On Rational Pricing of Derivative Securities For a Family of Non-Gaussian Processes

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Linear and non-linear analogues of the Black-Scholes equation are derived when shocks can be described by a truncated Lévy process.

A linear equation is derived under the perfect correlation assumption on returns for a derivative security and a stock, and its solutions for European put and call options are obtained. It is also shown that the solution violates the perfect correlation assumption unless a process is gaussian. Thus, for a family of truncated Lévy distributions, the perfect hedging is impossible even in the continuous time limit.

A second linear analogue of the Black-Scholes equation is obtained by constructing a portfolio which eliminates fluctuations of the first order and assuming that the portfolio is risk-free; it is shown that this assumption fails unless a process is gaussian.

It is shown that the difference between solutions to the linear analogues of the Black-Scholes equations and solutions to the Black-Scholes equations are sizable.

The equations and solutions can be written in a discretized approximate form which uses an observed probability distribution only.

Non-linear analogues for the Black-Scholes equation are derived from the non-arbitrage condition, and approximate formulas for solutions of these equations are suggested.

Assuming that a linear generalization of the Black-Scholes equation holds, we derive an explicit pricing formula for the perpetual American put option and produce numerical results which show that the difference between our result and the classical Merton's formula obtained for gaussian processes can be substantial. Our formula uses an observed distribution density, under very weak assumptions on the latter.

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I. Introduction

1.1. After seminal papers by Black and Scholes (1973) and Merton (1973), a geometrical Brownian motion model is widely being used as a standard reference model particularly in the context of option pricing and hedging, but empirically it is demostrated to be incorrect in number of ways. Main difficulties this model faces are systematic deviations of option prices from the ones predicted by the Black-Scholes formula, and a leptokurtic character of stock return variability.

The deviations may be due to various factors, for instance, due to an unrealistic assumption of continuous trading, at no cost, which is assumed in the Black-Scholes model, but the fact that real processes do not conform to the gaussian assumption certainly accounts for at least part of the deviation.

For models with non-zero cost of trading, see e.g. Morton and Pliska (1995), Grannan and Swindle (1996), Whalley and Wilmott (1997) and the bibliography there, and for different approaches to modelling of stock volatility, see e.g. Hull and White (1987), Merton (1976), Cox and Ross (1976), Rubinstein (1983), Taylor (1994), Duan (1995), Scott (1997), Björk, Kabanov and Runggaldier (1997), Renault and Touzi (1996), Rogers (1997) and bibliography there.

In some sense, almost all approaches to modelling of a stock volatility start with gaussian processes: one selects an appropriate mixture, with possible addition of jump components, or uses a convolution of the Brownian motion kernel with a polynomially decaying one, in models based on the Fractional Brownian motion (see e.g. Bouchaud and Sornette (1994) and Rogers (1997)).

In the paper, we suggest to use a family of "truncated Lévy processes" as reference models.[2] Truncated Lévy distributions were constructed by Mantegna and Stanley (1994), and Koponen (1995) suggested a family of infinitely divisible truncated Lévy distributions, which admit explicit description in terms of their Fourier transforms. Truncated Lévy distributions were observed in real financial markets (Mantegna and Stanley (1995), Cont et al. (1997)). Cont et al. (1997) gave a formula for the probability distribution of the Standard & Poor's 500 index futures, which explicitly describes the exponential fall-off in the tails of the distribution and fits the data.

By using the same simple heuristic ideas which are used to derive the Black-Scholes equation, we obtain their analogues and find the solutions for European call and put options.

The formulas are on almost the same level of complexity as the Black-Scholes formula, and hence admit simple adjustment so popular among practioners with the Black-Scholes formula. Possible advantages of the suggested approach are:

1) the basic processes have "fat tails", as empirical distributions do, so in applications, there may be no need to find an appropriate mixture of basic processes;

2) a basic process is characterized by three parameters, not by variance only, which entails additional possibilities of adjustment;

[2] After the first variant of this paper (without the last Section on the perpetual American put) had been prepared, Prof. Mantegna informed us about a paper by Matacz (1997) where the truncated Lévy distributions were used for similar purposes. The methods and results of this paper and a paper Matacz (1997) are different.

3) the equations and formulas admit natural approximate discrete versions which use an observed distribution only. (We illustrate the last point in Section 9, where we solve a discretized version of a generalized linear Black-Scholes equation for the perpetual American put; a continuous-time version of this result will be published elsewhere).

The simplest analogs of the Black-Scholes equation are linear pseudo-differential equations, which can easily be solved by means of the Fourier transform (in this respect, our approach is close to Scott (1997)). They are obtained under assumption that the returns on a stock and a derivative security are perfectly correlated, but we show that this assumption fails unless a process is gaussian.

We use the no-arbitrage approach, and derive two non-linear equations. They are rather involved, and we are unable to solve them as yet, though we suggest a scheme for an approximate solution.

1.2. Let $S = S(t)$ be a current or spot price of a stock S, and let F be a current price of a derivative security for the stock. Let r be the riskless rate. The celebrated Black-Scholes equation for the dynamics of F

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S - rF + F_t = 0 \tag{1}$$

was derived under two assumptions:

I. S follows a Geometric Brownian Motion, i.e. can be described by a stochastic differential equation

$$\frac{dS}{S} = \alpha dt + \sigma dz,$$

where dz is the increment of the standard Wiener process with zero mean and unit variance;

II. In the limit $\Delta t \to 0$, the returns on the stock and the derivative security are perfectly correlated: for some non-stochastic b,

$$\frac{\Delta F - E[\Delta F]}{F} = b\frac{\Delta S - E[\Delta S]}{S} + o(\Delta t) \quad \text{as} \quad \Delta t \to 0. \tag{2}$$

Instead of (1), we assume that the returns can be described by a stochastic process obeying rather general conditions which are satisfied e.g. by a family of truncated Lévy distributions which was constructed by Koponen (1995). For the sake of brevity, we consider only symmetric distributions of this family. In terms of the Fourier transform, these distributions are given by

$$\hat{p}_{\nu,\lambda,\sigma,\Delta t}(k) = \exp[-\Delta t\sigma^2\lambda^{2-\nu}[\lambda^\nu - (k^2+\lambda^2)^{\nu/2}\cos(\nu\arctan(k/\lambda))]/\nu(\nu-1)],$$

where $\sigma > 0, \lambda > 0$ and $\nu \in (0,2], \nu \neq 1$ are parameters. Note that the variance is independent of λ and ν.

For $\nu = 2$, we obtain $\hat{p}_{2,\lambda,\sigma,\Delta t}(k) = \exp(-\Delta t\sigma^2 k^2/2)$ which means that $p_{2,\lambda,\sigma,\Delta t}$ is a gaussian distribution. As ν moves from 2 down, $p_{\nu,\lambda,\sigma,\Delta t}$ deviates from a gaussian distribution, and for fixed $\nu \in (0,2), \nu \neq 1$, in the limit $\lambda \to +0$, $p_{\nu,\lambda,\sigma,\Delta t}$ becomes a Lévy distribution with $\hat{p}_{\nu,\Delta t}(k) = \exp(\Delta t c_\nu |k|^\nu)$.

Note that $\hat{p}_{\nu,\lambda,\sigma,\Delta t}(k)$ are holomorphic on a strip $|\Im k| < \lambda$, and we use this observation as the starting point.

1.3. In the paper, we impose the following condition on the behavior of

$$\Delta(\ln S)(t;\Delta t) = (\ln S)(t+\Delta t) - (\ln S)(t), \quad \text{as} \quad \Delta t \to 0:$$

$$\Delta(\ln S)(t;\Delta t) = \alpha\Delta t + Y_{t,\Delta t} + o(\Delta t), \tag{3}$$

where for fixed Δt, $Y_{t,\Delta t}$ are i.i.d. random variables with the distribution density $p_{\Delta t}$ given by

$$p_{\Delta t}(y) = (2\pi)^{-1}\int_{-\infty}^{+\infty} \exp\{iyk - \Delta t P(k)\}dk. \tag{4}$$

The main properties of the Koponen's (1995) family are

a)

$$P(k) = P(-k) > 0 \ \forall\ k \in R\setminus 0, \quad P(0) = 0, \quad P''(0) > 0; \tag{5}$$

b) there exist $\lambda > 0$ and $\nu \in (0, 2]$ such that

$$P \text{ is holomorphic on a strip } |\Im k| < \lambda, \tag{6}$$

and for any $\lambda' \in (0, \lambda)$ and $s = 0, 1$,

$$|P^s(k)| \le C_{\lambda',s}(1 + |k|)^{\nu-s}, \quad |\Im k| \le \lambda'; \tag{7}$$

c) there exist $P_\infty > 0$ and $\rho > 0$ such that for any $\lambda' \in (0, \lambda)$

$$P(k) = P_\infty|k|^\nu + O(|k|^{\nu-\rho}), \quad \text{as } k \to \pm\infty,\ |\Im k| < \lambda'. \tag{8}$$

In this paper, we shall use (7) with $s = 0$, and instead of (8), a weaker condition:

$$P(k) \to +\infty \text{ as } k \pm \infty,$$

but (7) with $s = 0, 1$ and (8) are needed for a continuous version of results of Section 9, which will be published elsewhere.

1.4. Clearly, gaussian processes satisfy (3)–(8) with $\lambda = +\infty$, but there are many other processes different from gaussian ones, say, the ones described by truncated Lévy distributions, which satisfy (3)–(8) with $\lambda < +\infty$ (and do not with $\lambda = +\infty$).

Under conditions (3)–(8), we derive non-gaussian analogs of the Black-Scholes equation (1). We use four approaches, which give the same equation (1) in the gaussian case. The approaches use, respectively:

1. The non-arbitrage condition

$$\frac{E[\Delta F] - rF\Delta t}{E[(\Delta F - E[\Delta F])^2]^{1/2}} = \frac{E[\Delta S] - rS\Delta t}{E[(\Delta S - E[\Delta S])^2]^{1/2}} + o(\Delta t), \quad \text{as } \Delta t \to 0. \tag{9}$$

2. The perfect correlation assumption (2).

3. The construction of a risk-minimizing portfolio consisting of shares of the stock and the derivative security. (This is in a spirit of Bouchaud and Sornette (1994))

4. The construction of a portfolio which eliminates fluctuations of order 1.

Let's call the corresponding equations the Generalized Black-Scholes equation I, II, III and IV, respectively, or GBSE-I, GBSE-II, GBSE-III and GBSE-IV.

As it turns out, these equations look different (though for gaussian processes, reduce to the same equation (1)), and for non-gaussian processes obeying (3)–(8), *are* different. To be more precise, let F be the spot price of an European call (or put) option. We have solved the GBSE-II and proved that

a) F satisfies the perfect correlation assumption (2) if and only if it satisfies both GBSE-I and GBSE-II;

b) F, a solution to GBSE-II (subject to appropriate boundary conditions) satisfies the perfect correlation assumption (2) if and only if P satisfies a certain very complicated non-linear pseudo-differential equation[3] (this is a condition (21) below), and we show that if the excess rate of return on the stock is not high then P does not satisfy condition (21) unless $P(k) = \frac{\sigma^2}{2}k^2$, i.e. the process is gaussian. We believe that non-gaussian P do not satisfy (21) in all cases;

c) the riskless portfolio consisting of shares of the stock and option exists if and only if F satisfies (21), and if F satisfies it then F satisfies both GBSE-II and GBSE-III.

Thus, in the case of non-gaussian processes obeying (3)–(8), the perfect correlation assumption fails, a riskless portfolio consisting of a stock and an European option does not exist, and the standard hedging is impossible. It is no suprize that the assumption and the riskless portfolio disappear simulteneously: the latter can be constructed if and only if the returns on the stock and the option are perfectly correlated.

1.5. GBSE-I and GBSE-III are non-linear pseudo-differential equations, and GBSE-II and GBSE-IV are linear pseudo-differential equations. We derive GBSE-I and GBSE-II in Sections 2 and 3, respectively. In Section 3, we also derive condition (21).

In Sections 4 and 5, we solve GBSE-II for European call and put options, and produce some numerical results which show that the difference between our result and the Black-Scholes formula can be sizable (assuming that the variance is the same), and in Section 6, we prove that the solutions does not obey the condition (21) unless the process is gaussian.

In Section 7, we derive GBSE-III, and in Section 8 – GBSE-IV; solutions to the latter are similar to the ones for GBSE-II.

Both GBSE-I and GBSE-III (especially the latter) are very complicated, and so far, we were unable to solve them.

In fact, we doubt that an analytical solution exists at all, and even the justification of an iteration procedure which we suggest in Section 8 for GBSE-I, seems to be very hard.

We suggest formulas for the first and the second approximation.

We suggest a way to rewrite our formulas in terms of an observed probability distribution. We hope that this can be used to produce appropriate computational schemes.

[3] An equation is called pseudo-differential if it involves pseudo-differential operators. A pseudo-differential operator $P(y, D_y)$ with the symbol $P = P(y, k)$ acts as follows:

$$u(x) \mapsto (2\pi)^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(i(y-z)k) P(y,k) u(y) dy dk.$$

If $P(y, k) = \sum p_j(y) k^j$ is a polynomial in k, then $P(y, D_y) = \sum p_j(y) D_y^j$, $D_y = -i\partial_y$, is a differential operator. If P is independent of y, one writes $P(D_y)$.

Finally, in Section 9, we derive a formula for the perpetual American put option, assuming that the dynamics of its price obeys a linear generalized Black-Scholes equation, and produce numerical examples showing that the difference between our formula and the Merton's one can be significant.

2. Derivation of the Generalized Black-Scholes Equation I: The Non-Arbitrage Approach

The nominators and denominators in Eq. (9) can be calculated by means of the following two lemmas.

Set $y = \ln S$, $f(y,t) = F(\exp y, t)$.

Lemma 2.1. *Let $f = f(y,t)$ be continuously differentiable and admit a bound*

$$|f(y,t)| \le C_t \exp(\lambda_1 |y|), \quad \forall\ y, t,$$

where $C_t > 0$ and $\lambda_1 \in (0, \lambda)$ are independent of y.

Then for all y, t, as $\Delta t \to 0$,

$$E_{t,y}[\Delta f] = (f_t + \alpha f_y - P(D_y)f)(y,t)\Delta t + o(\Delta t). \tag{10}$$

Proof. For small Δt and y,

$$f(y+\Delta y, t+\Delta t) - f(y,t) = (f(y+\Delta y, t+\Delta t) - f(y+\Delta y, t)) +$$

$$+ (f(y+\alpha\Delta t, t) - f(y,t)) + f(y+\Delta y, t) - f(y+\alpha\Delta t, t).$$

The first and second differences above being equal to

$$f_t(y+\Delta y, t)\Delta t + o(\Delta t) = f_t(y,t)\Delta t + o(\Delta t),$$

and

$$f_y(y+\alpha\Delta t, t)\Delta t + o(\Delta t) = f_y(y,t)\Delta t + o(\Delta t),$$

respectively, it remains to calculate

$$E_{t,y}[f(y+\alpha\Delta t+\Delta y, t) - f(y+\alpha\Delta t, t)] = \int_{-\infty}^{+\infty} f(y+\alpha\Delta t+z, t)p_{\Delta t}(z)dz - f(y+\alpha\Delta t, t) =$$

$$= \int_{-\infty}^{+\infty} f(y+\alpha\Delta t - z, t)p_{\Delta t}(z)dz - f(y+\alpha\Delta t, t).$$

(Here we have used the symmetry of p). By using one of the main properties of the Fourier transform $\widehat{f*g} = \hat{f}\cdot\hat{g}$: the convolution becomes the multiplication, and then Eq. (4), (6) and (7) (which allow us to change the order of calculation of the limit and the integral below), we obtain

$$\lim_{\Delta t\to 0} E_{t,y}[f(y+\Delta y, t) - f(y,t)]/\Delta t =$$

$$= \lim_{\Delta t\to 0} \frac{1}{2\pi\Delta t}\int_{-\infty}^{+\infty} \exp(ik(y+\alpha\Delta t)(\exp(-\Delta t P(k))) - 1)\hat{f}(k,t)dk =$$

$$= \lim_{\Delta t \to 0} \frac{1}{2\pi \Delta t} \int_{-\infty}^{+\infty} \exp(iky)(-\Delta t P(k)) + o(\Delta t))\hat{f}(k,t)dk =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(iky)(-P(k))\hat{f}(k,t)dk.$$

Thus, (10) has been proved.

Lemma 2.2. *Let f and g satisfy conditions of Lemma 2.1. Then*

$$E[(\Delta f - E[\Delta f])(\Delta g - E[\Delta g])] = (-P(D_y)(fg) + gP(D_y)f + fP(D_y)g)\Delta t + o(\Delta t). \quad (11)$$

Proof. Simple algebraic manipulations give

$$E[(\Delta f - E[\Delta f])(\Delta g - E[\Delta g])] = E(f + \Delta f - E[f + \Delta f])(g + \Delta g - E[g + \Delta g])] =$$

$$= E[(f + \Delta f)(g + \Delta g)] - E[(f + \Delta f)]E[(g + \Delta g)] =$$

(using (10) and the equality $(fg)_t = f_t g + f g_t$, we continue:)

$$= fg + ((fg)_t + \alpha(fg)_y - P(D_y)(fg))\Delta t + o(\Delta t) -$$

$$-(f + (f_t + \alpha f_y - P(D_y)f)\Delta t + o(\Delta t))(g + (g_t + \alpha g_y - P(D_y)g)\Delta t + o(\Delta t)) =$$

$$= (-P(D_y)(fg) + gP(D_y)f + fP(D_y)g)\Delta t + o(\Delta t).$$

Lemma has been proved.

Direct calculations show that for $|a| < \lambda$,

$$e^{-ay} P(D_y) e^{ay} = P(D_y - ia), \quad (12)$$

(as operators), and $P(D_y)$ acts on exponents as follows:

$$P(D_y)e^{ay} = P(-ia)e^{ay}. \quad (13)$$

Using (10), (11) and (13), we can rewrite (9) as follows:

$$\frac{(f_t + \alpha f_y - P(D_y)f - rf)\Delta t + o(\Delta t)}{[(-P(D_y)(f^2) + 2fP(D_y)f)\Delta t + o(\Delta t)]^{1/2}} + o((\Delta t)^{1/2}) =$$

$$= \frac{(\alpha - P(-i) - r)\Delta t + o(\Delta t)}{[(-P(-2i) + 2P(-i))\Delta t + o(\Delta t)]^{1/2}} + o((\Delta t)^{1/2}).$$

Passing to the limit, we obtain the Generalized Black-Scholes Equation-I:

$$f_t + \alpha f_y - P(D_y)f - rf = \frac{(\alpha - P(-i) - r)}{-P(-2i) + 2P(-i)}(-P(D_y)(f^2) + 2fP(D_y)f)^{1/2}. \quad (14)$$

Example 2.1. Let $p_{\Delta t}$ be gaussian. Then $\hat{p}(k) = \exp(-\Delta t \frac{\sigma^2}{2} k^2)$, and therefore

$$P(k) = \frac{\sigma^2}{2}k^2, \ P(-ia) = -\frac{\sigma^2}{2}a^2,$$

$$P(D_y) = -\frac{\sigma^2}{2}\partial_y^2, \ -P(-2i) + 2P(-i) = \frac{\sigma^2}{2}(4-2) = \sigma^2,$$

$$-P(D_y)(f^2) = \frac{\sigma^2}{2}(f^2)_{yy} = \sigma^2(ff_{yy} + (f_y)^2), \quad 2fP(D_y)f = -\sigma^2 ff_{yy}, \tag{15}$$

and (14) turns into

$$f_t + \alpha f_y + \frac{\sigma^2}{2} f_{yy} - rf = (\alpha + \frac{\sigma^2}{2} - r) f_y. \tag{16}$$

Since

$$f_t(y,t) = F_t(S,t), \quad f_y(y,t) = SF_S(S,t),$$
$$f_{yy}(y,t) = S^2 F_{SS}(S,t) + SF_S(S,t),$$

we obtain the Black-Scholes equation (1).

Thus, for gaussian p, Eq. (14) reduces to a linear differential equation, but for other p, it is a very complicated non-linear pseudo-differential equation.

3. Derivation of The Generalized Black-Scholes Equation II: The Perfect Correlation Assumption Approach

Now suppose that ΔF and ΔS are perfectly correlated in the limit $\Delta t \to 0$, i.e. (2) holds. Multiplying (2) by $(\Delta S - E[\Delta S])$, taking the expectation $E = E_{t,y}$, and applying Lemmas 2.1 and 2.2, we obtain

$$\frac{-P(D_y)(fe^y) + e^y P(D_y)f + fP(D_y)e^y}{f}\Delta t = b\frac{-P(D_y)e^{2y} + 2e^y P(D_y)e^y}{e^y}\Delta t + o(\Delta t). \tag{17}$$

Using (12) and (13), then dividing (17) by $e^{2y}\Delta t$ and passing to the limit as $\Delta t \to 0$, we obtain

$$\frac{-P(D_y - i)f + P(D_y)f + P(-i)f}{f} = b(-P(-2i) + 2P(-i)).$$

Thus,

$$b = \frac{-P(D_y - i)f + P(D_y)f + P(-i)f}{(-P(-2i) + 2P(-i))f}. \tag{18}$$

Consider forming a portfolio by investing a fraction w in the option and $1 - w$ in the stock. The return on this portfolio is

$$w\frac{\Delta F}{F} + (1-w)\frac{\Delta S}{S},$$

and its uncertain component is equal to

$$w\frac{\Delta F}{F} + (1-w)\frac{\Delta S}{S} - E[w\frac{\Delta F}{F} + (1-w)\frac{\Delta S}{S}] =$$

$$= w\frac{\Delta F - E[\Delta F]}{F} + (1-w)\frac{\Delta S - E[\Delta S]}{S} =$$

$$= (wb + 1 - w)\frac{\Delta S - E[\Delta S]}{S}.$$

The choice $w = 1/(1-b)$ makes the portfolio riskless, and since a riskless portfolio must earn the riskless rate of return, we obtain

$$r\Delta t = E[w\frac{\Delta F}{F} + (1-w)\frac{\Delta S}{S}] + o(\Delta t),$$

or

$$r\Delta t = \frac{f_t + \alpha f_y - P(D_y)f}{(1-b)f} - \frac{b}{1-b}\frac{\alpha e^y - P(D_y)e^y}{e^y}\Delta t + o(\Delta t).$$

Dividing by Δt, passing to the limit $\Delta t \to 0$, next using (13) and (18), and finally multiplying by $(1-b)F$, we obtain

$$[(-P(-2i) + 2P(-i))f - (-P(D_y - i)f + P(D_y)f + P(-i)f)]r =$$

$$= (-P(-2i) + 2P(-i))(f_t + \alpha f_y - P(D_y)f) -$$
$$- (-P(D_y - i)f + P(D_y)f + P(-i)f)(\alpha - P(-i)). \quad (19)$$

By using simple algebraic manipulations, we can rewrite (19) as

$$f_t + \alpha f_y - P(D_y)f - rf =$$

$$= \frac{\alpha - P(-i) - r}{-P(-2i) + 2P(-i)}(-P(D_y - i)f + P(D_y)f + P(-i)f). \quad (20)$$

Example 3.1. Let $p_{\Delta t}$ be a gaussian distribution. Then, using (15), we obtain

$$-P(D_y - i)f + P(D_y)f + P(-i)f =$$

$$= \frac{\sigma^2}{2}(-(D_y - i)^2 + D_y^2 - 1)f = \sigma^2 i D_y f = \sigma^2 f_y,$$

$$-P(-2i) + 2P(-i) = \sigma^2, \quad -P(-i) = \frac{\sigma^2}{2},$$

and therefore, (20) turns into (16), which is the Black-Scholes equation (1).

Thus, in the case of gaussian processes, Eq. (20) and (14) are identical and reduce to the Black-Scholes equation (1), as it should be the case since it is well-known that for gaussian processes, the approaches used in Sections 2 and 3 give the same result.

For a non-gaussian $p_{\Delta t}$, the RHS's of (14) and (20) differ: in the former, it is a non-linear in f, and in the latter – linear. Clearly, a linear equation (20) is much easier to solve, and we shall do it in the next two Sections for European call and put options, respectively.

But a linear equation (20) has been derived under the perfect correlation assumption (2) which implies a non-trivial restriction on P and F. To derive it, multiply (2) (with b defined by (18)) by $\Delta F - E[\Delta F]$, take the expectation and apply Lemmas 2.1 and 2.2:

$$\frac{-P(D_y)(f^2) + 2fP(D_y)f}{f}\Delta t = b\frac{-P(D_y)(fe^y) + e^y P(D_y)f + fP(D_y)e^y}{e^y}\Delta t + o(\Delta t).$$

Using (13), dividing by Δt and passing to the limit as $\Delta t \to 0$, then multiplying by f and using (18), we obtain

$$-P(D_y)(f^2) + 2fP(D_y)f =$$

$$= \frac{-P(D_y - i)f + P(D_y)f + P(-i)f}{-P(-2i) + 2P(-i)}(-P(D_y - i)f + P(D_y)f + P(-i)f),$$

or

$$(-P(D_y)(f^2) + 2fP(D_y)f)(-P(-2i) + 2P(-i)) = (-P(D_y - i)f + P(D_y)f + P(-i)f)^2. \tag{21}$$

Theorem 3.1. *The following statements are equivalent:*

a) $F(S,t) = f(\ln S, t)$ satisfies the perfect correlation assumption (2);

b) $F(S,t) = f(\ln S, t)$ satisfies (21);

c) f is a solution to Eq. (14) if and only if it is a solution to Eq. (20).

Proof. The equivalence of a) and b) has been proved already. Further, the LHS in (21) is non-negative, being the limit of non-negative functions, therefore (21) is equivalent to the statement: the RHS's of (14) and (20) are equal. Since the LHS's are identical, c) and b) are equivalent.

Theorem has been proved.

4. A Solution to The Generalized Black-Scholes Equation II for European Call Options

Rewrite (20) as

$$f_t = \mathcal{P}(D_y)f, \tag{22}$$

where

$$\mathcal{P}(D_y) = r - i\alpha D_y + P(D_y) + \frac{\alpha - P(-i) - r}{-P(-2i) + 2P(-i)}(-P(D_y - i) + P(D_y) + P(-i)).$$

Eq. (22) is valid for any derivative security of the stock S. If F is the spot price of an European call option, F satisfies the following boundary conditions

$$F(S,T) = \max(S - X, 0), \tag{23}$$

$$F(0,t) = 0, \quad F(S,t) \le S, \tag{24}$$

where T is the expiration date and X is the striking price. In terms of $f(y,t) = F(e^y, t)$, (24) can be rewritten as

$$f(y,T) = \max(e^y - e^x, 0), \tag{25}$$

where $x = \ln X$.

Take $\omega \in (1, \lambda - 1)$, and set $g(y,t) = e^{-\omega y} f(y,t)$. Then a problem (22), (25) for f is equivalent to the following problem for g:

$$g_t = \mathcal{P}(D_y - i\omega)g, \tag{26}$$

$$g(y,T) = \max(e^{(1-\omega)y} - e^{-\omega y + x}, 0). \tag{27}$$

By making the Fourier transform w.r.t. y, we see that a problem (26)–(27) is equivalent to

$$\hat{g}_t = \mathcal{P}(k - i\omega)\hat{g}, \tag{28}$$

$$\hat{g}(k, T) = \hat{h}(k), \tag{29}$$

where

$$\hat{h}(k) = \int_{-\infty}^{+\infty} e^{-iyk} \max(e^{(1-\omega)y} - e^{-\omega y + x}, 0)dy =$$

$$= \int_{x}^{+\infty} e^{-iyk}(e^{(1-\omega)y} - e^{-\omega y + x})dy =$$

$$= \frac{e^{(1-\omega-ik)y}}{1-\omega-ik}\Big|_x^{+\infty} + \frac{e^{(x+(-\omega-ik)y}}{-\omega-ik}\Big|_x^{+\infty} =$$

$$= -\frac{e^{(1-\omega-ik)x}}{1-\omega-ik} + \frac{e^{(1-\omega-ik)x}}{-\omega-ik} = -\frac{e^{(1-\omega-ik)x}}{(k-i\omega)(k-i\omega+i)}.$$

By solving the Cauchy problem (28)–(29), we obtain

$$\hat{g}(k, t) = -\frac{\exp(-\tau\mathcal{P}(k-i\omega) - ikx + (1-\omega)x)}{(k-i\omega)(k-i\omega+i)},$$

where $\tau = T - t$, and therefore,

$$g(y, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iyk}\hat{g}(k, t)dk =$$

$$= -\frac{e^{(1-\omega)x}}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp(i(y-x)k - \tau\mathcal{P}(k-i\omega))}{(k-i\omega)(k-i\omega+i)}dk.$$

Since $f(y, t) = e^{\omega y}g(y, t)$, we have

$$f(y, t) = -\frac{e^x}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp(i(y-x)(k-i\omega) - \tau\mathcal{P}(k-i\omega))}{(k-i\omega)(k-i\omega+i)}dk =$$

$$= -\frac{e^x}{2\pi}\int_{-\infty-i\omega}^{+\infty-i\omega} \frac{\exp(i(y-x)z - \tau\mathcal{P}(z))}{z(z+i)}dz,$$

and by returning to the initial variables $X = e^x, S = e^y, F(S, t) = f(y, t)$,

$$F(S, t) = -\frac{X}{2\pi}\int_{-\infty-i\omega}^{+\infty-i\omega} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}(z))}{z(z+i)}dz. \tag{30}$$

Here is another form of $F(S, t)$. Since the integrand in (30) is meromorphic on any strip $-\omega < \Im z < -\omega_1$, $\omega_1 \in (0, \lambda)$, with the only pole at $z = -i$, we have

$$F(S, t) = -\frac{X}{2\pi}\int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}(z))}{z(z+i)}dz -$$

$$- \frac{X}{2\pi}2\pi i\frac{\exp(i\ln(S/X)z - \tau\mathcal{P}(z))}{z}\Big|_{z=-i}$$

by the residue formula.

Since $P(0) = 0$, we have

$$\mathcal{P}(-i) = r - \alpha + P(-i) + \frac{\alpha - P(-i) - r}{-P(-2i) + 2P(-i)}(-P(-2i) + 2P(-i)) = 0,$$

and therefore,

$$F(S,t) = S - \frac{X}{2\pi}\int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}(z))}{z(z+i)}dz. \tag{31}$$

Under conditions (6)–(8), the integrals in (30) and (31) converge and decay faster than any power of $\ln(S/X)$, as $S + S^{-1} \to +\infty$. Hence, $F(0,t) = 0$. Below, we present some numerical results for a family (2)[4].

We fix $S = 100$, $r = 0.1$, and for $X = 80, 100, 120$ and $\nu = 2.0, 1.6, 1.2$ and different values of σ, λ and τ, compute $F(S,t)$. Recall that the variance is indepependent of λ and ν, and that for $\nu = 2$, a process is gaussian.

In the first series of examples, we take $\alpha = P(-i)$, which implies that the stock itself has zero drift.

Table 4.1.

Values of $F(S,t)$, *parameters*: $r = 0.1, \lambda = 3, \tau = 0.5$

σ	0.25	0.25	0.25	0.5	0.5	0.5	
X	120	100	80	120	100	80	
$\nu = 2.0$	2.48	9.58	24.30	8.98	16.26	27.76	
$\nu = 1.6$	2.83	9.46	24.28	9.11	16.23	27.67	
$\nu = 1.2$	3.36	9.28	24.26	9.32	16.16	27.59	

Table 4.2.

Values of $F(S,t)$, *parameters*: $r = 0.1, \lambda = 3, \tau = 0.25$

σ	0.25	0.25	0.25	0.5	0.5	0.5	
X	120	100	80	120	100	80	
$\nu = 2.0$	0.67	6.26	22.07	4.33	11.11	23.83	
$\nu = 1.6$	1.02	6.05	22.09	4.43	10.97	23.76	
$\nu = 1.2$	1.46	5.74	22.10	4.61	10.74	23.67	

We see that for stocks with zero drift, our formula gives higher price for options out of the money than the Black-Scholes one, and for options at the money (and usually, for options in the money) - lower price.

The next table shows that the results can change significantly if the drift changes. Here we take $\alpha = 3P(-i)/2$.

[4] The authors thank to Mitya Boyarchenko for the help with calculations.

Table 4.3.
Values of $F(S,t)$, *parameters*: $r = 0.1, \lambda = 3, \tau = 0.5$.

σ	0.25	0.25	0.25	0.5	0.5	0.5	
X	120	100	80	120	100	80	
$\nu = 2.0$	2.48	9.58	24.31	8.98	16.26	27.76	
$\nu = 1.6$	2.89	9.48	24.27	9.24	16.31	27.72	
$\nu = 1.2$	3.50	9.34	24.24	9.64	16.37	27.65	

5. A Solution to The Generalized Black-Scholes Equation II for European Put Options

For an European put option, the boundary conditions are

$$F(S,T) = \max(X - S, 0), \quad F(0,t) = Xe^{-r(T-t)}, \quad F(+\infty, t) = 0.$$

In terms of $y = \ln S, x = \ln X, f(y,t) = F(S,t), \tau = T - t$, they can be rewritten as

$$f(y,T) = \max(e^x - e^y, 0), \quad f(-\infty, t) = e^{x - r\tau}, \quad f(+\infty, t) = 0.$$

Take $\omega_2 \in (0, \lambda)$, and set $g(y,t) = e^{\omega_2 y} f(y,t)$. Then g is a solution to a problem

$$g_t = \mathcal{P}(D_y + i\omega_2)g, \tag{32}$$

$$g(y,t) = \max(e^{x+\omega_2 y} - e^{(1+\omega_2)y}, 0). \tag{33}$$

By making the Fourier transform w.r.t. y, we see that a problem (32)–(33) is equivalent to

$$\hat{g}_t(k,t) = \mathcal{P}(k + i\omega_2)\hat{g}(k,t), \tag{34}$$

$$\hat{g}(k,t) = \hat{h}(k), \tag{35}$$

where

$$\hat{h}(k) = \int_{-\infty}^{+\infty} e^{-iyk} \max(e^{x+\omega_2 y} - e^{(1+\omega_2)y}, 0)dy =$$

$$= \int_{-\infty}^{x} e^{-iyk}(e^{x+\omega_2 y} - e^{(1+\omega_2)y})dy =$$

$$= \left.\frac{e^{x+(\omega_2 - ik)y}}{\omega_2 - ik}\right|_{-\infty}^{x} - \left.\frac{e^{(1+\omega_2-ik)y}}{\omega_2 - ik}\right|_{-\infty}^{x} = -\frac{e^{(1+\omega_2-ik)x}}{(k+i\omega_2)(k+i\omega_2+i)}.$$

By solving the Cauchy problem (34)–(35), we obtain

$$\hat{g}(k,t) = -\frac{\exp(-\tau\mathcal{P}(k+i\omega_2) - ikx + (1+\omega_2)x)}{(k+i\omega_2)(k+i\omega_2+i)},$$

where $\tau = T - t$, and therefore,

$$g(y,t) = -\frac{e^{(1+\omega_2)x}}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp(-\tau\mathcal{P}(k+i\omega_2) + ik(y-x))}{(k+i\omega_2)(k+i\omega_2+i)}dk.$$

Since $f(y,t) = e^{-\omega_2 y} g(y,t)$, we have

$$f(y,t) = -\frac{e^x}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(-\tau \mathcal{P}(k+i\omega_2) + i(k+i\omega_2)(y-x))}{(k+i\omega_2)(k+i\omega_2+i)} dk =$$

$$= -\frac{e^x}{2\pi} \int_{-\infty+i\omega_2}^{+\infty+i\omega_2} \frac{\exp(-\tau \mathcal{P}(z) + iz(y-x))}{z(z+i)} dz.$$

By returning to the initial variables $X = e^x, S = e^y, F(S,t) = f(y,t)$, we obtain

$$F(S,t) = -\frac{X}{2\pi} \int_{-\infty+i\omega_2}^{+\infty+i\omega_2} \frac{\exp(-\tau \mathcal{P}(z) + iz\ln(S/X))}{z(z+i)} dz. \qquad (36)$$

The integrand in (36) being meromorphic on a strip $(-\omega_1, \omega_2)$, where $\omega_1 \in (0, \lambda)$, with the only pole at $z = 0$, we can apply the residue formula and obtain

$$F(S,t) = -\frac{X}{2\pi} \int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{\exp(-\tau \mathcal{P}(z) + iz\ln(S/X))}{z(z+i)} dz +$$

$$+\frac{X}{2\pi} 2\pi i \frac{\exp(-\tau \mathcal{P}(z) + iz\ln(S/X))}{z+i} \mid z = 0.$$

Since $P(0) = 0$, we have

$$\mathcal{P}(0) = r + P(0) + \frac{\alpha - P(-i) - r}{-P(-2i) + 2P(-i)}(-P(-i) + P(0) + P(-i)) = r,$$

therefore

$$F(S,t) = Xe^{-\tau r} - \frac{X}{2\pi} \int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{\exp(-\tau \mathcal{P}(z) + iz\ln(S/X))}{z(z+i)} dz. \qquad (37)$$

By comparing (31) and (37), we see that

$$F_{put}(S,t) = F_{call}(S,t) + Xe^{-r\tau} - S,$$

which is just the statement of the put-call pairity theorem.

Since $F_{call}(0,t) = 0$, Eq. (37) implies $F_{put}(0,t) = Xe^{-t\tau}$.

6. An Analysis of The Perfect Correlation Assumption

We have shown that the perfect correlation assumption (2) is equivalent to Eq. (21). Using Eq. (21), we derived the Generalized Black-Scholes Equation-II and found its solutions for European call and put options.

By denoting the last terms in (31) and (37) by $g(y,t)$, we can write

$$f_c(y,t) = e^y + g(y,t), \quad f_p(y,t) = Xe^{-r\tau} + g(y,t)$$

(c stands for call, and p – for put).

Let us check (21) for $f = f_p$ given by (37). Since $\int_{-\infty}^{+\infty} p_{\Delta t}(y) \cdot 1 dy = 1, \ \forall \ \Delta t > 0$, we have $P(D)1 = 0$, and therefore,

$$P(D_y)f_p = P(0)Xe^{-r\tau} + P(D_y)g = P(D_y)g,$$

$$P(D_y - i)f_p = P(-i)Xe^{-r\tau} + P(D_y - i)g, \quad f_p P(D_y)f_p = (Xe^{-r\tau} + g)P(D_y)g,$$

$$-P(D_y)f_p^2 = -P(D_y)(X^2e^{-r\tau} + 2Xe^{-r\tau}g + g^2) = -2Xe^{-r\tau}P(D_y)g - P(D_y)g^2.$$

It follows that (21) reduces to

$$(-2Xe^{-r\tau}P(D_y)g - P(D_y)g^2 + 2(Xe^{-r\tau} + g)P(D_y)g)(-P(-2i) + 2P(-i)) =$$

$$= (-P(-i)Xe^{-r\tau} - P(D_y - i)g + P(D_y)g + P(-i)Xe^{-r\tau} + P(-i)g)^2.$$

By simplifying, we see that Eq. (21) for f_p is equivalent to Eq. (21) for g.

Similar calculations show that Eq. (21) for f_c is equivalent to Eq. (21) for g.

Substituting

$$g(y,t) = -\frac{e^x}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp(i(y-x)(k-i\omega_1) - \tau\mathcal{P}(k-i\omega_1))}{(k-i\omega_1)(k-i\omega_1+i)}dk =$$

$$= -\frac{e^{x(1+\omega_1)+y\omega_1}}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp(i(y-x)k - \tau\mathcal{P}(k-i\omega_1))}{(k-i\omega_1)(k-i\omega_1+i)}dk$$

into Eq. (21), using the following equalities

$$P(D_y - ia)e^{y\omega_1}f = e^{y\omega_1}P(D_y - i(a+\omega_1))f, \quad (\mathcal{F}(P(D_y)f)(k,t) = P(k)\hat{f}(k,t),$$

$$\int_{-\infty}^{+\infty} e^{-iyk}\phi(y)\psi(y)dy = \int_{-\infty}^{+\infty}\hat{\phi}(k-k_1)\hat{\psi}(k_1)dk_1,$$

and cancelling the factor $\exp(2(x(1+\omega_1)+y\omega_1))$, we obtain an equivalent form of Eq. (21):

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iky}(-P(k-2i\omega_1) + 2P(k_1 - i\omega_1))\times$$

$$\times\frac{\exp(-\tau\mathcal{P}(k-k_1-i\omega_1) - \tau\mathcal{P}(k_1-i\omega_1))}{(k-k_1-i\omega_1)(k-k_1-i\omega_1+i)(k_1-i\omega_1)(k_1-i\omega_1+i)}dkdk_1\times$$

$$\times(-P(-2i) + 2P(-i)) =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}dk_1\int_{-\infty}^{+\infty}dke^{iky}(-P(k-k_1-i\omega_1-i) + P(k-k_1-i\omega_1) + P(-i))\times$$

$$\times\frac{(-P(k_1-i\omega_1-i) + P(k_1-i\omega_1) + P(-i))\exp(-\tau\mathcal{P}(k-k_1-i\omega_1) - \tau\mathcal{P}(k_1-i\omega_1))}{(k-k_1-i\omega_1)(k-k_1-i\omega_1+i)(k_1-i\omega_1)(k_1-i\omega_1+i)}.$$

By making the inverse Fourier transform, we obtain

$$\int_{-\infty}^{+\infty}\frac{(-P(k-2i\omega_1) + 2P(k_1-i\omega_1))\exp(-\tau\mathcal{P}(k-k_1-i\omega_1) - \tau\mathcal{P}(k_1-i\omega_1))}{(k-k_1-i\omega_1)(k-k_1-i\omega_1+i)(k_1-i\omega_1)(k_1-i\omega_1+i)}dk_1\times$$

$$\times(-P(-2i) + 2P(-i)) =$$

$$= \int_{-\infty}^{+\infty} (-P(k-k_1-i\omega_1-i)+P(k-k_1-i\omega_1)+P(-i))\times$$

$$\times(-P(k_1-i\omega_1-i)+P(k_1-i\omega_1)+P(-i))\times$$

$$\times \frac{\exp(-\tau\mathcal{P}(k-k_1-i\omega_1)-\tau\mathcal{P}(k_1-i\omega_1))}{(k-k_1-i\omega_1)(k-k_1-i\omega_1+i)(k_1-i\omega_1)(k_1-i\omega_1+i)} dk_1, \tag{38}$$

for all $k \in R, \tau > 0$, and $\omega_1 \in (0,1)$.

We have proved

Theorem 6.1. *Let (3)–(8) hold.*

Then the perfect correlation assumption (2) for an European call option holds if and only if Eq. (38) hold, and the same is true for an European put option.

The following theorem states that under reasonable conditions, which are certainly satisfied if the excess return on the stock is not high, Eq. (38), hence the perfect correlation assumption (2), fails.

Theorem 6.2. *Let $k = 0$ be the only point of minimum of $\mathcal{P}(k)$, and let it be non-degenerate.*

Then Eq. (38) fails unless

$$P(k) = -P(-i)k^2, \tag{39}$$

i.e. the process is gaussian.

Before proving Theorem 6.2, we note that the conditions of Theorem 6.2 are satisfied by P. Taking into account (3)– (8) and the definition of $\mathcal{P}$, we see that there exist $c > 0$ such if

$$\frac{\alpha - P(-i) - r}{-P(-2i)+2P(-i)} < c,$$

then $\mathcal{P}$ satisfies the condition of Theorem 6.2.

Proof of Theorem 6.2. By continuity of $\mathcal{P}$ and on the strength of (5)–(8), there exists $c > 0$ such that for any $\omega_1 \in (-c, c)$, the minimum of $\Re\mathcal{P}(k - i\omega_1)$ is attained at the only point $k = 0$ and is non-degenerate.

Fix $k = 0$ and $\omega_1 \in (0, c)$, and consider the asymptotics of integrals in (38) as $\tau \to +\infty$. In the both integrands, we see the same fast decaying exponential function, and the non-degeneracy condition stated above allows us to conclude that the leading term of the asymptotics of the LHS in (38) is of the form $const(\omega_1) \times d_l(\omega_1)\tau^{-1/2}$, where $const(\omega_1)$ depends on ω_1 and the fast decaying exponential function only, and

$$d_l(\omega_1) = (-P(-2i\omega_1) + 2P(-i\omega_1))(-P(-2i) + 2P(-i)).$$

Similarly, the leading term of the asymptotics of the RHS is of the form $const(\omega_1) \times d_r(\omega_1)\tau^{-1/2}$, where $const(\omega_1)$ is the same as above, and

$$d_r(\omega_1) = (-P(-i\omega_1 - i) + P(-i\omega_1) + P(-i))^2.$$

By comparing the leading terms, we see that if (38) holds then $\forall\ \omega \in (0, c)$,

$$(-P(-2i\omega)+2P(-i\omega))(-P(-2i)+2P(-i)) = (-P(-i\omega-i)+P(-i\omega)+P(-i))^2. \tag{40}$$

d_l and d_r coincide on a segment $(0, c)$, and since both are holomorphic on a strip $|\Re z| < \lambda/2$, they coincide on it. But d_r is holomorphic on a wider strip $|\Re z| < \lambda$, therefore d_l also is. Hence, a function $z \mapsto P(-i2z)$ is holomorphic on this strip, and therefore, P is holomorphic on a strip $|\Im k| < 2\lambda$. But by our assumption, $|\Im k| < \lambda$ was the widest strip with this property. Hence, $\lambda = +\infty$, and (40) holds for all ω_1.

Set $\omega = -1$. Since $P(0) = 0$, and $\overline{P(z)} = P(\bar{z})$, we obtain from (40)

$$(-P(-2i) + 2P(-i))^2 = (2P(-i))^2,$$

hence $P(-2i) = 4P(-i)$. Using the principle of mathematical induction, it is not difficult to show that

$$P(-ni) = n^2 P(-i), \quad \forall\, n \in Z.$$

Now we use the induction on $m = 1, 2, \ldots$, and for fixed m, the induction on $s = 0, \pm 1, \pm 2, \ldots$, to show that

$$P(-(\frac{1}{2} \pm s\frac{1}{2^m})i) = P(-i)(\frac{1}{2} \pm s\frac{1}{2^m})^2. \tag{41}$$

Eq. (41) means that Eq. (39) holds on a subset of R, which has an accumulation point; the both sides being holomorphic, Eq.(39) holds everywhere.

Theorem 6.2 has been proved.

7. Derivation of the Generalized Black-Scholes Equation III: The Risk-Minimization Approach

Consider an investor holding a fraction w of her/his wealth in the derivative security and a fraction $1 - w$ in the stock. Suppose that the investor wishes to minimize the variance of the portfolio

$$E\left[\left(w\frac{\Delta F - E[\Delta F]}{F} + (1-w)\frac{\Delta S - E[\Delta S]}{S}\right)^2\right].$$

The first order condition is

$$E\left[\left(\frac{\Delta F - E[\Delta F]}{F} - \frac{\Delta S - E[\Delta S]}{S}\right)\left(w\frac{\Delta F - E[\Delta F]}{F} + (1-w)\frac{\Delta S - E[\Delta S]}{S}\right)\right] =$$

$$= o(\Delta t),$$

as $\Delta t \to 0$. By using Lemma 2.2, we obtain

$$w(-f^{-2}P(D_y)f^2 + 2f^{-1}P(D_y)f) + (w-1)(-e^{-2y}P(D_y)e^{2y} + 2e^{-y}P(D_y)e^y) +$$

$$+(1-2w)(-f^{-1}e^{-y}P(D_y)(e^y f) + f^{-1}P(D_y)f + e^{-y}P(D_y)e^y) = 0.$$

Eq. (12) and (13) allow us to simplify

$$w(-f^{-2}P(D_y)f^2 + 2f^{-1}P(D_y)f) + (w-1)(-P(-2i) + 2P(-i)) +$$

$$+(1-2w)(-P(D_y-i)+f^{-1}P(D_y)f+P(-i)f)=0.$$

After rearranging, we arrive at

$$w(-f^{-2}P(D_y)f^2+2f^{-1}P(D_y)f-P(-2i))=$$

$$=f^{-1}P(D_y-i)f-f^{-1}P(D_y)f-P(-2i)+P(-i),$$

and therefore,

$$w=\frac{f^{-1}P(D_y-i)f-f^{-1}P(D_y)f-P(-2i)+P(-i)}{-f^{-2}P(D_y)f^2+2f^{-1}P(D_y)f-P(-2i)}=$$

$$=\frac{(P(D_y-i)-P(D_y)-P(-2i)+P(-i))f}{-f^{-1}P(D_y)f^2+2P(D_y-i)f-P(-2i)f}. \tag{42}$$

For a gaussian $p_{\Delta t}$, Eq. (15) gives

$$P(D_y)f=-\frac{\sigma^2}{2}f_{yy}, \quad -P(D_y)f^2=\frac{\sigma^2}{2}(2ff_{yy}+2f_y^2),$$

$$P(D_y-i)f=\frac{\sigma^2}{2}(-i\partial_y-i)^2f=-\frac{\sigma^2}{2}(\partial_y+1)^2f=-\frac{\sigma^2}{2}(f_{yy}+2f_y+f),$$

and

$$w=\frac{-f_{yy}-2f_y+f_{yy}+4f-f}{2f_{yy}+2f^{-1}f_y^2-2f_{yy}-4f_y-2f+4f}=$$

$$=\frac{f-f_y}{f-2f_y+f^{-1}f_y^2}=\frac{1-f^{-1}f_y}{1-2f^{-1}f_y+f^{-2}f_y^2}=\frac{1}{1-f^{-1}f_y}.$$

This is the fraction of wealth invested in the option in the Black-Scholes model.

Now we can calculate the minimal variance (attainable with the choice of w given by (42)).

By using Lemma 2.2 and (12)–(13), we obtain

$$v_{\min}:=\lim_{\Delta t\to 0} var_{\min}/\Delta t=$$

$$=w^2(-f^{-2}P(D_y)f^2+2f^{-1}P(D_y)f)+(1-w)^2(-P(-2i)+2P(-i))+$$
$$+2w(1-w)(-f^{-1}P(D_y-i)f+f^{-1}P(D_y)f+P(-i)).$$

After simplification, we obtain

$$v_{\min}=\frac{AB-C^2}{A+B-2C}, \tag{43}$$

where

$$A=-f^{-2}P(D_y)f^2+2f^{-1}P(D_y)f, \quad B=-P(-2i)+2P(-i),$$
$$C=-f^{-1}P(D_y-i)f+f^{-1}P(D_y)f+P(-i).$$

We see that the nominator in Eq. (43) is zero if and only if the perfect correlation condition (21) holds.

Thus, we have proven

Theorem 7.1. *Let conditions (3)–(8) hold.*

Then the riskless portfolio exists if and only if the perfect correlation assumption holds.

We proceed with the derivation of GBSE-III. w and $v_{\min}$ being found, we can use the non-arbitrage condition for the portfolio and the stock:

$$\frac{E\left[w\frac{\Delta F}{F}+(1-w)\frac{\Delta S}{S}\right]-r\Delta t}{var_{\min}^{1/2}}=\frac{E\left[\frac{\Delta S}{S}\right]-r\Delta t}{E\left[\left(\frac{\Delta S}{S}\right)^2\right]^{1/2}}+o(\Delta t^{1/2}).$$

By Lemma 2.1,

$$E[\Delta S/S]=(\alpha-P(-i))\Delta t+o(\Delta t),$$
$$E[(\Delta S/S)^2]=(-P(-2i)+2P(-i))\Delta t+o(\Delta t),$$

therefore we obtain

$$wf^{-1}\{f_t+\alpha f_y-P(D_y)f-rf\}+(1-w)(\alpha-P(-i)-r)=$$

$$=\frac{(\alpha-P(-i)-r)v_{\min}^{1/2}}{(-P(-2i)+2P(-i))^{1/2}},$$

and finally,

$$f_t+\alpha f_y-P(D_y)f-rf=$$
$$=(\alpha-P(-i)-r)f\times\left(1-w^{-1}+\frac{v_{\min}^{1/2}}{w(-P(-2i)+2P(-i))^{1/2}}\right). \tag{44}$$

This is the GBSE III. For gaussian $p_{\Delta t}$, $w=1/(1-f^{-1}f_y)$, $f(1-w^{-1})=f_y, v_{\min}=0$, and the RHS in (42) is equal to $(\alpha-P(-i)-r)f_y$.

Thus, Eq. (44) turns into (16), which is the Black-Scholes equation (1).

For other $p_{\Delta t}$, GBSE-III is a non-linear pseudo-differential equation, more complicated than GBSE-I.

8. A Scheme for Solving GBSE-I and GBSE-III

We can write both GBSE-I and GBSE-III in the form

$$f_t+(P(-i)+r)f_y-P(D_y)f-rf=(\alpha-P(-i)-r)\Phi(f), \tag{45}$$

where

$$\Phi(f)=\Phi_I(f)=(-P(D_y)f^2+2fP(D_y)f)/(-P(-2i)+P(-i))^{1/2}-f_y,$$

and

$$\Phi(f)=\Phi_{III}(f)=f\left(1-w^{-1}+\frac{v_{\min}^{1/2}}{w(-P(-2i)+2P(-i))^{1/2}}\right)-f_y,$$

respectively.

For gaussian $p_{\Delta t}$, the RHS in (45) is 0, therefore we may expect that if the process does not deviate too far from a gaussian one, then one can obtain the first approximation to the solution to (a boundary value problem for) Eq. (45) by solving the corresponding boundary-value problem for the following equation

$$f_t + (P(-i) + r)f_y - P(D_y)f - rf = 0. \tag{46}$$

We call it GBSE-IV.

Note that one can derive Eq. (46) by constructing a portfolio which eliminates fluctuations of order $e^{\Delta S} - 1$; the proof is similar to the one employed in Section 3.

Set

$$\mathcal{P}^1(k) = r + P(k) - (P(-i) + r)ik,$$

and write Eq. (46) in the form

$$f_t = \mathcal{P}^1(D_y)f.$$

A function $\mathcal{P}^1$ enjoys all the properties which we used when we derived formulas for European call and put options in Sections 4 and 5. Hence, we can write the corresponding solutions to Eq. (46) by replacing $\mathcal{P}$ with $\mathcal{P}^1$. The results are:

for an European call option:

$$F(S,t) = -\frac{X}{2\pi}\int_{-\infty-i\omega}^{+\infty-i\omega} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}^1(z))}{z(z+i)}dz, \tag{47}$$

where $\omega \in (1, \lambda)$ is arbitrary, or equivalently,

$$F(S,t) = S - \frac{X}{2\pi}\int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}^1(z))}{z(z+i)}dz, \tag{48}$$

where $\omega_1 \in (0, 1)$ is arbitrary;

for an European put option:

$$F(S,t) = -\frac{X}{2\pi}\int_{-\infty-i\omega}^{+\infty-i\omega} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}^1(z))}{z(z+i)}dz, \tag{49}$$

where $\omega \in (-\lambda, 0)$ is arbitrary, or equivalently,

$$F(S,t) = Xe^{-r\tau} - \frac{X}{2\pi}\int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{\exp(i\ln(S/X)z - \tau\mathcal{P}^1(z))}{z(z+i)}dz, \tag{50}$$

where $\omega_1 \in (0, 1)$ is arbitrary.

Here are several examples for an European call option.

Table 8.1.

Values of $F(S,t)$, *parameters*: $r = 0.1, \lambda = 3, \tau = 0.5$

σ	0.25	0.25	0.25	0.5	0.5	0.5	
X	120	100	80	120	100	80	
$\nu = 2.0$	2.48	9.58	24.30	8.98	16.26	27.76	
$\nu = 1.6$	2.41	9.27	24.34	2.42	9.33	24.34	
$\nu = 1.2$	2.37	8.81	24.40	2.39	8.95	24.39	

We see that the results differ from the ones for an European call option given by GBSE-II (see Table 4.1): here GBSE-IV gives higher values of $F(S,t)$ for options in the money, and lower for options at the money and out of the money.

Now we outline a numerical scheme for solving a non-linear GBSE for the case of an European call option; the case of a put one is similar. Set $f^0(y,t) = F(e^y,t)$, where F is given by either (47) or (48), and let f be a solution to Eq.(45) subject to boundary conditions

$$f(-\infty,t) = 0, \quad f(y,T) = \max(e^y - e^x, 0).$$

Set $g = f - f^0$. Then g is a solution to

$$g_t = \mathcal{P}^1(D_y)g + (\alpha - P(-i) - r)\Phi(f^0 + g), \tag{51}$$

subject to boundary conditions

$$g(-\infty,t) = 0, \quad g(y,T) = 0. \tag{52}$$

By applying $\mathcal{F}$, the Fourier transform w.r.t. y, we reduce a problem (51)–(52) to

$$\hat{g}_t(k,t) = \mathcal{P}^1(k)\hat{g}(k,t) + (\alpha - P(-i) - r)\mathcal{F}(\Phi(f^0 + g))(k,t), \tag{53}$$

with $\hat{g}$ subject to

$$\hat{g}(k,T) = 0. \tag{54}$$

(After a problem (53)–(54) is solved, one has to verify the first boundary condition in (52)). The well-known formula for the Cauchy problem allows us to reduce a problem (53)–(54) to a family of equations

$$\hat{g}(k,t) = (\alpha - P(-i) - r)\int_T^t \exp\{(t-t_1)\mathcal{P}^1(k)\}\mathcal{F}((\Phi(f^0+g))(k,t_1)dt_1.$$

By making the inverse Fourier transform, we arrive at

$$g(y,t) = (\alpha - P(-i) - r)\mathcal{F}^{-1}\int_T^t \exp\{(t-t_1)\mathcal{P}^1(k)\}\mathcal{F}((\Phi(f^0+g))(k,t_1)dt_1. \tag{55}$$

Suppose, that the simple iteration method is applicable to Eq. (55). Then we can take

$$g(y,t) = (\alpha - P(-i) - r)\mathcal{F}^{-1}\int_T^t \exp\{(t-t_1)\mathcal{P}^1(k)\}\mathcal{F}((\Phi(f^0))(k,t_1)dt_1 \tag{56}$$

as the first approximation to the solution of Eq.(55).

Thus, we suggest $f^0 + g$ as the second approximation to the solution of either GBSE-I and GBSE-III, where the first approximation, f, is given by (47)(=(48)), and a correction term, g, by (56).

Finally, recall that for small τ, approximately,

$$\exp(-\tau P(D_y))u(y,t) = (p_\tau * u)(y,t) := \int_{-\infty}^{+\infty} p_\tau(y-y_1)u(y_1,t)dy_1, \tag{57}$$

and therefore, the following approximate equalities hold:

$$P(D_y)u(y,t) = ((p_\tau * u)(y,t) - u(y,t))/\tau, \tag{58}$$

$$(\exp(-\tau \mathcal{P}^1(D_y))u)(y,t) = \exp(\tau r)(p_\tau * u)(y - \tau(r + P(-i)), t), \tag{59}$$

$$f^0(y,\tau) = (\exp(-\tau \mathcal{P}^1(D_y))h)(y,\tau) = \exp(\tau r)(p_\tau * h)(y - \tau(r + P(-i)), \tau), \tag{60}$$

where $h(y) = \max(e^y - e^x, 0)$, and

$$P(-ia) = \left(1 - \int_{-\infty}^{+\infty} p_{\Delta t}(y) e^{ay} dy\right) / \Delta t. \tag{61}$$

By using formulas (57)–(61), one can rewrite formulas (47), (49) and (56) for solutions of GBSE-I in terms of a given function h and an observed distribution $p_{\Delta t}$, and discretize them to develop a numerical scheme for computation the RHS's in these formulas.

Similarly one can rewrite formulas for solutions of GBSE-III (in this case they are much more involved).

9. The Pricing of The American Perpetual Put Option

9.1. Suppose that GBSE-IV, one of the linear generalizations of the Black-Scholes equation, hold, and consider the perpetual American put option (the case of GBSE-II can be considered similarly). Let X be the striking price, S – the level of the stock, and denote by $G(S, X)$ the rational put price. Then

$$G(X, S) = \max\{X - S, 0\}, \quad G(X, +\infty) = 0.$$

For a sufficiently low level of the stock price, it is advantageous to exercise the put. Define H to be the largest value of the stock such that the put holder is better off exercizing than continuing to hold the put, and set

$$x = \ln S, \quad h = \ln H, \quad g(x) := g(x, X) = G(S, X).$$

Then

$$g(x) = X - e^x, \ \forall\, x < h, \quad \text{and } g(+\infty) = 0. \tag{62}$$

Since $g(x)$ is independent of t, it obeys a stationary GBSE-IV:

$$((P_1(D) - i\gamma D + r)g)(x) = 0, \quad x > h, \tag{63}$$

where $\gamma = r + P(-i)$. As a proxy for $P(-i)$, in applications one may use Eq. (61), and it can be reasonable to adjust γ since the hedge used in deriving the GBSE-IV is not perfect and the portfolio constructed is not risk-free.

A discretized version of the GBSE-IV is

$$g(x) - e^{-r\Delta t} \int_{-\infty}^{+\infty} p_{\Delta t}(x + \gamma \Delta t - y) g(y) dy = 0, \tag{64}$$

The formula for h will be formulated in terms of an observed distribution density $p_{\Delta t}$, under fairly weak assumptions on (an even) $p_{\Delta t}$:

$$\int_{-\infty}^{+\infty} p_{\Delta t}(x)e^x dx < +\infty, \tag{65}$$

and there exists C and $\omega > 0$ such that $\hat{p} = \mathcal{F}p$, the Fourier transform of p, and its derivative satify bounds

$$\hat{p}(k) \le 1,\ \forall\ k \in R,$$
$$|\hat{p}(k)| + |\hat{p}'(k)| \le C(1 + |k|)^{-\omega},\ \forall\ k \in R. \tag{66}$$

The second bound is a weak form of a smoothness condition. For instance, for a piece-wise smooth p it holds with $\omega = 1$.

To simplify the notation, we normalize Δt to unity, and drop a subscript Δt. Set $A(k) = 1 - e^{-r+i\gamma k}\hat{p}(k)$. By the Taylor formula, $e^{i\gamma D}u(x) = u(x+\gamma)$, therefore (64) can be rewritten as

$$(A(D)g)(x) = 0, \quad \forall\ x < h. \tag{67}$$

On the strength of (62), we may look for solutions $g \in L_2(R)$ and rewrite (67) as

$$(A(D)g)(x) = \psi(x+h),\ \forall\ x, \tag{68}$$

where $\psi \in L_2(R_-)$. Here and below we identify $L_2(R_-)$ with a subspace of $L_2(R)$ by defining $\psi(x) = 0\ \forall\ x > 0$. Similarly, $L_2(R_+)$ is regarded as a subspace of $L_2(R)$.

We will solve (68) subject to (62) by the Wiener-Hopf (1931) method, in a bit more modern version (see e.g. Eskin (1973)). It is based on the factorization of $A(k)$.

The following lemma is a variant of standard factorization theorems (see e.g. Eskin (1973), Section 6).

Lemma 9.1. *Let (65) and (66) hold.*

Then $A(k)$ admits a factorization

$$A(k) = A_+(k)A_-(k) \tag{69}$$

with the $A_\pm(k)$ satisfying the following conditions:

a) A_+ (resp. A_-) is holomorphic in a half-plane $\Im k > 0$ (resp. $\Im k < 0$), and admits a continuous extension into the closed half-plane;

b) there exist $c > 0, C$ such that

$$c \le |A_\pm(k)| \le C, \quad \forall\ \pm\Im k \ge 0; \tag{70}$$

c) $A_\pm(k)^{-1}$ admits a representation

$$A_\pm(k)^{-1} = 1 + T_\pm(k), \tag{71}$$

where $T_\pm$ is holomorphic in a half-plane $\pm\Im k > 0$, and satisfies an estimate

$$|T_\pm(k)| \le C(1 + |k|)^{-\omega_1}, \quad \forall\ \pm\Im k \ge 0, \tag{72}$$

where $\omega_1 > 0$ and C are independent of k.

Proof. By (66), $\hat{p}(k) \le 1$, hence we have $\Re A(l) \ge 1 - e^{-r} > 0$, for all $l \in R$. Therefore, $\ln A(l)$ is well defined by a requirement: $\ln a$ is real for $a > 0$, and we may set, for $\tau > 0$ and $k \in R$,

$$b_{\pm}(k \pm i\tau) = \pm \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln A(l)}{k \pm i\tau - l} dl, \tag{73}$$

$$A_{\pm}(k \pm i\tau) = \exp(b_{\pm}(k \pm i\tau)).$$

The proof that $A_{\pm}$ satisfy (69) and a)–c) is a minor variation of the proof in Eskin (1973); for completeness, we give it in Appendix.

With a factorization (69) at our disposal, we return to (68) and multiply it by $A_+(D)^{-1}$:

$$(A_-(D)g)(x) = (A_+(D)^{-1}\phi)(h + x), \ \forall \ x.$$

Since $\mathrm{supp}\phi \subset \overline{R_-}$, and $A_+(k)$ satisfies (70) in a half-plane $\Im k \ge 0$, $\mathrm{supp} A_+(D)^{-1}\phi \subset \overline{R_-}$ (see e.g. Eskin (1973), Theorem 4.4). Hence,

$$(A_-(D)g)(x) = 0, \ \ x > h. \tag{74}$$

Set $u(x) = g(x + h) + e^{h+x} - X$. For $\beta \ge 0$,

$$A_-(D)e^{\beta x} = e^{\beta x}e^{-\beta x}A_-(D)e^{\beta x} \cdot 1 = e^{\beta x}A_-(D - i\beta) \cdot 1 = e^{\beta x}A_-(-i\beta),$$

since $\forall \ \psi \in \mathcal{S}(R)$,

$$\langle B(D) \cdot 1, \psi\rangle = \langle 1, \bar{B}(D)\psi\rangle = \langle \delta, \bar{B}\hat{\psi}\rangle = B(0)\hat{\psi}(0) = B(0)\langle \delta, \hat{\psi}\rangle = B(0)\langle 1, \psi\rangle.$$

Hence, we may rewrite (74) as

$$(A_-(D)u)(x) = \theta_+(x)(A_-(-i)e^{h+x} - A_-(0)X), \ \ x > 0, \tag{75}$$

where θ_+ is the characteristic function of R_+. Note that on the strength of (62), $\mathrm{supp} u \subset \overline{R_+}$, and $A_-(k)$ satisfies (70) in a half-plane $\Im k \le 0$. Therefore, the LHS of (75) is zero for $x < 0$, and hence, (75) holds for all x. By applying $A_-(D)^{-1}$ to (75), we obtain

$$u(x) = A_-(D)^{-1}\theta_+(A_-(-i)e^{h+x} - A_-(0)X), \ \ x > 0. \tag{76}$$

Take $\epsilon > 1$ and set $u^{\epsilon}(x) = e^{-\epsilon x}u(x)$, $f^{\epsilon}(x) = e^{-\epsilon x}f(x)$. Next, multiply (76) by $e^{-\epsilon x}$, and use an equality $e^{-\epsilon x}A_-(D)^{-1}e^{\epsilon x} = A_-(D - i\epsilon)^{-1}$. The result is

$$u^{\epsilon} = A_-(D - i\epsilon)^{-1}\theta_+ f^{\epsilon}, \tag{77}$$

where $f^{\epsilon}(x) = A_-(-i)e^{h+(1-\epsilon)x} - A_-(0)Xe^{-\epsilon x}$.

Lemma 9.2. *For $x < 0$, $u(x, h) = 0$, and for $x > 0$,*

$$u(x, h) = d(h)(1 + e^h\mu_1(x) + \mu_2(x)) +$$

$$+ \, xd(h)A_-(0)X + e^h\chi_1(x) + \chi_2(x), \tag{78}$$

where functions $\mu_1(x) = o(1), \mu_2(x) = o(1)$, $\chi_1(x) = o(x), \chi_1(x) = o(x)$, *as* $x \to +0$, *are independent of* h, *and* $d(h) = A_-(-i)e^h - A_-(0)X$.

Proof. The first statement is just (62), and (78) will be proved in Appendix.

Theorem 9.1. *Let (65) and (66) hold.*

Then a pricing formula for the American perpetual put option is given by: for any $\epsilon > 1$,

$$G(X, S) = X - S + A_-(0)XS^\epsilon(2\pi)^{-1}\int_{-\infty}^{+\infty}\frac{e^{ik\ln(S/H)}}{A_-(k - i\epsilon)(ik + \epsilon - 1)(ik + \epsilon)}dk, \quad (79)$$

where H, *the exercise price, is given by*

$$H = e^h = XA_-(0)/A_-(-i) = X(1 - e^{-r})^{1/2}\exp(I_2 - I_1), \quad (80)$$

and

$$I_1 = \frac{1}{2\pi}\int_0^{+\infty}\ln((1 - e^{-r}\hat{p}(l)\cos(\gamma l))^2 + (e^{-r}\hat{p}(l)\sin(\gamma l))^2)(1 + l^2)^{-1}dl,$$

$$I_2 = \frac{1}{\pi}\int_0^{+\infty}\arctan\left(\frac{\hat{p}(l)\sin(\gamma l)}{e^r - \hat{p}(l)\sin(\gamma l)}\right)l^{-1}(1 + l^2)^{-1}dl.$$

Proof. Direct calculations (see Appendix) show that $A_-(0), A_-(-i)$ are positive. Due to (78), if $d(h) < 0$, then a condition $G(X, e^x) \geq X - e^x$ is violated at $x = h$, and if $d(h) > 0$, then $G(X, S)$ is not decreasing w.r.t. S, which is also impossible. Hence, $d(h) = 0$, which is just (80), the calculations of $A_-(0)$ and $A_-(-i)$ in Appendix being taken into account.

Finally, Eq. (80) is equivalent to $A_-(-i)e^h = A_-(0)X$, therefore (77) can be rewritten as

$$u(x) = A_-(0)Xe^{\epsilon x}(2\pi)^{-1}\int_{-\infty}^{+\infty}e^{ikx}A_-(k - i\epsilon)^{-1}\int_0^{+\infty}\left(e^{-iky+(1-\epsilon)y} - e^{-iky-\epsilon y}\right)dydk =$$

$$= A_-(0)Xe^{\epsilon x}(2\pi)^{-1}\int_{-\infty}^{+\infty}e^{ikx}A_-(k - i\epsilon)^{-1}\left(\frac{1}{ik + \epsilon - 1} - \frac{1}{ik + \epsilon}\right)dk =$$

$$= A_-(0)Xe^{\epsilon x}(2\pi)^{-1}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{A_-(k - i\epsilon)(ik + \epsilon - 1)(ik + \epsilon)}dk,$$

and using equalities $g(x) = X - e^x + u(x - h)$, $x = \ln S$, $h = \ln H$, we obtain (79).

Note that due to (78),

$$\lim_{S\to H-0}G_S(X, S) = -1 < -1 + A_-(0)X = \lim_{S\to H+0}G_S(X, S),$$

which means that the smooth pasting condition valid in the standard continuous time Geometric Brownian motion model (Merton (1973)) fails in our discrete time model.

In Merton (1973), the exercise price $H_0 = X\beta/(1 - \beta)$, where $\beta = -2r/\sigma^2$ is the negative root of the characteristic equation $\sigma^2k^2/2 + (r - \sigma^2/2)k - r = 0$.

Numerical Examples. The first truncated Lévy distributions were constructed by Mantegna and Stanley (1994). Later, Koponen (1995) constructed a family of truncated

Lévy distributions which admit explicit description in terms of their Fourier transforms. For the sake of brevity, we consider only symmetric distributions of this family, with $\hat{p}_\nu$ defined by

$$\hat{p}_\nu(k) = \exp[-\sigma^2\lambda^2[1 - ((k/\lambda)^2 + 1)^{\nu/2} \cos(\nu \arctan(k/\lambda))]/\nu(\nu - 1)],$$

where $\sigma > 0, \lambda > 0$ and $\nu \in (0, 2], \nu \neq 1$ are parameters. We have chosen a normalization so that the variance is independent of ν and λ.

For $\nu = 2$, we obtain $\hat{p}_2(k) = \exp(-\sigma^2 k^2/2)$ which means that p_2 is a gaussian distribution. As ν moves from 2 down, p_ν deviates from a gaussian distribution, and for fixed $\nu \in (0, 2), \nu \neq 1$, in the limit $\lambda \to +0$, p_ν becomes a Lévy distribution with $\hat{p}_\nu(k) = \exp(-c_1|k|^\nu \cos(\nu\pi/2)/\nu(\nu - 1))$. Roughly speaking, $(-\lambda^{-1}, \lambda^{-1})$ is an interval where p_ν differs insignificantly from a Lévy distribution, and for $|x| >> \lambda^{-1}$, the distribution exhibits an exponential fall-off.

Here are some numerical examples.[5] In tables below, we fix r, σ, λ and see how the threshold H varies with ν. H_0, the threshold in Merton (1973), is independent of λ and ν.

Table 1. *Parameters:* $X = 1, r = 0.0005, \ \sigma = 0.002, \ \lambda = 1.5; \ H_0 = 0.333$

ν	2.0	1.8	1.6	1.4	1.2	0.8	0.6	0.4	0.2
H	0.337	0.341	0.346	0.351	0.356	0.364	0.368	0.369	0.370

Table 2. *Parameters:* $X = 1, r = 0.001, \ \sigma = 0.03, \ \lambda = 1.5; \ H_0 = 0.181$

ν	2.0	1.8	1.6	1.4	1.2	0.8	0.6	0.4	0.2
H	0.185	0.187	0.188	0.190	0.191	0.193	0.194	0.195	0.195

In these two examples, it is clearly seen that the threshold increases as ν goes from 2 down, i.e. as a process deviates from a gaussian one of the same variance.

Probably, this is a result of a smooth truncation. For instance, for a mixture with

$$\widehat{p}_\nu(x) = \nu \exp(-\sigma^2 k^2/2) + (1 - \nu)\sin(\sigma\sqrt{3})/(\sigma\sqrt{3}), \ \nu \in [0, 1],$$

the threshold decreases (though weakly) as ν goes from 1 down.

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Appendix

A1. Some basic facts of the theory of the Sobolev spaces and the theory of pseudo-differential operators (see e.g. Eskin (1973)

By $\mathcal{S}(R)$ one denotes the space of infinitely differentiable functions decaying at infinity faster than any power of x, together with all derivatives, and by $\mathcal{S}'(R)$ – its dual space.

Let $s \in R$. The Sobolev space $H^s(R)$ consists of $u \in \mathcal{S}'(R)$ with the finite norm

$$\| u \|_s = \left(\int_{-\infty}^{+\infty} (1+k^2)^s |\hat{u}(k)|^2 dk \right)^{1/2}.$$

The closure of $C_0^\infty(R_\pm)$ in $H^s(R)$ is denoted by $\overset{o}{H}{}^s(R_\pm)$. The spaces $H^s(R)$ and $\overset{o}{H}{}^s(R_\pm)$ are Hilbert spaces, and $H^0(R) = L_2(R)$, $\overset{o}{H}{}^0(R_\pm) = L_2(R_\pm)$.

For an integer $m \geq 0$ and $s > m + 1/2$, $H^s(R) \subset C^m(R)$, by the Sobolev embedding theorem.

The Dirac delta-function (a linear functional defined by $\delta(f) = f(0)$) belongs to $\overset{o}{H}{}^s(R_\pm)$, for any $s < -1/2$.

If the symbol of a PDO $A(D)$ is measurable and admits a bound

$$|A(k)| \leq C(1+|k|^2)^{m/2}, \ \forall\, k, \tag{81}$$

then $A(D)$ is said to be of order m. A PDO of order m is a bounded operator from $H^s(R)$ to $H^{s-m}(R)$. If $A(k)$ admits a holomorphic extension into a half-plane $\pm\Im k > 0$ and satisfies a bound (81) in the closed half-plane, then $A(D)$ is a bounded operator from $\overset{o}{H}{}^s(R_\mp)$ to $\overset{o}{H}{}^l(R_\mp)$, where $l = s - m$.

A2. Proof of Lemma 9.1. As $l \to \pm\infty$, $\ln A(l) = O(\hat{p}(l))$, and on the strength of (66), there exist $C, \omega > 0$ such that

$$|\ln A(l)| \leq C(1+|l|)^{-\omega}, \ \forall\, l \in R. \tag{82}$$

Fix $\tau_0 > 0$, and consider $b_\pm(k)$ in a half-plane $\pm\Im k \geq \tau_0$. Set $J_1 = \{l \mid |k-l| \geq |k|/2\}$, $J_2 = \{l \mid |k-l| \leq |k|/2\}$. On J_1, $|k| \leq 2|l-k|$ and $|l| \leq |k-l| + |k| \leq 3|l-k|$, hence

$$(1+|k-l|)^{-1}(1+|l|)^{-\omega} \leq 3(1+|k|)^{-\omega/2}(1+|l|)^{-1+\omega/2}(1+|l|)^{-\omega}, \tag{83}$$

and since $(1+|l|)^{-1-\omega/2} \in L_1(R)$, we deduce from (82)–(83) an estimate

$$\left| \int_{J_1} \frac{\ln A(l)}{k-l} dl \right| \leq C_{\tau_0}(1+|k|)^{-\omega/2}, \tag{84}$$

where a constant C_{τ_0} is independent of k and $\tau \geq \tau_0$. On J_2, $|l| \geq |k| - |k-l| \geq |k|/2 \geq |k-l|$, and hence,

$$(1+|k-l|)^{-1}(1+|l|)^{-\omega} \leq C(1+|k-l|)^{-1}(1+|k-l|)^{-\omega/2}(1+|k|)^{-\omega/2}.$$

Therefore,

$$\left|\int_{J_2} \frac{\ln A(l)}{k-l} dl\right| \le$$

$$\le C_{1\tau_0}(1+|k|)^{-\omega/2} \int_{-\infty}^{+\infty} (1+|k-l|)^{-1-\omega/2} dl \le C_{2\tau_0}(1+|k|)^{-\omega/2}.$$

Thus, (84) holds with R instead of J_1. Similar estimates hold for derivatives w.r.t. k, and in a region $\pm\Im k \ge \tau_0$, parts b), c) and the first part of a) have been proved.

To show that $A_\pm(k)$ admits a continuous extension up to the boundary of a half-plane $\pm\Im k > 0$, fix $k \in R$ and write, for $\tau > 0$,

$$b_\pm(k \pm i\tau) = b^1_\pm(k \pm i\tau) + b^2_\pm(k \pm i\tau),$$

where

$$b^1_\pm(k \pm i\tau) = \pm\frac{i}{2\pi} \int_{|k-l|>1} \frac{\ln A(l)}{k \pm i\tau - l} dl,$$

$$b^2_\pm(k \pm i\tau) = \pm\frac{i}{2\pi} \int_{|k-l|<1} \frac{\ln A(l)}{k \pm i\tau - l} dl.$$

The denominator of the integrand of $b^1_\pm$ being bounded away from zero, uniformly in $k \in R$ and $\tau \ge 0$, the proof for $\pm\Im k \ge \tau_0$ above shows that $b^1_\pm(k)$ is continuous in a closed half-plane $\pm\Im k \ge 0$ and satisfies all the necessary estimates there.

Consider $b^2_\pm(k)$. On the strength of (66), there exists $C > 0$ such that all for all $k \in R$ and $c \in (k-1, k+1)$,

$$|\ln A(c)| + |A'(c)/A(c)| \le C(1+|k|)^{-\omega}.$$

Hence, using the Lagrange formula

$$\ln A(l) = \ln A(k) + \frac{A'(c)}{A(c)}(l-k),$$

where $c \in (k, l)$ (or $c \in (l, k)$), and noticing that

$$\pm\frac{i}{2\pi} \int_{|k-l|<1} \frac{dl}{k \pm i\tau - l} = \pm\frac{i}{2\pi} \int_{|l|<1} \frac{dl}{i\tau - l} =$$

$$= \pm\frac{i}{2\pi} \int_{|l|<1} \frac{\mp i\tau - l}{l^2 + \tau^2} dl = \frac{1}{2\pi} \int_{|l|<1} \frac{\tau\, dl}{l^2 + \tau^2} \to 1/2,$$

as $\tau \to +0$, we obtain that $b^2_\pm(k)$ is continuous up to the boundary of a half-plane $\pm\Im k \ge 0$, and admits an estimate

$$|b^2_\pm(k \pm i\tau)| \le C(1+|\tau|+|k|)^{-\omega/2}.$$

This finishes the proof of Lemma 9.1.

A3. Proof of Eq. (78). Since $\epsilon > 1$, we have $f^\epsilon \in \mathcal{S}(\overline{R_+})$. Therefore, we may apply a formula (5.39) in Eskin (1973) and obtain

$$\theta_+ f^\epsilon = \sum_{s=1}^{m} (1+iD)^{-s}\delta \cdot ((1+iD)^{s-1} f^\epsilon)(0) + (1+iD)^{-m}\theta_+(1+iD)^m f^\epsilon. \qquad (85)$$

Here m is a positive integer, δ is the Dirac delta-function, and

$$((1+iD)^{-s}\delta)(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ixk}(1+ik)^{-s}dk.$$

By using (85) and (71)–(72), we can rewrite (77) as

$$u = e^{\epsilon x}(1+T_-(D-i\epsilon))[(1+iD)^{-1}\delta\cdot f^\epsilon(0)+$$

$$+(1+iD)^{-2}\delta\cdot((1+iD)f^\epsilon)(0)+(1+iD)^{-2}\theta_-(1+iD)^2 f^\epsilon].$$

By introducing the notation

$$\phi_s = (1+iD)^{-s}\delta, \quad w_1^\epsilon = T_-(D-i\epsilon)(1+iD)^{-1}\delta,$$

$$w_2^\epsilon = A_-(D-i\epsilon)^{-1}(1+iD)^{-2}\delta,$$

$$w_3^\epsilon = A_-(D-i\epsilon)^{-1}(1+iD)^{-2}A_-(-i)\theta_+((1+iD)^2 e^{(1-\epsilon)x}),$$

$$w_4^\epsilon = -A_-(D-i\epsilon)^{-1}(1+iD)^{-2}A_-(0)X\theta_+((1+iD)^2 e^{-\epsilon x}),$$

we can write

$$u = e^{\epsilon x}\{(\phi_1+w_1^\epsilon)f^\epsilon(0)+\phi_2\cdot(f^\epsilon(0)+(f^\epsilon)'(0))+ \tag{86}$$

$$+w_2^\epsilon\cdot(f^\epsilon(0)+(f^\epsilon)'(0))+e^h w_3^\epsilon+w_4^\epsilon\}.$$

Consider terms in (86).

1) We know that $\delta \in \overset{o}{H}{}^l(R)$, for any $l<-1/2$, and since $f^\epsilon\in\mathcal{S}(\overline{R_+})$, Theorem 5.1 in Eskin (1973) gives $\theta_+f^\epsilon, \theta_+(1+iD)^2 f^\epsilon \in H^s(R)$, for any $s\in(0,1/2)$. But $T_-(D-i\epsilon)(1+iD)^{-1}$ and $A_-(D-i\epsilon)(1-iD)^{-2}$ are PDO of order $-1-\omega$ and -2, respectively, and hence, $w_1^\epsilon \in H^{l+1+\omega}(R)\subset H^{1/2+\omega/2}(R)\subset C(R)$, $w_2^\epsilon\in H^{l+2}(R)\subset C^1(R)$. Similarly, $w_3^\epsilon, w_4^\epsilon \in C^1(R)$. The symbols of $A_-(D-i\epsilon)^{-1}$ and $T_-(D-i\epsilon)$ being holomorphic in a half-plane $\Im k<0$, we have $\mathrm{supp} w_j^\epsilon \subset \overline{R_+}$, $j=1,\cdots,4$. Hence, $w_1^\epsilon(0)=0$, and $w_j^\epsilon(0)=(w_j^\epsilon)'(0)=0$, $j=2,3,4$.

2) Consider ϕ_s, $s=1,2$. By the residue theorem, for $x<0$ and $\tau>0$,

$$\phi_s(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ixk}(1+ik)^{-s}dk = (2\pi)^{-1}\int_{-\infty-i\tau}^{+\infty-i\tau} e^{ixk}(1+ik)^{-s}dk \to 0,$$

as $\tau\to+\infty$, and hence, $\phi_s(x)=0$. Further,

$$\int_0^\infty e^{-ixk}e^x dx = (1+ik)^{-1},$$

therefore $\phi_1(x)=e^{-x}$ for $x>0$.

By differentiating ϕ_2 at $x>0$, we find

$$\phi_2'(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ixk}\frac{ik}{(1+ik)^2}dk = \phi_1(x)-\phi_2(x).$$

The general solution to an equation $\phi_2' = e^{-x}-\phi_2$ is $\phi_2(x)=xe^{-x}+Ce^{-x}$, but $\phi_2\in\overset{o}{H}{}^l(R_+)$, $\forall\, l<3/2$, is of the class $C^1(R)$, and hence, equal to 0 at $x=0$. Thus, we obtain $\phi_2(x)=xe^{-x}$, $\forall\, x>0$.

3) Now we calculate coefficients in (86):

$$f^{\epsilon}(0) = d(h) = A_-(-i)e^h - A_-(0)X,$$

$$(f^{\epsilon})'(0) = (1-\epsilon)A_-(-i)e^h + \epsilon A_-(0)X = (1-\epsilon)d(h) + A_-(0)X.$$

4) As $x \to +0$,

$$e^{\epsilon x}\phi_1(x)d(h) = (1 + (\epsilon - 1)x)d(h) + o(x),$$

$$e^{\epsilon x}\phi_2(x) = x + o(x),$$

$$e^{\epsilon x}\phi_1(x) + x[d(h) + (1-\epsilon)d(h) + A_-(0)X] =$$

$$= d(h) + xd(h)[(\epsilon - 1) + 1 + (1-\epsilon)] + xA_-(0)X + o(x) =$$

$$= d(h) + xd(h) + xA_-(0)X + o(x).$$

5) By gathering 1) – 4) and (86), we obtain (78).

A3. Calculation of $A_-(-i)$ and $A_-(0)$. By noticing that $\Re\hat{p}$ is even and $\Im\hat{p}$ is odd, and using (73), we obtain

$$b_-(-i) = -\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\ln(1-e^{-r+i\gamma l}\hat{p}(l))}{-i-l}dl =$$

$$= -\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{i-l}{l^2+1}\ln\left(((1-e^{-r}\hat{p}(l)\cos(\gamma l))^2 + (e^{-r}\hat{p}(l)\sin(\gamma l))^2)^{1/2}\right)dl -$$

$$-\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{i-l}{l^2+1}\left(-i\arctan\frac{e^{-r}\hat{p}(l)\sin(\gamma l)}{1-e^{-r}\hat{p}(l)\cos(\gamma l)}\right)dl =$$

$$= \frac{1}{2\pi}\int_0^{\infty}\frac{\ln((1-e^{-r}\hat{p}(l)\cos(\gamma l))^2 + (e^{-r}\hat{p}(l)\sin(\gamma l))^2)}{l^2+1}dl -$$

$$-\frac{1}{\pi}\int_0^{\infty}\frac{l}{l^2+1}\arctan\frac{e^{-r}\hat{p}(l)\sin(\gamma l)}{1-e^{-r}\hat{p}(l)\cos(\gamma l)}dl,$$

and

$$b_-(0) = -i\frac{1}{2\pi}\lim_{\tau\to+0}\int_{-\infty}^{+\infty}\frac{\ln(1-e^{-r+\gamma l}\hat{p}(l))}{-i\tau - l}dl =$$

$$= -i\frac{1}{2\pi}\lim_{\tau\to+0}\int_{-\infty}^{+\infty}\frac{(i\tau - l)\ln(1-e^{-r+\gamma l}\hat{p}(l))}{\tau^2+l^2}dl =$$

$$= \frac{1}{2\pi}\ln(1-e^{-r})\int_0^{\infty}\frac{dl}{l^2+1} - \frac{1}{\pi}\int_0^{\infty}l^{-1}\arctan\frac{\hat{p}(l)\sin(\gamma l)}{e^r - \hat{p}(l)\cos(\gamma l)}dl =$$

$$= \frac{1}{2}\ln(1-e^{-r}) - \frac{1}{\pi}\int_0^{\infty}l^{-1}\arctan\frac{\hat{p}(l)\sin(\gamma l)}{e^r - \hat{p}(l)\cos(\gamma l)}dl.$$

Since $A_- = \exp b_-$,

$$\frac{A_-(0)}{A_-(-i)} = \exp\{\frac{1}{2}\ln(1-e^{-r}) - I_1 + I_2\},$$

where I_1, I_2 are the same as in (79).