Asymptotics of Solutions of Differential Equations on Manifolds with Cusps

Y. Rebahi

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Contents

Introduction

Partial differential equations on closed manifolds with cuspidal singularities were investigated by Schulze and Tarkhanov in [7].

Let X be a C^∞ compact manifold and $\bar{\mathbb{R}}_+$ the set $\mathbb{R}_+ \cup \{0\}$. Denote by $(x_1, ..., x_n)$ local coordinates on X and by t the coordinate on $\bar{\mathbb{R}}_+$.

Close to a cuspidal singularity, a linear differential operator has the following form:

$$P(x,t,D_x,D_t) = (\delta'(t))^m \sum_{0\le\alpha\le m} a_{m-\alpha}(x,t,D_x)\left(\frac{1}{\delta'(t)}D_t\right)^\alpha, \qquad m \in \mathbb{N}, \quad (1)$$

where $\delta(t)$ is a diffeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}$, such that $\delta'(t) < 0$ for all $t \in \mathbb{R}_+$, and the coefficients $a_{m-\alpha}(x,t,D_x)$ are smooth up to $t = 0$ [7].

In case $\delta'(t)$ is equal $-t^{-k-1}$ near $t = 0$, with $k = 0, 1, \ldots$, the equality (1) gives typical differential operators on manifolds with power-like cusps. For the particular value $k = 0$, we get the general form of linear operators in a neighbourhood of a conical singularity (so-called Fuchs-type operators).

Our purpose is to study the elliptic equations

$$\left(\frac{1}{\delta'(t)}D_t\right)^m u + \sum_{0\le\alpha\le m-1} a_{m-\alpha}(x,t,D_x)\left(\frac{1}{\delta'(t)}D_t\right)^\alpha u = f(t) \qquad (2)$$

and to derive asymptotic formulas for solutions under reasonable assumptions on the coefficients $a_{m-\alpha}(x, t, D_x)$.

In this paper we will assume that the coefficients are just functions of t with complex values. The general case where the coefficients are differential operators on X will be investigated in a forthcoming paper.

On the other hand, we will consider a more general situation. We will require throughout this work that the limits of the functions $a_1(t), ..., a_m(t)$ exist when $t \longrightarrow 0$. The situation of cusps will appear as a particular case of the previous one.

It is more convenient to make the change of variables $s = \delta(t)$ to reduce (2) to the equation

$$D_s^m u + \sum_{0 \le \alpha \le m-1} b_{m-\alpha}(s)\, D_s^\alpha u = f(s) \tag{3}$$

and to describe afterwards the asymptotic behaviour of the solution u when $s \longrightarrow +\infty$.

Under this form, we can apply some results of Maz'ya and Plamenevskii [4] and Plamenevskii [5] about asymptotics for solutions of differential equations.

The solutions of (3) are considered in some weighted spaces of functions which show that the solution of the homogeneous equation is smooth everywhere, except at the cuspidal point.

In fact, via a special change of variable we reduce the above equation to a linear system

$$D_s U - A(s) U = F \tag{4}$$

and we derive asymptotic formulas for the solution $U(s)$. We will denote by A the limit of the matrix-valued function $A(s)$ as s tends to infinity.

Roughly speaking, Section 1 presents the necessary material to apply Theorem 3.1 of [5]. Some of the results we give here have analogues in [4] or [5]. However, since our situation is more explicit, we preferred – when we had the occasion – to proceed in a different manner.

It is worth noting that Subsection 1.3 is devoted to the concept of eigenchains and their meaning in our situation. We also give the relationship between the smoothness of the entries of the matrix $A(s)$ and the smoothness of the eigenchains.

In Subsection 2.1 we derive asymptotics for the solution $U(s)$ of (4) when $U(s)$ belongs to some weighted Sobolev space $\mathcal{W}_{m,\gamma}$, $\gamma \in \mathbb{R}$.

If $-\gamma$ does not coincide with the imaginary part of one of the eigenvalues of the matrix A, this problem is quite classical and the proof is given in Proposition 2.1.

On the other hand, if the line $\Im\lambda = -\gamma$ contains some eigenvalues of A, the situation becomes more difficult. In this case we derive two different asymptotic formulas for the solution $U(s)$. The first of the two is given by Proposition 2.2; it uses a "modified" version of Proposition 2.1. As for the second one, we apply Theorem 3.1 of Plamenevskii [5].

Note that instead of the lines $\Im\lambda = c$ we could consider those given by the equation $\Re\lambda = c$, since both contain only a finite number of eigenvalues of A.

In Subsection 2.2 we explain through the change of variable $s = \delta(t)$ how to apply the results of Subsection 2.1 to differential equations defined in a neighbourhood of the origin. The coefficients are reguired to meet some reasonable conditions near $t = 0$. Subsection 2.3 treats the case of coefficients smooth up to $t = 0$. It is proved in this situation that any solution of the homogeneous equation is the sum of several canonical singular terms and a function which behaves better close to the singularity $t = 0$. In particular, asymptotic formulas for solutions of Fuchs-type equations are derived.

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1 Some preliminary results

In this section we establish a number of results thanks to which Proposition 2.3 will be an immediate consequence of Theorem 3.1 of Plamenevskii [5]. In presenting them we follow Maz'ya and Plamenevskii [4], except the third part where we give another point of view.

1.1 Differential equations with constant coefficients

Consider the equation

$$P(D_s) = D_s^m u + \sum_{j=0}^{m-1} a_{m-j} D_s^j u = f, \qquad D_s = \frac{1}{i}\frac{d}{ds}, \tag{1.1}$$

for $s \in \mathbb{R}$, where $a_1, ..., a_m$ are some complex numbers and $a_m \neq 0$. Denote by $P(\lambda)$ the polynomial

$$P(\lambda) = \lambda^m + \sum_{j=0}^{m-1} a_{m-j}\lambda^j.$$

Since the operator $P(D_s)$ is elliptic, there exists $r > 0$ such that

$$\sum_{q=0}^{m} |\lambda|^q \leq c\ |P(\lambda)| \qquad \text{for all } \ |\lambda| > r. \tag{1.2}$$

Let us introduce the space $W_{m,\gamma}$ of functions of s with the norm

$$< u >_{m,\gamma}^2 = \int_{\mathbb{R}} e^{-2\gamma s} \sum_{j=0}^{m} \left|D_s^j u\right|^2 ds.$$

We first assume that $f \in W_{0,\gamma}$, and consider equation (1.1) in the space $W_{m,\gamma}$.

Thanks to the change of variables

$$\begin{array}{rcl} u & = & u_m, \\ u_{m-1} & = & D_s u_m + a_1 u_m, \\ \dots & \dots & \dots \\ u_1 & = & D_s u_2 + a_{m-1} u_m \end{array} \tag{1.3}$$

equation (1.1) is reduced to the system

$$D_s U - AU = F(s), \qquad s \in \mathbb{R}, \tag{1.4}$$

where $U = (u_1, ..., u_m)$, $F = (f, 0, ..., 0)$, and A is the following matrix:

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_m \\ 1 & 0 & \dots & 0 & -a_{m-1} \\ 0 & 1 & \dots & 0 & -a_{m-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_1 \end{pmatrix}.$$

Let us note that to reduce the equation (1.1) to the system (1.4), we could apply the usual change of variables in which the successive derivatives of a solution are regarded as new unknowns. When the coefficients $a_1, ..., a_m$ are differential operators, the usual change of variables - unlike to the change of variables that we use - fails to have an important property, namely the boundedness (in the terminology of [2]) of the matrix A.

We shall also introduce the spaces $\mathcal{W}_{m,\gamma}$ and $\mathcal{V}_{m-1,\gamma}$ of vector-valued functions with the norms

$$\begin{aligned} ||U||^2_{m,\gamma} &= \int_{\mathbb{R}} e^{-2\gamma s} \left(\sum_{q=0}^{m} |D_s^q u_m|^2 + \sum_{j=1}^{m-1} \sum_{q=0}^{j} |D_s^q u_j|^2 \right) ds, \\ |||F|||^2_{m-1,\gamma} &= \int_{\mathbb{R}} e^{-2\gamma s} \left(\sum_{j=1}^{m} \sum_{q=0}^{j-1} |D_s^q f_j|^2 \right) ds, \end{aligned}$$

respectively.

Lemma 1.1 *Let $\gamma \neq -\gamma_k$, $k = 1, ..., n$, where γ_k are the imaginary parts of the eigenvalues of the matrix A. Then, equation (1.4) has a unique solution $U \in \mathcal{W}_{m,\gamma}$ for any right-hand side $F = (f_1, ..., f_m) \in \mathcal{V}_{m-1,\gamma}$. Moreover, we have*

$$||U||_{m,\gamma} \leq c \; |||F|||_{m-1,\gamma},$$

with c a constant independent of F, and the solution is given by

$$U(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\gamma}^{+\infty - i\gamma} e^{i\lambda s} \, (\lambda - A)^{-1} \, \tilde{F}(\lambda) \, d\lambda$$

where $\tilde{F}(\lambda)$ is the Fourier transform of $F(s)$.

Proof. First, we apply the Fourier-Laplace transform to the system (1.4),

$$\tilde{U}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda s} \, U(s) \, ds, \qquad \Im\lambda = \gamma.$$

The change of variables (1.3) yields

$$\begin{aligned} \lambda \tilde{u_1} + a_m \tilde{u_m} &= \tilde{f_1}, \\ \lambda \tilde{u_j} - \tilde{u_{j-1}} + a_{m-j+1} \tilde{u_m} &= \tilde{f_j}, \qquad \text{for } \; 2 \leq j \leq m. \end{aligned}$$

By iteration, we arrive at the formulas

$$\lambda^q \tilde{u}_j + \left(\sum_{k=1}^{q} a_{m-j+k} \lambda^{q-k} \right) \tilde{u}_m - \tilde{u}_{j-q} \;\; = \;\; \sum_{k=0}^{q-1} \tilde{f}_{j-k} \, \lambda^{q-k-1},$$

$$\begin{aligned}
\lambda^j \tilde{u}_j + \left(\sum_{k=1}^{j} a_{m-j+k} \lambda^{j-k} \right) \tilde{u}_m &= \sum_{k=0}^{j-1} \tilde{f}_{j-k}\, \lambda^{j-k-1}, \\
P(\lambda) \tilde{u}_m &= \sum_{k=1}^{m} \tilde{f}_k\, \lambda^{k-1},
\end{aligned}$$

the first formula being valid for $q = 1, \ldots, j-1$.

Taking into account the estimate (1.2) and the above relations, we obtain easily

$$\begin{aligned}
&\left(\sum_{q=0}^{m} |\lambda|^q \right) |\tilde{u}_m| + \sum_{j=1}^{m-1} |\lambda|^j\, |\tilde{u}_j| \\
&\quad \le c_1\, |P(\lambda) \tilde{u}_m| + \sum_{j=1}^{m-1} \sum_{n=1}^{j} |\tilde{f}_n|\, |\lambda|^{n-1} + \left(\sum_{j=1}^{m-1} \sum_{n=0}^{j-1} |a_{m-n}|\, |\lambda|^n \right) |\tilde{u}_m| \\
&\quad \le c_1\, |P(\lambda) \tilde{u}_m| + c_2 \sum_{j=1}^{m-1} |\tilde{f}_j|\, |\lambda|^{j-1} + c_3 \left(\sum_{j=0}^{m-2} |a_{m-j}|\, |\lambda|^j \right) |\tilde{u}_m| \\
&\quad \le c_4 \sum_{j=1}^{m} |\tilde{f}_j|\, |\lambda|^{j-1},
\end{aligned}$$

the constant c_4 being independent of λ varying over any line in the complex plane where $P(\lambda) \neq 0$. Hence it follows that

$$\begin{aligned}
||U||^2_{m,\gamma} &= \sum_{q=0}^{m} ||e^{-\gamma s} D_s^q u_m||^2_{L^2} + \sum_{j=1}^{m-1} \sum_{q=0}^{j} ||e^{-\gamma s} D_s^q u_j||^2_{L^2} \\
&= \sum_{q=0}^{m} || \sum_{k=0}^{q} C_q^k \gamma^{q-k} D_s^k (e^{-\gamma s} u_m) ||^2_{L^2} + \sum_{j=1}^{m-1} \sum_{q=0}^{j} || \sum_{k=0}^{q} C_q^k \gamma^{q-k} D_s^k (e^{-\gamma s} u_j) ||^2_{L^2} \\
&\le \sum_{q=0}^{m} \sum_{k=0}^{q} C_q^k |\gamma|^{q-k} |\xi|^{2k} ||\widetilde{e^{-\gamma s} u_m}||^2_{L^2} + \sum_{j=1}^{m-1} \sum_{q=0}^{j} \sum_{k=0}^{q} C_q^k |\gamma|^{q-k} |\xi|^{2k} ||\widetilde{e^{-\gamma s} u_j}||^2_{L^2} \\
&= \int_{\mathbb{R}} \left(\sum_{q=0}^{m} (|\xi|^2 + |\gamma|)^q |\tilde{u_m}(\xi - i\gamma)|^2 + \sum_{j=1}^{m-1} \sum_{q=0}^{j} (|\xi|^2 + |\gamma|)^q |\tilde{u_j}(\xi - i\gamma)|^2 \right) d\xi \\
&\le \mathrm{const}(\gamma) \int_{-\infty - i\gamma}^{+\infty - i\gamma} \left(\sum_{q=0}^{m} |\lambda|^{2q} |\tilde{u_m}|^2 + \sum_{j=1}^{m-1} \sum_{q=0}^{j} |\lambda|^{2q} |\tilde{u_j}|^2 \right) d\lambda \\
&\le \mathrm{const}(\gamma) \int_{-\infty - i\gamma}^{+\infty - i\gamma} \left(\left(\sum_{q=0}^{m} |\lambda|^q \right) |\tilde{u_m}| + \sum_{j=1}^{m-1} \sum_{q=0}^{j} |\lambda|^q |\tilde{u_j}| \right)^2 d\lambda \\
&\le \mathrm{const}\; |||F|||^2_{m-1,\gamma},
\end{aligned}$$

where the constant does not depend on F.

This implies that the solution U belongs to the space $\mathcal{W}_{m,\gamma}$ and has the form

$$U(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\gamma}^{+\infty - i\gamma} e^{i\lambda s}\, (\lambda - A)^{-1}\, \tilde{F}(\lambda)\, d\lambda,$$

as desired.

$\square$

Let T be a real number and let $\eta_T(s)$ be an infinitely differentiable non-negative function such that $\eta_T(s) = 0$ for $s < T$ and $\eta_T(s) = 1$ for $s > T + 1$. The following lemma is useful in the sequel. It is a particular case of Lemma 2.3 of [4].

Lemma 1.2 *Let $F(s)$ be a vector-valued function defined on $(T, +\infty)$ and such that $\eta_T F \in \mathcal{V}_{m-1,\gamma'}$ for some $\gamma' \in (-\gamma_k, -\gamma_{k-1})$. Let $U(s)$ be a solution of the equation*

$$D_s U(s) - AU(s) = F(s), \qquad s > T, \tag{1.5}$$

such that $\eta_T U \in \mathcal{W}_{m,\gamma}$, where $\gamma \in (-\gamma_k, \gamma')$. Then $\eta_T U$ belongs to the space $\mathcal{W}_{m,\gamma'}$.

1.2 Differential equations with variable coefficients

Consider the equation with variable coefficients

$$P(s, D_s) = D_s^m u + \sum_{j=0}^{m-1} a_{m-j}(s) D_s^j u = f(s), \qquad s \in \mathbb{R}. \tag{1.6}$$

The coefficients $a_j(s)$, $j = 1, ..., m$, are assumed to be continuous complex-valued functions of $s \in \mathbb{R}$, satisfying

$$\lim_{s \longrightarrow +\infty} a_j(t) = a_j, \qquad j = 1, ..., m, \tag{1.7}$$

where $a_1, ..., a_m$ are constant and $a_m \neq 0$.

We shall reduce (1.6) to a first order system. This equation can be written as

$$P(s, D_s)u(s) = D_s^m u(s) + \sum_{j=0}^{m-1} D_s^j \left(b_{m-j} u\right)(s) = f(s), \qquad s \in \mathbb{R},$$

where

$$b_j(s) = a_j(s) + \alpha_{j,1} D_s a_{j-1}(s) + \ldots + \alpha_{j,j-1} D_s^{j-1} a_1(s)$$

and $\alpha_{j,k}$ are some numbers.

In fact (1.6) is equivalent to the system of equations

$$\left\{ \begin{array}{rcl} u_j(s) & = & D_s u_{j+1}(s) + b_{m-j}(s)\, u_m(s), \quad j = 1, ..., m-1, \\ D_s u_1(s) & = & -b_m(s) u_m(s) + f(s). \end{array} \right.$$

We are going to consider the system

$$D_s U - A(s) U = F(s), \qquad s \in \mathbb{R}, \tag{1.8}$$

where $F(s) = (f_1(s), ..., f_m(s))$ and

$$A(s) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -b_m \\ 1 & 0 & \ldots & 0 & -b_{m-1} \\ 0 & 1 & \ldots & 0 & -b_{m-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -b_1 \end{pmatrix}. \tag{1.9}$$

If $F(s) = (f(s), 0, ..., 0)$, equation (1.6) is equivalent to this system.

Moreover, a solution U of (1.8) belongs to $\mathcal{W}_{m,\gamma}$ if and only if the corresponding solution u of (1.6) belongs to $W_{m,\gamma}$.

Lemma 1.3 *If, for all $s \in (-\infty, +\infty)$,*

$$|D_s^q\,(b_j(s) - a_j)\,| \le \delta, \qquad q = 0, ..., m-j; \quad j = 1, ..., m,$$

where δ is a sufficiently small positive constant, then the system (1.8) has a unique solution $U \in \mathcal{W}_{m,\gamma}$, for each right-hand side $F \in \mathcal{V}_{m-1,\gamma}$ with $\gamma \in (-\gamma_k, -\gamma_{k-1})$.

Proof. The system (1.8) can be written as

$$U - (D_s - A)^{-1}\,(A(s) - A)\,U = (D_s - A)^{-1}\,F,$$

where the operator $(D_s - A)^{-1}$ is defined in Lemma 1.1. We further obtain

$$\begin{aligned}
&|||\,(A(s) - A)\,U\,|||^2_{m-1,\gamma} \\
&= \int_{\mathbb{R}} e^{-2\gamma s} \sum_{j=1}^{m}\sum_{q=0}^{j-1} |D_s^q\,((a_{m+1-j} - b_{m+1-j}(s))u_m)\,|^2 ds \\
&= \int_{\mathbb{R}} e^{-2\gamma s} \sum_{j=1}^{m}\sum_{q=0}^{j-1} |\sum_{k=0}^{q} C_q^k\, D_s^{q-k}\,(a_{m+1-j} - b_{m+1-j}(s))\,D_s^k u_m|^2\,ds \\
&\le c\,\delta^2 \int_{\mathbb{R}} e^{-2\gamma s} \sum_{j=1}^{m}\sum_{q=0}^{j-1} |\sum_{k=0}^{q} D_s^k u_m|^2\,ds \\
&\le C\,\delta^2\,||U||^2_{m,\gamma},
\end{aligned}$$

where C is a positive constant independent of δ and U. It means that

$$||A(s) - A||_{\mathcal{W}_{m,\gamma}\to\mathcal{V}_{m-1,\gamma}} \le \sqrt{C}\,\delta.$$

Since δ is sufficiently small, we can choose it such that

$$||(D_s - A)^{-1}||_{\mathcal{V}_{m-1,\gamma}\to\mathcal{W}_{m,\gamma}}\;||A(s) - A||_{\mathcal{W}_{m,\gamma}\to\mathcal{V}_{m-1,\gamma}} < 1.$$

Hence we deduce that equation (1.8) has a unique solution in $\mathcal{W}_{m,\gamma}$.

□

Proposition 1.4 *Let the following conditions hold:*

1. $\lim_{s\to+\infty} |D_s^q\,(b_j(s) - a_j)\,| = 0$, *for $q = 0, ..., m-j$ and $j = 1, ..., m$;*
2. *$F(s)$ is a vector-valued function on $(T, +\infty)$, such that $\eta_T F \in \mathcal{V}_{m-1,\gamma'}$ for some $\gamma' \in (-\gamma_k, -\gamma_{k-1})$.*

If moreover $\gamma \in (-\gamma_k, \gamma')$ and $U(s)$ is a solution of the equation

$$D_s U - A(s)U = F(s), \qquad s > T,$$

such that $\eta_T U \in \mathcal{W}_{m,\gamma}$, then $\eta_T U$ belongs to $\mathcal{W}_{m,\gamma'}$.

For the proof, see Lemma 3.2 of Maz'ya and Plamenevskii [4].

1.3 Spectral decomposition

In this section we assume that the functions $b_j(s)$, $j = 0, ..., m$, and the complex numbers a_j, $j = 0, ..., m$, satisfy the conditions of Proposition 1.4.

Denote by $\lambda_1(s), ..., \lambda_I(s)$ the eigenvalues of the matrix $A(s)$ and by $\alpha_1, ..., \alpha_I$ their multiplicities, respectively. We assume that these multiplicities do not depend on s, and we denote by $(\lambda - A(s))^{-1}$ - when it makes sense - the resolvent of $A(s)$.

Let Γ_i be a closed Jordan curve around the eigenvalue $\lambda_i(s)$ of $A(s)$. We define the spectral projection associated to $\lambda_i(s)$ by the matrix

$$P_i(s) = \frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - A(s))^{-1} \, d\lambda.$$

It is well-known that $(\lambda - A(s))^{-1}$ can be written in the form

$$(\lambda - A(s))^{-1} = \sum_{i=1}^{I} \sum_{k=0}^{\alpha_i - 1} (\lambda - \lambda_i(s))^{-k-1} (A(s) - \lambda_i(s)I)^k \, P_i,$$

with $(A(s) - \lambda(s)I)^0 = I$, the identity matrix.

Let us denote by

$$(A(s) - \lambda_i(s)I)^{\alpha_i - 1} v, \quad \ldots, \quad (A(s) - \lambda_i(s)I) \, v, \quad v$$

a basis of the kernel of the morphism $(A(s) - \lambda(s)I)^{\alpha_i}$ and by $\phi^i_k(s)$ the vectors

$$\phi^i_k(s) = (A(s) - \lambda_i(s)I)^{\alpha_i - 1 - k} \, v, \qquad k = 0, ..., \alpha_i - 1.$$

This set of vectors is called an *eigenchain* corresponding to the eigenvalue $\lambda_i(s)$.

Let $\psi^i_{\alpha_i - 1}(s)$ be an orthogonal vector to the family $\left(\phi^i_k(s)\right)_{k=0,\ldots,\alpha_i-1}$. It means that

$$\left(\psi^i_{\alpha_i - 1}(s), \phi^i_k(s)\right) = \delta_{\alpha_i - 1, \alpha_i - 1 - k},$$

where $\delta_{i,k}$ is the Kronecker symbol.

Denote by $\psi^i_{\alpha_i - 1 - k}(s)$, $j = 0, ..., \alpha_i - 1$, the vectors given by

$$\psi^i_{\alpha_i - 1 - k}(s) = \left(A^*(s) - \bar{\lambda}_i(s)I\right)^j \, \psi_{\alpha_i - 1}(s), \qquad k = 0, ..., \alpha_i - 1,$$

where $A^*(s)$ is the adjoint matrix of $A(s)$ and $\bar{\lambda}_i(s)$ is the conjugate of $\lambda_i(s)$. Note that

$$\begin{aligned}
\left(\phi^i_k(s), \psi^i_{\alpha_i - 1 - l}(s)\right) &= \left((A(s) - \lambda_i(s)I) \, \phi^i_k(s), \psi^i_{\alpha_i - l}(s)\right) \\
&= \left(\phi^i_{k-1}(s), \psi^i_{\alpha_i - l}(s)\right) \\
&= \ldots \\
&= \left(\phi^i_{k-l}(s), \psi^i_{\alpha_i - 1}(s)\right) \\
&= \delta_{k,l}.
\end{aligned}$$

Hence the set $\left\{\psi^i_{\alpha_i - 1}(s), ..., \psi^i_0(s)\right\}$ is an orthogonal system to the system of vectors $\left(\phi^i_k(s)\right)_{k=0,\ldots,\alpha_i-1}$.

Since the operator P_i maps a vector-valued function $F = (f_1, ..., f_m)$ into a linear combination of vectors $\phi_0^i(s), ..., \phi_{\alpha_i - 1}^i(s)$, it is easily verified that

$$(A(s) - \lambda_i(s)I)^N \, P_i F = \sum_{k=0}^{\alpha_i - 1 - N} c_k^i(s) \, \phi_k^i(s). \tag{1.10}$$

The coefficients $c_k^i(s)$ depend of course on F.

Therefore, if we multiply the equality (1.10) by the vector $\psi_{\alpha_i - k - 1}^i(s)$, we get

$$\begin{aligned} c_k^i(s) &= \left((A(s) - \lambda_i(s)I)^{N-1} P_i F, \, (A^*(s) - \bar{\lambda}_i(s)I)\psi_{\alpha_i - k - 1}^i(s)\right) \\ &= \left((A(s) - \lambda_i(s)I)^{N-1} P_i F, \, \psi_{\alpha_i - k - 2}^i(s)\right) \\ &= \dots \\ &= \left(F, \psi_{\alpha_i - 1 - k - N}^i(s)\right). \end{aligned}$$

Now consider the problem

$$\lambda U - A(s)U = F(s), \qquad s > T, \tag{1.11}$$

where $A(s)$ is the matrix (1.9) and T is sufficiently large.

Note that the eigenvalues of $A(s)$ are exactly the roots of the polynomial

$$P(s, \lambda) = \lambda^m + \sum_{j=0}^{m-1} b_{m-j}(s) \, \lambda^j.$$

Since $a_m \neq 0$, the value $\lambda = 0$ is regular for both the operators $(\lambda - A)^{-1}$ and $(\lambda - A(s))^{-1}$. Furthermore, any eigenvalue $\lambda_i(s)$ of $A(s)$ is bounded below by a non-negative constant. Thus, there exists $0 < c < 1$ such that

$$|\lambda_i(s)| > c \quad \text{for all} \quad s > T.$$

Consider now the polynomial

$$D(s, \lambda) = \lambda^{m-1} + \sum_{j=1}^{m-1} \frac{j}{m} \, b_{m-j}(s) \, \lambda^{j-1}.$$

We have

$$\begin{aligned} |D(s, \lambda_i)| &= c^{m-1} \left| (\lambda_i/c)^{m-1} + \sum_{j=1}^{m-1} \frac{j}{m} c^{m-j} \, b_{m-j}(s) \, (\lambda_i/c)^{j-1} \right| \\ &\geq c^{m-1} \left(|\lambda_i/c|^{m-1} - \sum_{j=1}^{m-1} \frac{j}{m} c^{m-j} \, |b_{m-j}(s)| \, |\lambda_i/c|^{j-1} \right) \\ &\geq \frac{c^{m-1}}{2} \, |\lambda_i/c|^{m-1}, \end{aligned}$$

where

$$\begin{aligned} c &< \min\left(1, \frac{1}{(m-1)\,M}\right), \\ M &= \sup_{\substack{j=1,\dots,m-1 \\ s \in (T, +\infty)}} |b_j(s)|. \end{aligned}$$

It follows that

$$
\begin{aligned}
1+|\lambda_i/c|+\ldots+|\lambda_i/c|^{m-1} &\leq m\ |\lambda_i c|^{m-1} \\
&\leq \frac{2m}{c^{m-1}}\,|D(s,\lambda_i)|.
\end{aligned}
$$

Now, if we differentiate the equation $P(s,\lambda_i(s))=0$ with respect to s, we obtain

$$
m\ D(s,\lambda_i)\,D_s\lambda_i(s) = -\sum_{j=0}^{m-1} D_s b_{m-j}(s)\,\lambda_i^j
$$

whence

$$
\begin{aligned}
|D_s\lambda_i(s)| &\leq \frac{2}{c^{m-1}}\,(\,1/\sum_{j=0}^{m-1}|\lambda_i/c|^j\,)\,(\,\sum_{j=0}^{m-1}|D_s b_{m-j}(s)|\,|\lambda_i|^j\,) \\
&\leq \frac{2}{c^{m-1}}\sup_{j=1,\ldots,m}|D_s b_j(s)|.
\end{aligned}
$$

The second derivative of $P(s,\lambda_i(s))=0$ gives

$$
\begin{aligned}
&m\ D(s,\lambda_i)\,D_s^2\lambda_i(s) \\
&\quad = -\,(\,m(m-1)\,\lambda_i^{m-2}+\sum_{j=2}^{m-1} j(j-1)\,b_{m-j}(s)\,\lambda_i^{j-2}\,)\,\left(D_s\lambda_i(s)\right)^2 \\
&\qquad -\,(\,\sum_{j=1}^{m-1} j\,D_s b_{m-j}(s)\,\lambda_i^{j-1}\,)\,D_s\lambda_i(s)-\sum_{j=0}^{m-1} D_s^2 b_{m-j}(s)\,\lambda_i^j.
\end{aligned}
$$

Proceeding as above, we get

$$
\begin{aligned}
|D_s^2\lambda_i(s)| &\leq C\left(\sup_{j=1,\ldots,m-1}|D_s b_j(s)|^2+\sup_{j=1,\ldots,m-1}|D_s^2 b_j(s)|\right) \\
&\leq C\sum_{n_1m_1+n_2m_2=2}\prod_{i=1}^{2}\sup_{j=1,\ldots,m-1}|D_s^{n_i}b_j(s)|^{m_i},
\end{aligned}
$$

the constant C being independent of s.

In fact, the derivative of order μ with respect to s of the eigenvalue $\lambda_i(s)$ satisfy the equality

$$
\begin{aligned}
&m\ D(s,\lambda_i)\,D_s^\mu\lambda_i(s) \\
&\quad = \sum_{\alpha_1\beta_1+\ldots+\alpha_{\mu-1}\beta_{\mu-1}+\beta_\mu=\mu}\left(D_s^{\alpha_1}\lambda_i(s)\right)^{\beta_1}\ldots\left(D_s^{\alpha_{\mu-1}}\lambda_i(s)\right)^{\beta_{\mu-1}}\ P^{(\beta_\mu)}(s)
\end{aligned}
$$

where $P^{(\beta_\mu)}(s)$ is a polynomial of order at most $m-1$ and whose coefficients involve derivatives of order β_μ of $b_j(s)$. By the same argument we get

$$
|D_s^\mu\lambda_i(s)|\leq C\ \sigma_\mu(s),\qquad \mu=1,\ldots,m,
$$

where

$$
\sigma_\mu(s)=\sum_{n_1m_1+\ldots+n_\mu m_\mu=\mu}\prod_{i=1}^{\mu}\sup_{j=1,\ldots,m-1}|D_s^{n_i}b_j(s)|^{m_i}.
$$

Let h be a vector independent of s, such that $(A(s) - \lambda_i(s)I)^{\alpha_i - 1} h \neq 0$ where α_i is the multiplicity of $\lambda_i(s)$. Denote by $\phi_k^i(s)$ the vectors

$$\phi_k^i(s) = (A(s) - \lambda_i(s)I)^{\alpha_i - 1 - k}\ h, \qquad j = 0, ..., \alpha_i - 1.$$

By the above estimate, the vector $\phi_{\alpha_i - 2}^i(s) = (A(s) - \lambda_i(s)I)\, h$ satisfies

$$\begin{aligned} |D_s^\mu \phi_{\alpha_i-2}^i(s)| &\leq c\ ||D_s^\mu\,(A(s) - A)\,|| \\ &\leq C\ \sigma_\mu(s). \end{aligned}$$

If $k = \alpha_i - 3$, we have

$$D_s^\mu \phi_{\alpha_i-3}^i(s) = \sum_{j=0}^{\mu} C_j^\mu\ D_s^j\ (A(s) - \lambda_i(s))\ D_s^{\mu-j}\, \phi_{\alpha_i-2}^i(s),$$

and so

$$\begin{aligned} |D_s^\mu \phi_{\alpha_i-3}^i(s)| &\leq c \sum_{n_1+n_2=\mu} \sigma_{n_1}(s)\ \sigma_{n_2}(s) \\ &\leq C\ \sigma_\mu(s). \end{aligned}$$

Repeating this argument, we obtain

$$|D_s^\mu \phi_k^i(s)| \leq C\ \sigma_\mu(s),$$

thus arriving at the following lemma.

Lemma 1.5 *Consider the problem $\lambda U - A(s)U = F(s)$, for $s > T$, where $A(s)$ is the matrix* (1.9) *and T is sufficiently large. Let $\lambda_i(s)$ be an eigenvalue of $A(s)$ to which there corresponds the eigenchain*

$$\phi_0^i(s),\ \phi_1^i(s),\ \ldots,\ \phi_{\alpha_i-1}^i(s), \qquad 1 \leq i \leq I.$$

Assume that the conditions on the coefficients $b_j(s)$ of Proposition 1.4 hold. Then we have the estimates

$$\begin{aligned} |D_s^\mu \lambda_i(s)| &\leq C\ \sigma_\mu(s), \qquad i = 1, ..., I;\quad \mu = 1, ..., m; \\ |D_s^\mu \phi_k^i(s)| &\leq C\ \sigma_\mu(s), \qquad i = 1, ..., I;\quad k = 0, ..., \alpha_i - 1. \end{aligned}$$

2 Asymptotic behaviour of solutions

We begin this section with the study of the asymptotic behaviour of solutions of equation (2.1). The results we obtain will be applied - via a change of variable - to get asymptotics for the solutions of equation (2.2). Hence the asymptotic behaviour of solutions of elliptic equations close to a cusp (see Corollary 2.8) will follow as a consequence of the above results.

2.1 Asymptotics for $s \to +\infty$

Let us consider the equation

$$D_s U - A(s)U = 0, \qquad s > T; \tag{2.1}$$

we confine ourselves to solutions U satisfying $\eta_T U \in \mathcal{W}_{m,\mu}$.

We have two situations. The first of the two is when the lines $\Im\lambda = -\gamma$ and $\Im\lambda = -\mu$ limiting the strip $-\gamma < \Im\lambda < -\mu$ do not involve any eigenvalue of the matrix A. The second situation appears if one of the above lines contains an eigenvalue of A. In this case we will derive two different asymptotic formulas for the solution $U(s)$. The first one contains eigenvalues of the limiting matrix A while the second one involves eigenvalues of the perturbed matrix $A(s)$.

Proposition 2.1 *Assume that:*

1. *Condition 1 of Proposition 1.4 is fulfilled.*
2. *N eigenvalues $\lambda_1, ..., \lambda_N$ (counting the multiplicities) to which correspond the eigenchains ϕ_k^i, $k = 0, ..., \alpha_i - 1$; $i = 1, ..., N$, are located in the strip $-\gamma < \Im\lambda_i < -\mu$.*
3. *There are no eigenvalues of the matrix A on the lines $\Im\lambda = -\gamma$ and $\Im\lambda = -\mu$.*

Then, any solution $U(s)$ of the equation $D_s U - A(s)U = 0$, $s > T$, such that $\eta_T U \in \mathcal{W}_{m,\mu}$ can be written as

$$U(s) = \sum_{i=1}^{N} \sum_{k=0}^{\alpha_i - 1} \sum_{l=0}^{\alpha_i - 1 - k} \sum_{n=l}^{\alpha_i - 1 - k} e^{i\lambda_i s} \frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!} \frac{(is)^l}{l!} \phi_k^i + Z(s),$$

where $\eta_T Z \in \mathcal{W}_{m,\mu}$.

Proof. We first consider the equation

$$D_s U - AU = F$$

where F is a function with support in $\mathbb{R}_+$.

Assume that in the strip $-\gamma < \Im\lambda_i < -\mu$ there exist only N eigenvalues $\lambda_1, ..., \lambda_N$ of A with multiplicities $\alpha_1, ..., \alpha_N$, respectively. Assume also that there are no eigenvalues of A on the lines $\Im\lambda = -\gamma$ and $\Im\lambda = -\mu$.

If F belongs to the space $\mathcal{V}_{m-1,\mu}$ (and therefore to $\mathcal{V}_{m-1,\gamma}$), then the solution U of our equation is unique in each of the spaces $\mathcal{W}_{m,\mu}$ and $\mathcal{W}_{m,\gamma}$. Moreover, the Fourier transform $\tilde{F}(\lambda)$ of $F(s)$ is analytic in the strip $-\gamma < \Im\lambda < -\mu$. Using the Cauchy formula, we get

$$\begin{aligned}
U(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\gamma}^{+\infty - i\gamma} e^{i\lambda s} (\lambda - A)^{-1} \tilde{F}(\lambda)\, d\lambda \\
&= \frac{1}{\sqrt{2\pi}} \lim_{X \to +\infty} \int_{-X - i\gamma}^{X - i\gamma} e^{i\lambda s} (\lambda - A)^{-1} \tilde{F}(\lambda)\, d\lambda \\
&= \frac{1}{\sqrt{2\pi}} \lim_{X \to +\infty} \left(\int_{-X - i\gamma}^{-X - i\mu} + \int_{-X - i\mu}^{X - i\mu} + \int_{X - i\mu}^{X - i\gamma} \right) + R \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\mu}^{+\infty - i\mu} e^{i\lambda s} (\lambda - A)^{-1} \tilde{F}(\lambda)\, d\lambda + R
\end{aligned}$$

where R stands for the sum of the residues of the function $e^{i\lambda s} (\lambda - A)^{-1} \tilde{F}(\lambda)$ in the strip $-\gamma < \Im\lambda < -\mu$ (cf. Fig. 1).

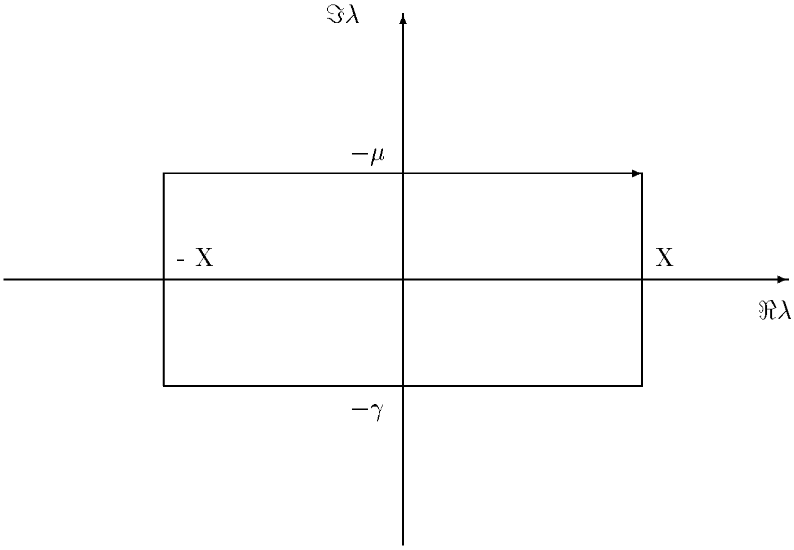

Fig. 1: The contour of integration.

Since the singularities of the function $e^{i\lambda s}(\lambda - A)^{-1}\tilde{F}(\lambda)$ are only the poles of $(\lambda - A)^{-1}$, i.e., the eigenvalues $\lambda_1, ..., \lambda_N$, it follows that

$$\begin{aligned}
&\text{Res}\left(e^{i\lambda s}(\lambda - A)^{-1}\tilde{F}(\lambda)\right)|_{\lambda=\lambda_i} \\
&= \frac{1}{(\alpha_i - 1)!}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k} c_l^i(\lambda)\,\phi_l^i\left(e^{i\lambda s}\,(\lambda - \lambda_i)^{\alpha_i-1-k}\right)^{(\alpha_i-1)}|_{\lambda=\lambda_i} \\
&= e^{i\lambda_i s}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i.
\end{aligned}$$

Hence,

$$\begin{aligned}
U(s) &= \sum_{i=1}^{N}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i s}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty-i\mu}^{+\infty-i\mu} e^{i\lambda s}\,(\lambda - A)^{-1}\,\tilde{F}(\lambda)\,d\lambda.
\end{aligned}$$

We now consider the equation

$$D_s U - A(s)U = 0, \qquad s > T,$$

and we introduce the function $V = \eta_T U$. This new function fulfills the equation

$$D_s V - A(s)V = F, \qquad s \in \mathbb{R},$$

where $F = (D_s\eta_T)\,U$ is a function with compact support.

Assume that $V \in \mathcal{W}_{m,\mu}$ and rewrite the above equation as

$$D_s V - AV = (A(s) - A)V + F.$$

Since both F and $(A(s)-A)V$ belong to the space $\mathcal{V}_{m-1,\mu}$, it follows from what has already been proved that the solution U can be represented in the form

$$U(s)=\sum_{i=1}^{N}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i s}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i+Z(s),$$

where $\eta_T Z\in\mathcal{W}_{m,\mu}$. This completes the proof.

□

Now, assume for simplicity that only the line $\Im\lambda=-\mu$ contains an eigenvalue of the matrix A. In this case, Proposition 2.1 can be applied but in a slightly different form.

Proposition 2.2 *Assume that:*

1. *Condition 1 of Proposition 1.4 is fulfilled.*
2. *N eigenvalues $\lambda_1,...,\lambda_N$ (counting the multiplicities) to which correspond the eigenchains ϕ_k^i, $k=0,...,\alpha_i-1$; $i=1,...,N$, are located in the strip $-\gamma<\Im\lambda_i<-\mu$.*
3. *There is only one eigenvalue λ_{N+1} of the matrix A on the line $\Im\lambda=-\mu$.*

Then, any solution $U(s)$ of the equation $D_sU-A(s)U=0$, $s>T$, such that $\eta_T U\in\mathcal{W}_{m,\mu}$, can be written as

$$\begin{aligned}U(s)\;=\;&\sum_{i=1}^{N}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i s}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i+e^{i\lambda_{N+1}s}\sum_{k=0}^{\alpha_{N+1}-1}c_k\frac{(is)^k}{k!}\\&+\;Z(s),\end{aligned}$$

where $\eta_T\left((D_s-\lambda_{N+1})^{\alpha_{N+1}}Z\right)\in\mathcal{W}_{m,\mu}$, and $c_0,\ldots,c_{\alpha_{N+1}-1}$ are some constant vectors.

Proof. Assume that the line $\Im\lambda=-\mu$ involves an eigenvalue λ_{N+1} of multiplicity α_{N+1} of A and consider the functions

$$\phi_k(s)=s^k\,e^{i\lambda_{N+1}s},\qquad 0\le k\le\alpha_{N+1}-1.$$

Let $\tilde{\mathcal{P}}(\lambda)$ be the polynomial $(\lambda-\lambda_{N+1})^{\alpha_{N+1}}$. Denote by $\mathcal{P}(D_s)$ the linear differential operator with constant coefficients, whose order is α_{N+1} and which vanishes on all the functions $\phi_k(s)$. This operator $\mathcal{P}(D_s)$ can be represented by

$$\begin{aligned}\mathcal{P}(D_s)\;&=\;(D_s-\lambda_{N+1})^{\alpha_{N+1}}\\&=\;\sum_{\alpha=0}^{\alpha_{N+1}}C_\alpha^{\alpha_{N+1}}\,(-\lambda_{N+1})^{\alpha_{N+1}-\alpha}\,D_s^\alpha.\end{aligned}$$

Consider now the equation

$$D_sU-AU=F,\qquad s\in\mathbb{R},$$

where F is a function with a support in $\mathbb{R}_+$.

Assume that $\lambda_1, ..., \lambda_N$ are N eigenvalues of A of multiplicities $\alpha_1, ..., \alpha_N$, respectively, lying in the strip $-\gamma < \Im\lambda_i < -\mu$. Assume also for simplicity that there are no eigenvalues of A on the line $\Im\lambda = -\gamma$.

If we apply the Fourier transform to the above equation, it gives

$$\begin{aligned}\tilde{\mathcal{P}}\tilde{U} &= \widetilde{\mathcal{P}U} \\ &= (\lambda - \lambda_{N+1})^{\alpha_{N+1}} (\lambda - A)^{-1} \tilde{F},\end{aligned}$$

where the resolvent $(\lambda - A)^{-1}$ has the form given in Subsection 1.3.

Note that the function $(\lambda - \lambda_{\alpha_{N+1}})^{N+1}(\lambda - A)^{-1}$ is meromorphic as $(\lambda - A)^{-1}$, but the point λ_{N+1} is now regular. Following the proof of Proposition 2.1, we get

$$\begin{aligned}\mathcal{P}(D_s)U(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty-i\gamma}^{+\infty-i\gamma} e^{i\lambda s}\,\tilde{\mathcal{P}}(\lambda)\,(\lambda - A)^{-1}\,\tilde{F}(\lambda)\,d\lambda \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty-i\mu}^{+\infty-i\mu} e^{i\lambda s}\,\tilde{\mathcal{P}}(\lambda)\,(\lambda - A)^{-1}\,\tilde{F}(\lambda)\,d\lambda \\ &+ \sum_{i=1}^{N} \mathrm{Res}\left(e^{i\lambda s}\tilde{\mathcal{P}}(\lambda)(\lambda - A)^{-1}\tilde{F}(\lambda)\right)|_{\lambda=\lambda_i}.\end{aligned}$$

As

$$\begin{aligned}&\mathrm{Res}\left(e^{i\lambda s}(\lambda - \lambda_{N+1})\,(\lambda - A)^{-1}\tilde{F}(\lambda)\right)|_{\lambda=\lambda_i} \\ &\quad = (D_s - \lambda_{N+1})\ \mathrm{Res}\left(e^{i\lambda s}(\lambda - A)^{-1}\tilde{F}(\lambda)\right)|_{\lambda=\lambda_i},\end{aligned}$$

it follows that

$$\begin{aligned}&\mathrm{Res}\left(e^{i\lambda s}\,\tilde{\mathcal{P}}(\lambda)\,(\lambda - A)^{-1}\,\tilde{F}(\lambda)\right)|_{\lambda=\lambda_i} \\ &\quad = \mathcal{P}(D_s)\left(\sum_{i=1}^{N}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i s}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i\right).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{P}(D_s)U(s) &= \mathcal{P}(D_s)\left(\sum_{i=1}^{N}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i s}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i\right) \\ &+ \mathcal{P}(D_s)Z(s).\end{aligned}$$

Since the functions

$$\frac{i^k}{k!}\,\phi_k(s), \quad k = 0, \ldots, \alpha_{N+1} - 1,$$

span the space of solutions of the equation $\mathcal{P}(D_s)u = 0$, we get

$$\begin{aligned}U(s) &= \sum_{i=1}^{N}\sum_{k=0}^{\alpha_i-1}\sum_{l=0}^{\alpha_i-1-k}\sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i s}\frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!}\frac{(is)^l}{l!}\phi_k^i + e^{i\lambda_{N+1}s}\sum_{k=0}^{\alpha_{N+1}-1} c_k\frac{(is)^k}{k!} \\ &+ Z(s),\end{aligned}$$

where $\eta_T\left(\mathcal{P}(D_s)Z\right) \in \mathcal{W}_{m,\mu}$, and $c_0, \ldots, c_{\alpha_{N+1}}$ are some constant vectors.

$\square$

As mentioned, we are going to give yet another result about asymptotics of solutions of equation (2.1) when one of the lines limiting the strip $-\gamma < \Im\lambda < -\mu$ contains an eigenvalue of A.

The results of Section 1 show that the following proposition is a consequence of Theorem 3.1 of Plamenevskii [5].

Proposition 2.3 *Let the following conditions hold:*

1. *On the line $\Im\lambda = -\gamma_i$, there is one eigenvalue $\lambda_0 = \sigma - i\gamma_i$ of the matrix A and only one eigenvalue $\lambda_i(s)$ (to which corresponds the eigenchain $\phi_k^i(s)$, $k = 1, ..., \alpha_i - 1$) of the matrix $A(s)$ tends to it as $s \to +\infty$, the integer α_i being independent of s.*
2. *$\lim_{s\to+\infty} \epsilon_q(s) = 0$, for $q = 0, \ldots, m$, where $\epsilon_q(s) = \max_{j=1,\ldots,m} |D_s^q(b_j(s) - a_j)|$.*
3. *$\int_T^{+\infty} s^{2r}\, \epsilon_1(s)ds < \infty$ and $\int_T^{+\infty} s^{2r}\, \left(\epsilon_q^2(s) + \sigma_q^2(s)\right) ds < \infty$, where r is equal to 1, if λ_0 is simple, and $\alpha_i - 1$, if λ_0 is multiple.*

If $U(s)$ is a solution of the system $D_sU - A(s)U = 0$, $s > T$, such that $\eta_T U \in \mathcal{W}_{m,\gamma}$, for $\gamma \in (-\gamma_i, -\gamma_{i-1})$, then

$$U(s) = \exp\left(i\int_T^s \lambda_i(\theta)d\theta\right)\left(\sum_{k=0}^{\alpha_i-1}\left(P^{(i)}_{\alpha_i-1-k}(s) + o(1)\right)\phi_k^i(s) + W(s)\right),$$

where $\eta_T W \in \mathcal{W}_{0,\gamma}$ and $P_k^{(i)}(s)$ denotes a polynomial of degree k whose coefficients depend on the solution $U(s)$.

2.2 Applications

Consider the differential equation

$$\left(\frac{1}{\delta'(t)}D_t\right)^m u + \sum_{0\le\alpha\le m-1} a_{m-\alpha}(t)\left(\frac{1}{\delta'(t)}D_t\right)^\alpha u = f(t), \qquad t < 1, \qquad (2.2)$$

where $\delta(t)$ is a smooth real-valued function on the interval $(0, 1]$, such that $\delta(t) \to +\infty$ as $t \to 0$ and $\delta'(t) < 0$ for $t \in (0, 1]$.

The coefficients $a_1(t), \ldots, a_m(t)$ are assumed to be continuous up to $t = 0$. We denote by $a_1, \ldots, a_m$ their values at $t = 0$. Without loss of generality we can assume that $a_m \neq 0$.

As mentioned, such differential equations appear in the analysis on manifolds with singular points. The derivative $\delta'(t)$ is determined close to $t = 0$ by geometry of singularities. It in turn determines the function $\delta(t)$ up to a constant, and it is a property of "real" singularities that $\delta(0+) = \infty$. The last condition is necessary in order that $\delta(t)$ could be extended to a diffeomerphism of $\mathbb{R}_+$ onto the whole real axis $\mathbb{R}$. Such is the case, in particular, if the derivative $\delta'(t)$ does change the sign for small $t > 0$.

Example 2.4 Let $\varphi(t) = -t^{k+1}$, for $t \in (0,1]$, where $k \geq 0$. Choose any C^∞ function $\varphi(t)$ on $\mathbb{R}_+$ with negative values, such that

$$\varphi(t) = \begin{cases} -t^{k+1}, & \text{if} \quad t \in (0,1]; \\ -1, & \text{if} \quad t \in [2,+\infty). \end{cases}$$

Set

$$\delta(t) = \int_{t_0}^{t} \frac{d\theta}{\varphi(\theta)}, \qquad t \in \mathbb{R}_+,$$

where $t_0 > 0$ is a fixed real number. Note that $\delta(t) \to +\infty$ as $t \to 0$, and $\delta(t) \to -\infty$ as $t \to +\infty$. Moreover, $\delta'(t) = 1/\varphi(t)$ is negative for all $t \in \mathbb{R}$, hence the function $t \mapsto \delta(t)$ defines a diffeomorphism between $\mathbb{R}_+$ and $\mathbb{R}$.

□

The change of variable $t = \delta^{-1}(s)$ yields

$$\begin{aligned} \lim_{s \to +\infty} \left(a_\alpha \circ \delta^{-1}\right)(s) &= a_\alpha, \\ D_s \left(a_\alpha \circ \delta^{-1}\right)(s) &= \left(\frac{1}{\delta'(t)}\, D_t a_\alpha\right)\left(\delta^{-1}(s)\right). \end{aligned}$$

Hence, to derive asymptotic formulas for solutions of equation (2.2) when $t \to 0$, we just need to apply the above change of variable and reduce the equation (2.2) to the form (1.6) which has been already investigated.

To simplify notation, we let $\mathbf{D}_t$ stand for the singular derivative occurring in (2.2), i.e.,

$$\mathbf{D}_t = \frac{1}{\delta'(t)}\, D_t.$$

Theorem 2.5 *Assume that:*

1. $\lim_{t \to 0} |\mathbf{D}_t^q\,(b_j(t) - a_j)\,| = 0$, *for* $q = 0, \ldots, m-j$ *and* $j = 1, \ldots, m$, *where* $b_j(t) = a_j(t) + \alpha_{j,1}\mathbf{D}_t a_{j-1}(t) + \ldots + \alpha_{j,j-1}\mathbf{D}_t^{j-1} a_1(t)$.

2. N *eigenvalues* $\lambda_1, \ldots, \lambda_N$ *(counting the multiplicities) of* A*, to which correspond the eigenchains* ϕ_k^i*,* $k = 0, \ldots, \alpha_i - 1$*;* $i = 1, \ldots, N$*, are located in the strip* $-\gamma < \Im\lambda_i < -\mu$.

3. *There are no eigenvalues of the matrix* A *on the lines* $\Im\lambda = -\gamma$ *and* $\Im\lambda = -\mu$.

Then, any solution $U(t)$ *of the equation* $\mathbf{D}_t U - A(t)U = 0$*,* $t < 1$*, such that* $\eta_T U(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$ *can be written as*

$$U(t) = \sum_{i=1}^{N} \sum_{k=0}^{\alpha_i - 1} \sum_{l=0}^{\alpha_i - 1 - k} \sum_{n=l}^{\alpha_i - 1 - k} e^{i\lambda_i \delta(t)} \frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!} \frac{(i\delta(t))^l}{l!} \phi_k^i + Z(t),$$

where $\eta_T Z(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$.

If the lines limiting the strip $-\gamma < \Im\lambda < -\mu$ meet the eigenvalues of the matrix A, Theorem 2.5 needs a slight modification (cf. Proposition 2.2).

Corollary 2.6 *Assume that:*

1. $\lim_{t\to 0} |\mathbf{D}_t^q (b_j(t) - a_j)| = 0$, *for* $q = 0, ..., m - j$ *and* $j = 1, ..., m$.

2. N *eigenvalues* $\lambda_1, ..., \lambda_N$ *(counting the multiplicities) of the matrix* A, *to which correspond the eigenchains* ϕ_k^i, $k = 0, ..., \alpha_i - 1$; $i = 1, ..., N$, *are located in the strip* $-\gamma < \Im\lambda_i < -\mu$.

3. *There is only one eigenvalue* λ_{N+1} *of the matrix* A *on the line* $\Im\lambda = -\mu$.

Then, any solution $U(t)$ *of the equation* $\mathbf{D}_t U - A(t)U = 0$, $t < 1$, *such that* $\eta_T U(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$, *can be written as*

$$U(t) = \sum_{i=1}^{N} \sum_{k=0}^{\alpha_i-1} \sum_{l=0}^{\alpha_i-1-k} \sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i\delta(t)} \frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!} \frac{(i\delta(t))^l}{l!} \phi_k^i + e^{i\lambda_{N+1}\delta(t)} \sum_{k=0}^{\alpha_{N+1}-1} c_k \frac{(i\delta(t))^k}{k!} + Z(t),$$

where $\eta_T\left((\mathbf{D}_t - \lambda_{N+1})^{\alpha_{N+1}} Z\right)(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$, *and* $c_0, \ldots, c_{\alpha_{N+1}-1}$ *are some constant vectors.*

Using again the change of variable $t = \delta^{-1}(s)$, we obtain the following consequence of Proposition 2.3. Set

$$\sigma_\mu(t) = \sum_{n_1 m_1 + \ldots + n_\mu m_\mu = \mu} \prod_{i=1}^{\mu} \sup_{j=1,\ldots,m-1} |\mathbf{D}_t^{n_i} b_j(t)|^{m_i}.$$

Theorem 2.7 *Let the following conditions hold:*

1. *On the line* $\Im\lambda = -\gamma_i$, *there is one eigenvalue* $\lambda_0 = \tau - i\gamma_i$ *of the matrix* A *and only one eigenvalue* $\lambda_i(t)$ *(to which corresponds the eigenchain* $\phi_k^i(t)$, $k = 1, ..., \alpha_i - 1$*) of the matrix* $A(t)$ *tends to it as* $t \to 0$, *the integer* α_i *being independent of* t.

2. $\lim_{t\to 0} \epsilon_q(t) = 0$, *for* $q = 0, \ldots, m$, *where* $\epsilon_q(t) = \max_{j=1,\ldots,m} |\mathbf{D}_t^q(b_j(t) - a_j)|$.

3. $\int_0^1 (\delta(t))^{2r}\, \epsilon_1(t) d\delta(t) < \infty$ *and* $\int_0^1 (\delta(t))^{2r}\, \left(\epsilon_q^2(t) + \sigma_q^2(t)\right) d\delta(t) < \infty$, *where* r *is equal to* 1, *if* λ_0 *is simple, and* $\alpha_i - 1$, *if* λ_0 *is multiple.*

If $U(t)$ *is a solution of* $\mathbf{D}_t U - A(t)U = 0$, $t < 1$, *satisfying* $\eta_T U(\delta^{-1}(s)) \in \mathcal{W}_{m,\gamma}$, *for* $\gamma \in (-\gamma_i, -\gamma_{i-1})$, *then*

$$U(t) = \exp\left(i \int_1^t \lambda_i(\theta) d\delta(\theta)\right) \left(\sum_{k=0}^{\alpha_i-1} \left(P_{\alpha_i-1-k}^{(i)}(\delta(t)) + o(1)\right) \phi_k^i(t) + W(t)\right),$$

where $\eta_T W(\delta^{-1}(s)) \in \mathcal{W}_{0,\gamma}$ *and* $P_k^{(i)}(s)$ *denotes a polynomial of degree* k *whose coefficients depend on the solution* $U(t)$.

2.3 The case of C^∞ coefficients

In this part we are going to derive asymptotic formulas for solutions of the equation (2.2) when the coefficients $a_1(t), \ldots, a_m(t)$ are smooth up to $t = 0$. Let us mention once again that such equations appear when we study linear differential operators close to cuspidal singularities.

By assumption, the derivatives of arbitrary order of $a_j(t)$ are continuous up to $t = 0$, and consequently the functions $D_t^\alpha a_j(t)$ are bounded on the interval $[0, 1]$.

Moreover,

$$\begin{aligned} b_j(t) &\to a_j(0) = a_j, \\ \mathbf{D}_t^q \left(b_j(t) - a_j\right) &\to 0 \end{aligned}$$

when t tends to 0, for all $q = 0, ..., m - j$ and $j = 1, ..., m$. It follows that the first condition of Theorem 2.5 is fulfilled. Hence Theorem 2.5 and Corollary 2.6 can be (simultaneously) rewritten in as follows.

Corollary 2.8 *Assume that:*

1. *The coefficients $a_1(t), \ldots, a_m(t)$ are smooth up to $t = 0$.*
2. *N eigenvalues $\lambda_1, ..., \lambda_N$ (counting the multiplicities) of the matrix A, to which correspond the eigenchains ϕ_k^i, $k = 0, ..., \alpha_i - 1$; $i = 1, ..., N$, are located in the strip $-\gamma < \Im\lambda_i < -\mu$.*
3. *There is only one eigenvalue λ_{N+1} of the matrix A on the line $\Im\lambda = -\mu$.*

Then, any solution $U(t)$ of the equation $\mathbf{D}_t U - A(t)U = 0$, $t < 1$, such that $\eta_T U(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$, can be written as

$$\begin{aligned} &U(t) \\ &= \sum_{i=1}^{N} \sum_{k=0}^{\alpha_i-1} \sum_{l=0}^{\alpha_i-1-k} \sum_{n=l}^{\alpha_i-1-k} e^{i\lambda_i\delta(t)} \frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!} \frac{(i\delta(t))^l}{l!} \phi_k^i + e^{i\lambda_{N+1}\delta(t)} \sum_{k=0}^{\alpha_{N+1}-1} c_k \frac{(i\delta(t))^k}{k!} \\ &+ Z(t), \end{aligned}$$

where $\eta_T \left((\mathbf{D}_t - \lambda_{N+1})^{\alpha_{N+1}} Z\right)(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$, and $c_0, \ldots, c_{\alpha_{N+1}-1}$ are some constant vectors. If there are no eigenvalues of A on the line $\Im\lambda = -\mu$, the vectors $c_0, \ldots, c_{\alpha_{N+1}-1}$ vanish and the function Z is such that $\eta_T Z(\delta^{-1}(s)) \in \mathcal{W}_{m,\mu}$.

If $\delta(t) = -\log t$, we have $\mathbf{D}_t = -tD_t$, i.e., (2.2) is a Fuchs-type equation. Hence, the asymptotic formula given in Theorem 2.5 is similar to the formula obtained by Kondrat'ev in [3], i.e.,

$$U(t) = \sum_{i=1}^{N} \sum_{k=0}^{\alpha_i-1} \sum_{l=0}^{\alpha_i-1-k} \sum_{n=l}^{\alpha_i-1-k} t^{-i\lambda_i} \frac{(c_k^i)^{(n-l)}(\lambda_i)}{(n-l)!} \frac{(-i\log t)^l}{l!} \phi_k^i + Z(t).$$

It also possible to derive asymptotics involving the eigenvalues of the matrix $A(t)$, for solutions of Fuchs-type equations. In fact, the integrals given in Theorem 2.7 converge because

$$\lim_{t\to 0} t^{1/2} \left(\log t\right)^{2r} = 0.$$

In this case, we have

$$U(t) = \exp\left(-i\int_1^t \frac{\lambda_i(\theta)}{\theta}d\theta\right)\left(\sum_{k=0}^{\alpha_i-1}\left(P^{(i)}_{\alpha_i-1-k}(-\log t)+o(1)\right)\phi^i_k(t)+W(t)\right).$$

Remark 2.9 *Unfortunately, when $\delta(t)$ is given by Example 2.4 and $k>0$, the mentioned integrals do not converge any more and Theorem 2.7 cannot be applied.*

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