A Semiclassical Quantization on Manifolds with Singularities and the Lefschetz Formula for Elliptic Operators

Bert-Wolfgang Schulze
Potsdam University
e-mail: schulze@mpg-ana.uni-potsdam.de
&
Vladimir Nazaikinskii* and **Boris Sternin***
Moscow State University
e-mail: nazay@ipmnet.ru
e-mail: sternin@mpg-ana.uni-potsdam.de

August 1, 1998

Abstract

For general endomorphisms of elliptic complexes on manifolds with conical singularities, the semiclassical asymptotics of the Atiyah–Bott–Lefschetz number is calculated in terms of fixed points of the corresponding canonical transformation of the symplectic space.

Keywords: elliptic operator, Fredholm property, conical singularities, pseudodifferential operators, Lefschetz fixed point formula, regularizer, Lefschetz number
1991 AMS classification: Primary 58G10, 58G12; Secondary 47A53, 47B25, 81Q60

*Supported by Institut für Mathematik, Universität Potsdam, and RFBR grants Nos. 96-01-00720 and 97-02-16722a.

Contents

Introduction

In the present paper we establish a semiclassical Atiyah–Bott–Lefschetz formula for general endomorphisms of elliptic complexes on manifolds with conical singularities.

Namely, we consider elliptic complexes of semiclassical (pseudo)differential operators with endomorphisms generated by semiclassical Fourier–Maslov integral operators. In this situation, the semiclassical Lefschetz number is defined in a natural way, and the main task of this paper is to express the semiclassical asymptotics of this Lefschetz number via some classical invariants of the endomorphism. Since the classical object associated with a Fourier integral operator is the corresponding Lagrangian manifold, it is natural to express these invariants in "Lagrangian" terms. In the present paper we consider the special class of Fourier integral operators associated with a Lagrangian manifold that is the graph of some canonical transformation and express the semiclassical asymptotics of the Lefschetz number in terms of fixed points of the transformation.

The present research was carried out on the basis of the papers [1, 2], where the semiclassical Lefschetz formula was established for general endomorphisms of elliptic complexes on smooth manifolds, the paper [3], where the semiclassical Lefschetz formula was established for geometric endomorphisms of elliptic complexes on manifolds with conical singularities, and the paper [4], where the theory of semiclassical quantization of canonical transformations was constructed for configuration spaces containing conical points.

Outline of the paper. In the first section, we introduce the main tools used in elliptic theory for singular manifolds in the semiclassical situation (semiclassical Sobolev spaces, semiclassical Mellin transform, and semiclassical pseudodifferential operators). The second section, where we recall the results of [4] in a refined form, deals with the main assertions concerning the semiclassical quantization on singular manifolds, including the theorem on the continuity of quantized canonical transformations in weighted Sobolev spaces. In the third section we state and prove our formula for the leading term of the asymptotics of the Lefschetz number $\mathcal{L}(h)$ as $h \to 0$.

1 Semiclassical Quantization of States and Observables

1.1 Conical and cylindrical coordinates

Let $(M, \{\alpha_1, \ldots, \alpha_N\})$ be a smooth manifold with conical singular points $\alpha_1, \ldots, \alpha_N$ (see [5]). In a neighborhood of each singular point α, the manifold can be represented as a cone

$$K = ([0,1) \times \Omega)/(\{0\} \times \Omega), \qquad (1)$$

where Ω, the base of the cone, is a smooth compact manifold. Throughout the following, we assume that the representation (1) is chosen and fixed in a neighborhood of each conical point. In this neighborhood we use two types of coordinates,[1] namely,

the *conical coordinates* (r, ω), $r \in [0,1)$, $\omega \in \Omega$;

the *cylindrical coordinates* (t, ω), $t \in (0, +\infty]$, $\omega \in \Omega$.

The latter are related to the conical coordinates by the change of variables

$$r = e^{-t}.$$

In fact, both coordinates are coordinates on the blow-up

$$K^\wedge = [0,1) \times \Omega$$

of the cone K.

[1] We use the same notation ω for a point $\omega \in \Omega$ (in this case our consideration can be global on Ω) and for its coordinates in some local coordinate system on Ω (then $\omega = (\omega_1, \ldots, \omega_{n-1}) \in \mathbf{R}^{n-1}$, where $n - 1 = \dim \Omega$). This will not lead to a misunderstanding.

We mainly use the cylindrical coordinates. (In this interpretation, the manifold M, or, more precisely, the blow-up $M^\wedge$, is represented as a manifold with cylindrical ends.) The conical coordinates will be used only for geometric visualization.

1.2 The semiclassical Mellin transform

This subsection, as well as several subsequent subsections, introduces the semiclassical counterparts of notions well-known in the theory of pseudodifferential operators on manifolds with singularities. Naturally, we focus on the dependence on a small parameter $h \geq 0$ occurring in semiclassical constructions.

Let $f(r) \in C_0^\infty(\mathbf{R}_+)$, $\gamma \in \mathbf{R}$. We define the *semiclassical Mellin transform* of the function f by the formula

$$\tilde{f}(p) = [\mathcal{M}_{h,\gamma} f](p) = \int_0^\infty r^{ip/h} f(r) \frac{dr}{r},$$

where p is a complex variable ranging on the *weight line*

$$\mathcal{L}_{h\gamma} = \{\operatorname{Im} p = h\gamma\}.$$

The inversion formula obviously has the form

$$f(r) = [\mathcal{M}_{h,\gamma}^{-1} \tilde{f}](r) = \frac{1}{2\pi h} \int_{\mathcal{L}_{h\gamma}} r^{-ip/h} \tilde{f}(p)\, dp.$$

In cylindrical coordinates $t = -\ln r$ ($r = e^{-t}$), the semiclassical Mellin transform becomes the semiclassical Fourier–Laplace transform (cf. [6])

$$\tilde{f}(p) = \int_{-\infty}^\infty e^{-ipt/h} \tilde{f}(t)\, dt, \quad p \in \mathcal{L}_{h\gamma},$$

and the inversion formula reads

$$f(t) = \frac{1}{2\pi h} \int_{\mathcal{L}_{h\gamma}} e^{ipt/h} \tilde{f}(p)\, dp.$$

1.3 Semiclassical Sobolev spaces

Let $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbf{R}^N$ and $s \in \mathbf{R}$. We introduce the Sobolev space $H_h^{s,\gamma}(M)$ as follows. Let $e_0, e_1, \dots, e_N$ be a partition of unity on M such that all the e_i are smooth functions independent of h, each function $e_j(x)$, $j = 1, \dots, N$, is supported in a neighborhood of the point α_j covered by the conical (and cylindrical) coordinate system, and $\operatorname{supp} e_0$ contains none of the singular points $\alpha_1, \dots, \alpha_N$. On the space $C_0^\infty(M)$ of smooth functions $u(x)$ on M such that $\operatorname{supp} u \cap \{\alpha_1, \dots, \alpha_N\} = \emptyset$, we define a norm $\|u\|_{s,\gamma,h}$ by setting

$$\|u\|_{s,\gamma,h}^2 = \sum_{j=1}^N \|e_j u\|_{s,\gamma_j,h}^2 + \|e_0 u\|_{s,h}^2.$$

Here $\|\cdot\|_{s,h}^2$ is one of the usual (pairwise equivalent) norms in the ordinary semiclassical Sobolev space $H_h^s(M \setminus \bigcup_{j=1}^N U_j)$ (e.g., see [6]), where U_j is a sufficiently small neighborhood of α_j. In local coordinates, up to equivalence, we have

$$\|u(x)\|_{s,h}^2 = \int_{\mathbf{R}^n} \left| \left(1 - h^2 \frac{\partial^2}{\partial x^2}\right)^{s/2} u(x) \right|^2 dx;$$

the norm on the right-hand side of this formula is just the norm in the Sobolev space $H_h^s(\mathbf{R}^n)$ [6]. Next, the norm $\|\cdot\|_{s,\gamma_j,h}$ is defined in the conical or cylindrical coordinates with the help of the semiclassical Mellin transform as follows:

$$\begin{aligned} \|u\|_{s,\gamma_j,h}^2 &= \int_0^\infty \frac{dr}{r} \left\| \left(1 + \left(ihr\frac{\partial}{\partial r}\right)^2 - h^2 \Delta_{\Omega_j}\right)^{s/2} r^{-2\gamma_j} u(r,\omega) \right\|_{L^2(\Omega_j)}^2 \\ &= \int_{-\infty}^\infty \left\| \left(1 - h^2 \frac{\partial^2}{\partial t^2} - h^2 \Delta_{\Omega_j}\right)^{s/2} e^{2\gamma_j t} u(e^{-t},\omega) \right\|_{L^2(\Omega_j)}^2 dt \\ &\sim \frac{1}{2\pi h} \int_{\mathcal{L}_{\gamma_j h}} \left\| (1 + |p|^2 - h^2 \Delta_{\Omega_j})^{s/2} \tilde{u}(p,\omega) \right\|_{L^2(\Omega_j)}^2 dp, \end{aligned}$$

where Ω_j is the base of the cone at the point α_j, Δ_{Ω_j} is the Beltrami–Laplace operator on Ω_j with respect to some Riemannian metric on Ω_j (independent of h), and $L^2(\Omega_j)$ is the space of functions on Ω_j square integrable with respect to the measure corresponding to this Riemannian metric. One can verify that up to equivalence $\|\cdot\|_{s,\gamma,h}$ is independent of the ambiguity in this construction, and moreover, the constants in the inequalities expressing the equivalence can be chosen to be independent of h.

For each h we complete the space $C_0^\infty(M)$ with respect to the norm $\|\cdot\|_{s,\gamma,h}$ and denote the space of functions $u(x,h)$ such that for each $h\in(0,1]$ the function $u(\cdot,h)$ belongs to the above-mentioned completion by $H_h^{s,\gamma}(M)$. Thus, the "norm" $\|\cdot\|_{s,\gamma,h}$ on the space $H_h^{s,\gamma}(M)$ is actually a family of seminorms indexed by $h\in(0,1]$.

For each given γ, the spaces $\{H_h^{s,\gamma}(M)\}_{s\in\mathbf{R}^n}$ form a scale. In particular,

$$H_h^{s,\gamma}(M)\subset H_h^{s',\gamma}(M)$$

for $s\ge s'$. Moreover,

$$H_h^{s,\gamma}(M)\subset H_h^{s',\gamma'}(M) \tag{2}$$

for $s\ge s'$ and $\gamma\ge\gamma'$ (the last inequality is understood in the sense that $\gamma_i\ge\gamma_i'$ for $i=1,\dots,N$). If $s>s'$ and $\gamma>\gamma'$ (that is, $\gamma_i>\gamma_i'$ for all $i=1,\dots,N$), then the embedding (2) is compact (for each fixed h).

We shall study operators acting in semiclassical Sobolev spaces. Let

$$A=A(h):\ H_h^{s,\gamma}\to H_h^{s_1,\gamma_1}$$

be some linear operator. We say that A is *bounded* with a norm of the order of $\varphi(h)$ if

$$\|A\,u\|_{s_1,\gamma_1,h}\le C\,\varphi(h)\,\|u\|_{s,\gamma,h}$$

with some constant C independent of h. We merely say that A is bounded if one can take $\varphi(h)=1$ in this inequality.

1.4 Semiclassical pseudodifferential operators

First, we introduce *symbol classes*.

Let $x=(x^1,\dots,x^n)\in\mathbf{R}^n$ and $\xi=(\xi_1,\dots,\xi_n)\in\mathbf{R}_n$. By

$$S^m=S^m(\mathbf{R}^n_x\times\mathbf{R}_{n\xi}\times(0,1]_h)$$

we denote the space of smooth functions $H(x,\xi,h)$ such that

$$\left|\frac{\partial^{|\alpha|+|\beta|+l}H}{\partial x^\alpha\partial\xi^\beta\partial h^l}(x,\xi,h)\right|\le C_{\alpha\beta l}(1+|\xi|)^{m-|\beta|},\quad |\alpha|+|\beta|+l=0,1,2,\dots. \tag{3}$$

For each function satisfying these estimates, the operator[2]

$$\hat{H}=H\left(\overset{2}{x},-ih\overset{1}{\frac{\partial}{\partial x}},h\right) \tag{4}$$

[2]We use Feynman ordering of noncommuting operators; see [7] for details.

is well defined and is bounded in the Sobolev spaces

$$\hat{H}\colon\ H_h^s(\mathbf{R}^n) \to H_h^{s-m}(\mathbf{R}^n)$$

for each $s \in \mathbf{R}$.

Next, let $x = (r, \omega_1, \ldots, \omega_{n-1}) \in \mathbf{R}_+^n = \{(r,\omega) | r \geq 0\}$ and $\xi = (p, q_1, \ldots, q_{n-1}) \in \mathbf{C}_p \times \mathbf{R}_q^{n-1}$. By

$$S_\varepsilon^m = S_\varepsilon^m(\mathbf{R}_x^n \times \mathbf{C}_p \times \mathbf{R}_q^{n-1} \times [0,1)_h)$$

we denote the space of functions $H(x, \xi, h)$ such that

i) $H(x,\xi,h)$ is defined in $\mathbf{R}_x^n \times \mathbf{C}_p \times \mathbf{R}_q^{n-1} \times [0,1)_h$ in the strip $\{|\operatorname{Im} p| < \varepsilon\}$ and is smooth with respect to all the variables and analytic in p in this strip;

ii) $H(x,\xi,h) = 0$ for $r > R$, where R is sufficiently large;

iii) the estimates (3) hold in this strip.

Under these conditions, the operator

$$\hat{H} = H\left(\overset{2}{r}, \overset{2}{\omega}, ihr\overset{1}{\frac{\partial}{\partial r}}, -\,ih\overset{1}{\frac{\partial}{\partial \omega}}\right) \equiv H\left(e^{\overset{2}{-t}}, \overset{2}{\omega}, -ih\frac{\partial}{\partial t}, -ih\frac{\partial}{\partial \omega}\right) \tag{5}$$

is well defined and bounded in the spaces

$$\hat{H}\colon\ H_h^{s,\gamma}(\mathbf{R}_+ \times \mathbf{R}^{n-1}) \to H_h^{s-m,\gamma}(\mathbf{R}_+ \times \mathbf{R}^{n-1})$$

for any $s \in \mathbf{R}$ and $\gamma \in \mathbf{R}$ and for sufficiently small h (it suffices to take $h < \varepsilon/|\gamma|$, so that the weight line $\mathcal{L}_{h\gamma}$ will lie in the strip $\{|\operatorname{Im} p| < \varepsilon\}$).

We now define a semiclassical pseudodifferential operator of order m on a manifold M with conical singularities as an operator which has the form (4) (respectively, (5)) in the local coordinates on the smooth part of M (respectively, near the conical singular points) modulo an integral operator $\hat{Q}$ with smooth kernel such that

$$\hat{Q}\colon\ H_h^{s,\gamma}(M) \to H_h^{s-N,\gamma}(M)$$

is compact for any s, γ, and N and has the norm $O(h^{N_1})$ for arbitrary N_1.

In the usual way we introduce the notion of the *conormal symbol* [5] of a semiclassical pseudodifferential operator $\hat{H}$ on M at each conical point $\alpha \in \{\alpha_1, \ldots, \alpha_N\}$. The conormal symbol is an analytic family $\hat{H}_0\,(p)$ of operators depending on the parameter p, $|\operatorname{Im} p| < \varepsilon$, and acting in spaces of functions on the base Ω of the corresponding cone.

2 Canonical Transformations and Their Quantization

2.1 The cotangent bundle

Using the cylindrical coordinates (t, ω) in a neighborhood of each conical point, we consider the blow-up $M^\wedge$ of the manifold M treating the latter as a manifold with conical ends. The manifold $M^\wedge$ is a manifold with boundary (the point $t = \infty$ is included), and by definition we set $T^*M = T^*M^\wedge$ (thus, T^*M is a manifold with boundary). In a neighborhood of a cylindrical end, T^*M has the form

$$T^*M \sim T^*\Omega \times (0, \infty] \times \mathbf{R} \ni (\omega, q; t, p)$$

and is equipped with symplectic form

$$\omega^2 = dp \wedge dt + dq \wedge d\omega$$

(here $dq \wedge d\omega$ is the standard symplectic form on $T^*\Omega$ and Ω is the base of the corresponding cone). In the conical coordinates near the singular point we have

$$T^*M \sim T^*\Omega \times [0, 1) \times \mathbf{R} \ni (\omega, q; r, p),$$

and the symplectic form becomes

$$\omega^2 = -\frac{dp \wedge dr}{r} + dq \wedge d\omega.$$

2.2 Canonical transformations

We consider canonical transformations of the space T^*M, that is, smooth mappings

$$g : T^*M \to T^*M$$

of manifolds with boundary such that

$$g^*\omega^2 = \omega^2.$$

Under the assumption that the bases of all cones corresponding to singular points are connected, it is obvious by definition that g is a diffeomorphism of the "fiber" of T^*M over each conical point α (the word "fiber" is used here to mean the corresponding component of the boundary of T^*M) onto the "fiber" over some (possibly, the same) conical point α_1. We write $\alpha_1 = g(\alpha)$; this notation will not lead to a misunderstanding.

Let us describe the structure of canonical transformations near the conical points. Let (t, ω, p, q) and (τ, ψ, ξ, η) be the cylindrical coordinates near the points α and $g(\alpha)$ on the first and the second copy of T^*M, respectively. Then y can be written as a mapping

$$g: (t, \omega, p, q) \mapsto (\tau, \psi, \xi, \eta).$$

The following properties of g were established in [4].

Theorem 1 1) *The mapping g near the conical point can be represented in the form*

$$\begin{aligned} \tau &= t + \chi(e^{-t}, \omega, p, q) \\ \psi &= \psi(e^{-t}, \omega, p, q), \\ \xi &= \xi(e^{-t}, \omega, p, q), \\ \eta &= \eta(e^{-t}, \omega, p, q), \end{aligned} \tag{6}$$

where χ, ψ, ξ, and η are smooth functions.

2) *The formulas*

$$\begin{aligned} \psi &= \psi(0, \omega, p, q), \\ \eta &= \eta(0, \omega, p, q) \end{aligned}$$

specify a family of canonical transformations

$$g(p): T^*\Omega_\alpha \to T^*\Omega_{g(\alpha)},$$

where Ω_α and $\Omega_{g(\alpha)}$ are the bases of the cones at the corresponding conical points. This family is called the conormal family *of g at the conical point α.*

3) *The mapping*

$$\xi = \xi(0, \omega, p, q),$$

has the form

$$\xi = p + c,$$

where c is some constant, which will be referred to as the conormal shift *of g at a the conical point α.*

4) *Let*

$$g(\infty, \omega_0, p_0, q_0) = (\infty, \psi_0, \xi_0, \eta_0).$$

Then there is a subset $I \subset \{1, \dots, n-1\}$ such that the functions $(\tau, \psi_I, \eta_{\overline{I}}, p, q)$, where $\overline{I} = \{1, \dots, n-1\} \backslash I$, form a system of local coordinates on the graph of g in a neighborhood of the point

$$(\infty, \omega_0, p_0, q_0; \infty, \psi_0, \xi_0, \eta_0).$$

Moreover, in the corresponding neighborhood of the point $(\infty, \omega_0, p_0, q_0)$ *the transformation is determined by a generating function of the form*

$$S_I(\tau, \psi_I, \eta_{\overline{I}}, p, q) = (p+c)\tau + S_{1I}(e^{-\tau}, \psi_I, \eta_{\overline{I}}, p, q)$$

by the usual formulas

$$t = \frac{\partial S_I}{\partial p}, \ \ \xi = \frac{\partial S_I}{\partial \tau}, \ \ \omega = \frac{\partial S_I}{\partial q}, \ \ \eta_I = \frac{\partial S_I}{\partial \psi_I}, \ \ \psi_{\overline{I}} = -\frac{\partial S_I}{\partial \eta_{\overline{I}}}.$$

2.3 Quantized canonical transformations

Let

$$g : T^*M \to T^*M$$

be a canonical transformation, and let a be a smooth function on T^*M. Under some additional assumptions we shall define an operator $T(g,a)$ (a quantized canonical transformation) acting in the semiclassical Sobolev spaces $H_k^{s,\gamma}(M)$.

Assumption 1 The transformation g is asymptotically first-order homogeneous with respect to the action of the group $\mathbf{R}_+$ of positive numbers on the fibers of T_0^*M.[3] The conormal shift of g is zero.

Assumption 2 In a neighborhood of the conical points, the generating functions of g are analytic in p in the strip $|\operatorname{Im} p| < \varepsilon$ for some $\varepsilon > 0$.

The graph

$$L_g \subset T^*M \times T^*M \tag{7}$$

of the canonical transformation g is a Lagrangian manifold with respect to difference of the symplectic forms on the first and the second copy of T^*M. It is equipped with the standard measure (volume form) that is the nth power of the symplectic form on either of the copies of T^*M lifted to L_g via the natural projection.

Assumption 3 The manifold L_g satisfies the quantization conditions [8, 6].

Assumption 4 The function a belongs to $S_\varepsilon^m(T^*M)$. (The space $S_\varepsilon^m(T^*M)$ is defined as the space of smooth functions on T^*M whose coordinate representatives belong to S^m for charts away from conical points and to S_ε^m for conical charts.)

[3]This requirement can be weakened and replaced by some finer conditions imposed on the behavior of g at infinity in the fibers. To clarify the exposition, we have chosen the simplest possibility.

Let K_g be the Maslov canonical operator on L_g [8, 6]. We define $\hat{T}(g,a)$ to be the integral operator with Schwartz kernel $[K_g(\pi_1^* a)](x,y)$ on $M \times M$, where $\pi_1 : L_g \to T^*M$ is the natural projection on the first factor on the right-hand side in (7). (It is assumed that a smooth measure dx on M is given such that in the cylindrical coordinates near each conical point we have

$$dx = dt \wedge d\omega,$$

where $d\omega$ is some smooth measure on the base Ω of the corresponding cone. Then we can treat Schwartz kernels as functions.)

If the support of the amplitude a is entirely contained in a cylindrical canonical chart with the coordinates $(\tau, \psi_I, \eta_{\overline{I}}, p, q)$, then the operator $\hat{T}(g,a)$ can be represented in the form

$$\begin{aligned}[\hat{T}(g,a)u](\tau,\psi) &= \left(\frac{i}{2\pi h}\right)^{n+|\overline{I}|/2} \iiint \exp\left\{\frac{i}{h}\left[S_I(\tau,\psi_I,\eta_{\overline{I}},p,q) + \psi_{\overline{I}}\eta_{\overline{I}}\right]\right\} \\ &\times (\pi_1^* a)\left(\frac{D(\xi,\eta_I,\psi_{\overline{I}})}{D(p,q)}\right)^{1/2} (e^{-\tau},\psi_I,\eta_{\overline{I}},p,q)\tilde{u}(p,q)\,dp\,dq\,d\eta_{\overline{I}},\end{aligned} \quad (8)$$

where $\tilde{u}(p,q)$ is the semiclassical Fourier–Laplace transform of the function $u(t,\omega)$, $\pi_1^* a$ is expressed via the local coordinates of the canonical chart, the integral with respect to p is taken over the weight line $\mathcal{L}_{h\gamma}$, and the argument of the Jacobian is chosen according to the construction of the canonical operator.

Theorem 2 *Under the above assumptions, the operator* $T(g,a)$ *is continuous in the spaces*

$$\hat{T}(g,a) : H_h^{s,\gamma}(M) \to H_h^{s-m,\gamma}(M)$$

for any $s \in \mathbf{R}$ *and* $\gamma \in \mathbf{R}$ *provided that* $h < \varepsilon/|\gamma|$.

Proof. Although the proof of this statement can be found in [4], here we outline a different proof. Without loss of generality (multiplying $\hat{T}(g,a)$ on the left by $e^{-\gamma t}$ and on the right by $e^{\gamma t}$) we can assume that $\gamma = 0$ (in this case, the coordinate p in the argument of the integrand is shifted by $i\gamma h$). Then $\hat{T}(g,a)$ becomes the usual Fourier–Maslov integral operator with a special behavior of the coefficients as $\tau \to \infty$. Using a technique similar to that in [9], one can show that

$$\overset{\wedge}{T^*}(g,a)\,\hat{T}(g,a),$$

where the star stands for the adjoint in $L^2(\mathbf{R}\times\Omega, dt\,d\omega)$, is a $2m$th-order pseudodifferential operator of the form described in Subsection 1.4. This readily implies the assertion of the theorem.

For the operator $\stackrel{\wedge}{T}=\stackrel{\wedge}{T}(g,a)$, we define the *conormal symbol* $\stackrel{\wedge}{T}_0(p)$ as follows. Let us represent $\stackrel{\wedge}{T}$ in a neighborhood of a conical point in the form

$$\stackrel{\wedge}{T}=\stackrel{\wedge}{\mathcal{T}}\left(\stackrel{2}{r}, ir\stackrel{1}{\frac{\partial}{\partial r}}\right),$$

where $\stackrel{\wedge}{\mathcal{T}}(r,p)$, the operator-valued symbol of $\stackrel{\wedge}{T}$, is a family of operators acting in function spaces on the base Ω of the corresponding cone. Then

$$\stackrel{\wedge}{T}_0(p)=\stackrel{\wedge}{\mathcal{T}}(0,p).$$

Equivalently, $\stackrel{\wedge}{T}_0(p)$ can be defined as the operator-valued symbol of the operator

$$\stackrel{\wedge}{T}_0\equiv\stackrel{\wedge}{T}_0\left(ir\frac{\partial}{\partial r}\right)\equiv\stackrel{\wedge}{\mathcal{T}}\left(0, ir\frac{\partial}{\partial r}\right)=\text{s-}\lim_{\lambda\to 0} U_\lambda^{-1}\stackrel{\wedge}{T}U_\lambda,$$

where

$$U_\lambda f(r)=f(r/\lambda)$$

and the strong limit is taken on the set of functions compactly supported in r.

Let us write out the conormal symbol for the case in which the operator $\stackrel{\wedge}{T}(g,a)$ is given by (8) (the general case can be reduced to that with the help of a partition of unity). Since the conormal shift is zero ($c=0$), we have

$$S_I(\tau,\psi_I,\eta_{\overline{I}},p,q)=p\tau+S_{1I}(e^{-\tau},\psi_I,\eta_{\overline{I}},p,q),$$

and the operator (8) can be rewritten in the cylindrical coordinates in the form

$$\begin{aligned}[T(g,a)u](r,\psi) &= \left(\frac{i}{2\pi h}\right)^{n-1+|\overline{I}|/2}\iint\exp\left\{\frac{i}{h}\left[S_{1I}\Big(\stackrel{2}{r},\psi_I,\eta_{\overline{I}}, ir\stackrel{1}{\frac{\partial}{\partial r}},q\Big)+\psi_{\overline{I}}\eta_{\overline{I}}\right]\right\}\\ &\quad\times(\pi_1^*a)\frac{D(\xi,\eta_I,\psi_{\overline{I}})}{D(p,q)}\Big(\stackrel{2}{r},\psi_I,\eta_{\overline{I}}, ir\stackrel{1}{\frac{\partial}{\partial r}},q\Big)\check{u}(r,q)\,dq\,d\eta_{\overline{I}}\end{aligned}$$

where $\check{u}(r,q)$ is the semiclassical Fourier transform of u with respect to ω. It follows that the conormal symbol $\stackrel{\wedge}{T}_0(p)$ is given by the formula

$$[\stackrel{\wedge}{T}_0(p)v](\psi) = \left(\frac{i}{2\pi h}\right)^{n-1+|\overline{I}|/2}\iint\exp\left\{\frac{i}{h}\left[S_{1I}\Big(0,\psi_I,\eta_{\overline{I}},p,q\Big)+\psi_{\overline{I}}\eta_{\overline{I}}\right]\right\}$$

$$\times(\pi_1^* a)\left(\frac{D(\xi,\eta_I,\psi_{\overline{I}})}{D(p,q)}\right)^{1/2}\left(0,\psi_I,\eta_{\overline{I}},p,q\right)\check{v}(q)\,dq\,d\eta_I,$$

or

$$\hat{T}_0\,(p) = \hat{T}\,(g(p),a(p)), \tag{9}$$

where $g(p)$ is the conormal family of g and $a(p)$ is produced by the restriction of a to the boundary $r=0$. To prove (9), it suffices to observe that the function $S_{1I}(0,\psi_I,\eta_{\overline{I}},p,q)$ (where p is a parameter) is just a generating function of $g(p)$ (we omit the computations with the Jacobian $D(\xi,\eta_I,\psi_{\overline{I}})/D(p,q)$).

3 The Atiyah–Bott–Lefschetz theorem

3.1 Statement of results

Let M be a compact manifold with conical singularities $\{\alpha_1,\ldots,\alpha_N\}$, and let

$$\hat{D}\colon\ C^\infty(M,E_1)\to C^\infty(M,E_2)$$

be a formally elliptic semiclassical pseudodifferential operator of order m on M (the formal ellipticity means that the principal symbol $\sigma(\hat{D})$ is invertible outside the zero section of T^*M). Then the conormal symbol $\hat{D}_0\,(p)$ is elliptic with parameter in the sense of Agranovich–Vishik [10] in some sector containing the real axis and is invertible in some sectors of the form shown in Fig. 1 and independent of $h\in(0,1]$. Outside these sectors, for each $h\in(0,1]$ the operator $\hat{D}{}_0^{-1}(p)$ has countably many poles with finite-dimensional principal parts of the Laurent series.

Thus, for each $h\in(0,1]$ there are at most finitely many values of γ such that the operator $\hat{D}$ is not elliptic in the scale $\{H_h^{s,\gamma}(M)\}$. Moreover, the set of pairs (γ,h) for which $\hat{D}{}_0^{-1}(p)$ has poles on the line $\mathcal{L}_{h\gamma}$ is one-dimensional (in the sense of measure theory), and hence for almost all γ the set of values of the parameter h for which $\hat{D}{}_0^{-1}(p)$ has poles on $\mathcal{L}_{h\gamma}$ is discrete. We only deal with such values of γ, which will be referred to as *admissible*. We denote the set of values of h for which the operator $\hat{D}$ is elliptic in the scale $\{H_h^{s,\gamma}(M)\}$ by $Z(\gamma)$; for each admissible γ, this is an everywhere dense subset of the interval $(0,1)$.

Now let $g : T^*M \to T^*M$ be a canonical transformation satisfying Assumptions 1– 3, and let $a_i\in S^0(T^*M)$, $i=1,2$, be amplitudes satisfying Assumption 4.

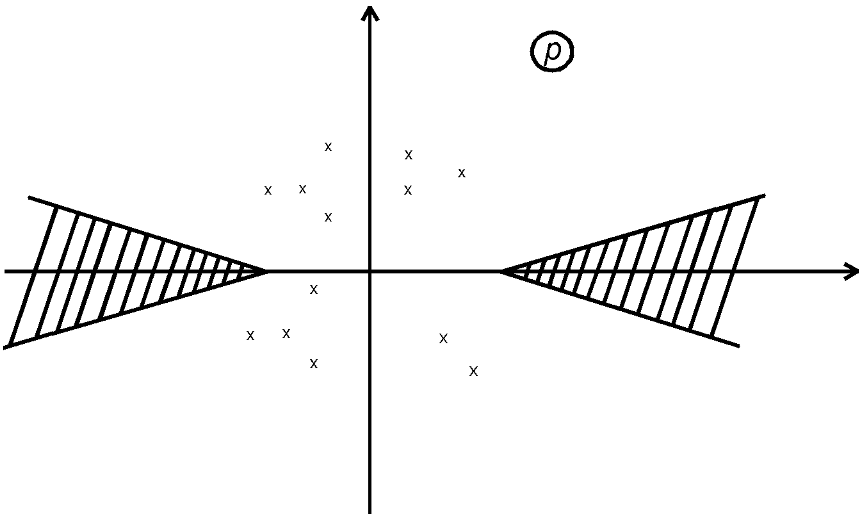

Figure 1. The conormal symbol has no poles in the dashed sectors

We set $\hat{T}_i = \hat{T}(g, a_i)$. Suppose that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\infty(E) & \xrightarrow{\hat{D}} & C^\infty(F) & \longrightarrow & 0 \\ & & \downarrow \hat{T}_1 & & \downarrow \hat{T}_2 & & \\ 0 & \longrightarrow & C^\infty(E) & \xrightarrow{\hat{D}} & C^\infty(F) & \longrightarrow & 0 \end{array}$$

commutes. Then so does the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_h^{s,\gamma}(E) & \xrightarrow{\hat{D}} & H_h^{s-m,\gamma}(F) & \longrightarrow & 0 \\ & & \downarrow \hat{T}_1 & & \downarrow \hat{T}_2 & & \\ 0 & \longrightarrow & H_h^{s,\gamma}(E) & \xrightarrow{\hat{D}} & H_h^{s-m,\gamma}(F) & \longrightarrow & 0 \end{array} \tag{10}$$

for all s, h, and $\gamma < \varepsilon/h$. For $h \in Z(\gamma)$, the operator $\hat{D}$ is elliptic, and hence the

Lefschetz number

$$\mathcal{L}(h) = \text{Trace } \hat{T}_1 |_{\text{Ker}\hat{D}} - \text{Trace } \hat{T}_2 |_{\text{Coker}\hat{D}}$$

is well defined.

We shall obtain the asymptotics of the Lefschetz number $\mathcal{L}(h)$ as $h \to 0$, $h \in Z(\gamma)$, under some additional assumptions about the canonical transformation g and the conormal symbol $\hat{D}_0(p)$.

Assumption 5 The transformation g is *nondegenerate* in the sense that

i) for each interior fixed point $z = g(z) \in T^*M$, one has

$$\det(1 - g_*(z)) \neq 0;$$

ii) for each conical fixed point α, one of the following conditions is satisfied.

Either

$$\chi(0, \omega, p, q) > 0$$

for all $p, \omega, q \in \mathbf{R} \times T^*\Omega$ (in this case the point is said to be *attractive*),

or

$$\chi(0, \omega, p, q) < 0$$

for all $p, \omega, q \in \mathbf{R} \times T^*\Omega$ (in this case the point is said to be *repulsive*).

Here $\chi(e^{-t}, \omega, p, q)$ is the function determining the t-component of g according to formulas (6) in Theorem 1.

Assumption 6 The conormal symbol $\hat{D}_0(p)$ at each conical point α satisfies the following conditions:

a) for each $\varepsilon > 0$, the number $N(h)$ of poles (with regard to multiplicity) of the family $\hat{D}_0^{-1}(p)$ in the strip $\{|\operatorname{Im} p| < \varepsilon\}$ satisfies the estimate

$$N(h) \leq C(\varepsilon) h^{-N_0}$$

for some $N_0 \in \mathbf{R}$;

b) for each compact subset $K \subset \mathbf{C}$, the family $\hat{D}_0^{-1}(p)$ satisfies the estimate

$$\|\hat{D}_0^{-1}(p)\|_{H^s(\Omega) \to H^s(\Omega)} \leq C \operatorname{dist}(p, \operatorname{spec}(\hat{D}_0))^{-N_1}, \quad p \in K,$$

where $\operatorname{dist}(p, \operatorname{spec}(\hat{D}_0))$ is the distance from the point p to the set $\operatorname{spec}(\hat{D}_0)$ of poles of the family $\hat{D}_0^{-1}(p)$.

Under these assumptions, the following theorem holds.

Theorem 3 *The Lefschetz number $\mathcal{L}(h)$ of the diagram* (10) *has the following asymptotics for given γ and $h \to 0$, $h \in Z(\gamma)$:*

$$\mathcal{L}(h) = \mathcal{L}_{\text{int}} + \sum_{\alpha_k} \mathcal{L}(\alpha_k) + O(h),$$

where $\mathcal{L}_{\text{int}}$ is the contribution of the interior fixed points, which can be calculated in the same way as in [1]*, $\sum\limits_{\alpha_k}$ extends over all conical fixed points α_k of the transformation g, and $\mathcal{L}(\alpha_k)$ is the contribution of the conical fixed point α_k, which is given by the formula*

$$\mathcal{L}(\alpha_k) = \pm \sum_{\pm h\gamma_k < \pm \operatorname{Im} p_j < \pm h\gamma_k + \rho_k(h)} \operatorname{Trace} \operatorname*{Res}_{p_j} \left\{ \hat{T}_{10}(p) \hat{D}{}_0^{-1}(p) \frac{\partial \hat{D}_0(p)}{\partial p} \right\} \tag{11}$$

(*the upper sign corresponds to attractive, and the lower, to repulsive points*)*. Here $\hat{T}_{10}(p)$ and $\hat{D}_0(p)$ are the conormal symbols of $\hat{T}_1(p)$ and $\hat{D}$, respectively, at the point α_k and $\rho_k(h)$ is some function of h such that $\varepsilon/2 \le \rho_k(h) \le \varepsilon$. If the coefficients of the principal parts of the Laurent series of $\hat{D}{}_0^{-1}(p)$ at the poles p_j are bounded in the L^2 norm by a constant of the order of h^{-N}, then in* (11) *we can take $\rho_k(h) = \text{const} = \varepsilon$.*

3.2 Proof of the theorem

To reduce the amount of subscripts in formulas, we carry out the proof for the case in which the manifold M has only one conical point α (which is necessarily a fixed point of g in this case). Without loss of generality, we can assume that $\gamma = 0$. The proof consists of two parts:

1) For each fixed $h \in Z(\gamma)$, we construct a regularizer of $\hat{D}$ with a special dependence on a parameter $\lambda \to \infty$; a preliminary formula for the Lefschetz number is obtained by passing to the limit as $\lambda \to \infty$.

2) We study the asymptotics of the expression thus obtained as $h \to 0$, $h \in Z(\gamma)$.

1) The Lefschetz number can be calculated by the well-known formula (e.g., see [11, 3])

$$\mathcal{L}(h) = \operatorname{Trace}(\hat{T}_1 (1 - \hat{R}\hat{D})) - \operatorname{Trace}(\hat{T}_2 (1 - \hat{D}\hat{R})), \tag{12}$$

where $\hat{R}$ is some global regularizer of the operator $\hat{D}$ on M. We construct $\hat{R}$ in the form

$$\hat{R} = \psi_1 \hat{R}_1 f_1 + \psi_2 \hat{R}_2 f_2, \tag{13}$$

where $\hat{R}_1$ is the exact local inverse of $\hat{D}$ in a neighborhood of the conical singular point, constructed in [3], and $\hat{R}_2$ is the "interior" (usual pseudodifferential) regularizer; f_1, f_2 is a partition of unity such that $f_1 \equiv 1$ in a neighborhood of the conical point and $f_1 \equiv 0$ outside a larger neighborhood; ψ_1 and ψ_2 are cutoff functions such that $\psi_i f_i = f_i$, $i = 1, 2$. We choose f_1, ψ_1 and ψ_2 to be functions that depend only on the cylindrical variable t in a neighborhood of the conical point; moreover,

$$f_1 = f_1(t - \lambda), \qquad \psi_1 = \psi_1(t - \lambda), \qquad \psi_2 = \psi_2(t - \lambda),$$

where λ is the above-mentioned large parameter. Note that while $\hat{R}_1$ is constructed for each fixed h separately and it is hard to say anything about the behavior of $\hat{R}_1$ as $h \to 0$, $\hat{R}_2$ is an semiclassical pseudodifferential operator and so is well-behaved as $h \to 0$ (see Lemma 3 below). In particular, it follows that the operators

$$\hat{Q}' = 1 - \hat{R}_2 \hat{D}, \quad \hat{Q} = 1 - \hat{D} \hat{R}_2$$

are semiclassical pseudodifferential operators.

Lemma 1 *The functions f_1, f_2, ψ_1 and ψ_2 can be chosen so that for all sufficiently large λ one has*

$$[\operatorname{supp} f_j \cup \pi g(\pi^{-1} \operatorname{supp} f_j)] \cap \operatorname{supp}(1 - \psi_j) = \emptyset, \quad j = 1, 2,$$

where $\pi : T^*M \to M$ is the natural projection.

The proof is the same as that of Lemma 2 in [3].

Let us substitute the regularizer (13) into (12). We have

$$\begin{aligned}
1 - \hat{D}\hat{R} &= 1 - \hat{D} \psi_1 \hat{R}_1 f_1 - \hat{D} \psi_2 \hat{R}_2 f_2 \\
&= 1 - [\hat{D}, \psi_1] \hat{R}_1 f_1 - f_1 - [\hat{D}, \psi_2] \hat{R}_2 f_2 - \psi_2 (1 - \hat{Q}) f_2 \\
&= \psi_2 \hat{Q} f_2 - [\hat{D}, \psi_1] \hat{R}_1 f_1 - [\hat{D}, \psi_2] \hat{R}_2 f_2, \\
1 - \hat{R}\hat{D} &= 1 - \psi_1 \hat{R}_1 f_1 \hat{D} - \psi_2 \hat{R}_2 f_2 \hat{D} \\
&= 1 - f_1 - \psi_2 (1 - \hat{Q}') f_2 - \psi_1 \hat{R}_1 [f_1, \hat{D}] - \psi_2 \hat{R}_2 [f_2, \hat{D}] \\
&= \psi_2 \hat{Q}' f_2 - \psi_1 \hat{R}_1 [f_1, \hat{D}] - \psi_2 \hat{R}_2 [f_2, \hat{D}],
\end{aligned}$$

and consequently,

$$\begin{aligned}\mathcal{L} &= \text{Trace}\,(\hat{T}_1\ \psi_2\hat{Q}' f_2) - \text{Trace}\,(\hat{T}_2\ \psi_2\ \hat{Q}\ f_2) \\ &\quad + \text{Trace}\,(\hat{T}_1\ \{\psi_1\ \hat{R}_1\ [\hat{D}, f_2] + \psi_2\ \hat{R}_2\ [\hat{D}, f_2]\}) \\ &\quad + \text{Trace}\,(\hat{T}_2\ \{[\hat{D}, \psi_1]\ \hat{R}_1\ f_1 + [\hat{D}, \psi_2]\ \hat{R}_2\ f_2).\end{aligned}$$

We rewrite this expression in the form

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,$$

where

$$\begin{aligned}\mathcal{L}_1 &= \text{Trace}\,(\hat{T}_1\ (\hat{R}_1 - \hat{R}_2)[\hat{D}, f_1]), \\ \mathcal{L}_2 &= \text{Trace}\,(\hat{T}_1\ \psi_2\hat{Q}' f_2) - \text{Trace}\,(\hat{T}_2\ \psi_2\hat{Q} f_2), \\ \mathcal{L}_3 &= \text{Trace}\,(\hat{T}_1\ \{(1-\psi_1)\ \hat{R}_1\ [f_1, \hat{D}] + (1-\psi_2)\ \hat{R}_2\ [\hat{D}, f_2]\}) \\ &\quad + \text{Trace}\,(\hat{T}_2\ \{[\hat{D}, \psi_1]\ \hat{R}_1\ f_1 + [\hat{D}, \psi_2]\ \hat{R}_2\ f_2\})\end{aligned}$$

(the argument of the functions f_i and ψ_i in this formula is $t - \lambda$).

Now let $\lambda \to \infty$. Just as in [3], we obtain

$$\lim_{\lambda\to\infty} \mathcal{L}_1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \text{Trace}\ \hat{T}_{10}\ (p)(\hat{R}_{10}\ (p) - \hat{R}_{20}\ (p))\frac{\partial\ \hat{D}_0\ (p)}{\partial p}\, dp. \tag{14}$$

(Here the additional subscript "0"indicates that the conormal symbol is taken. In particular, $\hat{R}_{10}\ (p) = \hat{D}_0^{-1}(p)$.)

To calculate $\lim_{\lambda\to\infty} \mathcal{L}_2$, let us consider the compactly supported function

$$f(t) = f_1(t) f_2(t-\lambda).$$

Lemma 2 Trace $(\hat{T}_{10}\ \hat{Q}_0' f) = \text{Trace}\,(\hat{T}_{20}\ \hat{Q}_0 f)$.

Proof.

$$\begin{aligned}&\text{Trace}\,(\hat{T}_{10}\ \hat{Q}_0' f) = \text{Trace}\,(\hat{T}_{10}\ (1 - \hat{R}_{20}\hat{D}_0) f) = \text{Trace}\,(\hat{T}_{10}\ (\hat{R}_{10} - \hat{R}_{20})\ \hat{D}_0\ f) \\ &= \text{Trace}\,(\hat{T}_{10}\ (\hat{R}_{10} - \hat{R}_{20})[\hat{D}_0, f]) + \text{Trace}\,(\hat{T}_{10}\ (\hat{R}_{10} - \hat{R}_{20}) f\ \hat{D}_0).\end{aligned}$$

The first term is zero (cf. [12]), and we can transform the second term by cyclically permuting the factors:

$$\begin{aligned}
&\mathrm{Trace}\,(\hat{T}_{10}\,(\hat{R}_{10}-\hat{R}_{20})f\,\hat{D}_0)\\
&=\mathrm{Trace}\,(\hat{D}_0\hat{T}_{10}\,(\hat{R}_{10}-\hat{R}_{20})f)\\
&=\mathrm{Trace}\,(\hat{T}_{20}\hat{D}_0\,(\hat{R}_{10}-\hat{R}_{20})f)\\
&=\mathrm{Trace}\,(\hat{T}_{20}\,(1-\hat{D}_0\hat{R}_{20})f)=\mathrm{Trace}\,(\hat{T}_{20}\,\hat{Q}_0 f).
\end{aligned}$$

The proof of the lemma is complete.

Now we obtain

$$\begin{aligned}
\lim_{\lambda\to\infty}\mathcal{L}_2 &= \lim_{\lambda\to\infty}\mathrm{Trace}\,(\hat{T}_1\,\psi_2(t-\lambda)\,\hat{Q}\,f_2(t-\lambda)-\hat{T}_{10}\,\hat{Q}'_0 f_1(t)f_2(t-\lambda))\\
&\quad-\lim_{\lambda\to\infty}\mathrm{Trace}\,(\hat{T}_2\,\psi_2(t-\lambda)\,\hat{Q}\,f_2(t-\lambda)-\hat{T}_{20}\,\hat{Q}_0 f_1(t)f_2(t-\lambda))\\
&= \mathrm{Trace}\,(\hat{T}_1\,\hat{Q}'-\hat{T}_{10}\,\hat{Q}'_0 f_1(t))-\mathrm{Trace}\,(\hat{T}_2\,\hat{Q}-\hat{T}_{20}\,\hat{Q}_0 f_1(t)).
\end{aligned}$$

(Both trace integrals converge and the passage to the limit is valid, since the difference $\hat{T}_1\,\hat{Q}'-\hat{T}_{10}\,\hat{Q}'_0$, as well as $\hat{T}_2\,\hat{Q}-\hat{T}_{20}\,\hat{Q}_0$, decays as $t\to\infty$ at the rate of e^{-t}).

Finally, the passage to the limit as $\lambda\to 0$ in $\mathcal{L}_3$ can be performed simply by freezing the coefficients of T_i, D, and R_i at $t=\infty$ ($r=0$). We obtain

$$\begin{aligned}
\lim_{\lambda\to\infty}\mathcal{L}_2 &= \mathrm{Trace}\,(\hat{T}_{10}\,\{(1-\psi_1)\,\hat{R}_{10}\,[f_1,\hat{D}_0]+(1-\psi_2)\,\hat{R}_{20}\,[\hat{D}_0,f_2]\})\\
&\quad+\mathrm{Trace}\,(\hat{T}_{20}\,\{[\hat{D}_0,\psi_1]\,\hat{R}_{10}\,f_1+[\hat{D}_0,\psi_2]\,\hat{R}_{20}\,f_2\})
\end{aligned}$$

(here the argument of ψ_i and f_i is t, not $t-\lambda$).

2) Next, we must find the asymptotics of the Lefschetz number as $h\to 0$, $h\in Z(\gamma)$. The contribution of the term

$$\mathcal{L}_{\mathrm{rem}}=\lim_{\lambda\to\infty}\mathcal{L}_3$$

is $O(h^\infty)$ by virtue of the assumptions imposed on the conormal symbol, the canonical transformation, and the supports of the functions f_i and ψ_i (we omit the corresponding, rather lengthy argument). It remains to calculate the asymptotics of the contributions

$$\mathcal{L}_{\mathrm{int}}=\lim_{\lambda\to\infty}\mathcal{L}_2$$

and

$$\mathcal{L}_{\text{cone}} = \lim_{\lambda\to\infty} \mathcal{L}_1.$$

Since $\hat{R}_2$ is a semiclassical pseudodifferential regularizer, it follows that the asymptotics as $h \to 0$ of the contribution $\mathcal{L}_{\text{int}}$ can be calculated according to the stationary phase method. This gives the standard expression for the contribution of the interior fixed points (see [1]). To calculate the asymptotics as $h \to 0$ of $\mathcal{L}_{\text{cone}}$, we take a special semiclassical pseudodifferential regularizer.

Lemma 3 *There exists a semiclassical pseudodifferential regularizer $\hat{R}_2$ such that in a neighborhood of the conical point one has $\hat{R}_2 = \hat{R}_2 (t, -ih\frac{\partial}{\partial t})$, where the symbol $\hat{R}_2 (t,p)$ is holomorphic in p in a sufficiently narrow strip $|\operatorname{Im} p| < \varepsilon$.*

Proof. Let $H(e^{-t}, \omega, p, q)$ be the principal symbol of $\hat{D}$ in a neighborhood of the conical point, and let

$$\overline{H}(e^{-t}, \omega, p, q) = \overline{H(e^{-t}, \omega, \overline{p}, q)},$$

where the bar on the right-hand side stands for complex conjugation. Then the function $\overline{H}(e^{-t}, \omega, p, q)$ is holomorphic in p in some strip and we can take the principal symbol of the regularizer in a neighborhood of the conical point in the form

$$R_2(e^{-t}, \omega, p, q) = \overline{H}(e^{-t}, \omega, p, q)(1 + H\overline{H}(e^{-t}, \omega, p, q))^{-1}.$$

The subsequent terms are constructed with the help of regular perturbation theory.

The regularizer thus constructed possesses the desired properties.

Now let us calculate the integral (14) using the residue formula. Suppose that the conical point is attractive. It follows from Assumption 6 that for each $h \in Z(\gamma)$ one can choose a number $\rho(h) \in [\varepsilon/2, \varepsilon]$ such that on the line $\operatorname{Im} p = \rho(h)$ there are no poles of the family $\hat{D}{}_0^{-1}(p)$ and $\hat{D}{}_0^{-1}(p) \le C \cdot h^{-N_0 N_1}$ on this line as $|p| < R$ (for $|p| > R$, where R is sufficiently large, the decay of $\hat{D}{}_0^{-1}(p)$ at infinity is guaranteed).

Now let us consider the integral over the contour shown in Fig. 2.

We have

$$\begin{aligned}\mathcal{L}_{\text{cone}} &= \sum_{0<\operatorname{Im} p_j<\rho(h)} \operatorname{Trace} \operatorname*{Res}_{p_j} \left\{ \hat{T}_{10}(p) \hat{D}{}_0^{-1}(p) \frac{\partial \hat{D}_0}{\partial p} \right\} \\ &\quad + \frac{1}{2\pi i} \int\limits_{\operatorname{Im} p=\rho(h)} \operatorname{Trace} \hat{T}_{10}(p)(\hat{R}_{10}(p) - \hat{R}_{20}(p)) \frac{\partial \hat{D}_0(p)}{\partial p}\, dp.\end{aligned}$$

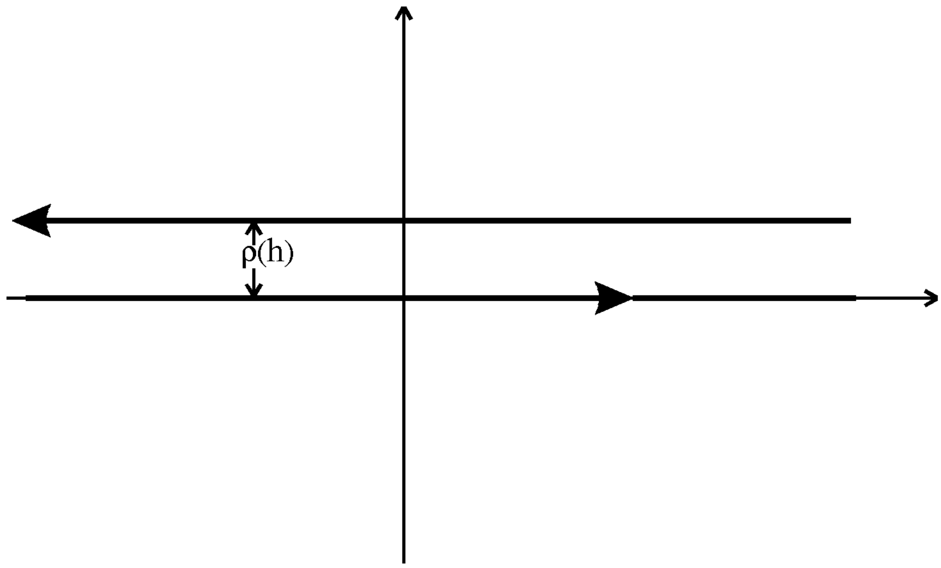

Figure 2. The integration contour

(We have used the fact that $\hat{R}_{20}(p)$ is holomorphic in the strip $0 \le \operatorname{Im} p \le \rho(h)$.)

It remains to estimate the integral over the line $\operatorname{Im} p = \rho(h)$.

We have

$$\|\hat{T}_{10}(p)\|_{L_2 \to L_2} \le C e^{-c_1/h}$$

for sufficiently small ε by virtue of the conditions imposed on the conical fixed point and on the canonical transformation (for small ε, the imaginary part of the generating function on the line Im $p = \rho(h)$ is estimated below by const $\cdot\varepsilon$ with a positive constant). Next,

$$\hat{R}_{10}(p) - \hat{R}_{20}(p) = \hat{D}_0^{-1}\hat{Q}_{20},$$

and hence

$$|||\hat{R}_{10}(p) - \hat{R}_{20}(p)||| \le \|\hat{D}_0^{-1}(p)\|_{L_2 \to L_2} |||\hat{Q}_{20}(p)|||,$$

where $|||\cdot|||$ is the trace norm of operators in L_2. Since

$$\|\hat{D}_0^{-1}(p)\|_{L_2 \to L_2} \le c h^{-N_1 N}$$

and

$$|||\hat{Q}_{20}(p)||| \le C(1 + |p|)^{-N_3},$$

where N_3 is arbitrarily large, we find that

$$\| \hat{T}_{10}(p)(\hat{R}_{10}(p) - \hat{R}_{20}(p))\frac{\partial \hat{D}_0(p)}{\partial p}\| \leq C(1+|p|)^{-N_3} e^{-c_1 h} h^{N_1 N}$$

(with some other constant C), whence it follows that the second integral is $O(h^\infty)$. The proof of the theorem is complete.

3.3 Example

In this subsection we shall briefly consider a simple example illustrating Theorem 3. In our example, the manifold M is the cylinder

$$C = S^1 \times \mathbf{R}^1$$

with two conical points $\{t = \pm\infty\}$ attached, and we take $\hat{D}$ to be the simplest first-order elliptic operator, namely, the Cauchy–Riemann operator

$$\hat{D} = -ih\frac{\partial}{\partial t} + h\frac{\partial}{\partial \varphi},$$

where φ is the standard angular coordinate on S^1. We shall consider this operator in the weighted Sobolev spaces

$$\hat{D}\colon H_h^{s,\gamma_1,-\gamma_2}(C) \to H_h^{s-1,\gamma_1,-\gamma_2}(C).$$

To avoid misunderstanding, recall that the weight function is $e^{\gamma_1\cdot(-t)} = e^{-\gamma_1 t}$ near $t = -\infty$ (where the standard cylindrical coordinate is $\tau = -t$) and $e^{-\gamma_2 t}$ near $t = \infty$ (where the standard cylindrical coordinate is t).

The conormal symbol

$$\hat{D}_0(p) = \pm p + h\frac{\partial}{\partial \varphi}$$

(the upper sign corresponds to the conical point $t = \infty$, and the lower, to $t = -\infty$) of the operator $\hat{D}$ has poles at the points

$$p_k = ikh.$$

Thus, $\hat{D}$ is Fredholm in the above spaces provided that neither γ_1 nor γ_2 is an integer. To be definite, we assume that $\gamma_1 < \gamma_2$. Then the cokernel of $\hat{D}$ is trivial, and the kernel is spanned by the functions

$$u_k(\varphi, t) = e^{k(t+i\varphi)}, \qquad \gamma_1 < k < \gamma_2.$$

Let $S(p,q)$ be a smooth function of at most linear growth at infinity, analytic in p in some strip $\operatorname{Im} p < \varepsilon$, and satisfying the condition

$$\frac{\partial S}{\partial p}(p,q) < 0 \quad \text{for real } p.$$

Consider the operator

$$\hat{T} = \exp\left\{\frac{i}{h} S(\hat{p}, \hat{q})\right\},$$

where

$$\hat{p} = -ih\frac{\partial}{\partial t}, \quad \hat{q} = -ih\frac{\partial}{\partial \varphi}.$$

For sufficiently small h the operator $\hat{T}$ is well defined in the scale $\{H_h^{s,\gamma_1,-\gamma_2}(C)\}$ and commutes with $\hat{D}$,

$$\hat{T}\hat{D} = \hat{D}\hat{T}$$

(indeed, both operators are operators with constant coefficients). The Lefschetz number

$$\mathcal{L} \equiv \mathcal{L}(\hat{D}, \hat{T}, \hat{T})$$

of the corresponding commutative diagram is given by

$$\mathcal{L} = \operatorname{Trace} \hat{T}\big|_{\operatorname{Ker}\hat{D}} = \sum_{\gamma_1 < k < \gamma_2} \exp\{\frac{i}{h} S(-ihk, hk)\}.$$

The same answer is provided by Theorem 3. Indeed, one can readily see that $\hat{T}$ is a Fourier–Maslov integral operator associated with the canonical transformation

$$g : (t, \varphi, p, q) \longmapsto (t', \varphi', p', q')$$

given by the formulas

$$\begin{aligned} t' &= t + \frac{\partial S}{\partial p}(p,q), \quad \varphi' = \varphi + \frac{\partial S}{\partial q}(p,q), \\ p' &= p, \qquad\qquad q' = q. \end{aligned}$$

Since $\partial S/\partial p < 0$, we see that the point $t = -\infty$ is attractive, the point $t = +\infty$ is repulsive, and there are no interior fixed points. Let us caclulate the contribution of the point $t = -\infty$. We have, according to (11),

$$\begin{aligned}\mathcal{L}_{-\infty} &= \sum_{h\gamma_1 < \operatorname{Im} p_k < h\gamma_1+\varepsilon} \operatorname{Trace} \operatorname*{Res}_{p_k} \frac{\exp\left\{\frac{i}{h} S(-p, \hat{q})\right\}}{p - i\,\hat{q}} \\ &= \sum_{\gamma_1 < k < \gamma_1+\varepsilon/h} \exp\{\frac{i}{h} S(-ihk, hk)\}.\end{aligned}$$

Similarly, for the point $t = \infty$ we have

$$\begin{aligned}\mathcal{L}_{\infty} &= -\sum_{-h\gamma_2-\varepsilon < \operatorname{Im} p_k < -h\gamma_2} \operatorname{Trace} \operatorname*{Res}_{p_k} \frac{\exp\left\{\frac{i}{h} S(p, \hat{q})\right\}}{p + i\,\hat{q}} \\ &= \sum_{\gamma_2 < k < \gamma_2+\varepsilon/h} \exp\{\frac{i}{h} S(-ihk, hk)\}.\end{aligned}$$

The sum of the last two expressions differs from the exact expression for the Lefschetz number by

$$\sum_{\varepsilon/h+\gamma_1 < k < \varepsilon/h+\gamma_2} \exp\{\frac{i}{h} S(-ihk, hk)\}.$$

The number of terms in this differnce does not exceed $[\gamma_2 - \gamma_1]$, and each term is exponentially small as $h \to 0$ (provided that ε is sufficiently small) by virtue of the conditions imposed on the function S. Indeed, we have

$$\operatorname{Im} S(-i\varepsilon + O(h), q) = -\varepsilon \frac{\partial S}{\partial p}(0, q) + O(\varepsilon + h) \geq C\varepsilon$$

with a positive constant independent of$h$ for sufficiently small h and ε. Hence we see that Theorem 3 provides the asymptotics of the Lefschetz number in this case.

References

[1] B. Sternin and V. Shatalov. *Quantization of Symplectic Transformations and the Lefschetz Fixed Point Theorem*, volume 92. Max-Planck-Institut für Mathematik, Bonn, 1994. Preprint.

[2] B. Sternin and V. Shatalov. Atiyah-Bott-Lefschetz fixed point theorem in symplectic geometry. *Dokl. Russ. Acad. Nauk*, **348**, No. 2, 1996, 165–168.

[3] V. Nazaikinskii, B.-W. Schulze, B. Sternin, and V. Shatalov. *A Lefschetz Fixed Point Theorem on Manifolds with Conical Singularities.* Univ. Potsdam, Institut für Mathematik, Potsdam, June 1997. Preprint N 97/20.

[4] V.Nazaikinskii, B.-W. Schulze, B. Sternin, and V. Shatalov. *Quantization of Symplectic Transformations on Manifolds with Conical Singularities.* Univ. Potsdam, Institut für Mathematik, Potsdam, July 1997. Preprint N 97/23.

[5] B.-W. Schulze. *Pseudodifferential Operators on Manifolds with Singularities.* North-Holland, Amsterdam, 1991.

[6] A. Mishchenko, V. Shatalov, and B. Sternin. *Lagrangian Manifolds and the Maslov Operator.* Springer-Verlag, Berlin–Heidelberg, 1990.

[7] V. Nazaikinskii, B. Sternin, and V. Shatalov. *Methods of Noncommutative Analysis. Theory and Applications.* Mathematical Studies. Walter de Gruyter Publishers, Berlin–New York, 1995.

[8] V.P. Maslov. *Perturbation Theory and Asymptotic Methods.* Moscow State Univ. Publ., Moscow, 1965. French transl.: *Théorie des perturbations et méthodes asymptotiques*, Dunod, Paris, 1972.

[9] V.P. Maslov. *Operator Methods.* Nauka, Moscow, 1973. English transl.: *Operational Methods*, Mir, Moscow, 1976.

[10] M. Agranovich and M. Vishik. Elliptic problems with parameter and parabolic problems of general type. *Uspekhi Mat. Nauk*, **19**, No. 3, 1964, 53–161. English transl.: Russ. Math. Surv. **19** (1964), N 3, p. 53–157.

[11] B. V. Fedosov. Trace formula for Schrödinger operator. *Russian Journ. of Math. Phys.*, **1**, No. 4, 1993, 447–463.

[12] V. Nazaikinskii, B.-W. Schulze, and B. Sternin. *The Index of Quantized Conact Transformations on Manifolds with Conical Singularities.* Univ. Potsdam, Institut für Mathematik, Potsdam, July 1998. Preprint N 98/16.

Moscow, Potsdam