On the Holomorphic Solution of Non-linear Totally Characteristic Equations with Several Space Variables [1]

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Abstract. In this paper we study a class of non-linear singular partial differential equation in complex domain $\mathbb{C}_t \times \mathbb{C}_x^n$. Under certain assumptions, we prove the existence and uniqueness of holomorphic solution near origin of $\mathbb{C}_t \times \mathbb{C}_x^n$.

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§1 Introduction and Main Result.

Let $(t,x) \in \mathbb{C}_t \times \mathbb{C}_x^n$, we consider the following non-linear singular partial differential equation

$$t\partial_t u = F(t,x,u,\bigtriangledown_x u), \qquad (t,x) \in \mathbb{C}_t \times \mathbb{C}_x^n. \tag{1}$$

where $u = u(t,x)$ is an unknown function, $\bigtriangledown_x = (\partial_{x_1}, \cdots, \partial_{x_n})$, $F(t,x,u,v)$ is a function with respect to the variables $(t,x,u,v) \in \mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_v^n$.

For the function $F(t,x,u,v)$, we suppose

(H1) $F(t,x,u,v)$ is a holomorphic function in a neighborhood of the origin $(0,0,0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_v^n$.

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(H2) $F(0,x,0,0)\equiv 0$ near $x=0$.

Thus we can expand $F(t,x,u,v)$ as the following form:

$$F(t,x,u,v)=a(x)t+b(x)u+\sum_{j=1}^{n}b_j(x)v_j+\sum_{p+q+|\gamma|\geq 2}a_{p,q,\gamma}(x)t^pu^qv^\gamma, \qquad (1')$$

where $a(x)=\partial_tF(0,x,0,0)$, $b(x)=\partial_uF(0,x,0,0)$, $b_j(x)=\partial_{v_j}F(0,x,0,0)$.

If for $1\leq j\leq n$, $b_j(x)\equiv 0$ near $x=0$, the linearlized equation of (1) is "Fuchsian type (cf. [1, 2]", so the equation (1) is called non-linear Fuchsian type PDE (or is called "Briot-Bouquet type equation" in [4, 5]); this situation has been discussed by [4-7]. If $b_j(0)\neq 0$ for some j, then we can use the implicit function theorem to solve v_j from the equation (1), then, by using Cauchy-Kowalewski theorem, we can easily deduce that (1) has a unique holomorphic solution $u(t,x)$ with $u(0,x)\equiv 0$ and $u(t,0)\equiv 0$ near $(0,0)\in\mathbb{C}_t\times\mathbb{C}_x^n$. So in this paper, we shall consider the case of $b_j(x)\not\equiv 0$ and $b_j(0)=0$, i.e. the indicial operator of (1) $b(x)+\sum_{j=1}^{n}b_j(x)\partial_{x_j}$ is a singular PDO. In this situation the equation (1) has been called totally characteristic type PDE by Chen-Tahara [8].

In this paper, we shall discuss the case, i.e. the indicial operator of (1) has regular singularity at $x=0$, we suppose

(H3) For $1\leq j\leq n$, $b_j(x)=x_jc_j(x)$, and $c_j(x)$ is a holomorphic function near $x=0$.

The situation of $b_j(x)=x_j^pc_j(x)$ for $p\geq 2$ will be studied in the forthcoming paper.

Actually, if we denote $C(t,x,\partial_t,\bigtriangledown_x)=t\partial_t-b(x)-\sum_{j=1}^{n}x_jc_j(x)\partial_{x_j}$, the equation (1) can be rewritten as

$$C(t,x,\partial_t,\bigtriangledown_x)u=a(x)t+\sum_{p+q+|\gamma|\geq 2}a_{p,q,\gamma}(x)t^pu^q(\bigtriangledown_xu)^\gamma. \qquad (2)$$

And the indicial polynomial of $C(t,x,\partial_t,\bigtriangledown_x)$ is defined as (cf. [1-3])

$$\begin{aligned}L(\theta,\lambda) &= [x^{-\lambda}t^{-\theta}C(t,x,\partial_t,\bigtriangledown_x)t^\theta x^\lambda]|_{(t,x)=(0,0)}\\ &= \theta-b(0)-\sum_{j=1}^{n}c_j(0)\lambda_j,\end{aligned}$$

where $\theta\in\mathbb{C}$, and $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n\mathbb{C}^n$, $x^\lambda=x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$.

Furthermore, we suppose

(H4) There exists a $\sigma > 0$, such that for any $(k, \alpha) \in \mathbf{N} \times \mathbf{Z}_+^n$, we have

$$|L(k, \alpha)| \geq \sigma(1 + |\alpha|).$$

We have the following result:

Theorem 1. *Under the conditions (H1), (H2), (H3) and (H4), the equation (1) has a unique holomorphic solution $u(t, x)$ near $(0, 0) \in \mathbf{C}_t \times \mathbf{C}_x^n$ with $u(0, x) \equiv 0$ near $x = 0$.*

Remark 1. Chen-Tahara [8] has studied a special case for non-totally characteristic PDE with one space variable $x \in \mathbf{C}^1$. Observe the situation with several space variables will be a non-trivial extension. Indeed we can not use the mathod in [8] directly, we use here the new idea to prove the result of Theorem 1. The result in the paper is new even in the case of space dimension $n = 1$. Actually the result in [8] can be easily deduced by Theorem 1 here.

§2 Proof of Main Result.

First we take a formal series

$$u(t, x) = \sum_{k=1}^{\infty} u_k(x) t^k. \tag{3}$$

And then introducing (3) into the equation (2) and comparing the coefficients of t^k in both sides of the equation, we have, for $k = 1$,

$$\left(1 - b(x) - \sum_{j=1}^{n} c_j(x) \left(x_j \frac{\partial}{\partial x_j}\right)\right) u_1 = a(x) \tag{4}$$

and for $k \geq 2$,

$$\begin{aligned} & \left(k - b(x) - \sum_{j=1}^{n} c_j(x) \left(x_j \frac{\partial}{\partial x_j}\right)\right) u_k \\ = & \sum_{2 \leq p+q+|\gamma| \leq k} a_{p,q,\gamma}(x) \sum_{(C1)} u_{m_1} \times \cdots \times u_{m_q} \times \partial_{x_1} u_{n_1^{(1)}} \times \cdots \\ & \times \partial_{x_1} u_{n_{\gamma_1}^{(1)}} \times \cdots \times \partial_{x_n} u_{n_1^{(n)}} \times \cdots \times \partial_{x_n} u_{n_{\gamma_n}^{(n)}} \end{aligned} \tag{5}$$

where $(C1)$ denotes a subset of $\mathbf{N} \times \mathbf{N}^q \times \mathbf{N}^{|\gamma|}$, in which $p + m_1 + \cdots + m_q + n_1^{(1)} + \cdots + n_{\gamma_1}^{(1)} + \cdots + n_1^{(n)} + \cdots + n_{\gamma_n}^{(n)} = k$.

We shall solve (4) and (5) formally in formal power series ring $\mathbb{C}[[x]]$ (cf. [7]) to get the formal solution of (1). Thus we expand $a(x)$, $b(x)$, $c_j(x)$ and $a_{p,q,\gamma}(x)$ into Taylor series in x :

$$\begin{cases} a(x) = \sum_{\alpha\in\mathbf{Z}_+^n} a_\alpha x^\alpha, \quad b(x) = \sum_{\alpha\in\mathbf{Z}_+^n} b_\alpha x^\alpha, \\ c_j(x) = \sum_{\alpha\in\mathbf{Z}_+^n} c_{j,\alpha} x^\alpha, \quad a_{p,q,\gamma}(x) = \sum_{\alpha\in\mathbf{Z}_+^n} a_{p,q,\gamma}^{(\alpha)} x^\alpha. \end{cases}$$

Also, expand unknown functions $u_k(x)$ $(k \geq 1)$ as a formal power series in x :

$$u_k(x) = \sum_{\alpha\in\mathbf{Z}_+^n} u_{k,\alpha} x^\alpha.$$

Then the equation (4) is equivalent to

$$\left(1 - b(0) - \sum_{j=1}^n \alpha_j c_j(0)\right) u_{1,\alpha} = a_\alpha + \sum_{\beta<\alpha} b_{\alpha-\beta} u_{1,\beta} + \sum_{j=1}^n \sum_{\beta<\alpha} \beta_j c_{j,\alpha-\beta} u_{1,\beta}, \quad \text{for any } \alpha \in \mathbf{Z}_+^n, \tag{4'}$$

where we know that only $u_{1,\beta}$ $(\beta < \alpha)$ appear in the right hand side of $(4')$. Since $\left(1 - b(0) - \sum_{j=1}^n \alpha_j c_j(0)\right) \neq 0$ (see (H4)), we can get $u_1(x) \in \mathbb{C}[[x]]$ from $(4')$, which is a unique formal solution of (4).

Moreover the equation (5) becomes (for $k \geq 2$) :

$$\begin{aligned} &\left(k - b(0) - \sum_{j=1}^n \alpha_j c_j(0)\right) u_{k,\alpha} \\ = &\sum_{\beta<\alpha} b_{\alpha-\beta} u_{k,\beta} + \sum_{j=1}^n \sum_{\beta<\alpha} \beta_j c_{j,\alpha-\beta} u_{k,\beta} \\ &+ \sum_{\substack{2\leq p+q+|\gamma|\leq k \\ \beta\leq\alpha}} a_{p,q,\gamma}^{(\beta)} \sum_{(C2)} u_{m_1,k_1} \times \cdots \times u_{m_q,k_q} \times ((m_{1,1}^{(1)}+1) u_{n_1^{(1)}, m_1^{(1)}+e_1}) \\ &\times \cdots \times ((m_{\gamma_1,1}^{(1)}+1) u_{n_{\gamma_1}^{(1)}, m_{\alpha_1}^{(1)}+e_1}) \times \cdots \times ((m_{1,n}^{(n)}+1) u_{n_1^{(n)}, m_1^{(n)}+e_n}) \\ &\times \cdots \times ((m_{\gamma_n,n}^{(n)}+1) u_{n_{\gamma_n}^{(n)}, m_{\gamma_n}^{(n)}+e_n}) \\ &\qquad \text{for any } \alpha \in \mathbf{Z}_+^n, \end{aligned} \tag{5'}$$

where $(C2)$ denotes a subset of $\mathbf{N} \times \mathbf{N}^q \times \mathbf{N}^{|\gamma|} \times \mathbf{Z}_+^{n(1+q+|\gamma|)}$, in which

$$\begin{aligned} p + m_1 + \cdots + m_q + n_1^{(1)} + \cdots + n_{\gamma_1}^{(1)} + \cdots + n_1^{(n)} + \cdots + n_{\gamma_n}^{(n)} &= k, \\ \beta + k_1 + \cdots + k_q + m_1^{(1)} + \cdots + m_{\gamma_1}^{(1)} + \cdots + m_1^{(n)} + \cdots + m_{\gamma_n}^{(n)} &= \alpha, \end{aligned}$$

and $e_j = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbf{Z}_+^n$ is j-th unit vector, and $m_j^{(k)} = (m_{j,1}^{(k)}, \cdots, m_{j,n}^{(k)})$.

From the condition in Theorem 1, we know $\left(k - b(0) - \sum_{j=1}^{n} \alpha_j c_j(0)\right) \neq 0$ (for any $(k,\alpha) \in \mathbf{N} \times \mathbf{Z}_+^n$) and only $\{u_{i,\beta}; 1 \le i \le k-1,\ \beta \in \mathbf{Z}_+^n\}$ and $\{u_{k,\beta},\ \beta < \alpha\}$ appear in the right hand side of $(5')$. So we can also solve $(5')$ inductively and get a unique formal solutions $u_k(x) \in \mathbb{C}[[x]]$ (for $k \ge 2$). Observe $u(t,x) = \sum_{(k,\alpha)\in \mathbf{N}\times\mathbf{Z}_+^n} u_{k,\alpha} t^k x^\alpha$ is a formal series solution of (1). It remains to prove the convergence of $u(t,x)$ near $(0,0)$.

Lemma 1. *The condition (H4) is equivalent to the following condition:*
(H4′) There exists a constant σ', such that for any $k \in \mathbf{N}$ and $\alpha \in \mathbf{Z}_+^n$, we have

$$\left| k - b(0) - \sum_{j=1}^{n} c_j(0)\alpha_j \right| \ge \sigma'(k + 1 + |\alpha|).$$

Proof: Observe the condition (H4′) implies the condition (H4). We only need to prove that (H4) implies (H4′).

We set $M = 1 + |\mathrm{Re} b(0)| + \sum_{j=1}^{n} |\mathrm{Re} c_j(0)|$, then if $k \ge 2M(|\alpha|+1)$, we have

$$\left| k - b(0) - \sum_{j=1}^{n} c_j(0)\alpha_j \right| \ge \frac{k}{2} \ge \frac{1}{4}(k + 1 + |\alpha|);$$

if $k < 2M(|\alpha|+1)$, we have

$$\begin{aligned} \left| k - b(0) - \sum_{j=1}^{n} c_j(0)\alpha_j \right| &\ge \sigma(1 + |\alpha|) \\ &\ge \tfrac{\sigma}{3M}(3M + 3M|\alpha|) \\ &\ge \tfrac{\sigma}{3M}(2M(1+|\alpha|) + 1 + |\alpha|) \\ &\ge \tfrac{\sigma}{3M}(k + 1 + |\alpha|). \end{aligned}$$

Set $\sigma' = \min\{\frac{1}{4}, \frac{\sigma}{3M}\}$, then we have

$$\left| k - b(0) - \sum_{j=1}^{n} c_j(0)\alpha_j \right| \ge \sigma'(k + 1 + |\alpha|).$$

Lemma 1 is proved.

From Lemma 1, we can define $U_{1,\alpha}$ (for $\alpha \in \mathbf{Z}_+^n$) as follows:

$$\begin{cases} U_{1,0} = \dfrac{1}{2\sigma'}|a_0|, \\ \text{for } \alpha > 0, \alpha \in \mathbf{Z}_+^n, \\ U_{1,\alpha} = \dfrac{1}{\sigma'(2+|\alpha|)}\left(|a_\alpha| + \sum_{\beta<\alpha} |b_{\alpha-\beta}| U_{1,\beta} + \sum_{j=1}^{n}\sum_{\beta<\alpha} \beta_j |c_{j,\alpha-\beta}| U_{1,\beta}\right), \end{cases}$$

where the constant $\sigma' > 0$ appeared in the condition (H4′).

Similarly, for $k \geq 2$, we define $U_{k,\alpha}$ (for $\alpha \in \mathbf{Z}_+^n$) by following recursive formula:

$$\begin{aligned} U_{k,\alpha} &= \frac{1}{\sigma'(k+1+|\alpha|)}\Big(\sum_{\beta<\alpha}|b_{\alpha-\beta}|U_{k,\beta} + \sum_{j=1}^{n}\sum_{\beta<\alpha}\beta_j|c_{j,\alpha-\beta}|U_{k,\beta} \\ &\quad + \sum_{2\leq p+q+|\gamma|\leq k}|a^{(\beta)}_{p,q,\gamma}|\sum_{(C2)}U_{m_1,k_1}\times\cdots\times U_{m_q,k_q}\times\cdots\times \\ &\quad ((m^{(1)}_{1,1}+1)U_{n^{(1)}_1,m^{(1)}_1+e_1})\times\cdots\times((m^{(1)}_{\gamma_1,1}+1)U_{n^{(1)}_{\gamma_1},m^{(1)}_{\alpha_1}+e_1})\times\cdots \\ &\quad \times((m^{(n)}_{1,n}+1)U_{n^{(n)}_1,m^{(n)}_1+e_n})\times\cdots\times((m^{(n)}_{\gamma_n,n}+1)U_{n^{(n)}_{\gamma_n},m^{(n)}_{\gamma_n}+e_n})\Big) \\ &= \frac{1}{\sigma'(k+1+|\alpha|)}\Big(\sum_{\beta<\alpha}|b_{\alpha-\beta}|U_{k,\beta} + \sum_{j=1}^{n}\sum_{\beta<\alpha}\beta_j|c_{j,\alpha-\beta}|U_{k,\beta} + g^{(\alpha)}_{k-1}\Big) \\ &\qquad \text{for any}\quad \alpha\in\mathbf{Z}^n_+, \end{aligned} \tag{6}$$

Then comparing with (4′) and (5′), we can easily deduce

Lemma 2. *Let $u_{k,\alpha}$ and $U_{k,\alpha}$ be defined as above, then we have*
(1) For any $(k,\alpha) \in \mathbf{N}\times\mathbf{Z}^n_+$, we have

$$|u_{k,\alpha}| \leq U_{k,\alpha}.$$

(2) Set

$$U(t,x) = \sum_{(k,\alpha)\in\mathbf{N}\times\mathbf{Z}^n_+} U_{k,\alpha}t^k x^\alpha. \tag{6}$$

Then $U = U(t,x)$ is the unique formal solution of the following equation:

$$\begin{aligned} \sigma' t\partial_t U &= A(x)t + [-\sigma' + B(x)]U + \sum_{j=1}^{n}[-\sigma' + C_j(x)]x_j\partial_{x_j}U \\ &\qquad + \sum_{p+q+|\gamma|\geq 2} A_{p,q,\gamma}(x)t^pU^q(\partial_x U)^\gamma, \end{aligned} \tag{7}$$

where

$$\begin{cases} A(x) = \sum_{\alpha\in\mathbf{Z}^n_+}|a_\alpha|x^\alpha, \qquad B(x) = \sum_{\alpha>0}|b_\alpha|x^\alpha, \\ C_j(x) = \sum_{\alpha>0}|c_{j,\alpha}|x^\alpha, \qquad A_{p,q,\gamma}(x) = \sum_{\alpha\in\mathbf{Z}^n_+}|a^{(\alpha)}_{p,q,\gamma}|x^\alpha. \end{cases}$$

So it would be enough if we can prove the convergence of $U(t,x)$. Let us rewrite

$U(t,x)$ as $U(t,x) = \sum_{k\in\mathbf{N}} U_k(x)t^k$ and from the equation (7), we have

$$\begin{aligned} \Big[\sigma' + \sigma' - B(x) + \sum_{j=1}^{n}\Big(\sigma' - C_j(x)\Big)x_j\partial_{x_j}\Big]U_1(x) &= A(x), \\ &\vdots \\ \Big[k\sigma' + \sigma' - B(x) + \sum_{j=1}^{n}\Big(\sigma' - C_j(x)\Big)x_j\partial_{x_j}\Big]U_k(x) &= g_{k-1}(x), \\ &\vdots \end{aligned} \tag{8}$$

where we also denote $g_0(x) = A(x)$, and for $k \geq 2$, $g_{k-1}(x) = \sum_{\alpha\in\mathbf{Z}_+^n} g_{k-1}^{(\alpha)}x^\alpha$ is a holomorphic function near $x = 0$. Actually, from following lemmas, we can prove the formal series solution $U(t,x)$, as mentioned in Lemma 2, is convergent near $(0,0)$.

Lemma 3 *For any $k \geq 1$, the formal solution $U_k(x)$ is a holomorphic function near $x = 0$, and there exist constant $C > 0$ and $R > 0$ small enough, such that for any $k \in \mathbf{N}$,*

$$\|U_k\|_R \leq \frac{C}{k}\|g_{k-1}\|_R,$$

where $\|f\|_R = \max_{|x_j|\leq R, 1\leq j\leq n}|f(x)|$.

Proof: From the definition of $U_{k,\alpha}$, we have

$$\begin{aligned} U_{k,\alpha} &= \frac{1}{\sigma'(k+1+|\alpha|)}\Big(\sum_{\beta<\alpha}|b_{\alpha-\beta}|U_{k,\beta} + \sum_{j=1}^{n}\sum_{\beta<\alpha}|c_{j,\alpha-\beta}|\beta_j U_{k,\beta} + g_{k-1}^{(\alpha)}\Big) \\ &\leq \frac{1}{\sigma'}\Big(\sum_{\beta<\alpha}|b_{\alpha-\beta}|U_{k,\beta} + \sum_{j=1}^{n}\sum_{\beta<\alpha}|c_{j,\alpha-\beta}|U_{k,\beta} + \frac{1}{k}g_{k-1}^{(\alpha)}\Big) \end{aligned}$$

which implies,

$$U_k(x) \ll G(x)U_k(x) + \frac{1}{\sigma' k}g_{k-1}(x), \tag{9}$$

where $g(x) \ll f(x)$ means $f(x)$ is a majorant series of $g(x)$ near $x = 0$, i.e. $|\partial_x^\beta g(0)| \leq \partial_x^\beta f(0)$; $G(x) = \frac{1}{\sigma'}\Big(B(x) + \sum_{j=1}^{n} C_j(x)\Big)$. Since $G(0) = 0$ and $g_{k-1}(x)$ is a holomorphic function near $x = 0$, then from (9) we can deduce that $U_k(x)$ is a holomorphic function near $x = 0$, and there exist $R > 0$ and $C > 0$, such that

$$\|U_k(x)\|_R \leq \frac{C}{k}\|g_{k-1}(x)\|_R, \quad \text{for any } k \in \mathbf{N}.$$

The proof of Lemma 3 is completed.

Lemma 4 *Let $R > 0$ and $f(x)$ be a holomorphic function on $D_R^n = \{x \in \mathbb{C}^n |\ \ |x_j| \leq R, 1 \leq j \leq n\}$. For any $r, 0 < r < R$, if $f(x)$ satisfies*

$$\max_{x \in D_r^n} |f(x)| \leq \frac{c}{(R-r)^a},$$

for some $c \geq 0$ and $a \geq 0$, then we have

$$\max_{x \in D_r^n} \left| \frac{\partial f}{\partial x_j}(x) \right| \leq \frac{(a+1)ec}{(R-r)^{a+1}}, \quad \text{for any } j\ (1 \leq j \leq n) \text{ and } r \in (0, R). \tag{10}$$

Proof: See [9, Lemma 5.1.3].

Now let us prove the convergence of the formal seires solution $U(t, x)$. Let $0 < R < 1$ small enough, such that

(i) $A_{p,q,\gamma}(x)$ is holomorphic on D_R^n;

(ii) $|A_{p,q,\gamma}(x)| \leq A_{p,q,\gamma}$ on D_R^n;

(iii) $\displaystyle\sum_{p+q+|\gamma| \geq 2} A_{p,q,\gamma} t^p u^q v^\gamma$ is a convergent power series in (t, u, v).

We choose $A > 0$, such that on D_R^n,

$$|U_1(x)| \leq A \ \text{ and } \ |\partial_{x_j} U_1(x)| \leq eA, \quad 1 \leq j \leq n.$$

Now we introduce a function $Y(t)$, satisfying the following equation:

$$Y = At + \frac{C}{R-r} \sum_{p+q+|\gamma| \geq 2} \frac{A_{p,q,\gamma}}{(R-r)^{(p+q+|\gamma|-2)}} t^p Y^q (eY)^{|\gamma|}, \tag{11}$$

where r is a parameter with $0 < r < R$, $C > 0$ is the constant appeared in Lemma 3.

Since the equation (11) is an analytic functional equation in Y, by the implicit function theorem we can easily prove that the equation (11) has a unique holomorphic solution $Y(t)$ in a neighborhood of $t = 0$ with $Y(0) = 0$.

Expanding $Y(t)$ into Taylor series in t,

$$Y(t) = \sum_{k=1}^{\infty} Y_k t^k. \tag{12}$$

From the equation (11), we know that the coefficients of (12) can be given by

$$Y_1 = A,$$

and for $k \geq 2$,

$$\begin{aligned} Y_k \quad = \quad & \frac{C}{R-r} \sum_{p+q+|\gamma| \geq 2} \sum_{(C3)} \frac{A_{p,q,\gamma}}{(R-r)^{p+q+|\gamma|-2}} Y_{m_1} \times \cdots \times Y_{m_q} \times \\ & (eY_{n_1}) \times \cdots \times (eY_{n_{|\gamma|}}), \end{aligned} \tag{13}$$

where $(C3)$ means $m_1 + \cdots + m_q + n_1 + \cdots + n_{|\gamma|} = k - p$.

Moreover we can deduce that Y_k is of the form

$$Y_k = \frac{C_k}{(R-r)^{k-1}}, \quad \text{for} \quad k = 1, 2, \cdots \tag{14}$$

where $C_1 = A$, and the constant $C_k \geq 0$, for $k \geq 2$, can be decided inductively from the equation (13), which is independent of r. Actually from (13), it is easy to check that the order of $\frac{1}{R-r}$ is $k-1$, i.e. $1 + (p + q + |\gamma| - 2) + (n_1 - 1) + \cdots + (n_q - 1) + (m_1 - 1) + \cdots + (m_{|\gamma|} - 1) = k - 1$, so the formula (14) holds.

Next, we prove that the series $\sum_{k\geq 1} Y_k t^k$ is a majorant series for the formal series solution $\sum_{k\geq 1} U_k(x) t^k$ near $x = 0$. In fact, we can prove, by induction, that for any $k \geq 1$ and $0 < r < R$, we have

$$|U_k(x)| \leq |kU_k(x)| \leq Y_k, \quad \text{on } D_R^n; \tag{15}$$

$$\left| \frac{\partial U_k}{\partial x_j}(x) \right| \leq eY_k, \ (1 \leq j \leq n) \quad \text{on } D_R^n. \tag{16}$$

Actually, since $Y_1 = A$, the estimates (15) and (16) hold for $k = 1$. We suppose that $k \geq 2$, and for any $1 \leq i < k$, (15) and (16) hold for i. Since $g_{k-1}^{(\alpha)}$ is decided by (6), $g_{k-1}(x) = \sum_\alpha g_{k-1}^{(\alpha)} x^\alpha$ then from Lemma 3 and (6), we have by induction that

$$\begin{aligned} |U_k(x)| \ &\leq \ \frac{C}{k} \sum_{p+q+|\gamma|\geq 2} \sum_{(C1)} A_{p,q,\gamma} \times U_{m_1} \times \cdots \times U_{m_q} \times \partial_{x_1} U_{n_1^{(1)}} \times \cdots \\ &\qquad \times \partial_{x_1} U_{n_{\gamma_1}^{(1)}} \times \cdots \times \partial_{x_n} U_{n_1^{(n)}} \times \cdots \times \partial_{x_n} U_{n_{\gamma_n}^{(n)}} \\ &\leq \ \frac{C}{k} \sum_{p+q+|\gamma|\geq 2} \sum_{(C3)} A_{p,q,\gamma} \times Y_{m_1} \times \cdots \times Y_{m_q} \times \\ &\qquad (eY_{n_1}) \times \cdots \times \cdots \times (eY_{n_{|\gamma|}}). \end{aligned}$$

Since $0 < r < R < 1$, thus $(R-r)^{p+q+|\gamma|-2} < 1$, then we have

$$\begin{aligned} |U_k(x)| \ &\leq \ \frac{C}{k} \sum_{p+q+|\gamma|\geq 2} \sum_{(C3)} \frac{A_{p,q,\gamma}}{(R-r)^{p+q+|\gamma|-2}} \times Y_{m_1} \times \cdots \\ &\qquad \times Y_{m_q} \times (eY_{n_1}) \times \cdots \times \cdots \times (eY_{n_{|\gamma|}}). \end{aligned}$$

From the formula (13) and (14), we have

$$|U_k(x)| \leq \frac{R-r}{k} Y_k = \frac{C_k}{k} \cdot \frac{1}{(R-r)^{k-2}}.$$

Thus

$$|U_k(x)| \leq |kU_k(x)| \leq \frac{C_k}{(R-r)^{k-2}} \leq \frac{C_k}{(R-r)^{k-1}} = Y_k,$$

the estimate (15) holds for k.

Next, by using Lemma 4, we have

$$\left|\frac{\partial U_k}{\partial x_j}(x)\right| \le \frac{k-1}{k}\cdot\frac{eC_k}{(R-r)^{k-1}} \le eY_k,$$

this implies the estimate (16) holds for k. Therefore we have proved that $\sum_{k\ge 1} Y_k t^k$ is the majorant series of the formal series solution $U(t,x)$ near $x=0$, which implies, by Lemma 2, that the formal series solution (3) is convergent near $(0,0)\in \mathbb{C}_t\times\mathbb{C}_x^n$, Theorem 1 is proved.

§3 Case of Higher Order Singular PDE

In this section, we shall extend the result of Theorem 1 to the case of higher order singular partial differential equation:

$$(t\partial_t)^m u = F\Big(t,x,\{(t\partial_t)^j\partial_x^\alpha u\}_{(j,\alpha)\in\mathcal{F}}\Big),\quad (t,x)\in\mathbb{C}_t\times\mathbb{C}_x^n, \tag{17}$$

where $\mathcal{F}=\{(j,\alpha)\ |j+|\alpha|\le m,\ j<m\}$.

Now we denote $(t\partial_t)^j\partial_x^\alpha u$ by notation $Z_{j,\alpha}$, i.e.

$$(t\partial_t)^j\partial_x^\alpha u \leftrightarrow Z_{j,\alpha},\quad\text{and } \{(t\partial_t)^j\partial_x^\alpha u\}_{(j,\alpha)\in\mathcal{F}} \leftrightarrow Z=\{Z_{j,\alpha}\}_{(j,\alpha)\in\mathcal{F}}.$$

For the function $F(t,x,Z)$, we suppose

(A1) $F(t,x,Z)$ is a holomorphic function in a neighborhood of origin $(0,0,0)\in \mathbb{C}_t\times\mathbb{C}_x^n\times\mathbb{C}^N$, where $N=\#\mathcal{F}$;

(A2) $F(0,x,0)\equiv 0$, near $x=0$;

(A3) $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0,x,0)=x^\alpha b_{j,\alpha}(x)$, and $b_{j,\alpha}(x)$ is a holomorphic function near $x=0$;

Thus we can rewrite $F(t,x,Z)$ as

$$F(t,x,Z)=a(x)t+\sum_{(j,\alpha)\in\mathcal{F}} x^\alpha b_{j,\alpha}(x)Z_{j,\alpha}+\sum_{p+|\gamma|\ge 2} a_{p,\gamma}t^pZ^\gamma,$$

where $a(x)=\frac{\partial F}{\partial t}(0,x,0)$.

Actually, if we denote

$$C(t,x,\{(t\partial_t)^j\partial_x^\alpha\}_{(j,\alpha)\in\mathcal{F}})=(t\partial_t)^m-\sum_{(j,\alpha)\in\mathcal{F}} x^\alpha b_{j,\alpha}(x)(t\partial_t)^j\partial_x^\alpha,$$

the equation (17) can be rewriten as

$$C(t,x,\{(t\partial_t)^j\partial_x^\alpha\}_{(j,\alpha)\in\mathcal{F}})u=a(x)t+\sum_{p+|\gamma|\ge 2} a_{p,\gamma}(x)t^p(\{(t\partial_t)^j\partial_x^\alpha u\}_{(j,\alpha)\in\mathcal{F}})^\gamma.$$

And the indicial polynomial of $C(t,x,\{(t\partial_t)^j\partial_x^\alpha\}_{(j,\alpha)\in\mathcal{F}})$ is defined as

$$\begin{aligned} L(\theta,\lambda) &= [x^{-\lambda}t^{-\theta}C(t,x,\{(t\partial_t)^j\partial_x^\alpha\}_{(j,\alpha)\in\mathcal{F}})t^\theta x^\lambda]|_{(t,x)=(0,0)} \\ &= \theta^m - \sum_{(j,\alpha)\in\mathcal{F}} b_{j,\alpha}(0)\theta^j \prod_{l=1}^{n}\Big(\prod_{m=1}^{\alpha_l}(\lambda_l-\alpha_l+m)\Big) \end{aligned}$$

where $(\theta,\lambda)\in\mathbf{C}_\theta\times\mathbf{C}_\lambda^n$.

Furthermore, we suppose

(A4) There exist a constant $\sigma>0$, such that for any $(k,\beta)\in\mathbf{N}\times\mathbf{Z}_+^n$, we have

$$|L(k,\beta)|\geq\sigma(1+|\beta|^m).$$

Similar to Lemma 1, we have

Lemma 5. *The condition (A4) is equivalent to the following condition:*
(A4′) There exist a constant $\sigma'>0$, such that for any $k\in\mathbf{N},\beta\in\mathbf{Z}_+^n$, we have

$$\Big|k^m-\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta} b_{j,\alpha}(0)k^j\frac{\beta!}{(\beta-\alpha)!}\Big|\geq\sigma'\Big(k^m+\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta}k^j\frac{\beta!}{(\beta-\alpha)!}\Big).$$

Proof: Here, we only prove (A4) implies (A4′).

We set $M=1+\sum_{(j,\alpha)\in\mathcal{F}}(1+|\mathrm{Re}b_{j,\alpha}(0)|)$, then we have

(a). for $k\geq 2M(|\beta|+1)$, we have

$$\begin{aligned} \Big|k^m-\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta} b_{j,\alpha}(0)k^j\frac{\beta!}{(\beta-\alpha)!}\Big| &\geq \frac{k^m}{2} \\ &\geq \frac{1}{4}(k^m+Mk^{k-1}|\beta|) \\ &\geq \frac{1}{4}\Big(k^m+\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta}k^j\frac{\beta!}{(\beta-\alpha)!}\Big); \end{aligned}$$

(b). for $k<2M(|\beta|+1)$, $N=\#\mathcal{F}$, we have

$$\begin{aligned} &\Big|k^m-\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta} b_{j,\alpha}(0)k^j\frac{\beta!}{(\beta-\alpha)!}\Big| \\ \geq\ & \sigma(1+|\beta|^m) \\ \geq\ & \tfrac{\sigma}{2N(4M)^m}(2N(4M)^m(1+|\beta|^m)+(2M)^m(2^m+(2|\beta|)^m)) \\ \geq\ & \tfrac{\sigma}{2N(4M)^m}(2N(4M)^m(1+|\beta|^m)+(2M)^m(1+|\beta|)^m) \\ \geq\ & \tfrac{\sigma}{2N(4M)^m}(2N(4M)^m(1+|\beta|^m)+k^m) \\ \geq\ & \tfrac{\sigma}{2N(4M)^m}\Big(k^m+\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta}k^j\frac{\beta!}{(\beta-\alpha)!}\Big). \end{aligned}$$

So we take $\sigma' = \min\{\frac{1}{4}, \frac{\sigma}{2N(4M)^m}\}$, then

$$\left|k^m - \sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta} b_{j,\alpha}(0)k^j \frac{\beta!}{(\beta-\alpha)!}\right| \geq \sigma'\left(k^m + \sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta} k^j \frac{\beta!}{(\beta-\alpha)!}\right).$$

Lemma 5 is proved.

The following is main result in this section.

Theorem 2. *Under the conditions (A1), (A2), (A3) and (A4), the equation (17) has a unique holomorphic solution* $u(t,x)$ *near* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$ *with* $u(0,x) \equiv 0$ *near* $x = 0$.

The proof of Theorem 2 is similar to the proof of Theorem 1. First we can expand $a(x)$, $b_{j,\alpha}(x)$, $a_{p,\gamma}(x)$ into Taylor series, i.e.

$$\begin{cases} a(x) = \sum_{\beta\in\mathbf{Z}_+^n} a_\beta x^\beta, \qquad b_{j,\alpha}(x) = \sum_{\beta\in\mathbf{Z}_+^n} b_{j,\alpha}^{(\beta)} x^\beta, \\ a_{p,\gamma}(x) = \sum_{\beta\in\mathbf{Z}_+^n} a_{p,\gamma}^{(\beta)} x^\beta. \end{cases}$$

Then, as similar to $(4')$ and $(5')$, we can obtain the unique formal solution (of equation (17)) $u(t,x) = \sum_{k\in\mathbf{N},\beta\in\mathbf{Z}_+^n} u_{k,\beta} t^k x^\beta$. And next we can construct a formal series $U(t,x) = \sum_{k\in\mathbf{N},\beta\in\mathbf{Z}_+^n} U_{k,\beta} t^k x^\beta$, which is a majorant series of $u(t,x)$ near $(0,0)$ and satisfies the following equation:

$$\begin{aligned} \sigma'(t\partial_t)^m U &= A(x)t + \sum_{(j,\alpha)\in\mathcal{F}} \left[(-\sigma' + B_{j,\alpha}(x))x^\alpha (t\partial_t)^j \partial_x^\alpha U\right] + \\ &\quad \sum_{p+|\gamma|\geq 2} A_{p,\gamma}(x) t^p \prod_{(j,\alpha)\in\mathcal{F}} \left((t\partial_t)^j \partial_x^\alpha U\right)^{\gamma_{j,\alpha}}, \end{aligned} \tag{18}$$

where

$$A(x) = \sum_{\beta\in\mathbf{Z}_+^n} |a_\beta| x^\beta, \quad B_{j,\alpha}(x) = \sum_{\beta>0} |b_{j,\alpha}^{(\beta)}| x^\beta, \quad A_{p,\gamma}(x) = \sum_{\beta\in\mathbf{Z}_+^n} |a_{p,\gamma}^{(\beta)}| x^\beta.$$

Thus we only need to prove the convergence of $U(t,x)$ near $(0,0)$. If we rewrite $U(t,x)$ as $U(t,x) = \sum_{k\in\mathbf{N}} U_k(x) t^k$, and introduce this formal solution into (18), we have

$$
\begin{aligned}
\Big[\sigma' + \sum_{(j,\alpha)\in\mathcal{F}} (\sigma' - B_{j,\alpha}(x))x^\alpha \partial_x^\alpha\Big] U_1(x) &= A(x), \\
&\vdots \\
\Big[\sigma' k^m + \sum_{(j,\alpha)\in\mathcal{F}} (\sigma' - B_{j,\alpha}(x))x^\alpha \partial_x^\alpha\Big] U_k(x) &= g_{k-1}(x), \\
&\vdots
\end{aligned}
\tag{19}
$$

where for $k \geq 2$, $g_{k-1}(x) = g_{k-1}(U_1, \cdots, U_{k-1}, \{(t\partial_t)^j \partial_x^\alpha U_l\}_{\substack{(j,\alpha)\in\mathcal{F} \\ 1\leq l\leq k-1}}) = \sum_{\beta\in\mathbf{Z}_+^n} g_{k-1}^{(\beta)} x^\beta$, and $g_0(x) = A(x)$.

From (19), We can solve $U_k(x)$ uniquely, which is holomorphic near $x = 0$. In fact, we have

Lemma 6 *For any $k \geq 1$, the formal solution $U_(x)$ is a holomorphic function near $x = 0$, and meanwhile there exist constants $C > 0$ and $R > 0$ small enough, such that for any $k \in \mathbf{N}$,*

$$\|U_k\|_R \leq \frac{C}{k^m}\|g_{k-1}\|_R,$$

Proof: From equation (18), we deduce

$$
\begin{aligned}
U_{k,\beta} &= \frac{1}{\sigma'\Big(k^m + \sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\beta} k^j \frac{\beta!}{(\beta-\alpha)!}\Big)} \times \\
&\qquad \Big(\sum_{(j,\alpha)\in\mathcal{F},\alpha\leq\mu<\beta} |b_{j,\alpha}^{(\beta-\mu)}| k^j \frac{\mu!}{(\mu-\alpha)!} U_{k,\mu} + g_{k-1}^{(\beta)}\Big) \\
&\leq \frac{1}{\sigma'}\Big(\sum_{(j,\alpha)\in\mathcal{F},\mu<\beta} |b_{j,\alpha}^{(\beta-\mu)}| U_{k,\mu} + \frac{1}{k^m} g_{k-1}^{(\beta)}\Big),
\end{aligned}
$$

which implies,

$$U_k(x) \ll G(x)U_k(x) + \frac{1}{\sigma' k^m} g_{k-1}(x),$$

where $g_0(x) = A(x)$, $G(x) = \frac{1}{\sigma'}\Big(\sum_{(j,\alpha)\in\mathcal{F}} B_{j,\alpha}(x)\Big)$, and $G(0) = 0$. Thus we can solve $U_k(x)$, which is a holomorphic function near $x = 0$, and satisfies

$$\|U_k(x)\|_R \leq \frac{C}{k^m}\|g_{k-1}(x)\|_R, \quad \text{for any } k \in \mathbf{N}. \tag{20}$$

Lemma 6 is proved.

Now let us prove the convergence of formal solution of the equation (18). We let $0 < R < 1$ small enough, such that

(i) $A_{p,\gamma}(x)$ is holomorphic on D_R^n;

(ii) $|A_{p,\gamma}(x)| \le A_{p,\gamma}$ on D_R^n;

(iii) $\sum_{p+|\gamma|\ge 2} A_{p,\gamma} t^p Z^\gamma$ is a convergent power series in (t, Z).

Then we choose $A > 0$, such that on D_R^n,

$$|(t\partial_t)^j \partial_x^\alpha U_1(x)| \le (me)^m A, \quad \text{for any } (j,\alpha) \in \mathcal{F}.$$

Next we introduce a function $Y(t)$, satisfying the following equation:

$$Y = At + \frac{C}{(R-r)^m} \sum_{p+|\gamma|\ge 2} \frac{A_{p,\gamma}}{(R-r)^{m(p+|\gamma|-2)}} t^p (BY)^{|\gamma|}, \tag{21}$$

where r is a parameter with $0 < r < R, C > 0$ is the constant appeared in the estimate (20), and $B = (me)^m$.

Similar to the proof in section 2, we know that the equation (21) has a unique holomorphic solution $Y(t)$ in a neighborhood of $t = 0$ with $Y(0) = 0$.

Expanding $Y(t)$ as a Taylor series in t,

$$Y(t) = \sum_{k=1}^{\infty} Y_k t^k, \tag{22}$$

then by the same argument as in the proof of Theorem 1, we can obtain, for any $k \ge 1$,

$$\left| k^j \partial_x^\alpha U_k(x) \right| \le (me)^{|\alpha|} Y_k \le BY_k \ \text{ on } \ D_R^n, \quad \text{for any } (j,\alpha) \in \mathcal{F}.$$

This implies that $Y(t) = \sum_{k\ge 1} Y_k t^k$ is a majorant series of the formal solution $U(t,x) = \sum_{k\ge 1} U_k(x) t^k$ near $x = 0$. Theorem 2 is prove.

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