

# THE HEAT CONTENT ASYMPTOTICS FOR VARIABLE GEOMETRIES

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ABSTRACT. We study the heat content asymptotics on a compact manifold with boundary defined by a time dependent family of operators of Laplace type.

Let  $M$  be a compact Riemannian manifold with smooth boundary  $\partial M$ . We denote the Riemannian measures on  $M$  and on  $\partial M$  by  $dx$  and  $dy$ . Let  $D_0$  be an operator of Laplace type on the space of smooth sections  $C^\infty(V)$  to a vector bundle over  $M$ . There exists a unique connection  $\nabla$  on  $V$  and a unique endomorphism  $E$  of  $V$  so that

$$D_0 = -\{\text{Trace } \nabla^2 + E\}.$$

(If  $D_0 = \Delta_0$ , then the connection  $\nabla$  is flat and  $E = 0$ ). We assume given a decomposition of the boundary  $\partial M$  as the disjoint union of two closed sets  $C_D$  and  $C_N$ . We consider the boundary operator

$$\mathcal{B}u := u|_{C_D} \oplus (u_{;m} + Su)|_{C_N}.$$

Here  $u_{;m}$  is the inward unit normal covariant derivative of  $u$  and  $S$  is an auxiliary endomorphism of  $V$ . This formalism permits us to treat both Robin and Dirichlet boundary conditions. Let  $\phi$  give the initial temperature distribution of the manifold and let  $u(x; t) = u_{\phi, D_0}(x; t)$  be the temperature distribution for  $t > 0$ ;

$$(\partial_t + D_0)u = 0, \quad \mathcal{B}u = 0, \quad \text{and } u|_{t=0} = \phi.$$

Let  $\rho$  be a smooth section to the dual bundle  $V^*$  giving the specific heat of the manifold. Then the total energy content of the manifold is given by:

$$\beta(\phi, D_0, \rho)(t) := \int_M \langle u(x; t), \rho(x) \rangle dx.$$

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In this expression,  $\langle \cdot, \cdot \rangle$  denotes the natural pairing between  $V$  and  $V^*$ . As  $t \downarrow 0$  there is an asymptotic series of the form

$$\beta(\phi, D_0, \rho)(t) \sim \sum_{n \geq 0} t^{n/2} \beta_n(\phi, D_0, \rho).$$

There exist invariants which are locally computable so that

$$\beta_n(\phi, D_0, \rho) = \int_M \beta_n^M(\phi, D_0, \rho) dx + \int_{\partial M} \beta_n^{\partial M}(\phi, D_0, \rho) dy.$$

These are the heat content asymptotics which describe the short time heat flow defined by the problem.

Let  $\tilde{D}$ ,  $\tilde{\nabla}$ , and  $\tilde{\mathcal{B}}$  be the adjoint operators on the dual bundle  $V^*$ . Let indices  $i$  and  $j$  range from 1 to  $m$  and index a local orthonormal frame field  $\{e_i\}$  for the tangent bundle of  $M$ ; on the boundary, we normalize the frame so that  $e_m$  is the inward unit normal and let indices  $a, b$ , and  $c$  range from 1 to  $m-1$  and index the induced orthonormal frame for the tangent bundle of the boundary. We adopt the Einstein convention and sum over repeated indices. Let ‘;’ and ‘:’ denote multiple covariant differentiation with respect to the Levi-Civita connections of  $M$  and of  $\partial M$ . Let  $R$  be the Riemann curvature tensor, let  $\mathcal{R}$  be the scalar curvature, and let  $L$  be the second fundamental form defined by the metric  $g_0$ . Let  $\Omega$  be the curvature of the connection  $\nabla$ . The following result follows from computations performed in [1, 3, 9]; we refer to [6, 7, 12, 13, 14, 15] for related work.

**Theorem 1.** *With the notation established above, we have:*

- (1)  $\beta_0(\phi, D_0, \rho) = \int_M \langle \phi, \rho \rangle dx.$
- (2)  $\beta_1(\phi, D_0, \rho) = -2\pi^{-1/2} \int_{C_D} \langle \phi, \rho \rangle dy.$
- (3)  $\beta_2(\phi, D_0, \rho) = -\int_M \langle D\phi, \rho \rangle dx + \int_{C_D} \{ \langle \frac{1}{2} L_{aa} \phi, \rho \rangle - \langle \phi, \rho;_m \rangle \} dy$   
 $+ \int_{C_N} \langle \mathcal{B}\phi, \rho \rangle dy.$
- (4)  $\beta_3(\phi, D_0, \rho) = -2\pi^{-1/2} \int_{C_D} \{ \frac{2}{3} \langle \phi;_{mm}, \rho \rangle + \frac{2}{3} \langle \phi, \rho;_{mm} \rangle - \langle \phi;_a, \rho;_a \rangle$   
 $+ \langle E\phi, \rho \rangle - \frac{2}{3} L_{aa} \langle \phi;_m, \rho \rangle - \frac{2}{3} L_{aa} \langle \phi, \rho;_m \rangle + \langle (\frac{1}{12} L_{aa} L_{bb} - \frac{1}{6} L_{ab} L_{ab}$   
 $+ \frac{1}{6} R_{amam}) \phi, \rho \rangle \} dy + \frac{2}{3} \cdot 2\pi^{-1/2} \int_{C_N} \langle \mathcal{B}\phi, \tilde{\mathcal{B}}\rho \rangle dy.$
- (5)  $\beta_4(\phi, D_0, \rho) = \frac{1}{2} \int_M \langle D\phi, \tilde{D}\rho \rangle + \int_{C_D} \{ \frac{1}{2} \langle (D\phi);_m, \rho \rangle + \frac{1}{2} \langle \phi, (\tilde{D}\rho);_m \rangle$   
 $- \frac{1}{4} \langle L_{aa} D\phi, \rho \rangle - \frac{1}{4} \langle L_{aa} \phi, \tilde{D}\rho \rangle + \langle (\frac{1}{8} E;_m - \frac{1}{16} L_{ab} L_{ab} L_{cc} + \frac{1}{8} L_{ab} L_{ac} L_{bc}$   
 $- \frac{1}{16} R_{ambm} L_{ab} + \frac{1}{16} R_{abcb} L_{ac} + \frac{1}{32} \mathcal{R};_m + \frac{1}{16} L_{ab;ab}) \phi, \rho \rangle - \frac{1}{4} L_{ab} \langle \phi;_a, \rho;_b \rangle$   
 $- \frac{1}{8} \langle \Omega_{am} \phi;_a, \rho \rangle + \frac{1}{8} \langle \Omega_{am} \phi, \rho;_a \rangle \} dy + \int_{C_N} \{ -\frac{1}{2} \langle \mathcal{B}\phi, \tilde{D}\rho \rangle$   
 $- \frac{1}{2} \langle D\phi, \tilde{\mathcal{B}}f_2 \rangle + \langle (\frac{1}{2} S + \frac{1}{4} L_{aa}) \mathcal{B}\phi, \tilde{\mathcal{B}}\rho \rangle dy.$

**Remark 2.** If  $D = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B)$  is an operator of Laplace type, then (see [10]) the connection 1 form  $\omega$  of  $\nabla$  and the endomorphism  $E$  are given by

$$\omega_\delta = \frac{1}{2}g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu), \text{ and}$$

$$E = b - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma).$$

Although the assumption that the underlying geometry is autonomous is natural in many situations, there are some physical situations in which the geometry is time dependent; the Universe evolves with time for example. If  $C_N$  is empty, then one can consider a time dependent boundary where the metric is autonomous. Since the underlying topology is not changed, this problem is equivalent to one where the boundary is fixed but the metric is varied.

We consider a time dependent family of operators of Laplace type:

$$D_t := D_0 + \sum_{r>0} t^r \left\{ \mathcal{G}_{r,ij}(x)\nabla_i\nabla_j + \mathcal{F}_{r,i}(x)\nabla_i + \mathcal{E}_r(x) \right\}.$$

(We expand relative to a frame for the tangent bundle which is orthogonal with respect to the original metric  $g_0$ ). Let  $u = u_{\phi,D}$  be the temperature distribution defined by the equations:

$$(\partial_t + D_t)u = 0, \quad \mathcal{B}u = 0 \text{ and } u|_{t=0} = \phi.$$

Let  $\beta$  and  $\beta_n$  be the associated heat content function and heat content asymptotics:

$$\beta(\phi, D, \rho)(t) := \int_M u(x; t)\rho(x; t)dx \sim \sum_{n \geq 0} \beta_n(\phi, D, \rho)t^{n/2}.$$

We expand the specific heat  $\rho(x; t) = \sum_{0 \leq k < k_0} \rho_k(x)t^k + O(t^{k_0})$  in a Taylor series. Then it is immediate that

$$\beta_n(\phi, D, \rho) = \sum_{2k+l=n} \beta_l(\phi, D, \rho_k).$$

Consequently, we assume that  $\rho = \rho(x)$  henceforth is autonomous. The following is the main theorem of this paper. It gives the new terms which appear in the asymptotic expansion when Laplacian is time dependent.

**Theorem 3.**

- (1)  $\beta_0(\phi, D, \rho) = \beta_0(\phi, D_0, \rho).$
- (2)  $\beta_1(\phi, D, \rho) = \beta_1(\phi, D_0, \rho).$
- (3)  $\beta_2(\phi, D, \rho) = \beta_2(\phi, D_0, \rho).$
- (4)  $\beta_3(\phi, D, \rho) = \beta_3(\phi, D_0, \rho) + \frac{1}{2\sqrt{\pi}} \int_{C_D} \mathcal{G}_{1,mm}\phi\rho dy.$
- (5)  $\beta_4(\phi, D, \rho) = \beta_4(\phi, D_0, \rho) - \frac{1}{2} \int_M \{ \mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{F}_{1,i}\phi_{;i} + \mathcal{E}_1\phi \} \rho dx$   
 $+ \int_{C_D} \{ (\frac{7}{16}\mathcal{G}_{1,mm;m} - \frac{1}{4}\mathcal{G}_{1,mm}L_{aa} - \frac{5}{16}\mathcal{F}_{1,m})\phi\rho - \frac{5}{16}\mathcal{G}_{1,am}\phi_{;a}\rho$   
 $+ \frac{1}{2}\mathcal{G}_{1,mm}\rho_{;m}\phi \} dy - \frac{1}{2} \int_{C_N} \mathcal{G}_{1,mm}\rho\mathcal{B}\phi dy.$

**Remark.** In this paper, we deal with homogeneous boundary conditions and zero heat source. However, the methods developed in [5] can easily be adapted to study variable geometry with inhomogeneous boundary conditions and non-trivial heat source.

We devote the remainder of this paper to the proof of Theorem 3. We begin with a technical Lemma involving products. We say that the structures split if  $M = M_1 \times M_2$  where  $M_1$  is closed, if  $\phi = \phi_1 \phi_2$ , if  $\rho = \rho_1 \rho_2$ , and if  $D_t = D_{1,t} + D_{2,t}$ . Then  $u = u_1 u_2$  so  $\beta(\phi, D, \rho)(t) = \beta(\phi_1, D_1, \rho_1)(t) \cdot \beta(\phi_2, D_2, \rho_2)(t)$ . This shows

**Lemma 4.** *If the structures split, then*

$$\beta_n(\phi, D, \rho) = \sum_{k+\ell=n} \beta_k(\phi_1, D_1, \rho_1) \beta_\ell(\phi_2, D_2, \rho_2).$$

There exist local invariants  $\beta_n^M$  and  $\beta_n^{\partial M}$  which are bilinear in the covariant derivatives of the functions  $\phi$  and  $\rho$  with coefficients which are invariant expressions in the covariant derivatives of the tensors  $L, S, R, \Omega, E, \mathcal{E}, \mathcal{F}$ , and  $\mathcal{G}$  so that:

$$\beta_n(\phi, D, \rho) = \int_M \beta_n^M(\phi, D, \rho) dx + \int_{\partial M} \beta_n^{\partial M}(\phi, D, \rho) dy.$$

We assign weight 0 to  $\phi$  and  $\rho$ ; we assign weight 1 to  $L$  and  $S$ ; we assign weight 2 to  $R, \Omega$ , and  $E$ ; we assign weight  $2k$  to  $\mathcal{G}_k$ ; we assign weight  $2k + 1$  to  $\mathcal{F}_k$ ; we assign weight  $2k + 2$  to  $\mathcal{E}_k$ . We increase the weight by 1 for every explicit covariant derivative. We established the following result in the autonomous case using dimensional analysis; the same argument extends immediately to the time dependent case so we omit details.

**Lemma 5.** *The local invariants  $\beta_n^M$  are homogeneous of weight  $n$  and the local invariants  $\beta_n^{\partial M}$  are homogeneous of weight  $n - 1$ .*

We use Lemma 5 to determine the general form of the invariants  $\beta_n$  for  $n \leq 4$ .

**Lemma 6.** *Let  $\rho = \rho(x)$ . Then there exist universal constants so that*

- (1)  $\beta_0(\phi, D, \rho) = \beta_0(\phi, D_0, \rho).$
- (2)  $\beta_1(\phi, D, \rho) = \beta_1(\phi, D_0, \rho).$
- (3)  $\beta_2(\phi, D, \rho) = \beta_2(\phi, D_0, \rho) + \int_M \{a_1 \mathcal{G}_{1,ii} \phi \rho\}.$
- (4)  $\beta_3(\phi, D, \rho) = \beta_3(\phi, D_0, \rho) + \int_{C_N} \{(a_2^N \mathcal{G}_{1,aa} + c_1^N \mathcal{G}_{1,mm}) \phi \rho\} dy.$   
 $+ \int_{C_D} \{(a_2^D \mathcal{G}_{1,aa} + c_1^D \mathcal{G}_{1,mm}) \phi \rho\} dy.$
- (5)  $\beta_4(\phi, D, \rho) = \beta_4(\phi, D_0, \rho) + \int_M \{(a_3 \mathcal{G}_{1,ii;jj} + a_4 \mathcal{G}_{1,ij;ij} + a_5 \mathcal{G}_{1,ii} \mathcal{G}_{1,jj}$   
 $+ a_6 \mathcal{G}_{1,ij} \mathcal{G}_{1,ij} + a_7 \mathcal{G}_{2,ii} \phi + a_8 \mathcal{G}_{1,ii} \phi_{;jj} + a_9 \mathcal{F}_{1,i;i} \phi + a_{13} \mathcal{G}_{1,ii} E \phi$   
 $+ (b_1 \mathcal{G}_{1,jj;i} + b_2 \mathcal{G}_{1,ij;j}) \phi_{;i} + c_2 \mathcal{G}_{1,ij} \phi_{;ij} + c_6 \mathcal{E}_1 \phi + d_2 \mathcal{F}_{1,i;\phi;i}\} \rho dx$

$$\begin{aligned}
& + \int_{C_N} \{a_{10}^N \mathcal{G}_{1,aa} \phi;_m \rho + a_{11}^N \mathcal{G}_{1,aa} \phi \rho;_m + a_{12}^N \mathcal{G}_{1,aa} L_{bb} \phi \rho \\
& + (a_{14}^N \mathcal{G}_{1,aa} S + c_7^N \mathcal{G}_{1,mm} S) \phi \rho \\
& + (b_3^N \mathcal{G}_{1,aa;m} + b_4^N \mathcal{G}_{1,am;a} + b_5^N \mathcal{G}_{1,ab} L_{ab}) \phi \rho \\
& + c_3^N \mathcal{G}_{1,mm} \phi;_m \rho + c_4^N \mathcal{G}_{1,mm} \phi \rho;_m + c_5^N \mathcal{G}_{1,mm} L_{aa} \phi \rho \\
& + d_3^N \mathcal{G}_{1,mm;m} \rho \phi + d_4^N \mathcal{G}_{1,am} \phi;_a \rho + d_5^N \mathcal{F}_{1,m} \phi \rho\} dy \\
& + \int_{C_D} \{a_{10}^D \mathcal{G}_{1,aa} \phi;_m \rho + a_{11}^D \mathcal{G}_{1,aa} \phi \rho;_m + a_{12}^D \mathcal{G}_{1,aa} L_{bb} \phi \rho \\
& + (b_3^D \mathcal{G}_{1,aa;m} + b_4^D \mathcal{G}_{1,am;a} + b_5^D \mathcal{G}_{1,ab} L_{ab}) \phi \rho \\
& + c_3^D \mathcal{G}_{1,mm} \phi;_m \rho + c_4^D \mathcal{G}_{1,mm} \phi \rho;_m + c_5^D \mathcal{G}_{1,mm} L_{aa} \phi \rho \\
& + d_3^D \mathcal{G}_{1,mm;m} \rho \phi + d_4^D \mathcal{G}_{1,am} \phi;_a \rho + d_5^D \mathcal{F}_{1,m} \phi \rho\} dy.
\end{aligned}$$

*Proof.* We use Weyl's theorem [16] on the invariants of the orthogonal group to express these invariants in terms of contractions of indices. We integrate by parts to exchange derivatives at the cost of introducing additional boundary terms to normalize the interior integrands so no covariant derivatives of  $\rho$  are present. Similarly, we integrate by parts on the boundary to normalize the boundary integrands so no tangential covariant derivatives of  $\rho$  are present. We write down a suitable spanning set and apply Lemma 5 to see that the  $\beta_n$  for  $n \leq 4$  have the form given in Lemma 6 where the constants a-priori depend on the dimension of the manifold.

Let  $(M_2, D_2, \phi_2, \rho_2)$  be given. Let  $M_1$  be the circle  $S^1$  with the usual periodic parameter  $y$ . Let  $D_1 = -\partial_y^2$  and  $\phi_1 = 1$ . Then  $u_1 = 1$  and thus we have that  $\beta(\phi_1, D_1, \rho_1) = \int_{M_1} \rho_1 dy_1$ . We use Lemma 4 to see

$$\beta_n(\phi, D, \rho) = \int_{M_1} \rho_1 dx_1 \cdot \beta_n(\phi_2, D_2, \rho_2).$$

It now follows that the coefficients which appear in Lemma 6 are independent of the dimension and are universal constants.  $\square$

The lack of commutativity in the vector valued case does not play a role in these expressions; we restrict henceforth therefore to the scalar setting. To simplify the notation, let  $\tilde{\beta}_n(\phi, D, \rho) := \beta_n(\phi, D, \rho) - \beta_n(\phi, D_0, \rho)$ . We begin the proof of Theorem 3 by determining some of the constants in Lemma 6.

**Lemma 7.**

$$\begin{aligned}
(1) \quad \tilde{\beta}_3(\phi, D, \rho) &= \frac{1}{2\sqrt{\pi}} \int_{C_D} \mathcal{G}_{1,mm} \phi \rho dy. \\
(2) \quad \tilde{\beta}_4(\phi, D, \rho) &= -\frac{1}{2} \int_M \{ \mathcal{G}_{1,ij} \phi;_{ij} + \mathcal{E}_1 \phi \} \rho dx \\
&+ \int_{C_D} \{ -\frac{1}{4} \mathcal{G}_{1,mm} L_{aa} \phi \rho + \frac{1}{2} \mathcal{G}_{1,mm} \rho;_m \phi \} dy - \frac{1}{2} \int_{C_N} \mathcal{G}_{1,mm} \rho \mathcal{B} \phi dy \\
&+ \int_M d_2 \mathcal{F}_{1,i} \phi;_i \rho dx + \int_{C_D} \{ d_3^D \mathcal{G}_{1,mm;m} \phi \rho + d_4^D \mathcal{G}_{1,am} \phi;_a \rho + d_5^D \mathcal{F}_{1,m} \phi \rho \} dy \\
&+ \int_{C_N} \{ d_3^N \mathcal{G}_{1,mm;m} \rho \phi + d_4^N \mathcal{G}_{1,am} \phi;_a \rho + d_5^N \mathcal{F}_{1,m} \phi \rho \} dy.
\end{aligned}$$

*Proof. Step 1:* We apply Lemma 4. Let  $(M_2, \phi_2, D_2, \rho_2)$  be arbitrary. Let

$$M_1 = \mathbb{T}^k := S^1 \times \dots \times S^1, \text{ let } \phi_1 = 1, \text{ and let}$$

$$D_{1,t} = \Delta_{M_1} + \sum_{r>0} t^r (\mathcal{G}_{r,ij} \nabla_i \nabla_j + \mathcal{F}_{r,i} \nabla_i).$$

Since  $D_{1,t} \phi_1 = 0$ ,  $u_1 = 1$  and  $\beta_n = 0$  for  $n > 0$ . Thus

$$\beta_n(\phi_2, D_1 + D_2, \rho_1 \rho_2) = \beta_0(1, D_1, \rho_1) \beta_n(\phi_2, D_2, \rho_2).$$

In particular  $\beta_n$  is independent of the tensors  $\mathcal{F}_{1,i}$  and  $\mathcal{G}_{1,ij}$  for  $i, j \leq k$ . This shows the following relations hold:

$$0 = a_1 = a_2^N = a_2^D = a_3 = a_4 = a_5 = a_6 = a_7 = a_8$$

$$= a_9 = a_{10}^N = a_{10}^D = a_{11}^N = a_{11}^D = a_{12}^N = a_{12}^D = a_{13} = a_{14}^N.$$

Consequently the higher order Taylor coefficients  $\mathcal{E}_r$ ,  $\mathcal{F}_{r,i}$ , and  $\mathcal{G}_{r,ij}$  do not play a role in the computation of  $\beta_n$  if  $n \leq 4$  and if  $r \geq 2$ . We may therefore restrict to first order deformations of the Laplacian  $\Delta$  henceforth and set

$$\mathcal{E} = \mathcal{E}_1, \mathcal{F}_i = \mathcal{F}_{1,i}, \mathcal{G}_{ij} = \mathcal{G}_{1,ij}.$$

**Step 2:** Let  $M := \mathbb{T}^k \times [0, 1]$ , let  $y_a$  be the periodic parameters on the torus for  $1 \leq a \leq k$ , and let  $z \in [0, 1]$  be the normal parameter. Let  $f_{ab}(z)$  be functions which are close in the  $C^\infty$  topology to the Kronecker symbol  $\delta_{ab}$  and let

$$ds^2 := f_{ab}(z) dy^a \circ dy^b + dz^2$$

define the Laplacian  $\Delta_0$ . Let  $\phi = \phi(z)$  and let  $u_0 = u_{\phi, \Delta_0}$  be defined by the trivial variation;  $u_0$  only depends on the normal parameter  $z$ . We take a variation where  $\mathcal{E} = 0$ ,  $\mathcal{F}_m = 0$ , and  $\mathcal{G}_{mm} = 0$ . Therefore:

$$(\mathcal{G}_{ij} \nabla_i \nabla_j + \mathcal{F}_i \nabla_i + \mathcal{E})u = 0$$

so  $u_{\phi, D} = u_0$ . Thus  $\beta$  is independent of the remaining  $\mathcal{F}$  and  $\mathcal{G}$  variables and

$$0 = b_1 = b_2 = b_3^N = b_3^D = b_4^N = b_4^D = b_5^N = b_5^D.$$

**Step 3:** Let  $s = s(t) := e^t - 1$ ;  $\partial_s = e^{-t} \partial_t$  and  $s(0) = 0$ . Consider the conformal deformation  $D_t = e^t D_0$ . Let  $u_0 := u_{\phi, D_0}$  and let  $u(x; t) = u_0(x; s(t))$ . Then:

$$(\partial_t + D_t)u = e^t (\partial_s + D_0)u_0 = 0, \mathcal{B}u = 0, \text{ and } u|_{t=0} = u_0|_{s=0} = \phi.$$

Consequently  $u_{\phi,D}(t) = u_{\phi,D_0}(s(t))$  and  $\beta(\phi, D, \rho)(t) = \beta(\phi, D_0, \rho)(s(t))$ . We have

$$\begin{aligned} s^{\frac{1}{2}} &= t^{\frac{1}{2}} + \frac{1}{4}t^{\frac{3}{2}} + O(t^{\frac{5}{2}}), \quad s = t + \frac{1}{2}t^2 + O(t^3), \\ s^{\frac{3}{2}} &= t^{\frac{3}{2}} + O(t^{\frac{5}{2}}), \quad s^2 = t^2 + O(t^3). \end{aligned}$$

We equate coefficients of  $t$  in the asymptotic expansions

$$\sum_n t^n \beta_n(\phi, D, \rho) \sim \beta(\phi, D, \rho)(t) = \beta(\phi, D_0, \rho)(s(t)) \sim \sum_n t^n \beta_n(\phi, D_0, \rho) s(t)^n$$

and use Theorem 1 to derive the relationships:

$$\begin{aligned} \tilde{\beta}_3(\phi, D, \rho) &= \frac{1}{4}\beta_1(\phi, D_0, \rho) = -\frac{1}{2\sqrt{\pi}} \int_{C_D} \phi \rho dy \\ \tilde{\beta}_4(\phi, D, \rho) &= \frac{1}{2}\beta_2(\phi, D_0, \rho) \\ &= \frac{1}{2} \int_M (\phi_{;ii} + E\phi) \rho dx + \frac{1}{2} \int_{C_N} \rho(\phi_{;m} + S\phi) dy \\ &\quad + \frac{1}{2} \int_{C_D} \left\{ \frac{1}{2} L_{aa} \phi \rho - \rho_{;m} \phi \right\} dy. \end{aligned}$$

In this setting, we have  $\mathcal{E} = -E$ ,  $\mathcal{F} = 0$ , and  $\nabla \mathcal{G} = -g_0$ . We may therefore complete the proof of the Lemma by deriving the relationships:

$$\begin{aligned} c_1^N &= 0, \quad c_1^D = \frac{1}{2\sqrt{\pi}}, \quad c_2 = -\frac{1}{2}, \quad c_3^N = -\frac{1}{2}, \quad c_3^D = 0, \\ c_4^N &= 0, \quad c_4^D = \frac{1}{2}, \quad c_5^N = 0, \quad c_5^D = -\frac{1}{4}, \quad c_6 = -\frac{1}{2}, \quad c_7^N = -\frac{1}{2}. \quad \square \end{aligned}$$

We continue the proof of Theorem 3 by determining additional coefficients:

**Lemma 8.** *We have  $\tilde{\beta}_4(\phi, D, \rho) = -\frac{1}{2} \int_M \{ \mathcal{G}_{1,ij} \phi_{;ij} + \mathcal{E}_1 \phi + \mathcal{F}_{1,i} \phi_{;i} \} \rho dx$   
 $-\frac{1}{2} \int_{C_N} \mathcal{G}_{1,mm} (\mathcal{B}\phi) \rho dy + \int_{C_D} \{ \frac{1}{2} \mathcal{G}_{1,mm} \phi \rho_{;m} - \frac{1}{4} \mathcal{G}_{1,mm} L_{aa} \phi \rho$   
 $+ d_3^D \mathcal{G}_{1,mm;m} \phi \rho - \frac{5}{16} \mathcal{G}_{1,am} \phi_{;a} \rho - \frac{5}{16} \mathcal{F}_{1,m} \phi \rho \} dy.$*

*Proof.* Through out the proof of this Lemma, we shall let  $M := S^1 \times [0, 1]$  with the flat metric and usual parameters  $(y, z)$ . Let  $\Delta_0 := -\partial_y^2 - \partial_z^2$  be the associated Laplacian.

**Step 1:** We use gauge invariance to determine the coefficients  $d_2$ , and  $d_5^D$ . For  $f \in C^\infty(M)$ , let  $\mathcal{D}_0 := \Delta_0 + f$  and let

$$D_t := e^{tf} (\partial_t + \mathcal{D}_0) e^{-tf} - \partial_t = \Delta_0 + 2tf_{;i} \nabla_i + tf_{;ii} - t^2 f_{;i}^2.$$

Here  $\nabla_i = \partial_i$ . We take pure Dirichlet boundary conditions so  $C_N = \emptyset$ . Let  $u_0 := u_{\phi,D}$  and let  $u := e^{tf} u_0$ . We compute:

$$(\partial_t + D_t)u = e^{tf} (\partial_t + \mathcal{D}_0)u_0 = 0, \quad \mathcal{B}u = 0, \quad \text{and } u|_{t=0} = u_0|_{t=0} = \phi.$$

Consequently  $u = u_{\phi,D}$  so  $\beta(\phi, D, \rho) = \int_M e^{tf} u_0 \rho$ . Therefore

$$\beta_4(\phi, D, \rho) = \beta_4(\phi, \mathcal{D}_0, \rho) + \beta_2(\phi, \mathcal{D}_0, f\rho) + \frac{1}{2}\beta_0(\phi, \mathcal{D}_0, f^2\rho).$$

We have  $\Omega = 0$  and  $E = -f$  for  $\mathcal{D}_0$ . We use Theorem 1 to see that

$$\begin{aligned} \beta_0(\phi, \mathcal{D}_0, f^2\rho) &= \int_M f^2 \phi \rho dx \\ \beta_2(\phi, \mathcal{D}_0, f\rho) &= -\int_M f\rho(\Delta_0 + f)\phi - \int_{C_D} \phi(f\rho)_{;m} dy \\ \beta_4(\phi, \mathcal{D}_0, \rho) &= \frac{1}{2} \int_M \{(\Delta_0 + f)\phi \cdot (\Delta_0 + f)\rho\} dx \\ &\quad + \int_{C_D} \left\{ \frac{1}{2}((\Delta_0 + f)\phi)_{;m} \rho + \frac{1}{2}\phi((\Delta_0 + f)\rho)_{;m} - \frac{1}{8}f_{;m}\phi\rho \right\} dy. \end{aligned}$$

We have  $D_0 = \Delta_0$ . We use Theorem 1 to compute  $\beta_4(\phi, \Delta_0, \rho)$  and to see that:

$$\begin{aligned} \tilde{\beta}_4(\phi, D, \rho) &= \frac{1}{2} \int_M f(\phi\Delta_0\rho - \rho\Delta_0\phi) dx \\ &\quad + \int_{C_D} \left\{ \frac{1}{2}f\phi_{;m}\rho - \frac{1}{2}f\phi\rho_{;m} - \frac{1}{8}f_{;m}\rho\phi \right\} dy. \end{aligned}$$

We use the Green's formula  $\int_M (\alpha\Delta_0\beta - \beta\Delta_0\alpha) dx = \int_{C_D} (\alpha\beta_{;m} - \beta\alpha_{;m})$  to see that

$$\begin{aligned} \int_M f(\phi\Delta_0\rho - \rho\Delta_0\phi) dx &= \int_M \{\rho(\Delta_0(f\phi) - f\Delta_0\phi)\} dx \\ &\quad + \int_{C_D} \{f\phi\rho_{;m} - \rho(f\phi)_{;m}\} dy. \end{aligned}$$

Consequently

$$\tilde{\beta}_4(\phi, D, \rho) = \frac{1}{2} \int_M \{(-2f_{;i}\phi_{;i} - f_{;ii}\phi)\rho\} dx - \frac{5}{8} \int_{C_D} f_{;m}\rho\phi dy.$$

Since  $\mathcal{F}_{1,i} = 2f_{;i}$  and  $\mathcal{E}_1 = f_{;ii}$ , we have

$$c_6 = -\frac{1}{2}, d_2 = -\frac{1}{2}, \text{ and } d_5^D = -\frac{5}{16}.$$

**Step 2:** We take pure Neumann boundary conditions. Let

$$D := \Delta_0 + at\partial_z\partial_y + btz\partial_z^2 + c\partial_z.$$

Let  $\phi = \phi(y)$  depend only on the tangential variable and let  $u_0 = u_{\phi, \Delta_0}$ . Then

$$(\partial_t + D)u_0 = 0, \quad \mathcal{B}u_0 = 0, \quad \text{and } u_0|_{t=0} = \phi.$$

Consequently  $u_{\phi,D} = u_0$ . This implies that  $\tilde{\beta}_4(\phi, D, \rho) = 0$  so:

$$0 = d_3^N = d_4^N = d_5^N.$$



**Step 3:** We take pure Dirichlet boundary conditions. Let

$$\mathcal{D} := e^{-\sqrt{-1}y} \Delta_0 e^{\sqrt{-1}y} = \Delta_0 - 2\sqrt{-1}\partial_y + 1.$$

Let  $u_0 = u_{1,\mathcal{D}}$ ; this function is independent of the angular parameter  $y$  and only depends on the normal parameter  $z$ . We use Remark 2 to compute  $\omega_y = \sqrt{-1}$ ,  $\omega_z = 0$ ,  $\Omega = 0$ , and  $E = 0$ . Let  $\rho = \rho(y)$ . We use Theorem 1 to see

$$\begin{aligned} \beta_4(1, \mathcal{D}, e^{\sqrt{-1}y} \rho) &= \frac{1}{2} \int_M (\Delta_0 + 1)(e^{\sqrt{-1}y} \rho) = \frac{1}{2} \int_M e^{\sqrt{-1}y} \rho \\ &= \beta_4(e^{\sqrt{-1}y}, \Delta_0, \rho). \end{aligned}$$

Let  $u := e^{\sqrt{-1}y} u_0$  and let  $D_t := \Delta_0 + t\partial_y\partial_z - \sqrt{-1}t\partial_z$ . Then

$$(\partial_t + D_t)u = (\partial_t + \Delta_0)e^{\sqrt{-1}y} u_0 + t\partial_z(u_0)(\partial_y - \sqrt{-1})e^{\sqrt{-1}y} = 0.$$

Let  $\phi := e^{\sqrt{-1}y}$ . Then  $u_{\phi,D} = u = e^{\sqrt{-1}y} u_0$  so

$$\beta_4(e^{\sqrt{-1}y}, D, \rho) = \beta_4(1, \mathcal{D}, e^{\sqrt{-1}y} \rho).$$

We show that  $d_4^D = d_5^D$  and complete the proof by computing:

$$\begin{aligned} 0 &= \beta_4(e^{\sqrt{-1}y}, D, \rho) - \beta_4(1, \mathcal{D}, e^{\sqrt{-1}y} \rho) \\ &= \beta_4(e^{\sqrt{-1}y}, D, \rho) - \beta_4(e^{\sqrt{-1}y}, \Delta_0, \rho) \\ &= \sqrt{-1} \int_{C_D} e^{\sqrt{-1}y} (d_4^D - d_5^D) \rho(y) dy. \quad \square \end{aligned}$$

To complete the proof of Theorem 3, it only remains to evaluate the coefficient  $d_3^D$ . This is a one dimensional problem. Let  $M := [0, 1]$ . We make a change of variables on the manifold  $M \times [0, \infty)$  to evaluate the coefficient of  $d_3^D$  that mixes up the space and time variables. Let

$$\begin{aligned} \tilde{z} &:= z + tz^2, & \tilde{t} &:= t, \\ d\tilde{z} &= (1 + 2tz)dz + z^2dt, & d\tilde{t} &= dt, \\ \partial_{\tilde{z}} &= (1 + 2tz)^{-1}\partial_z, & \partial_{\tilde{t}} &= \partial_t - z^2(1 + 2tz)^{-1}\partial_z. \end{aligned}$$

Let  $\tilde{D} := -\partial_{\tilde{z}}^2 + \tilde{z}^2\partial_{\tilde{z}}$ . Let

$$\begin{aligned} D &:= \partial_{\tilde{t}} + \tilde{D} - \partial_t \\ &= -z^2(1 + 2tz)^{-1}\partial_z - (1 + 2tz)^{-2}\partial_z^2 \\ &\quad + 2t(1 + 2tz)^{-3}\partial_z + (z + tz^2)^2(1 + 2tz)^{-1}\partial_z \\ &= \Delta_0 + t\{4z\partial_z^2 + (2z^3 + 2)\partial_z\} + O(t^2). \end{aligned}$$

Let  $\phi = 1$ , let  $\tilde{\rho}$  be identically 1 near  $\tilde{z} = 0$ , let  $\tilde{\rho}$  be identically 0 near  $\tilde{z} = 1$ , and let  $\rho(z; t) := \tilde{\rho}(z + tz^2)$ . We impose Dirichlet boundary conditions; they are preserved by this coordinate transformation. Since  $\rho$  is zero away from the left hand edge of the interval, the principal of not feeling the boundary shows we can neglect the right hand edge. Thus

$$\tilde{\rho}u_{\phi, \tilde{D}}(\tilde{z}; \tilde{t}) = \rho u_{\phi, D}(z; t) + \mathcal{E}(z, t)$$

where the error  $\mathcal{E}$  vanishes to infinite order in  $t$  as  $t \downarrow 0$ . Consequently, we have:

$$\begin{aligned} \beta(\phi, \tilde{D}, \tilde{\rho})(t) &= \int_0^\infty u_{\phi, \tilde{D}}(\tilde{z}; \tilde{t}) \tilde{\rho}(\tilde{z}) d\tilde{z} \\ &= \int_0^\infty u_{\phi, D}(z; t) \tilde{\rho}(z + tz^2) (1 + 2tz) dz + O(t^3) \\ &= \int_0^\infty u_{\phi, D}(z; t) \{ \tilde{\rho}(z) + \tilde{\rho}'(z)tz^2 + \frac{1}{2}\tilde{\rho}''(z)t^2z^4 \} \\ &\quad \cdot (1 + 2tz) dz + O(t^3) \\ &= \beta(\phi, D, \tilde{\rho})(t) + t\beta(\phi, D, \rho'z^2 + 2z\tilde{\rho})(t) \\ &\quad + t^2\beta(\phi, D, \frac{1}{2}\tilde{\rho}''z^4 + 2\rho'z^3)(t) + O(t^3). \end{aligned}$$

We expand both sides and compare the powers of  $t^2$  to see that:

$$\beta_4(1, \tilde{D}, \tilde{\rho}) = \beta_4(1, D, \tilde{\rho}) + \beta_2(1, D, \rho'z^2 + 2z\tilde{\rho}) + \beta_0(1, D, \frac{1}{2}\tilde{\rho}''z^4 + 2\rho'z^3).$$

We have  $D_0 = \Delta_0$ ,  $\mathcal{G}_{mm} = 4z$ , and  $\mathcal{F}_m = 2z^3 + 2$ . Recall that  $\tilde{\rho}$  is identically 1 near 0. We use Lemma 8 to compute  $\beta_n(1, D, \cdot)$ :

$$\begin{aligned} \beta_4(1, D, \tilde{\rho}) &= \int_{C_D} (4d_3^D - \frac{5}{8})\phi\tilde{\rho} \\ \beta_2(1, D, \rho'z^2 + 2z\tilde{\rho}) &= \int_{C_D} (-2)\phi\tilde{\rho} \\ \beta_0(1, D, \frac{1}{2}\tilde{\rho}'' + 2\rho'z^3) &= 0 \\ \beta_4(1, \tilde{D}, \tilde{\rho}) &= \int_{C_D} (4d_3^D - \frac{5}{8} - 2)\phi\tilde{\rho}. \end{aligned}$$

The operator  $\tilde{D}$  is autonomous. It is not self-adjoint so we must use a bit of care in applying Theorem 1. We use the formulas of remark 2. We complete the proof of Theorem 3 by computing:

$$\begin{aligned} \tilde{D} &= -\partial_{\tilde{z}}^2 + \tilde{z}^2\partial_{\tilde{z}}, & \tilde{D}^* &= -\partial_{\tilde{z}}^2 - \tilde{z}^2\partial_{\tilde{z}} - 2\tilde{z}, \\ \omega_m(\tilde{D}) &= -\frac{1}{2}\tilde{z}^2, & \omega_m(\tilde{D}^*) &= \frac{1}{2}\tilde{z}^2, \\ E(\tilde{D}) &= \tilde{z} - \frac{1}{4}\tilde{z}^4, & E(\tilde{D}^*) &= 2\tilde{z} - \tilde{z} - \frac{1}{4}\tilde{z}^4, \end{aligned}$$

Consequently, we compute that

$$\beta_4(\phi, \tilde{D}, \tilde{\rho}) = \int_{C_D} \{ \frac{1}{2}(-2\tilde{z}\rho);_m\phi + \frac{1}{8}(\tilde{z});_m\tilde{\rho}\phi \}.$$

This yields the relation

$$-1 + \frac{1}{8} = 4d_3^D - \frac{5}{8} - 2 \text{ so } d_3^D = \frac{7}{16}. \quad \square$$

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