On the Invariant Index Formulas for Spectral Boundary Value Problems

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Abstract

In the paper we study the possibility to represent the index formula for spectral boundary value problems as a sum of two terms, the first one being homotopy invariant of the principal symbol, while the second depends on the conormal symbol of the problem only. The answer is given in analytical, as well as in topological terms.

Keywords: spectral boundary value problems, Fredholm property, index of elliptic operator, Chern character, Atiyah–Patodi–Singer theory

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Contents

Introduction

In the present paper we consider the index problem for general spectral boundary value problems introduced in [1]. Recall, that these generalize the classical Atiyah-Patodi-Singer spectral boundary value problems (APS) [2] (that were constructed for the geometrical operators only) to the case of arbitrary elliptic differential operators. We remark also, that along with the statement of the boundary value problem, the paper [2] contains a certain formula for the index of the APS problem. This formula is a sum of two terms, the first one is the integral of the Atiyah-Singer form over the manifold with boundary, while the second is a certain nonlocal functional of the tangential operator (see below).

Unfortunatelly, the formulas presented in the paper [2], in principal solving the index problem, have, to our mind, one essential defect. Namely, the first (interior) summand entering the Atiyah-Patodi-Singer formula, that depends on the principal symbol of the operator is not in general its homotopy invariant. This happens, for example, when in the process of deformation of the Riemannian metric we can pass to a different representative of the cohomology class of the Atiyah-Singer form, this in its turn leads to the change of the value of the corresponding integral.

That is why there naturally arises the problem of extracting a homotopy invariant (with respect to the principal symbol) summand in such a way, that the remainder depends on the conormal symbol [3] only. We remark at once that on the space of all elliptic spectral boundary value problems this problem has a negative solution: the examples show, that there exist spectral boundary value problems, that do not admit such a decomposition of the index formula. In this connection there appears the problem of characterization of those sets of spectral boundary value problems, for which the index can be represented as a sum of a homotopy invariant term defined by the principal symbol and a functional of the conormal symbol. We find (this is the main result of the paper) the necessary and sufficient conditions for the index formula to possess the above described structure for a given set of operators. These conditions are given in the analytical terms (as a spectral flow of a certain special family of operators) as well as in topological terms – as a vanishing of a certain one-dimensional cohomology class representing the corresponding obstruction to the existence of the required decomposition.

Let us stop briefly on the contents of the paper. In the first section we recall the definition of spectral boundary value problems introduced in [1]. In the second section we present the statement of the problem. The third section is the central to the paper: here in the necessary generality the notion of a spectral flow is introduced and the main theorem on the existence of the required index formula is proved. In the following section the results obtained are generalized to the case of differential operators of arbitrary orders. Finally, the last section is devoted to (counter)examples. At the beginning the example showing the impossibility of separating (interior) homotopy invariant contribution in the set of *all* spectral boundary value problems is presented, and then we obtain the required index formula for the Hirzebruch operators with coefficient in a bundle flat or trivial on the boundary. In the obtained index formulas there appear Chern-Simons [4] secondary characteristic classes. Similar calculations can be found in the paper [5].

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1 Spectral boundary value problems

Consider a compact smooth manifold M with boundary ∂M, two smooth vector bundles E, F on it and an elliptic differential operator D of the first order [1] acting in the spaces of sections of these bundles. [2]

$$D : \Gamma(E) \to \Gamma(F).$$

In order to pose a spectral boundary value problem it is necessary to rewrite the operator in the collar neighborhood of the boundary as a linear function of the defivatives $\partial/\partial t$ with coefficients — differential operators on the boundary, where t is a certain normal coordinate to the boundary. Thus, a diffeomorphism

[1]The operators of arbitrary order are considered in section 4.

[2]In what follows the objects M, E, F will be considered as the fixed data of the problem.

$$g : U_{\partial M} \to \partial M \times [0,1)\,, \quad g|_{\partial M} : \partial M \to \partial M \times \{0\} \text{ identical}$$

is also included into the statement of the problem. Moreover, the derivative $\partial/\partial t$ presupposes that we already know the constant in the normal direction sections of the vector bundles. To do this an identification of the vector bundles in the collar with their pull-backs from the boundary of the cylinder $\partial M \times [0,1)$ is necessary

$$g_E : E|_{U_{\partial M}} \to \pi^* (E|_{\partial M})\,, \quad \text{where } \pi : \partial M \times [0,1) \to \partial M.$$

Here g_E is a fiberwise isomorphism of vector bundles that respects the diffeomorphism g: the fiber over the point x is mapped into the fiber over the point $g(x)$. In the sections of the bundle $\pi^*(E|_{\partial M})$ the operator $\partial/\partial t$ already makes sence. Consequently, under the transformations in the neighborhood of the boundary

$$\Gamma\left(\pi^*\left(E|_{\partial M}\right)\right) \xrightarrow{(g_E^*)^{-1}} \Gamma\left(E|_{U_{\partial M}}\right) \xrightarrow{D} \Gamma\left(E|_{U_{\partial M}}\right) \xrightarrow{g_F^*} \Gamma\left(\pi^*\left(F|_{\partial M}\right)\right)$$

the composition of operators $g_F^* D \left(g_E^*\right)^{-1}$ is uniquely represented in the form

$$D\left(t, -i\frac{\partial}{\partial t}\right) = D_1(t)\left(-i\frac{\partial}{\partial t}\right) + D_0(t) : \Gamma\left(\pi^*\left(E|_{\partial M}\right)\right) \to \Gamma\left(\pi^*\left(F|_{\partial M}\right)\right),$$

where the operators $D_{0,1}(t)$ have orders 1 and 0 correspondingly. From the ellipticity of the operator D it follows, that the operator $D_1(t)$ is an isomorphism and, without loss of generality can be considered to be equal to the identity operator. Thus we obtain the following family of operators with the parameter p

$$D(p) = D(0,p) = D_0(0) + p. \tag{1}$$

This operator family is called the *conormal family* (*symbol*) of the operator D. Elliptic differential operator of the first order

$$A \stackrel{\text{def}}{=} -D_0(0)$$

is called *tangential operator* for the operator D. From the ellipticity of D it follows, that the principal symbol $\sigma(A)(\xi)$ of the operator A has no real eigenvalues for the nonzero values of the cotangent variable ξ. Thus (see [6]) the operator A itself has at most finite number of eigenvalues of finite multiplicities in an angle of small opening with the vertex at the origin

$$\{p \in \mathbf{C} | \, |\arg p| < \varepsilon \ \text{ or } \ |\arg p - \pi| < \varepsilon\}.$$

It can be also shown (see [1]), that the projection P_+ in the space $H^{s-1/2}(\partial M,\, E|_{\partial M})$ projecting onto the subspace corresponding to the spectrum of the operator A eigenvalues with nonpositive imaginary parts along the corresponding positive subspace is a pseudodifferential operator with the principal symbol $\sigma(P_+)(\xi)$ — the projection onto the values at zero of decaying as $t \to \infty$ solutions of the ordinary differential equation

$$\left(-i\frac{d}{dt} - \sigma(A)(\xi)\right)\varphi(t) = 0$$

along the subspace of growing solutions. In this notation the *spectral boundary value problem* for the operator D corresponding to the choice of g and $g_{E,F}$ is the following nonhomogeneous system:

$$\begin{cases} Du = f, & u \in H^s(M,E)\,,\ \ f \in H^{s-1}(M,F)\,, \\ P_+(u|_{\partial M}) = h & h \in \operatorname{Im} P_+. \end{cases}$$

The spectral boundary value problem for the operator D is denoted by $\mathcal{D}$. It is proved in the paper [7] that this boundary value problem has the Fredholm property. Moreover, one can use an arbitrary pseudodifferential projection with the same principal symbol as P_+, and the finiteness theorem is valid in this case too. We remark also that the index of the problem is equal to the index of the corresponding homogeneous boundary value problem (since the operator P_+ is an isomorphism onto $\operatorname{Im} P_+$).

We are going to show now that the index of spectral problems is invariant under continuous deformations of boundary value problems.

Proposition 1 *For a smooth one-parameter deformation D_t of differential operators*

$$D_t : \Gamma(E) \to \Gamma(F)\,,$$

such that for any value of the parameter the strip $\{0 < \operatorname{Im} p < \varepsilon\}$ contains no eigenvalues of the family of tangential operators A_t, the index does not change.

Proof (cf. [8]). The index of the boundary value problem is equal to the index of the bounded operator

$$\begin{pmatrix} D_t \\ P_{+,t} \end{pmatrix} : H^s \to H^{s-1} \bigoplus \operatorname{Im} P_{+,t},$$

while from the assumptions of the theorem it follows that the family of projections $P_{+,t}$ (that are called *spectral*) is a continuous family. In this case the invariance of the index is proved by a standard technique, that we present here for the sake of completeness. Denote the subspace $H^{s-1} \bigoplus \operatorname{Im} P_{+,t} \subset H^{s-1}(M, F) \oplus H^{s-1/2}(\partial M, E|_{\partial M}) = H$ by L_t. Then the boundary value problem is rewritten in the form

$$C_t : H^s \to L_t,$$

where the family of subspaces L_t is defined by a continuous family of projections: $L_t = \operatorname{Im} \widetilde{P}_t$. Denote by

$$U_t : \ H \to H$$

a certain family of invertible operators having the property

$$\widetilde{P}_t U_t = U_t \widetilde{P_0}$$

One can take as U_t, for example, the solution of the equation

$$\dot{U_t} = \left[\dot{P}_t, P_t\right] U_t, \quad U_0 = 1.$$

Thus we obtain $L_t = U_t L_0.$ This implies

$$\operatorname{ind} C_t = \operatorname{ind} U_t^{-1} C_t.$$

A continuous family of Fredholm operators

$$U_t^{-1} C_t : H^s \to L_0$$

acts in the fixed Hilbert spaces, hence the index in this family is constant by the usual index theory.

2 Statement of the problem

At first we describe the classes of differential operators. To do this, recall that the principal symbol of the tangential operator A along with the usual ellipticity, has additional properties of its spectrum. These properties are formalized in the definition.

Definition 1 An elliptic pseudodifferential operator

$$A : \Gamma(E) \to \Gamma(E)$$

on a closed manifold is called *normally-elliptic,* if the eigenvalues of its principal symbol $\sigma(A)(\xi)$ do not lie on the real line for any $\xi \neq 0$.

Denote Smbl (X) — the set of normally elliptic principal symbols of differential operators on the manifold X. Let $\Sigma \subset \mathrm{Smbl}(X)$ be an arbitrary subset. Denote by Op (Σ) the set of elliptic differential operators on the manifold M with the boundary X, that have the form

$$-i\frac{\partial}{\partial t} - A\left(t, x, -i\frac{\partial}{\partial x}\right), \tag{2}$$

where the principal symbol of the operator A for $t = 0$ is an element of Σ. The set of operators $A(0, x, -i\partial/\partial x)$ in the formula (2) is denoted by Tan (Σ).

Definition 2 We say that the class Op (Σ) admits *a correct decomposition of the index formula,* if for any operator $D \in$ Op (Σ) there exist two numbers $i_f(D)$ and $i_b(D)$, such that the index of the corresponding spectral boundary value problem $\mathcal{D}$ has the form

$$\mathrm{ind}\,\mathcal{D} = i_f(D) + i_b(D), \tag{3}$$

and the numbers $i_f(D), i_b(D)$ satisfy the following properties :

1. $i_f(D)$ is determined by the principal symbol $\sigma(D)$ of the operator D and remains unchanged under continuous deformations of the operator D;

2. $i_b(D)$ is determined by the conormal family $D(p)$ (operator A).

Remark 1 If we consider deformations of the operator D, such that the corresponding deformation of the conormal family has no zeroes in a fixed strip

$$\{0 < \mathrm{Im}\, p < \delta,\ \delta > 0\},$$

the number $i_b(D)$ also does not change.

In order to prove this, one can show, that in the latter case the index of the spectral boundary value problem $\mathcal{D}$ is constant, hence $i_b(D)$ is constant too.

Remark 2 In the general case the deformations of the operator D do not induce continuous deformation of the spectral boundary value problem $\mathcal{D}$, hence the number $i_b(D)$ may change for such deformations.

Indeed, for arbitrary deformations of the operator D the spectral points of the operator A (the zeroes of the conormal symbol) can cross the real line. In this case the index of the boundary value problem, and consequently, the number $i_b(D)$ may change.

In the following section the conditions necessary and sufficient for the existence of the index formula in the class Op (Σ) in the above sense are given. These conditions are formulated in terms of the set of principal symbols Σ. It turns out that the answer is connected with the classical notion of spectral flow introduced in the paper [9]. However, since we consider general elliptic operators assuming no (skew)adjoincy of the operator A, as it was in [9], it is necessary to generalize the notion of the spectral flow to the general case. This is done in the following section following the ideas of the technique of spectral sections of R.Melrose and P.Piazza [11],[12] (see also [13]). From this theory we obtain a certain relative index formula, and the generalization of the (cohomological) Atiyah-Patodi-Singer formula, that they gave for the spectral flow of a periodic family of self-adjoint operators.

3 Spectral flows. The main theorem

In the present section the generalization of the spectral flow to the case of normally-elliptic operators is given.

Definition 3 For a normally elliptic operator A by its *odd symbol*

$$\sigma_+(A) \in \mathrm{Vect}\,(S^*M)$$

we denote the vector bundle generated by the rootspaces of the principal symbol $\sigma(A)(\xi)$ corresponding to the eigenvalues with negative imaginary part.

Let now A_t — be a continuous family of normally elliptic operators. The intuitive definition of the spectral flow of the family $\{A_t\}$ is almost the same as for self-adjoint operators [9] — the spectral flow is the algebraic number of eigenvalues of the operators A_t in the family that cross the real line in the upward direction during the homotopy of operators. This notion, however, can not in general serve as a rigorous definition. That is why, to define the spectral flow in the more general situation we will use an alternative method of spectral sections.

So, let $\{A_t\}_{t=0,1}$ be a continuous family of normally elliptic operators.

Definition 4 *The spectral section,* corresponding to the family $\{A_t\}_{t=0,1}$ is a continuous family of pseudodifferential projections $\{P_t\}_{t=0,1}$ such that their principal symbol $\sigma(P_t)$ coincides with the projection onto the subbundle $\sigma_+(A_t)$ for all values of the parameter t. Moreover, for a sufficiently big constant Λ the projection P_t has to be equal to the spectral projection of the operator A_t on the subspace corresponding to spectral points with modulus greater than Λ.

The following proposition contains the basic properties of spectral sections.

Proposition 2 *Spectral sections exist always. Two spectral sections P_t' and P_t'' are homotopic, provided $P_0' = P_0''$.*

Proof. We prove the existence first.

For any $t \in [0,1]$ there exist sufficiently small $\varepsilon_t > 0$ and λ_t such that on the segment $t' \in [t-\varepsilon, t+\varepsilon]$ the line $\{\operatorname{Im}\lambda = \lambda_t\}$ is free from eigenvalues of the operator $A_{t'}$. By the compactness of the segment we obtain a finite covering of the whole $[0,1]$, such that the segments of the covering cover $[0,1]$ from the left to the right, having in their intersection at most a single point. On any such interval I we define a spectral projection $P_t(I)$ onto the subspace corresponding to the spectral points satisfying the inequality Im $\lambda < \lambda_{I.}$ Then the spectral section P_t is constructed by consecutively attaching new segments to the already considered ones using the following procedure. On the first segment $I_0 = [0, t_0]$ let $P_t = P_t(I_0)$. Suppose now, that on the segment I_1 the section P_t is constructed already and we have to extend it on the union $I_1 \cup I_2$. Denote t' the intersection point of $I_{1,2}$. Then by the construction the projections $P_{t'}(I_1)$ and $P_{t'}(I_2)$ differ at most by a continuous projection, moreover, one of them is a subprojection in the other. For definiteness we consider the case $P_{t'}(I_1) \supset P_{t'}(I_2)$. The projection onto the complement $\operatorname{Im} P_{t'}(I_1) \ominus \operatorname{Im} P_{t'}(I_2)$ can be extended to a continuous family of projections $\widetilde{P}_t$ over the interval I_2 such that the $\widetilde{P}_t$ act in the kernel ker $P_t(I_2)$ for $t \in I_2$ and are equal to zero on the subspaces with sufficiently large eigenvalues. In this case the spectral section is extended to I_2 by the formula $P_t = P_t(I_2) \oplus \widetilde{P}_t$. For the opposite inclusion $P_{t'}(I_1) \subset P_{t'}(I_2)$, the modification takes place on the interval I_1. We note, that in order to preserve this monotone property it is necessary to extend the projections $\widetilde{P}_t$ till the ends of the segment $[0,1]$. This proves the first part of the proposition.

Remark 3 The modification of projections appearing in the proof can be of a different kind: instead of considering two cases it is sufficient to assume that $P_{t'}(I_1) \supset P_{t'}(I_2)$, choosing beforehand a sufficiently big negative number λ_{I_2}. This shows that the spectral section can be chosen to have an arbitrary spectral projection of the operator A_0 at its starting point.

The homotopy for the second part of the proposition is also constructed successively. Consider the square $\{0 \le t, \tau \le 1\}$. To prove the proposition we need to extend the homotopy $P_{t,\tau}$ from the three sides

$$P_{t,\tau=0} = P_t'; \quad P_{t,\tau=1} = P_t''; \quad P_{t=0,\tau} = P_0' = P_0''$$

inside the square. The extension procedure is similar to the considered in the above, but this time instead of segments we consider the covering by small squares. Then we extend the family of projections starting from the upper left square, passing to the next small diagonal down to the lower right corner. The modification involves all squares that are situated after the square considered. The details are left to the reader.

This proposition enables us to define the spectral flow. Before doing this we recall the following usefull notion.

Let $P_{1,2}$ be a pair of projections that differ by a compact operator (in our case these are projections with the same principal symbol).

Definition 5 The *relative index* of a pair (P_1, P_2) is the index of a Fredholm operator

$$\operatorname{ind}(P_1, P_2) \overset{\text{def}}{=} \operatorname{ind}(P_2 : \operatorname{Im} P_1 \to \operatorname{Im} P_2).$$

Now for a normally elliptic family $\{A_t\}$ and any of its spectral sections $\{P_t\}$ the *spectral flow* across the real line is by definition is the following number

$$\operatorname{sf}\{A_t\}_{t=0,1} = -\operatorname{ind}\left(P_0, P_0^+\right) + \operatorname{ind}\left(P_1, P_1^+\right),$$

where $P_{0,1}^+$ are nonpositive spectral projections of the operators $A_{0,1}$ correspondingly. The independence of the definition with respect to the choice of a spectral section follows from the homotopy invariance of the relative index and the logarithmic property

$$\operatorname{ind}(P_0, P_2) = \operatorname{ind}(P_0, P_1) + \operatorname{ind}(P_1, P_2).$$

The basic properties of the spectral flow are contained in the proposition.

Proposition 3 1. *The spectral flow depends only on the start and end points of the family and the homotopy of the principal symbols. In other words, for a normally elliptic family $\{A_t\}_{t=0,1}$ the spectral flow remains unchanged under continuous deformations of the family, provided the operators on the boundary $A_{0,1}$ do not change.*

2. *The spectral flow of a periodic family of normally elliptic operators*

$$\{A_t\}_{t=0,1}, A_0 = A_1$$

is completely determined by the family of odd symbols $\{\sigma_+(A_t)\} \in \operatorname{Vect}(S^1 \times S^*M)$.

Before proving the proposition, let us derive as a corollary a formula for the spectral flow of a periodic family.

Corollary 1

$$\operatorname{sf}\{A_t\} = -p_![\sigma_+(A_t)] \in K^1\left(S^1\right) \cong Z,$$

where $p : M \to \{pt\}$ — projection into a point, $[\sigma_+(A_t)] \in K^1(S^1 \times T^*M)$ — the Atiyah–Patodi–Singer element (see [9]) of the family A_t, and the mapping

$$p_! : K^1\left(S^1 \times T^*M\right) \to K^1\left(S^1\right)$$

— is the direct image in the K-theory with the parameter space S^1. For future references we present also a cohomological version of this formula

$$\mathrm{sf}\,\{A_t\} = -\left\langle \mathrm{ch}\,(\sigma_+(A_t))\,\pi^*\mathrm{Todd}\,(T^*M \otimes C)\,,\left[S^1 \times S^*M\right]\right\rangle, \tag{4}$$

where $\pi : S^1 \times S^*M \to M$ — the projection onto the base.

In order to prove the corollary note, that for an odd symbol $\sigma_+(A_t)$ it is possible to construct a certain (skew)adjoint family of operators having $\sigma_+(A_t)$ as its odd symbol. Then the right hand side of the formula (4) is equal to the spectral flow of this (skew)adjoint family by the classical Atiyah–Patodi–Singer spectral flow theorem [9], while the spectral flow of this family coincides with the spectral flow of the initial family of operators $\{A_t\}$ by the second part of the proposition.

The proof of the first part of the proposition consists of a reasoning similar to the one give in the proof of the proposition 2. Namely, the two-parameter homotopy of projections constructed can be regarded as a homotopy of spectral sections. This in its turn shows that the spectral flow is constant.

Now we prove the second part of the proposition dealing with periodic families. Let $A_{t,\tau}-$ be a homotopy of periodic families. Then due to the already proved first part of the proposition we obtain

$$\mathrm{sf}\,\{A_{t,1}\}_{t=0,1} = -\mathrm{sf}\,\{A_{0,\tau}\}_{\tau=0,1} + \mathrm{sf}\,\{A_{t,0}\}_{t=0,1} + \mathrm{sf}\,\{A_{1,\tau}\}_{\tau=0,1} = \mathrm{sf}\,\{A_{t,0}\}_{t=0,1}\,.$$

The independence of the spectral flow of a particular projection onto the subspace follows from the fact, that all such projections are linearly homotopic. This ends the proof of the proposition.

The next theorem shows how the index of spectral boundary value problems changes when we make deformations of the differential operator.

Theorem 1 *Let D_t be a continuous family of elliptic differential operators of the first order on the manifold M with boundary. Then the difference of the indices of spectral boundary value problems at the beginning and at the end of the homotopy is equal to the spectral flow of a family of tangential operators*

$$\mathrm{ind}\,\mathcal{D}_1 - \mathrm{ind}\,\mathcal{D}_0 = \mathrm{sf}\,\{A_t\}_{t=0,1.}$$

To prove the theorem consider P_t- a spectral section for A_t. Then the family

$$\begin{pmatrix} D_t \\ P_t \end{pmatrix} : H^s \to H^{s-1} \bigoplus \operatorname{Im} P_t \tag{5}$$

(denoted for short by (D_t, P_t)) is a continuous homotopy of boundary value problems. Consequently, the index in the family does not change. Hence, by the logarithmic property of the index, we obtain

$$\begin{aligned} \operatorname{ind}\left(D_1, P_1^+\right) &= \operatorname{ind}\left(D_1, P_1\right) + \operatorname{ind}\left(P_1, P_1^+\right) \\ &= \operatorname{ind}\left(D_0, P_0\right) + \operatorname{ind}\left(P_1, P_1^+\right) \\ &= \operatorname{ind}\left(D_0, P_0^+\right) - \operatorname{ind}\left(P_0, P_0^+\right) + \operatorname{ind}\left(P_1, P_1^+\right) \\ &= \operatorname{ind}\left(D_0, P_0^+\right) + \operatorname{sf}\left\{A_t\right\}_{t=0,1} \end{aligned}$$

as required

Denote by $\operatorname{Tan}(\Sigma)$ the set of normally elliptic differential operators on the boundary ∂M of the manifold M, with the principal symbols contained in the set Σ.

Theorem 2 *For a class* $\operatorname{Op}(\Sigma)$ *there exists a representation of the index formula in the form* (3) *if and only if for any periodic family of operators* $\{A_t\} \in \operatorname{Tan}(\Sigma)$ *the spectral flow is equal to zero.*

From the topological spectral flow formula (4) we obtain a usefull corollary.

Corollary 2 *For a class* $\operatorname{Op}(\Sigma)$ *there exists a decomposition* (3) *of the index formula if and only if for any periodic family of principal symbols on the boundary* $\{\sigma_t\}_{t=0,1}$, $\sigma_t \in \Sigma$, *the following equality is valid*

$$-\left\langle \operatorname{ch} \sigma_t^+ \pi^* \operatorname{Todd}\left(T^*M \otimes C\right), \left[S^1 \times S^*M\right] \right\rangle = 0;$$

(here σ_t^+ *is a vector bundle over* $S^1 \times S^*M$ *generated by eigenspaces of the symbols* σ_t *corresponding to eigenvalues with negative imaginary parts).*

Proof. We show first, that from the decomposition of the index formula the vanishing of the spectral flow follows. Let $\{A_t\}_{t=0,1}$, $\sigma(A_t) \in \Sigma$, $A_0 = A_1$ be a periodic family of tangential operators. There obviously exists an operator D on M such that its tangential operator is equal to A_0. Moreover, it can be assumed that the whole family $\{A_t\}_{t=0,1}$ is tangential for a certain family of differential operators D_t (we note that the family D_t need not be periodic). Such a family D_t can be constructed attaching to M a cylinder $\partial M \times [0,1]$ along the boundary. Then the operator D is extended to the operator $\widetilde{D}$ on $M \cup \partial M \times [0,1]$ such that the tangential operator of its restriction to $M \cup \partial M \times [0,t]$ is equal to A_t. Finally, the family D_t is constructed out of $\widetilde{D}$ as a composition with a smooth family of diffeomorphisms

$$g_t : M \cup \partial M \times [0,t] \to M$$

that are identical on the boundaries and outside a certain collar neighborhood of the boundary.

For the family D_t we obtain

$$i_f(D_0) = i_f(D_1),$$

since the operators are homotopic, as well as

$$i_b(D_0) = i_b(D_1),$$

due to the periodicity of the tangential operators A_t. Hence, by the assumed decomposition of the index formula we obtain

$$\operatorname{ind} \mathcal{D}_0 = \operatorname{ind} \mathcal{D}_1.$$

On the other hand, the relative index theorem Theorem 1 implies

$$\operatorname{ind} \mathcal{D}_1 - \operatorname{ind} \mathcal{D}_0 = \operatorname{sf} \{A_t\}.$$

Thus the spectral flow is equal to zero as required.

To prove the decomposition of the index formula we make a partition of all tangential operators $\operatorname{Tan}(\Sigma)$ into classes of homotopic operators inside the space $\operatorname{Tan}(\Sigma)$ (that is introduce an equivalence relation). In each class I we fix one representative of the class $A_I \in [A_I]$ – that is, we choose one tangential operator for every class. Then from the relative index theorem we obtain the following decomposition valid for any operator $D \in \operatorname{Op}(\Sigma)$ with a tangential operator A

$$\operatorname{ind} \mathcal{D} = \operatorname{ind}\left(\widetilde{\mathcal{D}}\right) - \operatorname{sf} \{A_t\}_{t=0,1}, \tag{6}$$

where it is supposed that the operator A is homotopic to a certain of chosen operators, say $A_J, \{A_t\}$ is an arbitrary homotopy connecting A with A_J, $\widetilde{D}$ — is an operator on the manifold with the attached cylinder (see first part of the proof). Now we prove that this formula is correctly defined (that is independent of the homotopy A_t chosen) and has the desired properties. Indeed, the correctness follows from the vanishing of spectral flow for periodic families in $\operatorname{Tan}(\Sigma)$: the difference

$$\operatorname{sf} \{A_t\} - \operatorname{sf} \{A'_t\}$$

corresponding to two homotopies in Tan (Σ) is equal to the flow of a periodic family $\{A_t\}_{t=0,1} \cup \{A'_t\}_{t=1,0}$ and, consequently, equal to zero by the assumption. The spectral flow in the formula (6), is readily seen to be the boundary contribution, while the first term is determined by the principal symbol of the operator and is homotopy invariant. This ends the proof of the theorem.

Remark 4 The arbitrariness in the choice of the representatives of the classes of homotopic operators $A_I \in [A_I]$ causes the absence of a canonical formula of the form (6).

The conditions obtained can be formulated in cohomological terms. Namely, considering the right hand side of the formula (4) for the spectral flow of periodic families of tangential operators as a homomorphism of groups

$$\mathrm{sf} : \pi_1 (\mathrm{Ell}\, (\partial M, E, F)) \longrightarrow Z,$$

where $\mathrm{Ell}\, (\partial M, E, F) \stackrel{\mathrm{def}}{=} \mathrm{Ell}$ — the space of normally elliptic principal symbols on the boundary ∂M. As any homomorphism into Z the spectral flow sf is determined by its values on the factor

$$\pi_1 (\mathrm{Ell}) / [\pi_1 (\mathrm{Ell}), \pi_1 (\mathrm{Ell})] \simeq H_1 (\mathrm{Ell}, Z).$$

Hence, the spectral flow of periodic families can be interpreted as a rational cohomology class of the space Ell :

$$\mathrm{sf} \in \mathrm{Hom}\, (H_1 (\mathrm{Ell}, Z), Z) \otimes Q \simeq H^1 (\mathrm{Ell}, Q).$$

The condition guaranteeing the existence of the index formula decomposition in cohomological terms takes the form: *the restriction of the cohomology class* sf *on the set* Σ *is zero*:

$$i^* \mathrm{sf} = 0 \in H^1 (\Sigma, Q), \text{ where } i : \Sigma \subset \mathrm{Ell}\, (\partial M, E, F).$$

4 Operators of higher orders

The theory constructed is generalized by a standard technique to the case of operators of arbitrary order m. We point on the changes necessary to extend the results to this situation.

So, consider an elliptic differential operator D of order m

$$D : \Gamma (E) \to \Gamma (F).$$

In a collar neighborhood of the boundary the operator is rewritten in the form

$$D\left(t,-i\frac{\partial}{\partial t}\right)=\sum_{j=0}^{m}D_j\left(t\right)\left(-i\frac{\partial}{\partial t}\right)^j:\Gamma\left(\pi^*\left(\left.E\right|_{\partial M}\right)\right)\to\Gamma\left(\pi^*\left(\left.F\right|_{\partial M}\right)\right),$$

where the operators $D_j\left(t\right)$ have orders $m-j$ correspondingly. The family of operators with parameter p

$$D\left(p\right)\stackrel{\text{def}}{=}D\left(0,p\right)=\sum_{j=0}^{m-1}D_j\left(0\right)p^j+p^m \tag{7}$$

is called *the conormal symbol* of the operator D (see [3]). In fact, our problem is reduced to the case of a first order operator. Indeed, consider the operator $j_{\partial M}^{m-1}$ defined by the formula

$$\begin{aligned} j_{\partial M}^{m-1}u\left(x,t\right) &= \left(u\left(x,0\right),-i\frac{\partial u}{\partial t}\left(x,0\right),...,-i^{m-1}\frac{\partial^{m-1}u}{\partial t^{m-1}}\left(x,0\right)\right), \\ j_{\partial M}^{m-1} &: \Gamma\left(M,E\right)\to\bigoplus_{j=0}^{m-1}\Gamma\left(\partial M,\left.E\right|_{\partial M}\right). \end{aligned}$$

Then the initial equation

$$Du=f \qquad \text{on } M$$

can be rewritten near the boundary as a first order in $\partial/\partial t$ equation in terms of jets $v=j_{\partial M}^{m-1}u$ of the function u in the form

$$\left(-i\frac{\partial}{\partial t}-A\left(t\right)\right)v=\left(0,...,0,f\right)^t,$$

where

$$A\left(t\right)=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -D_0\left(t\right) & -D_1\left(t\right) & -D_2\left(t\right) & \dots & -D_{m-1}\left(t\right) \end{pmatrix}. \tag{8}$$

The operator $A\left(0\right)=A$ is a *tangential operator*, it is elliptic in the Douglis–Nirenberg sense [14]. The above described theory, of course, can be generalized to the case of operators elliptic in the Douglis–Nirenberg sense. However, in this case it is possible to stay in the framework of the usual notion of ellipticity (however, passing to pseudodifferential operators). In fact, if we pass from the operator (8) to a first order operator with the help of a diagonal matrix of powers of the Laplace operator (on the boundary)

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & (1+\triangle)^{1/2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & (1+\triangle)^{(m-1)/2} \end{pmatrix},$$

then our operator occurs to be equivalent to an elliptic in the usual sense pseudodifferential operator. As earlier, from the ellipticity of D it follows that the operator A is normally elliptic. Thus it has at most a finite number of eigenvalues in the angle:

$$\{p \in \mathbf{C} | \, |\arg p| < \varepsilon \text{ or } \ |\arg p - \pi| < \varepsilon\},$$

and the nonpositive spectral projection P_+ for the operator A, acting in the space $\bigoplus_{j=0}^{m-1} H^{s-j-1/2}(\partial M, E|_{\partial M})$ is a pseudodifferential operator with the principal symbol $\sigma(P_+)(\xi)$ — the projection onto the space of the initial data of the decaying as $t \to \infty$ solutions of the ordinary differential equation

$$\sigma(D)\left(x, \xi, -i\frac{d}{dt}\right)\varphi(t) = 0$$

along the corresponding growing solutions.

The *spectral boundary value problem* for the operator D is the following nonhomogeneous system

$$\begin{cases} Du = f, & u \in H^s(M, E), \ f \in H^{s-m}(M, F), \\ P_+ j_{\partial M}^{m-1} u = h, & h \in \operatorname{Im} P_+ \subset \bigoplus_{j=0}^{m-1} H^{s-j-1/2}(\partial M, E|_{\partial M}). \end{cases}$$

The sets Σ appearing in the above in this case consist of vectors of the form

$$(\sigma_0, \sigma_1, ..., \sigma_{m-1}),$$

where σ_i — the polynomial symbol on ∂M of degree $m - j$ satisfies the condition

$$\det\left(\sum_{j=0}^{m-1} \sigma_i(\xi) p^j + p^m\right) \neq 0$$

for all $\xi \neq 0$ and $p \in \mathbf{R}$. The corresponding sets of operators Op (Σ) consist of all elliptic differential operators on M with the principal symbols of their coefficients in the decomposition (7) equal to $\sigma_0, \sigma_1, ..., \sigma_{m-1}$ correspondingly. In a similar fashion the set Tan (Σ) consists of the operators on the boundary of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots \\ -D_0 & -D_1 & -D_2 & \dots & -D_{m-1} \end{pmatrix}, \quad \deg D_j = m - j,$$

with the row of the principal symbols of the coefficints in Σ:

$$(-\sigma(D_0), -\sigma(D_1), \dots, -\sigma(D_{m-1})) \subset \Sigma.$$

Using this notation all of the above definitions, results and their proofs are extended to the considered case.

5 Examples

1. Our first example shows the necessity to consider classes of operators, instead of considering a single class consisting of all elliptic operators. Indeed, we show, that already *on a two-dimensional manifold in the class of all elliptic differential operators the index formula decomposition does not exist.* In fact, this statement follows from the existence of a periodic family of operators with a nonzero spectral flow. Indeed, let

$$A_t = (2P_{t,\tau} - 1)\frac{d}{d\tau},$$

($P_{t,\tau}$-a periodic family of orphogonal projections). It is easy to find the family of odd symbols

$$\sigma_+(A_t)(\xi) = \begin{cases} \operatorname{Im} P_{t,\tau} & \text{for } \xi > 0, \\ \operatorname{Im}(1 - P_{t,\tau}) & \text{for } \xi < 0. \end{cases}$$

Thus from the formula (4) for the spectral flow we obtain, due to the opposite orientations of the two connected components of S^*S^1, an expression of the flow in terms of the image of projections

$$\operatorname{sf}\{A_t\} = -2\left\langle \operatorname{ch}(\operatorname{Im} P_{t,\tau}), \left[S^1 \times S^1\right]\right\rangle.$$

In this way, taking as $P_{t,\tau}$ a projection, corresponding to a nontrivial vector bundle over the torus, we obtain the required family with nonzero spectral flow. For example, one can take a vector bundle on the cylinder $S^1 \times [0,1]$ and identify the fibers over the components of the boundary with the help of a nonzero complex-valued function on the circle of a nonzero degree. It is also possible to write the corresponding family of projections explicitly.

2. The second class of operators we consider concernes the class of geometrical operators — *Hirzebruch operators D with coefficients in bundles E.* These operators (denoted in what follows as $D \otimes 1_E$) are defined by a metric and a unitary connection ∇ in the bundle E (see [15]). Note, that this class of operators does not satisfy our requirements (the vector bundles, where the operator acts, depend on the choice of metric). However, the definitions given in the above can be corrected (that is, two Hirzebruch operators are supposed to be equal on the boundary, if the corresponding metrics are equal). Then it can be noticed that the spectral flow for a periodic family of operators in this class is zero. Indeed, if the periodic family of operators is constructed by a periodic family of metrics, then both families are homotopic to constant families, hence, the spectral flow is zero. Thus in this class the index formula decomposition exists.

We obtain such a formula for some of the classes of bundles E by transforming the Atiyah–Patodi–Singer formula. It should be emphasized that the results we obtain do not cover the whole class of operators in the sense of the definitions introduced, since on the one hand, a formula similar to the one proved by Atiyah–Patodi–Singer is not known, on the other hand, the methods we use to obtain explicit expressions in the index formula impose restrictions on the topology of auxiliary bundles E.

Recall the Atiyah–Patodi–Singer formula

$$\operatorname{ind}(D \otimes 1_E) = \int\limits_M L(M)\operatorname{ch}(E) - \frac{\dim\ker A \otimes 1_E + \eta(-iA \otimes 1_E)}{2}, \qquad (9)$$

where $\eta(-iA \otimes 1_E)$ — the Atiyah–Patodi–Singer η-invariant of a self-adjoint operator $-iA \otimes 1_E$.

1. We begin the considerations from the Hirzebruch operatod D itself, that is, there is no bundle E. The index of this operator, as shown by Atiyah–Patodi–Singer is connected with the signature of the manifold with boundary $\operatorname{sign}(M, \partial M)$:

$$\operatorname{ind} D = \operatorname{sign}(M, \partial M) - \frac{\dim\ker A}{2}.$$

2. A more interesting example appears in the case, when the vector bundle E, $\dim E = n$ is equipped with a flat connection in the neighborhood of the boundary. Suppose additionally, that the metric and the connection are of a product form near

the boundary and independent of t there. In this case we obtain

$$\begin{aligned}\operatorname{ind}(D\otimes 1_E) &= \int\limits_M L(M)\operatorname{ch}(E) - \frac{\dim\ker A\otimes 1_E + \eta(-iA\otimes 1_E)}{2},\\ \operatorname{sign}(M,\partial M) &= \int\limits_M L(M) - \frac{\eta(-iA)}{2},\end{aligned}$$

subtracting we obtain the formula

$$\begin{aligned}\operatorname{ind}(D\otimes 1_E) &= n\operatorname{sign}(M,\partial M) + \int\limits_M L(M)(\operatorname{ch}(E)-n) - \\ &\quad -\frac{\eta(-iA\otimes 1_E) - n\eta(-iA) + \dim\ker A\otimes 1_E}{2};\\ i_f &= n\operatorname{sign}(M,\partial M) + \int\limits_M L(M)(\operatorname{ch}(E)-n),\\ i_b &= -\frac{\eta(-iA\otimes 1_E) - n\eta(-iA) + \dim\ker A\otimes 1_E}{2}.\end{aligned}$$

This decomposition is an invariant one (the first two terms are finite-dimensional, while the rest is a boundary contribution). The homotopy invariant difference

$$\eta(-iA\otimes 1_E) - n\eta(-iA)$$

is known as the reduced η-invariant of flat bundles, it was studied in the paper [10]. The second term is invariant, since for flat connections the characteristic form $\operatorname{ch}(E)-n$ correctly defines a relative cohomology class $(\operatorname{ch}(E)-n)\in H^*(M,\partial M)$.

3. The most general operators we consider in the present paper are Hirzebruch operators with coefficients in a bundle E trivial at the boundary. In this case it is possible to write down an index formula, choosing a certain trivialization (actually its homotopy class) of E at the boundary:

$$G:\partial M\to P(E|_{\partial M}) - \text{section of the principal bundle}.$$

Subtracting as previously, we obtain

$$\begin{aligned}\operatorname{ind}(D\otimes 1_E) &= n\operatorname{sign}(M,\partial M) + \int\limits_M L(M)(\operatorname{ch}(E)-n) - \\ &\quad -\frac{\eta(-iA\otimes 1_E) - n\eta(-iA) + \dim\ker(A\otimes 1_E)}{2},\end{aligned}$$

but since the connection in E is not flat, the form $(\mathrm{ch}\,(E)-n)$ does not define a relative cohomology class. That is why, the formula has to be transformed once more. One can see, that the term in the fraction is already the boundary term, hence, to obtain an invariant decomposition of the formula we need to separate from the integral

$$\int\limits_{M} L\,(M,g)\,(\mathrm{ch}\;(E,\nabla)-n)$$

a homotopy invariant contribution. Namely, we are going to prove the following expression for this integral

$$\int\limits_{M} L\,(M,g)\,(\mathrm{ch}\;(E,\nabla)-n) = \int\limits_{M} L\,(M,g)\,(\mathrm{ch}\;(E,\nabla_{flat})-n)$$
$$+\int\limits_{G(\partial M)} \pi^{*}L\,(\partial M, g|_{\partial M})\,\mathrm{Tch}\;(E|_{\partial M},\nabla)\,. \tag{10}$$

As in the above described example, the first term of this formula defines a topological invariant independent of the choice of a metric and a particular connection ∇_{flat} associated with the trivialization G. Here π - is the projection in the principal bundle $P\,(E|_{\partial M})$, $\mathrm{Tch}\;(E|_{\partial M},\nabla)\in\Lambda^{*}\,(P\,(E|_{\partial M}))$ — Chern–Simons secondary characteristic form (see [16],[4]).

Before we prove this formula, let us write down the index formula for the Hirzebruch operator that follows from it

$$\begin{aligned}
\mathrm{ind}\;(D\otimes 1_E) \;=\; & \overbrace{n\,\mathrm{sign}\;(M,\partial M) + \int\limits_{M} L\,(M,g)\,(\mathrm{ch}\;(E,\nabla_{flat})-n)}^{i_f} \\
& + \overbrace{\int\limits_{G(\partial M)} \pi^{*}L\,(\partial M, g|_{\partial M})\,\mathrm{Tch}\;(E|_{\partial M},\nabla)}^{i_b} \\
& - \frac{\overbrace{\eta\,(-iA\otimes 1_E) - n\eta\,(-iA) + \dim\ker\,(A\otimes 1_E)}^{i_b}}{2}\,.
\end{aligned}$$

Note, that the difference

$$\eta\,(-iA\otimes 1_E) - n\eta\,(-iA)$$

here is not homotopy invariant.

In order to prove the formula (10) consider the difference of its first two terms

$$\int_M L(M)(\mathrm{ch}\ (E,\nabla)-n)-\int_M L(M)(\mathrm{ch}\ (E,\nabla_{flat})-n)$$
$$=\int_M L(M)(\mathrm{ch}\ (E,\nabla)-\mathrm{ch}\ (E,\nabla_{flat}))\stackrel{\mathrm{def}}{=}X.$$

The difference of differential forms representing the Chern character is equal to the following integral (t here is a parameter and not a normal coordinate!)

$$\mathrm{ch}\ (E,\nabla)-\mathrm{ch}\ (E,\nabla_{flat})=\int_0^1 \frac{d}{dt}\mathrm{ch}\ (E,\nabla_t)\,dt=\int_0^1 d\mathrm{ch}\ \left(E,\dot{\nabla}_t,\nabla_t\right)dt.$$

In the latter formula the interpretation of characteristic classes as symmetric polylinear functions of the connection forms is employed, while

$$\nabla_t=(1-t)\,\nabla_{flat}+t\nabla$$

is a linear homotopy of connections. Thus, by the Stokes formula we obtain

$$X=\int_M d\left(L(M)\int_0^1 \mathrm{ch}\ \left(E,\dot{\nabla}_t,\nabla_t\right)dt\right)=\int_{\partial M} L(\partial M)\int_0^1 \mathrm{ch}\ \left(E|_{\partial M},\dot{\nabla}_t,\nabla_t\right)dt.$$

Lifting the last integral to the principal bundle $P\left(E|_{\partial M}\right)$ using the trivialization G, we get

$$X=\int_{G(\partial M)} \pi^*L\left(\partial M,g|_{\partial M}\right)\int_0^1 \pi^*\mathrm{ch}\ \left(E|_{\partial M},\dot{\nabla}_t,\nabla_t\right)dt.$$

The secondary characteristic forms have the following property (see the above cited papers on secondary characteristic classes)

$$\int_0^1 \pi^*\mathrm{ch}\ \left(E|_{\partial M},\dot{\nabla}_t,\nabla_t\right)dt=\mathrm{Tch}\ (E|_{\partial M},\nabla)-\mathrm{Tch}\ (E|_{\partial M},\nabla_{flat})+dV$$

for a certain form V. On the other hand, the form $\mathrm{Tch}\ (E|_{\partial M},\nabla_{flat})$ is equal to zero, since its connection 1-form is identically zero in the trivializing basis. Hence, we finally obtain the required

$$X=\int_{G(\partial M)} \pi^*L\left(\partial M,g|_{\partial M}\right)\mathrm{Tch}\ \left(E|_{\partial M},\nabla_{flat}\right).$$

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