On the Holomorphic Solution of Non-linear Totally Characteristic Equations

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Abstract. The paper deals with a non-linear singular partial differential equation: (E) $t\partial/\partial t = F(t, x, u, \partial u/\partial x)$ in the holomorphic category. When (E) is of Fuchsian type, the existence of the unique holomorphic solution was established by Gérard-Tahara [2]. In this paper, under the assumption that (E) is of totally characteristic type, the authors give a sufficient condition for (E) to have a unique holomorphic solution. The result is extended to higher order case.

Keywords. Nonlinear, singular partial differential equation, holomorphic solution.

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1 Introduction and Main Result.

Let $(t, x) \in \mathbf{C}_t \times \mathbf{C}_x$, we consider the following non-linear first order singular partial differential equation

$$t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right), \tag{1}$$

where $u = u(t, x)$ is an unknown function, $F(t, x, u, v)$ is a function with respect to the variables $(t, x, u, v) \in \mathbf{C}_t \times \mathbf{C}_x \times \mathbf{C}_u \times \mathbf{C}_v$. Further we assume the following conditions:

(H1) $F(t,x,u,v)$ is a holomorphic function defined in a neighborhood of the origin $(0,0,0,0) \in \mathbf{C}_t \times \mathbf{C}_x \times \mathbf{C}_u \times \mathbf{C}_v$.
(H2) $F(0,x,0,0) \equiv 0$ near $x = 0$.

Thus the function $F(t,x,u,v)$ may be expressed in the form:

$$F(t,x,u,v) = \alpha(x)t + \beta(x)u + \gamma(x)v + R_2(t,x,u,v), \tag{2}$$

where

$$\alpha(x) = \frac{\partial F}{\partial t}(0,x,0,0), \quad \beta(x) = \frac{\partial F}{\partial u}(0,x,0,0), \quad \gamma(x) = \frac{\partial F}{\partial v}(0,x,0,0),$$

and the degree of $R_2(t,x,u,v)$ with respect to (t,u,v) is greater than or equal to 2.

If $\gamma(x) \equiv 0$ near $x = 0$, then $\beta(x)$ is the indicial exponent of (1), and Gérard-Tahara [2] gives that, if $\beta(0) \notin \{1,2,\cdots\}$, the equation (1) has a unique holomorphic solution $u(t,x)$ near a neighborhood of $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$ with $u(0,x) \equiv 0$ near $x = 0$. The condition "$\gamma(x) \equiv 0$ near $x = 0$" means that the linearlized equation of (1) is "Fuchsian type"; so, in this case, the equation (1) is called non-linear Fuchsian type partial differential equation (or is called "Briot-Bouquet type equation" in [1, 2]).

Remark. *Quite recently, Yamane [7] also studied the case for the nonlinear Fuchsian type PDE whose indicial exponent takes a positive integer value.*

If $\gamma(x) \not\equiv 0$ and $\gamma(0) \neq 0$, then we can use the implicit function theorem to solve v from the equation (1), then, by using Cauchu-Kowalewski Theorem, we can deduce easily that the equation (1) has a unique holomorphic solution $u(t,x)$ with $u(0,x) \equiv 0$ and $u(t,0) \equiv 0$ near $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$.

So we are only interested in the case of $\gamma(x) \not\equiv 0$ and $\gamma(0) = 0$, *i.e.* the case of $\gamma(x) = x^p c(x)$, where p is a positive integer and $c(0) \neq 0$. In this case, the equation (1) is called totally characteristic type; and $\beta(x) + \gamma(x)\partial_x$ is the indicial operator of (1). We know that there is an essential difference between the case of $p = 1$ and the case of $p \geq 2$, since, in the case of $p \geq 2$, the indicial operator $\beta(x) + \gamma(x)\partial_x$ has an irregular singularity at $x = 0$.

In this paper, we shall consider the simplest case, *i.e.*

(H3) $\gamma(x) = xc(x)$, and $c(0) \neq 0$

and the case of $\gamma(x) = x^p c(x)$, for $p \geq 2$ will be studied in the forthcoming paper.

We have the following result:

Theorem 1. *Under the conditions (H1), (H2) and (H3), if Re* $c(0) < 0$ *and* $\frac{\beta(0)-k}{c(0)} \notin \mathbf{Z}_- = \{0, -1, -2, \cdots\}$ *for any* $k \geq 1$*, then the equation (1) has a unique holomorphic solution* $u(t,x)$ *near* $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$ *with* $u(0,x) \equiv 0$ *near* $x = 0$.

2 Proof of Main Result.

First we put formally

$$u(t,x) = \sum_{k=1}^{\infty} u_k(x) t^k. \tag{3}$$

Then introduce the formal series (3) into the equation (1) and compare the coefficients of t^k in two sides of the equation, we have

$$\begin{array}{ll} u_1 & = \alpha(x) + \beta(x)u_1 + \gamma(x)\frac{\partial u_1}{\partial x}, \\ 2u_2 & = \beta(x)u_2 + \gamma(x)\frac{\partial u_2}{\partial x} + f_1(u_1, \frac{\partial u_1}{\partial x}), \\ 3u_3 & = \beta(x)u_3 + \gamma(x)\frac{\partial u_3}{\partial x} + f_2(u_1, u_2, \frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}), \\ & \vdots \end{array} \tag{4}$$

Thus we obtain the following recursive formula

$$\gamma(x)\frac{\partial u_k}{\partial x} + (\beta(x) - k)u_k = f_{k-1}\left(u_1, \cdots, u_{k-1}, \frac{\partial u_1}{\partial x}, \cdots, \frac{\partial u_{k-1}}{\partial x}\right), \text{ for } k \in \mathbb{N},$$

where $\gamma(x) = xc(x)$, and $f_0(x) = -\alpha(x)$, so

$$x\frac{\partial u_k}{\partial x} + \left(\frac{\beta(x) - k}{c(x)}\right) u_k = \frac{1}{c(x)} f_{k-1} = \tilde{f}_{k-1}, \text{ for } k \in \mathbb{N}. \tag{5}$$

Denoting $\frac{\beta(x)-k}{c(x)} = \lambda_k(x)$, we easily see that if $l_k(0) \notin \mathbf{Z}_- = \{0, -1, -2, \cdots\}$, then for any $k \geq 1$, the equation (5) has a unique holomorphic solution $u_k(x)$ near $x = 0$ and moreover we see that all $u_k(x)$ are holomorphic in a common neighborhood of $x = 0$. It remains to prove that the formal series solution (3) is convergent near $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$.

We need the following two lemmas, which will be useful for the proof of the main result.

Lemma 1. *If Re $c(0) < 0$, then there exists $K > 0$, large enough, such that*

$$\|u_k\|_r \leq \frac{C}{k}\|\tilde{f}_{k-1}\|_r, \quad \text{for } k \geq K \quad (\exists C > 0), \tag{6}$$

where u_k is the solution of (5), and the norm $\|\cdot\|_r$ (for $r > 0$ small enough) is defined by $\|f\|_r = \max_{|x|\leq r}|f(x)|$.

Proof. From the equation (5), we have

$$\frac{d}{dx}\left[e^{\int_a^x \frac{\lambda_k(s)}{s}ds}u_k(x)\right] = e^{\int_a^x \frac{\lambda_k(s)}{s}ds}\tilde{f}_{k-1}(x)\frac{1}{x},$$

where a is a point in the segment $l(0,x)$ joining the two points 0 and x in the complex plane. Thus

$$\int_a^x \frac{d}{dy}\left[e^{\int_a^y \frac{\lambda_k(s)}{s}ds}u_k(y)\right]dy = \int_a^x e^{\int_a^y \frac{\lambda_k(s)}{s}ds}\tilde{f}_{k-1}(y)\frac{1}{y}dy,$$

that is

$$e^{\int_a^x \frac{\lambda_k(s)}{s}ds}u_k(x) - u_k(a) = \int_a^x e^{\int_a^y \frac{\lambda_k(s)}{s}ds}\tilde{f}_{k-1}(y)\frac{1}{y}dy$$

and therefore

$$u_k(x) = e^{-\int_a^x \frac{\lambda_k(s)}{s}ds}u_k(a) + \int_a^x e^{-\int_y^x \frac{\lambda_k(s)}{s}ds}\tilde{f}_{k-1}(y)\frac{1}{y}dy.$$

Now, letting $a \to 0$ in $l(0,x)$ we have

$$e^{-\int_a^x \frac{\lambda_k(s)}{s}ds}u_k(a) \to 0,$$

because, for k large enough, we know *Re* $\lambda_k(0) > 0$.

Hence we have

$$u_k(x) = \int_0^x e^{-\int_y^x \frac{\beta(s)-k}{sc(s)}ds}\tilde{f}_{k-1}(y)\frac{1}{y}dy,$$

this implies, for $r > 0$ small enough, that

$$\|u_k\|_r \leq \max_{|x|\leq r}\left|\int_0^x e^{-\int_y^x \frac{\beta(s)-k}{sc(s)}ds}\frac{1}{y}dy\right| \cdot \|\tilde{f}_{k-1}\|_r. \tag{7}$$

Since $c(0) \neq 0$, then $\frac{\beta(x)}{c(x)}$ is bounded near $x = 0$, thus there exists a constant C_1, such that

$$\left| e^{-\int_y^x \frac{\beta(s)}{sc(s)} ds} \right| \leq e^{C_1 \int_{|y|}^{|x|} \frac{1}{\sigma} d\sigma} = e^{C_1 ln(\frac{|x|}{|y|})} = \left(\frac{|x|}{|y|} \right)^{C_1}.$$

Next, we set $C_2 = -Re\ c(0) > 0$, then near $x = 0$ we have

$$\left| e^{k \int_y^x \frac{1}{sc(s)} ds} \right| \approx e^{\frac{k}{Rec(0)} ln(\frac{|x|}{|y|})} = \left(\frac{|x|}{|y|} \right)^{-\frac{k}{C_2}}.$$

Thus we obtain, for a constant $c > 0$, that

$$\left| \int_0^x e^{-\int_y^x \frac{\beta(s)-k}{sc(s)} ds} \frac{1}{y} dy \right| \leq c \int_0^{|x|} \left(\frac{|x|}{\eta} \right)^{C_1 - \frac{k}{C_2}} \frac{1}{\eta} d\eta = c\ |x|^{C_1 - \frac{k}{C_2}} \int_0^{|x|} \eta^{\frac{k}{C_2} - C_1 - 1} d\eta. \tag{8}$$

Observe $C_2 > 0$, so if we choose $K > 0$ large enough, suth that $\frac{k}{C_2} - C_1 - 1 \geq 0$ for any $k \geq K$, then

$$\int_0^{|x|} \eta^{\frac{k}{C_2} - C_1 - 1} d\eta = \frac{C_2}{k - C_1 C_2} \cdot |x|^{\frac{k}{C_2} - C_1}.$$

Hence we can choose a constant $C > 0$, suth that for any $k \geq K$ we have $\frac{cC_2}{k - C_1 C_2} \leq \frac{C}{k}$, this implies, combining with (7) and (8), that Lemma 1 holds.

Lemma 2. *Let $R > 0$ and $f(x)$ be a holomorphic function on $D_R = \{x \in \mathbf{C} \mid |x| \leq R\}$. If for any r, $0 < r < R$, $f(x)$ satisfies*

$$\max_{|x| \leq r} |f(x)| \leq \frac{c}{(R-r)^a},$$

for some $c > 0$ and $a \geq 0$, then we have

$$\max_{|x| \leq r} |\frac{\partial f}{\partial x}(x)| \leq \frac{(a+1)ec}{(R-r)^{a+1}}, \text{ for } 0 < \forall r < R. \tag{9}$$

The proof of Lemma 2 is well-known, cf. Lemma 5.1.3 of [6] (also see [5]).

Now let us prove Theorem 1. First we expand the remainder term $R_2(t, x, u, v)$ of (2) into Taylor series with respect to (t, u, v), *i.e.*

$$R_2(t, x, u, v) = \sum_{p+q+\alpha \geq 2} a_{p,q,\alpha}(x) t^p u^q v^\alpha.$$

We let $R>0$ small enough, such that

(i) $\frac{a_{p,q,\alpha}(x)}{c(x)}$ is holomorphic on $D_R=\{x\in\mathbf{C}\mid |x|\le R\}$;
(ii) $|\frac{a_{p,q,\alpha}(x)}{c(x)}|\le A_{p,q,\alpha}$ on D_R;
(iii) $\sum_{p+q+\alpha\ge 2} A_{p,q,\alpha}t^pY^qY^\alpha$ is a convergent power series in (t,Y).

Secondly from the equation (4), we have

$$\begin{aligned}
(x\tfrac{\partial}{\partial x}+\lambda_1)u_1 &= -\tfrac{\alpha(x)}{c(x)},\\
&\vdots\\
(x\tfrac{\partial}{\partial x}+\lambda_k)u_k &= -\sum_{p+q+\alpha\ge 2}\ \sum_{\substack{p+k_1+\cdots+k_q\\+l_1+\cdots+l_\alpha=k}} \tfrac{a_{p,q,\alpha}(x)}{c(x)}\times u_{k_1}\times\cdots\times u_{k_q}\\
&\qquad\qquad\times\tfrac{\partial u_{l_1}}{\partial x}\times\cdots\times\tfrac{\partial u_{l_\alpha}}{\partial x}\\
&\vdots
\end{aligned} \tag{10}$$

Without loss of generality, we may assume that the estimate (6) in Lemma 1 holds for any $k\ge 1$, *i.e.*

$$\|u_k\|_r\le\frac{C}{k}\|\tilde f_{k-1}\|_r,\ \text{for } k\ge 1,$$

where C is independent of r. Then we choose $A>0$, suth that on D_R,

$$|u_1(x)|\le A \quad\text{and}\quad \left|\frac{\partial u_1}{\partial x}\right|\le A.$$

Now we introduce a function $Y(t)$, satisfying the following equation:

$$Y=At+\frac{C}{R-r}\sum_{p+q+\alpha\ge 2}\frac{A_{p,q,\alpha}}{(R-r)^{p+q+\alpha-2}}\,t^pY^q(eY)^\alpha, \tag{11}$$

where r is a parameter with $0<r<R$.

Since the equation (11) is an analytic functional equation in Y, then we can use the implicit function theorem to deduce that the equation (11) has a unique holomorphic solution $Y(t)$ in a neighborhood of $t=0$ with $Y(0)=0$.

Expanding $Y(t)$ into Taylor series in t,

$$Y(t)=\sum_{k\ge 1}Y_kt^k. \tag{12}$$

From the equation (11), we know that the coefficients of (12) can be given by

$$Y_1 = A,$$

and for $k \geq 2$

$$Y_k = \tfrac{C}{R-r} \sum_{p+q+\alpha \geq 2} \sum_{\substack{p+k_1+\dots+k_q \\ +l_1+\dots+l_\alpha = k}} \frac{A_{p,q,\alpha}}{(R-r)^{p+q+\alpha-2}} \times Y_{k_1} \times \cdots \times Y_{k_q} \times (eY_{l_1}) \times \cdots \times (eY_{l_\alpha}). \tag{13}$$

Moreover we can deduce that Y_k is of the form

$$Y_k = \frac{C_k}{(R-r)^{k-1}}, \text{ for } k = 1,\ 2,\ \cdots \tag{14}$$

where $C_1 = A$, and the constants $C_k > 0$, for $k \geq 2$, can be decided inductively from the equation (13), which are independent of r. Actually from (13), it is easy to check that the order of $\frac{1}{(R-r)}$ is $k-1$, *i.e.* $1 + (p+q+\alpha-2) + (k_1 - 1) + \cdots + (k_q - 1) + (l_1 - 1) + \cdots + (l_\alpha - 1) = k-1$, so the formula (14) will be hold.

Next, we prove that the series $\sum_{k \geq 1} Y_k t^k$ is a majorant series for the formal series solution $\sum_{k \geq 1} u_k(x) t^k$ near $x = 0$. In fact, we can prove, by induction, that for any $k \geq 1$ and $0 < r < R$, we have

$$|u_k(x)| \leq |k u_k(x)| \leq Y_k, \text{ on } D_r; \tag{I}$$

$$\left| \frac{\partial u_k}{\partial x}(x) \right| \leq eY_k, \text{ on } D_r. \tag{II}$$

Actually, since $Y_1 = A$, the estimates (I) and (II) hold for $k = 1$. Suppose that $k \geq 2$, and for any $1 \leq i < k$, (I) and (II) are all hold for i. Then for $i = k$, from the equation (10) and Lemma 1 (here the estimate (6) holds for any $k \geq 1$) and inductive assumptions, we have

$$\begin{aligned} |u_k(x)| &\leq \tfrac{C}{k} \sum_{p+q+\alpha \geq 2} \sum_{\substack{p+k_1+\dots+k_q \\ +l_1+\dots+l_\alpha = k}} A_{p,q,\alpha} \times |u_{k_1}(x)| \times \cdots \times |u_{k_q}(x)| \\ &\qquad \times |\tfrac{\partial u_{l_1}}{\partial x}(x)| \times \cdots \times |\tfrac{\partial u_{l_\alpha}}{\partial x}(x)| \\ &\leq \tfrac{C}{k} \sum_{p+q+\alpha \geq 2} \sum_{\substack{p+k_1+\dots+k_q \\ +l_1+\dots+l_\alpha = k}} A_{p,q,\alpha} \times Y_{k_1} \times \cdots \times Y_{k_q} \\ &\qquad \times (eY_{l_1}) \times \cdots \times (eY_{l_\alpha}). \end{aligned}$$

Since R is small enough, we may assume $0 < R < 1$, thus $(R-r)^{p+q+\alpha-2} < 1$, then we have

$$|u_k(x)| \le \tfrac{C}{k} \sum_{p+q+\alpha\ge 2} \sum_{\substack{p+k_1+\cdots+k_q \\ +l_1+\cdots+l_\alpha = k}} \frac{A_{p,q,\alpha}}{(R-r)^{p+q+\alpha-2}} \times Y_{k_1} \times \cdots \times Y_{k_q} \times (eY_{l_1}) \times \cdots \times (eY_{l_\alpha}).$$

From the formule (13) and (14), we have

$$|u_k(x)| \le \frac{R-r}{k} Y_k = \frac{C_k}{k} \cdot \frac{1}{(R-r)^{k-2}}.$$

Thus

$$|u_k(x)| \le |k u_k(x)| \le \frac{C_k}{(R-r)^{k-2}} \le \frac{C_k}{(R-r)^{k-1}} = Y_k,$$

the estimate (I) holds.

Next, by using Lemma 2, we have

$$\left| \frac{\partial u_k}{\partial x}(x) \right| \le \frac{k-1}{k} \cdot \frac{eC_k}{(R-r)^{k-1}} \le eY_k,$$

this implies the estimate (II). Therefore we have proved that $\sum_{k\ge 1} Y_k t^k$ is the majorant series of the formal series solution (3) near $x = 0$, thus the formal series solution (3) is convergent near $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$, Theorem 1 is proved.

3 Case of High Order Singular PDE

In this section, we shall extend the result of Theorem 1 to the case of high order singular partial differential equations. Let us consider the following high order singular partial differential equation:

$$(t\partial_t)^m u = F\left(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{\substack{j+\alpha\le m \\ j<m}}\right), \quad (t,x) \in \mathbf{C}_t \times \mathbf{C}_x. \tag{15}$$

Now we denote $(t\partial_t)^j \partial_x^\alpha u$ by notation $Z_{j,\alpha}$, *i.e.*

$$(t\partial_t)^j \partial_x^\alpha u \leftrightarrow Z_{j,\alpha}, \quad \text{and } \{(t\partial_t)^j \partial_x^\alpha u\}_{\substack{j+\alpha\le m \\ j<m}} \leftrightarrow Z = \{Z_{j,\alpha}\}_{\substack{j+\alpha\le m \\ j<m}}.$$

For the function $F(t,x,Z)$, we suppose

(H1)' $F(t,x,Z)$ is a holomorphic function defined in a neighborhood of the origin $(0,0,0) \in \mathbf{C}_t \times \mathbf{C}_x \times \mathbf{C}^N$, where $N = \#\{(j,\alpha) \mid j+\alpha \le m, j<m\}$.

(H2)' $F(0,x,0) \equiv 0$ near $x=0$.

Thus we rewrite $F(t,x,Z)$ near the origin as

$$F(t,x,Z) = \frac{\partial F}{\partial t}(0,x,0)t + \sum_{\substack{j+\alpha\le m\\ j<m}} \frac{\partial F}{\partial Z_{j,\alpha}}(0,x,0)Z_{j,\alpha} + R_2(t,x,Z), \tag{16}$$

where the degree of $R_2(t,x,Z)$ with respect to (t,Z) is greater than or equal to 2.

If $\frac{\partial F}{\partial Z_{j,\alpha}}(0,x,0) \equiv 0$ near $x=0$ for any $\alpha > 0$, then the equation (15) is called non-linear Fuchsian type partial differential equation. In this case the existence and uniqueness of holomorphic solution of (15) has been proved by Gérard and Tahara [3].

Here we shall consider the case for the equation (15) to be totally characteristic type. Denote $\frac{\partial F}{\partial t}(0,x,0)$ by $a(x)$, and $\frac{\partial F}{\partial Z_{j,\alpha}}(0,x,0)$ by $b_{j,\alpha}(x)$, then the equation (15) becomes

$$c(x,t\partial_t,\partial_x)u = a(x)t + R_2(t,x,Z), \tag{17}$$

where the operator

$$c(x,t\partial_t,\partial_x) = (t\partial_t)^m - \sum_{\substack{j+\alpha\le m\\ j<m}} b_{j,\alpha}(x)(t\partial_t)^j\partial_x^\alpha \tag{18}$$

is a linear singular partial differential operator, its symbol is given by

$$c(x,\rho,\xi) = \rho^m - \sum_{\substack{j+\alpha\le m\\ j<m}} b_{j,\alpha}(x)\rho^j\xi^\alpha.$$

We define the indicial operator of (15) as

$$c(x,\lambda,\partial_x) = [t^{-\lambda}c(x,t\partial_t.\partial_x)t^\lambda]|_{t=0}, \quad \text{for } \lambda \in \mathbf{C}.$$

Denoting $c_m(x,\rho,\xi)$ as principal symbol of $c(x,t\partial_t,\partial_x)$, *i.e.*

$$c_m(x,\rho,\xi) = \rho^m - \sum_{\substack{j+\alpha=m\\ j<m}} b_{j,\alpha}(x)\rho^j\xi^\alpha.$$

Thus we suppose the following condition:

(H3)' $b_{j,\alpha}(x) = x^\alpha \tilde{b}_{j,\alpha}(x)$ for $j+\alpha \le m$ and $j < m$, $\tilde{b}_{j,\alpha}(x)$ are holomorphic near $x = 0$ and $\tilde{b}_{0,m}(0) \ne 0$

Under the condition (H3)', the equation (15) is called totally characteristic type non-linear singular partial differential equation, and the operator $c(x, t\partial_t, \partial_x)$ is a linear partial differential operator with regular singularity at $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$. Furthermore, the principal symbol c_m can be factorized as

$$c_m(x,\rho,\xi) = \rho^m - \sum_{\substack{j+\alpha=m \\ j<m}} \tilde{b}_{j,\alpha}(x)\rho^j (x\xi)^\alpha = \tilde{b}_{0,m}(x) \prod_{j=1}^{m} (x\xi - \xi_j(x,\rho)), \quad (19)$$

where $\xi_j(x,\rho)$ is a continuous function near $x = 0$ and is homogeneous in ρ of degree 1, and here we suppose

(H4)' For every j, $1 \le j \le m$, $Re\ \xi_j(0,1) < 0$.

For any $k \ge 1$, operator $c(x,k,\partial_x)$ is a Fuchsian operator in x, its indicial polynomial is defined as

$$L(k,\lambda) = [x^{-\lambda} c(x,k,\partial_x) x^\lambda]|_{x=0},$$

thus roots of the indicial polynomial are called indicial exponents of $c(x,k,\partial_x)$. We have following result:

Theorem 2. *Under the conditions (H1)', (H2)', (H3)' and (H4)', if for any $k \ge 1$ $L(k,\lambda) \ne 0$ for any $\lambda \in \mathbf{Z}_+ = \{0,1,2,\cdots\}$, then the equation (15) has a unique holomorphic solution $u(t,x)$ near $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$ with $u(0,x) \equiv 0$ near $x = 0$.*

In order to prove Theorem 2, we let

$$u(t,x) = \sum_{k=1}^{\infty} u_k(x) t^k \quad (20)$$

be the formal series solution of the equation (15), then we introduce this formal series into the equation (17), and comparing the coefficients of t^k, for

$k \geq 1$, in both sides of the equation, we obtain

$$\begin{aligned} c(x, 1, \partial_x)u_1(x) &= a(x), \\ &\vdots \\ c(x, k, \partial_x)u_k(x) &= f_{k-1}\left(\{\partial_x^\alpha u_i; 1 \leq i \leq k-1, 0 \leq \alpha \leq m\}\right), \\ &\vdots \end{aligned} \tag{21}$$

where f_{k-1} is a holomorphic function near $x = 0$, only depending on $\{\partial_x^\alpha u_i; 1 \leq i \leq k-1, 0 \leq \alpha \leq m\}$.

Since $L(k, \lambda) \neq 0$, for any $k \geq 1$ and $\lambda \in \mathbf{Z}_+$, we can solve the functions $u_k(x)$ uniquely from the equation (21), which are holomorphic in a common neighborhood of $x = 0$. It remains to prove the convergence of the formal seires solution (20).

Similar to Lemma 1, for high order equation we also have the following lemma, which is important in the proof of Theorem 2.

Lemma 3. *There exists a constant $C > 0$, suth that for k large enough, we have*

$$\|u_k\|_r \leq \frac{C}{k^m}\|f_{k-1}\|_r, \tag{22}$$

where $r > 0$ small enough.

Proof: First, similar to (19), we can also use the condition (H3)' and factorize the operator $c(x, k, \partial_x)$ into

$$c(x, k, \xi) = \tilde{b}_{0,m}(x)\prod_{j=1}^{m}(x\xi - \tilde{\xi}_j(x, k)), \quad \tilde{b}_{0.m}(0) \neq 0.$$

Thus near $x = 0$, the equation (21) becomes

$$\prod_{j=1}^{m}(x\partial_x - \tilde{\xi}_j(x, k))u_k(x) = \tilde{f}_{k-1}, \tag{23}$$

where $\tilde{f}_{k-1} = \tilde{b}_{0,m}^{-1}(x)f_{k-1}$, and we know that $\xi_j(x, k)$ in the formula (19) is the main part of $\tilde{\xi}_j(x, k)$, which means

$$\frac{\tilde{\xi}_j(x, k)}{\xi_j(x, k)} \to 1, \quad \text{as} \quad k \to \infty. \tag{24}$$

We set

$$V_1 = \prod_{j=2}^{m} (x\partial_x - \tilde{\xi}_j(x,k))u_k,$$

then the equation (23) becomes

$$(x\partial_x - \tilde{\xi}_1(x,k))V_1 = \tilde{f}_{k-1}.$$

Thus we have

$$V_1 = \int_0^x e^{\int_y^x \frac{\tilde{\xi}_1(s,k)}{s}ds} \frac{\tilde{f}_{k-1}(y)}{y} dy.$$

Since $\xi_j(x,k)$ is homogenous of degree 1 with respect to k, from (24) we have

$$Re\ \tilde{\xi}_j(0,k) = kRe\ \xi_j(0,1) + o(k), \quad \text{for } 1 \le j \le m. \tag{25}$$

Thus from the condition (H4)' we have $Re\ \tilde{\xi}_j(0,k) < 0$ $(1 \le j \le m)$ for k large enough. So, similar to the proof of Lemma 1, near the origin and for large k we have for some positive constants c and C_1

$$\left| e^{\int_y^x \frac{\tilde{\xi}_1(s,k)}{s}ds} \right| \le c\ e^{C_1 Re\tilde{\xi}_1(0,k) ln(\frac{|x|}{|y|})}| = c\ \left(\frac{|x|}{|y|}\right)^{C_1 Re\tilde{\xi}_1(0,k)},$$

that is for large k, $-C_1\tilde{\xi}_1(0,k) > 0$ and we have

$$\begin{aligned} \|V_1\|_r &\le c \max_{|x|\le r} \left(|x|^{C_1 Re\tilde{\xi}_1(0,k)} \int_0^{|x|} \eta^{-C_1 Re\tilde{\xi}_1(0,k)-1} d\eta \right) \|\tilde{f}_{k-1}\|_r \\ &\le \frac{c}{-C_1 Re\tilde{\xi}_1(0,k)} \|\tilde{f}_{k-1}\|_r. \end{aligned}$$

Next, let $V_2 = \prod_{j=3}^{m}(x\partial_x - \tilde{\xi}_j(x,k))u_k$, then $(x\partial_x - \tilde{\xi}_2(x,k))V_2 = V_1$. It is the same we can deduce that there exists a constant $C_2 > 0$, such that for large k

$$\|V_2\|_r \le \frac{c}{-C_2 Re\tilde{\xi}_2(0,k)} \|V_1\|_r \le \frac{c^2}{C_1 C_2 Re\tilde{\xi}_1(0,k) Re\tilde{\xi}_2(0,k)} \|\tilde{f}_{k-1}\|_r.$$

So finally we have for k large enough,

$$\|u_k\|_r \le \prod_{j=1}^{m} \left(\frac{c}{-C_j Re\tilde{\xi}_j(0,k)} \right) \|\tilde{f}_{k-1}\|_r, \quad (\exists C_j > 0). \tag{26}$$

From (25), we know that there exists a constant $C' > 0$, such that for k large enough we have

$$\|u_k\|_r \leq \frac{C'c^m}{k^m}\|\tilde{f}_{k-1}\|_r \leq \frac{c_1 C' c^m}{k^m}\|f_{k-1}\|_r,$$

where

$$\max_{|x|\leq r}\left|\tilde{b}_{0,m}^{-1}(x)\right| \leq c_1,$$

for $r > 0$ small enough. Lemma 3 is proved.

By using Lemma 3, we can prove the convergence of the formal series solution (20) in the following way.

First we expand the remainder term $R_2(t, x, Z)$ of (16) into Taylor series with respect to (t, Z), *i.e.*

$$R_2(t,x,Z) = \sum_{p+|\nu|\geq 2} a_{p,\nu}(x) t^p Z^\nu,$$

where

$$\nu = \{\nu_{j,\alpha}\}_{\substack{j+\alpha\leq m \\ j<m}} \in \mathbb{N}^N, \quad |\nu| = \sum_{\substack{j+\alpha\leq m \\ j<m}} \nu_{j,\alpha}, \quad Z^\nu = \prod_{\substack{j+\alpha\leq m \\ j<m}} (Z_{j,\alpha})^{\nu_{j,\alpha}}.$$

We let $R > 0$ small enough such that $0 < R < 1$ and

(i) $a_{p,\nu}(x)$ is holomorphic on D_R;
(ii) $|a_{p,\nu}(x)| \leq A_{p,\nu}$ on D_R;
(iii) $\sum_{p+|\nu|\geq 2} A_{p,\nu} t^p Z^\nu$ is a convergent power series in (t, Z).

For simplicity we assume that the estimate (22) holds for any $k \geq 1$. Let us choose $A > 0$ so that

$$|\partial_x^\alpha u_1(x)| \leq A \quad \text{on } D_R, \qquad 0 \leq \forall\alpha \leq m.$$

Now, let us consider the following functional equation:

$$Y = At + \frac{C}{(R-r)^m} \sum_{p+|\nu|\geq 2} \frac{A_{p,\nu}}{(R-r)^{m(p+|\nu|-2)}} t^p (BY)^{|\nu|}, \tag{27}$$

where r is a parameter with $0 < r < R$, $C > 0$ is the constant in (22), and $B = (me)^m$.

Since the equation (27) is an analytic functional equation in Y, we see by the implicit function theorem that the equation (27) has a unique holomorphic solution $Y(t)$ in a neighborhood of $t = 0$ with $Y(0) = 0$.

If we expand $Y(t)$ into

$$Y = \sum_{k \geq 1} Y_k t^k,$$

then by the same argument as in the proof of Theorem 1 (and also in the discussion in chapter 6 of [4]) we can show that for any $k \geq 1$ and $0 < r < R$ we have

$$\left| k^j \partial_x^\alpha u_k(x) \right| \leq (me)^\alpha Y_k \leq BY_k \quad \text{on } D_r$$

$$\text{for any } (j, \alpha) \text{ with } j + \alpha \leq m \text{ and } j < m.$$

This implies that $Y = \sum_{k \geq 1} Y_k t^k$ is a majorant series of the formal solution (20). Thus, we have obtained the convergence of the formal series solution (20). Theorem 2 is proved.

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