

# ON A MATHEMATICAL MODEL OF A BAR WITH VARIABLE RECTANGULAR CROSS-SECTION

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## Abstract

Generalizing an idea of I. Vekua [1] who, in order to construct theory of plates and shells, fields of displacements, strains and stresses of three-dimensional theory of linear elasticity expands into the orthogonal Fourier-series by Legendre Polynomials with respect to the variable along thickness, and then leaves only first  $N+1$ ,  $N = 0, 1, \dots$ , terms, in the bar model under consideration all above quantities have been expanded into orthogonal double Fourier-series by Legendre Polynomials with respect to the variables along thickness, and width of the bar, and then first  $(N_3 + 1)(N_2 + 1)$ ,  $N_3, N_2 = 0, 1, \dots$ , terms have been left. This case will be called  $(N_3, N_2)$  approximation. Both in general  $(N_3, N_2)$  and in particular  $(0, 0)$   $(1, 0)$  cases of approximation, the question of wellposedness of initial and boundary value problems, existence and uniqueness of solutions have been investigated. The cases when variable cross-section turns into segments of straight line, and points have been also considered. Such bars will be called cusped bars (see also [2]).

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**Key words:** bar with variable cross-section, cusped bar, elastic bar.

## 1 Some Auxiliary Formulas

Let a domain of  $R^3$  occupied by an elastic bar be

$$V := \{(x_1, x_2, x_3) : 0 < x_1 < l, \quad \overset{(-)}{h_i}(x_1) \leq x_i \leq \overset{(+)}{h_i}(x_1), \quad i = 2, 3, \quad l = \text{const}\}, \quad (1.1)$$

$$2h_i(x_1) := \overset{(+)}{h_i} - \overset{(-)}{h_i} \geq 0, \quad h_i \in C^1(], 0, l[), \quad i = 2, 3.$$

$2h_3$  and  $2h_2$  be correspondingly the thickness and the width of the bar and their maximal quantities be essentially less than the length  $l$  of the bar.

Let further

$$f(x_1, x_2, x_3) \in C^1(V),$$

and define double moments of function  $f$  and its first derivatives  $f_{,j}$  as follows

$${}_{n_3,n_2}^{n_3,n_2}(x_1) := \int_{\substack{(-) \\ h_2}}^{\substack{(+)} \\ h_2} \int_{\substack{(-) \\ h_3}}^{\substack{(+)} \\ h_3} f(x_1, x_2, x_3) P_{n_2}(a_2 x_2 - b_2) P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3, \quad (1.2)$$

$${}_j {}_{n_3,n_2}^{n_3,n_2}(x_1) := \int_{\substack{(-) \\ h_2}}^{\substack{(+)} \\ h_2} \int_{\substack{(-) \\ h_3}}^{\substack{(+)} \\ h_3} f_{,j}(x_1, x_2, x_3) P_{n_2}(a_2 x_2 - b_2) P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3, \quad (1.3)$$

$$n_i = 0, 1, \dots, \quad i = 2, 3, \quad j = 1, 2, 3,$$

where

$$a_i := \frac{1}{h_i}, \quad b_i := \frac{\bar{h}_i}{h_i}, \quad 2\bar{h}_i := {}_{h_i}^{(+)} + {}_{h_i}^{(-)}, \quad i = 2, 3, \quad (1.4)$$

$P_{n_i}$ ,  $i = 2, 3$ , are Legendre Polynomials:

$$\int_{-1}^{+1} P_k(t) P_l(t) dt = \frac{2}{2k+1} \delta_{kl},$$

i.e., if  $t = a_i x_i - b_i$ ,

$$\left( k + \frac{1}{2} \right) \int_{\substack{(-) \\ h_i}}^{\substack{(+)} \\ h_i} P_k(a_i x_i - b_i) P_l(a_i x_i - b_i) a_i dx_i = \delta_{kl}, \quad (1.5)$$

$\delta_{kl}$  is Kronecker symbol, indeces after comma mean differentiation with respect to the corresponding variable.

### Lemma 1.1

$${}_{n_3,n_2}^{n_3,n_2} f = {}_{n_3,n_2}^{n_3,n_2} f_{,1} + \sum_{i=2}^3 \sum_{s=0}^{n_i} {}_{n_i}^{n_i} a_s^i {}_{n_i}^{n_i} f^{\delta_{i2} n_3 + \delta_{i3} s, \delta_{i3} n_2 + \delta_{i2} s} + F_1(f), \quad (1.6)$$

where

$${}_{n_i}^{n_i} a_i := n_i \frac{h_{i,1}}{h_i}, \quad {}_{n_i}^{n_i} a_s := (2s+1) \frac{{}_{h_i}^{(+)} h_{i,1} - (-1)^{n_i+s} {}_{h_i}^{(-)} h_{i,1}}{2h_i}, \quad s \neq n_i, \quad i = 2, 3, \quad (1.7)$$

$$\begin{aligned}
F_1(f) : &= \sum_{i=2}^3 \int_{\substack{(+ \\ h_i \\ - \\ h_i)}}^{(+ \\ h_i)} \left[ -h_{5-i,1} f \left( x_1, \delta_{i2}x_2 + \delta_{i3} \overset{(+)}{h}_2, \delta_{i3}x_3 + \delta_{i2} \overset{(+)}{h}_3 \right) \right. \\
&\quad \left. + (-1)^{n_5-i} h_{5-i,1} f \left( x_1, \delta_{i2}x_2 + \delta_{i3} \overset{(-)}{h}_2, \delta_{i3}x_3 + \delta_{i2} \overset{(-)}{h}_3 \right) \right] \\
&\quad \times P_{n_i}(a_i x_i - b_i) dx_i. \tag{1.8}
\end{aligned}$$

**Proof.** First of all let us recall (see [3], p. 197)

$$\begin{aligned}
P'_{n_i}(a_i x_i - b_i)(a_{i,1} x_i - b_{i,1}) \\
= A_0^i n_i P_{n_i}(a_i x_i - b_i) + \sum_{q=1}^{n_i} A_q^i (2n_i - 2q + 1) P_{n_i-q}(a_i x_i - b_i), \quad \sum_{q=1}^0 (\dots) := 0, \tag{1.9}
\end{aligned}$$

where

$$A_q^i := -\frac{h_{i,1} - (-1)^q h_{i,1}^{(-)}}{2h_i} = A_{q+2s}^i, \quad i = 2, 3, \quad q = 0, 1, 2, \dots, \tag{1.10}$$

and prime means differentiation with respect to the argument.

In view of (1.3), after integration by parts, since  $P_{n_2}(1) = 1$ , and  $P_{n_2}(-1) = (-1)^{n_2}$ , we have

$$\begin{aligned}
{}_{1f}^{n_3, n_2} &:= \int_{\substack{(- \\ h_3)}}^{(+ \\ h_3)} P_{n_3}(a_3 x_3 - b_3) dx_3 \left[ \frac{\partial}{\partial x_1} \int_{\substack{(- \\ h_2)}}^{(+ \\ h_2)} f P_{n_2}(a_2 x_2 - b_2) dx_2 - f \left( x_1, \overset{(+)}{h}_2, x_3 \right) \overset{(+)}{h}_{2,1} \right. \\
&\quad \left. + (-1)^{n_2} f \left( x_1, \overset{(-)}{h}_2, x_3 \right) \overset{(-)}{h}_{2,1} - \int_{\substack{(- \\ h_2)}}^{(+ \\ h_2)} f P'_{n_2}(a_2 x_2 - b_2)(a_{2,1} x_2 - b_{2,1}) dx_2 \right] \\
&= \frac{\partial}{\partial x_1} \left[ \int_{\substack{(- \\ h_3)}}^{(+ \\ h_3)} P_{n_3}(a_3 x_3 - b_3) dx_3 \int_{\substack{(- \\ h_2)}}^{(+ \\ h_2)} f P_{n_2}(a_2 x_2 - b_2) dx_2 \right] \\
&\quad - \overset{(+)}{h}_{3,1} \int_{\substack{(- \\ h_2)}}^{(+ \\ h_2)} f \left( x_1, x_2, \overset{(+)}{h}_3 \right) P_{n_2}(a_2 x_2 - b_2) dx_2 \\
&\quad + (-1)^{n_3} \overset{(-)}{h}_{3,1} \int_{\substack{(- \\ h_2)}}^{(+ \\ h_2)} f \left( x_1, x_2, \overset{(-)}{h}_3 \right) P_{n_2}(a_2 x_2 - b_2) dx_2
\end{aligned}$$

$$\begin{aligned}
& - \int_{\substack{(+)} \\ h_2}^{\substack{(+)} \\ h_3} \int_{\substack{(-)} \\ h_2}^{\substack{(-)} \\ h_3} f P_{n_2}(a_2 x_2 - b_2) P'_{n_3}(a_3 x_3 - b_{3,1})(a_{3,1} x_3 - b_{3,1}) dx_2 dx_3 \\
& - h_{2,1} \int_{\substack{(+)} \\ h_3}^{\substack{(+)} \\ h_3} f \left( x_1, \substack{(+)} \\ h_2, x_3 \right) P_{n_3}(a_3 x_3 - b_3) dx_3 \\
& + (-1)^{n_2} h_{2,1} \int_{\substack{(-)} \\ h_3}^{\substack{(-)} \\ h_3} f \left( x_1, \substack{(-)} \\ h_2, x_3 \right) P_{n_3}(a_3 x_3 - b_3) dx_3 \\
& - \int_{\substack{(+)} \\ h_2}^{\substack{(+)} \\ h_3} \int_{\substack{(-)} \\ h_2}^{\substack{(-)} \\ h_3} f P_{n_3}(a_3 x_3 - b_3) P'_{n_2}(a_2 x_2 - b_2)(a_{2,1} x_2 - b_{2,1}) dx_2 dx_3.
\end{aligned}$$

Further, substituting (1.9) here, and taking into account (1.2),

$$\begin{aligned}
{}_{1f}^{n_3, n_2} &= {}_{f,1}^{n_3, n_2} + \sum_{i=2}^3 \int_{\substack{(+)} \\ h_i}^{\substack{(+)} \\ h_i} \left[ - h_{5-i,1} f \left( x_1, \delta_{i2} x_2 + \delta_{i3} \substack{(+)} \\ h_2, \delta_{i3} x_3 + \delta_{i2} \substack{(+)} \\ h_3 \right) \right. \\
&+ \left. (-1)^{n_{5-i}} h_{5-i,1} f \left( x_1, \delta_{i2} x_2 + \delta_{i3} \substack{(-)} \\ h_2, \delta_{i3} x_3 + \delta_{i2} \substack{(-)} \\ h_3 \right) \right] \\
&\times P_{n_i}(a_i x_i - b_i) dx_i \\
&- \int_{\substack{(+)} \\ h_2}^{\substack{(+)} \\ h_3} \int_{\substack{(-)} \\ h_2}^{\substack{(-)} \\ h_3} f \left[ A_0^3 n_3 P_{n_3}(a_3 x_3 - b_3) + \sum_{q=1}^{n_3} A_q^3 (2n_3 - 2q + 1) P_{n_3-q}(a_3 x_3 - b_3) \right] \\
&\times P_{n_2}(a_2 x_2 - b_2) dx_2 dx_3 \\
&- \int_{\substack{(+)} \\ h_2}^{\substack{(+)} \\ h_3} \int_{\substack{(-)} \\ h_2}^{\substack{(-)} \\ h_3} f \left[ A_0^2 n_2 P_{n_2}(a_2 x_2 - b_2) + \sum_{q=1}^{n_2} A_q^2 (2n_2 - 2q + 1) P_{n_2-q}(a_2 x_2 - b_2) \right] \\
&\times P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3.
\end{aligned}$$

After substitution  $n_i - q = s$ , we obtain

$${}_{1f}^{n_3, n_2} = {}_{f,1}^{n_3, n_2} + F_1(f)$$

$$\begin{aligned}
& - \int_{\substack{(+)(+) \\ (-)(-) \\ h_2 \\ h_3}}^{\substack{(+) \\ h_2 \\ h_3}} f \left[ A_0^3 n_3 P_{n_3} (a_3 x_3 - b_3) + \sum_{s=0}^{n_3-1} A_{n_3-s}^3 (2s+1) P_s (a_3 x_3 - b_3) \right] \\
& \times P_{n_2} (a_2 x_2 - b_2) dx_2 dx_3 \\
& - \int_{\substack{(+)(+) \\ (-)(-) \\ h_2 \\ h_3}}^{\substack{(+) \\ h_2 \\ h_3}} f \left[ A_0^2 n_2 P_{n_2} (a_2 x_2 - b_2) + \sum_{s=0}^{n_2-1} A_{n_2-s}^2 (2s+1) P_s (a_2 x_2 - b_2) \right] \\
& \times P_{n_3} (a_3 x_3 - b_3) dx_2 dx_3, \quad \sum_{s=0}^{-1} (\dots) := 0.
\end{aligned}$$

Therefore, introducing notations (see also (1.10))

$$\begin{aligned}
a_{n_i}^{n_i i} &:= -n_i A_0^i, \\
a_s^{n_i i} &:= -(2s+1) A_{n_i-s}^i = -(2s+1) A_{(n_i-s)+2s}^i \\
&= -(2s+1) A_{n_i+s}^i, \quad s \neq n_i, \quad i = 2, 3,
\end{aligned}$$

follows

$$\begin{aligned}
{}_1 f^{n_3, n_2} &= {}_{f,1}^{n_3, n_2} + F_1(f) + \int_{\substack{(+)(+) \\ (-)(-) \\ h_2 \\ h_3}}^{\substack{(+) \\ h_2 \\ h_3}} f \sum_{s=0}^{n_3} {}_{a_s}^{n_3 3} P_s (a_3 x_3 - b_3) P_{n_2} (a_2 x_2 - b_2) dx_2 dx_3 \\
&+ \int_{\substack{(+)(+) \\ (-)(-) \\ h_2 \\ h_3}}^{\substack{(+) \\ h_2 \\ h_3}} f \sum_{s=0}^{n_2} {}_{a_s}^{n_2 2} P_s (a_2 x_2 - b_2) P_{n_3} (a_3 x_3 - b_3) dx_2 dx_3.
\end{aligned}$$

Thus, in virtue of (1.2),

$${}_1 f^{n_3, n_2} = {}_{f,1}^{n_3, n_2} + F_1(f) + \sum_{s=0}^{n_3} {}_{a_s}^{n_3 3} {}_f^{s, n_2} + \sum_{s=0}^{n_2} {}_{a_s}^{n_2 2} {}_f^{n_3, s}, \quad (1.11)$$

and hence (1.6) is valid.

### Lemma 1.2

$${}_i f^{n_3, n_2} = \sum_{s=0}^{n_i} {}_{a_{is}}^{n_i \delta_{i3}s + \delta_{i2}n_3, \delta_{i2}s + \delta_{i3}n_2} f + F_i(f), \quad i = 2, 3, \quad (1.12)$$

where

$${}_{a_{is}}^{n_i} := -(2s+1) \frac{1 - (-1)^{n_i+s}}{2h_i}, \quad i = 2, 3, \quad (1.13)$$

$$\begin{aligned}
F_i(f) &:= \int_{h_{5-i}}^{(+)h_{5-i}} \left[ f \left( x_1, \delta_{i3}x_2 + \delta_{i2} \overset{(+)h_i}{h_i}, \delta_{i2}x_3 + \delta_{i3} \overset{(+)h_i}{h_i} \right) \right. \\
&\quad - (-1)^{n_i} f \left( x_1, \delta_{i3}x_2 + \delta_{i2} \overset{(-)h_i}{h_i}, \delta_{i2}x_3 + \delta_{i3} \overset{(-)h_i}{h_i} \right) \Big] \\
&\quad \times P_{n_{5-i}}(a_{5-i}x_{5-i} - b_{5-i}) dx_{5-i}, \quad i = 2, 3.
\end{aligned} \tag{1.14}$$

**Prof.** First of all let us recall that

$$P'_{n_i}(a_i x_i - b_i) = \frac{1}{2} \sum_{s=0}^{n_i-1} (2s+1)[1 - (-1)^{n_i+s}] P_s(a_i x_i - b_i). \tag{1.15}$$

Obviously, in view of (1.15), (1.13),

$$\begin{aligned}
{}_2f^{n_3, n_2} &= \int_{h_3}^{(+)h_3} dx_3 P_{n_3}(a_3 x_3 - b_3) \int_{h_2}^{(+)h_2} [f P_{n_2}(a_2 x_2 - b_2)]_{,2} dx_2 \\
&\quad - \int_{h_3}^{(-)h_3} dx_3 P_{n_3}(a_3 x_3 - b_3) \int_{h_2}^{(-)h_2} f P'_{n_2}(a_2 x_2 - b_2) a_2 dx_2 \\
&= \int_{h_3}^{(+)h_3} P_{n_3}(a_3 x_3 - b_3) \left[ f \left( x_1, \overset{(+)h_2}{h_2}, x_3 \right) - (-1)^{n_2} f \left( x_1, \overset{(-)h_2}{h_2}, x_3 \right) \right] dx_3 \\
&\quad - \int_{h_2}^{(+)h_2} \int_{h_3}^{(-)h_3} f \frac{a_2}{2} \sum_{s=0}^{n_2-1} (2s+1)[1 - (-1)^{n_i+s}] P_s(a_2 x_2 - b_2) P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3 \\
&= \int_{h_3}^{(+)h_3} \left[ f \left( x_1, \overset{(+)h_2}{h_2}, x_3 \right) - (-1)^{n_2} f \left( x_1, \overset{(-)h_2}{h_2}, x_3 \right) \right] \\
&\quad \times P_{n_3}(a_3 x_3 - b_3) dx_3 + \sum_{s=0}^{n_2-1} \overset{n_2}{a_{2s}} \overset{n_3, s}{f}.
\end{aligned}$$

Similarly

$${}_3f^{n_3, n_2} = \int_{h_2}^{(+)h_2} \left[ f \left( x_1, x_2, \overset{(+)h_3}{h_3} \right) - (-1)^{n_3} f \left( x_1, x_2, \overset{(-)h_3}{h_3} \right) \right] P_{n_2}(a_2 x_2 - b_2) dx_2 + \sum_{s=0}^{n_3-1} \overset{n_3}{a_{3s}} \overset{s, n_2}{f}.$$

Therefore taking into account that  $\overset{n_i}{a}_{in_i} = 0$ ,  $i = 2, 3$ , we obtain (1.12).

## 2 Main Relations

The aim of this section is to reformulate the main relations of linear theory of elasticity in terms of double moments of the sought fields of displacements, strain and stress tensors.

All double moments at the points  $(x_1, 0, 0)$ ,  $x_1 \in [0, l]$ , where at least one of  $h_i = 0$ ,  $i = 2, 3$ , will be considered as limits of the double moments calculated at points where  $h_i > 0$ ,  $i = 2, 3$ .

To this end let us recall the main relations of the linear theory of elasticity in isotropic case

$$e_{ij} := \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3; \quad (2.1)$$

$$X_{ij} := \lambda \delta_{ij} \theta + 2\mu e_{ij}, \quad \theta = u_{kk}, \quad i, j = 1, 2, 3; \quad (2.2)$$

$$X_{ij,i} + X_j = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i, j = 1, 2, 3; \quad (2.3)$$

where we use Einstein convention, and assume that Latin indices run the values 1, 2, 3;  $u_i$ ,  $i = 1, 2, 3$ , are displacements,  $e_{ij}$  and  $X_{ij}$ ,  $i, j = 1, 2, 3$ , are strain and stress tensors correspondingly;  $\rho$  is density,  $\lambda$  and  $\mu$  are Lame constants;  $X_j$ ,  $j = 1, 2, 3$ , are components of volume force.

Multiplying both parts of equations (2.1), (2.2), (2.3) by  $P_{n_2}(a_2 x_2 - b_2)P_{n_3}(a_3 x_3 - b_3)$ , and then integrating them with respect to  $x_2$  and  $x_3$  within  $h_2^{(-)}$  and  $h_2^{(+)}$ ,  $h_3^{(-)}$  and  $h_3^{(+)}$  correspondingly, taking into account (1.2), we immediately obtain from (2.2)

$$\overset{n_3, n_2}{X}_{ij} = \lambda \delta_{ij} \overset{n_3, n_2}{\theta} + 2\mu \overset{n_3, n_2}{e}_{ij}, \quad \overset{n_3, n_2}{\theta} := \overset{n_3, n_2}{u}_{kk}, \quad i, j = 1, 2, 3. \quad (2.4)$$

From (2.1), we have

$$\begin{aligned} \overset{n_3, n_2}{e}_{ij} &= \frac{1}{2} \int_{h_2^{(-)}}^{h_2^{(+)}} \int_{h_3^{(-)}}^{h_3^{(+)}} u_{i,j} P_{n_2}(a_2 x_2 - b_2) P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3 \\ &+ \frac{1}{2} \int_{h_2^{(-)}}^{h_2^{(+)}} \int_{h_3^{(-)}}^{h_3^{(+)}} u_{j,i} P_{n_2}(a_2 x_2 - b_2) P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3, \quad i, j = 1, 2, 3. \end{aligned}$$

Hence, in virtue of (1.6),

$$\overset{n_3, n_2}{e}_{11} = \overset{n_3, n_2}{u}_{1,1} + \sum_{i=2}^3 \sum_{s=0}^{n_i} \overset{n_i}{a}_s \overset{\delta_{i2} n_3 + \delta_{i3} s, \delta_{i3} n_2 + \delta_{i2} s}{u}_1 + F_1(u_1); \quad (2.5)$$

in virtue of (1.12),

$$e_{\underline{i}}^{n_3, n_2} = \sum_{s=0}^{n_i} a_{\underline{i}s}^{n_i - \delta_{i3}s + \delta_{i2}n_3, \delta_{i2}s + \delta_{i3}n_2} u_i^s + F_{\underline{i}}(u_i), \quad i = 2, 3, \quad (2.6)$$

(hyphen under one of repeating indeces means that we do not sum with respect to these indeces)

$$2^{n_3, n_2} e_{23} = \sum_{s=0}^{n_3} a_{3s}^{n_3 - s, n_2} u_2^s + \sum_{s=0}^{n_2} a_{2s}^{n_2 - s, n_3} u_3^s + F_3(u_2) + F_2(u_3); \quad (2.7)$$

in virtue of both (1.6) and (1.12),

$$\begin{aligned} 2^{n_3, n_2} e_{1i} &= u_{i,1}^{n_3, n_2} + \sum_{k=2}^3 \sum_{s=0}^{n_k} a_s^{n_k - k, n_k} u_i^{s + \delta_{k2}n_3 + \delta_{k3}s, \delta_{k3}n_2 + \delta_{k2}s} \\ &+ \sum_{s=0}^{n_i} a_{\underline{i}s}^{n_i - \delta_{i3}s + \delta_{i2}n_3, \delta_{i2}s + \delta_{i3}n_2} u_1^s + F_1(u_i) + F_i(u_1), \quad i = 2, 3. \end{aligned} \quad (2.8)$$

Let  $u_j(x_1, x_2, x_3) \in C^2(V)$ ,  $j = 1, 2, 3$ , then we have uniformly convergent series as follows

$$\begin{aligned} u_j(x_1, x_2, x_3) &= \sum_{k,l=0}^{\infty} a_2 a_3 \left( k + \frac{1}{2} \right) \left( l + \frac{1}{2} \right) {}^{k,l} \hat{u}_j(x_1) P_k(a_3 x_3 - b_3) P_l(a_2 x_2 - b_2). \end{aligned} \quad (2.9)$$

Consequently,

$$\begin{aligned} u_j &\left( x_1, \delta_{i2}x_2 + \delta_{i3}h_2, \delta_{i3}x_3 + \delta_{i2}h_3 \right)^{(\pm)} \\ &= \sum_{k,l=0}^{\infty} a_2 a_3 \left( k + \frac{1}{2} \right) \left( l + \frac{1}{2} \right) {}^{k,l} \hat{u}_j(x_1) \\ &\times P_k \left( a_3 \left( \delta_{i3}x_3 + \delta_{i2}h_3 \right)^{(\pm)} - b_3 \right) P_l \left( a_2 \left( \delta_{i2}x_2 + \delta_{i3}h_2 \right)^{(\pm)} - b_2 \right) \\ &= \sum_{k,l=0}^{\infty} a_2 a_3 \left( k + \frac{1}{2} \right) \left( l + \frac{1}{2} \right) {}^{k,l} \hat{u}_j \\ &\times \left[ \delta_{i2}(\pm 1)^k P_l(a_2 x_2 - b_2) + \delta_{i3}(\pm 1)^l P_k(a_3 x_3 - b_3) \right], \quad j = 1, 2, 3, \quad i = 2, 3, \end{aligned}$$

similarly

$$\begin{aligned} u_j &\left( x_1, \delta_{i3}x_2 + \delta_{\underline{i}2}h_i, \delta_{i2}x_3 + \delta_{\underline{i}3}h_i \right)^{(\pm)} \\ &= \sum_{k,l=0}^{\infty} a_2 a_3 \left( k + \frac{1}{2} \right) \left( l + \frac{1}{2} \right) {}^{k,l} \hat{u}_j(x_1) \\ &\times \left[ \delta_{i2}(\pm 1)^l P_k(a_3 x_3 - b_3) + \delta_{i3}(\pm 1)^k P_l(a_2 x_2 - b_2) \right], \quad j = 1, 2, 3, \quad i = 2, 3, \end{aligned}$$

and, in virtue of (1.5),

$$\begin{aligned}
& (\pm 1)^{n_{5-i}} h_{5-\underline{i},1} \int_{\substack{(+ \\ h_i) \\ (- \\ h_i)}} u_j \left( x_1, \delta_{i2}x_2 + \delta_{i3}^{(\pm)} h_2, \delta_{i3}x_3 + \delta_{i2}^{(\pm)} h_3 \right) P_{n_i}(a_i x_i - b_i) dx_i \\
&= (\pm 1)^{n_{5-i}} h_{5-\underline{i},1} \\
&\quad \times \left[ \sum_{k,l=0}^{\infty} a_3 \left( k + \frac{1}{2} \right) u_j^{k,l} \delta_{i2} (\pm 1)^k \delta_{l n_2} + \sum_{k,l=0}^{\infty} a_2 \left( l + \frac{1}{2} \right) u_j^{k,l} \delta_{i3} (\pm 1)^l \delta_{k n_3} \right] \\
&= (\pm 1)^{n_{5-i}} h_{5-\underline{i},1} \\
&\quad \times \left[ \sum_{k=0}^{\infty} a_3 \left( k + \frac{1}{2} \right) u_j^{k,n_2} (\pm 1)^k \delta_{i2} + \sum_{l=0}^{\infty} a_2 \left( l + \frac{1}{2} \right) u_j^{n_3,l} (\pm 1)^l \delta_{i3} \right] \\
&= (\pm 1)^{n_{5-i}} h_{5-\underline{i},1} \sum_{s=0}^{\infty} a_{5-i} \left( s + \frac{1}{2} \right) (\pm 1)^s u_j^{\delta_{i2}s + \delta_{i3}n_3, \delta_{i2}n_2 + \delta_{i3}s}, \\
&\qquad\qquad\qquad j = 1, 2, 3, \quad i = 2, 3,
\end{aligned}$$

similarly

$$\begin{aligned}
& (\pm 1)^{n_{\underline{i}}} \int_{\substack{(+ \\ h_{5-i}) \\ (- \\ h_{5-i})}} u_j \left( x_1, \delta_{i3}x_2 + \delta_{i2}^{(\pm)} h_i, \delta_{i2}x_3 + \delta_{i3}^{(\pm)} h_i \right) P_{n_{5-i}}(a_{5-i} x_{5-i} - b_{5-i}) dx_{5-i} \\
&= (\pm 1)^{n_{\underline{i}}} \sum_{s=0}^{\infty} (\pm 1)^s \left( s + \frac{1}{2} \right) a_{5-i}^{\delta_{i3}s + \delta_{i2}n_3, \delta_{i2}s + \delta_{i3}n_2}, \quad j = 1, 2, 3, \quad i = 2, 3.
\end{aligned}$$

Thus, in view of (1.8), (1.14),

$$\begin{aligned}
F_1(u_j) &= \sum_{i=2}^3 \sum_{s=0}^{\infty} \left[ -h_{5-i,1}^{(+)} + (-1)^{s+n_{5-i}} h_{5-i,1}^{(-)} \right] \left( s + \frac{1}{2} \right) a_{5-i}^{\delta_{i2}s + \delta_{i3}n_3, \delta_{i2}n_2 + \delta_{i3}s} \\
&= - \sum_{i=2}^3 \sum_{s=0}^{\infty} b_{is}^{n_i} a_i^{\delta_{i3}s + \delta_{i2}n_3, \delta_{i3}n_2 + \delta_{i2}s}, \quad j = 1, 2, 3, \tag{2.10}
\end{aligned}$$

$$\begin{aligned}
F_i(u_j) &= \sum_{s=0}^{\infty} [1 - (-1)^{s+n_i}] \left( s + \frac{1}{2} \right) a_i^{\delta_{i3}s + \delta_{i2}n_3, \delta_{i3}n_2 + \delta_{i2}s} \\
&= - \sum_{s=0}^{\infty} b_{is}^{n_i} a_i^{\delta_{i3}s + \delta_{i2}n_3, \delta_{i3}n_2 + \delta_{i2}s}, \quad j = 1, 2, 3, i = 2, 3, \tag{2.11}
\end{aligned}$$

where

$$b_{is}^{n_i} := \left( s + \frac{1}{2} \right) \left[ h_{i,1}^{(+)} - (-1)^{s+n_i} h_{i,1}^{(-)} \right], \quad b_{is}^{n_i} := - \left( s + \frac{1}{2} \right) [1 - (-1)^{n_i+s}], \tag{2.12}$$

i.e., in virtue of (1.7), (1.13),

$$\underline{a}_s^i = \begin{cases} n_i h_{i,1} \frac{1}{h_i}, & s = n_i, \\ b_s^i a_i, & s \neq n_i, \end{cases} \quad \underline{a}_{is}^i = \begin{cases} 0 = b_{in_i}^{n_i}, & s = n_i, \\ b_{is}^i a_i, & s \neq n_i. \end{cases} \quad (2.13)$$

So in (2.5)-(2.8),  $F_i(u_j)$ ,  $i, j = 1, 2, 3$ , are given by (2.10), (2.11) i.e., taking into account (2.13),

$$\begin{aligned} e_{11}^{n_3, n_2} &= u_{1,1}^{n_3, n_2} + \sum_{i=2}^3 \sum_{s=0}^{n_i} \underline{a}_s^i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i3} n_2 + \delta_{i2} s u_1 - \sum_{i=2}^3 \sum_{s=0}^{\infty} b_s^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_1 \\ &= u_{1,1}^{n_3, n_2} + \sum_{i=2}^3 \left( \underline{a}_{n_i}^i u_1^{n_3, n_2} - \sum_{s=n_i}^{\infty} b_s^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_1 \right), \end{aligned} \quad (2.14)$$

$$\begin{aligned} e_{ii}^{n_3, n_2} &= \sum_{s=0}^{n_i} \underline{a}_{is}^i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_i - \sum_{s=0}^{\infty} b_{is}^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_i \\ &= - \sum_{s=n_i+1}^{\infty} b_{is}^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_i, \quad i = 2, 3, \end{aligned} \quad (2.15)$$

$$\begin{aligned} 2^{n_3, n_2} e_{23} &= \sum_{s=0}^{n_3} b_{3s}^{n_3} a_3^{s, n_2} u_2 + \sum_{s=0}^{n_2} b_{2s}^{n_2} a_2^{n_3, s} u_3 - \sum_{s=0}^{\infty} b_{3s}^{n_3} a_3^{s, n_2} u_2 - \sum_{s=0}^{\infty} b_{2s}^{n_2} a_2^{n_3, s} u_3 \\ &= - \sum_{i=2}^3 \sum_{s=n_i+1}^{\infty} b_{is}^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_{5-i}, \end{aligned} \quad (2.16)$$

$$\begin{aligned} 2^{n_3, n_2} e_{1i} &= u_{i,1}^{n_3, n_2} + \sum_{k=2}^3 \sum_{s=0}^{n_k} \underline{a}_s^k \delta_{k2} n_3 + \delta_{k3} s, \delta_{k2} s + \delta_{k3} n_2 u_i + \sum_{s=0}^{n_i} \underline{a}_{is}^i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_1 \\ &\quad - \sum_{k=2}^3 \sum_{s=0}^{\infty} b_s^k a_k \delta_{k2} n_3 + \delta_{k3} s, \delta_{k2} s + \delta_{k3} n_2 u_i - \sum_{s=0}^{\infty} b_{is}^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_1 \\ &= u_{i,1}^{n_3, n_2} + \sum_{k=2}^3 \left( \underline{a}_{nk}^k u_i^{n_3, n_2} - \sum_{s=n_k}^{\infty} b_s^k a_k \delta_{k2} n_3 + \delta_{k3} s, \delta_{k2} s + \delta_{k3} n_2 u_i \right) \\ &\quad - \sum_{s=n_i+1}^{\infty} b_{is}^i a_i \delta_{i2} n_3 + \delta_{i3} s, \delta_{i2} s + \delta_{i3} n_2 u_1, \quad i = 2, 3. \end{aligned} \quad (2.17)$$

From (2.3), we have

$$\int_{\substack{(+) \\ h_2}}^{\substack{(+) \\ h_3}} \int_{\substack{(-) \\ h_2}}^{\substack{(-) \\ h_3}} X_{ij,i} P_{n_2}(a_2 x_2 - b_2) P_{n_3}(a_3 x_3 - b_3) dx_2 dx_3 + \int_{\substack{(+) \\ h_2}}^{\substack{(+) \\ h_3}} \int_{\substack{(-) \\ h_2}}^{\substack{(-) \\ h_3}} X_j^{n_3, n_2} = \rho \frac{\partial^2}{\partial t^2} \int_{\substack{(+) \\ h_2}}^{\substack{(+) \\ h_3}} \int_{\substack{(-) \\ h_2}}^{\substack{(-) \\ h_3}} u_j^{n_3, n_2} dx_2 dx_3, \quad j = 1, 2, 3.$$

Further, in virtue of (1.6), (1.12),

$$\begin{aligned} X_{1j,1}^{n_3, n_2} &+ \sum_{i=2}^3 \sum_{s=0}^{n_i} a_s^{i,j} X_{1j}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i3}n_2 + \delta_{i2}s} + F_1(X_{1j}) + \sum_{i=2}^3 \sum_{s=0}^{n_i} a_{is} X_{ij}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \\ &+ \sum_{i=2}^3 F_i(X_{ij}) + X_j^{n_3, n_2} = \rho \frac{\partial^2 u_j^{n_3, n_2}}{\partial t^2}, \quad j = 1, 2, 3, \quad n_2, n_3 = 0, 1, \dots \quad (2.18) \end{aligned}$$

But, in view of (1.8), (1.14),

$$\begin{aligned} &\sum_{i=1}^3 F_i(X_{ij}) \\ &= \int_{\substack{(+)} \\ h_2}}^{(+)} \left[ -h_{3,1}^{(+)} X_{1j} \left( x_1, x_2, h_3 \right) + (-1)^{n_3} h_{3,1}^{(-)} X_{1j} \left( x_1, x_2, h_3 \right) \right] P_{n_2}(a_2 x_2 - b_2) dx_2 \\ &+ \int_{\substack{(-)} \\ h_2}}^{(+)} \left[ -h_{2,1}^{(+)} X_{1j} \left( x_1, h_2, x_3 \right) + (-1)^{n_2} h_{2,1}^{(-)} X_{1j} \left( x_1, h_2, x_3 \right) \right] P_{n_3}(a_3 x_3 - b_3) dx_3 \\ &+ \int_{\substack{(+)} \\ h_3}}^{(+)} \left[ X_{2j} \left( x_1, h_2, x_3 \right) - (-1)^{n_2} X_{2j} \left( x_1, h_2, x_3 \right) \right] P_{n_3}(a_3 x_3 - b_3) dx_3 \\ &+ \int_{\substack{(-)} \\ h_3}}^{(+)} \left[ X_{3j} \left( x_1, x_2, h_3 \right) - (-1)^{n_3} X_{3j} \left( x_1, x_2, h_3 \right) \right] P_{n_2}(a_2 x_2 - b_2) dx_2 \\ &= \int_{\substack{(-)} \\ h_2}}^{(+)} \left[ \frac{(\nu_{31})}{\sqrt{h_3^+}} X_{1j} \left( x_1, x_2, h_3 \right) + \frac{(\nu_{33})}{\sqrt{h_3^+}} X_{3j} \left( x_1, x_2, h_3 \right) \right] P_{n_2}(a_2 x_2 - b_2) dx_2 \\ &+ (-1)^{n_3} \int_{\substack{(-)} \\ h_2}}^{(+)} \left[ \frac{(-\nu_{31})}{\sqrt{h_3^-}} X_{1j} \left( x_1, x_2, h_3 \right) + \frac{(-\nu_{33})}{\sqrt{h_3^-}} X_{3j} \left( x_1, x_2, h_3 \right) \right] P_{n_2}(a_2 x_2 - b_2) dx_2 \\ &+ \int_{\substack{(+)} \\ h_3}}^{(+)} \left[ \frac{(\nu_{21})}{\sqrt{h_2^+}} X_{1j} \left( x_1, h_2, x_3 \right) + \frac{(\nu_{22})}{\sqrt{h_2^+}} X_{2j} \left( x_1, h_2, x_3 \right) \right] P_{n_3}(a_3 x_3 - b_3) dx_3 \end{aligned}$$

$$+(-1)^{n_2} \int_{\frac{(-)}{h_3}}^{\frac{(+)}{h_3}} \left[ \frac{(\pm)}{\sqrt{h_2^-}} X_{1j} \left( x_1, \frac{(-)}{h_2}, x_3 \right) + \frac{(\pm)}{\sqrt{h_2^+}} X_{2j} \left( x_1, \frac{(-)}{h_2}, x_3 \right) \right] P_{n_3}(a_3 x_3 - b_3) dx_3,$$

$j = 1, 2, 3,$

because of

$$\begin{aligned} \frac{(\pm)}{\nu_{ii}} &= \cos \left( \frac{(\pm)}{\nu_i}, x_i \right) = \pm \sqrt{h_i^\pm}, \quad i = 2, 3, \\ \frac{(\pm)}{\nu_{ij}} &= \cos \left( \frac{(\pm)}{\nu_i}, x_j \right) = \mp h_{i,j} \sqrt{h_i^\pm}, \quad i \neq j, \quad i = 2, 3, \quad j = 1, 2, 3, \end{aligned}$$

where

$$\sqrt{h_i^\pm} := \left[ \sqrt{1 + \left( \frac{(\pm)}{h_{i,1}} \right)^2 + \left( \frac{(\pm)}{h_{i,j}} \right)^2} \right]^{-1}, \quad i \neq j, \quad i, j = 2, 3,$$

$\frac{(\pm)}{\nu_i}$  is normal of the surface  $x_i = h_i(x_1)$ ,  $i = 2, 3$ , directed exterior to the bar, i.e.

$$\sqrt{h_i^\pm} = \left[ \sqrt{1 + \left( \frac{(\pm)}{h_{i,1}} \right)^2} \right]^{-1}, \quad \frac{(\pm)}{\nu_{23}} = 0, \quad \frac{(\pm)}{\nu_{32}} = 0.$$

So,

$$\begin{aligned} \sum_{i=1}^3 F_i(X_{ij}) &= \frac{1}{\sqrt{h_3^+}} \int_{\frac{(-)}{h_2}}^{\frac{(+)}{h_2}} X_{\frac{(+)}{\nu_{3j}}} \left( x_1, x_2, \frac{(+)}{h_3} \right) P_{n_2}(a_2 x_2 - b_2) dx_2 \\ &+ \frac{(-1)^{n_3}}{\sqrt{h_3^-}} \int_{\frac{(-)}{h_2}}^{\frac{(+)}{h_2}} X_{\frac{(-)}{\nu_{3j}}} \left( x_1, x_2, \frac{(-)}{h_3} \right) P_{n_2}(a_2 x_2 - b_2) dx_2 \\ &+ \frac{1}{\sqrt{h_2^+}} \int_{\frac{(-)}{h_3}}^{\frac{(+)}{h_3}} X_{\frac{(+)}{\nu_{2j}}} \left( x_1, \frac{(+)}{h_2}, x_3 \right) P_{n_3}(a_3 x_3 - b_3) dx_3 \\ &+ \frac{(-1)^{n_2}}{\sqrt{h_2^-}} \int_{\frac{(-)}{h_3}}^{\frac{(+)}{h_3}} X_{\frac{(-)}{\nu_{2j}}} \left( x_1, \frac{(-)}{h_2}, x_3 \right) P_{n_3}(a_3 x_3 - b_3) dx_3 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=2}^3 \int_{\frac{(-)}{h_i}}^{\frac{(+)}{h_i}} \left[ \frac{1}{\sqrt{h_{5-i}^+}} X_{\nu_{5-i}^{(\pm)} j} \left( x_1, \delta_{i2}x_2 + \delta_{i3}h_2, \delta_{i2}h_3 + \delta_{i3}x_3 \right) + \right. \\
&\quad \left. + \frac{(-1)^{n_{5-i}}}{\sqrt{h_{5-i}^-}} X_{\nu_{5-i}^{(\pm)} j} \left( x_1, \delta_{i2}x_2 + \delta_{i3}h_2, \delta_{i2}h_3 + \delta_{i3}x_3 \right) \right] \\
&\times P_{n_i}(a_i x_i - b_i) dx_i, \quad j = 1, 2, 3,
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
X_{\nu_i^{(\pm)} j} \left( x_1, \delta_{i2}h_2, \delta_{i3}x_2, \delta_{i2}x_3 + \delta_{i3}h_3 \right) &= X_{kj} \nu_{ik}^{(\pm)}, \quad i = 2, 3, \quad j = 1, 2, 3, \\
X_{\nu_i^{(\pm)} j} \left( x_1, \delta_{i2}h_2 + \delta_{i3}x_2, \delta_{i2}x_3 + \delta_{i3}h_3 \right), \quad j = 1, 2, 3, \text{ are components of surface force} \\
\text{acting on the surface } x_i = h_i(x_1), i = 2, 3, \text{ from the outside of the bar.}
\end{aligned}$$

From (2.18), (2.19) follows

$$\begin{aligned}
X_{1j,1}^{n_3, n_2} &+ \sum_{i=2}^3 \sum_{s=0}^{n_i} \left( \begin{matrix} n_i & \delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2 \\ a_s & X_{1j} \end{matrix} + \begin{matrix} n_i & \delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2 \\ a_{is} & X_{ij} \end{matrix} \right) \\
&+ X_j^0 = \rho \frac{\partial^2 u_j^{n_3, n_2}}{\partial t^2}, \quad j = 1, 2, 3, \quad n_3, n_2 = 0, 1, \dots,
\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}
X_j^0 &:= \sum_{i=2}^3 \int_{\frac{(-)}{h_i}}^{\frac{(+)}{h_i}} \left[ \sqrt{1 + \left( \frac{(+)}{h_{5-i,1}} \right)^2} X_{\nu_{5-i}^{(\pm)} j} \left( x_1, \delta_{i2}x_2 + \delta_{i3}h_2, \delta_{i2}h_3 + \delta_{i3}x_3 \right) \right. \\
&\quad \left. + (-1)^{n_{5-i}} \sqrt{1 + \left( \frac{(-)}{h_{5-i,1}} \right)^2} X_{\nu_{5-i}^{(\pm)} j} \left( x_1, \delta_{i2}x_2 + \delta_{i3}h_2, \delta_{i2}h_3 + \delta_{i3}x_3 \right) \right] \\
&\times P_{n_i}(a_i x_i - b_i) dx_i + X_j^{n_3, n_2}, \quad j = 1, 2, 3, \quad n_3 n_2 = 0, 1, \dots,
\end{aligned}$$

i.e.

$$\begin{aligned}
X_j^0 &:= \sum_{i=2}^3 \left[ \sqrt{1 + \left( \frac{(+)}{h_{5-i,1}} \right)^2} Q_{\nu_{5-i}^{(\pm)} j}(x_1) + (-1)^{n_{5-i}} \sqrt{1 + \left( \frac{(-)}{h_{5-i,1}} \right)^2} Q_{\nu_{5-i}^{(\pm)} j} \right] \\
&+ X_j^{n_3, n_2}, \quad j = 1, 2, 3, \quad n_3 n_2 = 0, 1, \dots,
\end{aligned} \tag{2.21}$$

with obvious expressions for  $Q_{\nu_i^{(\pm)} j}(x_1)$ .

From (2.20), taking into account (2.12), (2.13), we obtain

$$\begin{aligned} X_{1j,1}^{n_3, n_2} &+ \sum_{s=0}^{n_2} \left( \overset{n_2}{a}_s^{n_3, s} X_{1j} + \overset{n_2}{a}_{2s}^{n_3, s} X_{2j} \right) + \sum_{s=0}^{n_3} \left( \overset{n_3}{a}_s^{s, n_2} X_{1j} + \overset{n_3}{a}_{3s}^{s, n_2} X_{3j} \right) \\ &+ X_j^0 = \rho \frac{\partial^2}{\partial t^2} \overset{n_3, n_2}{u}_j, \quad j = 1, 2, 3, \quad n_3, n_2 = 0, 1, \dots; \end{aligned}$$

$$\begin{aligned} X_{1j,1}^{n_3, n_2} &+ \frac{1}{h_2} \left[ n_2 h_{2,1}^{n_3, n_2} X_{1j} + \sum_{s=0}^{n_2-1} \left( \overset{n_2}{b}_s^{n_3, s} X_{1j} + \overset{n_2}{b}_{2s}^{n_3, s} X_{2j} \right) \right] \\ &+ \frac{1}{h_3} \left[ n_3 h_{3,1}^{n_3, n_2} X_{1j} + \sum_{s=0}^{n_3-1} \left( \overset{n_3}{b}_s^{s, n_2} X_{1j} + \overset{n_3}{b}_{3s}^{s, n_2} X_{3j} \right) \right] \\ &+ X_j^0 = \rho \frac{\partial^2}{\partial t^2} \overset{n_3, n_2}{u}_j, \quad j = 1, 2, 3, \quad n_3, n_2 = 0, 1, \dots. \end{aligned}$$

Further, after multiplying of both sides by  $h_2^{n_2} h_3^{n_3}$ ,

$$\begin{aligned} h_2^{n_2} h_3^{n_3} X_{1j,1}^{n_3, n_2} &+ n_2 h_2^{n_2-1} h_{2,1} h_3^{n_3} X_{1j}^{n_3, n_2} + n_3 h_2^{n_2} h_3^{n_3-1} h_{3,1} X_{1j}^{n_3, n_2} \\ &+ h_2^{n_2-1} h_3^{n_3} \sum_{s=0}^{n_2-1} \left( \overset{n_2}{b}_s^{n_3, s} X_{1j} + \overset{n_2}{b}_{2s}^{n_3, s} X_{2j} \right) \\ &+ h_2^{n_2} h_3^{n_3-1} \sum_{s=0}^{n_3-1} \left( \overset{n_3}{b}_s^{s, n_2} X_{1j} + \overset{n_3}{b}_{3s}^{s, n_2} X_{3j} \right) \\ &+ h_2^{n_2} h_3^{n_3} X_j^0 = \rho h_2^{n_2} h_3^{n_3} \frac{\partial^2}{\partial t^2} \overset{n_3, n_2}{u}_j, \quad j = 1, 2, 3, \quad n_3, n_2 = 0, 1, \dots. \end{aligned}$$

Finally

$$\begin{aligned} &\left( h_2^{n_2} h_3^{n_3} X_{1j}^{n_3, n_2} \right),_1 \\ &+ h_2^{n_2-1} h_3^{n_3} \sum_{s=0}^{n_2-1} \left( \overset{n_2}{b}_s^{n_3, s} X_{1j} + \overset{n_2}{b}_{2s}^{n_3, s} X_{2j} \right) + h_2^{n_2} h_3^{n_3-1} \sum_{s=0}^{n_3-1} \left( \overset{n_3}{b}_s^{s, n_2} X_{1j} + \overset{n_3}{b}_{3s}^{s, n_2} X_{3j} \right) \\ &+ h_2^{n_2} h_3^{n_3} X_j^0 = \rho h_2^{n_2} h_3^{n_3} \frac{\partial^2}{\partial t^2} \overset{n_3, n_2}{u}_j, \quad j = 1, 2, 3, \quad n_3, n_2 = 0, 1, \dots, \quad (2.22) \end{aligned}$$

i.e.

$$\begin{aligned} &\left( h_2^{n_2} h_3^{n_3} X_{1j}^{n_3, n_2} \right),_1 \\ &+ \sum_{i=2}^3 h_2^{n_2-\delta_{i2}} h_3^{n_3-\delta_{i3}} \sum_{s=0}^{n_i-1} \left( \overset{n_i}{b}_s^{n_3, n_2} X_{1j} + \overset{n_i}{b}_{is}^{n_3, n_2} X_{ij} \right) \\ &+ h_2^{n_2} h_3^{n_3} X_j^0 = \rho h_2^{n_2} h_3^{n_3} \frac{\partial^2}{\partial t^2} \overset{n_3, n_2}{u}_j, \quad j = 1, 2, 3, \quad n_3, n_2 = 0, 1, \dots. \quad (2.23) \end{aligned}$$

The system (2.20), i.e. (2.22), is exact since it has been obtained as consequence from exact equilibrium equations of linear theory of elasticity. Joining to the above system, the system, obtained from (2.4) after substitution of (2.14)-(2.17), which is exact up to the Hook's law, we have full system of equations containing as unknown double moments  ${}^{n_3, n_2} \dot{u}_j$ ,  $j = 1, 2, 3$ ,  $n_3, n_2 = 0, 1, \dots$ , of components of displacement vector.

### 3 $(N_3, N_2)$ Approximation

Now, we assume

$${}^{n_3, n_2} \dot{u}_j = 0, \quad j = 1, 2, 3, \quad (3.1)$$

if at least one of the following conditions

$$n_i > N_i, \quad i = 2, 3,$$

is fulfilled. Then from (2.9), we get

$$\begin{aligned} u_j(x_1, x_2, x_3) &\approx \\ &\sum_{n_2=0}^{N_2} \sum_{n_3=0}^{N_3} \left( n_2 + \frac{1}{2} \right) \left( n_3 + \frac{1}{2} \right) \frac{1}{2^{n_2+n_3} n_2! n_3!} \\ &\times \frac{d^{n_2} \left[ (x_2 - \bar{h}_2)^2 - h_2^2 \right]^{n_2}}{dx_2^{n_2}} \cdot \frac{d^{n_3} \left[ (x_3 - \bar{h}_3)^2 - h_3^2 \right]^{n_3}}{dx_3^{n_3}} {}^{n_3, n_2} v_j(x_1), \end{aligned} \quad (3.2)$$

where

$${}^{n_3, n_2} v_j(x_1) := \frac{{}^{n_3, n_2} u_j(x_1)}{h_2^{n_2+1}(x_1) h_3^{n_3+1}(x_1)}, \quad (3.3)$$

since

$$\begin{aligned} P_{n_i}(a_i x_i - b_i) &= P_{n_i} \left( \frac{x_i - \bar{h}_i}{h_i} \right) = \frac{1}{2^{n_i} n_i!} \frac{d^{n_i} \left[ \left( \frac{x_i - \bar{h}_i}{h_i} \right)^2 - 1 \right]^{n_i}}{d \left( \frac{x_i - \bar{h}_i}{h_i} \right)^n} \\ &= \frac{1}{2^{n_i} n_i!} h_i^{n_i} \frac{d^{n_i} \left[ (x_i - \bar{h}_i)^2 - 1 \right]^{n_i}}{dx_i^{n_i}} = \frac{1}{2^{n_i} n_i!} h_i^{n_i} \frac{d^{n_i} \left[ (x_i - \bar{h}_i)^2 - 1 \right]^{n_i}}{dx_i^{n_i}} \end{aligned}$$

because of

$$\frac{d}{d \left( \frac{x_i - \bar{h}_i}{h_i} \right)} = h_i \frac{d}{dx_i}, \quad i = 2, 3.$$

Thus, from (3.3), (2.14)-(2.17), (2.4), we obtain:

$$\begin{aligned}
 e_{11}^{n_3, n_2} &= u_{1,1}^{n_3, n_2} + n_2 \frac{h_{2,1}}{h_2} u_1^{n_3, n_2} + n_3 \frac{h_{3,1}}{h_3} u_1^{n_3, n_2} - \sum_{i=2}^3 \sum_{s=n_i+1}^{\infty} b_s^{n_i} a_i^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i3}n_2+\delta_{i2}s} u_1^{n_3, n_2} \\
 &- (2n_2 + 1) \frac{h_{2,1}}{h_2} u_1^{n_3, n_2} - (2n_3 + 1) \frac{h_{3,1}}{h_3} u_1^{n_3, n_2} = h_2^{n_2+1} h_3^{n_3+1} v_{1,1}^{n_3, n_2} \\
 &- \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} \frac{h_2^{\delta_{i2}s+\delta_{i3}n_2+1} h_3^{\delta_{i2}n_3+\delta_{i3}s+1}}{h_i} b_s^{n_i} a_i^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2} v_1^{n_3, n_2}, \sum_{s=N_i+1}^{N_i} (\dots) = 0,
 \end{aligned} \tag{3.4}$$

since

$$a_i = \frac{1}{h_i}, \quad b_s^{n_i} = \left( n_i + \frac{1}{2} \right) \left[ \begin{array}{c} (+) \\ h_{i,1} - (-1)^{2n_i} h_{i,1} \end{array} \right] = (2n_i + 1) h_{i,1},$$

$$\begin{aligned}
 h_2^{n_2+1} h_3^{n_3+1} &\left( \frac{1}{h_2^{n_2+1} h_3^{n_3+1}} \right)_1 \\
 &= \frac{-(n_2 + 1) h_2^{n_2} h_{2,1} h_3^{n_3+1} - (n_3 + 1) h_2^{n_2+1} h_3^{n_3} h_{3,1}}{h_2^{2(n_2+1)} h_3^{2(n_3+1)}} h_2^{n_2+1} h_3^{n_3+1} \\
 &= -(n_2 + 1) \frac{h_{2,1}}{h_2} - (n_3 + 1) \frac{h_{3,1}}{h_3};
 \end{aligned}$$

$$e_{i\underline{i}}^{n_3, n_2} = - \sum_{s=n_i+1}^{N_i} \frac{h_2^{\delta_{i2}s+\delta_{i3}n_2+1} h_3^{\delta_{i2}n_3+\delta_{i3}s+1}}{h_i} b_{\underline{i}s}^{n_i} a_i^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2}, \quad i = 2, 3; \tag{3.5}$$

$$2^{n_3, n_2} e_{23} = - \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} \frac{h_3^{\delta_{i2}n_3+\delta_{i3}s+1} h_2^{\delta_{i2}s+\delta_{i3}n_2+1}}{h_i} b_{is}^{n_i} a_i^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2}, \tag{3.6}$$

$$\begin{aligned}
 2^{n_3, n_2} e_{1i}^{n_3, n_2} &= u_{i,1}^{n_3, n_2} + \left[ n_2 \frac{h_{2,1}}{h_2} + n_3 \frac{h_{3,1}}{h_3} - (2n_2 + 1) \frac{h_{2,1}}{h_2} - (2n_3 + 1) \frac{h_{3,1}}{h_3} \right] u_i^{n_3, n_2} \\
 &- \sum_{k=2}^3 \sum_{s=n_k+1}^{N_k} b_s^{n_k} a_k^{\delta_{k2}n_3+\delta_{k3}s, \delta_{k2}s+\delta_{k3}n_2} u_i^{n_3, n_2} - \sum_{s=n_i+1}^{N_i} b_{\underline{i}s}^{n_i} a_i^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2} u_1^{n_3, n_2} \\
 &= h_2^{n_2+1} h_3^{n_3+1} v_{i,1}^{n_3, n_2} - \sum_{k=2}^3 \sum_{s=n_k+1}^{N_k} \frac{h_2^{\delta_{k2}s+\delta_{k3}n_2+1} h_3^{\delta_{k2}n_3+\delta_{k3}s+1}}{h_k} b_s^{n_k} \\
 &\times v_i^{\delta_{k2}n_3+\delta_{k3}s, \delta_{k2}s+\delta_{k3}n_2} - \sum_{s=n_i+1}^{N_i} \frac{h_2^{\delta_{i2}s+\delta_{i3}n_2+1} h_3^{\delta_{i2}n_3+\delta_{i3}s+1}}{h_i} b_{\underline{i}s}^{n_i} \\
 &\times v_1^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2}, \quad i = 2, 3,
 \end{aligned} \tag{3.7}$$

$$n_i = \overline{0, N_i}, \quad i = 2, 3;$$

$$\begin{aligned} {}^{n_3, n_2} X_{11} &= (\lambda + 2\mu) {}^{n_3, n_2} e_{11} + \lambda \sum_{i=2}^3 {}^{n_3, n_2} e_{ii} = (\lambda + 2\mu) \left( h_2^{n_2+1} h_3^{n_3+1} v_{1,1} \right. \\ &\quad - \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} h_i^{-1} b_s^{n_i} b_{is}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} v_1 \\ &\quad - \lambda \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} h_i^{-1} b_{is}^{n_i} b_{is}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} v_i \left. \right), \end{aligned} \quad (3.8)$$

$$\begin{aligned} {}^{n_3, n_2} X_{ii} &= \lambda \left( {}^{n_3, n_2} e_{11} + {}^{n_3, n_2} e_{5-i, 5-i} \right) + (\lambda + 2\mu) {}^{n_3, n_2} e_{ii} = \lambda \left( h_2^{n_2+1} h_3^{n_3+1} v_{1,1} \right. \\ &\quad - \sum_{k=2}^3 \sum_{s=n_k+1}^{N_k} h_2^{\delta_{k2}s + \delta_{k3}n_2 + 1} h_3^{\delta_{k2}n_3 + \delta_{k3}s + 1} h_k^{-1} b_s^{n_k} b_{ks}^{\delta_{k2}n_3 + \delta_{k3}s, \delta_{k2}s + \delta_{k3}n_2} v_1 \\ &\quad - \sum_{s=n_{5-i}+1}^{N_{5-i}} h_2^{\delta_{5-i2}s + \delta_{5-i3}n_2 + 1} h_3^{\delta_{5-i2}n_3 + \delta_{5-i3}s + 1} h_{5-i}^{-1} b_{5-i}^{n_{5-i}} b_{5-is}^{\delta_{5-i2}n_3 + \delta_{5-i3}s, \delta_{5-i2}s + \delta_{5-i3}n_2} v_{5-i} \left. \right) \\ &\quad - (\lambda + 2\mu) \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} h_i^{-1} b_{is}^{n_i} b_{is}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} v_i, \quad i = 2, 3, \end{aligned} \quad (3.9)$$

$$\begin{aligned} {}^{n_3, n_2} X_{23} &= 2\mu {}^{n_3, n_2} e_{23} \\ &= -\mu \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} h_i^{-1} b_{is}^{n_i} b_{is}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} {}^{n_3, n_2} X_{1i} &= 2\mu {}^{n_3, n_2} e_{1i} = \mu \left( h_2^{n_2+1} h_3^{n_3+1} v_{1,1} \right. \\ &\quad - \sum_{k=2}^3 \sum_{s=n_k+1}^{N_k} h_2^{\delta_{k2}s + \delta_{k3}n_2 + 1} h_3^{\delta_{k2}n_3 + \delta_{k3}s + 1} h_k^{-1} b_s^{n_k} b_{ks}^{\delta_{k2}n_3 + \delta_{k3}s, \delta_{k2}s + \delta_{k3}n_2} v_i \\ &\quad - \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} h_i^{-1} b_{is}^{n_i} b_{is}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2}, \quad i = 2, 3, \left. \right) \end{aligned} \quad (3.11)$$

$$n_i = \overline{0, N_i}, \quad i = 2, 3.$$

Substituting (3.8)-(3.11) into first  $(N_3+1)(N_2+1)$  equations of the system (2.23) for fixed  $j = 1, 2, 3$ , we will have the following system of  $3(N_3+1)(N_2+1)$  equations with respect to  $3(N_3+1)(N_2+1)$  unknown functions  $\overset{r,s}{v}_k(x_1, t)$ ,  $r = \overline{0, N_3}$ ,  $s = \overline{0, N_2}$ ,  $k = 1, 2, 3$ :

$$\begin{aligned} & \Lambda_j \left( h_2^{2n_2+1} h_3^{2n_3+1} \overset{n_3, n_2}{v}_{j,1} \right)_{,1} \\ & + E_j^{\overset{n_3, n_2}{v}} \left( \cdots \overset{r,s}{v}_{k,1} \cdots \right) + M_j^{\overset{n_3, n_2}{v}} \left( \cdots \overset{r,s}{v}_k \cdots \right) + h_2^{n_2} h_3^{n_3} X_j^{\overset{n_3, n_2}{v}} \\ & = \rho h_2^{n_2} h_3^{n_3} \frac{\partial^2 h_2^{n_2+1} h_3^{n_3+1} \overset{n_3, n_2}{v}_j}{\partial t^2}, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \quad (3.12) \end{aligned}$$

$$\Lambda_j = \begin{cases} \lambda + 2\mu, & j = 1, \\ \mu, & j = 2, 3, \end{cases}$$

$E_j^{\overset{n_3, n_2}{v}}$ ,  $M_j^{\overset{n_3, n_2}{v}}$  are certain linear functions of their arguments, where  $r = \overline{0, N_3}$ ,  $s = \overline{0, N_2}$ ,  $k = 1, 2, 3$ , in general, with unbounded integrable coefficients depending on  $x_1, t$ .

The explicit form of (3.11) is as follows:

If  $j = 1$ ,

$$\begin{aligned} & (\lambda + 2\mu) \left( h_2^{2n_2+1} h_3^{2n_3+1} \overset{n_3, n_2}{v}_{1,1} \right)_{,1} - (\lambda + 2\mu) \\ & \times \left( \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + n_3 + 1} h_i^{-1} b_s^{n_i} v_1^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \right)_{,1} \\ & - \lambda \left( \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + n_3 + 1} h_i^{-1} b_{is}^{n_i} v_i^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \right)_{,1} \\ & + \sum_{i=2}^3 h_2^{n_2 - \delta_{i2}} h_3^{n_3 - \delta_{i3}} \sum_{s=0}^{n_i-1} \left\{ b_s^{n_i} [(\lambda + 2\mu) (h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} \right. \\ & \times v_{1,1}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} - \sum_{k=2}^3 \sum_{r=\delta_{k2}(s+n_2)+\delta_{k3}(s+n_3)+1}^{N_k} h_2^{\delta_{k2}r + \delta_{k3}(\delta_{i2}s + \delta_{i3}n_2) + 1} \right. \\ & \times h_3^{\delta_{k2}(s+n_3 + \delta_{i3}s) + \delta_{k3}r + 1} h_k^{-1} b_r^{k} \\ & \times v_1^{\delta_{k2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{k3}r, \delta_{k2}r + \delta_{k3}(\delta_{i2}s + \delta_{i3}n_2)} \Big) \\ & - \lambda \sum_{k=2}^3 \sum_{r=\delta_{k2}(s+n_2)+\delta_{k3}(s+n_3)+1}^{N_k} h_2^{\delta_{k2}r + \delta_{k3}(\delta_{i2}s + \delta_{i3}n_2) + 1} h_3^{\delta_{k2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{k3}r + 1} h_k^{-1} \end{aligned}$$

$$\begin{aligned}
 & \times \left[ b_{kr}^{\delta_{k2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{k3}(\delta_{i2}n_3+\delta_{i3}s)} v_i^{\delta_{k2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{k3}r, \delta_{k2}r+\delta_{k3}(\delta_{i2}s+\delta_{i3}n_2)} \right] \\
 & + \mu b_{is}^{n_i} \left( h_2^{\delta_{i2}s+\delta_{i3}n_2+1} h_3^{\delta_{i2}n_3+\delta_{i3}s+1} v_{i,1}^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2} \right. \\
 & - \sum_{k=2}^3 \sum_{r=\delta_{k2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{k3}(\delta_{i2}n_3+\delta_{i3}s)+1}^{N_k} h_2^{\delta_{k2}r+\delta_{k3}(\delta_{i2}s+\delta_{i3}n_2)+1} h_3^{\delta_{k2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{k3}r+1} h_k^{-1} \\
 & \times \left. b_r^{\delta_{k2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{k3}(\delta_{i2}n_3+\delta_{i3}s)} v_i^{\delta_{k2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{k3}r, \delta_{k2}r+\delta_{k3}(\delta_{i2}s+\delta_{i3}n_2)} \right) \\
 & - \sum_{r=s+1}^{N_i} h_2^{\delta_{i2}r+\delta_{i3}(\delta_{i2}s+\delta_{i3}n_2)+1} h_3^{\delta_{i2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{i3}r+1} h_i^{-1} \\
 & \times b_{ir}^{\delta_{i2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{i3}r, \delta_{i2}r+\delta_{i3}(\delta_{i2}s+\delta_{i3}n_2)} \Big) \Big\} + h_2^{n_2} h_3^{n_3} X_1^{n_3, n_2} \\
 & = \rho h_2^{2n_2+1} h_3^{2n_3+1} \frac{\partial^2 v_1^{n_3, n_2}}{\partial t^2}, \quad n_i = \overline{0, N_i}, \quad i = 2, 3; \tag{3.13}
 \end{aligned}$$

If  $j = 2, 3$ ,

$$\begin{aligned}
 & \mu \left( h_2^{2n_2+1} h_3^{2n_3+1} v_{j,1}^{n_3, n_2} \right)_{,1} - \mu \left( \sum_{k=2}^3 \sum_{s=n_k+1}^{N_k} h_2^{\delta_{k2}s+\delta_{k3}n_2+n_2+1} h_3^{\delta_{k2}n_3+\delta_{k3}s+n_3+1} h_k^{-1} b_s^{n_k} \right. \\
 & \times \left. v_j^{\delta_{k2}n_3+\delta_{k3}s, \delta_{k2}s+\delta_{k3}n_2} + \sum_{s=n_j+1}^{N_j} h_2^{\delta_{j2}s+\delta_{j3}n_2+n_2+1} h_3^{\delta_{j2}n_3+\delta_{j3}s+n_3+1} h_j^{-1} \right. \\
 & \times b_{js}^{n_j} v_1^{\delta_{j2}n_3+\delta_{j3}s, \delta_{j2}s+\delta_{j3}n_2} \Big)_{,1} + \sum_{i=2}^3 h_2^{n_2-\delta_{i2}} h_3^{n_3-\delta_{i3}} \sum_{s=0}^{n_i-1} \mu b_s^{n_i} \left[ h_2^{\delta_{i2}s+\delta_{i3}n_2+1} h_3^{\delta_{i2}n_3+\delta_{i3}s+1} \right. \\
 & \times v_{j,1}^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2} - \sum_{k=2}^3 \sum_{r=\delta_{k2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{k3}(\delta_{i2}n_3+\delta_{i3}s)+1}^{N_k} h_2^{\delta_{k2}r+\delta_{k3}(\delta_{i2}s+\delta_{i3}n_2)+1} \\
 & \times h_3^{\delta_{k2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{k3}r+1} h_k^{-1} b_r^{\delta_{k2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{k3}(\delta_{i2}n_3+\delta_{i3}s)} \\
 & \times \left. v_j^{\delta_{k2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{k3}r, \delta_{k2}r+\delta_{k3}(\delta_{i2}s+\delta_{i3}n_2)} - \sum_{r=\delta_{j2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{j3}(\delta_{i2}n_3+\delta_{i3}s)+1}^{N_j} h_2^{\delta_{j2}r+\delta_{j3}(\delta_{i2}s+\delta_{i3}n_2)+1} \right. \\
 & \times h_3^{\delta_{j2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{j3}r+1} h_j^{-1} b_{jr}^{\delta_{j2}(\delta_{i2}s+\delta_{i3}n_2)+\delta_{j3}(\delta_{i2}n_3+\delta_{i3}s)}
 \end{aligned}$$

$$\begin{aligned} & \times \left[ v_1^{\delta_{j2}(\delta_{i2}n_3+\delta_{i3}s)+\delta_{j3}r,\delta_{j2}r+\delta_{j3}(\delta_{i2}s+\delta_{i3}n_2)} \right] + H_j + h_2^{n_2} h_3^{n_3} X_j^{n_3,n_2} \\ & = \rho h_2^{2n_2+1} h_3^{2n_3+1} \frac{\partial^2 v_j}{\partial t^2}, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \end{aligned} \quad (3.14)$$

where

$$H_j := \sum_{i=2}^3 h_2^{n_2-\delta_{i2}} h_3^{n_3-\delta_{i3}} \sum_{s=0}^{n_i-1} b_{is}^{\delta_{i2}n_3+\delta_{i3}s, \delta_{i2}s+\delta_{i3}n_2} X_{ij}^{n_3,n_2}, \quad j = 2, 3,$$

i.e.

$$\begin{aligned} H_2 &= h_2^{n_2-1} h_3^{n_3} \sum_{s=0}^{n_2-1} b_{2s}^{n_2, n_3, s} X_{22} + h_2^{n_2} h_3^{n_3-1} \sum_{s=0}^{n_3-1} b_{3s}^{n_3, s, n_2} X_{32} \\ &= h_2^{n_2-1} h_3^{n_3} \sum_{s=0}^{n_2-1} b_{2s}^{n_2} \left[ \lambda \left( h_2^{s+1} h_3^{n_3+1} v_{1,1}^{s, n_2} - \sum_{k=2}^3 \sum_{r=\delta_{k2}s+\delta_{k3}n_3+1}^{N_k} h_2^{\delta_{k2}r+\delta_{k3}s+1} \right. \right. \\ &\quad \times h_3^{\delta_{k2}n_3+\delta_{k3}r+1} h_k^{-1} b_r^{\delta_{k2}s+\delta_{k3}n_3} v_1^{\delta_{k2}n_3+\delta_{k3}r, \delta_{k2}r+\delta_{k3}s} \\ &\quad \left. \left. - \sum_{r=n_3+1}^{N_3} h_2^{s+1} h_3^{r+1} h_3^{-1} b_{3r}^{r, s} v_3 \right) - (\lambda + 2\mu) \sum_{r=s+1}^{N_2} h_2^{r+1} h_3^{n_3+1} h_2^{-1} b_{2r}^{s, n_3, r} v_2 \right] \\ &\quad - \mu h_2^{n_2} h_3^{n_3-1} \sum_{s=0}^{n_3-1} b_{3s}^{n_3} \sum_{i=2}^3 \sum_{r=\delta_{i2}n_2+\delta_{i3}s+1}^{N_i} h_2^{\delta_{i2}r+\delta_{i3}n_2+1} h_3^{\delta_{i2}s+\delta_{i3}r+1} \\ &\quad \times h_i^{-1} b_{ir}^{\delta_{i2}n_2+\delta_{i3}s} v_{5-i}^{\delta_{i2}s+\delta_{i3}r, \delta_{i2}r+\delta_{i3}n_2}, \end{aligned}$$

$$\begin{aligned} H_3 &= h_2^{n_2-1} h_3^{n_3} \sum_{s=0}^{n_2-1} b_{2s}^{n_2, n_3, s} X_{23} + h_2^{n_2} h_3^{n_3-1} \sum_{s=0}^{n_3-1} b_{3s}^{n_3, s, n_2} X_{33} \\ &= -\mu h^{n_2-1} h_3^{n_3} \sum_{s=0}^{n_2-1} b_{2s}^{n_2} \sum_{i=2}^3 \sum_{r=\delta_{i2}s+\delta_{i3}n_3+1}^{N_i} h_2^{\delta_{i2}r+\delta_{i3}s+1} \\ &\quad \times h_3^{\delta_{i2}n_3+\delta_{i3}r+1} h_i^{-1} b_{ir}^{\delta_{i2}s+\delta_{i3}n_3} v_{5-i}^{\delta_{i2}n_3+\delta_{i3}r, \delta_{i2}r+\delta_{i3}s} \\ &\quad + h_2^{n_2} h_3^{n_3-1} \sum_{s=0}^{n_3-1} b_{3s}^{n_3} \left[ \lambda \left( h_2^{n_2+1} h_3^{s+1} v_{1,1}^{s, n_2} - \sum_{k=2}^3 \sum_{r=\delta_{k2}n_2+\delta_{k3}s+1}^{N_k} h_2^{\delta_{k2}r+\delta_{k3}s+1} \right. \right. \\ &\quad \times h_3^{\delta_{k2}s+\delta_{k3}r+1} h_k^{-1} b_r^{\delta_{k2}n_2+\delta_{k3}s} v_1^{\delta_{k2}s+\delta_{k3}r, \delta_{k2}r+\delta_{k3}s} \end{aligned}$$

$$-\sum_{r=n_2+1}^{N_2} h_2^{r+1} h_3^{s+1} h_2^{-1} b_{2r}^{n_2} v_2^r \Big) - (\lambda + 2\mu) \sum_{r=s+1}^{N_3} h_2^{n_2+1} h_3^{r+1} h_3^{-1} b_{3r}^s v_3^{r,n_2} \Big] .$$

(3.13) can be rewritten in the form as follows

If  $j = 1$ ,

$$\begin{aligned} & (\lambda + 2\mu) \left( h_2^{2n_2+1} h_3^{2n_3+1} v_{1,1}^{n_3,n_2} \right)_{,1} - \left( \sum_{i=2}^3 \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + \delta_{i3}n_2 + n_2 + 1} \right. \\ & \times h_3^{\delta_{i2}n_3 + \delta_{i3}s + n_3 + 1} h_i^{-1} \left[ (\lambda + 2\mu) b_s^{n_i} v_1^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \right. \\ & \left. \left. + \lambda b_{is}^{n_i} v_i^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \right] \right)_{,1} + \sum_{i=2}^3 h_2^{n_2 - \delta_{i2}} h_3^{n_3 - \delta_{i3}} \sum_{s=0}^{n_i-1} \left\{ h_2^{\delta_{i2}s + \delta_{i3}n_2 + 1} \right. \\ & \times h_3^{\delta_{i2}n_3 + \delta_{i3}s + 1} \left[ (\lambda + 2\mu) b_s^{n_i} v_{1,1}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} + \mu b_{is}^{n_i} v_{i,1}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \right] \\ & - \sum_{k=2}^3 \sum_{r=\delta_{k2}(\delta_{i2}s + \delta_{i3}n_2) + \delta_{k3}(\delta_{i2}n_3 + \delta_{i3}s) + 1}^{N_k} h_2^{\delta_{k2}r + \delta_{k3}(\delta_{i2}s + \delta_{i3}n_2) + 1} h_3^{\delta_{k2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{k3}r + 1} h_k^{-1} \\ & \times \left[ (\lambda + 2\mu) b_s^{n_i} b_r^{\delta_{k2}(\delta_{i2}s + \delta_{i3}n_2) + \delta_{k3}(\delta_{i2}n_3 + \delta_{i3}s)} v_1^{\delta_{k2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{k3}r, \delta_{k2}r + \delta_{k3}(\delta_{i2}s + \delta_{i3}n_2)} \right. \\ & \left. + \left( \lambda b_s^{n_i} b_{kr}^{\delta_{k2}(\delta_{i2}s + \delta_{i3}n_2) + \delta_{k3}(\delta_{i2}n_3 + \delta_{i3}s)} + \mu b_{is}^{n_i} b_r^{\delta_{k2}(\delta_{i2}s + \delta_{i3}n_2) + \delta_{k3}(\delta_{i2}n_3 + \delta_{i3}s)} \right) \right. \\ & \left. \times h_i^{\delta_{i2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{i3}r, \delta_{k2}r + \delta_{k3}(\delta_{i2}s + \delta_{i3}n_2)} \right] - \mu \sum_{r=s+1}^{N_i} h_2^{\delta_{i2}r + \delta_{i3}(\delta_{i2}s + \delta_{i3}n_2) + 1} \\ & \times h_3^{\delta_{i2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{i3}r + 1} h_i^{-1} b_{ir}^{n_i} b_{is}^{\delta_{i2}(\delta_{i2}n_3 + \delta_{i3}s) + \delta_{i3}r, \delta_{i2}r + \delta_{i3}(\delta_{i2}s + \delta_{i3}n_2)} \Big\} \\ & + h_2^{n_2} h_3^{n_3} X_1^{n_3, n_2} = \rho h_2^{2n_2+1} h_3^{2n_3+1} \frac{\partial^2 v_1^{n_3, n_2}}{\partial t^2}, \quad n_i = \overline{0, N_i}, \quad i = 2, 3. \end{aligned} \tag{3.15}$$

## 4 Initial and Boundary Value Problems

We assume that on surfaces  $h_i$ ,  $i = 2, 3$ , of the bar surface forces (which appear in (2.21)), and on the ends of the bar either the components of displacement vector or the components of the surface forces are given, or both are given mixed.

In  $(N_3, N_2)$  approximation if on the end of the bar  $2h_i > 0$ ,  $i = 2, 3$ , then the boundary conditions on the above end have the forms as follows.

1. Boundary conditions in displacements:

$$\overset{r,s}{v}_j = f_j, \quad j = 1, 2, 3, \quad r = \overline{0, N_3}, \quad s = \overline{0, N_2}, \quad (4.1)$$

where  $\overset{r,s}{f}_j$  are given on the end constants. In dynamical case we have to add the initial conditions as well:

$$\overset{r,s}{v}_j \Big|_{t=0} = \overset{r,s}{\varphi}_j(x_1), \quad \frac{\partial \overset{r,s}{v}_j}{\partial t} \Big|_{t=0} = \overset{r,s}{\psi}_j(x_1), \quad (4.2)$$

$$x_1 \in ]0, l[, \quad j = 1, 2, 3, \quad r = \overline{0, N_3}, \quad s = \overline{0, N_2},$$

where  $\overset{r,s}{\varphi}_j, \overset{r,s}{\psi}_j$  are given functions, and  $\overset{r,s}{f}_j$  should be, in general, functions of  $t$ .

2. Boundary conditions in stresses:

$$\overset{r,s}{X}_{j1} = g_j, \quad j = 1, 2, 3, \quad r = \overline{0, N_3}, \quad s = \overline{0, N_2}, \quad (4.3)$$

where  $\overset{r,s}{g}_j$  are given on the end constants.

3. Mixed boundary condition are called such boundary conditions when either on the one end the conditions (4.1), and on the another one the conditions (4.3), or on the both ends for some indices the conditions (4.1) and for the remained ones the conditions (4.3) are given.

If on the end of a bar at least one of  $2h_i, i = 2, 3$ , vanishes then in the conditions (4.1), (4.3) the left-hand side terms should be understood as limits from the inside of the bar provided that these limits exist. In some cases conditions (4.1) should be replaced by conditions of boundedness of  $\overset{r,s}{v}_j$ .

To investigate the above problems one can use wide literature in ordinary differential equations (see e.g. [4]), and in hyperbolic partial differential equations (see e.g. [5]). But to apply them one needs some efforts especially in case  $2h_i(x_1) \geq 0, i = 2, 3$ , on  $[0, l]$ .

If in the statical case  $h_i > 0, i = 2, 3$ , on  $[0, l]$  then the existence of a regular solution in usual sense, taking into account that the system (3.14), (3.15) can be reduced to the system of first order ordinary differential equations, in virtue of well-known theorem (see [4], p. 146), follows from uniqueness of solution of above problems which can be proved by means of potential energy similarly to [3].

If in the statical case  $2h_i(x_1) \geq 0, i = 2, 3$ , on  $[0, l]$ , the following boundary conditions can be set in the  $(N_3, N_2)$  approximation:

$$\overset{n_3, n_2}{v}_j(0) = \overset{n_3, n_2}{\varphi}_j^o, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \quad \text{if } I_0 < +\infty; \quad (4.4)$$

$$\overset{n_3, n_2}{v}_j(\ell) = \overset{n_3, n_2}{\varphi}_j^\ell, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \quad \text{if } I_\ell < +\infty; \quad (4.5)$$

$$\overset{n_3, n_2}{v}_j(x_1) = 0(1), \quad x_1 \rightarrow 0_+, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \quad \text{if } I_0 = +\infty; \quad (4.6)$$

$$\overset{n_3, n_2}{v}_j(x_1) = 0(1), \quad x_1 \rightarrow \ell_-, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \text{ if } \overset{N_3, N_2}{I}_\ell = +\infty; \quad (4.7)$$

$$\begin{aligned} & \lim_{x_1 \rightarrow 0+} h_2^{n_2} h_3^{n_3} X_{1j}(x_1) \\ &= \lim_{x_1 \rightarrow 0+} \left\{ \Lambda_j h_2^{2n_2+1} h_3^{2n_3+1} \overset{n_3, n_2}{v}_{j,1} - \Lambda_j \sum_{k=2}^3 \sum_{s=n_k+1}^{N_k} h_2^{\delta_{k2}s + (\delta_{k3}+1)n_2+1} \right. \\ & \quad \times h_3^{\delta_{k3}s + (\delta_{k2}+1)n_3+1} h_k^{-1} b_s^{n_k} v_j^{\delta_{k2}n_3 + \delta_{k3}s, \delta_{k2}s + \delta_{k3}n_2} - \left[ \delta_{j1}\lambda \sum_{i=2}^3 + \delta_{ij}(\delta_{j2} + \delta_{j3})\mu \right] \\ & \quad \times \sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s + (\delta_{i3}+1)n_2+1} h_3^{(\delta_{i2}+1)n_3 + \delta_{i3}s+1} h_i^{-1} b_{is}^{n_i} v_{\delta_{j1}i + \delta_{j2} + \delta_{j3}}^{\delta_{i2}n_3 + \delta_{i3}s, \delta_{i2}s + \delta_{i3}n_2} \Big\} \\ &= \overset{n_3, n_2}{\psi}_j^0, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \text{ if } \overset{N_3, N_2}{I}_0 \leq +\infty; \end{aligned} \quad (4.8)$$

$$\lim_{x_1 \rightarrow l_-} h_2^{n_2} h_3^{n_3} X_{1j}(x_1) = \overset{n_3, n_2}{\psi}_j^\ell, \quad j = 1, 2, 3, \quad n_i = \overline{0, N_i}, \quad i = 2, 3, \text{ if } \overset{N_3, N_2}{I}_\ell \leq +\infty; \quad (4.9)$$

where  $\overset{n_3, n_2}{\varphi}_j^0, \overset{n_3, n_2}{\varphi}_j^\ell, \overset{n_3, n_2}{\psi}_j^0, \overset{n_3, n_2}{\psi}_j^\ell$  are given constants,

$$\overset{N_3, N_2}{I}_0 := \int_0^\varepsilon h_2^{-2N_2-1}(\tau) h_3^{-2N_3-1}(\tau) d\tau, \quad \varepsilon = const > 0,$$

$$\overset{N_3, N_2}{I}_\ell := \int_{\ell-\varepsilon}^\ell h_2^{-2N_2-1}(\tau) h_3^{-2N_3-1}(\tau) d\tau, \quad \varepsilon = const > 0.$$

The boundary value problems (3.14), (3.15), (4.4), (4.5); (3.14), (3.15), (4.4), (4.9); (3.14), (3.15), (4.5), (4.8); (3.14), (3.15), (4.4), (4.7); (3.14), (3.15), (4.5), (4.6) are uniquely solvable (here we do not bother to make precise the appropriate classes of solutions). The problem (3.14), (3.15), (4.6), (4.7) is solvable up to the rigid motion.

If the bar has cusped end (i.e. at this end at least one of  $h_i, i = 2, 3$ , vanishes), and  $N_2, N_3 \rightarrow +\infty$ , in limit case (which obviously coincides with the three-dimensional case) the boundary conditions (4.4), or (4.5), or both disappear, and will be replaced by boundedness of  $\overset{n_3, n_2}{v}_j, j = 1, 2, 3, n_2, n_3 = 0, 1, \dots$ , i.e. by boundedness of displacement vector in a neighbourhood of the corresponding end of the bar.

If in a neighbourhood of a cusped end stresses are bounded then at the above end all moments of stress vector will be equal to zero. Non-zero stress vector moments

given at a cusped bar end mean that this end in the three-dimensional case is loaded by concentrated surface force, and concentrated moments of corresponding order.

In the dynamical case to the boundary conditions (4.4)-(4.9) we have to add initial conditionss (4.2) since  $h_i(x_1) > 0$ ,  $i = 2, 3$ , by  $t = 0$ ,  $x_1 \in ]0, l[$ , and therefore the system (3.14), (3.15) is not degenerate for such  $(x_1, t)$ .

The (0.0) and (1.0) approximations have been considered in [6] and [7] respectively.

Finally let us note that Mellin pseudo-differential operator theory developed by Prof. B.-W. Schulze (see e.g. [8]) and his disciples has a good chance of success as an important tool of investigation of mathematical models of cusped bars, and in general, of bodies with non-smooth boundaries.

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