BENDING OF AN ORTHOTROPIC CUSPED PLATE

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ABSTRACT

The bending of an orthotropic cusped plate in energetic and weighted Sobolev spaces has been considered. The existence and uniqueness of generalized and weak solutions of admissible boundary value problems (BVPs) have been investigated.

AMS subject CLASSIFICATION: primary 35J40; secondary 73C99

Key Words - Elliptic equation with order degeneration, energetic space, weighted Sobolev space, bending of an orthotropic cusped plate

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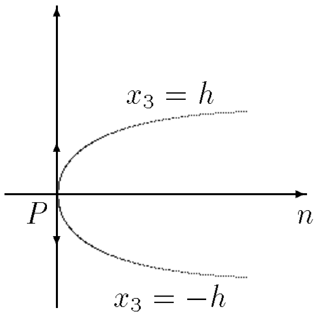
INTRODUCTION

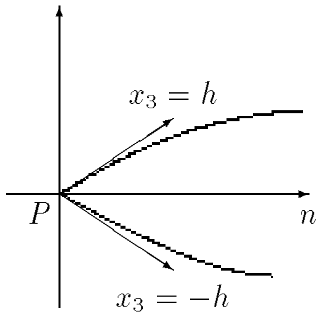
Let $Ox_1x_2x_3$ be the Cartesian coordinate system, and Ω be a domain in the plane Ox_1x_2 with a piecewise smooth boundary. The body bounded from upper by the surface $x_3 = h(x_1, x_2) \geq 0$, $(x_1, x_2) \in \Omega$, from lower by the surface $x_3 = -h(x_1, x_2)$, $(x_1, x_2) \in \Omega$, from the side by a cylindrical surface parallel to the x_3-axis, will be called cusped plate. The points $P \in \partial\Omega$, at which s.c. plate thickness $2h(x_1, x_2) = 0$, will be called plate cusps. If $h \in C^1(\Omega)$, obviously,

$$0 \leq L := \lim_{Q \to P} \frac{\partial 2h(Q)}{\partial n} \leq +\infty, \quad Q \in \Omega, \quad P \in \partial\Omega,$$

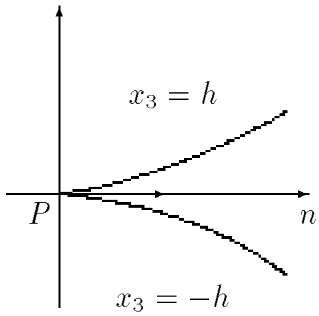
provided that the finite or infinite limit L exists; if P is an angular point of the boundary $\partial\Omega$ then under inward to $\partial\Omega$ normal n we mean bisectriss of angle between unilateral tangents to $\partial\Omega$ at P. Ω will be called the projection of the plate. $\partial\Omega$ will be called the plate boundary. On the figures 1-3 are represented the possible normal sections (profiles) of a symmetric plate at the point P in its neighbourhood.



$L = +\infty$
fig.1



$0 < L < +\infty$
fig.2



$L = 0$
fig.3

Let us now consider an orthotropic cusped plate.

The equation of the classical bending theory of the orthotropic plates has the form as follows (see [1], p. 364)

$$\begin{aligned} Jw &:= (D_1 w,_{11}),_{11} + (D_2 w,_{22}),_{22} + (D_3 w,_{22}),_{11} + \\ &+ (D_3 w,_{11}),_{22} + 4(D_4 w,_{12}),_{12} = f(x_1, x_2) \ \text{ in } \ \Omega \subset R^2, \end{aligned} \tag{0.1}$$

where w is a deflection; f is a lateral load; $D_i \in C^2(\Omega)$, $i = 1,2,3,4$, and

$$D_i := \frac{2\,E_i\,h^3}{3}, \quad i = 1,2,3, \quad D_4 := \frac{2\,G\,h^3}{3};$$

$$D_\alpha - D_3 > 0, \quad \alpha = 1,2 \quad \text{if } h > 0 \tag{0.2}$$

(for all known orthotropic plates these last conditions are fulfilled (see [1])); E_i, $i = 1,2,3$, and G are elastic constants for the orthotropic case; indices after comma mean differentiation with respect to corresponding variables.

In particular, if the plate is isotropic,

$$E_\alpha = \frac{E}{1-\sigma^2}, \quad \alpha = 1,2, \quad E_3 = \frac{\sigma E}{1-\sigma^2}, \quad G = \frac{E}{2(1+\sigma)},$$

where E is Young's modulus and σ is Poisson's ratio.

Let $\partial\Omega$ be the piecewise smooth boundary of the domain Ω with a part Γ_1 lying on the axis Ox_1 and a part Γ_2 lying in the upper half-plane $x_2 > 0$ ($\partial\Omega \equiv \overline{\Gamma}_1 \cup \overline{\Gamma}_2$).

Let further the thickness $2h > 0$ *in* $\Omega \cup \Gamma_2$, and $2h \geq 0$ on Γ_1. Therefore

$$D_i(x_1, x_2) > 0 \quad \text{in} \quad \Omega \cup \Gamma_2, \quad D_i(x_1, x_2) \geq 0 \quad \text{on} \quad \Gamma_1, \quad i = 1,2,3,4. \tag{0.3}$$

In particular case let

$$D_{1i} x_2^{\varkappa} \leq D_i(x_1, x_2) \leq D_{2i} x_2^{\varkappa}, \quad i = 1,2,3,4, \quad \text{in } \Omega, \tag{0.4}$$

where

$$D_{\alpha i} = const > 0, \quad \alpha = 1,2, \quad i = 1,2,3,4, \quad \varkappa = \text{ const} \geq 0,$$

i.e.

$$D_i(x_1, x_2) = \tilde{D}_i(x_1, x_2) x_2^{\varkappa}, \quad D_{1i} \leq \tilde{D}_i(x_1, x_2) \leq D_{2i},$$

$$D_{1\alpha} > D_{23}, \quad \alpha = 1,2, \tag{0.5}$$

(otherwise there would exist such points of Ω where (0.2) will be violated). In the case under consideration, (0.1) is an elliptic equation, in general, with order degeneration on Γ_1.

We recall (see [1]) that

$$M_\alpha = -(D_\alpha w,_{\alpha\underline{\alpha}} + D_3 w,_{\beta\underline{\beta}}), \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \tag{0.6}$$

$$M_{12} = -M_{21} = 2D_4 w,_{12} \tag{0.7}$$

$$Q_\alpha = M_{\alpha,\underline{\alpha}} + M_{21,\beta}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \tag{0.8}$$

$$Q^*_\alpha = Q_\alpha + M_{21,\beta}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \tag{0.9}$$

where M_α are bending moments, $M_{\alpha\beta}$, $\alpha \neq \beta$, are twisting moments, Q_α are shearing forces and Q^*_α are generalized shearing forces (bar under repeated indices means that we do not sum with respect to these indices).

In points of the plate boundary, where the thickness vanishes, all quantities will be defined as limits from inside of Ω.

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In this case all quantities depend e.g. only on x_2. Hence (0.1) will have the form as follows:

$$(D_2 w_{,22})_{,22} = f(x_2), \quad O < x_2 < l, \tag{1.1}$$

where l is width of the plate. In any section $x_1 = const$ we have the same deformation. Therefore the length of the plate plays no role. From (0.6)-(0.9) we have in Ω:

$$M_2 = -D_2 w_{,22}, \quad M_1 = -D_3 w_{,22} = \frac{D_3}{D_2} M_2, \quad M_{12} = -M_{21} = 0 \tag{1.2}$$

$$Q_1 = 0, \quad Q_2 = M_{2,2}, \quad Q^*_\alpha = Q_\alpha, \quad \alpha = 1, 2. \tag{1.3}$$

From (1.1)-(1.3) follows

$$Q_{2,2} = -f(x_2), \quad M_{2,22} = -f(x_2),$$

$$D_2 w_{,22} = \int_l^{x_2} (x_2 - t) f(t) dt - C_1 (x_2 - l) - C_2, \quad C_1, C_2 = \text{ const.}$$

Hence

$$Q_2 = -\int_l^{x_2} f(t) dt + C_1, \tag{1.4}$$

$$M_2 = -\int_l^{x_2} (x_2 - t) f(t) dt + C_1 (x_2 - l) + C_2, \tag{1.5}$$

$$w_{,2} = \int_l^{x_2} K_1(\tau) D_2^{-1}(\tau) d\tau + \int_l^{x_2} K_2(\tau) \tau D_2^{-1}(\tau) d\tau + C_3, \tag{1.6}$$

$$\begin{aligned} w = &\int_l^{x_2} (x_2 - \tau) K_1(\tau) D_2^{-1}(\tau) d\tau + \\ + &\int_l^{x_2} (x_2 - \tau) K_2(\tau) \tau D_2^{-1}(\tau) d\tau + C_3 (x_2 - l) + C_4, \end{aligned} \tag{1.7}$$

where

$$K_1(\tau) := C_1 l - C_2 - \int_l^\tau f(t) t dt, \tag{1.8}$$

$$K_2(\tau) := -C_1 + \int_l^\tau f(t) dt. \tag{1.9}$$

For f summable on [0,l], obviously,

$$Q_2, \quad M_2 \in C([0,l]); \quad w, \ w_{,2} \in C(]0,l]);$$

the behaviour of

$$w_{,2} \quad \text{and} \quad w \quad \text{when} \quad x_2 \to 0+$$

depends, in view of (1.6), (1.7), on convergence of

$$I_i := \int_0^l \tau^i D_2^{-1}(\tau) d\tau, \quad i = 0, 1, 2, \cdots .$$

Obviously,

$$I_i < +\infty \ \Rightarrow \ I_{i+1} < +\infty, \quad i \geq 0; \quad \text{and} \quad I_i = +\infty \ \Rightarrow \ I_{i-1} = +\infty, \ i \geq 0.$$

STATEMENT 1.1.

$$w, \ \ w_{,2} \in C([0,l]) \quad \text{if} \quad I_0 < +\infty;$$

$$w \in C([0,l]) \quad \text{if} \quad I_0 = +\infty, \quad I_1 < +\infty \tag{1.10}$$

(provided, f is bounded with its derivative in some neighbourhood $]0,\varepsilon]$ of 0); in this case ($I_0 = +\infty, \ \ I_1 < +\infty$) $w_{,2}$ is bounded as $x_2 \to 0+$ iff (if and only if)

$$K_1(0) = 0, \tag{1.11}$$

i.e., in virtue of (1.5), (1.8), iff $M_2(0) = 0$, otherwise it will be unbounded.

If $I_1 = +\infty$ (hence $I_0 = +\infty$), $I_2 < +\infty$, then w is bounded (provided, $D_2 \in C^3([0,l])$, f is continuous in 0, and has bounded first and second derivatives in $]0,\varepsilon]$) iff (1.11) is fulfilled, otherwise w will be unbounded.

If $I_2 = +\infty$, then w will be bounded (provided, for fixed $k \geq 2$

$$I_k = +\infty, \quad I_{k+1} < +\infty; \tag{1.12}$$

$$f^{(j)}(0) = 0, \quad j = 0, 1, ..., k-2, \quad f^{k-1}(x_2) \quad \text{is continuous in } 0) \tag{1.13}$$

iff (1.11) is fulfilled, and

$$K_2(0) = 0, \tag{1.14}$$

i.e., in virtue of (1.4), (1.9), $Q_2(0) = 0$, otherwise, i.e. either $K_1^2(0) + K_2^2(0) \neq 0$ or $I_k = +\infty \ \ \forall k$, w will be unbounded;

If $I_1 = +\infty$, then $w_{,2}$ is bounded (provided, (1.12) and (1.13) are fulfilled, when $k \geq 2$, and, when $k = 1$, (1.12) is fulfilled and $f(x_2)$ is continuous in 0) iff (1.11), (1.14) are fulfilled, otherwise, i.e. either $K_1^2(0) + K_2^2(0) \neq 0$ or $I_k = +\infty \ \ \forall k$, $w_{,2}$ will be unbounded.

PROOF. Let $I_0 = +\infty, \quad I_1 < +\infty$. Then

$$\begin{aligned} \left| x_2 \int\limits_{x_2}^{l} K_1(\tau) D_2^{-1}(\tau) d\tau \right| &= \left| \int\limits_{x_2}^{l} K_1(\tau) \frac{x_2}{\tau} \tau D_2^{-1}(\tau) d\tau \right| \le \\ \le C \int\limits_{0}^{l} \tau D_2^{-1}(\tau) d\tau &< +\infty \qquad \text{for } \; x_2 \in]0,l] \end{aligned} \tag{1.15}$$

because of

$$|K_1(\tau)| \le C, \quad \tau \in [0,l]; \quad \left| \frac{x_2}{\tau} \right| \le 1, \quad x_2 \in]0,l], \quad \tau \in [x_2, l].$$

Further, in virtue of (1.8),

$$\lim_{x_2 \to 0+} x_2 \int\limits_{x_2}^{l} K_1(\tau) D_2^{-1}(\tau) d\tau = \lim_{x_2 \to 0+} \frac{x_2^2 K_1(x_2)}{D_2(x_2)} = \lim_{x_2 \to 0+} \frac{2x_2 K_1(x_2) - f(x_2) x_2^3}{D_2'(x_2)} =$$

$$= \begin{cases} 0 \quad \text{if } \; D_2'(0) \ne 0 \quad \text{or} \quad D_2'(0) = +\infty \\ \lim\limits_{x_2 \to 0+} \dfrac{2K_1(x_2) - 2x_2^2 f(x_2) - f'(x_2) x_2^3 - 3f(x_2) x_2^2}{D_2''(x_2)} = \quad (\text{if } \; D_2'(0) = 0) \end{cases}$$

$$= \left\{ \begin{array}{l} 0 \qquad \text{if } \; D_2''(0) = +\infty \\ \\ \dfrac{2K_1(0)}{D_2''(0)} = -\dfrac{2M_2(0)}{D_2''(0)} = \dfrac{2(C_1 l - C_2 - \int\limits_{l}^{0} f(t) t dt)}{D_2''(0)} \quad \text{if } D_2''(0) \ne 0 \\ \\ \infty \qquad \text{if } \; D_2''(0) = 0, \quad K_1(0) \ne 0 \end{array} \right\} \text{ when } D_2'(0) = 0. \tag{1.16}$$

But $D_2''(0)$ can not be equal to 0, when $K_1(0) \ne 0$, otherwise (1.15) and (1.16) will contradict each other.

If $K_1(0) = 0$, then

$$\begin{aligned} \left| \int\limits_{x_2}^{l} K_1(\tau) D_2^{-1}(\tau) d\tau \right| &= \left| \int\limits_{x_2}^{l} \frac{K_1(\tau)}{\tau} \tau D_2^{-1}(\tau) d\tau \right| \le \\ \le C \int\limits_{x_2}^{l} \tau D_2^{-1}(\tau) d\tau \le C \int\limits_{0}^{l} \tau D_2^{-1}(\tau) d\tau &< +\infty, \quad x_2 \in]0,l], \end{aligned} \tag{1.17}$$

since

$$\lim_{\tau \to 0+} \frac{K_1(\tau)}{\tau} = -\lim_{\tau \to 0+} f(\tau) \tau = 0,$$

and hence

$$\left| \frac{K_1(\tau)}{\tau} \right| \le C, \quad \tau \in]0,l].$$

Thus,

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{l} K_1(\tau) D_2^{-1}(\tau)d\tau = 0.$$

In (1.7), obviously, other terms have limits when $x_2 \to 0+$, and (1.10) has been proved.

If (1.11) is fulfilled, then in view of (1.17), obviously, $w_{,2}$ is bounded on $]0, l]$. Otherwise, if $K_1(0) \neq 0$, it will be unbounded since in this case, without loss of generality, we can take $K_1(0) > 0$, and therefore $K_1(\tau) > \tilde{C}$ =const> 0 in some right neighbourhood $[0, \varepsilon]$ of 0, and if we suppose that

$$\left| \int\limits_{x_2}^{\varepsilon} K_1(\tau) D_2^{-1}(\tau)d\tau \right| < +\infty \quad \text{for} \quad x_2 \in]0, \varepsilon],$$

then

$$\tilde{C} \left| \int\limits_{x_2}^{\varepsilon} D_2^{-1}(\tau)d\tau \right| \leq \left| \int\limits_{x_2}^{\varepsilon} K_1(\tau) D_2^{-1}(\tau)d\tau \right| < +\infty.$$

Hence

$$\left| \int\limits_{x_2}^{\varepsilon} D_2^{-1}(\tau)d\tau \right| < +\infty, \quad \text{for} \quad x_2 \in]0, \varepsilon], \tag{1.18}$$

which would be contradiction with $I_0 = +\infty$.

Let $I_1 = +\infty$, $I_2 < +\infty$. Then

$$\begin{aligned} \left| x_2 \int\limits_{x_2}^{l} K_2(\tau)\tau D_2^{-1}(\tau)d\tau \right| = \left| \int\limits_{x_2}^{l} K_2(\tau) \frac{x_2}{\tau} \tau^2 D_2^{-1}(\tau)d\tau \right| \leq \\ \leq C \int\limits_{0}^{l} \tau^2 D_2^{-1}(\tau)d\tau < +\infty, \quad \text{for} \quad x_2 \in]0, l] \end{aligned} \tag{1.19}$$

because of

$$|K_2(\tau)| \leq C, \quad \tau \in [0, l]; \quad \left| \frac{x_2}{l} \right| \leq 1, \quad x_2 \in]0, l], \quad \tau \in [x_2, l].$$

Further, in virtue of (1.9),

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{l} K_2(\tau)\tau D_2^{-1}(\tau)d\tau = \lim_{x_2\to 0+} \frac{x_2^3 K_2(x_2)}{D_2(x_2)} = \lim_{x_2\to 0+} \frac{3x_2^2 K_2(x_2) + x_2^3 f(x_2)}{D_2'(x_2)} =$$

$$= \begin{cases} 0 \quad \text{if} \quad D_2'(0) \neq 0 \quad \text{or} \quad D_2'(0) = +\infty \\ \lim\limits_{x_2\to 0+} \dfrac{6x_2 K_2(x_2) + 6x_2^2 f(x_2) + x_2^3 f'(x_2)}{D_2''(x_2)} = \quad (\text{ if } D_2'(0) = 0) \end{cases}$$

$$= \left\{ \begin{array}{ll} 0 \quad \text{if} \quad D_2''(0) \neq 0 \quad \text{or} \quad D_2''(0) = +\infty & \\ \lim\limits_{x_2 \to 0+} \dfrac{6K_2(x_2) + 18x_2 f(x_2) + 9x_2^2 f'(x_2) + x_2^3 f''(x_2)}{D_2'''(x_2)} = \ (\text{ if } D_2''(0) = 0) & \end{array} \right\} \begin{array}{c} \text{when} \\ D_2'(0) = 0 \end{array}$$

$$= \left\{ \begin{array}{ll} 0 \quad \text{if} \quad D_2'''(0) = +\infty & \\ \dfrac{6K_2(0)}{D_2'''(0)} = -\dfrac{6Q_2(0)}{D_2'''(0)} = -\dfrac{6\left[C_1 - \int\limits_l^0 f(t)tdt\right]}{D_2'''(0)} & \text{if } D_2'''(0) \neq 0 \\ \infty \quad \text{if} \quad D_2'''(0) = 0, \quad K_2(0) \neq 0 & \end{array} \right\} \begin{array}{c} \text{when } D_2''(0) = \\ = D_2'(0) = 0. \end{array} \tag{1.20}$$

But $D_2'''(0)$ can not be equal to 0 when $K_2(0) \neq 0$, otherwise (1.19) and (1.20) will contradict each other.

If $K_2(0) = 0$,

$$\begin{aligned} \left| \int\limits_{x_2}^{l} K_2(\tau)\tau D_2^{-1}(\tau)d\tau \right| &= \left| \int\limits_{x_2}^{l} \frac{K_2(\tau)}{\tau}\tau^2 D_2^{-1}(\tau)d\tau \right| \leq \\ &\leq C \int\limits_0^l \tau^2 D_2^{-1}(\tau)d\tau < +\infty, \quad x_2 \in]0, l], \end{aligned} \tag{1.21}$$

since

$$\lim_{\tau \to 0+} \frac{K_2(\tau)}{\tau} = \lim_{\tau \to 0+} f(\tau) = f(0),$$

and hence

$$\left| \frac{K_2(\tau)}{\tau} \right| \leq C, \quad \tau \in]0, l].$$

Thus,

$$\lim_{x_2 \to 0+} x_2 \int\limits_{x_2}^{l} K_2(\tau)\tau D_2^{-1}(\tau)d\tau = 0.$$

The boundedness of other terms of (1.7) on $]0, l]$ is clear (see below (1.22), (1.23) by $k = 1$), as well as validity of other assertions of statement 1.1 which are either obvious or should be shown in analogous way as above taking into account that, if $K_i(0) = 0, \ \ i = 1, 2$, and (1.12), (1.13) are fulfilled, then

$$\begin{aligned} \left| \int\limits_{x_2}^{l} K_2(\tau)\tau D_2^{-1}(\tau)d\tau \right| &= \left| \int\limits_{x_2}^{l} \frac{K_2(\tau)}{\tau^k}\tau^{k+1} D_2^{-1}(\tau)d\tau \right| \leq \\ &\leq C \int\limits_0^l \tau^{k+1} D_2^{-1}(\tau)d\tau < +\infty, \quad x_2 \in]0, l], \end{aligned} \tag{1.22}$$

since

$$\lim_{\tau\to 0+}\frac{K_2(\tau)}{\tau^k}=\lim_{\tau\to 0+}\frac{f^{(k-1)}(\tau)}{k!}=\frac{1}{k!}f^{(k-1)}(0),\quad \text{i.e.}\quad \left|\frac{K_2(\tau)}{\tau^k}\right|\le C,\quad \tau\in]0,l];$$

$$\begin{gathered}\left|\int\limits_{x_2}^{l}K_1(\tau)D_2^{-1}(\tau)d\tau\right|=\left|\int\limits_{x_2}^{l}\frac{K_1(\tau)}{\tau^{k+1}}\tau^{k+1}D_2^{-1}(\tau)d\tau\right|\le\\ \le C\int\limits_0^l\tau^{k+1}D_2^{-1}(\tau)d\tau<+\infty,\qquad x_2\in]0,l],\end{gathered}\tag{1.23}$$

since

$$\lim_{\tau\to 0+}\frac{K_1(\tau)}{\tau^{k+1}}=\lim_{\tau\to 0+}\frac{-f(\tau)\tau}{(k+1)\tau^k}=-\frac{f^{(k-1)}(0)}{(k+1)(k-1)!},\quad \text{i.e.}\quad \left|\frac{K_1(\tau)}{\tau^{k+1}}\right|\le C,\quad \tau\in]0,l];$$

from (1.23), (1.22) follows, correspondingly,

$$\lim_{x_2\to 0+}x_2\int\limits_l^{x_2}K_{i+1}(\tau)\tau^iD_2^{-1}(\tau)d\tau=0,\qquad i=0,1,\tag{1.24}$$

and also convergence of

$$\int\limits_l^{x_2}K_{i+1}(\tau)\tau^{i+1}D_2^{-1}(\tau)d\tau,\qquad i=0,1.$$

If $I_k=+\infty\ \ \forall k$, and $K(\tau):=K_1(\tau)+\tau K_2(\tau)$ is analytic in a right neighbourhood of $\tau=0$, then, obviously, w and $w_{,2}$ are unbounded when $x_2\to 0+$. Indeed, e.g. (1.6) can be rewritten in the following form (let it be bounded when $x_2\to 0+$, and $K_i(0)=0,\ \ i=1,2$; the last conditions are necessary for it)

$$w_{,2}(x_2)=\int\limits_l^{x_2}K(\tau)D_2^{-1}(\tau)d\tau+C_3,$$

where $K(0)=0$. Since analytic $K(\tau)\not\equiv\ 0$, there exists such k that

$$K^{(j)}(0)=0,\quad j=0,1,...,k-1,\quad K^{(k)}(0)\neq 0.$$

Further

$$w_{,2}(x_2)=\int\limits_l^{x_2}\frac{K(\tau)}{\tau^k}\tau^kD_2^{-1}(\tau)d\tau+C_3$$

where

$$\lim_{\tau\to 0+}\frac{K(\tau)}{\tau^k}=\frac{K^{(k)}(0)}{k!}\neq 0.$$

Then, taking into account boundedness of $w_{,2}$, similarly to (1.18) we can show

$$\left|\int\limits_{x_2}^{\varepsilon}\tau^kD_2^{-1}(\tau)d\tau\right|<+\infty\qquad\text{for}\qquad x_2\in]0,\varepsilon]$$

which would be contradiction with $I_k=+\infty\ \ \forall k$.

From the statement 1.1. follows that on the cusped edge $x_2 = 0$ admissible are only four different pairs of the boundary data as follows:

$$w(0) = w_0, \qquad w'(0) = w'_0 \qquad \text{iff } I_0 < +\infty; \tag{1.25}$$

$$\begin{array}{lll} w'(0) = w'_0, & Q_2(0) = Q(0) & \text{iff } I_0 < +\infty; \\ w(0) = w_0, & M_2(0) = M_0 & \text{iff } I_1 < +\infty; \end{array}$$

$$M_2(0) = M_0, \qquad Q_2(0) = Q_0 \qquad \text{always,} \quad \text{i.e. if } I_i \leq +\infty, \quad i = 0, 1,$$

where $w_0, \;\; w'_0,$

$$M_0 \begin{cases} \text{is arbitrary} & \text{if } \;\; I_0 < +\infty, \\ = 0, & \text{if } \;\; I_0 = +\infty, \end{cases}$$

$$Q_0 \begin{cases} \text{is arbitrary} & \text{if } \;\; I_1 < +\infty, \\ = 0, & \text{if } \;\; I_1 = +\infty, \end{cases}$$

are given constants.

On the edge $x_2 = l$ we always can give each of the above four boundary data taking into account peculiarities of cylyndrical bending (see (1.4), (1.5)) that by arbitrary load $f, \;\; Q_2$ can be given only on a one edge; from $Q_2(0)$ (or $Q_2(l)$), $M_2(0), \;\; M_2(l)$ only two can participate in boundary conditions on the both edges (these peculiarities are not caused by cusps they arise already in case of cylyndrical bending of a plate of a constant thickness). If we choose f corrispondingly (see (1.4), (1.5)), we could avoid these peculiarities but restriction on choice of f would be artificial. Nevertheless also such posed problems can have practical sense. Obviously, solutions of all these problems can be constructed in the explicite forms. Some of them are unique, some defined either up to rigid translating or rigid rotating or general rigid motion.

Let $I_0 < +\infty$, and e.g. solve BVP with boundary conditions (1.25), and

$$w(l) = w_l, \;\; w'(l) = w'_l, \tag{1.26}$$

where $w_l, \;\; w'_l$ are also given constants.

In view of (1.6), (1.7), from (1.26) follow

$$C_4 = w_l, \;\; C_3 = w'_l.$$

For determination of constants $C_1, \;\; C_2$, from (1.25) we have the algebraic system as follows:

$$C_1 \int\limits_0^l \tau(\tau - l) D_2^{-1}(\tau) d\tau + C_2 \int\limits_0^l \tau D_2^{-1}(\tau) d\tau = \int\limits_0^l \tau D_2^{-1}(\tau) \int\limits_l^\tau f(t)(\tau - t) dt d\tau -$$
$$-l w'_l + w_l - w_0,$$
$$-C_1 \int\limits_0^l (\tau - l) D_2^{-1}(\tau) d\tau - C_2 \int\limits_0^l D_2^{-1}(\tau) d\tau =$$
$$= - \int\limits_0^l D_2^{-1}(\tau) \int\limits_l^\tau f(t)(\tau - t) dt d\tau + w'_l - w'_0$$

which is solvable as its determinant

$$\Delta := \left[\int_0^l \tau D_2^{-1}(\tau)d\tau\right]^2 - \int_0^l \tau^2 D_2^{-1}(\tau)d\tau \cdot \int_0^l D_2^{-1}(\tau)d\tau < 0$$

since Hoelder inequality is strong because $\tau D_2^{-\frac{1}{2}}(\tau)$ and $D_2^{-\frac{1}{2}}(\tau)$ are positive on $]0,l[$, and $\tau^2 D_2^{-1}(\tau)$ and $D_2^{-1}(\tau)$ differ from each other with nonconstant factor τ^2.

Other problems can be solved in analogous way taking into account (1.16), (1.20) and (1.24) in corresponding cases.

2- BENDING IN THE ENERGETIC SPACE

Let $D_i \in C^2(\Omega \cup \Gamma_2)$, $i = 1,2,3,4$. Let us consider the operator J (acting in $L_2(\Omega)$) on D_J :

1. $w \in C^4(\Omega \cup \Gamma_2)$;
   $Jw \in L_2(\Omega)$;

$$w \begin{cases} \in C(\overline{\Omega}) & \text{when} \quad I_{1i} < +\infty \ \text{in case (0.3)} \ (0 \le \varkappa < 2 \ \text{in case (0.4)}), \\ = O(1), \quad x_2 \to 0+, & \text{when} \quad I_{1i} = +\infty \ (\varkappa \ge 2); \end{cases} \tag{2.1}$$

$$w_{,\alpha} \begin{cases} \in C(\overline{\Omega}) & \text{when} \quad I_{0i} < +\infty \ (0 \le \varkappa < 1), \\ = \mathrm{O}(1), \quad x_2 \to 0+, & \text{when} \ I_{0i} = +\infty \ (1 \le \varkappa < +\infty), \ \ \alpha = 1,2; \end{cases} \tag{2.2}$$

$$I_{ki} := \int_0^l x_2^k D_i^{-1}(x_1,x_2)dx_2, \ \ i = 1,2,3,4, \ \ k = 0,1,...,$$

$$(D_2 - D_3)^{\frac{1}{2}} w_{,22} \in L_2(\Omega) \tag{2.3}$$

(this restriction can be avoided when we consider only solutions with finite energy); the bending moment, and the generalized shearing force

$$M_2 = -(D_2 w,_{22} + D_3 w,_{11}) \in C(\overline{\Omega}), \tag{2.4}$$

$$Q_2^* = -[(D_2 w,_{22} + D_3 w,_{11}),_2 + 4(D_4 w,_{12}),_1] \in C(\overline{\Omega}); \tag{2.5}$$

2.

$$w|_{\Gamma_2} = 0, \quad \left.\frac{\partial w}{\partial n}\right|_{\Gamma_2} = 0 \tag{2.6}$$

where n is the inward normal;

3. On Γ_1 one of the following pairs of boundary value conditions (BVCs) is fulfilled:

$$w = 0, \ \ w,_2 = 0 \quad \text{if} \quad I_{0i} < +\infty, \ \ i = 1,2,3,4, \ \ (0 \le \varkappa < 1); \tag{2.7}$$

$$w,_2 = 0, \ \ Q_2^* = 0 \quad \text{if} \quad I_{0i} < +\infty, \ \ i = 1,2,3,4, \ \ (0 \le \varkappa < 1); \tag{2.8}$$

$$w = 0, \ \ M_2 = 0 \quad \text{if} \quad I_{1i} < +\infty, \ \ i = 1,2,3,4, \ \ (0 \le \varkappa < 2); \tag{2.9}$$

$$M_2 = 0, \ \ Q_2^* = 0 \quad \text{if} \quad I_{0i} \le +\infty, \ \ i = 1,2,3,4, \ \ (0 \le \varkappa < +\infty). \tag{2.10}$$

REMARK 2.1. How it follows from the case of cylindrical bending (see section 1, and also [2], p. 96), the BVCs (2.7)-(2.9) can not be posed (in sense of correctness) for other values of $\varkappa$ except of indicated ones, or in general case (0.3) if $I_{0i} = +\infty$, and $I_{1i} = +\infty$, correspondingly.

STATEMENT 2.1. The operator J is linear, symmetric, and positive on the lineal D_J, and

$$\begin{gathered}(Jw, v) := \int\limits_{\Omega} vJw d\Omega = \int\limits_{\Omega} [D_1 v,_{11} w,_{11} + D_2 v,_{22} w,_{22} + D_3(v,_{11} w,_{22} + \\ + v,_{22} w,_{11}) + 4D_4 v,_{12} w,_{12}] d\Omega =: \int\limits_{\Omega} B(v,w) d\Omega \quad \forall v, w \in D_J.\end{gathered} \tag{2.11}$$

In particular, if $v = w$,

$$\begin{gathered}(Jw, w) := \int\limits_{\Omega} [D_1 (w,_{11})^2 + D_2 (w,_{22})^2 + 2D_3\ w,_{11} w,_{22} + \\ + 4D_4 (w,_{12})^2] d\Omega = \int\limits_{\Omega} [D_3 (w,_{11} + w,_{22})^2 + (D_1 - D_3)(w,_{11})^2 + \\ + 4D_4 (w,_{12})^2 + (D_2 - D_3)(w,_{22})^2] d\Omega.\end{gathered} \tag{2.12}$$

PROOF. It is obvious that J is linear operator on the lineal D_J (the latter about D_J easyly follows from the linearity of J on $C^4(\Omega \cup \Gamma_2)$). Since $D_J \subset L_2(\Omega)$ and $Jw \in L_2(\Omega)$, we can consider the following scalar product in $L_2(\Omega)$

$$(Jw, v) := \int\limits_{\Omega} vJw d\Omega = \lim_{\delta \to 0} \int\limits_{\Omega_\delta} vJw d\Omega_\delta, \quad \forall v, w \in D_J,$$

where

$$\Omega_\delta := \{(x_1, x_2) \in \Omega\ :\ x_2 > \delta = \text{ const} > 0\}.$$

After integration by parts twice and using formulas (d), (c) on page 87 of [1] we have

$$(Jw, v) := \lim_{\delta \to 0} \left[\int\limits_{\partial\Omega_\delta} \left(vQ_n - \frac{\partial v}{\partial n} M_n + \frac{\partial v}{\partial s} M_{ns} \right) ds + \int\limits_{\Omega_\delta} B(v, w) d\Omega_\delta \right],$$

where ds is the arc element, Q_n is the shearing force, M_n is the bending moment, M_{ns} is the twisting moment.

But

$$\int\limits_{\partial\Omega_\delta} \frac{\partial v}{\partial s} M_{ns} ds = \int\limits_{\partial\Omega_\delta} \frac{\partial v M_{ns}}{\partial s} ds - \int\limits_{\partial\Omega_\delta} v \frac{\partial M_{ns}}{\partial s} ds = - \int\limits_{\partial\Omega_\delta} v \frac{\partial M_{ns}}{\partial s} ds$$

as $v,\ M_{ns} \in C(\overline{\Omega}_\delta)$. Hence

$$(Jw, v) := \lim_{\delta \to 0} \int\limits_{\partial\Omega_\delta} \left(vQ_n^* - \frac{\partial v}{\partial n} M_n \right) ds + \lim_{\delta \to 0} \int\limits_{\Omega_\delta} B(v, w) d\Omega_\delta, \tag{2.13}$$

where

$$Q_n^* := Q_n - \frac{\partial M_{ns}}{\partial s}.$$

In view of (2.6)

$$\int\limits_{\partial\Omega_\delta} \left(vQ_n^* - \frac{\partial v}{\partial n} M_n \right) ds = \int\limits_{\Gamma_1^\delta} (vQ_2^* - v,_2 M_2) ds,$$

where

$$\Gamma_1^\delta := \{(x_1, x_2) \in \Omega \ : \ x_2 = \delta = \text{ const} > 0\}.$$

In virtue of (2.1), (2.2), (2.4), (2.5), (2.7)-(2.10), $\forall \varepsilon$ =const>0 $\exists \delta(\varepsilon)$ =const>0 such that

$$|vQ_2^* - v,_2 M_2| \le |v||Q_2^*| + |v,_2||M_2| < \varepsilon, \quad \text{when } \ 0 < x_2 < \delta,$$

i.e., taking into account (2.6),

$$\left| \int\limits_{\partial\Omega_\delta} \left(vQ_n^* - \frac{\partial v}{\partial n} M_n \right) ds \right| = \left| \int\limits_{\Gamma_1^\delta} (vQ_2^* - v,_2 M_2) ds \right| < \varepsilon |\Gamma_1^\delta| < \varepsilon |\partial\Omega_\delta| \le \varepsilon |\partial\Omega|$$

($|\partial\Omega|$ is the length of the curve $\partial\Omega$). So that

$$\lim_{\delta\to 0} \int\limits_{\partial\Omega_\delta} \left(vQ_n^* - \frac{\partial v}{\partial n} M_n \right) ds = \lim_{\delta\to 0} \int\limits_{\Gamma_1^\delta} (vQ_2^* - v,_2 M_2) ds = 0.$$

Therefore, because of existence of integral on the left side of (2.13), limit of the second addend on the right hand side of (2.13) also exists, and (2.11) is valid. (2.12) is obvious.

From (2.11) follows

$$(Jw, v) = (Jv, w) = (w, Jv), \ \ \forall v, w \in D_J.$$

Hence the operator J is symmetric.

From (2.12), taking into account (0.2), we have

$$(Jw, w) \ge 0.$$

But

$$(Jw, w) = 0, \ \ w \in D_J,$$

iff

$$w,_{11} = 0, \ \ w,_{22} = 0, \ \ w,_{12} = 0 \ \ \text{ in } \ \Omega,$$

i.e.

$$w = k_1 x_1 + k_2 x_2 + k_3, \ \ k_i = \text{ const}, \ i = 1, 2, 3, \ \ \text{ in } \ \Omega.$$

The latter, in virtue of (2.6), should be zero on Γ_2 and therefore on $\overline{\Omega}$, because of its linearity.

STATEMENT 2.2. The operator J is positive definite if only $D_0 > 0$ $(0 \le \varkappa \le 4)$,

$$0 \le D_0 := \inf_\Omega \frac{D_2 - D_3}{x_2^4}$$

PROOF. Let $D_0 = 0$, and consider particular case (0.4). Hence

$$D_0 = \inf_\Omega (\tilde{D}_2 - \tilde{D}_3) x_2^{\varkappa - 4} = 0 \ \ \text{ if } \ \text{ only } \ \varkappa > 4.$$

Then J is not positive definite. Indeed, let the rectangle

$$\Pi_0 := \{(x_1, x_2): \ a < x_1 < b, \ 0 < x_2 < \delta\}$$

be cut out from Ω. Let (see [3])

$$w_\delta(x_1, x_2) := \begin{cases} (\delta - x_2)^5 sin^5 \dfrac{\pi(x_1 - a)}{b - a} & \text{when } \ (x_1, x_2) \in \Pi_0; \\ O & \text{when } \ (x_1, x_2) \in \Omega \backslash \Pi_0. \end{cases}$$

Obviously $w_\delta \in D_J$, and because of $\varkappa > 4$, (2.10) should be and, in fact, is fulfilled by w_δ. It is easy to see that

$$0 \leq \frac{(Jw_\delta, w_\delta)}{\|w_\delta\|^2_{L_2(\Omega)}} \leq \overset{*}{C} \delta^{\varkappa - 4}, \ \ \overset{*}{C} = \text{ const} > 0,$$

since

$$\|w_\delta\|^2_{L_2(\Omega)} = \frac{1}{11} \ \frac{b-a}{\pi} \frac{1}{2^{10}} \binom{10}{5} \pi \delta^{11},$$

and, in view of (0.4),

$$0 \leq (Jw_\delta, w_\delta) \leq \int\limits_{\Pi_0} \underset{i \in \{1,2,3,4,\}}{max} \{D_{2i}\} x_2^{\varkappa} [(w_{\delta,11})^2 + (w_{\delta,22})^2 +$$

$$+ 2w_{\delta,11} w_{\delta,22} + 4(w_{\delta,12})^2] dx_1 dx_2 \leq \overset{**}{C} (\delta^{\varkappa+11} + \delta^{\varkappa+7} + \delta^{\varkappa+9})$$

$$\overset{**}{C} = \text{const} > 0,$$

Hence J is not positive definite on D_J.

Now let us return to the general case (0.3). Further $D_0 > 0$ $(0 \leq \varkappa \leq 4)$, and prove that J is positive definite.

From (2.12), taking into account (2.3), (0.2) and (0.3), we obtain

$$(Jw, w) \geq \int\limits_\Omega (D_2 - D_3)(w,_{22})^2 d\Omega \geq D_0 \int\limits_\Omega \ x_2^4 (w,_{22})^2 d\Omega = D_0 \int\limits_\Pi \ x_2^4 (w,_{22})^2 dx_1 dx_2,$$

where

$$\Pi := \{(x_1, x_2): \ a < x_1 < b, \ 0 < x_2 < 1\}, \tag{2.14}$$

and without loss of generality, is supposed that the domain Ω lies inside of the rectangle Π, and a definition of the function w is completed assuming w equal to zero outside of Ω. Then w will be continuous in Π with its first derivatives, and its derivatives of second order, in general, will have discontinuity of first kind on the arc Γ_2. Further

$$(Jw, w) \geq D_0 \int\limits_a^b \int\limits_0^1 x_2^4 (w,_{22})^2 dx_1 dx_2 \geq \frac{9D_0}{16} \int\limits_a^b \int\limits_0^1 w^2 dx_1 dx_2 =$$

$$= \gamma \int\limits_\Pi \ w^2 dx_1 dx_2 = \gamma \int\limits_\Omega \ w^2 dw = \gamma \|w\|^2_{L_2(\Omega)},$$

where

$$\gamma := \frac{9}{16} D_0.$$

In the previous reasonings we have used the following

LEMMA 2.1. Let $w(x_1, .)$ be real function of x_2 for fixed x_1 satisfying the following conditions:

1.) w and $w,_2$ are absolutly continuous on $[\delta, 1] \quad \forall \delta \in]0, 1[$;
2.) $w,\ w,_2 = 0(1)$ when $x_2 \to 0+$;
3.) $x_2^2 w,_{22} \in L_2(\,]0, 1[\,)$;
4.) $w(x_1, 1) = w,_2(x_1, 1) = 0$.

Then

$$\int_0^1 x_2^4 (w,_{22})^2 dx_2 \geq \frac{9}{16} \int_0^1 w^2 dx_2, \tag{2.15}$$

$$\int_0^1 x_2^4 (w,_{22})^2 dx_2 \geq \frac{9}{4} \int_0^1 x_2^2 (w,_2)^2 dx_2. \tag{2.16}$$

PROOF is similar to one used in [3] for the case $\delta = 0$ but we have to consider all integrals from δ to 1 and then let δ tend to zero.

Let H_J be the energetic space (see e.g. [3,4]) corresponding to the operator J defined on D_J and acting in $L_2(\Omega)$.

THEOREM 2.1. Let $f \in L_2(\Omega)$. if $D_0 > 0$ $(0 \leq \varkappa \leq 4)$, there exists unique generalized solution of (1.1) in the energetic space H_J. If $D_0 = 0$ $(\varkappa > 4)$, and $f(x_1, x_2) = 0$ in $\Omega \backslash \Omega_\delta$ then there exists unique generalized solution of the finite energy.

PROOF. Firstly let us prove that the solution with the finite energy exists for $D_0 \geq 0$ $(\varkappa \geq 0)$, if $f = 0$ in $\Omega \backslash \Omega_\delta$ (the last restriction of f can be weakened). Let $w \in D_J$. Then

$$w|_{\Gamma_2} = 0, \quad \frac{\partial w}{\partial n}|_{\Gamma_2} = 0,$$

and exist continuous on Γ_2 from inside derivatives $w,_{\alpha\beta}$. Let us put again the domain Ω inside of the rectangle (2.14) and complete a definition of the function w assuming it equal to zero outside of Ω.

We have

$$\begin{aligned} |(w, f)|^2 = \left| \int_\Omega w f dx_1 dx_2 \right|^2 = \left| \int_{\Omega_\delta} w f dx_1 dx_2 \right|^2 \leq \\ \leq \int_{\Omega_\delta} f^2 dx_1 dx_2 \int_{\Omega_\delta} w^2 dx_1 dx_2 = C \int_{\Pi_\delta} w^2 dx_1 dx_2 \quad \forall w \in D_J, \end{aligned} \tag{2.17}$$

where

$$C := \int_\Omega f^2 dx_1 dx_2 \geq 0,$$

$$\Pi_\delta := \{(x_1, x_2) \in \Pi \ : \ x_2 > \delta = \text{ const¿0}\}.$$

Obviously, when $x_2 > 0$

$$\int\limits_a^{x_1} w,_1\, dx_1 = w(x_1, x_2) - w(a, x_2) = w(x_1, x_2) \tag{2.18}$$

since $w(a, x_2) = 0$ as $(a, x_2) \in \Pi \backslash \Omega$. Acording to Cauchy - Bunyakovskii inequality, from (2.18), we have

$$w^2 \le \int\limits_a^{x_1} 1^2 dx_1 \int\limits_a^{x_1} (w,_1)^2 dx_1 \le (b-a) \int\limits_a^{b} (w,_1)^2 dx_1.$$

Integrating both sides in limits $a \le x_1 \le b,\ \delta \le x_2 \le 1$,

$$\begin{gathered}\int\limits_{\Pi_\delta} w^2 dx_1 dx_2 \le (b-a)^2 \int\limits_{\Pi_\delta} (w,_1)^2 dx_1 dx_2 \le \\ \le (b-a)^4 \int\limits_{\Pi_\delta} (w,_{11})^2 dx_1 dx_2 = (b-a)^4 \int\limits_{\Omega_\delta} (w,_{11})^2 dx_1 dx_2 =\end{gathered} \tag{2.19}$$

(in the second inequality the first inequality is applied to $w,_1$)

$$\begin{gathered}= (b-a)^4 \int\limits_{\Omega_\delta} \frac{(D_1 - D_3)(w,_{11})^2}{D_1 - D_3} dx_1 dx_2 \le \\ \le \frac{(b-a)^4}{D_\delta} \int\limits_{\Omega_\delta} (D_1 - D_3)(w,_{11})^2 dx_1 dx_2 \le \\ \le \frac{(b-a)^4}{D_\delta} \int\limits_{\Omega_\delta} [(D_1 - D_3)(w,_{11})^2 + \\ + D_3(w,_{11} + w,_{22})^2 + 4D_4(w,_{12})^2 + (D_2 - D_3)(w,_{22})^2] dx_1 dx_2 \le \\ \le \frac{(b-a)^4}{D_\delta}(Jw, w) = \frac{(b-a)^4}{D_\delta} \|w\|^2_{H_J}, \\ D_\delta := \min_{\Omega_\delta}(D_1 - D_3).\end{gathered}$$

From (2.17) and (2.19) follows

$$|(w, f)|^2 \le \frac{C(b-a)^4}{D_\delta} \|w\|^2_{H_J},$$

i.e. (w, f) can be considered as a linear bounded functional with respect to the energetic norm. But then, according to the well-known theory [3], there exists solution of the finite energy.

In case $D_0 > 0$ $(0 \le \varkappa \le 4)$, moreover, according to the general theory [3,4], there exists generalized solution since J is positive definite (see statement 2.2).

REMARK 2.2. In particular case (0.4):

$$I_{0i}(x_1) \int\limits_0^l \tilde{D}_i^{-1}(x_1, \tau)\tau^{-\varkappa} d\tau \le D_{2i} \frac{\tau^{1-\varkappa}}{1-\varkappa} \Big|_0^l = D_{2i} \frac{l^{1-\varkappa}}{1-\varkappa} < +\infty \quad \text{if } \varkappa < 1,$$

and, when $\varkappa \geq 1$,

$$I_{0i}(x_1, x_2) \geq \ D_{1i} \lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{l} \tau^{-\varkappa} d\tau = +\infty.$$

Similarly

$$I_{1i}(x_1) \begin{cases} < +\infty & \text{if} \quad \varkappa < 2, \\ = +\infty & \text{if} \quad \varkappa \geq 2. \end{cases}$$

3. ON A MODIFICATION OF THE LAX-MILGRAM THEOREM

The section deals with a modification of the Lax-Milgram theorem as follows:

THEOREM. 3.1. Let V be a real Hilbert space, and $J(u,v)$ be a bilinear form defined on $V \times V$. Let there exist a constant $k > 0$ such that

$$|J(u,v)| \leq k\|u\|_V \|v\|_V \ \ \forall u, v \in V, \tag{3.1}$$

and let

$$J(v,v) = 0 \ \Rightarrow \ v = \theta \text{ in } V \tag{3.2}$$

(θ is the zero element of V). Then for any bounded linear functional F defined on V there exists the unique functional $F_{z_0} \in V^*$ (V^* is the space conjugate to V) such that

$$Fv = F_{z_0} v := \lim_{k \to \infty} J(z_k, v) \ \ \forall v \in V, \tag{3.3}$$

where

$$z_k := C^{-1} t_k \tag{3.4}$$

for any sequence $t_k \in C(V) \subset V$ converging to t_0 uniquely defined by F in view of Riesz theorem. C^{-1} is the inverse operator of the bounded linear operator C:

$$t = Cz \tag{3.5}$$

defined in the space V by the relation

$$J(z,v) = (v,t) \ \ \forall v \in V, \tag{3.6}$$

and fixed $z \in V$.

PROOF. In view of Riesz theorem, it is possible to express every bounded linear functional F in V in the following form

$$Fv = (v, t_0) \ \ \forall v \in V, \tag{3.7}$$

where the element $t_0 \in V$ is uniquely determined by the functional F and $\|t_0\|_V = \|F\|_{V^*}$.

If $z \in V$ is fixed, then the bilinear form $J(z,v)$ represents, obviously, a linear functional in V. This functional is bounded since by (3.1)

$$|J(z,v)| \leq \tilde{k}\|v\|_V, \ \ \tilde{k} = K\|z\|_V = const > 0. \tag{3.8}$$

Then according to above Riesz theorem, there exists a unique $t \in V$ such that holds (3.6), and also, in virtue of (3.6), (3.8),

$$\|t\|_V \leq K\|z\|_V. \tag{3.9}$$

By the relation (3.6) to every $z \in V$ a unique $t \in V$ is assigned. This defines by (3.5) an operator C in V. C is, obviously, linear one, and, in view of (3.9), also bounded. The range $L \equiv C(V)$ of this operator C is a certain linear set in V.More precisly, let L be the metric space whose elements are the elements of that linear set L with the metric of the space V.

We will prove that the mapping (3.5) from V onto L is one-to-one, i. e. $L \sim V$. To this end it is sufficient to prove that to the zero-element of L there corresponds the zero-element of V. Thus, let $\theta = Cz$, i. e., in virtue of (3.6),

$$J(z, v) = (v, \theta) = 0 \ \ \forall v \in V. \tag{3.10}$$

In particular, for $v = z$, (3.10) yields

$$J(z, z) = 0.$$

But then, according to (3.2), $z = \theta$. Hence $\exists C^{-1}$:

$$z = C^{-1}t. \tag{3.11}$$

Let $\{t_k\}$ be a fundamental sequence in L, and thus also in V. Since V is complete, $\exists t_0 \in V$ such that

$$\lim_{k\to\infty} t_k = t_0, \ \ \text{in } V. \tag{3.12}$$

Therefore complete $\overline{L}$ is subspace of V.

Now we will prove that $\overline{L} \equiv V$. The proof will be performed by contradiction. Let $\overline{L} \neq V$. Then there exists an element $w \neq \theta$ in V orthogonal to the subspace $\overline{L}$, so that

$$(w, t) = 0 \tag{3.13}$$

holds $\forall t \in \overline{L}$. Since $w \in V$, in view of (3.6), a unique $t_* \in L \subset \overline{L}$ exists such that

$$J(w, v) = (v, t_*) \ \ \forall v \in V.$$

In particular, for $v = w$, we have

$$J(w, w) = (w, t_*) = 0$$

because of (3.13). Therefore, in virtue of (3.2), $w = \theta$ in V, which is contradiction with assumption $w \neq \theta$. Hence $\overline{L} \equiv V$.

For any bounded linear functional F in V we have (3.7), where $t_0 \in V \equiv \overline{L}$ is uniquely determined by F. For the above $t_0 \in \overline{L}$ there exists sequence $t_k \in L$ which is convergent to t_0 in V. According to (3.11) $\forall t_k \in L \ \ \exists z_k \in V$ such that

$$J(z_k, v) = (v, t_k) \ \ \forall v \in V. \tag{3.14}$$

Functionals $J(z_k, v)$ and (v, t_k) are bounded linear functionals from V^* for fixed k. Now tending $k \to \infty$ in (3.14), since, in view of (3.12), there exists limit (which is equal to (v, t_0) because of continuity of scalar product) in the right hand side, the limit of the left side will also exist, and

$$\lim_{k\to\infty} J(z_k, v) = (v, t_0) \ \ \forall v \in V. \tag{3.15}$$

Then, in virtue of an immediate corollary of the Banach-Steinhaus theorem (see e. g. [5], p. 277, Corollary 1), linear form

$$F_{z_0} : \ v \to \lim_{k\to\infty} J(z_k, v) \tag{3.16}$$

is bounded linear functional on V, which does not depend on choise of $\{z_k\}$, i.e. of $\{t_k\}$ since for any sequence $t_k \to t_0$ in V, on the right hand side of (3.15) we have the same limit (v, t_0). Thus, from (3.7), (3.15) and (3.16) follows (3.3).

REMARK 3.1. If the sequence $\{z_k\}$, $z_k \in V$, corresponding to $\{t_k\}(t_k \in L$ is from (3.12)) is fundamental in V, then because of completness of V $\exists z_0 \in V$ such that

$$\lim_{k\to\infty} z_k = z_0 \text{ in } V.$$

Therefore, taking into account (3.1), we have

$$F_{z_0} v := \lim_{k\to\infty} J(z_k, v) = J(z_0, v)$$

(this is justification of notation F_{z_0}), and from (3.3) follows that there exists unique $z_0 \in V$ such that

$$Fv = J(z_0, v) \ \ \forall v \in V$$

which coincides with the assertion of the Lax-Milqram theorem (see e. g. [6], chapter III, §7, and section 4 below). Therefore $F_{z_0} \in V^*$ can be identified with $z_0 \in V$. If the sequence $\{z_k\}$ is not fundamental in V (let us note that the number sequence $\{J(z_k, v)\}$ is fundamental for fixed $v \in V$), then $F_{z_0} \in V^*$ will be identified with the ideal element z_0 which does not belong to V. Let us denote by V_i set of the ideal elements z, and by $\tilde{V} := V \cup V_i$.

Under product λz_0, $\lambda \in R$, $z_0 \in \tilde{V}$, we understand the ζ_0 identified with the functional

$$F_{\zeta_0} v := \lim_{k\to\infty} J(\lambda z_k, v) = \lim_{k\to\infty} \lambda J(z_k, v) =: \lambda F_{z_0} v$$

Under sum $z_0' + z_0''$ of $z_0', z_0'' \in \tilde{V}$ we understand ζ_0, identified with the functional

$$F_{\zeta_0} v := \lim_{k\to\infty} J(z_k' + z_k'', v) = \lim_{k\to\infty} J(z_k', v) + \lim_{k\to\infty} J(z_k'', v) =: F_{z_0'} v + F_{z_0''} v,$$

where

$$z_k' := C^{-1} t_k', \quad z_k'' := C^{-1} t_k'',$$
$$\lim_{k\to\infty} t_k' = t_0', \quad \lim_{k\to\infty} t_k'' = t_0'' \text{ in } V,$$

t_0' and t_0'' are uniquely defined, in view of Riesz theorem, by bounded linear functionals $F' := (v, t_0')$ and $F'' := (v, t_0'')$ corespondingly. Obviously $\tilde{V}$ is linear vector space.

Now introducing in $\tilde{V}$ a norm as

$$\|z_0\|_{\tilde{V}} := \|F_{z_0}\|_{V^*}, \tag{3.17}$$

$\tilde{V}$ will be Banach, and moreover Hilbert space since such is V^*. Indeed,

$$\|z_0' + z_0''\|_{\tilde{V}}^2 + \|z_0' - z_0''\|_{\tilde{V}}^2 := \|F_{z_0'} + F_{z_0''}\|_{V^*}^2 + \|F_{z_0'} - F_{z_0''}\|_{V^*}^2 =$$
$$= 2(\|F_{z_0'}\|_{V^*}^2 + \|F_{z_0''}\|_{V^*}^2) =: 2(\|z_0'\|_{\tilde{V}}^2 + \|z_0''\|_{\tilde{V}}^2).$$

Therefore scalar product can be defined as

$$(z_0', z_0'')_{\tilde{V}} := 4^{-1}(\|z_0' + z_0''\|_{\tilde{V}}^2 + \|z_0' - z_0''\|_{\tilde{V}}^2).$$

Completness of $\tilde{V}$ is obvious from (3.17).

REMARK 3.2. If $C^{-1}$is bounded operator then from (3.11), (3.12) follows that $\{z_k\}$ is fundamental sequence.

REMARK 3.3. If J is coercive, i. e.

$$|J(v,v)| \geq c\|v\|_V^2, \quad c = const > 0, \quad \forall v \in V,$$

then C^{-1} is bounded operator (see above reference on [6]).

REMARK 3.4. If (3.2) then either $J(v,v) \geq 0 \;\; \forall v \in V$ or $J(v,v) \leq 0 \;\; \forall v \in V$.

PROOF (belongs to S.S. Kharibegashvili). Let us take arbitrary fixed $v_0 \in V$, $v_0 \neq \theta$, from $J(v_0, v_0) \neq 0$ we have either

$$J(v_0, v_0) > 0 \tag{3.18}$$

or

$$J(v_0, v_0) < 0 \tag{3.19}$$

Let us now show that if (3.18) then from $\forall v \in V$, $v \neq \theta$ follows $J(v,v) > 0$, but if (3.19) then $J(v,v) < 0$.

Let first $v \in V$, $v \neq \theta$ be not lineary dependent on v_0 then for $\forall t \in]-\infty, +\infty[$, we have

$$0 \neq J(v_0 + tv, v_0 + tv) = J(v_0, v_0) + [J(v_0, v) + J(v, v_0)]t + J(v,v)t^2, \tag{3.20}$$

since $v_0 + tv \neq \theta \;\; \forall t \in V] - \infty, +\infty[$. Therefore according to well-known property of quadratic trinomial

$$J(v_0, v_0) \cdot J(v,v) > \frac{1}{4}[J(v_0, v_0) + J(v,v)]^2 \geq 0. \tag{3.21}$$

But if (3.18), then in view of (3.21), obviously $J(v,v) > 0$, for arbitrary $v \in V \backslash \{\theta\}$, which is lineary independent of v_0; if (3.19) then from (3.21), we get similary $J(v,v) < 0$, $\forall v \in V \backslash \{\theta\}$, which is lineary independent of v_0.

Let now $v \in V$, $v \neq \theta$, and be lineary dependent on v_0, i.e. $\exists t_0 \in]-\infty, +\infty[$, such that $v_0 + t_0 v = \theta$. Obviously, such t_0 is unique, i.e. the equation $v_0 + tv = \theta$ with respect to t has unique solution $t = t_0$. On the other hand from

$$J(v_0 + tv, v_0 + tv) = 0 \;\Leftrightarrow\; v_0 + tv = \theta$$

follows that the trinomial (3.20) has unique zero $t = t_0$. This is equivalent with the assertion that the discriminant of the trinomial (3.20) is equal to zero:

$$J(v_0, v_0)J(v,v) = \frac{1}{4}[J(v_0, v) + J(v, v_0)]^2 > 0 \tag{3.22}$$

(the last equality is strong since $J(v_0, v_0) \neq 0, \;\; J(v,v) \neq 0$). Finaly from (3.22) follows $J(v,v) > 0$ and $J(v,v) < 0$ when correspondingly (3.18) and (3.19) are fulfilled. Thus the remark is proved.

4 - BENDING IN THE WEIGHTED SOBOLEV SPACE

Let us consider for the equation (0.1) the inhomogenuoes BVCs as follows: on Γ_2

$$w = g_{12}, \quad \frac{\partial w}{\partial n} = g_{22}, \tag{4.1}$$

and on Γ_1 either

$$w = g_{11}, \quad w,_2 = g_{21} \quad \text{if} \quad I_{0i} < +\infty \;\; (0 < \varkappa < 1), \tag{4.2}$$

or

$$w,_2 = g_{21}, \quad Q_2^* = h_2 \quad \text{if} \quad I_{0i} < +\infty \ \ (0 < \varkappa < 1), \tag{4.3}$$

or

$$w = g_{11},$$

$$M_2 = h_1 \begin{cases} \not\equiv 0 & \text{when} \ \ I_{0i} < +\infty \ \ (0 \le \varkappa < 1), \\ \equiv 0 & \text{when} \ \ I_{0i} = +\infty \ \ (1 \le \varkappa < 2) \end{cases} \ \ \text{if } I_{1i} < +\infty \ \ (0 \le \varkappa < 2), \tag{4.4}$$

or

$$M_2 = h_1 \begin{cases} \not\equiv \ 0 \text{ when } I_{0i} < +\infty \ \ (0 \le \varkappa < 1), \\ \equiv 0 \text{ when } I_{0i} = +\infty \ \ (1 \le \varkappa < +\infty), \end{cases}$$
$$\text{if } I_{0i} \le +\infty \ (0 \le \varkappa < +\infty). \tag{4.5}$$
$$Q_2^* = h_2 \begin{cases} \not\equiv \ 0 \text{ when } I_{1i} < +\infty \ (0 \le \varkappa < 2), \\ \equiv 0 \text{ when } I_{1i} = +\infty \ (2 \le \varkappa < +\infty) \end{cases}$$

Let

$$g_{\alpha\beta}, \ h_\alpha \in L_2(\Gamma_1), \quad \alpha, \beta = 1, 2, \tag{4.6}$$

and g_{11}, g_{21}, g_{12}, g_{22} be traces of certain given function $u \in W_2^2(\Omega, \tilde{D})$ (see below (4.7), (4.10)).

REMARK 4.1. Conditions $h_\alpha = 0$, $\alpha = 1, 2$, in (4.4), (4.5) are necessary conditions (see section 1) of boundedness of deflection w and $w,_2$ correspondingly when $I_{1i} = +\infty$ $(2 \le \varkappa < +\infty)$, and $I_{0i} = +\infty$ $(1 \le \varkappa < +\infty)$. The demand of boundedness of w and $w,_2$ is natural in mechanical point of view since we do not consider the case of concentrated shearing forces and moments, when w and $w,_2$ should be, in general, unbounded.

REMARK 4.2. In particular case (0.4), let

$$g_{12} \in W_2^{\frac{3}{2}}(\Gamma_2), \ g_{22} \in W_2^{\frac{1}{2}}(\Gamma_2); \ g_{11} \in W_2^{\frac{3-\varkappa}{2}}(\Gamma_1),$$
$$g_{21} \in W_2^{\frac{1-\varkappa}{2}}(\Gamma_1), \quad h_1, \ h_2 \in L_2(\Gamma_1),$$

and g_{11}, g_{21}, g_{12}, g_{22} be traces of certain given function $u \in W_2^2(\Omega, \tilde{D})$ (see below (4.15), and remark 4.5) and its derivative of first order (if $\partial\Omega$ is of the class C^3, they exist, on Γ_2 always, and on Γ_1 when $0 < \varkappa < 2$ and $0 < \varkappa < 1$ respectively (see [7.8], and [9], section 10).

Let further

$$W_2^2(\Omega, \ D) \tag{4.7}$$

be the set of all measurable functions $u = u(x_1, x_2)$ defined on Ω which have on Ω generalized derivatives $D_{x_1,x_2}^{(\alpha_1,\alpha_2)} u$ for $\alpha_1 + \alpha_2 \le 2$, $\alpha_1, \alpha_2 \in \{0, 1, 2\}$ such that

$$\int\limits_\Omega \left| D_{x_1,x_2}^{(\alpha_1,\alpha_2)} u \right|^2 \rho_{\alpha_1,\alpha_2}(x_1, x_2) d\Omega < +\infty \tag{4.8}$$

for $\rho_{0,0} := 1$, $\rho_{2,0} := D_1(x_1, x_2)$, $\rho_{1,1} := D_4(x_1, x_2)$, $\rho_{0,2} := D_2(x_1, x_2)$. D_i, $i = 1, 2, 3, 4$, are bounded measurable on Ω functions satisfyng (0.2),(0.3). Therefore, since $D_\alpha \ge D_3$ in $\bar{\Omega}$,

$$\int\limits_\Omega D_3 (u,_{\alpha\alpha})^2 d\Omega \le \int\limits_\Omega D_\alpha (u,_{\alpha\alpha})^2 d\Omega < +\infty, \quad \alpha = 1, 2, \tag{4.9}$$

$$\int_\Omega D_3(u,_{11}+u,_{22})^2 d\Omega \le \int_\Omega D_1(u,_{11})^2 d\Omega + 2\int_\Omega D_1^{\frac{1}{2}} u,_{11} \cdot D_2^{\frac{1}{2}} u,_{22}\, d\Omega +$$

$$+\int_\Omega D_2(u,_{22})^2 d\Omega \le \left\{ \left[\int_\Omega D_1(u,_{11})^2 d\Omega\right]^{\frac{1}{2}} + \left[\int_\Omega D_2(u,_{22})^2 d\Omega\right]^{\frac{1}{2}} \right\}^2 < +\infty.$$

Let

$$D := \{\rho_{0,0}, \ \ \rho_{2,0}, \rho_{1,1}, \rho_{0,2}\},$$

and

$$\tilde{D} := D \cup \{\rho_{0,1} := x_2^2\}.$$

Then, in view of (4.7), (4.8), the sense of the notation $W_2^2(\Omega, \tilde{D})$ is clear. Obviously,

$$W_2^2(\Omega, \tilde{D}) \ \ \subset W_2^2(\Omega, D). \tag{4.10}$$

From (0.3), it is clear that

$$\rho^{-1}_{\alpha_1,\alpha_2} \in L_1^{loc}(\Omega).$$

Hence according to [10] $W_2^2(\Omega, D)$ and $W_2^2(\Omega, \tilde{D})$, in virtue of (4.8), (4.9), will be a Banach spaces under the norms

$$\begin{gathered} ||u||^2_{W_2^2(\Omega,D)} := \int_\Omega \big[u^2 + D_3(u,_{11}+u,_{22})^2 + \\ +(D_1 - D_3)(u,_{11})^2 + 4D_4(u,_{12})^2 + (D_2 - D_3)(u,_{22})^2\big]\, d\Omega, \end{gathered} \tag{4.11}$$

$$||u||^2_{W_2^2(\Omega,\tilde{D})} := ||u||^2_{W_2^2(\Omega,D)} + \int_\Omega x_2^2 (u,_2)^2 d\Omega \tag{4.12}$$

respectively, and moreover, Hilbert spaces under the scalar products

$$(u,v)_{W_2^2(\Omega,D)} := \int_\Omega \ [uv + D_3(u,_{11}+u,_{22})(v,_{11}+v,_{22}) + (D_1 - D_3)u,_{11} v,_{11} +$$

$$+4D_4 u,_{12} v,_{12} + (D_2 - D_3)u,_{22} v,_{22}]d\Omega,$$

$$(u,v)_{W_2^2(\Omega,\tilde{D})} := (u,v)_{W_2^2(\Omega,D)} + \int_\Omega x_2^2 u,_2 v,_2\, d\Omega$$

respectively. Let further $f \in L_2(\Omega)$, and

$$V := \overset{0}{W}{}_2^2(\Omega, \tilde{D}) = \overline{C_0^\infty(\Omega)} \quad \text{in the norm of} \quad W_2^2(\Omega, \tilde{D}). \tag{4.13}$$

Since $\rho_{\alpha_1,\alpha_2} \in L_1^{loc}(\Omega)$ we have $C_0^\infty(\Omega) \subset W_2^2(\Omega, D)$, and (4.13) has the sense. In particular case (0.4), we can take as V also

$$\begin{gathered} V := \Big\{ v \in W_2^2(\Omega, \tilde{D}) : \ v|_{\Gamma_2} = 0, \quad \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \quad \text{and either} \\ v\Big|_{\Gamma_1} = 0, \ \ v,_2\Big|_{\Gamma_1} = 0 \qquad \text{if} \quad (4.2) \quad \text{or} \quad v,_2\Big|_{\Gamma_1} = 0 \quad \text{if} \quad (4.3) \\ \text{or} \quad v|_{\Gamma_1} = 0 \quad \text{if } (4.4) \ \text{ in sense of traces} \Big\}. \end{gathered} \tag{4.14}$$

In case (0.4) we could introduce weights and norm as follows:

$$\rho_{0,0} := 1, \ \ \rho_{2,0} \equiv \rho_{1,1} \equiv \rho_{0,2} := x_2^{\varkappa},$$

$$||u||^2_{W^2_2(\Omega, x_2^{\varkappa})} := \int\limits_\Omega \{u^2 + x_2^{\varkappa}\left[(u,_{11} + u,_{22})^2 + (u,_{11})^2 + (u,_{12})^2 + (u,_{22})^2\right]\}d\Omega. \quad (4.15)$$

It is obvious, in view of (0.4), that the latter norm and (4.11) are equivalent in $W^2_2(\Omega, D)$. But we prefer (4.11) since the above resonings are valid for the more general case (0.3).

DEFINITION 4.1. A function $w \in W^2_2(\Omega, \tilde{D})$ will be called a weak solution of the BVP (0.1), (0.3), (4.1)-(4.5) in the space $W^2_2(\Omega, \tilde{D})$ if it satisfies conditions as follows:

$$w - u \in V, \quad (4.16)$$

and $\forall v \in V$

$$J(w, v) := \int\limits_\Omega B(w, v)d\Omega = \int\limits_\Omega vfd\Omega, \quad (4.17)$$

where defined in (2.11)

$$\begin{aligned} B(v, w) := D_3(w,_{11} + w,_{22})(v,_{11} + v,_{22}) + \\ +(D_1 - D_3)w,_{11} v,_{11} + 4D_4 w,_{12} v,_{12} + (D_2 - D_3)w,_{22} v,_{22}, \end{aligned} \quad (4.18)$$

or corespondingly, for particular case (0.4),

$$J(w, v) := \int\limits_\Omega B(w, v)d\Omega = \int\limits_\Omega vfd\Omega + \gamma_2 \int\limits_{\Gamma_1} h_2 v dx_1 - \gamma_1 \int\limits_{\Gamma_1} h_1 v,_2 dx_1, \quad (4.19)$$

where $\gamma_1 = \gamma_2 = 0$ if (4.2); $\gamma_1 = 0$, $\gamma_2 = 1$ if (4.3); $\gamma_1 = 1$, $\gamma_2 = 0$ if (4.4); and if (4.5) then $\gamma_1 = 1$ when $0 \le \varkappa < 1$, and $\gamma_1 = 0$ when $1 \le \varkappa < +\infty$; $\gamma_2 = 1$ when $0 \le \varkappa < 2$, and $\gamma_2 = 0$ when $2 \le \varkappa < +\infty$.

REMARK 4.3. The BVCs (4.1), (4.2), the firsts of (4.3), (4.4) and (4.5), the seconds of (4.3), (4.4) are specified in (4.16) and (4.17), (4.19) correspondingly.

REMARK 4.4. Obviously, if the solution of above problem exists in the classical sense then (4.16), (4.17) and (4.19) will be fulfilled.

THEOREM 4.1. In case (0.3), if $D_0 > 0$, under other above conditions there exists the unique weak solution of the BVP (0.1), (0.3), (4.1)-(4.5). This solution is such that

$$||w||_{W^2_2(\Omega, D)} \le \tilde{C}\left[||f||_{L_2(\Omega)} + ||u||_{W^2_2(\Omega, D)}\right], \quad (4.20)$$

where constant $\tilde{C}$ is independent of f, u.

THEOREM 4.2. In case (0.4), if $0 < \varkappa \le 4$, under other above conditions there exists the unique weak solution of the BVP (0.1), (0.4), (4.1)-(4.5). This solution is such that

$$||w||_{W^2_2(\Omega, D)} \le \tilde{C}\left[||f||_{L_2(\Omega)} + ||u||_{W^2_2(\Omega, D)} + \gamma_1||h_1||_{L_2(\Gamma_1)} + \gamma_2||h_2||_{L_2(\Gamma_1)}\right], \quad (4.21)$$

where constant $\tilde{C}$ is independent of f, u, h_1, h_2.

PROOF of the theorems 4.1 and 4.2. First of all, let us prove that V is a subspace of $W^2_2(\Omega, \tilde{D})$. In case (0.3), it is obvious. In case (0.4), for it we have to show its completeness. Because of linearity of the trace operators and operators in (4.1)-(4.4), obviously, V is a lineal. Since $u \in W_2^2(\Omega, \tilde{D})$ has the traces [7-9]

$$u|_{\Gamma_1} \in W_2^{\frac{3-\varkappa}{2}}(\Gamma_1) \quad \text{for } \ 0 \le \varkappa < 2,$$
$$u|_{\Gamma_2} \in W_2^{\frac{3}{2}}(\Gamma_2) \quad \text{for } \ 0 \le \varkappa < +\infty,$$
$$u,_2|_{\Gamma_1} \in W_2^{\frac{1-\varkappa}{2}}(\Gamma_1) \quad \text{for } \ 0 \le \varkappa < 1,$$
$$\frac{\partial u}{\partial n}\Big|_{\Gamma_2} \in W_2^{1/2}(\Gamma_2) \quad \text{for } \ 0 \le \varkappa < +\infty,$$

then $\exists C_1 =$const> 0 such that

$$||u||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \le C_1 ||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < 2, \tag{4.22}$$
$$||u||_{W_2^{\frac{3}{2}}(\Gamma_2)} \le C_1 ||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < +\infty, \tag{4.23}$$
$$||u,_2||_{W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)} \le C_1 ||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < 1, \tag{4.24}$$
$$\Big|\Big|\frac{\partial u}{\partial n}\Big|\Big|_{W_2^{\frac{1}{2}}((\Gamma_2)} \le C_1 ||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < +\infty. \tag{4.25}$$

Let $v_m \in V$ be fundamental sequence. It will be also fundamental sequence in $W_2^2(\Omega, \tilde{D})$. But the latter is complete set, i. e. $\exists v \in W_2^2(\Omega, \tilde{D})$ such that

$$||v_m - v||_{W_2^2(\Omega,D)} \underset{m \to +\infty}{\longrightarrow} 0.$$

Then, in virtue of (4.22)-(4.25), respectively,

$$||v_m - v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \le C_1 ||v_m - v||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < 2,$$
$$||v_m - v||_{W_2^{\frac{3}{2}}(\Gamma_2)} \le C_1 ||v_m - v||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < +\infty,$$
$$||v_{m,2} - v,_2||_{W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)} \le C_1 ||v_m - v||_{W_2^2(\Omega,D)} \quad \text{for } 0 \le \varkappa < 1,$$
$$\Big|\Big|\frac{\partial v_m}{\partial n} - \frac{\partial v}{\partial n}\Big|\Big|_{W_2^{\frac{1}{2}}(\Gamma_2)} \le C_1 ||v_m - v||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \le \varkappa < +\infty.$$

Therefore

$$||v_m - v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \le \varkappa < 2, \tag{4.26}$$
$$||v_m - v||_{W_2^{\frac{3}{2}}(\Gamma_2)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \le \varkappa < +\infty, \tag{4.27}$$
$$||v_{m,2} - v,_2||_{W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \le \varkappa < 1, \tag{4.28}$$
$$\Big|\Big|\frac{\partial v_m}{\partial n} - \frac{\partial v}{\partial n}\Big|\Big|_{W_2^{\frac{1}{2}}(\Gamma_2)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } \ 0 \le \varkappa < +\infty. \tag{4.29}$$

But since

$$v_m|_{\Gamma_1} = 0 \quad \text{for } \ 0 \le \varkappa < 2, \tag{4.30}$$
$$v_m|_{\Gamma_2} = 0 \quad \text{for } \ 0 \le \varkappa < +\infty, \tag{4.31}$$
$$v_{m,2}|_{\Gamma_1} = 0 \quad \text{for } \ 0 \le \varkappa < 1, \tag{4.32}$$
$$\frac{\partial v_m}{\partial n}\Big|_{\Gamma_2} = 0 \quad \text{for } \ 0 \le \varkappa < +\infty, \tag{4.33}$$

from (4.30) follows

$$||v_m||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} = 0.$$

Then, taking into account (4.26),

$$0 \le ||v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} = |\ ||v_m||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} - ||v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)}| \le ||v_m - v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \underset{m \to +\infty}{\longrightarrow} 0$$

i.e. almost everywhere (a.e.)

$$v|_{\Gamma_1} = 0 \quad \text{for } \ 0 \le \varkappa < 2.$$

Similarly, in view of (4.27)-(4.29), (4.31)-(4.33), we have a.e.

$$v|_{\Gamma_2} = 0 \quad \text{for } \ 0 \le \varkappa < +\infty,$$
$$v_{,2}|_{\Gamma_1} = 0 \quad \text{for } \ 0 \le \varkappa < 1,$$

$$\left.\frac{\partial v}{\partial n}\right|_{\Gamma_2} = 0 \quad \text{for } \ 0 \le \varkappa < +\infty.$$

Thus V is complete i.e. it is a Hilbert space and a subspace of $W_2^2(\Omega, \tilde{D})$.

Further the proof of theorems 4.1 and 4.2 will be realized by means of (see[4,6]) The Lax-Milgram theorem. Let V be a real Hilbert space and $J(w,v)$ be a bilinear form defined on $V \times V$. Let this form be continuous –i.e. let there exist a constant $K > 0$ such that

$$|J(w,v)| \le K \|w\|_V \|v\|_V \tag{4.34}$$

holds $\forall w, v \in V-$ and V-elliptic –i.e. let there exist a constant $\alpha > 0$ such that

$$J(w,w) \ge \alpha \|w\|_V^2 \tag{4.35}$$

holds $\forall w \in V$. Further F be a bounded linear functional from V^* dual of V. Then there exists one and only one element $z \in V$ such that

$$J(z,v) = < F, v > \equiv Fv \quad \forall v \in V \tag{4.36}$$

and

$$\|z\|_V \le \alpha^{-1} \|F\|_{V^*}. \tag{4.37}$$

Obviously, for bilinear form (4.17), in view of (4.18),

$$|J(w,v)| \le \int_\Omega (D_1-D_3)^{\frac{1}{2}}|w,_{11}| \cdot (D_1-D_3)^{\frac{1}{2}}|v,_{11}|d\Omega +$$
$$+\int_\Omega (D_2-D_3)^{\frac{1}{2}}|w,_{22}| \cdot (D_2-D_3)^{\frac{1}{2}}|v,_{22}|d\Omega +$$
$$+\int_\Omega D_3^{\frac{1}{2}}|w,_{11}+w,_{22}| \cdot D_3^{\frac{1}{2}}|v,_{22}+v,_{11}|d\Omega +$$
$$+4\int_\Omega D_4^{\frac{1}{2}}|w,_{12}| \cdot D_4^{\frac{1}{2}}|v,_{12}|d\Omega \le$$
$$\le \left[\int_\Omega (D_1-D_3)(w,_{11})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega (D_1-D_3)(v,_{11})^2 d\Omega\right]^{\frac{1}{2}} + \tag{4.38}$$
$$+\left[\int_\Omega (D_2-D_3)(w,_{22})^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega (D_2-D_3)(v,_{22})^2 d\Omega\right]^{\frac{1}{2}} +$$
$$+\left[\int_\Omega D_3(w,_{11}+w,_{22})^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega D_3(v,_{11}+v,_{22})^2 d\Omega\right]^{\frac{1}{2}} +$$
$$+4\left[\int_\Omega D_4(w,_{12})^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega D_4(w,_{12})^2 d\Omega\right]^{\frac{1}{2}} \le 7||w||_{W_2^2(\Omega,D)}||v||_{W_2^2(\Omega,D)}$$

and, in particular (see below remark 4.5),

$$|J(w,v)| \le 7||w||_V||v||_V \quad \forall w,v \in V. \tag{4.39}$$

Hence (4.34) is fulfilled.

Taking into account that $(D_2-D_3)(v,_{22})^2 \in L_1(\Omega)$, because of

$$x_2^4 \le \frac{D_2-D_3}{D_0}, \tag{4.40}$$

obviously,

$$x_2^4(v,_{22})^2 \in L_1(\Omega)$$

Without loss of generality, we can suppose that Ω lies in Π (see (2.14)), and let $v \in V$ and $v \equiv 0$ in $R_+^2 \backslash \Omega$. Then for fixed x_1

$$v(x_1,\cdot) \in W_2^2(]0,1[,\ x_2^4),\ ||v||^2_{W_2^2(]0,1[,x_2^4)} := \int_0^1 [v^2 + x_2^4(v,_{22})^2]dx_2,$$
$$v(x_1,1)=0, \quad v,_2(x_1,1)=0,$$

and if we suppose that $(D_2-D_3)^{\frac{1}{2}}(v,_2)^2 \in L_1(\Omega)$, i.e. $x_2^2(v,_2)^2 \in L_1(\Omega)$ since $x_2^2 \le \dfrac{(D_2-D_3)^{\frac{1}{2}}}{D_0^{\frac{1}{2}}}$ because of (4.40), it is easy to show (see below lemma 4.1) that the inequalities (2.15), (2.16) are valid for such functions $v \in W_2^2(\Omega,\tilde{D}),\ \Omega \subset \Pi$.

REMARK 4.5. In viev of (2.16), (4.40), when $D_0 > 0$ the norms (4.12), and (4.11) are equivalent in $W_2^2(\Omega, \tilde{D})$, $\Omega \subset \Pi$. Consequently (4.38) holds also for $W_2^2(\Omega, \tilde{D})$.

LEMMA 4.1. If $v \in W_2^2(]0,1[, x_2^4)$, $x_2^2(v,_2)^2 \in L_1(]0,1[)$ and

$$v(x_1, 1) = 0, \quad v,_2(x_1, 1) = 0,$$

then (2.15), (2.16) are valid, i.e.

$$\int\limits_0^1 x_2^4(v,_{22})^2 dx_2 \geq \frac{9}{16} \int\limits_0^1 v^2 dx_2,$$

$$\int\limits_0^1 x_2^4(v,_{22})^2 dx_2 \geq \frac{9}{4} \int\limits_0^1 x_2^2(v,_2)^2 dx_2.$$

PROOF. In the case under consideration $v(x_1, \cdot) \in W_2^2(]c,1[)$ for $\forall [c,1] \subset]0,1]$. Therefore (see [4], Remark 29.6) $v(x_1,\cdot)$ and $v,_2(x_1,\cdot)$ are absolutely continuous on $[\varepsilon, 1]$ for arbitrarily small $\varepsilon =$const> 0. Now we have to repeat proof of lemma 2.1 considering all integrals in limits $\varepsilon \leq x_2 \leq 1$, and then tending ε to $0+$ taking into account that from square summability of $v(x_1,\cdot)$ and $x_2 v,_2(x_1,\cdot)$ follows respectively

$$\lim_{x_2 \to 0+} x_2 v^2(x_1, x_2) = 0, \quad \lim_{x_2 \to 0+} x_2^3[v,_2(x_1, x_2)]^2 = 0.$$

Othervise if we assume $\lim\limits_{x_2 \to 0+} x_2 v^2(x_1, x_2) = c_0(x_1) > 0$, $\lim\limits_{x_2 \to 0+} x_2^3[v,_2(x_1, x_2)]^2 = c_1(x_1) > 0$ then in some right neighbourhood of point $(x_1, 0)$

$$v^2(x_1, x_2) > \frac{c_0(x_1)}{2x_2}, \quad x_2^2[v,_2(x_1, x_2)]^2 > \frac{c_1(x_1)}{2x_2}.$$

But this is contradiction since on the left hand sides we have integrable functions while on the right hand sides we have nonintegrable functions.

In view of (2.15), as $0 < D_0 \leq \frac{D_2 - D_3}{x_2^4}$, for $v \in W_2^2(\Omega, \tilde{D})$, we have

$$\int\limits_\Omega v^2(x_1, x_2) d\Omega = \int\limits_\Pi v^2(x_1, x_2) dx_1, dx_2 = \int\limits_a^b dx_1 \int\limits_0^1 v^2 dx_2 \leq$$

$$\leq \frac{16}{9} \int\limits_a^b dx_1 \int\limits_0^1 x_2^4(v,_{22})^2 dx_2 \leq \frac{16}{9D_0} \int\limits_a^b dx_1 \int\limits_0^1 D_0 x_2^{\varkappa}(v,_{22})^2 dx_2 \leq$$

$$\leq \frac{16}{9D_0} \int\limits_\Omega (D_2 - D_3)(v,_{22})^2 d\Omega.$$

Hence

$$\begin{aligned} ||v||_V^2 := \int\limits_\Omega [v^2 + x_2^2(v,_2)^2 + D_3(v,_{11} + v,_{22})^2 + (D_1 - D_3)(v,_{11})^2 + \\ +4D_4(v,_{12})^2 + (D_2 - D_3)(v,_{22})^2] d\Omega \leq \\ \leq \frac{16}{9D_0} \int\limits_\Omega (D_2 - D_3)(v,_{22})^2 d\Omega + J(v,v) = \overset{*}{C} J(v,v), \end{aligned} \tag{4.41}$$

where

$$\overset{*}{C} := 1 + \frac{16}{9D_0}.$$

(4.41) means V-ellipticity of the bilinear form J. Thus (4.35) is also fulfilled.

Now let us consider the following functional

$$Fv := (v, f) - J(u, v) + \gamma_2 \int\limits_{\Gamma_1} v h_2 dx_1 - \gamma_1 \int\limits_{\Gamma_1} v,_2 h_1 dx_1, \quad v \in V \tag{4.42}$$

(For case (0.3) we have to take $\gamma_1 = \gamma_2 = 0$).

Further

$$|(v, f)| \le ||v||_{L_2(\Omega)} ||f||_{L_2(\Omega)} \le ||v||_V ||f||_{L_2(\Omega)}, \tag{4.43}$$

and, since in case (0.4) traces belonging to $W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)$, $0 \le \varkappa < 2$; $W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)$, $0 \le \varkappa < 1$; are also traces belonging to $L_2(\Gamma_1)$,

$$\left| \int\limits_{\Gamma_1} v\, h_2 dx_1 \right| \le ||v||_{L_2(\Gamma_1)} ||h_2||_{L_2(\Gamma_1)} \le C ||v||_V ||h_2||_{L_2(\Gamma_1)}, \tag{4.44}$$

$$C = const > 0, \quad 0 \le \varkappa < 2,$$

$$\left| \int\limits_{\Gamma_1} v,_2\, h_1 dx_1 \right| \le ||v,_2||_{L_2(\Gamma_1)} ||h_1||_{L_2(\Gamma_1)} \le C ||v||_V ||h_1||_{L_2(\Gamma_1)}, \tag{4.45}$$

$$0 \le \varkappa < 1.$$

After substitution of (4.43), (4.38), (4.44), (4.45) in (4.42), we obtain

$$|Fv| \le \big[||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} + C(\ \gamma_2 ||h_2||_{L_2(\Gamma_1)} +$$

$$+ \gamma_1 ||h_1||_{L_2(\Gamma_1)}) \big] ||v||_V. \tag{4.46}$$

Let us note that by demonstration of boundedness of the functional F defined by (4.42), we did not use that $D_0 > 0$ $(0 \le \varkappa \le 4)$, i.e. the assertion is true for $D_0 \ge 0$ $(0 \le \varkappa < +\infty)$. Therefore the linear functional (4.42) is bounded in V. So in view of (4.39), (4.41), (4.46), according to the Lax-Milgram theorem $\exists z \in V-$ unique such that, in virtue of (4.36), we have

$$J(z, v) = Fv := (v, f) - J(u, v) + \gamma_2 \int\limits_{\Gamma_1} v h_2 dx_1 - \gamma_1 \int\limits_{\Gamma_1} v,_2 h_1 dx_1 \ \forall v \in V,$$

i.e.

$$J(w, v) = (v, f) + \gamma_2 \int\limits_{\Gamma_1} v h_2 dx_1 - \gamma_1 \int\limits_{\Gamma_1} v,_2 h_1 dx_1 \ \ \forall v \in V, \tag{4.47}$$

where

$$w := u + z \in W_2^2(\Omega, \tilde{D}). \tag{4.48}$$

So

$$w - u = z \in V,$$

and (4.16) is fulfilled. (4.47) coincides with (4.19) (in case (0.3) with (4.17)). Thus the existence of the unique weak solution $w \in W_2^2(\Omega, \tilde{D})$ of the BVP (0.1), (0.4) or (0.3), (4.1)-(4.5), has been proved.

From (4.46) it follows that

$$||F||_{V*} \leq ||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} + C(\gamma_2||h_2||_{L_2(\Gamma_1)} + \gamma_1||h_1||_{L_2(\Gamma_1)}). \tag{4.49}$$

In virtue of (4.48), (4.37), (4.49),

$$||w||_{W_2^2(\Omega,D)} \leq ||u||_{W_2^2(\Omega,D)} + ||z||_V \leq ||u||_{W_2^2(\Omega,D)} +$$

$$+\alpha^{-1}\left[||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} + C(\gamma_2||h_2||_{L_2(\Gamma_1)} + \gamma_1||h_1||_{L_2(\Gamma_1)})\right] \leq$$

$$\leq \tilde{C}\left[||f||_{L_2(\Omega)} + ||u||_{W_2^2(\Omega,D)} + \gamma_1||h_1||_{L_2(\Gamma_1)} + \gamma_2||h_2||_{L_2(\Gamma_1)}\right],$$

where

$$\tilde{C} := max\{7\alpha^{-1} + 1, \ \ \alpha^{-1}C\},$$

i.e. (4.20), and (4.21) are valid in cases (0.3) and (0.4) respectively.

REMARK 4.6. Instead of V defined by (4.14), we could consider the space

$$\overset{0}{W}{}_2^2(\Omega, \tilde{D}).$$

Then taking into account that (2.15) is, obviously, valid for $v \in C_0^\infty(\,]0.1[\,)$, the condition (4.41) will be fulfilled for $v \in C_0^\infty(\Omega)$ and hence for $v \in \overset{0}{W}{}_2^2(\Omega, D)$. The condition (4.39) will be also realized on $\overset{0}{W}{}_2^2(\Omega, \tilde{D})-$ subspace of $W_2^2(\Omega, \tilde{D})$. (4.46) (where $\gamma_1 = \gamma_2 = 0$) will be also carried out for $v \in \overset{0}{W}{}_2^2(\Omega, \tilde{D})$. Therefore Theorem 4.2 will be valid if in the definition 4.1 the space V is replaced by the space $\overset{0}{W}{}_2^2(\Omega, \tilde{D}) \subset V$, and (4.19) by (4.17).

5- THE CASE $D_0 = 0$ $(\varkappa > 4)$

In this case only the BVP (0.1), (0.3), (4.1), (4.5), can be correctly posed. Let

$$V \equiv \overset{0}{W}{}_2^2(\Omega, D) := \overline{C_0^\infty(\Omega)}$$

with the norm of $W_2^2(\Omega, D)$.

DEFINITION 5.1. Let $u \in W_2^2(\Omega, D)$ be given, and

$$Fv := (v, f) - J(u, v), \quad v \in V, \tag{5.1}$$

where J is defined by (4.17). $z_0 + u$, where $z_0 \in \tilde{V}$ is identified with $F_{z_0} \in V^*$ (see the modification of the Lax-Milgram theorem in section 3), will be called the ideal solution of the BVP (0.1), (0.3), (4.1), (4.5), if it satisfies condition as follows:

$$F_{z_0}v := \lim_{k\to\infty} J(z_k, v) = \int_\Omega fvd\Omega - J(u,v) \ \ \forall v \in V \equiv \overset{0}{W}{}_2^2(\Omega, D). \tag{5.2}$$

THEOREM 5.1. There exists the unique ideal solution of the BVP (0.1), (0.3), (4.1), (4.5).

PROOF. Obviously,

$$|Fv| \leq ||v||_{L_2(\Omega)}||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} \cdot ||v||_V \leq$$

$$\leq ||f||_{L_2(\Omega)}||v||_V + 7||u||_{W_2^2(\Omega,D)} \cdot ||v||_V,$$

since (4.38) is all the more fulfilled for $v \in V \equiv \overset{0}{W}{}_2^2(\Omega, D) \subset W_2^2(\Omega, D)$. Hence F defined by (5.1) is a bounded linear functional on V. In view of (4.39), which is all the more valid for $V \equiv \overset{0}{W}{}_2^2(\Omega, D)$, (3.1) holds.

From $v \in V$ and

$$J(v, v) = 0$$

follows

$$v = k_1 x_1 + k_2 x_2 + k_3, \quad k_i = \text{ const}, \quad i = 1, 2, 3, \quad \text{a.e. in} \quad \Omega,$$

since from (4.17), (4.18) we have

$$J(v, v) = \int\limits_{\Omega} \left[D_1 (v,_{11})^2 + D_2 (v,_{22})^2 + 2D_3 v,_{11} \cdot v,_{22} + 4D_4 (v,_{12})^2 \right] d\Omega =$$

$$= \int\limits_{\Omega} \Big[D_3 (v,_{11} + v,_{22})^2 + (D_1 - D_3)(v,_{11})^2 +$$

$$+ (D_2 - D_3)(v,_{22})^2 + 4D_4 (v,_{12})^2 \Big] d\Omega = 0,$$

and hence a.e. in Ω

$$v,_{11} = 0, \quad v,_{22} = 0 \quad v,_{12} = 0.$$

On the other hand, it is obvious that

$$u \in W_2^2(\Omega, D) \Rightarrow u \in W_2^2(\Omega_\delta, D) \equiv W_2^2(\Omega_\delta).$$

Hence u and $\dfrac{\partial u}{\partial n}$ have traces on $\Gamma_2 \cap \bar{\Omega}_\delta$ $\forall \delta > 0$, and since $v \in V \equiv \overset{0}{W}{}_2^2(\Omega, D)$, similarly, in sense of traces,

$$v \Big|_{\Gamma_2 \cup \bar{\Omega}_\delta} = \frac{\partial v}{\partial n} \Big|_{\Gamma_2 \cup \bar{\Omega}_\delta} = 0.$$

Therefore $v = 0$ a.e. in Ω i.e. $v \equiv \theta$ in V. Hence (3.2) is fulfilled.

Thus, we can apply the modified Lax-Milgram theorem. Which asserts the existence of the unique ideal element z_0 such that (5.2) is fulfilled.

REMARK 5.2. If, in particular, $z_0 \in V$ then $z_0 + u \in W_2^2(\Omega, D)$, and on Γ_2 the traces of $z_0 + u$, $\dfrac{\partial z_0 + u}{\partial u}$ and u, $\dfrac{\partial u}{\partial n}$ coincide.

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