# ITERATIONS OF SELF-ADJOINT OPERATORS AND THEIR APPLICATIONS TO ELLIPTIC SYSTEMS

Alexander Shlapunov

Krasnoyarsk State University
pr. Svobodnyi, 79
660041 Krasnoyarsk
Russia

**Abstract**

Let $H_0$, $H_1$ be Hilbert spaces and $L : H_0 \to H_1$ be a linear bounded operator with $\|L\| \leq 1$. Then $L^\star L$ is a bounded linear self-adjoint non-negative operator in the Hilbert space $H_0$ and one can use the Neumann series $\sum_{\nu=0}^{\infty}(I - L^\star L)^\nu L^\star f$ in order to study solvability of the operator equation $Lu = f$.

In particular, applying this method to the ill-posed Cauchy problem for solutions to an elliptic system $Pu = 0$ of linear PDE's of order $p$ with smooth coefficients we obtain solvability conditions and representation formulae for solutions of the problem in Hardy spaces whenever these solutions exist. For the Cauchy-Riemann system in $\mathbb{C}$ the summands of the Neumann series are iterations of the Cauchy type integral.

We also obtain similar results 1) for the equation $Pu = f$ in Sobolev spaces, 2) for the Dirichlet problem and 3) for the Neumann problem related to operator $P^*P$ if $P$ is a homogeneous first order operator and its coefficients are constant. In these cases the representations involve sums of series whose terms are iterations of integro-differential operators, while the solvability conditions consist of convergence of the series together with trivial necessary conditions.

## 1 Introduction

Let $H_0$, $H_1$ be Hilbert spaces and $L : H_0 \to H_1$ be a linear bounded operator. Consider an operator equation $Lu = f$ where $f$ is a given element of $H_1$. It may happen that the image of the operator $L$ is not dense and is not closed in $H_1$. In particular, this means that in such a case solutions to

$Lu = f$ do not exist for some data $f \in H_1$ and do not depend continuously on the data. A (trivial) necessary condition for solvability of $Lu = f$ is that $f$ should belong to the orthogonal complement $(\ker L^\star)^\perp$ of the kernel $\ker L^\star$ of the adjoint operator $L^\star : H_1 \to H_0$ for the operator $L$. It was noted in [16] that, generally speaking, the solvability conditions for $Lu = f$ can not be described in terms of continuous linear functionals.

Construction of regularization operators (cf., for instance, [8]) is one of the effective methods of investigation of problems of this kind.

For example, let $B$ be a Banach space and $A$ be a linear bounded operator $A : B \to B$ satisfying $\|I - A\| < 1$. It is well-known (see, for instance, [18], Ch. II, §2) that in this case the Neumann series $S = \sum_{\nu=0}^{\infty}(I - A)^\nu$ converges to the inverse operator $A^{-1}$ in the strong operator topology of $B$.

If $\|I - A\| = 1$ then it is not clear how to use this idea in a Banach space, because nothing guarantees the convergence of the Neumann series. However, due to the spectral theorem, we can do it in a Hilbert space $H_0$ for a linear bounded self-adjoint non-negative operator $L^\star L : H_0 \to H_0$ if $\|L\| \leq 1$. Then the solvability of the operator equation $Lu = f$ is equivalent to the convergence of the Neumann series $\sum_{\nu=0}^{\infty}(I - L^\star L)^\nu L^\star f$ if $f \in (\ker L^\star)^\perp$ (see Section 2). This fact is well-known for compact operators (cf., for instance, [8], pp. 47–50). The case where $A$ ($\|A\| \leq 1$) is a continuous self-adjoint positive operator with a spectrum $\mathrm{sp}(A)$ containing zero was considered in [1].

Though the proof of a similar statement for an arbitrary continuous non-negative self-adjoint operator $A$ ($\|A\| \leq 1$) is rather trivial we could not find a proper reference for it; we sketch the proof in Section 2 (cf. [9]).

The aim of the paper is to illustrate how this method works in theory of elliptic systems of PDE's (see Sections 4 and 5). We discuss both well-posed and ill-posed problems related to elliptic systems.

Let us describe the contents of the paper in detail.

Let $P$ be an elliptic differential operator of order $p$ with smooth coefficients on an open set $X \subset \mathbb{R}^n$ (see Section 3 for more information about elliptic systems). In Section 4 we consider the ill-posed Cauchy problem for solutions to the system $Pu = 0$ in Hardy spaces (cf. [4], [5], [14], [17]). Using the Neumann series we obtain a representation formula for its solutions whenever they exist and obtain criterion for the solvability of the problem. For the Cauchy-Riemann system in $\mathbb{C}$ the entries of the Neumann series are iterations of the Cauchy type integrals.

Another part of the paper is devoted to elliptic homogeneous first order systems with constants coefficients. Namely, in Section 5 for such a system we use this approach in order to study 1) an equation $Pu = f$ in Sobolev spaces, 2) the Dirichlet problem and 3) the Neumann problem related to the operator $P^*P$. In these cases the summands of the Neumann series are iterations of integro-differential operators related to Green integrals of the operator $P^*P$. In the case of the Cauchy-Riemann system in $\mathbb{C}^n$ $(n > 1)$ these integro-differential operators are related to the Martinelli-Bochner integral in $\mathbb{C}^n$ and the corresponding results were obtained by Romanov [13]. For general elliptic systems similar results were obtained in [9].

# 2 Linear problems with continuous operators in Hilbert spaces

## 2.1 Bounded self-adjoint operators

Let $H$ be a Hilbert space with a scalar product $(.,.)_H$ and $A : H \to H$ be a linear bounded self-adjoint operator. We also assume that $A$ is non-negative i.e., $(Au, u) \geq 0$ for every $u \in H$. If $H$ is a complex Hilbert space then any non-negative operator is well-known to be self-adjoint. Without loss of generality we consider operators $A$ with $\|A\| \leq 1$.

**Problem 2.1** *Let $v \in H$ be a given element. Find (if possible) an element $u \in H$ such that $Au = v$.*

Generally speaking, Problem 2.1 is ill-posed, i.e., it may happen that the image of the operator $A$ is not closed. In particular, this means that in such a case solutions do not exist for some data $v \in H$ and do not depend continuously on the data. It was noted in [16] that, generally speaking, the solvability conditions for Problem 2.1 can not be described in terms of continuous linear functionals.

Let $I_H$ stand for the identity operator in $H$. If it can not cause any misunderstanding we will write simply $I$ instead of $I_H$. For an operator $B : H \to H$ we denote by $\ker B$ the kernel of $B$. If $\Sigma$ is a closed subspace of $H$, we denote by $\Pi(\Sigma)$ the orthogonal projection from $H$ to $\Sigma$.

**Theorem 2.2** *In the strong operator topology in $H$ we have*

$$\lim_{\nu\to\infty} A^\nu = \Pi(\ker(I-A)), \qquad \lim_{\nu\to\infty} (I-A)^\nu = \Pi(\ker A).$$

**Proof.** Since the operator $A$ is continuous, the kernel $\ker(I-A)$ of the operator $(I-A)$ is a closed subspace of $H$. Therefore it is a Hilbert space (with the Hermitian structure induced from $H$).

The spectral theorem for bounded self-adjoint operators yields

$$A^\nu = \int_{-0}^{1+0} \lambda^\nu dE(\lambda) \tag{2.1}$$

where $\{E(\lambda)\}_{0\le\lambda\le 1}$ is a resolution of the identity in the Hilbert space $H$ corresponding to the self-adjoint operator $0 \le A \le I$ (see, for instance, [18], Ch. XI, §§5,6).

Passing to the limit in (2.1) one obtains $\lim_{\nu\to\infty} A^\nu = E(1+0)-E(1-0)$. Because $E(\lambda)$ is a spectral function (see, for instance, [18], Ch. XI, §§5,6) the operator $E(1+0) - E(1-0)$ is an orthogonal projection from $H$ onto a (closed) subspace $V(1) \subset H$. Obviously, $(I - A)\lim_{\nu\to 0} A^\nu v = 0$ for every $v \in H$, i.e., $V(1) \subset \ker(I - A)$. Finally, if $v \in \ker(I - A)$ then, for every $\nu \ge 0$, we have $v = Av + (I - A)v = Av = A^\nu v = \lim_{\nu\to 0} A^\nu v$. Therefore $V(1) = \ker(I - A)$.

In order to finish the proof it is sufficient to note that, since $A$ is a self-adjoint non-negative operator with $\|A\| \le 1$, the operator $(I - A)$ has the same properties.

□

**Corollary 2.3** *In the strong operator topology in $H$ we have:*

$$I = \lim_{\nu\to\infty} A^\nu + \sum_{\mu=0}^{\infty} A^\mu (I - A), \tag{2.2}$$

$$I = \lim_{\nu\to\infty} (I - A)^\nu + \sum_{\mu=0}^{\infty} (I - A)^\mu A. \tag{2.3}$$

**Proof.** The formula $A + (I - A) = I$ implies

$$I = A^\nu + \sum_{\mu=0}^{\nu-1} A^\mu (I - A) = (I - A)^\nu + \sum_{\mu=0}^{\nu-1} (I - A)^\mu A. \tag{2.4}$$

for every $\nu \in \mathbb{N}$. Now using Theorem 2.2 we can pass to the limit for $\nu \to \infty$ in (2.4), thus obtaining (2.2) and (2.3).

□

We use Corollary 2.3 in order to obtain a solvability condition for Problem 2.1.

**Theorem 2.4** *Problem* 2.1 *is solvable if and only if the Neumann series*

$$u_0 = \sum_{\nu=0}^{\infty} (I - A)^\nu v$$

*converges in $H$. Moreover, if Problem* 2.1 *is solvable then $Au_0 = v$.*

**Proof.** Let Problem 2.1 have a solution $u \in H$. Then Corollary 2.3 implies that the series

$$\sum_{\nu=0}^{\infty}(I-A)^{\nu}Au = \sum_{\nu=0}^{\infty}(I-A)^{\nu}v = u_0$$

converges in $H$.

Back, let the series $u_0$ converge in $H$. Then, since the operator $(I-A)$ is continuous, we have

$$u_0 - Au_0 = (I-A)u_0 = \sum_{\nu=1}^{\infty}(I-A)^{\nu}v = u_0 - v.$$

Hence $Au_0 = v$ and Problem 2.1 is solvable.

□

**Remark 2.5** *Theorem* 2.2 *implies that the solution* $u_0$ *in Theorem* 2.4 *is a unique solution of Problem* 2.1 *orthogonal to* $\ker A$.

**Remark 2.6** *Note that the elements*

$$u_0^{(N)} = \sum_{\nu=0}^{N}(I-A)^{\nu}v$$

*can be regarded as approximate solutions to Problem* 2.1.

## 2.2 Bounded operators

Consider now a more general situation. Let $H_0$, $H_1$ be Hilbert spaces and $L : H_0 \to H_1$ be a bounded linear operator. Again without loss of generality we can consider operators $L$ with $\|L\| \leq 1$.

**Problem 2.7** *Let* $f \in H_1$ *be a given element. Find (if possible) an element* $u \in H_0$ *such that* $Lu = f$.

Let $L^{\star} : H_1 \to H_0$ be the adjoint operator of $L : H_0 \to H_1$ in the sense of Hilbert spaces.

**Proposition 2.8** *Problem* 2.7 *is solvable if and only if*

(1) *there exists* $u \in H$ *such that* $L^{\star}Lu = L^{\star}f$;

(2) $(f,g)_{H_1} = 0$ *for all* $g \in \ker L^{\star}$.

**Proof.** This follows immediately from the definition of $L^\star$.

$\square$

It is easy to see that $L^\star L : H_0 \to H_0$ is a bounded non-negative self-adjoint operator with $\|L^\star L\| \le 1$. Therefore Problem 2.7 is equivalent to Problem 2.1 with $f \in (\ker L^\star)^\perp$ and $v = L^\star f$.

**Corollary 2.9** *In the strong operator topology in $H_1$ we have*

$$\Pi((\ker L^\star)^\perp) = \sum_{\mu=0}^{\infty} L(I - L^\star L)^\mu L^\star. \tag{2.5}$$

**Proof.** It is easy to see that $(I_{H_1} - LL^\star)L = L(I_{H_0} - L^\star L)$ and then $(I_{H_1} - LL^\star)^\mu LL^\star = L(I_{H_0} - L^\star L)^\mu L^\star$ for every $\mu \ge 0$.

Note that the operator $LL^\star : H_1 \to H_1$ is bounded self-adjoint non-negative and $\|LL^\star\| \le 1$. Then, according to Theorem 2.2 and Corollary 2.3, we have

$$\Pi((\ker L^\star)^\perp) = \sum_{\mu=0}^{\infty} (I_{H_1} - LL^\star)^\mu LL^\star = \sum_{\mu=0}^{\infty} L(I_{H_0} - L^\star L)^\mu L^\star.$$

$\square$

**Corollary 2.10** *Problem* 2.7 *is solvable if and only if*

(1) *the series $u_1 = \sum_{\nu=0}^{\infty} (I - L^\star L)^\nu L^\star f$ converges in $H_0$;*

(2) *$(f, g)_{H_1} = 0$ for all $g \in \ker L^\star$.*

*Moreover, if Problem* 2.7 *is solvable then the series $u_1$ is one of its solutions.*

**Proof.** This follows immediately from Proposition 2.8, Theorem 2.4 and Corollary 2.10.

In the sequel we will discuss some applications of this approach to elliptic PDE's.

# 3 Elliptic differential operators. Preliminaries

Assume that $X$ is an open set in $\mathbb{R}^n$, and $E = X \times \mathbb{C}^k$, $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over $X$. Sections of $E$ and $F$ of a class $\mathfrak{C}$ on an open set $U \subset X$ can be interpreted as columns of complex-valued functions from $\mathfrak{C}(U)$, that is, $\mathfrak{C}(E|_U) \cong [\mathfrak{C}(U)]^k$, and similarly for $F$. Throughout the paper we will usually write the letters $u$, $v$ for sections of $E$, and $f$, $g$ for sections of $F$.

A differential operator $P$ of order $p \geq 1$ and of type $E \to F$ can be written in the form $P(x, D) = \sum_{|\alpha| \leq p} P_\alpha(x) D^\alpha$, with suitable $(l \times k)$-matrices $P_\alpha(x)$ of smooth functions on $X$. The *principal symbol* $\sigma(P)$ of $P$ is a function on the cotangent bundle of $X$ with values in the space of bundle morphisms $E \to F$. Given any $(x, \xi) \in X \times \mathbb{C}^n$, we have $\sigma(P)(x, \xi) = \sum_{|\alpha|=p} P_\alpha(x) \xi^\alpha$. We say that $P$ is *elliptic* if the mapping $\sigma(P)(x, \xi) : \mathbb{C}^k \to \mathbb{C}^l$ is injective for every $x \in X$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Hence it follows that $l \geq k$; we say that $P$ is *determined* elliptic if $l = k$, and *overdetermined* elliptic if $l > k$.

Every elliptic operator is hypo elliptic, i.e., all distribution sections satisfying $Pu = 0$ on an open set $U$ of $X$ are infinitely differentiable there. If $U$ is an open subset of $X$, then we denote by $S_P(U)$ the vector space of all $C^\infty$ solutions to the equation $Pu = 0$ on $U$.

Here and in the sequel "domain in $X$" means an open connected subset of $X$.

Denote by $E^* = X \times (\mathbb{C}^k)'$ the conjugate bundle of $E$, and similarly for $F$. For the operator $P$, we define the transpose ${}^tP$ as usual, so that ${}^tP$ is a differential operator of type $F^* \to E^*$ and order $p$ on $X$:

$$ {}^tPg\,(x) = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^\alpha ({}^tP_\alpha(x)\, g(x)) $$

for $g \in C^\infty_{comp}(F)$.

Fix the standard Hermitian structure in the fibers $E_x = \mathbb{C}^k$ $(x \in X)$ of $E$, i.e., $(u, v)_x = \sum_{j=1}^k u_j \overline{v_j} = v^* u$ for $u, v \in \mathbb{C}^k$ where $v^*$ is the conjugate vector.

Let $L^2(E_{|D}) = [L^2(D)]^k$ be the Hilbert space of all measurable functions defined on $D$, for which

$$ \|u\|_{L^2(E_{|D})} = \left( \int_D (u, u)_x dx \right)^{1/2} < \infty. $$

We also denote by $W^{m,2}(E_{|D}) = [W^{m,2}(D)]^k$ $(m \in \mathbb{N})$ the Sobolev space of distribution sections of $E$ over $D$ having weak derivatives up to order $m$ in the Lebesgue space $L^2(E_{|D})$. The space $W^{m,2}(E_{|D})$ is a Hilbert space under the scalar product

$$ (u, v)_{W^{m,q}(E_{|D})} = \sum_{|\alpha| \leq m} \int_D (D^\alpha u, D^\alpha v)_x dx. $$

For $m \geq 0$ we define the Sobolev spaces $W^{m,2}(E_{|D}) = [W^{m,2}(D)]^k$ by one of the usual interpolation methods. As usual, we use the notation $W^{m,2}_{loc}(E_{|D})$

for the Sobolev space of functions belonging to $W^{m,2}(E_{|K})$ for any compact set $K \Subset D$ ($L^2_{loc}(E_{|D}) = W^{0,2}_{loc}(E_{|D})$). The symbol $S^{m,2}_P(D)$ will stand for $S_P(D) \cap W^{m,2}(E_{|D})$.

Let us endow the spaces $C^\infty_{comp}(E)$ and $C^\infty_{comp}(F)$, consisting of infinitely differentiable sections with compact supports of $E$ and $F$, respectively, with ($L^2-$) pre-Hilbert structures by $(u,v)_X = \int_X (u,v)_x \, dx$. Under these structures, the operator $P$ has a formal adjoint which is denoted by $P^*$. This is the differential operator of type $F \to E$ and order $p$ on $X$ given by

$$P^* g\,(x) = \sum_{|\alpha| \le p} (-1)^{|\alpha|} D^\alpha (P_\alpha(x)^* \, g(x))$$

for $g \in C^\infty_{comp}(F)$).

The operator $\Delta = P^*P$ is usually referred to as the generalized *Laplacian* associated to $P$. It is easy to see that $\Delta$ is an elliptic differential operator of type $E \to E$ and order $2p$ on $X$. If $P$ is the gradient operator in $\mathbb{R}^n$, then $\Delta = P^*P$ is the usual Laplace operator, up to the factor $-1$. On the other hand, if $P$ is the Cauchy-Riemann operator in $\mathbb{C}^n$, then $\Delta = P^*P$ coincides with the usual Laplace operator on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ up to the factor $-\frac{1}{4}$.

Every determined elliptic operator with smooth coefficients has locally a bilateral (i.e., left and right) fundamental solution, and hence every overdetermined elliptic operator with smooth coefficients has locally a left fundamental solution. If the coefficients of $P$ are real analytic, there exist global fundamental solutions of $P$ on $X$ (cf., for example, [15], §8). In fact, for the existence of left (right) fundamental solutions of the operator $P$, the so-called Uniqueness Condition $(U)_S$ for the Cauchy problem in the small on $X$ for the operator $P$ ($P'$) is important (see [15], Corollary 27.8):

**Property 3.1** *If for a domain $\mathcal{O} \subset X$ we have $Pu = 0$ in $\mathcal{O}$, and $u = 0$ on a non-empty open subset of $\mathcal{O}$ then $u \equiv 0$ in $\mathcal{O}$.*

Of course, this property holds for $P$, if the coefficients of $P$ are real analytic and $P$ is elliptic. If $P$ is overdetermined, it may happen that there are no right (in particular bilateral) fundamental solutions.

From now on we will assume that the operator $P$ is elliptic and has a (left) fundamental solution.

# 4 The Cauchy problem for elliptic systems in Hardy spaces

In this section we consider the Cauchy problem for elliptic systems of PDE's in Hardy spaces. Let $P$ be a determined elliptic differential operator on an open set $X \subset \mathbb{R}^n$ of order $p \geq 1$, $D \Subset X$ be a domain with smooth boundary $\partial D$ and $\{B_j\}$ be a Dirichlet system of order $(p-1)$ on $\partial D$ ($B_j \in do_j(E_{|U} \to F_j)$).

In order to study the Cauchy problem for solutions of elliptic system we need an information about boundary behavior of the solutions. In [16] the maximal subclasses of solutions $u \in S_P(D)$, for which one can speak of the boundary values of the expressions $B_j u$ $(0 \leq j \leq p-1)$ on $\partial D$ in the class of usual (not generalized) sections of $F_j$, was distinguished (cf. [14]). These are the so-called Hardy spaces $H^2_{P,B}(D)$ $(1 < q < \infty)$ modelled on the pattern of the classical Hardy spaces of holomorphic functions. One can say that $H^2_{P,B}(D)$ consists of all solutions $u \in S_P(D)$ for which the weak limit values of the expressions $B_j u$ $(0 \leq j \leq p-1)$ on $\partial D$ exist and belong to $L^2(F_{j|\partial D})$. In particular, with the topology induced by $L^2(\oplus F_{j|\partial D})$, i.e., with the scalar product

$$(u,v) = \sum_{j=0}^{p-1} \int_{\partial D} (B_j u, B_j v)_x ds \quad (u, v \in H^2_{P,B}(D)), \tag{4.1}$$

the space $H^2_{P,B}(D)$ is a separable Hilbert space (see [14]).

**Problem 4.1** *Let $S$ be a measurable subset of $\partial D$ of positive $(2n-1)$-measure, $u_j \in L^2(F_{j|S})$ $(0 \leq j \leq p-1)$ be known sections on $S$. Find a section $u \in H^2_{P,B}$, satisfying $B_j u = u_j$ $(0 \leq j \leq p-1)$ on $S$.*

Problem 4.1 is well known to be ill-posed (see, for example, [4], [5], [14], [17])). So we have no hope to obtain any solvability conditions in terms of continuous linear functionals.

We assume that $P$ is a determined elliptic differential operator such that both $P$ and the transposed operator ${}^tP$ satisfy the Uniqueness Condition for the Cauchy problem in the small on $X$ (see Property 3.1). A solvability criterion for Problem 4.1 in terms of surface bases with double orthogonality was found in [14]. However in order to use this criterion one needs first to find a generalized system of eigen functions of a non-compact self-adjoint operator. In this section, we obtain solvability conditions for Problem 4.1 which do not require this procedure.

**Lemma 4.2** *If $S$ has at least one interior point then Problem 4.1 has no more than one solution. Moreover, if the complement of $S$ on $\partial D$ has at least one interior point then Problem 4.1 is densely solvable.*

**Proof.** See, for instance, [14].

□

According to the general scheme of Section 2 we set $H_0 = H^2_{P,B}(D)$, $H_1 = L^2(\oplus F_{j|S})$. Further $L : H_0 \to H_1$ is the operator given by $Lu = (\oplus B_j u)_{|S}$ for an element $u \in H^2_{P,B}$; obviously, $\|L\| \leq 1$.

To apply the results of Section 2 to Problem 4.1 we need some information about the orthogonal projection in $L^2(\oplus F_{j|\partial D})$ on the subspace formed by elements of the form $\oplus B_j u$, where $u \in H^2_{P,B}(D)$. Let us briefly sketch the corresponding results from [14].

Let $x$ be a fixed point of the domain $D$. We consider the functional $\delta^{(j)}_x$ $(1 \leq j \leq k)$ on $H^2_{P,B}(D)$ given by $\delta^{(j)}_x u = u^{(j)}(x)$ $(1 \leq j \leq k)$ where $u^{(j)}(x)$ is the $j$-th component of $u$ at the point $x$. This functional is continuous on $H^2_{P,B}(D)$; moreover, a stronger property than continuity holds (see [14]). Namely, for any compact $K \subset D$ there is a constant $C_K$ such that $\|\delta^{(j)}_x\| < C_K$ for $x \in K$. Hence, $H^2_{P,B}(D)$ is a space with a reproducing kernel (see Aronszajn [2]). One can now use the Riesz theorem on the general form of a continuous linear functional on a Hilbert space and thus find (unique) elements $\mathcal{K}^{(j)}_x \in H^2_{P,B}(D)$ $(1 \leq j \leq k)$ such that $u^{(j)}(x) = (u, \mathcal{K}^{(j)}_x)_{H_0}$ for all $u \in H_0$. We denote by $\mathcal{K}^{(i,j)}_x$ $(1 \leq j, i \leq k)$ the i-th component of the vector-valued function $\mathcal{K}^{(j)}_x$. The (well-defined) matrix $\mathcal{K}(x,y) = \|\mathcal{K}^{(i,j)}_x(y)\|$ is called the reproducing kernel of the domain $D$ relative to $H^2_{P,B}(D)$. Its properties are well-known.

**Proposition 4.3** *The matrix $\mathcal{K}$ is Hermitian i.e., $\mathcal{K}(x,y)^* = \mathcal{K}(y,x)$. Moreover $tr\mathcal{K}(x,x) = \sum_{j=1}^{k} \|\delta^{(j)}_x\|$, and if $\{e_\nu\}$ is an orthonormal basis of the space $H^2_{P,B}(D)$ then for all $x \in D$ we have $\mathcal{K}^{(j)}_x = \sum_{\nu=1}^{\infty} \overline{e^{(j)}_\nu(x)} e_\nu$ $(1 \leq j \leq k)$ where the series converges in the norm of $H^2_{P,B}(D)$. As a series of (vector-) functions of two variables $(x,y) \in D \times D$, it converges uniformly on compact subsets of $D \times D$.*

**Proof.** See, for instance, [14].

□

The formula for the reproducing kernel mentioned in Proposition could be written in the form $\mathcal{K}(x,y) = \sum_{\nu=1}^{\infty} e_\nu(x)^* \otimes e_\nu(y)$. A priori estimates for a solution of an elliptic system imply that this series here converges uniformly together with all its derivatives on compact subsets of $D \times D$, that is, $\mathcal{K}$ is an infinitely differentiable section of $E \boxtimes E$ over $D \times D$.

**Theorem 4.4** *For all solutions $u \in H^2_{P,B}(D)$ the following formula holds*

$$u(x) = \int_{\partial D} \sum_{j=0}^{p-1} (B_j u, B_j \mathcal{K}(x,.))_y ds \quad (x \in D). \tag{4.2}$$

**Proof.** We simply rewrite the reproducing property of the kernel $\mathcal{K}$ in detail (for holomorphic functions of several complex variables Theorem 4.4 is due to Bungart [3]).

□

**Corollary 4.5** *In the space $L^2(\partial D)$ the operator of the orthogonal projection on the subspace $\Sigma_1$ formed by elements of the form $\oplus B_j u$ where $u \in H^2_{P,B}(D)$, has the form*

$$\Pi(\oplus u_j) = \oplus B_j \left( \int_{\partial D} \sum_{i=0}^{p-1} (u_i, B_i \mathcal{K}(x,.))_y ds \right) \quad (\oplus u_j \in L^2(\partial D)). \qquad (4.3)$$

**Proof.** See [14].

□

Now we can formulate the main results of this section.

Let $\sigma$ be a measurable subset of $\partial D$. Denote by $K_\sigma : L^2(\oplus F_{j|\sigma}) \to H^2_{P,B}(D)$ the bounded operator defined by

$$\left(K_\sigma(\oplus u_j)\right)(x) = \int_\sigma \sum_{j=0}^{p-1} (u_j, B_j \mathcal{K}(x,.))_y ds, \quad (\oplus u_j \in L^2(\oplus F_{j|\sigma})).$$

**Lemma 4.6** *We have $L^* = K_S$.*

**Proof.** Let $\oplus u_j \in \oplus L^2(F_{j|S})$ and $u \in H^2_{P,B}(D)$. Denote by $\tilde{u}_j$ an element of $L^2(F_{j|\partial D})$ such that $\tilde{u}_j = u_j$ on $S$ and $\tilde{u}_j = 0$ on $\partial D \setminus S$. Then

$$(\oplus u_j, Lu)_{H_1} = (\oplus \tilde{u}_j, \oplus B_j u)_{L^2(\oplus F_{j|\partial D})} = (\oplus \tilde{u}_j, \Pi(\oplus B_j u)_{L^2(\oplus F_{j|\partial D})} =$$

$$= (\oplus B_j \Pi(\oplus \tilde{u}_j), \oplus B_j u)_{L^2(\oplus F_{j|\partial D})} = (K_S(\oplus u_j), u)_{H^2_{P,B}(D)},$$

which was to be proved.

□

**Theorem 4.7** *Assume that the complement of $S$ in $\partial D$ has at least one interior point. Then for the solvability of Problem 4.1 it is necessary and sufficient that the series*

$$u_0 = \sum_{\nu=0}^{\infty} \left(K_{\partial D \setminus S}(\oplus B_j\, .)\right)^\nu K_S(\oplus u_j) \qquad (4.4)$$

*converges in $H^2_{P,B}(D)$. Moreover the series $u_0$, if converges, is a solution of Problem 4.1.*

**Proof.** Lemma 4.2 yields that $\ker L^* = 0$. Then the statement follows from Lemma 4.6 and Corollary 2.10 because $\|L\| \leq 1$, $L^*Lv = K_S(\oplus B_j v)$ and $(I - L^*L)v = K_{\partial D \setminus S}(\oplus B_j\, v)$ for every $v \in H^2_{P,B}(D)$.

□

**Remark 4.8** *Note that the elements*

$$u_0^{(N)} = \sum_{\nu=0}^{N} \left(K_{\partial D \setminus S}(\oplus B_j\, .)\right)^\nu K_S(\oplus u_j)$$

*can be regarded as approximate solutions to Problem* 4.1.

**Example 4.9** Let $D = \mathcal{B}_1$ be the unit ball in $\mathbb{C}$ and $S$ be a closed measurable subset of $\partial\mathcal{B}_1$ with *meas* $S > 0$ and *meas* $\partial\mathcal{B}_1 \setminus S > 0$. Let $P$ be the Cauchy-Riemann operator in $\mathbb{C}$. As a Dirichlet system we take $B_0 = 1$. Then the corresponding Hardy space $H^2_{P,B}(D)$ is the usual Hardy space $H^2(\mathcal{B}_1)$ of holomorphic functions in $\mathcal{B}_1$.

The system of monomials $\{z^\nu/\sqrt{2\pi}\}_{\nu=0}^\infty$ is an orthonormal basis in $H^2(\mathcal{B}_1)$ and therefore, for $|\zeta| = 1$ and $|z| < 1$, we have

$$\mathcal{K}(z,\zeta) = \sum_{\nu=0}^{\infty} \frac{\overline{z}^\nu \zeta^\nu}{2\pi} = \frac{1}{2\pi} \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}}.$$

Hence, for $u_0 \in L^2(S)$, $v \in H^2(\mathcal{B}_1)$ and $z \in \mathcal{B}_1$, we obtain that

$$(K_S u_0)(z) = \frac{1}{2\pi\sqrt{-1}} \int_S \frac{u_0(\zeta) d\zeta}{\zeta - z},$$

$$(K_{\partial D \setminus S} v)(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial D \setminus S} \frac{v(\zeta) d\zeta}{\zeta - z}$$

are given by the Cauchy integrals.

# 5 Applications to systems with constant coefficients

In this section we consider applications of the spectral theorem to elliptic homogeneous differential first order operators $P$ with constant coefficients in $\mathbb{R}^n$ $(n \geq 2)$, i.e.,

$$P = \sum_{j=1}^{n} P_j \frac{\partial}{\partial x_j}.$$

Obviously, the differential operator $P$ defines a bounded linear operator $\mathcal{P} : [W^{1,2}(D)]^k \to [L^2(D)]^l$.

In this section we construct a scalar product in the space $[W^{1,2}(D)]^k$ which is equivalent to the original one. This scalar product gives a possibility to find easily the adjoint operator $\mathcal{P}^\star : [L^2(D)]^l \to [W^{1,2}(D)]^k$ (in the sense of Hilbert spaces) for $\mathcal{P}$. Similar scalar products and adjoint operator were constructed, for example, in [13] (for the multidimensional Cauchy–Riemann system), and in [9] (for general linear elliptic operators with smooth coefficients).

As $P^*P$ is a homogeneous elliptic second order operator in $\mathbb{R}^n$ with constant coefficients it has a fundamental solution $\Phi_n(x-y)$ of convolution type (see, for example, [15], p. 74):

$$\Phi_n(x-y) = A\left(\frac{x-y}{|x-y|}\right)|x-y|^{2-n} + \delta_{n2} B\, ln|x|, \tag{5.1}$$

where the elements of the $(k\times k)$-matrix $A(\zeta)$ are real analytic in a neighborhood of the sphere $\{|\zeta| = 1\}$, the elements of the $(k\times k)$-matrix $B$ belong to $\mathbb{C}$, and $\delta_{n2}$ is the Kronecker delta. For example, if $P$ is the gradient operator in $\mathbb{R}^n$ then $P^*P = -\Delta_n$ where $\Delta_n$ is the Laplace operator in $\mathbb{R}^n$, and $\Phi_n(x-y)$ is the standard fundamental solution $\phi_n(x-y)$ of $\Delta_n$; if $P$ is the Cauchy–Riemann system in $\mathbb{C}^m$ $(\cong \mathbb{R}^{2m})$ then $P^*P = -\frac{\Delta_{2m}}{4}$, $\Phi_{2m}(x-y) = 4\,\phi_{2m}(x-y)$.

Let us construct a scalar product in the space $[W^{1,2}(D)]^k$ such the adjoint operator $\mathcal{P}^\star$ for $\mathcal{P}$ is given by the integral

$$Tf(x) = \int_D ({}^tP^*(y)\Phi_n(x,y))' f(y)dy \quad (x \in \mathbb{R}^n \setminus \partial D,\ f \in [L^2(D)]^l). \tag{5.2}$$

## 5.1 Adjoint operator

For this aim we need an information on solvability of the "exterior" Dirichlet problem.

Denote by $\tilde{S}^{1,2}_{P^*P}(\mathbb{R}^n \setminus \overline{D})$ the set of sections such that

1) $P^*Pu = 0$ (weakly) in the domain $\mathbb{R}^n \setminus \overline{D}$;

2) $u \in [W^{1,2}(Y \setminus \overline{D})]^k$ for every bounded domain $Y \subset \mathbb{R}^n$ such that $D \Subset Y$;

3) there exists a limit $\lim_{|x|\to\infty} u(x) = 0$ for $n > 2$, or $\lim_{|x|\to\infty} u(x) \in \mathbb{C}^k$ for $n = 2$.

Let $\tilde{S}^{1,2}_{P}(\mathbb{R}^n \setminus \overline{D})$ be a closed subspace of $\tilde{S}^{1,2}_{P^*P}(\mathbb{R}^n \setminus \overline{D})$ consisting of elements $u \in S_P(\mathbb{R}^n \setminus \overline{D})$ such that $\lim_{|x|\to\infty} u(x) = 0$.

Fix a point $x_0$ in the domain $D$. Then Theorem 7.25 from [15] implies that for every function $u \in \tilde{S}^{1,2}_{P^*P}(\mathbb{R}^n \setminus \overline{D})$ we have:

$$u(x) = \delta_{n2} K_0(u, x_0) + (1 - \delta_{n2}) \Phi_n(x - x_0) C_0(u, x_0) + \sum_{|\alpha| \geq 1} D^\alpha \Phi_n(x - x_0) C_\alpha(u, x_0) \qquad (5.3)$$

in the complement of the ball $B(x_0, r)$ ($r = \sup_{x \in D} |x - x_0|$), where the series converges absolutely and uniformly on compact subsets in $\mathbb{R}^n \setminus \overline{B(x_0, r)}$, and the coefficients $K_0(u, x_0)$, $C_0(u, x_0)$, $C_\alpha(u, x_0) \in \mathbb{C}^k$ are uniquely defined.

**Lemma 5.1** *For every function $u_0 \in [W^{1/2,2}(\partial D)]^k$ there exists a solution $S(u_0) \in \tilde{S}^{1,2}_{P^*P}(\mathbb{R}^n \setminus \overline{D})$ such that $S(u_0) = u_0$ on $\partial D$. Moreover*

$$\int_{\mathbb{R}^n \setminus D} \sum_{j=1}^n \left| \frac{\partial S(u_0)}{\partial x_j}(x) \right|^2 dx + \left| \lim_{|x| \to \infty} S(u_0)(x) \right|^2 \leq C \|u_0\|^2_{[W^{1/2,2}(\partial D)]^k}$$

*with a positive constant $C$ independent on $u_0$.*

**Proof.** Note that because of the ellipticity of the operator $P$ the operator $P^*P$ is strongly elliptic, i.e., for every non-zero vector $z \in \mathbb{C}^k$ we have:

$$Re \left( z^* \sum_{i=1}^n \sum_{j=1}^n P_j^* P_i \zeta_j \zeta_i z \right) \neq 0 \text{ for all } \zeta \in \mathbb{R}^n \setminus \{0\}.$$

Therefore the statement of the lemma follows from classical results on solvability of the Dirichlet problem for strongly elliptic systems (see, for example, [12], 7.4). □

**Theorem 5.2 (adjoint operator).** *The Hermitian form*

$$H_P(u, v) = \int_D (Pv)^*(y)(Pu)(y) dy +$$

$$\int_{\mathbb{R}^n \setminus D} PS(v))^*(y) PS(u)(y) dy + \left( \lim_{|x| \to \infty} S(v)(x) \right)^* \lim_{|x| \to \infty} S(u)(x)$$

*is a scalar product on the space $[W^{1,2}(D)]^k$ inducing the topology equivalent to the original one. Moreover the adjoint operator $\mathcal{P}^\star$ of $\mathcal{P}$ with respect to $H_P(.,.)$ is given by the integral* (5.2).

**Proof.** The proof is similar to the proof of Theorem 3.13 in [9].

Let us formulate now some direct corollaries of Theorem 5.2. Denote by $\mathcal{G}$ the following integral

$$\mathcal{G}w(x) = -\int_{\partial D} {}^t({}^tP_y{}^*\Phi_n(x,y)) \sum_{j=1}^n P_j(-1)^{j-1} w(y)dy[j],$$

where $dy[j] = dy_1 \wedge dy_2 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n$. Using the Stokes formula one easily obtains

$$TPu + \mathcal{G}u = \chi_D u \text{ for all } u \in [W^{1,2}(D)]^k$$

where $\chi_D$ is the characteristic function of the domain $D$.

**Corollary 5.3** *The integrals $TP$ and $\mathcal{G}$ define bounded non-negative self-adjoint operators in the space $[W^{1,2}(D)]^k$ (with the scalar product $H_P(.,.)$ from Theorem* 5.2) *and their norms are less or equal than one.*

**Proof.** This follows from Theorem 5.2. □

**Corollary 5.4** *In the strong operator topology in $[W^{1,2}(D)]^k$ we have*

$$\lim_{\nu\to\infty} \mathcal{G}^\nu = \Pi(S_P^{1,2}(D)),$$

$$\lim_{\nu\to\infty} (TP)^\nu = \Pi(\widetilde{S}_P^{1,2}(\mathbb{R}^n \backslash \overline{D})).$$

**Proof.** This follows from Theorem 5.3 and Theorem 2.2. □

Because of Lemma 5.1 we can regard $\widetilde{S}_P^{1,2}(\mathbb{R}^n\backslash\overline{D}))$ as a closed subspace $\Sigma$ in $[W^{1,2}(D)]^k$. Namely, $\Sigma$ consists of functions $u \in [W^{1,2}(D)]^k$ such that $S(u) \in \widetilde{S}_P^{1,2}(\mathbb{R}^n\backslash\overline{D}))$. Thus, in Corollary 5.4 the notation $\Pi(\widetilde{S}_P^{1,2}(\mathbb{R}^n\backslash\overline{D}))$ is understood as $\Pi(\Sigma)$.

**Corollary 5.5** *In the strong operator topology in $[W^{1,2}(D)]^k$ we have*

$$I = \lim_{\nu\to\infty} \mathcal{G}^\nu + \sum_{\mu=0}^\infty \mathcal{G}^\mu TP, \tag{5.4}$$

$$I = \lim_{\nu\to\infty} (TP)^\nu + \sum_{\mu=0}^\infty (TP)^\mu \mathcal{G}. \tag{5.5}$$

**Proof.** This follows from Theorem 5.3 and Corollary 2.3. □

## 5.2 On formulae for solutions of $Pu = f$

As we have seen in Section 2 we may use Corollaries 5.4 and 5.5 in order to obtain formulae for equation $\mathcal{P}u = f$ whenever these solutions exist.

**Corollary 5.6** *Let $f \in [L^2(D)]^l$ belong to the image of the operator $\mathcal{P} : [W^{1,2}(D)]^k \to [L^2(D)]^l$. Then the series $\tilde{u} = \sum_{\mu=0}^{\infty} \mathcal{G}^{\mu} T f$ converges in the space $[W^{1,2}(D)]^k$ and satisfies $P\tilde{u} = f$ in $D$.*

Note that Corollary 5.4 implies that the series $\tilde{u}$ is the unique solution to $\mathcal{P}u = f$ that belongs to the orthogonal complement of $S_P^{1,2}(D)$ in the space $[W^{1,2}(D)]^k$ (with respect to the scalar product $H_P(.,.)$).

**Example 5.7** Let $P$ be a determined elliptic operator, i.e., $l = k$. Then it has a bilateral fundamental solution of the convolution type (see, for example, [15], p. 74) and boundedness theorems for potential operators (see [11], 2.2.2 and 2.3.2.4) imply that every section $f \in [L^2(D)]^l$ belongs to the image of the operator $\mathcal{P}$. Obviously, in this case we need not use Corollary 5.6 because a solution to the equation $\mathcal{P}u = f$ could be easily obtained by means of the fundamental solution.

**Example 5.8** Let $P$ be overdetermined i.e., $l > k$. Since the coefficients of the operator $P$ are constant it can be included into a compatibility complex $\{E^i, P^i\}$ (Hilbert complex), i.e., there are differential operators $P^i$ $(0 \leq i \leq N < \infty)$ of orders $p_i > 0$ such that $P^{i+1}P^i = 0$, $P^0 = P$ and the sequence

$$0 \longrightarrow \mathbb{C}^{k_0} \overset{\sum_{j=1}^{n} P_j \zeta_j}{\longrightarrow} \mathbb{C}^{k_1} \overset{\sum_{|\alpha|=p_1} P^1_{\alpha} \zeta^{\alpha}}{\longrightarrow} \mathbb{C}^{k_2} \longrightarrow \dots$$

is exact for all $\zeta \in \mathbb{R}^n \setminus \{0\}$. Then $P^1 f = 0$ in $D$ if the equation $\mathcal{P}u = f$ is solvable. Moreover if the domain is convex, and $f \in [C^{\infty}(D)]^l$ satisfies $P^1 f = 0$ in $D$ then there exists a section $u \in [C^{\infty}(D)]^k$ such that $Pu = f$ in $D$ (see, for example, [10]). If the domain is not convex then, in general, the condition $P^1 f = 0$ is not sufficient for the solvability of $Pu = f$. Moreover, even in convex domains there are functions $f \in [L^2(D)]^l$ satisfying $P^1 f = 0$ in $D$, which do not belong to the image of $\mathcal{P} : [W^{1,2}(D)]^k \to [L^2(D)]^l$ (see, for example, [9]), example 8.4). In any case the elements $\tilde{u}^{(N)} = \sum_{\mu=0}^{N} \mathcal{G}^{\mu} T f$, can be regarded as approximate solutions to $\mathcal{P}u = f$ (even if we have no information on the solvability of the equation).

**Example 5.9** Let $P$ be the gradient operator in $\mathbb{R}^n$. Then $(-P^*P)$ is the Laplace operator $\Delta_n$ in $\mathbb{R}^n$ and the compatibility complex is de Rham complex. It is well-known that for every $f \in [L^2(D)]^n$ satisfying the compatibility conditions in a convex domain $D$ there exists a solution of $Pu = f$ in $W^{1,2}(D)$. In this case $S_P(D) = \mathbb{C}$,

$$\Phi_n(x,y) = \phi_n(x-y) = \begin{cases} \frac{1}{(2-n)\ \sigma_n} \frac{1}{|x-y|^{n-2}}, & n > 3, \\ \frac{1}{2\pi} \ln\left(|x-y|\right) & n = 2, \end{cases}$$

$$Tf(x) = \frac{1}{\sigma_n} \int_D \sum_{j=1}^{n} \frac{y_j - x_j}{|x-y|^n} f_j(y) dy,$$

and

$$\mathcal{G}v(x) = -\frac{1}{\sigma_n} \int_{\partial D} \sum_{j=1}^{n} (-1)^{j-1} \frac{y_j - x_j}{|x-y|^n} v(y) dy[j], \tag{5.6}$$

where $x \in \mathbb{R}^n \setminus \partial D$, $v \in W^{1,2}(D)$, and $\sigma_n$ is the area of the unit sphere in $\mathbb{R}^n$.

**Example 5.10** Let $P$ be the Cauchy-Riemann operator in $\mathbb{C}^n$. Then the operator $(-4P^*P)$ is the Laplace operator $\Delta_{2n}$ in $\mathbb{R}^{2n}$, the compatibility complex is the Dolbeault complex. It is well-known that for every $f \in [W^{1,2}(D)]^n$ satisfying the compatibility conditions in a pseudoconvex domain $D$ there exists a solution of $Pu = f$ in $W^{1,2}(D)$. In this case $S_P(D)$ is the space of holomorphic functions in the domain $D$,

$$Tf(z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int_D \sum_{j=1}^{n} \frac{\zeta_j - z_j}{|z-\zeta|^{2n}} f_j(\zeta) d\zeta,$$

and $Gv$ is the Bochner-Martinelli integral:

$$\mathcal{G}v(z) = -\frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int_{\partial D} \sum_{j=1}^{n} (-1)^{j-1} \frac{\zeta_j - z_j}{|z-\zeta|^{2n}} v(\zeta) d\zeta[j], \tag{5.7}$$

where $z \in \mathbb{C}^n \setminus \partial D$, $v \in W^{1,2}(D)$. Solutions of that kind were obtained in [13].

**Example 5.11** Let $\partial_j$ stand for $\frac{\partial}{\partial x_j}$. Consider the following systems $P_{(n)}$ in $\mathbb{R}^n$ ($n = 2$ or $n = 3$):

$$P_{(2)} = \begin{pmatrix} \sqrt{2\mu}\partial_1 & 0 \\ 0 & \sqrt{2\mu}\partial_2 \\ \sqrt{\lambda}\partial_1 & \sqrt{\lambda}\partial_2 \\ \sqrt{\mu}\partial_2 & \sqrt{\mu}\partial_1 \end{pmatrix}, \qquad P_{(3)} = \begin{pmatrix} \sqrt{2\mu}\partial_1 & 0 & 0 \\ 0 & \sqrt{2\mu}\partial_2 & 0 \\ 0 & 0 & \sqrt{2\mu}\partial_3 \\ \sqrt{\lambda}\partial_1 & \sqrt{\lambda}\partial_2 & \sqrt{\lambda}\partial_3 \\ \sqrt{\mu}\partial_2 & \sqrt{\mu}\partial_1 & 0 \\ \sqrt{\mu}\partial_3 & 0 & \sqrt{2\mu}\partial_1 \\ 0 & \sqrt{2\mu}\partial_3 & \sqrt{2\mu}\partial_2 \end{pmatrix}$$

with $\lambda \geq 0$, $\mu > 0$. Then $(-P^*_{(n)} P_{(n)})$ is the Lamé operator $\mathcal{L}_n$ in $\mathbb{R}^n$,

$$\mathcal{L}_n = \mu \Delta_n + (\lambda + \mu) \nabla_n \, div_n$$

with the Lamé constants $\mu$, $\lambda$, and $\Phi_n(x-y)$ is the Kelvin-Somigliana matrix $(\Phi^{(i,j)}_n)_{i,j=1,2,\dots n}$ (see, for example, [6]) with components

$$\Phi^{(i,j)}_n = \frac{1}{2\mu(\lambda+2\mu)} \left( \delta_{ij} (\lambda + 3\mu) \phi_n(x-y) - (\lambda+\mu) x_j \frac{\partial}{\partial x_i} \phi_n(x-y) \right).$$

In this case $S_P(D)$ consists of (not all!) polynomials of the first degree and the corresponding compatibility operators $P^1_{(n)}$ are

$$P^1_{(2)} = \begin{pmatrix} \sqrt{\lambda} & \sqrt{\lambda} & -\sqrt{2\mu} & 0 \\ \partial_{2,2} & \partial_{1,1} & 0 & -\sqrt{2}\partial_{1,2} \end{pmatrix},$$

$$P^1_{(3)} =$$

$$\begin{pmatrix} \sqrt{\lambda} & \sqrt{\lambda} & \sqrt{\lambda} & -\sqrt{2\mu} & 0 & 0 & 0 \\ \partial_{2,2} & \partial_{1,1} & 0 & 0 & -\sqrt{2}\partial_{1,2} & 0 & 0 \\ \partial_{3,3} & 0 & \partial_{1,1} & 0 & 0 & -\sqrt{2}\partial_{1,3} & 0 \\ 0 & \partial_{3,3} & \partial_{2,2} & 0 & 0 & 0 & -\sqrt{2}\partial_{2,3} \\ 0 & 0 & -\sqrt{2}\partial_{1,2} & 0 & -\partial_{3,3} & \partial_{2,3} & \partial_{1,3} \\ 0 & -\sqrt{2}\partial_{1,3} & 0 & 0 & \partial_{2,3} & -\partial_{2,2} & \partial_{1,2} \\ -\sqrt{2}\partial_{3,2} & 0 & 0 & 0 & \partial_{1,3} & \partial_{1,2} & -\partial_{1,1} \end{pmatrix}$$

where $\partial_{i,j} = \frac{\partial^2}{\partial x_i \partial x_j}$. In the linear elasticity theory the equation $P_{(n)} u = f$ reads as follows: given components of the deformation $f$ of an isotropic elastic body $D$, find the components of the displacement $u(x)$ for all the points $x \in D$. By simple calculations one obtains that for every $f \in [L^2(D)]^n$ satisfying the compatibility conditions in a convex domain $D$ there exists a solution of $Pu = f$ in $W^{1,2}(D)$.

## 5.3 The Neumann problem

In this subsection using the iterations of the integrals $TP$ and $\mathcal{G}$ we construct formulae for solutions of the $P$-Neumann problem in $D$ whenever these solutions exist.

Consider the following problem (cf. [9], Problem 7.1).

**Problem 5.12** (**$P$-Neumann problem.**) *Let $w_0 \in [W^{-1/2,2}(\partial D)]^k$ be a given section. Find a section $u \in [W^{1,2}(D)]^k$ such that*

$$\begin{cases} P^* P u = 0 & in\ D, \\ \sum_{j=1}^n P^*_j \frac{\partial \rho}{\partial x_j}(x) P u = w_0 & on\ \partial D. \end{cases},$$

*where $\rho \in C^\infty$ is a real-valued function definig the domain $D$ with $\nabla \rho \neq 0$ on $\partial D$.*

The equation $P^* P u = 0$ in $D$ is understood in the sense of distributions, while the boundary values are interpreted in the variational sense:

$$\int_D (Pv(y))^* Pu(y) dy = \int_{\partial D} v^*(y) w_0(y) ds(y) \text{ for every } v \in [C^\infty(\overline{D})]^k.$$

It is clear that the difference between two solutions to the Problem 5.12 belongs to $S_P^{1,2}(D)$. The Stokes formula implies that a necessary condition for the solvability of Problem 5.12, for given $w_0$, is

$$\int_{\partial D} v^*(y) w_0(y) ds(y) = 0 \text{ for all } v \in S_P^{1,2}(D). \tag{5.8}$$

Hence the Problem 5.12 is not elliptic in general. For instance, it is not elliptic if the dimension of the space $S_P^{1,2}(D)$ is not finite. It is known (see, for example, [9], Proposition 7.12) that the problem is solvable for all $w_0 \in [W^{-1/2,2}(\partial D)]^k$, satisfying the condition (5.8) if and only if the image of the operator $\mathcal{P} : [W^{1,2}(D)]^k \to [L^2(D)]^l$ is closed.

Set

$$\tau w_0(x) = \int_{\partial D} (\Phi_n(x,y))' w_0(y) ds(y).$$

**Corollary 5.13** *Let Problem 5.12 be solvable for $w_0 \in [W^{-1/2,2}(D)]^k$. Then the series $\tilde{v} = \sum_{\mu=0}^\infty \mathcal{G}^\mu \tau w_0$, converging in $[W^{1,2}(D)]^k$, is one of the solutions to this problem.*

**Proof.** Under the hypothesis of the corollary there exists a solution $v \in [W^{1,2}(D)]^k$ to the Problem 5.12. Then the Stokes formula implies $\tau w_0 = TPv$. Moreover using formula (2.2) and Corollary 5.4 we see that $\tilde{v}$ is a solution to the Problem 5.12 too. □

Note that according to Corollary 5.4 the series $\tilde{v}$ is a unique solution to Problem 5.12 belonging to the orthogonal complement (with respect to the scalar product $H_P(.,.)$) of the space $S_P^{1,2}(D)$ in $[W^{1,2}(D)]^k$.

**Example 5.14** Let $P$ be the gradient operator in $\mathbb{R}^n$ (see example 5.9). Then Problem 5.12 is the classical Neumann problem for the Laplace operator. It is well-known that this problem is elliptic. In this case the operator $\mathcal{G}v$ is defined by (5.6).

**Example 5.15** Let $P$ be the Cauchy-Riemann system in $\mathbb{C}^n$ (see example 5.10). Then Problem 5.12 is $\overline{\partial}$-Neumann problem, and it is not elliptic. The operator $Gv$ is defined by (5.7). Solutions of that kind were obtained in [7] (see. §18, p. 177).

**Example 5.16** Let $P_{(n)}$ be the systems of example 5.11. Then the boundary operator $\sum_{j=1}^{n} P^*_{(n)_j} \frac{\partial \rho}{\partial x_j}(x) P_{(n)} = T_n = (T_n^{(i,j)})_{i,j=1,2,\ldots n}$ is the stress operator with the components

$$T_n^{(i,j)} = \left( \delta_{ij}\mu \frac{\partial}{\partial n} + \lambda n_i(x) \frac{\partial}{\partial x_j} + \mu n_j(x) \frac{\partial}{\partial x_i} \right),$$

where $n_j(x)$ is the $j-$th component of the outward normal vector $n(x)$ to $\partial D$ at the point $x$ and $\frac{\partial}{\partial n}$ is the normal derivative with respect to $\partial D$ (see [6]).

In the linear elasticity theory Problem 5.12 reads as follows: given stress vector $w_o$ on the boundary of an isotropic elastic body $D$ under 0 (zero) load, find the components of the displacement $u(x)$ for all points $x \in D$. As the image of the operator $\mathcal{P}_{(n)}$ is closed this problem is elliptic.

## 5.4 The Dirichlet problem

In this subsection we obtain formulae for solutions to the Dirichlet problem for elliptic systems.

**Corollary 5.17** *The series $u = \sum_{\mu=0}^{\infty} (TP)^\mu \mathcal{G} u_0$ converges in $[W^{1,2}(D)]^k$ for every function $u_0 \in [W^{1/2,2}\partial D)]^k$, and, moreover, if $\widetilde{S}_P^{1,2}(\mathbb{R}^n \backslash \overline{D}) = \{0\}$ then it is a solution of the Dirichlet problem for the operator $P^*P$ in $D$ with data $u_0$, i.e.,*

$$\begin{cases} P^*Pu = 0 & \textit{in } D, \\ u = u_0 & \textit{on } \partial D. \end{cases} \tag{5.9}$$

**Proof.** It follows from classical results on the solvability of the Dirichlet problems for strongly elliptic systems that a solution of such a problem exists and is unique in $[W^{1,2}(D)]^k$, for every section $u_0 \in [W^{1/2,2}(\partial D)]^k$ (see, for instance, [12], 7.4). Hence the statement of the corollary follows from Corollary 5.4 and formula (2.3). □

Using Corollary 5.17 one can obtain solutions to Dirichlet Problem (5.9) in any domain $D$ with a sufficiently smooth boundary (for instance, if $\partial D \in C^\infty$). In particular, these formulae may be useful in the case where the Green function of the Dirichlet problem is not calculated in a precise form.

Note that the condition "$\widetilde{S}_P^{1,2}(\mathbb{R}^n \backslash \overline{D}) = \{0\}$" means that the operator $P$ is sufficiently overdetermined, as is , for instance, the gradient operator or the multidimensional Cauchy-Riemann system. It does not hold for $l = k$, because in this case the operator $P$ has a bilateral fundamental solution of convolution type, say, $\phi_n(x-y) = a\left(\frac{x-y}{|x-y|}\right) |x-y|^{1-n}$, where the $(k \times k)$-matrix $a(\zeta)$ has real analytic entries in a neighborhood of the sphere $\{|\zeta| =$

1}. However, it is not true that $\widetilde{S}^{1,2}_P(\mathbb{R}^n \backslash \overline{D}) = \{0\}$ if $l > k$. For example, if $n = 3$, $l = 4$, $k = 3$, and $P$ is given by the matrix

$$P = \begin{pmatrix} \partial_2 & -\partial_1 & 0 \\ \partial_3 & 0 & -\partial_1 \\ 0 & \partial_3 & -\partial_2 \\ -\partial_1 & -\partial_2 & -\partial_3 \end{pmatrix},$$

then any vector $x \in \mathbb{R}^3$ belongs to $\widetilde{S}^{1,2}_P(\mathbb{R}^n \backslash \overline{D})$, because $S(x) = \frac{x}{|x|^3}$.

Let us consider examples.

**Example 5.18** Let $L$ be a differential operator of the second order with constant coefficients, such that $L = \sum_{i=1}^n Q_i^* Q_i \frac{\partial^2}{\partial x_i^2}$, where $Q_i$ are some $(l_i \times k)$-matrices over $\mathbb{C}$, of rank $k$ (here $l_i \geq k$, $1 \leq i \leq n$). Then the homogeneous $((\sum_{i=1}^n l_i) \times k)$-matrix operator

$$P = \begin{pmatrix} Q_1 \partial_1 \\ \cdots \\ Q_n \partial_n \end{pmatrix}$$

has constant coefficients and is elliptic. Moreover, it is easy to see that $P^* P = L$, and $Pu = 0$ if and only if the section $u$ is constant. Therefore $L$ is strongly elliptic, $\widetilde{S}^{1,2}_P(\mathbb{R}^n \backslash \overline{D}) = \{0\}$, and one may use Corollary 5.17 to obtain solutions to the Dirichlet problem for the operator $L$ in $D$. In particular, if $L = \Delta_n$ is the Laplace operator in $\mathbb{R}^n$ then for the gradient operator $P$ in $\mathbb{R}^n$ we have $P^* P = -\Delta$ and the operators $T$ and $\mathcal{G}$ are given by (5.6).

**Example 5.19** Let $\mathcal{L}_n$ be the Lamé system in $\mathbb{R}^n$ ($n = 2, 3$) (see example 5.11). Then $-\mathcal{L}_n = P^*_{(n)} P_{(n)}$ and $\widetilde{S}^1_P(\mathbb{R}^n \backslash \overline{D}) = \{0\}$ because the elements of this space are polynomials.

In the linear elasticity theory the Dirichlet problem for $\mathcal{L}_n$ reads as follows: given displacement $u$ on the boundary of an isotropic elastic body $D$ under zero load, find the components of the displacement $u(x)$ for all points $x \in D$.

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