangleleft$

In the same way as Proposition 3.4 we can show the following.

3.5 Proposition. *Let $p,\tilde{p},\omega,\tilde{\omega}$ be as in 3.4, $\varphi,\tilde{\varphi}\in C_0^\infty(\mathbb{R}^n)$. Fix $\varepsilon>0$. Suppose ψ is a function in $C_b^\infty(\mathbb{R}\times\mathbb{R})$, with $\psi(t,\tilde{t})=0$ for $|t-\tilde{t}|\leq\varepsilon$ and $\psi(t,\tilde{t})=1$ for $|t-\tilde{t}|\geq 2\varepsilon$.*

Then the push-forward (3.9) is an integral operator with a rapidly decreasing integral kernel taking values in $\mathcal{L}(E,F)$.

In the following, we shall often restrict the variables to the set

$$S = \{(y,t,\tilde{y},\tilde{t},\xi,\tau) : t,\tilde{t}\geq 1, |t/\tilde{t}-1|<1/2, |y/t|\leq C, |\tilde{y}/\tilde{t}|\leq C\}. \qquad (3.10)$$

Note that, on S, we have $[t]\sim[\tilde{t}]\sim[y,t]\sim[\tilde{y},\tilde{t}]$.

3.6 Lemma. *Let c,T be as in 3.2 and 3.3. On the set S, we have*
(a) $|D_t^k D_{\tilde{t}}^l T(t,\tilde{t})| = O(t^{1-k-l})$ *and* $T(t,\tilde{t})\geq c_0 t$ *for some constant* $c_0>0$.
(b) $|D_y^\alpha D_t^k D_{\tilde{y}}^\beta D_{\tilde{t}}^l c(y,t,\tilde{y},\tilde{t})| = O(t^{-|\alpha|-k-|\beta|-l})$. *The left hand side is zero for* $|\alpha|+|\beta|>1$.

On S, the function c hence satisfies SG^0-estimates and T those for $SG^{(0,1,0)}$.

Proof. (a) Write $T(t,\tilde{t}) = ts(\tilde{t}/t)$, where $s(r)=(r-1)/\ln r$. Clearly, $T(t,\tilde{t})=O(t)$ and $T(t,\tilde{t})\geq c_0 t$. Moreover, $(t\partial_t)T(t,\tilde{t}) = t(s(r)-(r\partial_r)s(r))|_{r=\tilde{t}/t} = O(t)$; in an analogous way, one sees that $(\tilde{t}\partial_{\tilde{t}})T = O(t)$. Since $t^kD_t^k$ can be written as a linear combination of terms of the form $(tD_t)^j, 1 \le j \le k$, we see that $t^k D_t^k \tilde{t}^l D_{\tilde{t}}^l T(t,\tilde{t}) = O(t)$.

(b) $|c(y,t,\tilde{y},\tilde{t})| \le \max\{|y|,|\tilde{y}|\} \int_0^1 (t+\sigma(\tilde{t}-t))^{-2} d\sigma = \max\{|y|,|\tilde{y}|\}(\tilde{t}t)^{-1}$. Since y/t and $\tilde{y}/t$ both are bounded and since $T(t,\tilde{t}) = O(t)$, we conclude that $c = O(1)$. Next we observe that

$$D_t^k D_{\tilde{t}}^l c(y,t,\tilde{y},\tilde{t}) = c_{kl} \int_0^1 (1-\sigma)^k \sigma^l \frac{y+\sigma(\tilde{y}-y)}{(t+\sigma(\tilde{t}-t))^{2+k+l}} d\sigma.$$

The same estimate as above shows that this term is $\le C \max\{|y|,|\tilde{y}|\} t^{-2-k-l}$. Now (a) and Leibniz' rule imply the assertion for $\alpha = \beta = 0$. In case $|\alpha|+|\beta| = 1$, the integrand is $(1-\sigma)^{k+|\alpha|}\sigma^{l+|\beta|}(t+\sigma(\tilde{t}-t))^{-2-k-l}$ and we conclude as before. ◁

3.7 Lemma. *Let $\eta \neq 0$ be fixed. There exist constants $C, \delta > 0$, such that*

$$\delta t[\xi,\tau] \le [T\xi, c\xi + t\tau, t\eta] \le Ct[\xi,\tau] \text{ on } S. \tag{3.11}$$

Proof. If $|\xi,\tau| \le 1$, then the middle term side is $\ge [t\eta] \ge c_1 t \ge c_2 t[\xi,\tau]$. So we may assume that $|\xi,\tau| > 1$. We let $\gamma = \sup|c| + 1$ and distinguish the cases where $|\tau| \le 2\gamma|\xi|$ and $|\tau| \ge 2\gamma|\xi|$, respectively. Note that in the former, we have $|\xi| \ge (2\gamma+1)^{-1}$, in the latter $|\tau| \ge 1/2$.

For $|\tau| \le 2\gamma|\xi|$, the middle term in (3.11) is $\ge c_3|T\xi| \ge c_4 t|\xi| \ge c_5 t[\xi,\tau]$. The last inequality is due to the fact that $|\xi| \ge (2\gamma+1)^{-1}$. For $|\tau| \ge 2\gamma|\xi|$, we note that

$$|c\xi + t\tau| \ge |t\tau| - |c\xi| \ge \frac{t}{2}|\tau| + \frac{1}{2}|\tau| - \gamma|\xi| \ge \frac{t}{2}|\tau| \ge c_6[\xi,\tau].$$

We thus obtain the estimate from below.

For the second inequality we first note that $T \le c_0 t$ and therefore $[T\xi] \le c_7 t[\xi] \le c_8 t[\xi,\tau]$. Similarly $[c\xi + t\tau] \le c_9([c\xi] + [t\tau]) \le c_{10} t[\xi,\tau]$. This yields the estimate from above. ◁

3.8 Theorem. *Let $\tilde{p} \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^{1+q}; E, F)$, $\mu \in \mathbb{R}$, let $\varphi, \tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ and $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega(t) = \tilde{\omega}(t) = 1$ for $t \le 1$. Fix $\eta \neq 0$ and define*

$$p(t,x,\tau,\xi) = \tilde{p}(x,\tilde{x},\xi,t\tau,t\eta).$$

Then the push-forward $\chi_*(\varphi(x)(1-\omega(t))(\mathrm{op}_{x,t}p)(1-\tilde{\omega}(t))\tilde{\varphi}(x))$ has a symbol in $SG^{(\mu,\mu,0)}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1}; E, F)$. Its symbol semi-norms can be estimated by those for $\tilde{p}$.

Proof. Since ω and $\tilde{\omega}$ vanish for $t \le 1$, we deduce from Lemma 3.3 that the symbol $q = q(y,t,\tilde{y},\tilde{t},\xi,\tau)$ of the push-forward is given by

$$\tilde{p}\left(y/t, \tilde{y}/\tilde{t}, T\xi, c\xi + t\tau, t\eta\right)(1-\omega(t))(1-\tilde{\omega}(\tilde{t}))\varphi(y/t)\tilde{\varphi}(\tilde{y}/\tilde{t})T^n[\tilde{t}]^{-n}.$$

The compactness of $\operatorname{supp}\varphi$ and $\operatorname{supp}\tilde{\varphi}$ shows that y/t and $\tilde{y}/\tilde{t}$ may be assumed to be bounded. Moreover, we may suppose that $|t-\tilde{t}|$ is small, since we may multiply by a function supported near $t = \tilde{t}$, at the expense of an error whose push-forward is an integral operator with a rapidly decreasing kernel, as we saw in 3.5.

In fact, since $t, \tilde t \geq 1$, we may assume in particular that $|t/\tilde t - 1| < 1/2$. We therefore only have to establish the symbol estimates on the set S of (3.10), for q vanishes on the complement.

We next note the following identities. In order to save space we shall write $(\dots)$ instead of $(y/t, \tilde y/\tilde t, T(t,\tilde t)\xi, c(y,t,\tilde y,\tilde t)\xi + t\tau, t\eta)$

$$
\begin{aligned}
D_{y_j}\{\tilde p(\dots)\} &= (D_{y_j}\tilde p)(\dots)t^{-1} + (D_\tau \tilde p)(\dots)\sum_{k=1}^n \partial_{y_j} c_k \xi_k,\\
D_t\{\tilde p(\dots)\} &= \sum_{k=1}^n \{(D_{y_k}\tilde p)(\dots)(-y_k/t^2) + (D_{\xi_k}\tilde p)(\dots)\partial_t T\xi_k + \qquad (3.12)\\
&\quad + (D_\tau \tilde p)(\dots)(\partial_t c_k \xi_k + \tau) + (D_{\eta_k}\tilde p)(\dots)\eta_k\},\\
D_{\xi_j}\{\tilde p(\dots)\} &= (D_{\xi_j}\tilde p)(\dots) + (D_\tau \tilde p)(\dots)c_j,\\
D_\tau\{\tilde p(\dots)\} &= (D_\tau \tilde p)(\dots)t.
\end{aligned}
$$

Here c_k and ξ_k denote the components of c and ξ, respectively. The derivatives with respect to $\tilde y$ and $\tilde t$ are easily deduced from those. If we restrict the attention to S, then $\partial_{y_j} c_k\xi_k$ satisfies the estimates for an $SG^{(1,-1)}$-function, $\partial_t c_k \xi_k + \tau$ and $\partial_t T\xi_k$ those for a $SG^{(1,0)}$-function, while t^{-1} and y_k/t^2 satisfy those for an $SG^{(0,-1)}$-function. Also $(t,y) \mapsto \varphi(y/t)$ and $(\tilde t, \tilde y) \mapsto \varphi(\tilde y/\tilde t)$ are SG^0 functions on S.

According to Lemma 3.7 we may estimate $D_{y_k}\tilde p(\dots)$ by $[t]^\mu[\xi,\tau]^\mu$ while $D_{\xi_j}\tilde p$, $D_\tau \tilde p$, and $D_{\eta_k}\tilde p$ are $O([t]^{\mu-1}[\xi,\tau]^{\mu-1})$. Since $[y,t] \sim [t] \sim T \sim [\tilde t] \sim [\tilde y, \tilde t]$ on S, we obtain the estimate

$$
D_\xi^\alpha D_\tau^r D_y^\beta D_{\tilde y}^{\beta'} D_t^k D_{\tilde t}^{k'} q(y,t,\tilde y,\tilde t,\xi,\tau) = O\Big([y,t]^{\mu-|\beta|-k}[\tilde y,\tilde t]^{-|\beta'|-k'}[\tau,\xi]^{\mu-|\alpha|-r}\Big)
$$

provided the total number of derivatives is ≤ 1. The form of the derivatives in (3.12), however, together with the above observations on the functions $\partial_{y_j} c_k\xi_k, \dots, y_k/t^2$ shows that the general result follows in the same way. ◁

We shall now apply this to boundary value problems in Boutet de Monvel's calculus. We consider the half-space $\overline{\mathbb{R}}^{n+1}_+ = \{(t, x_1, \dots, x_n) : t, x_1, \dots, x_{n-1} \in \mathbb{R}, x_n \in \overline{\mathbb{R}}_+\}$. We shall see that the blow-up induced by χ transforms an ordinary t-independent operator in Boutet de Monvel's calculus into an SG operator.

3.9 The situation. We let $\widetilde A \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}^n_+; \mathbb{R}^{1+q})$,

$$
\widetilde A = \begin{pmatrix} \widetilde P_+ + \widetilde G & \widetilde K \\ \widetilde T & \widetilde S \end{pmatrix}. \qquad (3.13)
$$

We suppose that $\widetilde P = \operatorname{op}_x \tilde p$, $\widetilde G = \operatorname{op}_{\tilde x}\tilde g$, $\widetilde K = \operatorname{op}_{\tilde x}\tilde k$, $\widetilde T = \operatorname{op}_{\tilde x}\tilde t$ and $\widetilde S = \operatorname{op}_{\tilde x}\tilde s$. Now we fix $\eta \neq 0$ and define $A \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}^{n+1}_+)$ by

$$
A = \begin{pmatrix} \operatorname{op}^+ p + \operatorname{op} g & \operatorname{op} k \\ \operatorname{op} t & \operatorname{op} s \end{pmatrix},
$$

where $p(t,x,\tau,\xi) = \tilde p(x,\xi,t\tau,t\eta), g(t,x',\tau,\xi') = \tilde g(x',\xi',t\tau,t\eta), \dots, s(t,x',\tau,\xi') = \tilde s(x',\xi',t\tau,t\eta)$. Here, $x' = (x_1,\dots,x_{n-1})$, $\xi' = (\xi_1,\dots\xi_{n-1})$.

3.10 Theorem. *Let A be as in 3.9, $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbb{R}}_+), \omega_1(t) = \omega_2(t) = 1$ for $t \le 1$; let $\varphi_1, \varphi_2 \in C_0^\infty(\overline{\mathbb{R}}^n_+)$, and $\Phi_j, j = 1, 2$, the operator of multiplication by $\operatorname{diag}\{\varphi_j, \varphi_j|_{\mathbb{R}^{n-1}\times\{0\}}\}$. Then the push-forward*

$$B := \chi_*((1-\omega_1)\Psi_1 A \Psi_2 (1-\omega_2)) \tag{3.14}$$

is an element of $\mathcal{B}^{\mu,d}_{SG}(\overline{\mathbb{R}}^{n+1}_+)$.

Proof. For each entry of B we may apply Theorem 3.8. We start with the pseudodifferential part. Here we choose $E = F = \mathbb{C}$. The push-forward then is an SG-symbol. We still have to check the transmission property. Denoting as in 3.8 the symbol of the push-forward by q, we have on S

$$\begin{aligned}
&D^k_{y_n} D^{k'}_{\tilde y_n} q(y, t, \tilde y, \tilde t, \xi', [\xi', \tau]\xi_n, \tau)|_{y_n=\tilde y_n=0} \\
&= t^{-k}\tilde t^{-k'}\Big((D^k_{x_n} D^{k'}_{\tilde x_n}\tilde p)(y/t, \tilde y/\tilde t, T\xi', T[\xi', \tau]\xi_n, c\cdot(\xi', [\xi', \tau]\xi_n + t\tau, t\eta) \\
&\qquad \varphi_1(y/t)\,\varphi_2(\tilde y/\tilde t)\Big)\Big|_{y_n=\tilde y_n=0} T^n[\tilde t]^{-n}(1-\omega_1(t))(1-\omega_2(\tilde t)).
\end{aligned}$$

We note that $c_n(y, t, \tilde y, \tilde t) = 0$ for $y_n = \tilde y_n = 0$, so that $c\cdot(\xi', [\xi', \tau]\xi_n) = c'\cdot\xi'$, where $c' = (c_1, \ldots, c_{n-1})|_{y_n=\tilde y_n=0}$. Since $\tilde p$ satisfies the transmission property

$$D^k_{x_n} D^{k'}_{\tilde x_n}\tilde p(x, \tilde x, \xi', [\xi', \tau, \eta]\xi_n, \tau, \eta)|_{x_n=\tilde x_n=0} = \sum \lambda_j p_j(x', \tilde x', \xi', \tau, \eta) h_j(\xi_n),$$

where $\{\lambda_j\} \in \ell^1$, $\{p_j\}$ is a null sequence in $S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \mathbb{R}^{n-1}\times\mathbb{R}^{1+q})$ and h_j is a null sequence in H. Therefore

$$\begin{aligned}
&D^k_{x_n} D^{k'}_{\tilde x_n}\tilde p(y/t, \tilde y/\tilde t, T\xi', T[\xi', \tau]\xi_n, c\cdot(\xi', [\xi', \tau]\xi_n) + t\tau, t\eta)|_{y_n=\tilde y_n=0} \\
&= \sum \lambda_j p_j\left(\frac{y'}{t}, \frac{\tilde y'}{\tilde t}, T\xi', c'\xi' + t\tau, t\eta\right) h_j\left(\frac{T[\xi', \tau]\xi_n}{[T\xi', c'\xi' + t\tau, t\eta]}\right).
\end{aligned}$$

According to 3.8 the functions

$$p_j\Big(\frac{y'}{t}, \frac{\tilde y'}{\tilde t}, T\xi', c'\xi' + t\tau, t\eta\Big)T^n[\tilde t]^{-n}\varphi_1\Big(\frac{y'}{t}, 0\Big)\,\varphi_2\Big(\frac{\tilde y'}{\tilde t}, 0\Big)\,(1-\omega_1(t))\,(1-\omega_2(\tilde t))$$

belong to $SG^{(\mu,\mu,0)}$; they form a null sequence, since we saw in 3.8 that the mapping is continuous.

We finally deduce from Lemmata 3.11 and 3.12, below, that $h_j(\ldots) \in SG^0\hat\otimes_\pi H$. This shows the SG-transmission property for q.

Next consider the push-forward associated with the singular Green symbol $\tilde g = \sum_{j=0}^d \tilde g_j \partial_r^j$. Focus first on one of the $\tilde g_j \in S^{\mu-j}(\mathbb{R}^{n-1}, \mathbb{R}^{n+q}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$. We let E run over the scale of spaces $H_0^\sigma(\overline{\mathbb{R}}_+)$, $\sigma \in \mathbb{R}^2$, and F over the scale $H^\sigma(\mathbb{R}_+)$, and deduce that the symbol of the corresponding push-forward in (3.14) is an element of $SG^{(\mu-j,\mu-j)}(\mathbb{R}^n\times\mathbb{R}^n; \mathbb{R}^{n+1+q}, \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$. Since the push-forward of the normal derivative is $t\partial_r$, the argument is complete. ◁

It remains to establish the two following technical lemmata used above.

3.11 Lemma. *Let $h \in H$ and $f \in SG^0$ with $|f(x, \xi)| \ge c > 0$. Then*

$$g(x, \xi, \nu) = h(f(x, \xi)\nu) \in SG^0\hat\otimes_\pi H.$$

Proof. Write $h = p + h_0$ with p polynomial and $h_0 \in H_0$. Then the assertion is trivial for p. Without loss of generality we may therefore treat the case where $h \in H^+$. The topology of H^+ is that inherited from $\mathcal{S}(\mathbb{R}_+)$ via the Fourier transform. Writing $h = (e^+u)^\wedge$, we have

$$g(x, \xi, \nu) = \big(\mathcal{F}_{s \to \nu}(e^+u)(s/f(x, \xi))\big)/f(x, \xi).$$

Since $f(x, \xi)^{-1} \in SG^0$ it suffices to check that $(x, \xi, s) \mapsto (e^+u)(s/f(x, \xi)) \in SG^0 \hat{\otimes}_\pi \mathcal{S}(\mathbb{R}_+)$. This space coincides with $\mathcal{S}(\mathbb{R}_+, SG^0)$. Hence we have to check that for all α, β, k, k' and $s > 0$ the estimate

$$D_\xi^\alpha D_x^\beta s^k \partial_s^{k'} \{u(s/f(x, \xi))\} = O([\xi]^{-|\alpha|}[x]^{-|\beta|})$$

holds. We observe that $s^k \partial_s^{k'} \{u(s/f(x, \xi))\} = f(x, \xi)^{k-k'}(s^k \partial_s^{k'} u)(s/f(x, \xi))$. It is therefore sufficient to treat the case $k = k' = 0$. Now

$$D_{\xi_j}\{u(s/f(x, \xi))\} = -(\partial_s u)(s/f(x, \xi))f(x, \xi)^{-2}\partial_{\xi_j} f(x, \xi).$$

An analogous relation holds for x-derivatives. Since $f(x, \xi)^{-2}\partial_{\xi_j} f(x, \xi) \in SG^{(-1,0)}$ the case of higher order derivatives presents no further difficulty. $\triangleleft$

3.12 Lemma. *On the set S, the function*

$$G(y, t, \tilde{y}, \tilde{t}, \xi, \tau) = T[\xi, \tau]/[T\xi, c\xi + t\tau, t\eta]$$

is an element of SG^0. As before $\eta \neq 0$ is fixed.

Proof. In view of Lemmata 3.6 and 3.7 it is sufficient to show that $[T\xi, c\xi + t\tau, t\eta] \in SG^{(1,1,0)}$. Since we are only interested in the case where $t + |\xi, \tau|$ is large, we may replace $[\ldots]$ by $|\ldots|$. Writing $\ldots$ instead of $T\xi, c\xi + t\tau, t\eta$, we have

$$\begin{aligned}
\partial_t |\ldots| &= \frac{(\ldots)}{|\ldots|}(\partial_t T\xi, \partial_t c\xi + \tau, \eta) = O([\xi, \tau]), \\
\partial_{\tilde{t}} |\ldots| &= \frac{(\ldots)}{|\ldots|}(\partial_{\tilde{t}} T\xi, \partial_{\tilde{t}} c, 0) = O([\xi, \tau]), \\
\partial_y |\ldots| &= \frac{(\ldots)}{|\ldots|}(0, \partial_y c\xi, 0) = O([\xi, \tau]), \\
\partial_{\xi_j} |\ldots| &= \frac{(\ldots)}{|\ldots|}(Te_j, ce_j, 0) = O([t]), \\
\partial_\tau |\ldots| &= \frac{(\ldots)}{|\ldots|}(0, t, 0) = O([t]).
\end{aligned}$$

By Lemma 3.7 again all first order derivatives satisfy the desired estimates; the task for checking higher derivatives is the same as before. $\triangleleft$

3.13 The manifold case. Let X be closed compact with boundary, and let $\widetilde{A} \in \mathcal{B}^{\mu,d}(X, \mathbb{R}^{1+q})$ have entries as in (3.13). We fix $0 \neq \eta \in \mathbb{R}^q$ and define $A \in \mathcal{B}^{\mu,d}(X \times \mathbb{R})$ by the process in 3.9.

From Theorem 3.10 and Proposition 3.4 we immediately get the following result:

3.14 Theorem. *Let* $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbb{R}}_+), \omega_1(t) = \omega_2(t) = 1$ *for* $t \leq 1$, *let* χ, A *be as above. Then the push-forward*

$$\chi_*((1-\omega_1)A(1-\omega_2))$$

is an element of $\mathcal{B}_{SG}^{(\mu,\mu),d}(X^{\asymp})$. *Here,* $X^{\asymp}$ *is as in 2.5.*

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