Moduli Spaces and Deformation Quantization in Infinite Dimensions

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Abstract

We construct a deformation quantization on an infinite-dimensional symplectic space of regular connections on an SU(2)-bundle over a Riemannian surface of genus $g \geq 2$. The construction is based on the normal form thoerem representing the space of connections as a fibration over a finite-dimensional moduli space of flat connections whose fibre is a cotangent bundle of the infinite-dimensional gauge group. We study the reduction with respect to the gauge groupe both for classical and quantum cases and show that our quantization commutes with reduction.

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Introduction

The problem of existence and classification of deformation quantizations is now completely solved for finite-dimensional Poisson manifolds [1]. In contrast, in infinite dimensions there is even no satisfactory definition of deformation quantization. The matter is that the natural dictionary

points	functions
functions	functionals
summation	integration
partial derivatives	variational derivatives

utilized to pass from finite-dimensional mechanics to field theory should be completed by precise description of the class of *admissible functionals* (*observables*).

A formal translation with the help of this dictionary leads to the following definition of $*$-product on functionals.

Definition 0.1 *A $*$-product is a formal power series defined for any two admissible functionals $F(u(x)),\ G(u(x))$*

$$F * G = \sum_{k=0}^{\infty} h^k C_k(F, G)$$

with the properties

1. *the coefficients C_k are bilinear functionals in variational derivatives of F and G of order $\leq k$,*
2. $$C_0 = FG,$$
3. $$C_1(F, G) - C_1(G, F) = -ih\{F, G\},$$
4. *extended by linearity, the $*$-product defines an associative algebra structure on the space of formal functionals*

$$F(u, h) = \sum_{k=0}^{\infty} h^k F_k(u).$$

Here h is a formal parameter, $\{F, G\}$ means the Poisson bracket of functionals. The variational derivative of order k

$$\frac{\delta^k F}{\delta u(x_1) \ldots \delta u(x_k)}$$

is a function, symmetric in $x_1, \ldots, x_k$, defined by the identity

$$\begin{aligned} \delta^k F &:= \frac{d^k}{dt^k} F(u + t\delta u)\Big|_{t=0} \\ &= \int \ldots \int \frac{\delta^k F}{\delta u(x_1) \ldots \delta u(x_k)} \delta u(x_1) \ldots \delta u(x_k) dx_1 \ldots dx_k. \end{aligned}$$

For $k > 1$ the variational derivatives are mostly distributions rather than smooth functions, while the bilinear functionals C_k may be ill-defined for distributions, whence a difficulty comes of defining a class of observables.

Consider, for example, usual local functionals of variational calculus

$$F = \int f(x, u(x), u'(x), u''(x), \ldots) dx. \tag{0.1}$$

They are good enough for classical mechanics because they have smooth first-order variational derivative, thus, the Poisson bracket is well-defined for them and is of the same class (for local Poisson brackets). But the higher-order variational derivatives of (0.1) contain δ-functions $\delta(x_1 - x_2) \ldots \delta(x_{k-1} - x_k)$ unless F is linear

$$F = \langle f, u \rangle := \int f(x) u(x) dx,$$

thus, in general, the functionals (0.1) are not admissible for deformation quantization. Of course, we always have a minimal admissible class consisting of *tensors*, that is of finite sums of finite products

$$\langle f_1, u \rangle \ldots \langle f_k, u \rangle$$

since variational derivatives of any order are smooth for them, but this class is too restrictive.

Anyway, passing from classical to quantum mechanics in infinite dimensions we have to restrict significantly the set of observables. As a consequence, the class of quantum-mechanical observables is *not invariant* with respect to canonical transformations, that is under changes of variables preserving the Poisson bracket. Indeed, even if we start with an invariant class of classical observables, the restriction to quantum-mechanical observables may spoil this invariance. For example, the classes of tensors are quite different for different choices of independent variables $u(x)$. Thus, the proper choice of independent variables may be crucial for deformation quantization.

In this paper we construct a $*$-product on the space of connections on an $SU(2)$ vector bundle E over a Riemannian surface X of genus $g \geq 2$. The space M of all connections is a symplectic space with a symplectic form

$$\omega = \frac{1}{2} \int_X \operatorname{tr} \delta D \wedge \delta D, \tag{0.2}$$

where $D \in M$ denotes a connection on E and δD its variation. The group G of automorphisms of E called the gauge group acts on M

$$g : D \mapsto g^{-1} \circ D \circ g \tag{0.3}$$

preserving the symplectic form. The *moment map* of this action turns out to be

$$\mu(\alpha) = \int_X \operatorname{tr} \kappa\alpha \tag{0.4}$$

where α belongs to the Lie algebra $\mathcal{G}$ of G and

$$\kappa = d\Gamma + \frac{1}{2}[\Gamma, \Gamma] \tag{0.5}$$

is the curvature of D.

The Marsden–Weinstein symplectic reduction [2] may be applied to this action. The zero level set $M_0 \subset M$ of the moment map consists of all connections ∂ which are flat, that is $\kappa \equiv 0$ for them. The group G preserves M_0 and the orbit space

$$B = M_0/G \tag{0.6}$$

turns out to be a *finite-dimensional* manifold (with singularities) called *moduli space of flat connections.*

The functionals (0.4) are Hamiltonians of the infinitesimal gauge action, and we would like our admissible set of quantizable functionals to contain all of them. Moreover, we would like the relation

$$\{\mu(\alpha_1), \mu(\alpha_2)\} = \int_X \operatorname{tr} \kappa[\alpha_1, \alpha_2] = \mu([\alpha_1, \alpha_2]) \tag{0.7}$$

which holds for the classical moment map to admit the corresponding quantum analog

$$[\widehat{\mu}(\alpha_1), \widehat{\mu}(\alpha_2)] = -ih\widehat{\mu}([\alpha_1, \alpha_2]). \tag{0.8}$$

In other words, we would like to quantize the classical moment map.

In finite-dimensional case we know [3] that the reduction commutes with the canonical G-invariant quantization, that is the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{Q_A} & \widehat{A} \\ \downarrow & & \downarrow \\ R & \xrightarrow{Q_R} & \widehat{R} \end{array} . \tag{0.9}$$

Here A is the algebra of classical observables, R is the reduced algebra consisting of functions on the reduced manifold B, Q_A and Q_R mean canonical G-invariant deformation quantizations of these algebras. The vertical arrows mean reductions: classical (left) and quantum (right).

Now, in the infinite-dimensional case the left part of the diagram (0.9) is known. A crucial fact is that the reduced manifold B (the moduli space of flat connections) is finite-dimensional. Thus, the lower part of the diagram (0.9) is also known since for finite- dimensional manifolds there is a canonical deformation quantization. Moreover, whatever the algebra $\widehat{A}$ should be, the quantum reduction is also defined, provided the moment map is quantized, so we know the right part of the diagram (0.9). It remains to reconstruct the quantization map Q_A to fill the diagram to a commutative one.

Our construction of deformation quantization is based on a special choice of independent variables given by the so-called *normal form theorem*. In the finite-dimensional space it is well-known [4], here we prove it for the space of connections. Roughly speaking, this theorem allows one to consider all the functionals (0.4) as independent variables. More precisely, it gives a representation of M as a fibering over B with a fiber $F = G \times \mathcal{G}^*$. Thus, the quantization procedure consists of two steps:

1. quantization of the fiber resulting in a bundle $\widehat{A}_F$ over B whose fibres are algebras of quantum observables on F,

2. a canonical deformation quantization of the finite-dimensional manifold B with coefficients in the bundle $\widehat{A}_F$.

For a finite-dimensional Lie group G a $*$-product on $G \times \mathcal{G}^* = T^*(G)$ was constructed in [5]. In our case G is an infinite-dimensional Lie group, nevertheless a modification of this construction works. As for the second step, it goes similarly to the finite-dimensional case (cf. [6, Theorem 6.5.1], [7]). Our algebra of quantum observables is *by definition* the result of these two steps, so commutativity of the diagram (0.9) is an almost tautological assertion.

Let us briefly describe the content of the paper. Trying to make it self-contained, we consider in section 1 standard geometrical facts concerning connections, flat connections and their moduli spaces. The most important is the *regularity* (*irreducibility*) condition. In section 2 we prove the normal form theorem for regular connections with sufficiently small curvature. In section 3 the quantization procedure is exposed including quantum reduction.

In conclusion we would like to mention a large paper [8] where geometric quantization on moduli spaces is considered in detail from the point of view of symplectic reduction.

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1 The Geometry of Connections

Let X be a Riemannian surface of genus $g \geq 2$. We consider connections on $SU(2)$ vector bundle E over X. Since the structure group $SU(2)$ is connected and simply connected, the bundle E is trivial, so any section s may be treated as a column vector-function

$$s = \begin{pmatrix} s_1(x) \\ s_2(x) \end{pmatrix} \tag{1.1}$$

and any connection on the bundle E has the form

$$Ds = ds + \Gamma s \tag{1.2}$$

where d is the usual differential and $\Gamma \in C^\infty(X, su(2) \otimes \Lambda^1)$ is a 1-form on X with values in the Lie algebra $su(2)$. The curvature of the connection D is defined by (0.5). This is a 2-form on X with values in $su(2)$.

Let M denote the space of all connections (1.2), $M_0 \subset M$ the space of flat connections ($\kappa \equiv 0$). Treating the infinite-dimensional geometrical objcts we always deal with their restrictions to some finite-dimensional parameter space Λ. By $\delta D = \delta\Gamma$ we denote the differential (variation) with respect to $\lambda \in \Lambda$. It is convenient to treat $\delta\Gamma$ as a differential 2-form on $X \times \Lambda$. We have two anticommuting defferentials on $X \times \Lambda$: d with respect to $x \in X$ and δ with respect to $\lambda \in \Lambda$, so differential forms on $X \times \Lambda$ have double grading. Thus, $\delta\Gamma$ is a $(1,1)$-form on $X \times \Lambda$. There is a *symplectic form* on M given by (0.2) or

$$\omega = \frac{1}{2}\int_X \operatorname{tr} \delta\Gamma \wedge \delta\Gamma. \tag{1.3}$$

Note, that $\delta\Gamma \wedge \delta\Gamma$ is a $(2,2)$-form on $X \times \Lambda$, so we may integrate it partially over X obtaining a 2-form on Λ, which is closed and non-degenerate. For partial integration we need to fix the order of differentials: we assume that first go dx and then $d\lambda$.

Definition 1.1 *A gauge group G is the group of automorphisms of the bundle E.*

Thus, $g \in G$ is given by a function $g(x) \in C^\infty(X, SU(2))$. The corresponding Lie algebra $\mathcal{G}$ is the algebra of functions $a(x) \in C^\infty(X, su(2))$, the commutator $[a, b]$ is defined point-wise. The dual space $\mathcal{G}^*$ consists of $su(2)$-valued 2-forms ξ and the pairing is given by

$$\langle \xi, a \rangle = \int_X \operatorname{tr} \xi a.$$

The gauge group acts from the right on the space of connections: for any $D \in M$ any $g \in G$ defines a new connection $g(D)$ by

$$g(D)s = g^{-1}D(gs) = ds + (g^{-1}\Gamma g + g^{-1}dg)s. \tag{1.4}$$

Clearly,

$$g : \kappa \mapsto g^{-1}\kappa g, \tag{1.5}$$

thus, the subspace M_0 is invariant.

Lemma 1.2 *The action* (1.4) *preserves the form* ω *and its moment map is*

$$\mu(a) = \int_X \operatorname{tr} \kappa a \tag{1.6}$$

for $a \in \mathcal{G}$.

Proof. For fixed $g(x) \in \mathcal{G}$ we have $\delta(g^{-1}dg) = 0$, so G acts on variations $\delta\Gamma$ by conjugation:

$$g : \delta\Gamma \mapsto g^{-1}\delta\Gamma g. \tag{1.7}$$

Since the trace is not affected by conjugations, the invariance of ω is evident.

For $a \in \mathcal{G}$ the infinitesimal action corresponding to

$$g : \Gamma \mapsto g^{-1}\Gamma g + g^{-1}dg$$

is

$$a : \Gamma \mapsto [\Gamma, a] + da = Da.$$

Let V_a be the correspondinfg vector field on M, so that

$$i(V_a)\delta\Gamma = Da.$$

Then we have

$$i(V_a)\omega = \int \operatorname{tr} Da \wedge \delta\Gamma = -\int \operatorname{tr} aD(\delta\Gamma).$$

But

$$D(\delta\Gamma) = d(\delta\Gamma) + [\Gamma, \delta\Gamma] = -\delta(d\Gamma + \Gamma^2) = -\delta\kappa.$$

Thus, for fixed $a \in \mathcal{G}$

$$i(V_a)\omega = \int_X \operatorname{tr} a\delta\kappa = \delta(\int_X \operatorname{tr} a\kappa),$$

which means that V_a is a Hamiltonian vector field and the functional

$$\mu(a) = \int_X \operatorname{tr} a\kappa$$

is the corresponding Hamiltonian.

$\square$

The Marsden-Weinstein reduction consists of two steps: the restriction to the *zero level set* of the moment map and then the projection to the *orbit space*. From (1.6) it is seen immediately that $\mu(a) = 0$ for any $a \in \mathcal{G}$ implies $\kappa \equiv 0$, so the level set $\mu = 0$ is the subspace $M_0 \subset M$ of flat connections. In the sequel we use the symbols ∂, γ for flat connections to distinguish them from general ones. We know that the gauge group acts on M_0, the orbit space $B = M_0/G$ is called the *moduli space* of flat connections.

To describe the space B, we consider the monodromies of a flat connection ∂. Given a cycle C on X and a connection (not necessarily flat), the *monodromy* $T(C)$ is the isomorphism of the fiber E_{x_0} at a base point $x_0 \in X$ defined by the parallel transport along the closed curve C. For a flat connection the monodromy depends on the homotopy class of C only. Thus, all the monodromies of a flat connection ∂ define a homomorphism of the fundamental group $\pi_1(X)$ into $SU(2)$. The group $\pi_1(X)$ is generated by the canonical cycles $a_i, b_i,\ i = 1, 2 \ldots, g$ satisfying the only relation (see e.g. [9])

$$\prod_{i=1}^{g} b_i^{-1} a_i^{-1} b_i a_i = 1,$$

hence the monodromies are defined by $2g$ matrices

$$A_i = T(a_i); \quad B_i = T(b_i) \tag{1.8}$$

satisfying the relation

$$\prod_{i=1}^{g} B_i^{-1} A_i^{-1} B_i A_i = 1. \tag{1.9}$$

Clearly, this matrices are defined up to conjugations by a matrix $U \in SU(2)$ corresponding to a change of frames in E_{x_0}.

Definition 1.3 *A flat connection ∂ is called regular if al least for one i the monodromies A_i, B_i do not commute:*

$$B_i^{-1} A_i^{-1} B_i A_i \neq 1. \tag{1.10}$$

We denote by M_0^{reg} the space of regular flat connections and by $B^{reg} = M_0^{reg}/G$ the corresponding moduli space.

Theorem 1.4 *The moduli space B^{reg} of regular flat connections is a smooth manifold of dimension* $6(g-1)$.

Proof. Passing to the universal covering $\widetilde{X}$, we would have for any flat connection ∂ a family of parallel frames of the bundle E lifted to $\widetilde{X}$. These frames are defined by a matrix function $U(\widetilde{X})$ satisfying

$$dU + \gamma U = 0. \tag{1.11}$$

By the definition of the monodromy we have

$$U(\widetilde{x}c) = U(\widetilde{x})T(c) \tag{1.12}$$

for any $c \in \pi_1(X)$ acting from the right on $\widetilde{x} \in \widetilde{X}$. Vice versa, any family of matrices $U(\widetilde{x})$ on $\widetilde{X}$ satisfying (1.12) defines a flat connection $\partial = d + \gamma$ with

$$\gamma = -dUU^{-1}$$

on $\widetilde{X}$ which is invariant under the action of π_1, the matrices $T(c)$ in (1.12) being the monodromies of ∂. The action (1.4) of the gauge group on the space of flat connections corresponds to the adjoint action on $U(\widetilde{x})$;

$$g : U(\widetilde{x}) \to g^{-1}(x)U(\widetilde{x})g(x),$$

giving for monodromies

$$g : T(c) \to g^{-1}(x_0)T(c)g(x_0).$$

Therefore two flat connections belong to the same orbit of the gauge group if their monodromies coincide (up to conjugation).

For a given representation T of the group $\pi_1(X)$ the family $U(\widetilde{x})$ satisfying (1.12) may be constructed using a standard description of a fundamental domain $\Pi \subset \widetilde{X}$ (see e.g. [9]). Taking $\widetilde{x}_0 \in \widetilde{X}$ arbitrarily, we set

$$U(\widetilde{x}_0 c) = T(c).$$

The cycle a_i lifted to $\widetilde{X}$ with the initial point $\widetilde{x}_0$ gives an edge $[\widetilde{x}_0, \widetilde{x}_0 a_i] \subset \widetilde{X}$ and we define $U(\widetilde{x})$ for $\widetilde{x} \in [\widetilde{x}_0, \widetilde{x}_0 a_i]$ taking any curve in $SU(2)$ connecting 1 with $T(a_i) = A_i$. For other liftings $\widetilde{x} \in [\widetilde{x}_0 p, \quad \widetilde{x}_0 a_i p]$ we set

$$U(\widetilde{x}) = U(\widetilde{x}p^{-1})T(p) \tag{1.13}$$

(here $\widetilde{x}p^{-1} \in [\widetilde{x}_0, \widetilde{x}_0 a])$. Similarly, we proceed with the b-cycles. The liftings of canonical cycles define a $\pi_1(X)$-invariant subdivision of $\widetilde{X}$ into fundamental domains. Taking one of them, we obtain a polygon Π whose successive edges are the liftings of the cycles

$$a_1, b_1, a_1^{-1}, b_1^{-1} \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

The relation (1.9) ensures that the functions $U(\widetilde{x})$ already defined on edges give a cycle $\partial\Pi \to SU(2)$ which may be contracted since $SU(2)$ is simply connected. This gives a function $U(\widetilde{x})$ in the interior of Π. Any other fundamental domain Π' has the form Πc for some $c \in \pi_1(X)$ and we set

$$U(\widetilde{x}) = U(\widetilde{x}c^{-1})T(c).$$

This definition is consistent due to (1.9) and we obtain a function $\widetilde{U}(\widetilde{x})$ on the whole $\widetilde{X}$ satisfying (1.12). It remains to apply a $\pi_1(X)$-invariant smoothing procedure to obtain a smooth function $U(\widetilde{x})$.

Thus, we have proved that the moduli space of flat connections is a compact topological space consisting of conjugacy classes of unitary matrices $A_i, B_i, \quad i = 1, 2 \ldots, g$ satisfying (1.9). Now, consider regular flat connections. Supposing that $A_1B_1 \neq B_1A_1$, we may reduce these matrices uniquely to the form

$$A_1 = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \; B_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \bar{\alpha} \end{pmatrix} \tag{1.14}$$

with $0 < \varphi < \pi$ and β real positive, $|\alpha^2| + \beta^2 = 1$. Then

$$ABA^{-1}B^{-1} = \begin{pmatrix} 1 + \beta^2(e^{2i\varphi} - 1) & \alpha\beta(e^{2i\varphi} - 1) \\ -\bar{\alpha}\beta(e^{-2i\varphi} - 1) & 1 + \beta^2(e^{-2i\varphi} - 1) \end{pmatrix} \tag{1.15}$$

Any unitary matrix

$$A = \begin{pmatrix} 1 + a & b \\ -\bar{b} & 1 + \bar{a} \end{pmatrix} \in SU(2) \tag{1.16}$$

with $\Re a < 0$ may be uniquely represented in the form (1.15) taking

$$\begin{gathered} \varphi = \arg a + \frac{\pi}{2} \in (0, \pi); \\ \beta = \left(\frac{|a|}{2 \sin \varphi} \right)^{1/2}; \quad \alpha = \frac{b}{a}\beta. \end{gathered} \tag{1.17}$$

It means that for matrices sufficiently close to a given regular ones with non-commuting A_1, B_1 we may take $A_2, B_2, \ldots, A_g, B_g$ arbitrarily and find A_1, B_1 from (1.17). Since $\dim SU(2) = 3$, we obtain $6(g-1)$ independent real parameters giving a coordinate chart for the manifold B^{reg}.

□

The regularity property for flat connections admits another characterization which may be generalized to any connection, not necessarily flat. To this end we introduce the Laplace operators associated to a given flat connection ∂. Consider differential forms on X with values in $su(2)$. For a given Riemannian metric $ds^2 = f|dz|^2, \quad f > 0$, where $z = x + iy$ is a local complex parameter, define the Hodge operator $*$ by

$$*a = a^* f dx \wedge dy = -af dx \wedge dy; \quad *a dx \wedge dy = \frac{a^*}{f} = -\frac{a}{f}$$

$$*a(adx + bdy) = a^* dy - b^* dx = -ady + bdx$$

where a, b are functions with values in $su(2)$, $a^* = -a$ means hermitian conjugation. Then we have a positive definite scalar product on forms

$$(\alpha, \beta) = \int_X \operatorname{tr} \alpha \wedge *\beta. \tag{1.18}$$

For $su(2)$-valued differential forms on X the differential ∂ is defined by

$$\partial\alpha = d\alpha + [\gamma, \alpha]. \tag{1.19}$$

Its adjoint with respect to the scalar product (1.18) is equal to $\partial^* = -*\partial*$ For flat connections we have $\partial^2 = (\partial^*)^2 = 0$, so the Laplace operator on $su(2)$-valued differential forms may be defined in a usual way

$$\Delta = (\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial.$$

Actually, we have three operators

$$\Delta_0 = \partial^*\partial; \quad \Delta_1 = \partial\partial^* + \partial^*\partial; \quad \Delta_2 = \partial\partial^* = *\Delta_0*$$

on $0, 1$ and 2-forms.

Lemma 1.5 *If ∂ is a regular flat connection, then the operators Δ_0 and Δ_2 are invertible.*

Proof. It is sufficient to consider Δ_0. Show that $\operatorname{Ker}\Delta_0 = 0$. Let u be a harmonic $su(2)$-valued function, that is $\partial u = 0$. Passing to the universal covering $\widetilde{X}$ and using parallel frames $U(\widetilde{x})$, we may write

$$\partial u = Ud(U^{-1}uU)U^{-1} = 0,$$

so that $U^{-1}uU = \text{const}$ on $\widetilde{X}$ or

$$u(\widetilde{x}) = U(\widetilde{x})cU^{-1}(\widetilde{x}) \tag{1.20}$$

where $c \in su(2)$ is a constant matrix. To define a function on X, it must be invariant with respect to the action of $\pi_1(X)$. It means that for any monodromy T

$$TcT^{-1} = c.$$

Reducing $c \neq 0$ to the diagonal form $\operatorname{diag}\{i\mu, \ -i\mu\}$, $\mu > 0$, we see that all the monodromies must be diagonal matrices which contradicts to the assumption that ∂ is a regular connection, hence $c = 0$. The spectrum of Δ_0 is discrete, so Δ_0^{-1} is a bounded operator in L^2.

$\square$

Thus, for regular flat connections the operator $\partial : \Lambda^0 \to \Lambda^1$ has a left inverse $\Delta_0^{-1}\partial^* = (\partial^*\partial)^{-1}\partial^*$ while $\partial : \Lambda^1 \to \Lambda^2$ has a right inverse $\partial^*\Delta_2^{-1} = \partial^*(\partial\partial^*)^{-1}$. Moreover, the operators $\partial\Delta_0^{-1}\partial^*$ and $\partial^*\Delta_2^{-1}\partial$ on Λ^1 are orthogonal projectors in $L^2(X, \Lambda^1 \otimes su(2))$ giving the Hodge decomposition, that is $\partial\Delta_0^{-1}\partial^*$ projects L^2 onto the subspace of exact forms, while $\partial^*\Delta_2^{-1}\partial$ projects L^2 onto the subspace of coexact forms. The complementary orthogonal projector

$$p = 1 - \partial^*\Delta_2^{-1}\partial - \partial\Delta_0^{-1}\partial^* \tag{1.21}$$

maps $L^2(X, \Lambda^1 \otimes su(2))$ onto a finite-dimensional subspace of harmonic forms.

In other words, we have a family of elliptic operators $\partial + \partial^*$ on a manifold M_0^{reg} of regular flat connections. The cokernels of these operators are trivial, so their kernels defined by the projector p give us the index bundle of $\partial+\partial^*$ over M_0^{reg}. The index of $\partial+\partial^*$ is independent of a particular flat connection ∂ (not necessarily regular), so we obtain taking $\partial = d$

$$\dim p = \mathrm{ind}\,(\partial+\partial^*) = \mathrm{ind}\,(d+d^*) = 2(g-1)\dim su(2) = 6(g-1). \quad (1.22)$$

Lemma 1.6 *The tangent bundle TB^{reg} of the moduli manifold B^{reg} is the bundle p of harmonic forms.*

Proof. Consider a regular flat connection $\partial = d+\gamma$. We have

$$d\gamma + \gamma^2 = 0$$

since the curvature $\kappa(\partial) = 0$. Taking variations and using that d and δ anticommute, we obtain

$$-d\delta\gamma + \delta\gamma\wedge\gamma + \gamma\wedge\delta\gamma = -\partial(\delta\gamma) = 0$$

for a variation $\delta\gamma$ of a flat connection. In other words, the tangent space to M_0^{reg} at a "point" $\partial \in M_0^{reg}$ consists of ∂-closed 1-forms $\delta\gamma$. To obtain TB^{reg}, we must take the quotient of the space of ∂-closed forms by the infinitesimal action of the gauge group. Taking $g = 1+a$, $a \in \mathcal{G}$, we get from (1.4)

$$\delta\gamma = \gamma a - a\gamma + da = \partial a.$$

This means that the variation $\delta\gamma$ corresponding to the infinitesimal action of the gauge group are precisely a ∂-exact 1-form. The cohomology space $\mathrm{Ker}\,\partial/\mathrm{Im}\,\partial$ may be identified by the Hodge theorem with the space of ∂-harmonic forms, proving the lemma.

□

In view of this lemma the expression (1.22) is nothing but the dimension of B^{reg} given by Theorem 1.4.

Finally, we introduce the following definition.

Definition 1.7 *A connection D (not necessarily flat) is called regular if the operator DD^* on 2-forms is invertible.*

The operator

$$D^*D = - * D * D = *DD^* *$$

on 0-forms is automatically invertible for regular connections. We denote by $M^{reg} \subset M$ the space of regular connections. Lemma 1.5 implies that for flat connections definitions 1.3 and 1.7 are equivalent, so $M_0^{reg} \subset M^{reg}$.

2 The Normal Form Theorem

We will consider $su(2)$-valued differential forms on X in Hölder spaces C^r, $r > 0$, $r \notin \mathbb{Z}$. The Hölder spaces (versus Sobolev ones) have a nice property: for any $u_1 \in C^{r_1}$, $u_2 \in C^{r_1}$ we have

$$u_1 u_2 \in C^{r_3},\ r_3 = \min\{r_1, r_2\}$$

and

$$\|u_1 u_2\|_{r_3} \le C\|u_1\|_{r_1}\|u_2\|_{r_2}. \tag{2.1}$$

For this reason the Hölder spaces are more appropriate for non-linear problems. On the contrary, the action of classical pseudo-differential operators on a compact manifold is more complicated in Hölder spaces then in the Sobolev ones. We will need the following result (see e.g. [10]): a classical pseudo-differential operator P of order m acts continuously from C^r to C^{r-m}, provided both $r, r-m$ are positive and non-integer, that is

$$\|Pu\|_{r-m} \le C\|u\|_r. \tag{2.2}$$

Saying that α is a small form we mean the smallness of its Hölder norm $\|\alpha\|_r$ for some fixed positive non-integer r.

The purpose of this section is the following *normal form* thorem (cf. [4]).

Theorem 2.1 *There exists a continuous one-to-one correspondence*

$$F : (\partial, \xi) \mapsto D \tag{2.3}$$

between regular connections $D = d + \Gamma$, $\Gamma \in C^r$, $r > 1$, $r \notin \mathbb{Z}$ with a small curvature $\kappa(D) \in C^{r-1}$, and pairs $(\partial,\ \xi)$ where $\partial = d + \gamma$, $\gamma \in C^r$ is a regular flat connection and $\xi \in C^{r-1}$ is a small two-form. The map (2.3) *is equivariant with respect to the usual action of the gauge group on connections D and ∂ and the adjoint action $\xi \mapsto g^{-1}\xi g$ on ξ.*

The symplectic form (0.2) *on M^{reg} is pulled back by* (2.3) *to the form*

$$\omega_0 = F^*\omega = \frac{1}{2}\int_X \operatorname{tr} \delta\gamma \wedge \delta\gamma + \delta \int_X \operatorname{tr} \xi (\partial^*\partial)^{-1}\partial^*(\delta\gamma). \tag{2.4}$$

Before proving we would like to make some comments. Roughly speaking, the theorem gives a representation

$$M^{reg} = M_0^{reg} \times \mathcal{G}^*$$

similar to that in a finite-dimensional case (cf., e.g. [3]). More precisely, this representation is valid for a small neighbourhood of M_0^{reg} in M^{reg} consisting of connctions with sufficiently small curvature. On the right-hand side one should take a sufficiently small neighbourhood of $0 \in \mathcal{G}^*$ (rather than the whole $\mathcal{G}^*$).

Further, the space M_0^{reg} is a principal G-bundle over B^{reg} (Theorem 1.4) and the expression $(\partial^*\partial)^{-1}\partial^*(\delta\gamma)$ may be considered as a connection one-form on the principal bundle. Indeed, if V_a is a vector field on M^{reg} corresponding to the infinitesimal action of $a \in \mathcal{G}$ then $i(V_a)\delta\gamma = \partial a$, thus

$$i(V_a)(\partial^*\partial)^{-1}\partial^*(\delta\gamma) = -(\partial^*\partial)^{-1}\partial^*(\partial a) = -a. \tag{2.5}$$

The first term in (2.4) may be considered as a symplectic form on the base B^{reg} (the Marsden-Weinstein reduced form). Indeed, for a flat connection $\partial = d + \gamma$ the form $\delta\gamma$ is ∂-closed and its exact component does not affect the integral while its harmonic component defines a form on B^{reg} according to Lemma 1.6.

The moment map for the symplectic form ω_0 has an extremely simple form. Indeed, using (2.5), we obtain

$$i(V_a)\omega_0 = -\delta i(V_a)\int_X \operatorname{tr}\xi(\partial^*\partial)^{-1}\partial^*(\delta\gamma) = \delta\int_X \operatorname{tr}\xi a,$$

so that $\mu(\partial, \xi) = \xi$.

Proof of Theorem 2.1. The proof is divided into several lemmas.

Lemma 2.2 *Any regular connection D having sufficiently small curvature may be uniquely represented in the form*

$$D = \partial + \Gamma \tag{2.6}$$

where ∂ is a regular flat connection and Γ is a ∂-coexact 1-form with values in $su(2)$.

Proof. If we have (2.6), then taking curvatures we get

$$\kappa(D) = \kappa(\partial) + \partial\Gamma + \Gamma^2.$$

For a flat connection ∂ we have $\kappa(\partial) = 0$, so the last equation may be rewritten in the form

$$\partial\Gamma = \kappa - \Gamma^2$$

where $\kappa = \kappa(D)$. Applying to both sides the operator $\partial^*(\partial\partial^*)^{-1}$ and using that $\partial^*(\partial\partial^*)^{-1}\partial$ is the projector to coexact forms, we obtain

$$\Gamma = \partial^*(\partial\partial^*)^{-1}(\kappa - \Gamma^2). \tag{2.7}$$

The flat connection ∂ may by eliminated from (2.7) as $\partial = D - \Gamma$. On sections of $\operatorname{Hom}(E, E)$ it acts as

$$\partial = D + \operatorname{ad}\Gamma = D - [\Gamma, \cdot\,].$$

This leads to a non-linear equation for Γ

$$\Gamma = (D - \operatorname{ad}\Gamma)^*((D - \operatorname{ad}\Gamma)(D - \operatorname{ad}\Gamma)^*)^{-1}(\kappa - \Gamma^2). \tag{2.8}$$

of the form

$$\Gamma = A(\Gamma) \tag{2.9}$$

with a non-linear operator A given by the right-hand side of (2.8). We will show that Equation (2.8) has a unique sulution $\Gamma \in C^r$, $r > 1$, $r \notin \mathbb{Z}$ provided the curvature $\kappa \in C^{r-1}$ has sufficiently small norm $\|\kappa\|_{r-1}$. To this end we show that the operator $A(\Gamma)$ is a contraction in some ball $\|\Gamma\| < \varepsilon$ in C^r.

Lemma 2.3 *There exists an $\varepsilon > 0$ such that the operator*

$$P^{-1} = ((D - \operatorname{ad}\Gamma)(D - \operatorname{ad}\Gamma)^*)^{-1} : C^{r-1} \to C^{r+1}$$

is uniformly bounded in the ball $\|\Gamma\|_r < \varepsilon$, $r > 1$, $r \notin \mathbb{Z}$.

Proof. Consider

$$\begin{aligned} &P = (D - \operatorname{ad}\Gamma)(D - \operatorname{ad}\Gamma)^* = DD^* - D(\operatorname{ad}\Gamma)^* - (\operatorname{ad}\Gamma)D + (\operatorname{ad}\Gamma)(\operatorname{ad}\Gamma)^* \\ &= (DD^*)^{1/2}\left\{1 - (DD^*)^{-1/2}(D(\operatorname{ad}\Gamma)^* + (\operatorname{ad}\Gamma)D\right. \\ &\left. -(\operatorname{ad}\Gamma)(\operatorname{ad}\Gamma)^*)(DD^*)^{-1/2}\right\}(DD^*)^{1/2}. \end{aligned} \tag{2.10}$$

Thus,

$$P^{-1} = (DD^*)^{-1/2}(1 - Q)^{-1}(DD^*)^{-1/2}$$

where

$$Q = (DD^*)^{-1/2}(D(\operatorname{ad}\Gamma)^* + (\operatorname{ad}\Gamma)D - (\operatorname{ad}\Gamma)(\operatorname{ad}\Gamma)^*)(DD^*)^{-1/2}. \tag{2.11}$$

The operator $(DD^*)^{-1/2}$ acts boundedly from C^{r-1} to C^r and from C^r to C^{r+1} since it is a classical pseudo-differential operator of order -1 and $r > 1$ is not integer. It remains to prove that

$$1 - Q : C^r \to C^r$$

has a uniformly bounded inverse which will follow from the norm estimate $\|Q\| < 1/2$. Consider for example the first summand in (2.11)

$$(DD^*)^{-1/2}D(\operatorname{ad}\Gamma)(DD^*)^{-1/2}.$$

For a 2-form $u \in C^r$ we have

$$U = (DD^*)^{-1/2}u \in C^{r+1} \subset C^r$$

and

$$\|v\|_r \le C_1\|u\|_r.$$

Next, if $\|\Gamma\|_r < \varepsilon$ then

$$\|(\operatorname{ad}\Gamma)v\| = \|[\Gamma, v]\| \le C_2\|\Gamma\|_r\|v\|_r \le C_3\varepsilon\|v\|_r.$$

Finally, since $(DD^*)^{-1/2}D$ has order 0, it acts continuously in C^r, so

$$\|(DD^*)^{-1}D\mathrm{ad}\,\Gamma v\|_r \leq C_4\varepsilon\|u\|_r.$$

Similar estimates are valid for other summands in (2.11). This leads to the following norm estimate of the operator Q

$$\|Q\| \leq C_5\varepsilon,$$

proving the lemma.

□

The previous lemma shows that the operator $A(\Gamma)$ given by the right-hand side of (2.8) is defined correctly on the sphere $\|\Gamma\| < \varepsilon$ in $C^{(r)}$ (for a fixed $\kappa \in C^{r-1}$).

Lemma 2.4 *There exists $\varepsilon_1 > 0$ such that for any $\kappa \in C^{r-1}$ with $\|\kappa\|_{r-1} < \varepsilon_1^2$ the operator*

$$\Gamma \to A(\Gamma)$$

maps the sphere $\|\Gamma\|_r < \varepsilon_1$ into itself.

Proof. Supposing that $\varepsilon_1 < \varepsilon$, so that we may use Lemma 2.3, let us estimate the norm $\|A(\Gamma)\|_r$ for $\kappa \in C^{r-1}$ with $\|\kappa\|_{r-1} < \varepsilon_1^2$ and $\Gamma \in C^r$ with $\|\Gamma\|_r < \varepsilon_1$. We have

$$\|\Gamma^2\|_{r-1} \leq \|\Gamma^2\|_r \leq C_1\|\Gamma\|_r^2 \leq C_1\varepsilon_1^2$$

and

$$\|\kappa - \Gamma^2\|_{r-1} \leq (1 + C_1)\varepsilon_1^2.$$

Next, by Lemma 2.3

$$\|P^{-1}(\kappa - \Gamma^2)\|_{r+1} \leq \|P^{-1}\|(1 + C_1)\varepsilon_1^2 \leq C_r\varepsilon_1^2. \tag{2.12}$$

Finally, the norm of the operator

$$(D - \mathrm{ad}\,\Gamma)^* : C^{r+1} \to C^r$$

is uniformly bounded if $\|\Gamma\|_r < \varepsilon_1$, so that

$$\|A(\Gamma)\|_r \leq C_3\varepsilon_1^2.$$

Choosing ε_1 such that $C_3\varepsilon_1 < 1$, we would have $\|A(\Gamma)\|_r < \varepsilon_1$ proving the lemma.

□

Finally, show that A is a contraction, that is

$$\|A(\Gamma_1) - A(\Gamma_2)\|_r \leq q\|\Gamma_1 - \Gamma_2\|_r$$

with $q<1$, on the sphere $\|\Gamma\|<\varepsilon_1$ provided ε_1 is small enough. Using the relation

$$\Gamma_1\wedge\Gamma_1-\Gamma_2\wedge\Gamma_2=\frac{1}{2}[\Gamma_1+\Gamma_2,\Gamma_1-\Gamma_2]\in C^r$$

we have an estimate

$$\|\Gamma_1^2-\Gamma_2^2\|_r\le C_5\varepsilon_1\|\Gamma_1-\Gamma_2\|_r.$$

The operator $(D-\mathrm{ad}\Gamma_2)^*P^{-1}(\Gamma_2)$ is uniformly bounded in C^r, so for the third summand in (2.10) we have the norm estimate $C_6\varepsilon_1\|\Gamma_1-\Gamma_2^2\|_r$. Gathering all the three estimates, we obtain

$$\|A(\Gamma_1)-A(\Gamma_2)\|_r\le(C_2\varepsilon_1^2+C_4\varepsilon_1^2+C_5\varepsilon_1)\|\Gamma_1-\Gamma_2\|_r.$$

The factor $q=C_2\varepsilon_1^2+C_4\varepsilon_1^2+C_5\varepsilon_1$ is less then 1 for ε_1 small enough. This completes the proof of Lemma 2.2.

□

Conversely, a solution Γ of (2.8) defines a connection $\partial=D-\Gamma$ which is necessarily flat. Indeed, in terms of ∂ we may rewrite (2.8) in the form (2.7). Applying ∂ to both sides of (2.7), we get

$$\partial\Gamma=\kappa(D)-\Gamma^2.$$

On the other hand, taking the curvatures of both sides of the equation $D=\partial+\Gamma$ yields

$$\kappa(D)=\kappa(\partial)+\partial\Gamma+\Gamma^2$$

implying $\kappa(\partial)=0$.

The symplectic form ω in new coordinates (2.6) becomes

$$\omega=\frac{1}{2}\int\mathrm{tr}\,\delta\gamma\wedge\delta\gamma+\int\mathrm{tr}\,\delta\Gamma\wedge\delta\gamma+\frac{1}{2}\int\mathrm{tr}\,\delta\Gamma\wedge\delta\Gamma \tag{2.13}$$

where γ is a connection 1-form for the flat connection ∂ (so that $\partial=d+\gamma$) and $\Gamma=\partial^*b$ is a coexact form with sufficiently small norm $\|\Gamma\|_r$. Note, that representation (2.6) is equivariant with respect to G-action on connections D and ∂ and adjoint action on Γ

$$g:\Gamma\mapsto g^{-1}\Gamma g.$$

Our next goal is to find other coordinates (∂_0,Γ_0) and a non-linear operator

$$(\partial,\Gamma)=f(\partial_0,\Gamma_0)$$

equivariant under the G-action transforming ω to the form

$$\omega_0=\frac{1}{2}\int\mathrm{tr}\,\delta\partial_0\wedge\delta\partial_0+\int\mathrm{tr}\,\delta\Gamma_0\wedge\delta\gamma_0. \tag{2.14}$$

This may be done precisely as in the finite-dimensional case using Weinstein's arguments (see e.g. [6]). To this end introduce a family of symplectic forms

$$\omega_t = \frac{1}{2}\int \operatorname{tr}\delta\gamma\wedge\delta\gamma + \int \operatorname{tr}\delta\Gamma\wedge\delta\gamma + \frac{t}{2}\int \operatorname{tr}\delta\Gamma\wedge\delta\Gamma, \tag{2.15}$$

and a time-dependent vector field $V_t = (V_t\gamma, V_t\Gamma)$ on the space of pairs (∂, Γ) consisting of a flat regular connection ∂ and a coexact form $\Gamma = \partial^* b$ in such a way that

$$i(V_t)\omega_t + \int \operatorname{tr}\Gamma\wedge\delta\Gamma = 0. \tag{2.16}$$

Note that the form

$$\varphi = \frac{1}{2}\int \operatorname{tr}\Gamma\wedge\delta\Gamma$$

has the property

$$\delta\varphi = \frac{\partial}{\partial t}\omega_t.$$

Having found the vector field, we solve a differential equation

$$\dot\gamma = V_t\gamma,\ \dot\Gamma = V_t\Gamma \tag{2.17}$$

with the initial condition

$$(\gamma, \Gamma)|_{t=0} = (\gamma_0, \Gamma_0).$$

The solution defines a flow

$$f_t : (\gamma_0, \Gamma_0) \mapsto (\gamma, \Gamma)$$

such that

$$f_t^*\omega_t \equiv \omega_0. \tag{2.18}$$

Indeed,

$$\frac{d}{dt}F_t^*\omega_t = f_t^*\left(\mathcal{L}_{V_t}\omega_t + \frac{\partial\omega_t}{\partial t}\right) = f_t^*(\delta(i(V_t)\omega_t + \varphi)) \equiv 0.$$

Here $\mathcal{L}_{V_t}$ means the Lie derivative

$$\mathcal{L}_{V_t} = \delta i(V_t) + i(V_t)\delta.$$

Taking $t = 1$ in (2.18), we obtain the desired transformation reducing the symplectic form $\omega = \omega_1$ to ω_0. The infinite-dimensional specific is essential only when solving Equations (2.16) and (2.17) in Hölder spaces.

Consider first Equation (2.16). For ω_t given by (2.15) this equation takes the form

$$\int \operatorname{tr}(V_t\gamma + V_t\Gamma)\wedge\delta\gamma + \int \operatorname{tr}\left(V_t\gamma + tV_t\gamma + \frac{1}{2}\Gamma\right)\wedge\delta\Gamma = 0. \tag{2.19}$$

Here $\partial = d + \gamma$ is a regular flat connection and $\Gamma = \partial^* b = - * \partial * b$ is a coexact form. Because of the flatness, we have $\partial(\partial\gamma) = \partial(V_t\gamma) = 0$.

For a given 1-form a

$$\begin{aligned}\int \mathrm{tr}\ a \wedge \delta\Gamma &= \int \mathrm{tr}\ a \wedge \delta(\partial^* b) \\ &= \int \mathrm{tr}\ * a \wedge [\delta\gamma, *b] - \int \mathrm{tr}\ a \wedge \partial^*(\delta b) \\ &= \int \mathrm{tr}\, [*b, *a] \wedge \delta\gamma + \int \mathrm{tr}\ \partial^* a \wedge \delta b,\end{aligned}$$

so that (2.19) becomes

$$\begin{aligned}&\int \mathrm{tr}\ \left(V_t\gamma + V_t\Gamma + \left[*b, * \left(V_t\gamma + tV_t\Gamma + \frac{1}{2}\Gamma \right) \right] \right) \delta\gamma \\ &+ \int \mathrm{tr}\ \partial^* \left(V_t\gamma + tV_t\Gamma + \frac{1}{2}\Gamma \right) \wedge \delta b = 0. \qquad (2.20)\end{aligned}$$

Since δb may be *any* 2-form (with sufficiently small Hölder norm in C^{r+1}) and since $\partial\Gamma = 0$ we conclude from (2.20)

$$\partial^*(V_t\gamma + tV_t\Gamma) = 0, \qquad (2.21)$$

so that the second term in (2.20) vanishes identically. Now, since $\delta\gamma$ in the first term of (2.20) may be *any* closed form, we conclude that

$$V_t\gamma + V_t\Gamma + \left[*b, * \left(V_t\gamma + tV_t\Gamma + \frac{1}{2}\Gamma \right) \right] = \partial X \qquad (2.22)$$

that is the form on the left-hand side is exact.

We first study the solvability of the system (2.22). Let

$$P = \partial(\partial^*\partial)^{-1}\partial^*, \quad Q = \partial^*(\partial\partial^*)^{-1}\partial$$

be projectors onto exact and coexact 1-forms. Being pseudo-differential operators of order 0 they act boundedly in any Hölder space C^r, $r > 0$, $r \notin \mathbb{Z}$. We will consider flat connections $\partial \in M_0^{reg}$ for which these projectors are uniformly bounded in C^r with some fixed $r > 0$, $r \notin \mathbb{Z}$.

Lemma 2.5 *Let $\|P\|, \|Q\| < C$ for P, Q considered as operators in C^r and $\|\Gamma\|_r < \varepsilon$ with ε sufficiently small. Then for any $t \in [0, 1]$ the system* (2.21), (2.22) *has a unique solution $V_t\gamma, V_t\Gamma$ satisfying the estimate*

$$\|V_t\gamma\|_r + \|V_t\Gamma\|_r < C_1\varepsilon^2. \qquad (2.23)$$

Proof. Let us apply Q to both sides of (2.22). Since the form $V_t\gamma$ is closed, we have $Q(V_t) = 0$, the same is true for the form ∂X on the right -hand side. Now,

$$\begin{aligned} QV_t\Gamma &= Q(\partial^* V_t b - [*V_t\gamma, *b]) \\ &= \partial^* V_t b - Q[*V_t\gamma, *b] \\ &= V_t\Gamma + (1-Q)[*V_t\gamma, *b]. \end{aligned}$$

Thus, we obtain

$$V_t\Gamma = (1-Q)[*b, *V_t\gamma] - Q\left[*b, *\left(V_t\gamma + tV_t\Gamma + \frac{1}{2}\Gamma\right)\right]. \qquad (2.24)$$

Next we apply $1-P$ to both sides of (2.22). By (2.21) the form $V_t\gamma + tV_t\Gamma$ is closed, so

$$(1-P)(V_t\gamma + tV_t\Gamma) = V_t\gamma + tV_t\Gamma.$$

On the other hand

$$\begin{aligned} (1-P)V_t\gamma &= (1-P)(\partial^* V_t b - [*V_t\gamma, *B]) \\ &= \partial^* V_t b - (1-P)[*V_t\gamma, *b] = V_t\Gamma + P[*V_t\gamma, *b]. \end{aligned}$$

Thus, using that $(1-P)\partial X = 0$, we obtain

$$V_t\gamma + V_t\Gamma - (1-t)P[*b, *V_t\gamma] + (1-P)\left[*b, *\left(V_t\gamma + tV_t\Gamma + \frac{1}{2}\Gamma\right)\right] = 0,$$

or substituting (2.24)

$$\begin{aligned} V_t\gamma &= ((1-t)P - (1-Q))[*b, *V_t\gamma] \\ &- \Big(1 - P - Q\Big)\left[*b, *(V_t\gamma + tV_t\Gamma + \frac{1}{2}\Gamma\Big)\right]. \end{aligned} \qquad (2.25)$$

The system (2.24), (2.25) with respect to unknowns $A = (V_t\gamma, V_t\Gamma)$ has a standard form

$$A = KA + B \qquad (2.26)$$

where K is a linear operator in C^r with a small norm since

$$\|b\|_r = \|(\partial\partial^*)^{-1}\partial\Gamma\|_r \leq C_2\|\Gamma\|_r < C_2\varepsilon.$$

The constant term B is the projection of the vector $[*b, \Gamma]$ onto coexact forms (in (2.24)) and onto harmonic forms (in (2.25)). Thus,

$$\|[*b, \Gamma]\|_r \leq C_3\varepsilon^2,$$

and the same estimate is valid for the solution. From (2.25) it follows $\partial(V_t\gamma) = 0$ since the right-hand side is, clearly, closed. So, the system

(2.24), (2.25) is equivalent to the original system (2.21), (2.22) with an additional condition $\partial(V_t\gamma) = 0$.

□

We now turn to the system (2.17) of ordinary differential equations in Hölder spaces and prove its unique solvability. The treatment is similar to that in finite-dimensional vector spaces, so we omit standard details. We rewrite the system in an integral form

$$y(t) = y_0 + \int_0^t A(\tau, y(\tau))d\tau \tag{2.27}$$

where $y(t) = (\gamma(t), \Gamma(t))$ is considered as a function on the interval $t \in [0,1]$ with values in C^r and $A = (V\gamma, V\Gamma)$ is a non-linear operator

$$A = (1-K)^{-1}B$$

satisfying (2.26). Here K is a linear operator in C^r and $B \in C^r$, they both depend on $y(t)$. If $\|y(t) - y_0\| < \varepsilon$ with ε small enough, then from the explicit expressions for K and B from (2.24), (2.25) we have estimates

$$\|K\| \leq C_1\varepsilon, \quad \|B\|_r \leq C_2\varepsilon^2.$$

Moreover, the Frechét differential of $A(t,y)$ with respect to y admits an estimate

$$\|d_yA(t,y)\| \leq C_3\varepsilon.$$

In a standard way these estimates imply that the right-hand side of (2.27) is a contraction on the space of continuous functions $y(t),\ \ t \in [0,1]$ with values in C^r with the norm $\max_t \|y(t)\|_r$, provided ε is small enough, so the unique solvability follows.

We are able now to complete the proof of Theorem 2.1. To find a desired representation (∂, ξ) of the regular connection D we first represent it in the form $D = \partial + \Gamma,\ \Gamma = \partial^* b$ by Lemma 2.2 Then using Lemma 2.5 and solving ordinary differential equations, we find another pair (∂_0, Γ_0) with $\Gamma_0 = \partial_0^* b_0$ and a map

$$(\partial_0, \Gamma_0) \mapsto (\partial, \Gamma)$$

which pulls back the form

$$\omega = \frac{1}{2}\int_X \mathrm{tr}\ \delta\gamma \wedge \delta\gamma + \int_X \mathrm{tr}\ \delta\Gamma \wedge \delta\gamma + \frac{1}{2}\int_X \mathrm{tr}\ \delta\Gamma \wedge \delta\Gamma$$

to

$$\omega_0 = \frac{1}{2}\int_X \mathrm{tr}\ \delta\gamma_0 \wedge \delta\gamma_0 + \delta\int_X \mathrm{tr}\ \Gamma_0\delta\gamma_0.$$

All these maps are equivariant by construction. The form ω_0 may be reduced to (2.4) if we put $\Gamma_0 = \partial_0^*(\partial_0\partial_0^*)^{-1}\xi$ (in the final expression we drop the subscripts 0).

□

3 Quantization

In this section which is divided into three subsections we perform a quantization program in the spirit of [6]. In general, the canonical geometrical construction of deformation quantization is meaningless in infinite dimensions, the Abelian connection needed for quantization does not exist. However, for the space of connections M^{reg} the quantization program of [6] is possible because the non-trivial part of the program deals with a finite-dimensional manifold B^{reg}.

We use the representation

$$M^{reg} = M_0^{reg} \times \mathcal{G}^*$$

given by the normal form theorem and the symplectic form (2.4). Since M_0^{reg} is a fibration over B^{reg} with a fibre G, the space M^{reg} becomes a fibration over B^{reg} with a fibre $F = G \times \mathcal{G}^*$. At the first step we construct an algebra $\mathcal{A}_F$ of quantum observables on the fibre F. We simply use the explicit formulas from [5] checking that they have sense in infinite-dimensional case after restriction to a suitable class of functionals. At the second step we consider canonical quantization on a finite-dimensional manifold B^{reg} with coefficients in the bundle $\mathcal{A}_F$. The result of these two steps is *by definition* the algebra of quantum observables on M^{reg}.

So, let us consider a coordinate chart $U_i \subset B^{reg}$ with local coordinates $z^1, \ldots, z^{2n}$, and let ∂_i be a family of flat connections on X depending on parameters $z^1, \ldots, z^{2n}$, so that ∂_i defines a section of the principal bundle $M_0^{reg} \to B^{reg}$ over U_i. Any flat connection ∂ over U_i may be represented in the form $\partial = g^{-1} \circ \partial_i \circ g$ where $g = g(x) \in \mathcal{G}$. A functional $A(D)$ for $D \in M^{reg}$ may be written as

$$A(D) = A(\partial, \xi) = A(g^{-1} \circ \partial_i \circ g, \xi) := a_i(z, g, \xi). \tag{3.1}$$

Fixing $z \in U_i$ in (3.1), we obtain a functional on $F = G \times \mathcal{G}^*$, the fibre of a fibration $M^{reg} \to B^{reg}$. Along with the two-form $\xi \in \mathcal{G}^*$ we will consider a two-form $\eta = g\xi g^{-1} \in \mathcal{G}^*$.

3.1 Quantization of the Fibre

Taking fixed functions $a, b \in \mathcal{G}$ define linear functionals [1]

$$\langle \xi, a \rangle = \int_X \operatorname{tr} \xi a \tag{3.2}$$

$$\langle \eta, b \rangle = \langle g\xi g^{-1}, b \rangle = \langle \xi, g^{-1} b g \rangle. \tag{3.3}$$

[1] In the finite-dimensional case the functionals (3.3) are finite linear combinations of the functionals (3.2) and vice versa. Thus, it is sufficient to use either (3.2) or (3.3). In infinite dimensions we need both of them.

The functionals (3.2) are Hamiltonian functionals for the right action of the group G, that is

$$\{\langle\xi,a\rangle,f(g,\xi)\}=\frac{d}{dt}f\ \left(ge^{ta},g^{-ta}\xi e^{ta}\right)\big|_{t=0}\,. \tag{3.4}$$

In particular,

$$\{\langle\xi,a\rangle,\langle\xi,b\rangle\}=\langle\xi,[a,b]\rangle \tag{3.5}$$

$$\{\langle\xi,a\rangle,\langle\eta,b\rangle\}=0. \tag{3.6}$$

Next,

$$\begin{aligned}&\{\langle\eta,b\rangle,f(g)\}=\{\langle\xi,g^{-1}bg\rangle,f(g)\}\\&=\frac{d}{dt}f\ \left(ge^{tg^{-1}bg}\right)\Big|_{t=0}=\frac{d}{dt}f(e^{tb}g)\Big|_{t=0},\end{aligned} \tag{3.7}$$

so that $\langle\eta,b\rangle$ is the Hamiltonian functional for the infinitesimal left action of the group G. Using (3.6) and (3.7), we obtain

$$\{\langle\eta,b\rangle,f(g,\xi)\}=\frac{d}{dt}\ f(e^{tb}g,\xi)\big|_{t=0}\,, \tag{3.8}$$

in particular,

$$\begin{aligned}&\{\langle\eta,b\rangle,\langle\eta,c\rangle\}=\frac{d}{dt}\ \langle\xi,g^{-1}e^{-tb}ce^{tb}g\rangle\big|_{t=0}\\&=\langle\xi,g^{-1}[c,b]g\rangle=-\langle\eta,[b,c]\rangle.\end{aligned} \tag{3.9}$$

Now, as a first step to quantization we consider an associative algebra generated by $\langle\xi,a\rangle,\langle\eta,b\rangle$ with commutation relation similar to (3.5), (3.6), (3.9), namely

$$[\langle\xi,a\rangle,\langle\xi,b\rangle]=-ih\langle\xi,[a,b]\rangle \tag{3.10}$$

$$[\langle\eta,a\rangle,\langle\eta,b\rangle]=ih\langle\eta,[a,b]\rangle \tag{3.11}$$

$$[\langle\xi,a\rangle,\langle\eta,b\rangle]=0. \tag{3.12}$$

In other words, the linear functionals $\langle\xi,a\rangle,\ \langle\eta,b\rangle$ form an infinite-dimensional Lie algebra defined by commutators (3.10)–(3.12), and we take its universal enveloping algebra $\mathcal{U}$. The product $\circ$ in the algebra $\mathcal{U}$ may be transported to a $*$-product on polynomials (tensors) in $\langle\xi,a\rangle,\langle\eta,b\rangle$ using symmetric ordering. To this end, for any monomial

$$A=\langle\xi,a_1\rangle\dots\langle\xi,a_m\rangle\langle\eta,a_{m+1}\rangle\dots\langle\eta,a_{m+n}\rangle$$

define an element $\varphi(A)\in\mathcal{U}$ by

$$\varphi(A)=\frac{1}{(m+n)!}\sum\langle\xi,a_{i_1}\rangle\circ\dots\circ\langle\xi,a_{i_m}\rangle\circ\langle\eta,a_{i_{m+1}}\rangle\circ\dots\circ\langle\eta,a_{i_{m+n}}\rangle$$

where the sum runs over all permutations$(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n})$. The map φ gives a linear isomorphism between polynomials and the elements of $\mathcal{U}$ (the Poincare-Birkhoff-Witt theorem), and we set

$$A * B = \varphi^{-1}(\varphi(A) \circ \varphi(B)). \tag{3.13}$$

This construction was used in [5] to define a $*$-product on T^*G in a finite-dimensional case. We need to generalize it for a larger class of functionals.

Let $P(y_1, y_2, \ldots, y_n)$ be an analytic function, that is a power series converging in a polydisc $|y_1| < R, \ldots, |y_n| < R$. Then for $a_1, \ldots, a_n, \in \mathcal{G}$ the functional

$$P(\langle\eta, a_1\rangle, \ldots, \langle\eta, a_n\rangle) \tag{3.14}$$

is defined for sufficiently small η such that $|\langle\eta, a_i\rangle| < R$. In particular, these conditions are fulfilled if the L^2-norm of $\eta = g\xi g^{-1}$ is small enough. Since

$$(\eta, \eta) = (g\xi g^{-1}, g\xi g^{-1}) = (\xi, \xi)$$

this is equivalent to the smallness of the L^2-norm of ξ.

Lemma 3.1 *Let*

$$P = P(\langle\eta, a_i\rangle, \ldots, \langle\eta, a_n\rangle)$$

$$Q = Q(\langle\eta, b_1\rangle, \ldots, \langle\eta, b_m\rangle)$$

be two functionals of the type (3.14)*, Then the formula*

$$P * Q = \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} P^{(\alpha)} Q^{(\beta)} \left(\frac{\partial}{\partial t}\right)^{\alpha} \left(\frac{\partial}{\partial \tau}\right)^{\beta} \exp\left\{-\frac{1}{h}\langle\eta,\right.$$
$$\left. CH(ih(t_1a_1 + \ldots + t_na_n), ih(\tau_1 b_1 + \ldots + \tau_m b_m))\rangle\right\}\Big|_{t=\tau=0} \tag{3.15}$$

gives an extension of the $*$-product (3.13)*.*

Here CH means the non-linear terms in the Campbell-Hausdorff formula

$$CH(a, b) = \frac{1}{2}[a, b] + \frac{1}{12}[[a, b], b] + \frac{1}{12}[a, [a, b]] + \ldots.$$

Proof. For the linear form $\exp\langle\eta, ta\rangle = \langle\eta, t_1a_1 + \ldots + t_na_n\rangle$ its k-th power is equal to its k-th power in the algebra $\mathcal{U}$, that is

$$\langle\eta, ta\rangle^k = \underbrace{\langle\eta, ta\rangle \circ \langle\eta, ta\rangle \circ \ldots \circ \langle\eta, ta\rangle}_{k}.$$

It implies that $\exp\langle\eta, ta\rangle$ coincides with the exponential function with respect to $\circ$-product in the algebra $\mathcal{G}$. Given another linear functional

$$\langle\eta, \tau b\rangle = \langle\eta, \tau_1 b_1 + \ldots, +\tau_m b_m\rangle,$$

we obtain

$$\begin{aligned}&\exp\langle\eta, ta\rangle \circ \exp\langle\eta, \tau b\rangle\\&= \exp\{\langle\eta, ta\rangle + \langle\eta, \tau b\rangle + CH(\langle\eta, ta\rangle\langle\eta, \tau b\rangle)\}\end{aligned}$$

By the Campbell-Hausdorff formula in the algebra $\mathcal{U}$. But

$$\begin{aligned}[\langle\eta, ta\rangle, \langle\eta, \tau b\rangle] &= ih\langle\eta, [ta, \tau b]\rangle\\&= -\frac{i}{h}\langle\eta, [ihta, ih\tau b]\rangle,\end{aligned}$$

thus

$$\begin{aligned}&CH(\langle\eta, ta\rangle, \langle\eta, \tau b\rangle)\\&= -\frac{i}{h}\langle\eta, CH(ihta, ih\tau b)\rangle.\end{aligned}$$

Finally,

$$\begin{aligned}&\exp\langle\eta, ta\rangle \circ \exp\langle\eta, \tau b\rangle\\&= \exp\langle\eta, ta + \tau b\rangle \exp\left\{-\frac{i}{h}\langle\eta, CH(ihta, ih\tau b)\rangle\right\}. \qquad (3.16)\end{aligned}$$

Let first P, Q be polynomials. Applying to both sides of (3.16) a differential operator

$$P\left(\frac{\partial}{\partial t}\right) Q\left(\frac{\partial}{\partial \tau}\right) = P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) Q\left(\frac{\partial}{\partial \tau_1}, \dots, \frac{\partial}{\partial \tau_m}\right)$$

and putting $t = \tau = 0$, we obtain on the left-hand side $P(\langle\eta, a\rangle) * Q(\langle\eta, b\rangle)$ since

$$\langle\eta, a_1\rangle \circ \exp\langle\eta, ta\rangle = \frac{\partial}{\partial t_i}\exp\langle\eta, ta\rangle.$$

On the right-hand side we treat the exponential functions with respect to the usual commutative product. Applying $P\left(\frac{\partial}{\partial t}\right) Q\left(\frac{\partial}{\partial \tau}\right)$, we use the Leibniz rule

$$\begin{aligned}&P\left(\frac{\partial}{\partial t}\right) Q\left(\frac{\partial}{\partial \tau}\right) f_1 f_2\\&= \sum_{\alpha,\beta} P^{(\alpha)}\left(\frac{\partial}{\partial t}\right) Q^{(\beta)}\left(\frac{\partial}{\partial \tau}\right) f_1 \frac{1}{\alpha!}\frac{1}{\beta!}\left(\frac{\partial}{\partial t}\right)^{\alpha}\left(\frac{\partial}{\partial \tau}\right)^{\beta} f_2\end{aligned}$$

and

$$\begin{aligned}&P^{(\alpha)}\left(\frac{\partial}{\partial t}\right) Q^{\beta}\left(\frac{\partial}{\partial \tau}\right) \exp\,\langle\eta, ta + \tau b\rangle|_{t=\tau=0}\\&= P^{(\alpha)}(\langle\eta, a\rangle) Q^{(\beta)}(\langle\eta, b\rangle).\end{aligned}$$

Thus, we have proved (3.15) for polynomials. But the right-hand side of (3.15) is meaningful for analytical functions P and Q (as a formal expansion in powers of h). The properties of the $*$-product follow for analytical functions since they are valid for polynomials.

$\square$

Remark 3.2 A similar formula may be obtained for functionals depending on $\langle \xi, a\rangle$ or for mixed functionals depending on $\langle \xi, a\rangle$ and $\langle \eta, b\rangle$. The only difference consists in the Campbell-Hausdorf formula for different Lie algebras corresponding to the commutation relations (3.10)–(3.12). For example,

$$P(\langle \xi, a\rangle) * Q(\langle \eta, b\rangle) = P(\langle \xi, a\rangle)Q(\langle \eta, b\rangle)$$

because of (3.12).

Our next step is to include in the algebra of observables functionals depending on $g \in G$. We wish the following relations to be satisfied:

$$A(g) * B(g) = A(g)B(g) \tag{3.17}$$

$$A(g) * P(\langle \xi, a\rangle, \langle \eta, b\rangle) = A(g)P(\langle \xi, a\rangle, \langle \eta, b\rangle) \tag{3.18}$$

$$[\langle \xi, a\rangle, A(g)] = -ih\{\langle \xi, a\rangle, A(g)\} = -ih\frac{d}{dt}A\left(ge^{ta}\right)\Big|_{t=0} \tag{3.19}$$

$$[\langle \eta, b\rangle, A(g)] = -ih\{\langle \eta, a\rangle, A(g)\} = -ih\frac{d}{dt}A\left(e^{tb}g\right)\Big|_{t=0}. \tag{3.20}$$

Using these equations and reasoning similarly to lemma 3.1, we may extend the $*$-product

$$P(\langle \xi, a\rangle, \langle \eta, b\rangle) * A(g)$$

for functionals P which are analytical functions in

$$\langle \xi a_1\rangle, \ldots, \langle \xi, a_n\rangle, \langle \eta, b_1\rangle \ldots \langle \eta, b_m\rangle.$$

As for $A(g)$, it may be a quite general functional. The only condition is that the function

$$A(e^{\tau_1, b_1+\ldots+\tau_m b_m} g e^{t_1 a_1+\ldots+t_n a_n})$$

be smooth in τ, t in a neighborhood of $t = \tau = 0$.

Lemma 3.3 *Let*

$$P(\langle \xi, a\rangle) = P(\langle \xi, a_1\rangle, \ldots, \langle \xi, a_n\rangle)$$

be an analytical function in the variables $\langle \xi, a\rangle = (\langle \xi, a_1\rangle, \ldots, \langle \xi, a_n\rangle)$. *Then the relations* (3.18), (3.19) *imply the following formula*

$$\begin{aligned} &P(\langle \xi, a\rangle) * A(g) \\ &= \sum_\alpha \frac{1}{\alpha!} P^{(\alpha)}(\langle \xi, a\rangle) \left(\frac{\partial}{\partial t}\right)^\alpha A\left(ge^{\langle \xi, -ihta\rangle}\right)\Big|_{t=0} \end{aligned} \tag{3.21}$$

compatible with the natural $*$-product for polynomials P.

Proof. Similarly to lemma 3.1 we use again the crucial observation concerning the exponential function $\exp\langle\xi, a\rangle$, namely, it is defined by the same power series for both products: the usual commutative product and the $\circ$-product in the algebra $\mathcal{U}$. This implies

$$\exp\langle\xi, ta\rangle * A(g) = A(ge^{-ihta}) \exp\langle\xi, ta\rangle.$$

Applying to both sides a differential operator $P\left(\frac{\partial}{\partial t}\right)$ where P is a polynomial, we come to (3.21). Now, the right-hand side of this formula is meaningful for analytical functions P also. This proves the lemma.

□

Remark 3.4 Similarly,

$$P(\langle\eta, b\rangle) * A(g)$$
$$\sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{\partial\tau}\right)^\alpha A(e^{-ih\tau b}g)P^{(\alpha)}(\langle\eta, b\rangle)\bigg|_{\tau=0}, \tag{3.22}$$

and the obvious generalization may be written for mixed analytic functions $P(\langle\xi, a\rangle, \langle\eta, b\rangle)$.

We summarize the considerations of this subsection as a theorem.

Theorem 3.5 *There exists a $*$-product on functionals $\Phi(g, \xi)$ of the type*

$$\Phi(g, \xi) = A(g)P(\langle\xi, a\rangle, \langle g\xi g^{-1}, b\rangle) \tag{3.23}$$

where P is an analytical function.

Proof. Consider two functionals

$$\Phi_1(g, \xi) = A_1(g)P_1 := A_1(g) * P_1$$
$$\Phi_2(g, \xi) = A_2(g)P_2 := A_2(g) * P_2.$$

Then we set

$$\Phi_1 * \Phi_2 = A_1(g) * P_1 * A_2(g) * P_2$$

and use first lemma 3.3 to compute

$$P_1 * A_2(g) = \sum_\alpha A_\alpha(g, h)P_1^{(\alpha)} :$$
$$= \sum_\alpha A_\alpha(g, h) * P_1^{(\alpha)}.$$

Next, we use lemma 3.1 to compute $P_1^{(\alpha)} * P_2$. The result is of the form

$$\Phi_1 * \Phi_2 = \sum_\alpha A_1(g)A_\alpha(g, h)(P_1^{(\alpha)} * P_2)$$

which is again a formal expansion in powers of h whose coefficients are functionals of the type (3.23).

□

3.2 Quantization of the Moduli Space

Consider a fibering $\pi : M_0^{reg} \to B^{reg}$. Let $O_i \subset B^{reg}$ be a coordinate chart with local coordinates $z = (z^1, \ldots, z^{2n})$. Let $\partial_i = \partial_i(z)$ be a family of flat connections depending on a parameter $z \in O_i$ giving a section of the fibering over O_i, That is $\pi\partial_i(z) = z$.

Any functional $\Phi(D)$ defined on regular connections D with sufficiently small curvature may be written by the normal form theorem as

$$\Phi(D) = \Phi(\partial, \xi)$$

where ∂ is a regular flat connection and $\xi \in \mathcal{G}^*$ is a two-form on X with values in $su(2)$. If ∂ lies over O_i, that is $\pi\partial = z \in O_i$, the connection ∂ has a representation $\partial = g^{-1} \circ \partial_i(z) \circ g$, so, our functional becomes

$$\Phi(\partial, \xi) = \Phi(g^{-1} \circ \partial_i(z) \circ g, \xi) := a_i(z, g, \xi). \tag{3.24}$$

For another local section $\partial_j(z)$ in a local chart $O_j \in B^{reg}$ the same functional $\Phi(\partial, \xi)$ gives another representation

$$a_j(z, g, \xi) := \Phi(g^{-1} \circ \partial_j(z) \circ g, \xi).$$

In $O_i \cap O_j$ where $\partial_i(z) = g_{ij}(z) \circ \partial_j(z) \circ g_{ji}(z)$ these local representations satisfy the transition rule

$$a_i(z, g, \xi) = a_j(z, g_{ji}(z)g, \xi). \tag{3.25}$$

Thus, any functional $\Phi(\partial, \xi)$ may be treated as a section of a bundle over B^{reg} whose fibres consist of functionals $a(g, \xi)$ on $F = G \times \mathcal{G}^*$. Taking $a(g, \xi)$ from the class considered in the preceding subsection:

$$a(g, \xi) = A(g)P(\langle \xi, a\rangle, \langle \xi, g^{-1}bg\rangle), \tag{3.26}$$

we obtain a bundle over B^{reg} denoted by K. Its sections are given by local representations $a_i(z, g, \xi)$ of the form (3.26) and transition rules (3.25) (clearly, the form (3.26) is invariant under transition rules). The fibres K_z may be considered as algebras with respect to the $*$-product of the preceding subsection. This $*$-product is also invariant under transition rules (3.25), thus, the space of sections $C^\infty(B^{reg}, K)$ becomes an algebra with respect to the fibrewise $*$-product.

The bundle K may be equipped with a connection defined by the following local expression

$$\nabla a_i(z, g, \xi) = \delta a_i(z, g, \xi) + \frac{i}{h}[\langle \eta, (\partial_i^* \partial_i)^{-1} \partial_i^* (\delta \partial_i)\rangle, a_i]. \tag{3.27}$$

The geometrical meaning of (3.27) may be clarified using definition (3.24) of the local representation $a_i(z, g, \xi)$ of the functional $\Phi(\partial, \xi)$. Denoting by

v a vector field on the base B^{reg} consider

$$\begin{aligned}\nabla_v a_i &= va_i + \frac{i}{h} i(v)[\langle \eta, (\partial_i^* \partial_i)^{-1} \partial_i^*(\delta\partial)\rangle, a] \\ &= va_i - \frac{i}{h}[\langle \eta, (\partial_i^* \partial_i)^{-1} \partial - i^*(v\partial_i)\rangle, a_i] \\ &= \frac{d}{dt} a_i(z+vt, g, \xi)\Big|_{t=0} + \frac{d}{dt} a_i(z, e^{-(\partial_i^*\partial_i)^{-1}\partial_i^*(v\partial_i)t} g, \xi)\Big|_{t=0}.\end{aligned}$$

We have made use of (3.20). Now, by (3.24) we may rewrite the last expression in the form

$$\frac{d}{dt}\Phi(g^{-1} e^{(\partial_i^*\partial_i)^{-1}\partial_i^*(v\partial_i)t} \circ \partial_i(z+vt) \circ e^{-(\partial_i^*\partial_i)^{-1}\partial_i^*(v\partial_i)t} g, \xi)\Big|_{t=0}$$
$$\frac{d}{dt}\Phi(g^{-1}(\partial_i + (v\partial_i)t - \partial_i(\partial_i^*\partial_i)^{-1}\partial_i^*(v\partial_i)t)g, \xi)\Big|_{t=0}.$$

This is nothing but the derivation of the functional $\Phi(\partial, \xi)$ along the horizontal lifting v_{hor} of the vector field v on B^{reg} to M_0^{reg} defined as the harmonic component of

$$\widetilde{v} = g^{-1}(v\partial_i)g,$$

that is

$$v_{hor} = \widetilde{v} - p\widetilde{v} = \widetilde{v} - \partial(\partial^*\partial)^{-1}\partial\widetilde{v}. \tag{3.28}$$

Note that v_{hor} does not depend on the choice of the lifting $\widetilde{v}$ because the ambiguety is of the form ∂e whose harmonic component is equal to 0.

We thus are at a starting point for "deformation quantization with twisted coefficients" considered in the book [6]. Indeed, we have a coefficient bundle K over a finite-dimensional symplectic manifold B^{reg} whose fibres K_z are algebras with respect to the fibrewise $*$-product. There is a connection ∇ on K preserving the fibrewise $*$-product:

$$\nabla a = \delta a_i + \frac{i}{h}[\gamma_i, a_i]$$

with K-valued local connection one-forms

$$\gamma_i = \langle \eta, (\partial_i^*\partial_i)^{-1}\partial_i^*(\delta\partial_i)\rangle \tag{3.29}$$

The only difference is that the fibres K_z are $*$-product algebras on an *infinite-dimensional manifold* $F = G \times \mathcal{G}^*$. This, however, does not matter and the quantization procedure of [6, Theorem 6.5.1] may be carried out (it is important that the base is *finite-dimensional*). We expose here this construction for the sake of completeness and to convince the reader that infinite-dimensionality of fibres is really irrelevant.

We start with the *Weyl algebras bundle* $W = W(B^{reg}, K)$ over B^{reg} with coefficients in K, or, more generally, with differential forms on B^{reg}

with values in W. The sections of the bundle $W \otimes \Lambda$ are of the form

$$a = \sum_{k,p=0}^{\infty} h^k a_{ki_1\ldots i_p j_1\ldots j_q}(z) u^{i_1} \ldots u^{i_p} \delta z^{j_1} \wedge \ldots \wedge \delta z^{j_q}. \tag{3.30}$$

Here $z = (z^1, \ldots, z^{2n}) \in B^{reg}, u = (u^1, \ldots, u^{2n}) \in T_z B^{reg}, h$ is a formal parameter. We prescribe degrees: $\deg u^i = 1$, $\deg h = 2$ and order the terms of the formal series (3.30) by their total degree $p+2k$. The coefficients $a_{ki_1\ldots i_p j_1\ldots j_q}(z)$ are tensors on B^{reg} symmetric in $i_1, \ldots, i_p$ and antisymmetric in $j_1, \ldots, j_q$ with values in the algebra $\mathcal{A}_F$ ($*$-product algebra on the fibre $F = G \times \mathcal{G}^*$ constructed in the subsection 3.1). We will also use a shorter notation

$$a = a(z, u, \delta z, h) = \sum_{k,|\alpha|=0}^{\infty} h^k a_{k,\alpha,\beta}(z) u^\alpha \delta z^{\wedge\beta}, \tag{3.31}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2n})$, $\beta = (\beta_1, \ldots, \beta_{2n})$ are multiindices with $\beta_i = 0, 1$. A product on $W \otimes \Lambda$ denoted by $\circ$ is defined as

$$a \circ b = \sum h^{k+l} a_{k,\alpha,\beta}(z) b_{l,\gamma,\delta}(z)(u^\alpha \circ u^\gamma) \delta z^{\wedge\beta} \wedge \delta z^{\wedge\delta}.$$

Here $a_{\ldots}(z) b_{\ldots}(z)$ means a $*$-product on a fibre $K_z \equiv \mathcal{A}_F$ and

$$u^\alpha \circ u^\gamma = \exp\left(-\frac{ih}{2}\omega^{ij}\frac{\partial}{\partial u^i}\frac{\partial}{\partial v^j}\right) u^\alpha u^\gamma|_{v=u}.$$

Let ∂^s be a symplectic connection on the manifold B^{reg} (the connection coefficients Γ_{ijk} in the Darboux local coordinates are completely symmetric). Along with the connection ∇ on the coefficient bundle K the connection ∂^s defines a connection ∂_W on the bundle $W(B^{reg}, K) = W(B^{reg}) \otimes K$. The local expression for ∂_W is

$$\begin{aligned}\partial_W a &= \nabla a + \frac{i}{h}\left[\frac{1}{2}\Gamma_{ijk} u^i u^j dx^k, a\right] \\ &= \delta a + \frac{i}{h}\left[\frac{1}{2}\Gamma_{ijk} u^i u^j dx^k - \gamma, a\right]\end{aligned}$$

where γ is a local one-form (3.29) and δ means a differential with respect to z.

Lemma 3.6 *For any section $a \in C^\infty(B^{reg}, W)$*

$$\partial^2 a = \frac{i}{h}[R + \langle\eta, \kappa\rangle, a] \tag{3.32}$$

where

$$R = \frac{1}{4} R_{ijkl} u^i u^j dx^k \wedge dx^l, \tag{3.33}$$

R_{ijkl} being the curvature tensor of the symplectic connection ∂^s, and

$$\kappa = \kappa_i = (\partial_i^* \partial_i)^{-1} * [*p_i(\delta\partial_i), p_i(\delta\partial_i)] \tag{3.34}$$

where p_i means a projection on the harmonic one-forms with respect to ∂_i.

Proof. Since $\Gamma = 1/2\Gamma_{ijk}u^i u^j dx^k$ and $\gamma = \langle \eta, (\partial_i^* \partial_i)^{-1}\partial_i^*(\delta\partial_i)\rangle$ commute, we have

$$\partial_W^2 a = \frac{i}{h}\left[\delta\Gamma + \frac{1}{2}\Gamma^2 + \delta\gamma + \frac{i}{h}\gamma^2, a\right].$$

The terms $\delta\Gamma + i/h\Gamma^2$ give the curvature form R in (3.32).

Denoting $(\partial_i^*\partial_i)^{-1}\partial_i^*(\delta\partial_i)$ by λ, we obtain using (3.11)

$$\begin{aligned}&\delta\gamma + \frac{i}{h}\gamma^2 = \langle \eta, \delta\lambda\rangle + \frac{i}{2h}[\langle\eta,\lambda\rangle, \langle\eta,\lambda\rangle]\\ &= \langle \eta, \delta\lambda - \frac{1}{2}[\lambda,\lambda]\rangle.\end{aligned}$$

Next, suppressing the subscript i and using the relation $\partial^* = - * \partial *$, we write

$$\begin{aligned}&\delta\lambda = -(\partial^*\partial)^{-1} * ad_{\delta\partial} * (\delta\partial)\\ &+(\partial^*\partial)^{-1} * ad_{\delta\partial} * \partial(\partial^*\partial)^{-1}\partial^*(\delta\partial)\\ &+(\partial^*\partial)^{-1}\partial^* ad_{\delta\partial}(\partial^*\partial)^{-1}\partial^*(\delta\partial)\\ &= -(\partial^*\partial)^{-1} * [\delta\partial, *p(\delta\partial)]\\ &+(\partial^*\partial)^{-1}\partial^*[\delta\partial, (\partial^*\partial)^{-1}\partial^*(\delta\partial)].\end{aligned}$$

Further,

$$\begin{aligned}&\frac{1}{2}[\lambda,\lambda] = (\partial^*\partial)^{-1}\partial^*[\partial\lambda, \lambda]\\ &= (\partial^*\partial)^{-1}\partial^*[\partial(\partial^*\partial)^{-1}\partial^*(\delta\partial), (\partial^*\partial)^{-1}\partial^*(\delta\partial)].\end{aligned}$$

Thus,

$$\begin{aligned}&\delta\lambda - \frac{1}{2}[\lambda,\lambda] = -(\partial^*\partial)^{-1} * [\delta\partial, *p(\delta\partial)]\\ &+(\partial^*\partial)^{-1}\partial^*[p(\delta\partial), (\partial^*\partial)^{-1}\partial^*(\delta\partial)].\end{aligned}$$

The second summand may be rewritten as

$$\begin{aligned}&-(\partial^*\partial)^{-1} * \partial[*p(\delta\partial), (\partial^*\partial)^{-1}\partial^*(\delta\partial)]\\ &= -(\partial^*\partial)^{-1} * [*p(\delta\partial), \partial(\partial^*\partial)^{-1}\partial^*(\delta\partial)]\end{aligned}$$

since the form $*p(\delta\partial)$ is a harmonic one. Finally,

$$\delta\lambda - \frac{1}{2}[\lambda,\lambda] = (\partial^*\partial)^{-1} * [*p(\delta\partial), p(\delta\partial)],$$

proving the lemma

□

As described in [6], we need to construct an Abelian connection D_W on the bundle $W(B^{reg}, K)$. We look for it in the form

$$D_W^a = \partial_W a + \frac{1}{h}[s, a] \tag{3.35}$$

where s is a globally defined one-form on B^{reg} with values in K.

Applying D_W twice and using Lemma 3.6, we obtain

$$D_W^2 a = \frac{i}{h}[\partial_W s + \frac{i}{h}s^2 + R + \langle \eta, \kappa \rangle, a].$$

The property $D_W^2 \equiv 0$ (Abelian property) will be satisfied if we set

$$\partial_W s + \frac{i}{h}s^2 + R + \langle \eta, \kappa \rangle = -\omega \tag{3.36}$$

where

$$\omega = \frac{1}{2}\omega_{ij}\delta z^i \wedge \delta z^j$$

is the symplectic form on B^{reg}. We extract the leading term of s (of degree 1) writing

$$s = \omega_{ij}u^i\theta_k^j(\eta)\delta z^k + r \tag{3.37}$$

with $\deg r \geq 2$. Substituting in (3.36), we have

$$\begin{aligned}\omega_{ij}\theta_k^i(\eta)\theta_l^j(\eta)\delta z^k \wedge \delta z^l &= \omega_{kl}\delta z^k \wedge \delta z^l \\ &+\langle \eta, \kappa_{kl}\rangle \delta z^k \wedge \delta z^l\end{aligned} \tag{3.38}$$

where κ_{kl} are the entries of curvature form (3.34)

$$\kappa = \frac{1}{2}\kappa_{kl}\delta z^k \wedge \delta z^l.$$

Introducing the matrices Ω with the entries ω_{ij}, $\Theta(\eta)$ with the entries $\theta_j^i(\eta)$ and K with the entries κ_{kl} rewrite (3.38) in the form

$$\Theta^t(\eta)\Omega\Theta(\eta) = \Omega + \langle \eta, \mathrm{K}\rangle.$$

This equation has a unique solution

$$\Theta(\eta) = (1 + \Omega^{-1}\langle \eta, K\rangle)^{1/2} \tag{3.39}$$

satisfying the relation

$$\Theta^t\Omega = \Omega\Theta, \tag{3.40}$$

provided the entries $\langle \eta, \kappa_{ij}\rangle$ are small enough.

Now, substitute (3.37) into (3.36) and obtain the following equation for r

$$\theta_j^i(\eta)\delta z^j \wedge \frac{\partial r}{\partial u^i} = Ar + \frac{i}{h}r^2 + B \tag{3.41}$$

where A is a linear operator *not raising degrees* and $\deg r \geq 2$. The explicit expressions for A and B are quite cumbersome:

$$\begin{aligned}Ar &= \partial_W r + \theta_j^i(\eta)\delta z^j \wedge \frac{\partial r}{\partial u^i} - \theta_j^i(\eta)\delta z^j \circ \frac{\partial}{\partial u^i} \\ &+\frac{i}{h}\omega_{ik}[\theta_j^i(\eta)\delta z^j, r] \circ u^k\end{aligned}$$

$$B = -\omega - R - \langle\eta, \kappa\rangle - \frac{i}{h}(\omega_{ij}\theta^j_k(\eta)\delta z^k) \circ (\omega_{ij}\theta^j_k(\eta)\delta z^k),$$

but they are irrelevant for the sequel.

Let us denote the operator on the left-hand side of (3.41) by θ,

$$\theta r = \theta^i_j(\eta)\delta z^j \wedge \frac{\partial r}{\partial u^i}$$

and introduce an operator θ^* by

$$\theta^* r = u^k i\left(\frac{\partial}{\partial z^k}\right) r.$$

Then the following theorem holds (cf. [6, Theorem 6.5.1]).

Theorem 3.7 *Equation* (3.41) *has a unique solution, such that*

$$\theta^* r = 0 \tag{3.42}$$

Proof. First we would like to invert the operator θ. We have $\theta^2 = (\theta^*)^2 \equiv 0$, so it is natural to consider the Laplacian $\Delta = \theta\theta^* + \theta^*\theta$. On tensors

$$a_{pq} = a_{i_1\ldots i_p j_1\ldots j_q}(\eta)u^{i_1}\ldots u^{i_p}\delta z^{j_1} \wedge \ldots \wedge \delta z^{j_q} \tag{3.43}$$

Δa_{pq} has the same form (3.43) with coefficients

$$\begin{aligned}&\theta^i_{i_1}(\eta)a_{i\ldots i_p j_1\ldots j_q}(\eta) + \ldots + \theta^i_{i_p}(\eta)a_{i_1\ldots i j_1\ldots j_q}(\eta)\\ &+\theta^i_{j_1}(\eta)a_{i_1\ldots i_p i\ldots j_q}(\eta) + \ldots + \theta^i_{j_q}a_{a_{i_1}\ldots i_p j_1\ldots i}(\eta)\end{aligned} \tag{3.44}$$

If the matrix $\Theta(\eta)$ is sufficiently close to identity, so that $\|\theta(\eta) - 1\| < 1$, then (3.44) implies

$$\|\frac{\Delta_{pq}}{p+q} - 1\| < 1$$

so that

$$\Delta^{-1}_{pq} = (p+q)^{-1}\left(1 + \left(\frac{\Delta_{pq}}{p+q} - 1\right)\right)^{-1}$$

exsists for $p + q > 0$. We define an operator θ^{-1} by

$$\theta^{-1}a_{pq} = \Delta^{-1}_{pq}\theta^* a_{pq}, \quad p+q > 0$$

and

$$\theta^{-1}a_{00} = 0.$$

Now, applying θ^{-1} to both sides of (3.41) and using that $\theta^* r = 0$, we obtain an equation

$$r = \theta^{-1}(B + Ar + \frac{i}{h}r^2)$$

which can be solved by iterations.

□

Having constructed the Abelian connection on the bundle $W(B^{ref}, K)$, we define the corresponding quantization Q as a linear map from sections of K to flat sections of $W(B^{reg}, K)$. Thus, $\widehat{a} = Qa$ means that

$$D_W \widehat{a} \equiv 0; \quad \widehat{a}|_{u=0} = a. \tag{3.45}$$

Theorem 3.8 *For any $a \in C^\infty(K)$ there exists a unique section $\widehat{a} \in C^\infty$ satisfying* (3.45).

Proof. The condition $D_W \widehat{a} \equiv 0$ is equivalent to

$$\theta\widehat{a} = \partial\widehat{a} + A\widehat{a} + \frac{i}{h}[r\widehat{a}]$$

where θ and A are the same as before. Applying θ^{-1}, we get

$$\widehat{a} = a + \theta^{-1}\left(\partial\widehat{a} + A\widehat{a} + \frac{i}{h}[r, \widehat{a}]\right),$$

so the iterations give a unique solution.

$\square$

3.3 Quantum Reduction

In this subsection we describe once more the algebras of classical and quantum observables and consider the reduction procedure for both algebras.

The Poisson-Lie algebra of classical observables consists of functionals $\Phi(D)$ on regular connections D with sufficiently small curvature (in a C^r norm with $r > 0$, $z \notin \mathbb{Z}$). The functionals $\Phi(D)$ are supposed to be *smooth* in the sense that their variations admit a representation by variational derivatives:

$$\delta\Phi := \frac{d}{dt}\Phi(D + t\delta D) = \int_X \mathrm{tr}\ \delta D \wedge \frac{\delta\Phi}{\delta D}$$

where $\delta\Phi/\delta D$ is a smooth $su(r)$-valued one-form on X called the variational derivative of Φ. The action of the gauge group G is defined by

$$g : \Phi(D) \mapsto \Phi(g^{-1} \circ D \circ g). \tag{3.46}$$

The Poisson bracket is

$$\{\Phi_1, \Phi_2\} = \int_X \mathrm{tr}\ \frac{\delta\Phi_1}{\delta D} \wedge \frac{\delta\Phi_2}{\delta D}.$$

The action (3.46) is Hamiltonian with the moment map

$$\mu(e) = \langle\kappa(D), e\rangle = \int_X \mathrm{tr}\ \kappa(D)e$$

where $\kappa(D)$ is the curvature of D and $s \in \mathcal{G}$ is an $su(2)$-valued function on X.

The normal form theorem (Theorem 2.1) gives a representation of D by a pair ∂, ξ where ∂ is a regular flat connection and ξ is a $su(2)$-valid 2-form with the action of G

$$g : (\partial, \xi) \mapsto (g^{-1} \circ \partial \circ g,\ g^{-1}\xi g)$$

whose moment map is

$$\mu(e) = \langle \xi, e \rangle.$$

The functionals $\Phi(D)$ become functionals on pairs $\Phi(\partial, \xi)$. Further, over a coordinate neighborhood U_i of the moduli space B^{reg} with local coordinates $z = (z^1, \ldots, z^{2n})$ any flat connection may be written as $\partial = g^{-1} \circ \partial_i(z) \circ g$ where $\partial_i(z)$ is a section of a principal bundle $M_0^{reg} \to B^{reg}$ over U_i. So, our functional $\Phi(\partial, \xi)$ may be written in the form

$$\Phi(\partial, \xi) = a(z, g, \xi) \tag{3.47}$$

with the action of $v \in G$

$$v : a(z, g, \xi) \mapsto a(z, gv, v^{-1}\xi v). \tag{3.48}$$

The classical reduction in terms of the algebra A of classical observables consists of two steps. First we define a subalgebra $A_0 \subset A$ of G-invariant functionals using local representations (3.47), (3.48). We introduce a new 2-form $\eta = g\xi g^{-1}$ which is clearly G-invariant and rewrite the functional (3.47) in the form

$$a(z, g, \xi) = a(z, g, g^{-1}\eta g) = b(z, g, \eta) \tag{3.49}$$

with the action of G

$$v : b(z, g, \eta) \mapsto b(z, gv, \eta). \tag{3.50}$$

From (3.50) it follows that G-invariant functionals are those which are independent of g in any local representation (3.49). Thus, the subalgebra A_0 consists of functionals with the local representation of the form

$$A_0 = \{b(z, \eta) = b(z, g\xi g^{-1})\}. \tag{3.51}$$

The next step is to restrict the functionals (3.51) to the zero level set of the moment map. So, we put $\xi = 0$ in (3.51) which is equivalent to $\eta = 0$. We thus obtain a homomorphism $\pi : A_0 \to R$

$$\pi : b(z, \eta) \mapsto b(z, 0) \tag{3.52}$$

to the reduced Poisson-Lie algebra R.

The algebra $\widehat{A}$ of quantum observables has also a local description given by Theorem 3.8. So, any $\widehat{a} \in \widehat{A}$ is a section of the Weyl algebras bundle $W(B^{reg}, K)$ with coefficients in the bundle K, the section $\widehat{a}$ is flat with respect to the Abelian connection D_W constructed in Theorem 3.7. Recall that a section $a \in C^\infty(W(B^{reg}, K))$ over a local chart $U \in B^{reg}$ has the form

$$a(z, u, g, \xi, h) = \sum_{k,|\alpha|=0}^{\infty} h^k a_{k\alpha}(z, g, \xi) u^\alpha$$

where $z \in U$, $u \in T_z B^{reg}$ and $a_{k\alpha}$ are functionals of the form

$$a_{k\alpha} = A(z, g) P(\xi, \eta)$$

where $\eta = g\xi g^{-1}$ and P is an analytical function in a finite set of variables

$$\langle \xi, a_1 \rangle, \ldots \langle \xi, a_m \rangle, \langle \eta, a_{m+1} \rangle, \ldots \langle \eta, a_{m+n} \rangle$$

with $a_1, \ldots, a_{m+n} \in \mathcal{G}$. For a section $a(z, g, \xi, h) \in C^\infty(K)$ we denote by

$$\widehat{a} = \widehat{a}(z, u, g, \xi, h) := Qa \in \widehat{A}$$

a flat section of $W(B^{reg}, K)$ such that

$$\widehat{a}\big|_{u=0} = a.$$

The gauge group G acts on the coefficient bundle K and $W(B^{reg}, K)$. For $a \in C^\infty(W(B^{reg}, K))$ this action reads:

$$v : a(z, u, g, \xi, h) \mapsto a(z, u, gv, v^{-1}\xi v, h). \tag{3.53}$$

Similar to the classical case we denote by π the restriction of sections to $\xi = \eta = 0$. So,

$$\pi a = a\big|_{\xi=0}$$

for $a \in C^\infty(K)$ or $a \in C^\infty(W(B^{reg}, K))$.

Now, we introduce two subbundles $K_0 \subset K$ and $K_J \subset K_0$ and the corresponding subbundles $W(B^{reg}, K_0) \subset W(B^{reg}, K)$ and $W(B^{reg}, K_J) \subset W(B^{reg}, K_0)$. The first subbundle K_0 consists of G-invariants of K. The same considerations as in the classical case show that the sections of K_0 are functionals depending on η only:

$$C^\infty(K_0) = \{a(z, g, \xi, h) = b(z, \eta, h)\}$$

and

$$C^\infty(W(B^{reg}, K_0)) = \{a(z, u, g, \xi, h) = b(z, u, \eta, h)\}.$$

The second subbundle $K_J \subset K_0$ is the kernel of the map π, thus

$$C^\infty(K_J) = \{b(z, \eta, h) : b(z, 0, h) = 0\}$$

and

$$C^\infty(W(B^{reg}, K_J)) = \{b(z, u, \eta, h) : b(z, u, 0, h) = 0\}.$$

Lemma 3.9 *The subbundles $K_J \in K_0$ and $W(B^{reg}, K_J) \subset W(B^{reg}, K_0)$ are two-sided ideals.*

Proof. It is sufficient to consider two sections of K_0 of the form

$$a(z,\eta) = A(z)P(\eta), \quad b(z,\eta) = B(z)Q(\eta).$$

Then

$$a * b = A(z)B(z)(P(\eta) * Q(\eta))$$

where $P * Q$ is given by (3.15). The explicit form of (3.15) implies that the higher-order terms vanish at $\eta = 0$, so

$$P(\eta) * Q(\eta)|_{\eta=0} = P(\eta)Q(\eta)|_{\eta=0}\,. \tag{3.54}$$

Thus, if πa or πb is equal to zero, then by (3.54) $\pi(a * b) = 0$.

□

In other words, the map π defines bundle homomorphisms

$$\pi : K_0 \to \mathbb{C}(h) \cong K_0/K_J$$

$$\pi : W(B^{reg}, K_0) \to W(B^{reg}) \tag{3.55}$$

where $\mathbb{C}(h)$ means a trivial bundle over B^{reg} whose fibres are formal power series in h with constant coefficients, $W(B^{reg})$ means the Weyl algebras bundle over B^{reg} with scalar coefficients. Consider now the Abelian connection

$$D_W a = \partial_s a + \frac{i}{h}[\gamma + \omega_{ij}\theta^j_k(\eta)u^i\delta z^k + r, a] \tag{3.56}$$

on $W(B^{reg}, K)$. From the explicit construction it follows that $r = r(\eta)$ depends on z and η only, so D_W is G-invariant. It implies that the quantization map Q is also G-invariant, that is

$$v(Qa) = Q(va)$$

where $v \in G$ and the action of v is given by (3.53). As an immediate consequence we have the following lemma.

Lemma 3.10 *The subalgebra $\widehat{A}_0 \subset \widehat{A}$ of G-invariant flat sections coincides with the image of $C^\infty(K_0)$ under the quantization map:*

$$\widehat{A}_0 = Q(C^\infty(K_0)).$$

The proof is straightforward.

Finally, we study the relations between the Abelian connection (3.56) and the map (3.55). We write D_W in the form:

$$D_W a = \partial_W a + \frac{i}{h}[\sigma, a] = \partial_s a + \frac{i}{h}[\gamma + \sigma, a]$$

where

$$\sigma = \omega_{ij}\theta^j_k(\eta)u^i\delta z^k + r.$$

The form σ satisfies the normalization condition

$$\theta^*\sigma = \omega_{ij}\theta^j_k u^i u^j + \theta^* = 0$$

since $\theta^* r = 0$ by construction and the matrix $\omega_{ij}\theta^j_k$ is skew-symmetric. Denote by $\pi(D_W)$ a connection on the bundle $W(B^{reg})$ with scalar coefficients obtained by substitution $\eta = 0$ into the connection one-form $\gamma + \sigma$. Since $\pi\gamma = 0$ we obtain

$$(\pi D_W)a = \partial_s a + \frac{i}{h}[\pi\sigma, a].$$

Lemma 3.11 *The connection $\pi(D_W)$ is a standard Abelian connection on the bundle $W(B^{reg})$.*

Proof. The form r is uniquely defined by two conditions:

$$\partial_s(\gamma + \sigma) + \frac{i}{h}(\gamma + \sigma)^2 = -\omega,$$

$$\theta^*\sigma = 0$$

(see Theorem 3.2). Since π is a homomorphism and $\pi\gamma = 0$ we get

$$\partial(\pi\sigma) + \frac{i}{h}(\pi\sigma)^2 = -\omega,$$

$$\theta^*(\pi\sigma) = 0.$$

But

$$\pi\sigma = \omega_{ij}u^i\delta z^j + \pi r$$

since $\theta^i_j(0) = \delta^i_j$. It means that πr satisfies the usual conditions for the standard Abelian connection on $W(B^{reg})$ which define πr uniquely.

$\square$

Now, we may complete the reduction procedure in the quantum case. Define the reduced quantum algebra as a quotient

$$\widehat{R} = \widehat{A}_0/\widehat{J} = \pi\widehat{A}_0$$

where $\widehat{J}$ is the kernel of the map π. Then we have the following reduction theorem.

Theorem 3.12 *The algebra $\widehat{R}$ coincides with the algebra of flat sections of $W(B^{reg})$ with respect to the Abelian connection πD_W.*

Proof. Let $\widehat{a} \in \widehat{A}_0$. Applying π to both sides of the equation $D_W\widehat{a} = 0$ and using that π is a homomorphism, we get

$$\pi(D_W\widehat{a}) = (\pi D_W)\pi\widehat{a} = 0.$$

Thus, $\pi\widehat{a} \in C^\infty(W(B^{reg}))$ is a flat section with respect to $\pi(D_W)$. $\square$

References

[1] Kontsevich, M.: Deformation quantization of Poisson manifolds, Preprint q-alg/9709040, 1997.

[2] Marsden, J. E. and Weinstein, A.: Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5**(1974), 121 - 130.

[3] Fedosov, B. V.: Non-Abelian reduction in deformation quantization, *Lett. Math. Phys.* **43**(1998), 137–154.

[4] Gotay, M. J.: On coisotropic embeddings of presymplectic manifolds, *Proc. Am. Math. Soc.* **84**(1982), 111 - 114.

[5] Gutt, S.: An explicit $*$-product on the cotangent bundle of a Lie group, *Lett. Math. Phys.* **7**(1983), 249 - 258.

[6] Fedosov B. V.: *Deformation Quantization and Index Theory*, Akademie Verlag, Berlin, 1996.

[7] Kravchenko, O.: Deformation quantization of symplectic fibrations, Preprint IHES, Bures-sur-Yvette, 1997.

[8] Axelrod, S., Della Pietra, S. and Witten, E.: Goemetric quantization of Chern-Simons gauge theory, *J. Diff. Geom.*

[9] Springer, G.: *Introduction to Riemann surfaces,* Addison-Wesley, 1957.

[10] Taylor, M. E.: *Pseudodifferential Operators*, Princeton University Press, Princeton, New Jersey, 1983.