On the Existence of Smooth Solutions of the Dirichlet Problem for Hyperbolic Differential Equations

B. Paneah, B.-W. Schulze

Preprint 98/5

February 1988

Universitat Potsdam

Institut fur Mathematik

Postfach 60 15 53

14415 Potsdam

On the existence of smooth solutions of the Dirichlet problem for hyperbolic differential equations

B. Paneah, B.-W. Schulze

Contents

1 Introduction

The Dirichlet problem for hyperbolic differential equations in a bounded domain is usually regarded as an "unnatural" problem of mathematical physics. Its solution may neither exist, nor be uniquely determined, nor depend continuously on the data. Nevertheless, beginning with the Thirties from time to time there appeared papers in which a character of this "unnaturalness" was investigated under various points of view (cf. [1], [5], [7]). One of the typical results of this kind is the following. Consider the equation $u_{xy} = 0$ in a rectangle with sides of slopes ± 1 and let ξ be the ratio of the sides of the rectangle. Then the solution of the Dirichlet problem is uniquely determined if and only if ξ is irrational. the solution exists for all sufficiently smooth boundary values if ξ can be approximated "sufficiently fast" by rationals [5].

On this background the following facts which have been established by one of the authors [6] turned out to be unexpected to some extent. For any linear 3rd-order hyperbolic differential operator $P(\partial_x, \partial_y)$ with constant coefficients and for a wide class of domains $D \subset \mathbb{R}^2$ intimately connected to the operator the Dirichlet problem

$$P(\partial_x, \partial_y)u = f \quad \text{in } D, \qquad u = g \quad \text{on } \partial D \tag{1.1}$$

is uniquely solvable for all functions $f \in C(\overline{D})$, $g \in C^2(\partial D)$, and the inverse operator of the problem is bounded.

If we say "solvability" here and in what follows we have in mind the existence of a generalized (in the sense of the distribution theory) solution $u(x, y) \in C^2(\overline{D})$.

In the case of third order hyperbolic equations the Dirichlet problem in special domains is motivated by equivalent interesting problems in integral geometry or functional equations, cf. [6]. These connections turn the problem into a natural one and there arise new questions in this context that require more complete information.

The main goal of the present paper is to describe conditions under which any generalized solution to problem (1.1) is a classical one.

2 Third order hyperbolic operators in the plane

In what follows we consider a linear differential operator $\mathcal{P}$ in the (x, y)-plane of the form

$$\mathcal{P}u = Pu + Qu,$$

where $P = P(\partial_x, \partial_y)$ is an arbitrary homogeneous x-hyperbolic operator of 3-rd order with constant coefficients, and $Q = Q(x, y, \partial_x, \partial_y)$ is an arbitrary smooth linear differential operator of second order. The x-hyperbolicity of the operator P means that the characteristic polynomial $P(\tau, \lambda)$ has, for any $\lambda \neq 0$, three different real roots in τ. It follows that for some real constants M, a_1, a_2, a_3 with $a_j \neq a_k$ for $j \neq k$

$$P(\tau, \lambda) = M(\tau - a_1\lambda)(\tau - a_2\lambda)(\tau - a_3\lambda).$$

In turn this means that straight lines

$$y - a_1x \equiv \text{const}, \qquad y - a_2x \equiv \text{const}, \qquad y - a_1x \equiv \text{const},$$

are characteristics of the operator $\mathcal{P}$. Denote by $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_6$, characteristic rays beginning at some point O. Choose any triple of neighboring rays $\mathcal{R}_j$, say $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. Let $\mathcal{R}_3$ be the ray lying between $\mathcal{R}_1$ and $\mathcal{R}_2$. Consider a curvilinear triangle D with sides $OR \subset \mathcal{R}_1$, $OQ \subset \mathcal{R}_2$ and $\Gamma = RQ$, where Γ is an supposed to satisfy the $\mathcal{R}_1\mathcal{R}_2$-*angle condition.* This means that if $q \in \Gamma$ and $q_1 \in \mathcal{R}_1$ (resp. $q_2 \in \mathcal{R}_2$) is the projection of q along $\mathcal{R}_2$ (resp. along $\mathcal{R}_1$) then the parallelogram Oq_1qq_2 lies in D. The Dirichlet problem that we treat is as follows:

Given functions $G \in C(\overline{D})$ and $h \in C(\partial D)$, find a solution of the boundary value problem

$$\mathcal{P}u = G \quad \text{in } D; \qquad u = h \quad \text{on } \partial D. \tag{2.1}$$

Denote by $C^k(\partial D)$, $k \geq 1$, the space of continuous functions on ∂D whose restrictions to OR, OQ and Q, R are k times differentiable.

The following result was obtained in the paper [6] by one of the authors:

Theorem 2.1 *Assume that the curve Γ has no points of tangency with characteristics which are parallel to $\mathcal{R}_1$ or $\mathcal{R}_2$. Then for any functions $G \in C(\overline{D})$ and $h \in C^2(\partial\overline{D})$, there exists a unique generalized solution $u(x, y)$ of the problem* (1.1). *The inverse operator: $(G, h) \to u$ is continuous: $C(\overline{D}) \times C^2(\partial D) \to C^2(\overline{D})$.*

However the existece of a *classical* solution $u(x, y) \in C^3(\overline{D})$ was given in [6] only in the case of a straight line Γ.

In the present paper we generalize this last result to a more general class of curves Γ.

3 Classical solutions to a hyperbolic Dirichlet problem

Theorem 3.1 *Assume that the curve* Γ *satisfies the conditions of Theorem* 2.1 *and that the curvature* $\mathcal{C}(\Gamma)$ *is sufficiently small. If* $G \in C^1(\overline{D})$ *and* $h \in C^3(\partial D)$, *then the generalized solution* $u(x,y)$ *of the problem* 1.1 *is the classical one and we have* $u \in C^3(\overline{D})$.

Proof. The proof consists of two parts. In part I we prowe Theorem 3.1 for the case $G = 0$. After this, in part II, we construct a solution of the problem (1.1) which belongs to the space $C^3(\overline{D})$ and vanishes on ∂D. We will restrict ourselves to the homogeneous operator $\mathcal{P} = P$. The general case can be considered in the framework of perturbation theory.

Part I. It is obvious that there exists such a linear change of variables in the space $\mathbb{R}^2$ which reduces the problem (1.1) (with $G = 0$) to the problem

$$(l_1\partial_x + l_2\partial_y)\partial_x\partial_y u = 0 \text{ in } D, \qquad u = h \text{ on } \partial D. \tag{3.1}$$

Here D is a domain in $\mathbb{R}^2$ whose boundary ∂D consists of three parts $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ such that

$$\begin{aligned}
\Gamma_1 &= \{(x,y) \mid y = 0, \quad 0 \le x \le 1\},\\
\Gamma_2 &= \{(x,y) \mid x = 0, \quad 0 \le y \le 1\},\\
\Gamma_3 &= \{(x,y) \mid x = x(t),\ y = y(t); \quad 0 \le t \le 1\},
\end{aligned}$$

where $l_1 > 0,\ l_2 > 0$ and

$$x(0) = 0, \quad x(1) = 1, \quad y(0) = 1, \quad y(1) = 0. \tag{3.2}$$

Let $h = h_1(x)$ on Γ_1, $h = h_2(y)$ on Γ_2 and $h = h_3(x,y)$ on Γ_3. The continuity of the function h leads to the natural compatibility conditions

$$h_1(0) = h_2(0), \quad h_1(1) = h_3(1,0), \quad h_2(1) = h_3(0,1). \tag{3.3}$$

Since the domain $D = \{(x,y) \mid 0 \le x \le x(t),\ 0 \le y \le y(t),\ 0 \le t \le 1\}$ is supposed to satisfy the $\mathcal{R}_1\mathcal{R}_2$- angle condition (with $\mathcal{R}_1$ as x- and $\mathcal{R}_2$ as y- axes), an arbitrary generalized solution $u \in C^2(\overline{D})$, satisfying the boundary conditions on $\Gamma_1 \cup \Gamma_2$ is nothing bit

$$u(x,y) = \int_0^x \left(\int_0^y F(m_1 s + m_2 t)dt \right) ds + h_1(x) + h_2(y) - h_1(0), \tag{3.4}$$

$(x,y) \in \overline{D}$. Here $m = (m_1, m_2)$ is the unit vector which is orthogonal to the vector $l = (l_1, l_2)$ and $m_1 > 0,\ m_2 < 0$. As for the function F, this is an arbitrary continuous function on the interval $I = (m_2, m_1)$. The necessity of satisfying the boundary condition $u = h_3$ on Γ_3 leads naturally to the following integral equation for the unknown function $F \in C(I)$

$$\int_0^{x(t)} \left(\int_0^{y(t)} F(m_1 x + m_2 y) dy \right) dx = H(t), \qquad 0 \le t \le 1, \tag{3.5}$$

where $H(t) = -h_1\big(x(t)\big) - h_2\big(y(t)\big) + h_3\big(x(t), y(t)\big) + j_1(0)$. What is essential here is that the function $H(t)$, generated by an arbitrary function $h \in C^3(\partial D)$, belongs to the space $C^3_0([0,1]) = (C^3 \cup C_0)([0,1])$. This follows from the compatibility conditions (3.3).

Thus, in order to prove that the function $u(x,y)$ (which is defined by (3.4)) belongs to the space $C^3(\overline{D})$ it is sufficient to prove that *if* $H(t) \in C^3([0,1])$, *then* $F \in C^1(I)$.

Introduce the new variable

$$z = m_1 x(t) + m_2 y(t), \qquad 0 \le t \le 1.$$

Since the domain D satisfies the angle condition, we have $x'(t) \ge 0$, $y'(t) \le 0$ for any $t \in [0,1]$. Therefore, $m_1 x'(t) + m_2 y'(t) > 0$ for all t, and there exists the inverse function

$$t = \sigma(z), \quad z \in I$$

for which $= \sigma'(z)$. Introduce the function

$$\delta(z) = m_1 (x \circ \sigma)(z), \qquad \varrho(z) = m_2 (y \circ \sigma)(z),$$

where $f \circ g$ denotes the composition of two maps f and g. The following properties of these functions which follow their definition will be important for us:

$$\delta(z) + \varrho(z) \equiv z, \quad 0 \le \delta(z) \le m_1, \quad m_2 \le \varrho(z) \le 0; \qquad 0 \le z \le 1.$$

Substituting $\sigma(z)$ instead of t in the equation (3.5) we get

$$\int_0^{\delta(z)/m_1} \left(\int_0^{\varrho(z)/m_2} F(m_1 x + m_2 y) dx \right) dy = \mathcal{H}(z), \qquad z \in I, \tag{3.7}$$

where $\mathcal{H} = H \circ \sigma \in C^3_0(I)$. According to conditions (3.2) the right and the left hand sides of this equation vanish at the ends of the interval I. This means that differentiating twice this equation we arrive at an *equivalent equation* that looks as follows

$$(3.8)\qquad F(z)-\delta'^2(z)(F\circ\delta)(z)-\varrho'^2(z)(F\circ\varrho)(z)+\delta''(z)\int\limits_{\varrho(z)}^{z}F(s)ds$$
$$+\varrho''(z)\int\limits_{\delta(z)}^{z}F(s)ds=\mathcal{H}''(z).$$

Denote by $T_k,\ k=1,2,\dots$, the linear operator in the space $C(I)$ given by

$$T_k: F\to\delta'^k(F\circ\delta)+\varrho'^k(F\circ\varrho).$$

Since the functions $\mathcal{H}''(z),\ \delta''(z),\ \varrho''(z)$ belong to the space $C^1(I)$ and $F(z)\in C(I)$, the function

$$\Phi=m_1m_2\mathcal{H}''-\delta''\int\limits_{\varrho}^{z}F(s)ds-\varrho''\int\limits_{\delta}^{z}F(s)ds$$

belongs to the space $C^1(I)$. Therefore, what remains to be checked is that the relation

$$(3.9)\qquad F-T_2F=\Phi,$$

implies $F\in C^1(I)$. In order to do this consider the "differentiated" equation (3.9):

$$(3.10)\qquad G-\delta'^3(G\circ\delta)-\varrho'^3(G\circ\varrho)-2\delta'\delta''\int\limits_0^{\delta}G(s)ds-2\varrho'\varrho''\int\limits_0^{\varrho}G(s)ds=\Phi'.$$

Denote by K the linear operator in $C(I)$

$$K: G\to 2\delta'\delta''\int\limits_0^{\delta}G(s)ds+-2\varrho'\varrho''\int\limits_0^{\varrho}G(s)ds.$$

Then the equation (3.10) takes the form

$$G-T_3G-KG=\Phi'.$$

Note that because the curvature $\mathcal{C}$ of the curve Γ equals

$$\mathcal{C}(z)=|\delta'(z)\varrho''(z)-\delta''(z)\varrho'(z)|/\left(\delta'^2(z)+\varrho'^2(z)\right)^{3/2},$$

from (3.6) it follows that

$$|\delta''(z)| = \mathcal{C}(z)\Big(\delta'^2(z) + \varrho'^2(z)\Big)^{3/2}.$$

This means that for any $\varepsilon > 0$ the relation

$$\mathcal{C}(z) < \max\Big(\delta'^2(z) + \varrho'^2(z)\Big)^{3/2}$$

entails

$$|\delta''(z)| < \varepsilon \qquad \text{for all } \; z \in I.$$

Since $\delta'(z) + \varrho'(z) \equiv 1$ and $\delta'(z)\varrho'(z) > 0$, we have

$$\sup\Big(\delta'^3(z) + \varrho'^3(z)\Big) < 1,$$

and consequently the norm of the operator T_3 is less than one. The same is also true for the operator $T_3 + K$ if the number ε is sufficiently small. Thus, the equation (3.10) has the unique solution $G \in C(I)$. It is clear that the function

$$\Psi = \int_0^z G(s)ds$$

satisfies the equation

$$\frac{d}{dx}(\Psi - T_2\Psi) = \Phi'$$

and consequently

$$\Psi - T_2\Psi = \Phi + m,$$

where m is a constant. Set $F - \Psi = \chi m$. Then the function χ belongs to the space $C(I)$ and satisfies the equation

$$\chi - T_2\chi = 1. \tag{3.11}$$

To prove that $F \in C^1(I)$ we have to check that $\chi \in C^1(I)$.

Since the norm of the operator T_2 in $C(I)$ is less than one, the solution χ of the equation (3.11) can be written in the form

$$\chi = \sum_{n=0}^{\infty} T_2^n 1. \tag{3.12}$$

The functional series in (3.12) converges uniformly on the interval I. It remains to prove that the differentiated series $\sum_{n=0}^{\infty}(T_2^n 1)'(z)$ also converges uniformly on I. Denote

$$(T_2 1)(z) = \delta'^2(z) + \varrho'^2(z) := r(z)$$

and let

$$\max_I r(z) = \overline{r}, \qquad 2\max_I |\delta''(z)| = \overline{r}'.$$

Since

$$|(T_2^n 1)(z)| = |T(T_2^{n-1}1)(z)| \le \overline{r}\max_I |T^{n-1}1|,$$

the inequality

$$\max_I |T_2^n 1| \le) = \overline{r}^n \tag{3.13}$$

holds. Further,

$$\frac{d}{dz}(T_2^n 1) = 2\delta'\delta''(T_2^{n-1}1)\circ\delta + 2\varrho'\varrho''(T_2^{n-1}1)\circ\varrho$$

$$+\delta'^3\left(\frac{d}{dz}T_2^{n-1}1\right)\circ\delta + \varrho'^3\left(\frac{d}{dz}T_2^{n-1}1\right)\circ\varrho.$$

Since $\delta' + \varrho' = 1$, $\delta'\varrho' > 0$ and $|\delta''| = |\varrho''|$ (cf. (6)) we get the inequality

$$\sup\left|\frac{d}{dz}T_2^n 1\right| \le \overline{r}\prime \sup|T_2^{n-1}1| + \overline{r}\sup\left|\frac{d}{dz}T_2^{n-1}1\right|$$

which implies, taking into account the inequality (3.13), the inequality

$$\sup\left|\frac{d}{dz}T_2^n 1\right| \le \overline{r}'\overline{r}^{n-1} + \overline{r}\sup\left|\frac{d}{dz}T_2^{n-1}1\right|.$$

By iterating this inequality we arrive at the estimate

$$\sup\left|\frac{d}{dz}T_2^n 1\right| \le n\overline{r}'\overline{r}^{n-1}$$

which together with the inequality $\overline{r} < 1$ leads to the uniform convergence of the series $\sum|\frac{d}{dz}T_2^n 1|$. Thus, the differentiability of the function χ is proved, and together with this Theorem 3.1 is proved for $G = 0$.

Part II. In this part of the proof we present a C^3-solution u of the equation

$$(l_1\partial_x + l_2\partial_y)\partial_x\partial_y u = G \qquad \text{in } \ D,$$

where G is an arbitrary C^1-function in $overline D$. Changing x on $l_1 x$ and y on $l_2 y$ and using again the notation u, G and D for the new functions and domain, we reduce our problem to the equation

$$(\partial_x + \partial_y)\partial_x\partial_y u = G.$$

It has an obvious solution

$$u(x,y) = \frac{1}{2}\int_0^{x+y} H\left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) dz,$$

where

$$H(x,y) = \int_0^x \left(\int_0^y G(s,t)dt\right) ds.$$

We are going to show that $u \in C^3(\overline{D})$. It is clear that

$$\partial_x u = \frac{H}{2} + \frac{1}{4}\int_0^{x+y} \Big[H_x\left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) - H_y\left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right)\Big] dz, \tag{3.14}$$

and

$$\partial_x^2 u = \frac{3}{4}H_x - \frac{1}{4}H_y + \frac{1}{8}\int_0^{x+y} [H_{xx} - 2H_{xy} + H_{yy}] \left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) dz, \tag{3.15}$$

where $H_x = \partial_x H,\ H_{xx} = \partial_x^2 H$ etc. Since $G \in C^1(\overline{D})$, we have

$$\partial_x^2 \partial_y H(x,y) \in C(\overline{D}), \qquad \partial_x \partial_y^2 H(x,y) \in C(\overline{D}),$$

and therefore the function

$$\frac{3}{4}H_x - \frac{1}{4}H_y + \frac{1}{8}\int_0^{x+y} (-2H_{xy})\left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) dz$$

belongs to the space $C^1(D)$. Assume for a moment that both functions

$$\Phi_1(x,y) = \int_0^{x+y} H_{xx}\left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) dz,$$

$$\Phi_2(x,y) = \int_0^{x+y} H_{yy}\left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) dz,$$

belong to the space $C^1(\overline{D})$. Then by virtue of (3.15) we find that $\partial_x^2 u \in C^1(\overline{D})$ and consequently $\partial_x^3 u \in C(\overline{D})$. Because of the symmetry between the variables x

and y, we verify in just the same way that $\partial_y^3 u \in C(\overline{D})$, and, finally, $u \in C^3(\overline{D})$. Thus it remains to check that $\Phi_j(x,y) \in C^1(\overline{D})$, $j = 1, 2$. A direct calculation shows that

$$H_{xx} = \left(\frac{z+x-y}{2}, \frac{z-x+y}{2}\right) = \int\limits_0^{\frac{z-x+y}{2}} G_x\left(\frac{z+x-y}{2}, t\right) dt.$$

This means that

$$\Phi_1(x,y) =) = \int\limits_0^{x+y} \left(\int\limits_0^{\frac{z-x+y}{2}} G_x\left(\frac{z+x-y}{2}, t\right) dt \right) dz.$$

Introducing new variables $z + x - y = 2s; t = t$, we get

$$\Phi_1(x,y) =) = 2 \int\limits_{\frac{x-y}{2}}^{x} \left(\int\limits_0^{s-x+y} G_x(s,t) dt \right) ds.$$

In this form the differentiability of the function $\Phi_1(x,y)$ is guaranteed. In just the same way one can check the differentiability of the function $\Phi_2(x,y)$. This proves Theorem 3.1.

References

[1] D. Bourgin and R. Duffin. The Dirichlet problem for the vibrating string equation. *Bull. Amer. Math. Soc.,* 45:851-859, 1939.

[2] D.W. Fox and C. Pucci. The Dirichlet problem for the wave equation. *Ann. Math. Pura Appl.,* 46:155-182, 1958.

[3] J. Hadamard. On the Dirichlet problem for the hyperbolic case. *Proc. Nat. Acad. Sci., U.S.A.*, 28:258-263, 1942

[4] A.Huber. Die erste Randwertaufgabe für geschlossene Bereiche bei der Gleichung $\partial^2 z/\partial x \partial y = f(x,y)$. *Monatsbl. Math. Phys.*, 39:70-100, 1932.

[5] F. John. The Dirichlet problem for a hyperbolic equation. *Amer. J. Math.*, 63:141-154, 1941.

[6] B. Paneah. On a problem in integral geometry connected to the Dirichlet problem for hyperbolic equations. *Internat. Math. Res. Notices*, 5:213-222, 1997.

[7] B.-W. Schulze. Mengen der Kapazität Null fr nicht-elliptische Differentialgleichungen; das Dirichlet Problem fur $u_{xy} = 0$. In Elliptische Differentialgleichungen, volume 8 of *Schriftenreihe der Institute für Mathematik*, pages 217-257. Akademie-Verlag, Berlin, 1971.